Exceptional Times for the Discrete Web and Predictability in Ising Models

by

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Abstract

The dynamical discrete web (DyDW) is a system of one-dimensional coalescing random walks that evolves in an extra dynamical time parameter, τ. At any deterministic τ the paths behave as coalescing simple symmetric random walks. It has been shown by Fontes, Newman, Ravishankar and Schertzer that there exist exceptional dynamical times, τ, at which the path from the origin, S_0^τ, is K-subdiffusive, meaning $S_0^\tau(t) \leq j+K\sqrt{t}$ for all t, where t is the random walk time, and j is some constant. In this thesis we consider for the first time the existence of superdiffusive exceptional times. To be specific, we consider τ such that $\limsup_{t\to\infty} S_0^\tau(t)/\sqrt{t\log(t)} \geq C$. We show that such exceptional times exist for small values of C, but they do not exist for large C. Another goal of this thesis is to establish the existence of exceptional times for which the path from the origin is K-subdiffusive in both directions, i.e., τ such that $|S_0^\tau(t)| \leq j + K\sqrt{t}$ for all t. We also obtain upper and lower bounds for the Hausdorff dimensions of these two-sided subdiffusive exceptional times. For the superdiffusive exceptional times we are able to get a lower bound on Hausdorff dimension but not an upper bound. This thesis concludes with a brief description of recent joint work with Charles Newman and Daniel Stein on dynamical Ising models. We consider Ising models with symmetric i.i.d. initial conditions evolving under zero temperature dynamics. The main goal is to examine the relative importance of the initial conditions versus the dynamics in determining the state of the system at large times.

Contents

List of Figures

Chapter 1

Introduction

This thesis is about random graphical models which evolve in time. A graph, $G = (V, E)$, consists of a set of vertices, V, and a set of edges, E, which are connections between pairs of vertices. Graphs are typically used to represent connections between a group of objects or people. As such, graphs are studied in a variety of fields, including mathematics, physics, computer science and the social sciences. A social network, for example, can be represented as a graph. Each user would correspond to a vertex, and the edges would represent "friendships" between users. We are interested in graphical models which are both random and "dynamical", meaning they evolve in time. Such models are constructed by introducing a time parameter and then randomly updating the model as time increases.

The models considered in this thesis are inspired by physics. Graphs are used in a variety of areas within physics, often to provide a convenient framework for representing small-scale, localized interactions between collections of particles or other discrete systems. Using the tools of probability theory, we can see how relatively simple microscopic interactions can lead to complex macroscopic and statistical properties for the system as a whole. Our main focus will be on the dynamical discrete web, which will be covered in the remainder of this chapter through Chapter 7. We conclude with a short discussion of recent work on dynamical Ising models, see Chapter 8.

We now examine the dynamical discrete web (DyDW), a system of coalescing random walks that evolves in a continuous dynamical time parameter. The dynamical discrete web was introduced by Howitt and Warren in [10]. The DyDW and related systems have been considered as models for erosion and drainage networks (see [14],[2]). We examine "exceptional times" for the DyDW. These are dynamical times at which paths from the DyDW display behavior that would have probability zero for a standard random walk, or for the DyDW observed at a deterministic time.

First we define the dynamical discrete web, and briefly describe our main results.

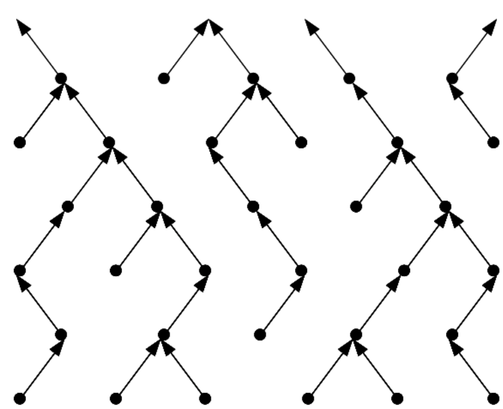

Figure 1.1: A partial realization of the discrete web. Each arrow independently points left or right with probability 1/2. In the dynamical discrete web, each arrow has an independent Poisson clock and resets whenever it rings.

We will follow [3] closely; see Section 1 of that paper for further introductory material.

To discuss the DyDW, we first define the discrete web (DW). The discrete web is a system of coalescing one-dimensional simple symmetric random walks. To construct it, we independently assign to each point in $\mathbb{Z}^2_{\text{even}} := \{(x,t) \in \mathbb{Z}^2 : \ x+t \text{ is even}\}$ a symmetric, ± 1-valued Bernoulli random variable, $\xi_{(x,t)}$. We then draw an arrow from (x,t) to $(x+\xi_{(x,t)}, t+1)$ (see Figure 1.1). For each $(x,t) \in \mathbb{Z}^2_{\text{even}}$, we let $S_{(x,t)}(\cdot)$ be the path that starts at (x,t) and follows the arrows from there. The discrete web is the collection of all such paths for $(x,t) \in \mathbb{Z}^2_{\text{even}}$. Although our use of arrows gives a representation of the DW as a directed graph, since all arrows point forward in time one could replace the arrows by undirected edges with no loss of information. As the figures and the ordering of (x,t) suggest, we let the path time coordinate, t, run vertically, and the space coordinate, x, run horizontally. Future references to left/right or up/down in $\mathbb{Z}^2_{\text{even}}$ should be understood according to this convention.

The DyDW was first introduced by Howitt and Warren in [10]. It is a discrete web that evolves in an extra dynamical time parameter, τ, by letting the arrows independently switch directions as τ increases. To accomplish this, we assign to each $(x,t) \in \mathbb{Z}^2_{\text{even}}$ an independent, rate one Poisson clock. When the clock at (x,t) rings, we reset the arrow at (x,t) by replacing it with a new, independent arrow (which may or may not agree with the previous arrow). Note that this gives the same distribution as if we had forced the arrows to switch at half the rate. These dynamics correspond to replacing the $\xi_{(x,t)}$ from the DW with right-continuous τ-varying versions, $\xi^\tau_{(x,t)}$. We then let $\mathcal{W}(\tau)$ denote the discrete web constructed from the $\xi^\tau_{(x,t)}$'s, and let $S^\tau_{(x,t)}(\cdot)$ denote the path from $\mathcal{W}(\tau)$ starting at (x,t). Note that these dynamics are stationary. This means that at any deterministic τ, $\mathcal{W}(\tau)$ is distributed as a discrete web, and thus all paths in $\mathcal{W}(\tau)$ behave as simple symmetric random walks.

An "exceptional time" refers to a random dynamical time at which the DyDW behaves in a way that would have probability zero for the DW. Any deterministic dynamical time will have probability zero of being an exceptional time, due to the fact that the dynamics are stationary. The same reasoning implies that any countable set of dynamical times which is deterministic or independently random will have probability zero of containing an exceptional time. Furthermore, the set of all such exceptional times will have measure zero, due to Fubini's Theorem. However, as we will see, such exceptional times do exist in the DyDW. This is possible because we assume that our dynamical time parameter is continuous, taking uncountably many values. Considering that the continuity of physical time is debated by physicists (see [9], for example), it is an interesting question what role, if any, such exceptional times might play in physics.

The study of exceptional times for the DyDW has been motivated by earlier work on dynamical percolation, see [8], [15]. Similarly to the DyDW, dynamical percolation consists of a lattice of Bernoulli random variables which reset according to independent Poisson processes. For static (non-dynamical) percolation with critical edge probabilities it is believed that no infinite cluster should exist. This is proven for dimension two and large dimensions (see [7], for example). In [15] it was shown that critical two-dimensional dynamical percolation has exceptional times where this fails, i.e. where an infinite cluster exists. However, no such exceptional times exist for large dimensions, see [8].

Exceptional times for the DyDW were first studied by Fontes, Newman, Ravishankar and Schertzer in [3]. They use techniques similar to those used for dynamical percolation to show that there exist exceptional times for the DyDW. Their paper shows the existence of τ at which the path from the origin, S_0^τ, is subdiffusive in one direction, growing slower than allowed by the classical law of the iterated logarithm. To be specific, they show that for sufficiently large K, j:

$$\mathbb{P}\left(\exists \tau \in [0,1] \text{ s.t. } S_0^\tau(t) \leq j + K\sqrt{t} \text{ for all } t \geq 0\right) > 0. \tag{1.1}$$

In this thesis we will carry out a similar analysis of the following related question: do there also exist exceptional times at which the path from the origin grows *faster* than allowed by the law of iterated logarithm? In this case we say S_0^τ is superdiffusive and call τ a superdiffusive exceptional time. The question of the existence of such superdiffusive exceptional times has not been studied previously. We will show that such exceptional times do in fact exist, and give a bound on how large such superdiffusive paths can get (see Theorems 2 and 4). We are also able to extend the subdiffusive results from [3], showing the existence of exceptional times at which S_0^τ is subdiffusive in both directions, meaning $|S_0^\tau(t)| \leq j + K\sqrt{t}$ for all t.

Now we will state our main results in the order in which they will appear. The subdiffusive results will be presented first. Chapters 2 to 4 are devoted to the proof of:

Theorem 1. *For K, j sufficiently large:*

$$\mathbb{P}\left(\exists \tau \in [0,1] \ s.t. \ |S_0^\tau(t)| \leq j + K\sqrt{t} \ for \ all \ t \geq 0\right) > 0. \tag{1.2}$$

An immediate consequence of this is:

Corollary 1. *For K sufficiently large:*

$$\mathbb{P}\left(\exists \tau \in [0,1] \ s.t. \ \limsup_{t\to\infty} \frac{|S_0^\tau(t)|}{\sqrt{t}} \leq K\right) = 1. \tag{1.3}$$

Our study of superdiffusive exceptional times begins in Chapter 5, where we prove:

Theorem 2. *For $C > 0$ sufficiently small:*

$$\mathbb{P}\left(\exists \tau \in [0,1] \ s.t. \ \limsup_{t\to\infty} \frac{S_0^\tau(t)}{\sqrt{t\log(t)}} \geq C\right) = 1. \tag{1.4}$$

In Chapter 5.1 we sketch a proof of the two-sided analogue of this theorem:

Theorem 3. *For $C > 0$ sufficiently small:*

$$\mathbb{P}\left(\exists \tau \in [0,1] \ s.t. \ \limsup_{t\to\infty} \frac{S_0^\tau(t)}{\sqrt{t\log(t)}} \geq C \ and \ \liminf_{t\to\infty} \frac{S_0^\tau(t)}{\sqrt{t\log(t)}} \leq -C\right) = 1. \tag{1.5}$$

The choice of $[0,1]$ for the interval of dynamical time is arbitrary. The events in (1.3), (1.4) and (1.5) are (almost surely equal to) tail events with respect to the arrow processes. This means that those sets of exceptional times will be a.s. empty or a.s. dense. To see that Theorem 1 still holds for any other choice of interval, first note that the process is stationary in τ so all that matters is the length of the interval. If the probability in (1.2) is zero for a given choice of interval, clearly it must also be zero for any shorter interval. However, any larger interval could be covered by multiple copies of the original interval, each of which would have probability zero of containing an exceptional time. Thus the probability in (1.2) is zero for our choice of interval if and only if it is zero for all non-degenerate intervals.

Theorem 2 is in some sense optimal, in that such exceptional times do not exist for large values of C. In Chapter 6 we will prove:

Theorem 4. *For $C > 0$ sufficiently large:*

$$\mathbb{P}\left(\exists \tau \in [0,1] \ s.t. \ \limsup_{t\to\infty} \frac{S_0^\tau(t)}{\sqrt{t\log(t)}} \geq C\right) = 0. \tag{1.6}$$

In Chapter 7 of the thesis we study the Hausdorff dimensions of these various sets of exceptional times. Chapter 7.1 is devoted to two-sided subdiffusive exceptional times. We look at the sets:

$$\{\tau \in [0,\infty) : \exists j \text{ s.t. } |S_0^\tau(t)| \leq j + K\sqrt{t} \text{ for all } t \geq 0\}, \tag{1.7}$$

$$\{\tau \in [0,\infty) : \limsup_{t\to\infty} |S_0^\tau(t)|/\sqrt{t} \leq K\}, \tag{1.8}$$

and derive upper and lower bounds for their Hausdorff dimensions, as functions of K. Our bounds are analogous to, and motivated by, those from [3] for the one-sided case. As in the one-sided case, the dimensions tend to 1 as K goes to ∞. In other words, the set of all two-sided subdiffusive exceptional times has Hausdorff dimension equal to one. For small K it is known that (1.7) is empty, see Proposition 5.8 of [3]. This implies (1.8) is also empty for small K, see Chapter 7. Our analysis of (1.8) is helped by noting that (1.8) only depends on arrows with arbitrarily large time coordinate (almost surely). This means (1.8) can be analysed using tail events, allowing us to improve the lower bound slightly relative to the methods of [3]. The two sets (1.7) and (1.8) have the same dimensions, except for at most countably many values of K (see Chapter 7 for details). In Chapter 7.2 we look at the sets of superdiffusive exceptional times:

$$\{\tau \in [0,\infty) : \limsup_{t\to\infty} S_0^\tau(t)/\sqrt{t\log(t)} \geq C\}. \tag{1.9}$$

For these sets we are able to get a lower bound on Hausdorff dimension, but we do not have an upper bound at this time. As a consequence of our lower bound we see that the dimension of the superdiffusive exceptional times tends to 1 as C goes to 0, i.e. the set of all superdiffusive exceptional times has dimension one.

Chapter 2

Structure of the Proof of Theorem 1

As in [3], we show that subdiffusivity occurs by showing that a series of "rectangle events" occur. First, we define our rectangles. Take $\gamma > 1$ and let $d_k = 2(\lfloor \frac{\gamma^k}{2} \rfloor + 1)$. Let R_0 be the rectangle with vertices $(-d_0, 0)$, $(+d_0, 0)$, $(-d_0, d_0^2)$ and $(+d_0, d_0^2)$. Given R_k we take R_{k+1} to be the rectangle of width $2d_{k+1}$ and height d_{k+1}^2, that is centered about the t-axis, and stacked on top of R_k (see Figure 2.1). An easy computation shows that the entire stack of rectangles lies between the curves defined by $x = -j - K\sqrt{t}$ and $x = j + K\sqrt{t}$, where j, K depend on γ. For example, we can take $j = 2, K = \gamma$, see Proposition 3 of Chapter 7. Thus if S_0^τ stays within the stack, it will be subdiffusive in both directions.

Let t_k denote the time coordinate of the lower edge of R_k (i.e. $t_k = d_0^2 + d_1^2 + ... + d_{k-1}^2$). For $k \geq 1$, let l_k denote the upper left vertex of R_{k-1} and r_k the upper right vertex of R_{k-1}. We would like to define our rectangle events, B_k^τ, as:

$$\begin{aligned} B_0^\tau :=& \{|S_0^\tau(t)| \le d_0 \ \ \forall t \in [0, t_1]\}, \\ B_k^\tau :=& \{|S_{l_k}^\tau(t)| \le d_k \text{ and } |S_{r_k}^\tau(t)| \le d_k \ \ \forall t \in [t_k, t_{k+1}]\} \ \text{ for } k \ge 1. \end{aligned}$$

Then on the event $\bigcap_{k\ge 0} B_k^\tau$, S_0^τ will stay in the stack of rectangles, and thus be subdiffusive in both directions. This follows from the discussion above, combined with the fact that paths in the discrete web do not cross. Thus if for some γ we can show:

$$\mathbb{P}\left(\exists \tau \in [0,1] \text{ s.t. } \bigcap_{k\ge 0} B_k^\tau(\gamma) \text{ occurs}\right) > 0, \tag{2.1}$$

then Theorem 1 will follow immediately.

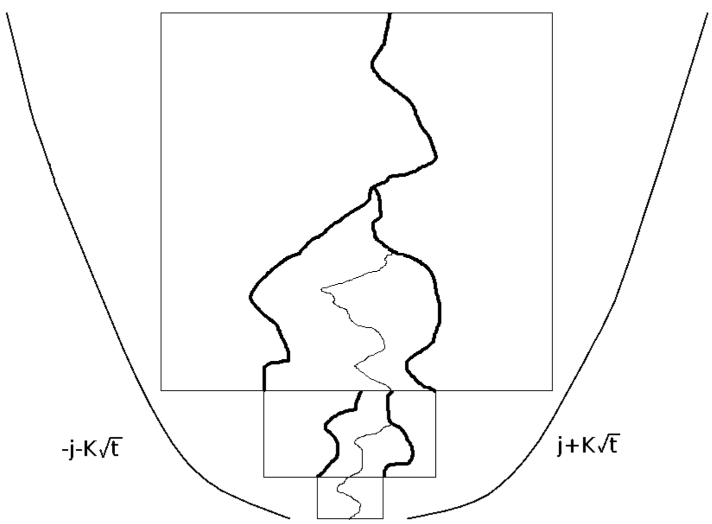

Figure 2.1: Rough sketch of the first three rectangles and paths for which the B_k's occur. The darker paths are the $S_{l_k}^\tau$'s and $S_{r_k}^\tau$'s. The lighter path is S_0^τ.

Remark 1. *Notice that this gives a bound with left-right symmetry. If we wish to study exceptional times where* $-j_L - K_L\sqrt{t} \leq S_0^\tau(t) \leq j_R + K_R\sqrt{t}$*, we can skew our rectangles. This can be accomplished by horizontally scaling the left and right halves of each rectangle by* C_L *and* C_R*, respectively (and rounding out to the nearest point in* $\mathbb{Z}^2_{even}$*). For the sake of simplicity of our arguments (and notation) we will largely ignore the asymmetrical case. However, it should be noted that our results easily extend to the asymmetrical case, using the above construction.*

To prove (2.1), we will need to understand the interaction between pairs of paths from the DyDW. This can be described as a combination of coalescing (if the paths have the same dynamical time) and sticking (if the dynamical times differ). Let S_z^τ be the path from $z = (x,t) \in \mathbb{Z}^2_{\text{even}}$ at dynamical time τ, and let $S_{z'}^{\tau'}$ be the path from $z' = (x', t')$ at dynamical time τ'. The paths will evolve independently until they meet at some time $t^* \geq \text{Max}(t, t')$. If $\tau = \tau'$, the paths coalesce when they meet, otherwise they "stick". To be precise, let $x^* := S_z^\tau(t^*) = S_{z'}^{\tau'}(t^*)$ and let $z^* = (x^*, t^*)(\in \mathbb{Z}^2_{\text{even}})$. Then if the clock at z^* has not rung in $[\tau, \tau']$ (WLOG assume $\tau < \tau'$), the two paths will follow the same arrow on $[t^*, t^*+1]$. We will say the paths are sticking on $[t^*, t^*+1]$. The paths continue to stick until they reach a site whose clock has rung, at which point they follow independent arrows. Note that these independent arrows may agree, but this will not be considered sticking.

To prove Theorem 1, we would like to show (2.1). Unfortunately, we are not able to prove (2.1) directly. The problem arises in the interaction between sticking and coalescing (to be specific, (3.5)-(3.7) fail for B_k^τ, so we are unable to establish (3.9)). To get around this, we construct a larger system where the relevant paths do not coalesce. In addition to the main DyDW, $\mathcal{W}(\tau)$, we will need an independent, secondary DyDW, $\hat{\mathcal{W}}(\tau)$. From now on, all "arrows", "clock rings", etc. should be understood to refer to $\mathcal{W}(\tau)$ (the main DyDW), unless otherwise specified.

Given $S^\tau_{l_k}$ and $S^\tau_{r_k}$ we want to construct non-coalescing versions, $X^\tau_{l_k}$ and $X^\tau_{r_k}$. We accomplish this by letting $X^\tau_{l_k} = S^\tau_{l_k}$, and taking $X^\tau_{r_k}$ to be the path from r_k that follows the arrows (from $\mathcal{W}(\tau)$) unless it meets $X^\tau_{l_k}$. If $X^\tau_{r_k}$ meets $X^\tau_{l_k}$ at space-time $z^* = (x^*, t^*) \in \mathbb{Z}^2_{\text{even}}$, then on $[t^*, t^*+1]$ we let $X^\tau_{r_k}$ follow the arrow at z^* from $\hat{\mathcal{W}}(\tau)$ (at dynamical time τ). At time t^*+1 we repeat this, following $\hat{\mathcal{W}}(\tau)$ if the paths are together, but following $\mathcal{W}(\tau)$ otherwise. Continuing in this manner we get an independent pair of non-coalescing simple symmetric random walks $X^\tau_{l_k}$ and $X^\tau_{r_k}$. Now we define new rectangle events, C^τ_k:

$$\begin{aligned} C^\tau_0 :=& B^\tau_0, \\ C^\tau_k :=& \{|X^\tau_{l_k}(t)| \le d_k \text{ and } |X^\tau_{r_k}(t)| \le d_k \;\; \forall t \in [t_k, t_{k+1}]\} \;\text{ for } k \ge 1. \end{aligned}$$

Notice that C^τ_k implies B^τ_k. This is because the only difference between $X^\tau_{l_k}$,$X^\tau_{r_k}$ and $S^\tau_{l_k}$,$S^\tau_{r_k}$ is the (possible) extension of $X^\tau_{r_k}$ beyond the initial meeting point. So if we can show:

$$\mathbb{P}\left(\exists \tau \in [0,1] \text{ s.t. } \bigcap_{k \ge 0} C^\tau_k \text{ occurs}\right) > 0, \tag{2.2}$$

then (2.1), and thus Theorem 1, will follow immediately. The next two chapters will be devoted to proving (2.2).

Chapter 3

A Decorrelation Bound

Throughout this chapter we assume $\tau, \tau' \in [0,1], \tau < \tau'$ and we fix arbitrary $k \geq 1, \gamma > 1$. We also translate the paths to start at $t = 0$. That is, we set $Y_l^\tau(t) := X_{l_k}^\tau(t_k + t)$ and $Y_r^\tau(t) := X_{r_k}^\tau(t_k + t)$ (k is fixed so we drop it from the notation). We will also consider diffusively rescaled versions of these paths, $\tilde{Y}_l^\tau(t) := Y_l^\tau(td_k^2)/d_k$ and $\tilde{Y}_r^\tau(t) := Y_r^\tau(td_k^2)/d_k$. The relevant "rectangle event" is then:

$$\begin{aligned} C^\tau :=& \{|Y_l^\tau(t)| \leq d_k \text{ and } |Y_r^\tau(t)| \leq d_k \;\; \forall t \in [0, d_k^2]\} \\ =& \{|\tilde{Y}_l^\tau(t)| \leq 1 \text{ and } |\tilde{Y}_r^\tau(t)| \leq 1 \;\; \forall t \in [0,1]\}. \end{aligned}$$

Similarly to [3] we define $\Delta := \dfrac{1}{d_k|\tau - \tau'|}$ (take their $\delta = d_k^{-1}$). As in [3], the key ingredient for the proof of (2.2) is a decorrelation bound for the rectangle events:

Proposition 1. *There exist $c, a \in (0, \infty)$ such that:*

$$\mathbb{P}(C^\tau \cap C^{\tau'}) \leq \mathbb{P}(C^0)^2 + c\,(\Delta)^a \leq \mathbb{P}(C^0)^2 + c\left(\frac{1}{\gamma^k|\tau - \tau'|}\right)^a,$$

with a, c independent of k, τ and τ'.

Note that the second inequality is follows immediately from the definitions of Δ, d_k. The remainder of this chapter is devoted to proving the first inequality, and thus Proposition 1. The structure is similar to the proof of Proposition 3.1 from [3], with a few necessary modifications.

As discussed in the previous chapter, paths from the DyDW at different dynamical times interact by sticking. This sticking leads to dependence between the web paths. Our modified paths (the Y_τ's) have their own version of sticking that is slightly more complicated. To prove Proposition 1 we will prove bounds for the amount of sticking, which will allow us to bound the dependence between the C^τ's. We begin with some notation and definitions.

We call $n \in \mathbb{Z}$ a "sticking time" if a Y^τ-path and a $Y^{\tau'}$-path follow the same arrow at time n. For this to occur, a pair of paths from $Y_l^\tau, Y_l^{\tau'}, Y_r^\tau, Y_r^{\tau'}$ need to be at the same space-time location and follow the arrow from the same web ($\mathcal{W}$ or $\hat{\mathcal{W}}$). In addition, this arrow must not have been updated in $[\tau, \tau']$. This can happen in five ways:

(i) $Y_l^\tau(n) = Y_l^{\tau'}(n)$ no ring in $[\tau, \tau']$,

(ii) $Y_l^\tau(n) = Y_r^{\tau'}(n) \neq Y_l^{\tau'}(n)$ no ring in $[\tau, \tau']$,

(iii) $Y_l^{\tau'}(n) = Y_r^\tau(n) \neq Y_l^\tau(n)$ no ring in $[\tau, \tau']$,

(iv) $Y_r^\tau(n) = Y_r^{\tau'}(n) \neq Y_l^\tau(n), Y_l^{\tau'}(n)$ no ring in $[\tau, \tau']$,

(v) $Y_r^\tau(n) = Y_r^{\tau'}(n) = Y_l^\tau(n) = Y_l^{\tau'}(n)$ no $\hat{\mathcal{W}}$-ring in $[\tau, \tau']$.

We will call (i) an ll(left-left)-sticking time, (ii) an lr-sticking time, (iii) an rl-sticking time, and (iv),(v) will both be rr-sticking times. These names refer to the pair(s) of paths that are sticking at time n.

Given $s \in [0, \infty)$ let n_s be the unique $n \in \mathbb{Z}$ such that $s \in [n, n+1)$. We define:

$$g(s) := \begin{cases} 0 & \text{if } n_s \text{ is a sticking time} \\ 1 & \text{otherwise} \end{cases}$$

$$G(t) := \int_0^t g(s)ds.$$

We will also need:

$$g_{ll}(s) := \begin{cases} 0 & \text{if } n_s \text{ is an } ll\text{-sticking time} \\ 1 & \text{otherwise} \end{cases}$$

$$G_{ll}(t) := \int_0^t g_{ll}(s)ds,$$

and G_{lr}, G_{rl}, G_{rr}, which are defined analogously.

Notice that $t - G(t)$ is the amount of time spent sticking up to time t. So if we make the time change $t \to t - G(t)$ we will include only the steps where a pair of paths from $Y_l^\tau, Y_l^{\tau'}, Y_r^\tau, Y_r^{\tau'}$ are sticking. Similarly if we make the time change $t \to G(t)$ we will include only non-sticking steps. This allows us to decompose the paths as:

$$\begin{aligned} Y_l^\tau(t) =&Y_{l_d}^\tau(G(t)) + Y_{l_s}^\tau(t - G(t)),\\ Y_r^\tau(t) =&Y_{r_d}^\tau(G(t)) + Y_{r_s}^\tau(t - G(t)),\\ Y_l^{\tau'}(t) =&Y_{l_d}^{\tau'}(G(t)) + Y_{l_s}^{\tau'}(t - G(t)),\\ Y_r^{\tau'}(t) =&Y_{r_d}^{\tau'}(G(t)) + Y_{r_s}^{\tau'}(t - G(t)), \end{aligned} \tag{3.1}$$

with $Y_{l_d}^\tau(0) = Y_l^\tau(0) = -d_{k-1}$, $Y_{r_d}^\tau(0) = Y_r^\tau(0) = d_{k-1}$, and $Y_{l_s}^\tau(0) = Y_{r_s}^\tau(0) = 0$ (similarly for τ'). Recall that the Y_{l_d}'s and Y_{r_d}'s include only non-sticking steps of each walk. This means that the τ-paths and the τ'-paths follow different, independent arrows, and thus are independent.

To make the above splitting work for the $\tilde{Y}$'s the appropriate rescaling of G is $\bar{G}(t) := G(td_k^2)/d_k^2$. We then make the time changes $t \to t - \bar{G}(t)$ and $t \to \bar{G}(t)$. We would like a bound for $t - \bar{G}(t)$, the total amount of sticking for the rescaled paths in $[0, t]$. This is given by the following adaptation of Lemma 3.4 from [3]. The original lemma follows from a bound on the expected number of sticking steps combined with the Markov inequality, see [3] for the details.

Lemma 1. *For any* $0 < \beta < 1$

$$\mathbb{P}\left(\sup_{t\in[0,1]}(t - \bar{G}(t)) \geq \Delta^\beta\right) \leq c''\,\Delta^{1-\beta},$$

where $c'' \in (0, \infty)$ *is independent of* k, τ *and* τ'.

Proof. Notice that by definition:

$$t - G(t) \leq \underset{(a)}{(t - G_{ll}(t))} + \underset{(b)}{(t - G_{lr}(t))} + \underset{(c)}{(t - G_{rl}(t))} + \underset{(d)}{(t - G_{rr}(t))}. \tag{3.2}$$

Let $C(t)$ be defined as in [3], i.e. such that $t-C(t)$ is the sticking time for S_0^τ and $S_0^{\tau'}$. Lemma 3.4 from [3] gives a bound for the rescaled sticking time, $t-\bar{C}(t)$, which is identical to the bound for $t-\bar{G}(t)$ in Lemma 1. Our strategy will be to use Lemma 3.4 from [3] to bound the terms on the right side of (3.2). We claim that each of $(a),(b),(c),(d)$ is stochastically bounded by $t-C(t)$ (given random variables X,Y, X is said to stochastically bound Y if $\mathbb{P}(Y>x) \leq \mathbb{P}(X>x)$ for all $x \in \mathbb{R}$). For (a) this is obvious, since $t - G_{ll}(t) \overset{d}{=} t - C(t)$ (equal in distribution). This is because the Y_l's are just translated web paths and the DyDW is invariant under space-time translations. We now concentrate on (d); (b) and (c) can be handled similarly.

We'd like to compare $t-G_{rr}(t)$, the amount of sticking for Y_r^τ and $Y_r^{\tau'}$, to $t-C(t)$, the amount of sticking for S_0^τ and $S_0^{\tau'}$. We'll accomplish this by constructing coupled versions of the two processes. In both cases there are two paths that alternate between identical sticking sections and independent non-sticking sections. To be specific, we take $T_0 = T_0^* := 0$ and for $k \geq 0$ define:

$$\begin{aligned} T_{2k+1} :=& \inf\{k \geq T_{2k} : \text{ The clock at } S_0^\tau(k) = S_0^{\tau'}(k) \text{ rings in } [\tau,\tau']\}, \\ T_{2k+2} :=& \inf\{k > T_{2k+1} : \ S_0^\tau(k) = S_0^{\tau'}(k)\}, \\ \Delta_k :=& T_{2k+1} - T_{2k} \geq 0, \ \Gamma_k := T_{2k+2} - T_{2k+1} \geq 1, \end{aligned}$$

and:

$$\begin{aligned}
T^*_{2k+1} &:= \inf\{k \geq T_{2k}:\ k \text{ is not an } rr\text{-sticking time}\},\\
T^*_{2k+2} &:= \inf\{k > T_{2k+1}:\ Y^\tau_r(k) = Y^{\tau'}_r(k)\},\\
\Delta^*_k &:= T^*_{2k+1} - T^*_{2k} \geq 0,\ \Gamma^*_k := T^*_{2k+2} - T^*_{2k+1} \geq 1.
\end{aligned}$$

Then on $[T^{(*)}_{2k}, T^{(*)}_{2k+1}]$ we have S^τ_0 and $S^{\tau'}_0 (Y^\tau_r$ and $Y^{\tau'}_r)$ sticking for $\Delta^{(*)}_k$ steps, while on $[T^{(*)}_{2k+1}, T^{(*)}_{2k+2}]$ they move independently until meeting at $T^{(*)}_{2k+2}$. Notice that Γ_k and Γ^*_k have the same distribution, they are both excursion times for pairs of independent random walks. So we may take $\Gamma_k = \Gamma^*_k$ for our coupled versions. To compare Δ_k, Δ^*_k, notice that:

$$\mathbb{P}\left(\Delta^{(*)}_k \geq j\right) = \prod_{i=1}^{j} \mathbb{P}\left(\Delta^{(*)}_k \geq i | \Delta^{(*)}_k \geq i-1\right)$$

and:

$$\mathbb{P}\left(\Delta^*_k \geq i | \Delta^*_k \geq i-1\right) \leq \mathbb{P}\left(\Delta_k \geq i | \Delta_k \geq i-1\right) \text{ for all } i \geq 1, \tag{3.3}$$

so:

$$\mathbb{P}\left(\Delta^*_k \geq j\right) \leq \mathbb{P}\left(\Delta_k \geq j\right) \text{ for all } j, k \geq 0. \tag{3.4}$$

To see (3.3), consider that $\mathbb{P}\left(\Delta_k \geq i | \Delta_k \geq i-1\right)$ is just the probability of no clock ring in $[\tau, \tau']$. For Δ^*_k, we have the probability that $Y^\tau_r = Y^{\tau'}_r \neq Y^\tau_l, Y^{\tau'}_l$ and there is no $\mathcal{W}$-ring, or $Y^\tau_r = Y^{\tau'}_r = Y^\tau_l = Y^{\tau'}_l$ and there is no $\hat{\mathcal{W}}$-ring. These are disjoint events and the clocks are independent of the positions of previous arrows, so this is bounded by the probability of no clock ring.

Combining this with the above observations, we can couple Δ_k, Δ_k^* and Γ_k, Γ_k^* such that $\Delta_k^* \leq \Delta_k$ and $\Gamma_k = \Gamma_k^*$. This means that the rr-sticking sections are shorter than the $S_0^\tau, S_0^{\tau'}$ sticking sections, while the independent sections have the same length. This implies $t - G_{rr}(t) \leq t - C(t)$ for the coupled versions, which shows (d) is stochastically bounded by $t - C(t)$. This can be proven for $(b), (c)$ by a nearly identical coupling argument, where the portion of the left/right paths after their first meeting is coupled with $S_0^\tau, S_0^{\tau'}$. So we've shown that $(a), (b), (c), (d)$ are each stochastically bounded by $t - C(t)$. Combining this with (3.2) we get:

$$
\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,1]} (t - \bar{G}(t)) \geq \Delta^\beta\right) &= \mathbb{P}\left(\sup_{t\in[0,d_k^2]} (t - G(t)) \geq d_k^2 \Delta^\beta\right) \\
&\leq 4\mathbb{P}\left(\sup_{t\in[0,d_k^2]} (t - C(t)) \geq d_k^2 \frac{\Delta^\beta}{4}\right) \\
&\quad \text{(using (3.2) and above paragraph)} \\
&= 4\mathbb{P}\left(\sup_{t\in[0,1]} (t - \bar{C}(t)) \geq \frac{\Delta^\beta}{4}\right) \\
&\leq 4\tilde{c}\left(\frac{\Delta}{4^{1/\beta}}\right)^{1-\beta} \quad \text{(by Lemma 3.4 from [3])} \\
&= c''\,\Delta^{1-\beta}.
\end{aligned}
$$

This completes the proof since $\tilde{c}$, and thus c'', is independent of k, τ and τ'. □

Now we define C_d^τ to be the rectangle event for $Y_{l_d}^\tau, Y_{r_d}^\tau$. That is:

$$\begin{aligned} C_d^\tau :=& \{|Y_{l_d}^\tau(t)| \le d_k \text{ and } |Y_{r_d}^\tau(t)| \le d_k \ \ \forall t \in [0, d_k^2]\} \\ =& \{|\tilde{Y}_{l_d}^\tau(t)| \le 1 \text{ and } |\tilde{Y}_{r_d}^\tau(t)| \le 1 \ \ \forall t \in [0,1]\}. \end{aligned}$$

Given $r > 0$ we define the r-approximations of our rectangle events as:

$$\begin{aligned} \{C_{(d)}^\tau + r\} :=& \{|Y_{l_{(d)}}^\tau(t)| \le (1+r)d_k \text{ and } |Y_{r_{(d)}}^\tau(t)| \le (1+r)d_k \ \ \forall t \in [0, d_k^2]\} \\ =& \{|\tilde{Y}_{l_{(d)}}^\tau(t)| \le 1+r \text{ and } |\tilde{Y}_{r_{(d)}}^\tau(t)| \le 1+r \ \ \forall t \in [0,1]\}. \end{aligned}$$

Recall that $Y_{l_d}^\tau, Y_{r_d}^\tau$ are independent of $Y_{l_d}^{\tau'}, Y_{r_d}^{\tau'}$, and therefore:

$$C_d^\tau(\{C_d^\tau + r\}) \text{ is independent of } C_d^{\tau'}(\{C_d^{\tau'} + r\}). \tag{3.5}$$

We also have:

$$(Y_{l_d}^\tau, Y_{r_d}^\tau) \overset{d}{=} (Y_l^\tau, Y_r^\tau), \tag{3.6}$$

since both are just pairs of independent random walks. So:

$$\mathbb{P}(C_d^\tau) = \mathbb{P}(C^\tau) = \mathbb{P}(C^0). \tag{3.7}$$

We will need the following adaptation of Lemma 3.3 from [3]:

Lemma 2. *Given any $\alpha < 1/2$, there is $c' \in (0, \infty)$ independent of Δ, k such that:*

$$\mathbb{P}\left(\{C_d^\tau + \Delta^\alpha\} \setminus C_d^\tau\right) \le c' \Delta^\alpha.$$

Proof.

$$\begin{aligned}\mathbb{P}\left(\{C_d^\tau + \Delta^\alpha\} \setminus C_d^\tau\right) \le\ & \mathbb{P}\left(\inf_{t\in[0,1]} \tilde{Y}_{l_d}^\tau(t) \in [-1-\Delta^\alpha, -1)\right) \\ & + \mathbb{P}\left(\sup_{t\in[0,1]} \tilde{Y}_{l_d}^\tau(t) \in (1, 1+\Delta^\alpha]\right) \\ & + \mathbb{P}\left(\inf_{t\in[0,1]} \tilde{Y}_{r_d}^\tau(t) \in [-1-\Delta^\alpha, -1)\right) \\ & + \mathbb{P}\left(\sup_{t\in[0,1]} \tilde{Y}_{r_d}^\tau(t) \in (1, 1+\Delta^\alpha]\right).\end{aligned}$$

Now each of the four terms on the right is bounded by $c\,\Delta^\alpha$. This follows exactly as in the proof of Lemma 3.3 in [3]. To see this, note that the $\tilde{Y}$'s are simple symmetric random walks started at $\pm d_{k-1}/d_k \in [-1, 1]$, diffusively rescaled by $\delta = d_k^{-1}$. We can thus approximate the $\tilde{Y}$'s by Brownian motion paths (for details see [4] and [3], Lemma 3.3). The result then follows, as the maximum (minimum) process of a Brownian motion has a bounded probability density function. □

The final ingredient for the proof of Proposition 1 is a bound on the modulus of continuity of a random walk. This is given by Lemma 3.5 from [3]:

Lemma 3. *(Lemma 3.5, [3]) Let $S(t)$ be a simple symmetric random walk and define $\tilde{S}(t) := S(t/\delta^2)\delta$. Let $\omega_{\tilde{S}}(\epsilon) := \sup_{s,t\in[0,1],|s-t|<\epsilon} |\tilde{S}(t) - \tilde{S}(s)|$ be the modulus of continuity of $\tilde{S}$. Let $\alpha, \beta \in (0,\infty)$ be such that $\beta/2 > \alpha$. For any $r \ge 0$, there exists c (independent of Δ and δ) such that:*

$$\mathbb{P}\left(\omega_{\tilde{S}}(\Delta^\beta) \ge \frac{\Delta^\alpha}{2}\right) \le c\,\Delta^r.$$

This is a consequence of the Garsia-Rodemich-Rumsey inequality [5]. For a proof see [3].

We may now prove Proposition 1. The remaining steps are nearly identical to the proof of Proposition 3.1 from [3] (see the end of Section 3). We include them for the sake of completeness.

For any $0 < \alpha < 1/2$, we have:

$$\begin{aligned}\mathbb{P}\left(C^\tau \cap C^{\tau'}\right) \leq & \mathbb{P}\left(\{C_d^\tau + \Delta^\alpha\} \cap \{C_d^{\tau'} + \Delta^\alpha\}\right) \\ & + 2\mathbb{P}\left(C^\tau \setminus \{C_d^\tau + \Delta^\alpha\}\right),\end{aligned} \tag{3.8}$$

where we used the equidistribution of $(C^\tau, \{C_d^\tau + \Delta^\alpha\})$ and $(C^{\tau'}, \{C_d^{\tau'} + \Delta^\alpha\})$. Using (3.5)-(3.7) we get:

$$\begin{aligned}\mathbb{P}(\{C_d^\tau + \Delta^\alpha\} \cap \{C_d^{\tau'} + \Delta^\alpha\}) = & \mathbb{P}(\{C_d^\tau + \Delta^\alpha\})\mathbb{P}(\{C_d^{\tau'} + \Delta^\alpha\}) \\ \leq & \mathbb{P}(C_d^\tau)^2 + 2\mathbb{P}(\{C_d^\tau + \Delta^\alpha\} \setminus C_d^\tau) \\ = & \mathbb{P}(C^0)^2 + 2\mathbb{P}(\{C_d^\tau + \Delta^\alpha\} \setminus C_d^\tau).\end{aligned} \tag{3.9}$$

Combined with Lemma 2 this gives:

$$\mathbb{P}(\{C_d^\tau + \Delta^\alpha\} \cap \{C_d^{\tau'} + \Delta^\alpha\}) \leq \mathbb{P}(C^0)^2 + 2c'\Delta^\alpha. \tag{3.10}$$

Now that we have (3.8) and (3.10) we just need $\hat{c}, a'$ such that:

$$\mathbb{P}\left(C^\tau \setminus \{C_d^\tau + \Delta^\alpha\}\right) \leq \hat{c}\,\Delta^{a'}. \tag{3.11}$$

Recall the splitting of the Y^τ's given by (3.1). Analogous considerations for the $\tilde{Y}^\tau$'s gives:

$$\begin{aligned}\tilde{Y}^\tau_l(t) =& \tilde{Y}^\tau_{l_d}(\bar{G}(t)) + \tilde{Y}^\tau_{l_s}(t-\bar{G}(t)) \\ =& \tilde{Y}^\tau_{l_d}(t) + [\tilde{Y}^\tau_{l_d}(\bar{G}(t)) - \tilde{Y}^\tau_{l_d}(t)] + \tilde{Y}^\tau_{l_s}(t-\bar{G}(t)), \qquad (3.12) \\ \tilde{Y}^\tau_r(t) =& \tilde{Y}^\tau_{r_d}(t) + [\tilde{Y}^\tau_{r_d}(\bar{G}(t)) - \tilde{Y}^\tau_{r_d}(t)] + \tilde{Y}^\tau_{r_s}(t-\bar{G}(t)). \qquad (3.13)\end{aligned}$$

Notice that all the $\tilde{Y}$'s appearing in (3.12), (3.13) are simple symmetric random walks rescaled by $\delta = d_k^{-1}$, as in Lemma 3. Also, we've taken $\alpha < 1/2$, so we may choose $0 < \beta < 1$ such that $\beta/2 > \alpha$. Then:

$$\begin{aligned}\mathbb{P}\left(C^\tau \setminus \{C^\tau_d + \Delta^\alpha\}\right) \leq& \mathbb{P}\left(|\tilde{Y}^\tau_l - \tilde{Y}^\tau_{l_d}|_\infty \geq \Delta^\alpha\right) + \mathbb{P}\left(|\tilde{Y}^\tau_r - \tilde{Y}^\tau_{r_d}|_\infty \geq \Delta^\alpha\right) \\ \leq& \mathbb{P}\left(|\tilde{Y}^\tau_{l_d}(\bar{G}(t)) - \tilde{Y}^\tau_{l_d}(t)|_\infty \geq \frac{\Delta^\alpha}{2}\right) \qquad (3.14) \\ &+ \mathbb{P}\left(|\tilde{Y}^\tau_{l_s}(t-\bar{G}(t))|_\infty \geq \frac{\Delta^\alpha}{2}\right) \\ &+ \mathbb{P}\left(|\tilde{Y}^\tau_{r_d}(\bar{G}(t)) - \tilde{Y}^\tau_{r_d}(t)|_\infty \geq \frac{\Delta^\alpha}{2}\right) \\ &+ \mathbb{P}\left(|\tilde{Y}^\tau_{r_s}(t-\bar{G}(t))|_\infty \geq \frac{\Delta^\alpha}{2}\right) \\ \leq& 4\mathbb{P}\left(\omega_{\tilde{S}}(\Delta^\beta) \geq \frac{\Delta^\alpha}{2}\right) + 4\mathbb{P}\left(\sup_{t\in[0,1]}(t-\bar{G}(t)) \geq \Delta^\beta\right) \\ \leq& 4c\,\Delta^r + 4c''\,\Delta^{1-\beta}, \qquad (3.15)\end{aligned}$$

where $|\cdot|_\infty$ denotes the sup norm restricted to $[0,1]$. The last inequality follows from Lemmas 1 and 3. This completes the proof of Proposition 1.

Chapter 4

Proof of Theorem 1

Now that we have Proposition 1 we are almost ready to prove Theorem 1. We'd like to show the existence of exceptional times at which $\bigcap_{k\geq 0} C_k^\tau$ occurs. We just need one more Lemma from [3]:

Lemma 4. *(Lemma 4.3, [3]) There exists* $c \in (0,\infty)$ *such that for* $\tau, \tau' \in [0,1]$, $\forall n \geq 0$*:*

$$\prod_{k=0}^{n} \frac{\mathbb{P}(C_k^\tau \cap C_k^{\tau'})}{\mathbb{P}(C_k)^2} \leq c \frac{1}{|\tau - \tau'|^b},$$

where $C_k := C_k^0$ *and* $b = \log(\sup_k[\mathbb{P}(C_k)^{-1}])/\log\gamma > 0$.

This was established in [3] for a different collection of rectangle events, A_k. The key idea is to split the product at $N_0 = \lfloor -\log(|\tau - \tau'|)/\log(\gamma) \rfloor + 1$. In order to make their proof work for C_k, we just need a, c such that:

$$\mathbb{P}(C_k^\tau \cap C_k^{\tau'}) \leq \mathbb{P}(C_k)^2 + c\left(\frac{1}{\gamma^k|\tau - \tau'|}\right)^a \quad \forall \tau, \tau' \in [0,1], k \geq 0, \tag{4.1}$$

and:

$$\sup_k [\mathbb{P}(C_k)^{-1}]) < \infty. \tag{4.2}$$

(4.1) follows from Proposition 1. To see (4.2), notice that the rectangles R_k grow diffusively, and therefore $\mathbb{P}(C_k) \longrightarrow \mathbb{P}(C_\infty)$, the probability of the corresponding rectangle event for Brownian motion paths. So Lemma 4 follows exactly as in [3]; see [3] for the details.

Theorem 1 now follows as in [3],[15]. We will repeat their arguments for the sake of completeness. The Cauchy-Schwartz inequality and Lemma 4 give, $\forall n \geq 0$:

$$\mathbb{P}\left(\int_0^1 \prod_{k=0}^n \mathbb{1}_{C_k^\tau} d\tau > 0\right) \geq \frac{\left(\mathbb{E}\left[\int_0^1 \prod_{k=0}^n \mathbb{1}_{C_k^\tau} d\tau\right]\right)^2}{\mathbb{E}\left[\left(\int_0^1 \prod_{k=0}^n \mathbb{1}_{C_k^\tau} d\tau\right)^2\right]} \tag{4.3}$$

$$= \left[\int_0^1 \int_0^1 \prod_{k=0}^n \frac{\mathbb{P}(C_k^\tau \cap C_k^{\tau'})}{\mathbb{P}(C_k)^2} d\tau d\tau'\right]^{-1} \tag{4.4}$$

$$\geq c^{-1}\left[\int_0^1 \int_0^1 \frac{1}{|\tau - \tau'|^b} d\tau d\tau'\right]^{-1}, \tag{4.5}$$

where (4.4) comes from the independence of the arrow configurations in different R_k's and the stationarity of $\tau \longrightarrow \mathcal{W}(\tau)$. We would like to show that (4.5) is strictly positive. Lemma 4 gave:

$$b = \log(\sup_k [\mathbb{P}(C_k)^{-1}]) / \log \gamma.$$

Recall that the R_k's, and thus the $\mathbb{P}(C_k)$'s, depend on γ. As γ increases, R_0 remains the same, while for $k \geq 1$, R_k scales diffusively. The size of R_{k-1} relative to R_k also tends to zero, so the starting points of $X^{\tau}_{l_k}, X^{\tau}_{r_k}$ converge to the center of the rectangle when diffusively rescaled. This implies that as γ goes to infinity, $\sup_k[\mathbb{P}(C_k)^{-1}]$ converges to $\max\{\mathbb{P}(C_0)^{-1}, \mathbb{P}(C^*)^{-1}\}$, where C^* is the rectangle event for two independent Brownian motions started in the center. So for γ sufficiently large we have $b < 1$, and thus $|\tau - \tau'|^{-b}$ integrable on $[0,1] \times [0,1]$. (4.3)-(4.5) then imply:

$$\inf_n \mathbb{P}\left(\int_0^1 \prod_{k=0}^{n} \mathbb{1}_{C^{\tau}_k} d\tau > 0 \right) \geq p > 0. \tag{4.6}$$

Letting $E_n := \{\tau \in [0,1] : \bigcap_{k=0}^{n} C^{\tau}_k \text{ occurs}\}$, (4.6) then implies $\mathbb{P}(\bigcap_{n=0}^{\infty}\{E_n \neq \varnothing\}) \geq p > 0$. Notice that the E_n are decreasing in n. So if the E_n were closed, this would imply $\mathbb{P}((\bigcap_{n=0}^{\infty} E_n) \neq \varnothing) \geq p > 0$ and (2.2), and thus Theorem 1, would follow.

Unfortunately, the E_n are not closed. This is handled as in [3],[8](Lemma 3.2) by noting that the E_n are nested collections of intervals, and their endpoints must be switching times for some arrow in $\mathcal{W}$ or $\hat{\mathcal{W}}$. There are only countably many switching times, and the locations of an arrow's switching times are independent of the configuration of the rest of the arrows. This means that $\mathcal{W}(\tau)$ and $\hat{\mathcal{W}}(\tau)$ are distributed as discrete webs for all switching times, τ. This will imply that, almost surely:

$$\bigcap_{n=0}^{\infty} E_n = \bigcap_{n=0}^{\infty} \bar{E}_n. \tag{4.7}$$

To see this, assume (4.7) is not true, i.e., that with positive probability there exists a τ that is in the right set but not the left. Then for some m, τ is in $\bar{E}_m$, but not E_m. This implies that τ is the right endpoint to an interval from E_m (by right continuity), and thus must be a switching time for exactly one arrow, ξ_*^τ, from $\mathcal{W}$ or $\hat{\mathcal{W}}$. As discussed above, this means that $\mathcal{W}(\tau)$ and $\hat{\mathcal{W}}(\tau)$ are distributed as discrete webs. Now, since the E_n are nested and τ is in $\bar{E}_n$ for all n, τ must also be a right endpoint to an interval from E_k, for all $k \geq m$. This means that for all n, either $\bigcap_{k=0}^{n} C_k^\tau$ occurs, or there is an $\epsilon > 0$ such that $\bigcap_{k=0}^{n} C_k^{\tau'}$ occurs for $\tau' \in [\tau - \epsilon, \tau)$. In the second case, $\bigcap_{k=0}^{n} C_k^\tau$ ceases to occur only due to the resetting of ξ_*^τ (since no other arrow could switch at the same time). This means that by switching the value of ξ_*^τ we will cause $\bigcap_{k=0}^{\infty} C_k^\tau$ to occur. However, $\mathcal{W}(\tau)$ and $\hat{\mathcal{W}}(\tau)$ are distributed as discrete webs, and switching a single arrow cannot cause a probability zero event to occur. This is a contradiction, and it came from the assumption that (4.7) was not true. So we've proven (4.7) and Theorem 1 then follows from the discussion in the previous paragraph.

Chapter 5

Proof of Theorem 2

In this chapter we prove Theorem 2, which states that for $C > 0$ sufficiently small we have:

$$\mathbb{P}\left(\exists \tau \in [0,1] \text{ s.t. } \limsup_{t\to\infty} \frac{S_0^\tau(t)}{\sqrt{t\log(t)}} \geq C\right) = 1.$$

We begin by defining new rectangle events, $\hat{A}_k^\tau$. The new rectangles, $\hat{R}_k$, will be similar to the R_k defined in Chapter 2 (see Figure 2.1), but they will be wider and grow faster in k. We will let $\gamma > 1$ and define:

$$\hat{d}_k = 2\left(\left\lfloor \frac{\gamma^{\gamma^k}}{2} \right\rfloor + 1\right).$$

It is possible to prove Theorem 2 using the same d_k from Chapter 2. However, the faster growth of the $\hat{d}_k$ is necessary for the results of Chapters 5.1 and 7.2.

Now let $C > 0$ and introduce $\hat{w}_k = 2\left(\left\lfloor C\sqrt{\log(\hat{d}_k^2)\hat{d}_k^2}/2 \right\rfloor + 1\right)$. Take $\hat{R}_0$ to be the rectangle with vertices $(\hat{w}_0, 0)$, $(-\hat{w}_0, 0)$, $(\hat{w}_0, \hat{d}_0^2)$ and $(-\hat{w}_0, \hat{d}_0^2)$. $\hat{R}_{k+1}$ will be the rectangle of width $2\hat{w}_{k+1}$ and height $\hat{d}_{k+1}^2$, stacked on top of $\hat{R}_k$ and centered about the t-axis. Let $\hat{l}_k, \hat{r}_k$ be the upper left and upper right corners of $\hat{R}_{k-1}$. $\hat{t}_k$ will be the time coordinate of the lower edge of $\hat{R}_k$. Then for $k \geq 1$ we define:

$$\hat{A}_k^\tau := \{S_{\hat{l}_k}^\tau(\hat{t}_{k+1}) > \hat{w}_k\}.$$

Notice that on $\hat{A}_k^\tau$ we must have either:

$$S_0^\tau(\hat{t}_k) < -\hat{w}_{k-1} \text{ or } S_0^\tau(\hat{t}_{k+1}) > \hat{w}_k.$$

Now $\hat{w}_k \geq C_k' \sqrt{\log(\hat{t}_{k+1})\hat{t}_{k+1}}$ with $C_k' < C$. Furthermore, $\hat{t}_{k+1}/\hat{d}_k^2 \to 1$ when $k \to \infty$, so we may choose C_k' such that $C_k' \to C$. So for a given τ, if $\hat{A}_k^\tau$ occurs for infinitely many k this will imply:

$$\limsup_{t\to\infty} \frac{S_0^\tau(t)}{\sqrt{t\log(t)}} \geq C \text{ or } \liminf_{t\to\infty} \frac{S_0^\tau(t)}{\sqrt{t\log(t)}} \leq -C. \tag{5.1}$$

Using symmetry we could then say that both types of exceptional times must in fact exist. So our strategy for proving the existence of superdiffusive exceptional times will be to show that for C small, we have:

$$\mathbb{P}\left(\exists \tau \in [0,1] \text{ s.t. } \hat{A}_k^\tau \text{ occurs infinitely often}\right) = 1. \tag{5.2}$$

To begin, we define $\hat{E}_k := \{\tau \text{ s.t. } \hat{A}_k^\tau\}$ and examine $\mathbb{P}(\hat{E}_k \cap [a,b] \neq \varnothing)$.

Proposition 2. *For C sufficiently small, there exists $\mathcal{K}_C : (0,\infty) \to (0,1)$ such that for all $a \neq b$ in $[0,\infty)$ we have:*

$$\mathbb{P}(\hat{E}_k \cap [a,b] \neq \varnothing) \geq \mathcal{K}_C(|a-b|) > 0 \tag{5.3}$$

for all k.

This Proposition implies Theorem 2. Notice that the $\hat{E}_k$ are independent, and each $\hat{E}_k$ is a disjoint union of half open intervals. Let $[a_0, b_0] = [0,1]$. Proposition 2 implies $\mathbb{P}(\hat{E}_k \cap [a_0,b_0] \neq \varnothing) \geq \mathcal{K}_C(1) > 0$ for all $k > 0$. So, almost surely we will have some k_1 such that $\hat{E}_{k_1} \cap [a_0,b_0] \neq \varnothing$ and we can choose $[a_1,b_1] \subset \hat{E}_{k_1} \cap [a_0,b_0]$. We then have $\mathbb{P}(\hat{E}_k \cap [a_1,b_1] \neq \varnothing) \geq \mathcal{K}_C(|a_1 - b_1|) > 0$ for all $k > k_1$. Then for some $k_2 > k_1$, we'll have $[a_2,b_2] \subset \hat{E}_{k_2} \cap [a_1,b_1]$. Continuing in this manner we get a nested sequence of non-empty closed intervals, $\{[a_k,b_k]\}_{k\geq 0}$. So $\bigcap_{k\geq 0}[a_k,b_k] \neq \varnothing$ almost surely, and for $\tau \in \bigcap_{k\geq 0}[a_k,b_k]$ we know that $\hat{A}_k^\tau$ occurs for infinitely many k. This proves (5.2), and thus Theorem 2.

Now let $\hat{\Delta} := \dfrac{1}{\hat{d}_k|\tau - \tau'|}$. To prove Proposition 2 we need the following decorrelation bound, which is the natural analog of Proposition 3.1 from [3], and Proposition 1 above. The decorrelation bound is essentially the same as in the subdiffusive case, but we will need to use it in a different way to account for the fact that $\mathbb{P}(\hat{A}_k^\tau) \to 0$.

Lemma 5. *There exist $c', a' \in (0,\infty)$ such that:*

$$\mathbb{P}(\hat{A}_k^\tau \cap \hat{A}_k^{\tau'}) \leq \mathbb{P}(\hat{A}_k^0)^2 + c'\left(\hat{\Delta}\right)^{a'} \leq \mathbb{P}(\hat{A}_k^0)^2 + c'\left(\frac{1}{\gamma^{\gamma^k}|\tau - \tau'|}\right)^{a'},$$

with a', c' independent of k, τ and τ'.

Lemma 5 follows from the same arguments used to establish Proposition 3.1 in [3]. We presented a modified version of these arguments in the proof of Proposition 1 above. The $\hat{A}_k^\tau$ in Lemma 5 only depend on one path in each rectangle, which means we don't have many of the difficulties encountered in Chapters 2 and 3 of this thesis. In fact, the original proof from [3] goes through without any significant modification. We will not repeat the proof here, just outline the main steps. For more details see Section 3 of [3]. To prove Lemma 5, we need analogues of Lemmas 3.2-3.5 from [3]. Lemma 3.5 is a general result about random walks, and Lemmas 3.2 and 3.4 are just statements about sticking between pairs of paths in the DyDW (see the discussion following (2.1) for an explanation of "sticking" in the DyDW). Since these three Lemmas are not specific to the rectangle events they consider, they can be used in their orignal form. Lemma 3.3 is specific to their rectangle events, but the proof only relies on approximating the random walk by a Brownian motion and the boundedness of the density of the normal distribution. Combining these lemmas as in [3] gives the result. A modified version of this argument was presented above, see the end of Chapter 3.

Now we prove Proposition 2. Using the Cauchy-Schwartz inequality as in (4.3)-(4.4), we have:

$$\begin{aligned}\mathbb{P}(\hat{E}_k \cap [a,b] \neq \varnothing) &\geq \mathbb{P}\left(\int_a^b \mathbb{1}_{\hat{A}_k^\tau} d\tau > 0\right) \\ &\geq (b-a)^2 \left[\int_a^b \int_a^b \frac{\mathbb{P}(\hat{A}_k^\tau \cap \hat{A}_k^{\tau'})}{\mathbb{P}(\hat{A}_k)^2} d\tau d\tau'\right]^{-1} \qquad (5.4)\end{aligned}$$

So we need a bound on $\int_a^b \int_a^b \mathbb{P}(\hat{A}_k^\tau \cap \hat{A}_k^{\tau'})/\mathbb{P}(\hat{A}_k)^2 d\tau d\tau'$ that is uniform in k. We split the integral in to two parts:

(i) $\{\tau,\tau' \in (a,b)\times(a,b) : \gamma^{\gamma^{k-1}}|\tau-\tau'| \le 1\}$

(ii) $\{\tau,\tau' \in (a,b)\times(a,b) : \gamma^{\gamma^{k-1}}|\tau-\tau'| > 1\}$

On (i) we use the bound:

$$\frac{\mathbb{P}(\hat{A}_k^\tau \cap \hat{A}_k^{\tau'})}{\mathbb{P}(\hat{A}_k)^2} \le \frac{1}{\mathbb{P}(\hat{A}_k)}$$

to get:

$$\int\int_{(i)} \frac{\mathbb{P}(\hat{A}_k^\tau \cap \hat{A}_k^{\tau'})}{\mathbb{P}(\hat{A}_k)^2} d\tau d\tau' \le \frac{1}{\mathbb{P}(\hat{A}_k)} \frac{2(b-a)}{\gamma^{\gamma^{k-1}}}. \tag{5.5}$$

On (ii) we use Lemma 5 and $\gamma^{\gamma^{k-1}}|\tau-\tau'| > 1$ to get:

$$\frac{\mathbb{P}(\hat{A}_k^\tau \cap \hat{A}_k^{\tau'})}{\mathbb{P}(\hat{A}_k)^2} \le 1 + \frac{c'}{\gamma^{\gamma^k a'}|\tau-\tau'|^{a'}\mathbb{P}(\hat{A}_k)^2} \le 1 + \frac{c'}{\gamma^{(\gamma^k-\gamma^{k-1})a'}\mathbb{P}(\hat{A}_k)^2}.$$

So:

$$\int\int_{(ii)} \frac{\mathbb{P}(\hat{A}_k^\tau \cap \hat{A}_k^{\tau'})}{\mathbb{P}(\hat{A}_k)^2} d\tau d\tau' \le (b-a)^2 \left[1 + \frac{c'}{\gamma^{(\gamma^k-\gamma^{k-1})a'}\mathbb{P}(\hat{A}_k)^2}\right]. \tag{5.6}$$

Now, $\mathbb{P}(\hat{A}_k)$ is the probability that a random walk started at $-\hat{w}_{k-1}$ exceeds $\hat{w}_k$ after $\hat{d}_k^2$ steps. If we let $\epsilon_k = \hat{w}_{k-1}/\hat{w}_k$ then this is the same as the probability that a random walk started at 0 exceeds $(1+\epsilon_k)\hat{w}_k$ after $\hat{d}_k^2$ steps. We will bound this probability by arguing that the random walk is closely approximated by a Brownian motion and then using standard bounds on the tail of a normal distribution. To accomplish this we'll use the main result of [4]. For simplicity of notation we'll consider simple symmetric random walks, $S(t)$, started at $x = 0$, $t = 0$. $S_\delta(t)$ will denote the diffusive rescaling of such a path by δ, i.e. $S_\delta(t) = S(t/\delta^2)\delta$. The main theorem in [4] says that there exists a Brownian motion, $B(t)$, and a sequence of rescaled random walks, $\{S_\delta(t)\}_{\delta>0}$, such that for any $\alpha < 1/2$ $\mathbb{P}(|S_\delta - B|_\infty > \delta^\alpha)$ decays faster than any power of δ (where the $|\cdot|_\infty$ norm is restricted to $[0,1]$). Then by taking $\delta = 1/\hat{d}_k$ and $\alpha = 1/3$ we have:

$$\begin{aligned}
\mathbb{P}(\hat{A}_k) =& \mathbb{P}\left(S(\hat{d}_k^2) > (1+\epsilon_k)\hat{w}_k\right) \\
\geq& \mathbb{P}\left(S_{1/\hat{d}_k}(1) > (1+\epsilon_k)C\left(\sqrt{\log(\hat{d}_k^2)} + c/\hat{d}_k\right)\right) \\
\geq& \mathbb{P}\left(B(1) > (1+\epsilon_k)C\left(\sqrt{\log(\hat{d}_k^2)} + c/\hat{d}_k\right) + (1/\hat{d}_k)^{1/3}\right) \qquad (5.7) \\
&- \mathbb{P}\left(|S_{1/\hat{d}_k} - B|_\infty > (1/\hat{d}_k)^{1/3}\right). \qquad (5.8)
\end{aligned}$$

Absorb the $1/\hat{d}_k$ terms from (5.7) in to ϵ_k, and use the Theorem from [4] to bound (5.8) by $(1/\hat{d}_k)^{1000C^2}$. This gives, for k sufficiently large:

$$\begin{aligned}
\mathbb{P}(\hat{A}_k) \geq& \frac{K}{\sqrt{\log(\hat{d}_k^2)}} \exp\left[-\left((1+\epsilon_k')C\sqrt{\log(\hat{d}_k^2)}\right)^2/2\right] - (1/\hat{d}_k)^{1000C^2} \\
\geq& \frac{K'}{\sqrt{\log(\gamma^{2\gamma^k})}}(\gamma^{\gamma^k})^{-(1+\epsilon_k')^2C^2} \\
\geq& K''(\gamma^{\gamma^k})^{-(1+\epsilon_k'')^2C^2} \qquad (5.9)
\end{aligned}$$

Now $\epsilon_k'' \to 0$, so for C sufficiently small (5.5)$\to 0$ and (5.6)$\to (a-b)^2$ as $k \to \infty$. This gives the bound needed in (5.4), which completes the proof of Theorem 2. Notice that we have in fact proven a stronger result than needed, $\mathbb{P}(\hat{E}_k \cap [a,b] \neq \varnothing) \to 1$. This is not surprising, since the set of exceptional times will be dense, as discussed in the introduction.

5.1 Two-Sided Superdiffusivity

As in the subdiffusive case, one can obtain a two-sided version of this result. Theorem 3 states that for $C > 0$ sufficiently small,

$$\mathbb{P}\left(\exists \tau \in [0,1] \text{ s.t. } \limsup_{t\to\infty} \frac{S_0^\tau(t)}{\sqrt{t\log(t)}} \geq C \text{ and } \liminf_{t\to\infty} \frac{S_0^\tau(t)}{\sqrt{t\log(t)}} \leq -C\right) = 1.$$

This follows from a straightforward extension of the results above, combining the reasoning from Chapters 2 and 3 with the bounds in this chapter. We just need to consider rectangle events with two paths in each rectangle that trap the path from the origin in to a zig-zag pattern. This is illustrated in figure 5.1.

A full proof would involve repeating nearly all the arguments of Chapters 2-5 for a new set of rectangle events. Instead we give only a quick sketch of the proof and leave the details to the reader. Use the same definitions for $\hat{R}_k, \hat{t}_k, \hat{l}_k, \hat{r}_k$ as above and let $\hat{l}_k^*, \hat{r}_k^*$ denote the lower left and lower right corners of $\hat{R}_k$, respectively. Then let

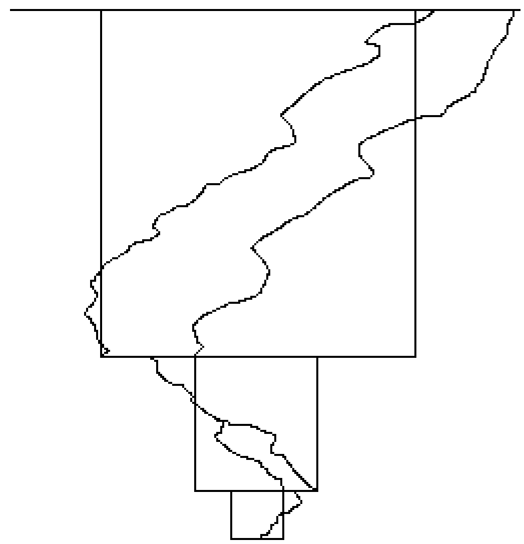

Figure 5.1: Rough sketch of the first three rectangles and paths for which the $\hat{B}_k^\tau$ occur.

$\hat{B}_0^\tau = S_0^\tau(\hat{t}_1) \in [\hat{w}_0, \hat{w}_1]$, and for $k \geq 1$:

$$\begin{aligned}\hat{B}_{2k-1} =& \{S_{\hat{r}_k}^\tau(\hat{t}_{2k}) \in [-\hat{w}_{2k}, -\hat{w}_{2k-1}], S_{\hat{r}_k^*}^\tau(\hat{t}_{2k}) \in [-\hat{w}_{2k}, -\hat{w}_{2k-1}]\} \\ \hat{B}_{2k} =& \{S_{\hat{l}_k}^\tau(\hat{t}_{2k+1}) \in [\hat{w}_{2k}, \hat{w}_{2k+1}], S_{\hat{l}_k^*}^\tau(\hat{t}_{2k+1}) \in [\hat{w}_{2k}, \hat{w}_{2k+1}]\}\end{aligned}$$

One can check that if $\bigcap_{k=0}^\infty \hat{B}_k^\tau$ occurs then S_0^τ will be superdiffusive in both directions, see figure 5.1.

To handle two paths in each rectangle, we then consider analogous rectangle events, $\hat{C}_k^\tau$, for a larger system in which paths do not coalesce, as in Chapter 2. Using the techniques from Chapters 2 and 3 we can get a decorrelation bound analogous to Lemma 5 above. For C sufficiently small we can then get an integrable bound on:

$$\prod_{k=0}^{n} \frac{\mathbb{P}(\hat{C}_k^{\tau} \cap \hat{C}_k^{\tau'})}{\mathbb{P}(\hat{C}_k)^2}$$

using a similar strategy as in Chapter 7.2. An application of the Cauchy-Schwartz inequality then implies the existence of two-sided superdiffusive exceptional times, see Chapter 4 for a similar argument.

Chapter 6

Proof of Theorem 4

In this chapter we prove Theorem 4, which says for $C > 0$ sufficiently large we have:

$$\mathbb{P}\left(\exists \tau \in [0,1] \text{ s.t. } \limsup_{t \to \infty} \frac{S_0^\tau(t)}{\sqrt{t \log(t)}} \geq C\right) = 0.$$

This is a natural counterpoint to Theorem 2, showing that there do not exist exceptional times where the paths are substantially larger. The proof is inspired by [15], Theorem 8.1. Let $\gamma > 1$ and $d_k = 2(\lfloor \frac{\gamma^k}{2} \rfloor + 1)$, $t_k = d_0^2 + d_1^2 + ... + d_{k-1}^2$ as in Chapter 2, and let $w_k = 2(\lfloor \alpha \sqrt{\log(d_k^2) d_k^2}/2 \rfloor + 1)$. We do not use the $\hat{d}_k, \hat{w}_k$ from Chapter 5 because we need t_k and t_{k+1} to be of the same order for the following arguments. We begin by introducing events Υ_k^τ:

$$\Upsilon_k^\tau := \left\{ \sup_{t \in [t_k, t_{k+1}]} S_0^\tau(t) - S_0^\tau(t_k) \geq w_k \right\}.$$

Suppose that Υ_k^τ occurs for only finitely many k. It then follows easily that we must have:

$$\limsup_{t\to\infty} \frac{S_0^\tau(t)}{\sqrt{t\log(t)}} \le C, \tag{6.1}$$

for some $C > \alpha$. So if for some α we can show $\mathbb{P}\left(\exists\tau \in [0,1] \text{ s.t. } \Upsilon_k^\tau\right)$ is summable in k we will have:

$$\mathbb{P}\left(\exists\tau \in [0,1] \text{ s.t. } \Upsilon_k^\tau \text{ occurs i.o.}\right) = 0. \tag{6.2}$$

Theorem 4 then follows from (6.1), (6.2). All that is left is to show is:

Claim 1. *For α sufficiently large $\mathbb{P}\left(\exists\tau \in [0,1] \text{ s.t. } \Upsilon_k^\tau\right)$ is summable in k.*

Proof. Consider the set of τ such that Υ_k^τ occurs. It is a union of disjoint, half-open intervals, let υ_k denote the set of its endpoints in $[0,1]$. We have:

$$\mathbb{P}\left(\exists\tau \in [0,1] \text{ s.t. } \Upsilon_k^\tau\right) \le \mathbb{P}\left(\forall\tau \in [0,1]\ \Upsilon_k^\tau \text{ occurs}\right) + \mathbb{P}\left(\upsilon_k \ne \varnothing\right). \tag{6.3}$$

Now,

$$\mathbb{P}\left(\forall\tau \in [0,1]\ \Upsilon_k^\tau \text{ occurs}\right) \le \mathbb{P}\left(\Upsilon_k^0\right).$$

We can bound $\mathbb{P}\left(\Upsilon_k^0\right)$ by approximating the random walk by a Brownian motion. A similar argument is carried out above to bound $\mathbb{P}(\hat{A}_k)$, see the end of Chapter 5.

This gives:

$$\mathbb{P}\left(\Upsilon_k^0\right) \le C_1 \exp\left(-\left(\alpha\sqrt{\log(d_k^2)}\right)^2/2\right)$$
$$\le C_2 d_k^{-\alpha^2}$$

which is summable since $d_k \sim \gamma^k$. So we just need to bound the second term on the right side of (6.3). Let Ω_k denote the sites of $\mathbb{Z}^2_{\text{even}}$ on which Υ_k^τ depends. Notice that $|\Omega_k| \le C_3 d_k^4$, since the path can cover a spatial area of at most d_k^2 steps during the d_k^2 time steps in $[t_k, t_{k+1}]$. Each endpoint in υ_k comes from an arrow switching at some $x \in \Omega_k$. So $\upsilon_k = \bigcup_{x\in\Omega_k} \upsilon_k(x)$, where $\upsilon_k(x)$ denotes the endpoints arising from arrow switches at x. Then:

$$\begin{aligned}
\mathbb{P}\left(\upsilon_k \neq \varnothing\right) &\le \mathbb{E}(|\upsilon_k|) \\
&= \sum_{x\in\Omega_k} \mathbb{E}(|\upsilon_k(x)|) \\
&= \sum_{x\in\Omega_k} \mathbb{E}(\text{Number of arrow switches at } x)\mathbb{P}(x \text{ is pivotal for } \Upsilon_k^\tau) \\
&\le C_4|\Omega_k|\mathbb{P}(\Upsilon_k) \\
&\le C_5 d_k^4 d_k^{-\alpha^2}.
\end{aligned}$$

The τ in the third line is a switching time for x, it is omitted in the fourth line because τ does not affect the probability. For α sufficiently large $d_k^{4-\alpha^2}$ will be summable, so the proof of the claim, and thus Theorem 4, is complete. □

Chapter 7

Hausdorff Dimensions of the Sets of Exceptional Times

In this chapter we look at the sets of exceptional times and examine their Hausdorff dimensions. In Chapter 7.1 we extend the Hausdorff dimension bounds from [3] to the sets of two-sided subdiffusive exceptional times and examine the relationship between the dimensions of various related sets of exceptional times. In Chapter 7.2 we discuss the sets of superdiffusive exceptional times. We are able to get a lower bound on the Hausdorff dimension of these sets using similar techniques as in the subdiffusive case. However, we are not able to get an upper bound at this time.

7.1 Two-Sided Subdiffusive Exceptional Times

First we look at subdiffusive exceptional times. We've shown the existence of exceptional times at which $|S_0^\tau(t)|$ remains bounded by $j + K\sqrt{t}$ and exceptional times at which $\limsup_{t\to\infty} |S_0^\tau(t)|/\sqrt{t} \le K$. The strategy was to show that S_0^τ remained within a stack of diffusively growing rectangles. The size of the rectangles, and thus the values of K, j in our bounds, were determined by a parameter, γ. This next proposition attempts to capture the relationships between K, j and γ.

Proposition 3. *Let $\sigma_\gamma(t)$ denote the right edge of $\bigcup_{k\ge 0} R_k(\gamma)$. We have:*

$$\sigma_\gamma(t) \le 2 + \gamma\sqrt{t} \quad \text{for all } t \ge 0, \tag{7.1}$$

$$\limsup_{t\to\infty} \frac{\sigma_\gamma(t)}{\sqrt{t}} \le \sqrt{\gamma^2 - 1}. \tag{7.2}$$

Proof. Recall that $d_k = 2(\lfloor \frac{\gamma^k}{2} \rfloor + 1)$, so $\gamma^k \le d_k \le \gamma^k + 2$. This gives:

$$t_k = d_0^2 + d_1^2 + ... + d_{k-1}^2 \ge \gamma^0 + \gamma^2 + ... + \gamma^{2(k-1)} = \frac{\gamma^{2k} - 1}{\gamma^2 - 1}.$$

Now, for $t_k \le t < t_{k+1}$, $k \ge 1$ we have:

$$\begin{aligned} \sigma_\gamma(t) =& d_k \le \gamma^k + 2 \\ \le& \gamma^k \left(\frac{\gamma^{2k} - 1}{\gamma^2 - 1} \right)^{-\frac{1}{2}} \sqrt{t} + 2 = \sqrt{\frac{\gamma^2 - 1}{1 - \gamma^{-2k}}} \sqrt{t} + 2, \end{aligned} \tag{7.3}$$

and (7.2) follows immediately. To see (7.1), notice that for $t < t_1$ we have $\sigma_\gamma(t) = d_0 = 2 \le 2 + \gamma\sqrt{t}$. For $t \ge t_1$, we see $\sqrt{(\gamma^2 - 1)/(1 - \gamma^{-2k})} \le \gamma$ when $k \ge 1$, so (7.1) follows from (7.3). □

Now we'd like to consider various sets of exceptional times. For non-negative $j \in \mathbb{Z}$, we define:

$$\begin{aligned}
T_j^{\pm}(K) :=& \{\tau \in [0,\infty) : |S_0^{\tau}(t)| \leq j + K\sqrt{t} \quad \forall t\}, \; T_{\infty}^{\pm}(K) := \bigcup_{j\geq 0} T_j^{\pm}(K), \\
T_j^{+}(K) :=& \{\tau \in [0,\infty) : S_0^{\tau}(t) \leq j + K\sqrt{t} \quad \forall t\}, \quad T_{\infty}^{+}(K) := \bigcup_{j\geq 0} T_j^{+}(K), \\
T_j^{-}(K) :=& \{\tau \in [0,\infty) : S_0^{\tau}(t) \geq -j - K\sqrt{t} \quad \forall t\}, \; T_{\infty}^{-}(K) := \bigcup_{j\geq 0} T_j^{-}(K).
\end{aligned}$$

We are interested in the Hausdorff dimensions of these sets. As in [3],[8], ergodicity of the DyDW in τ implies the dimension of any set of exceptional times will be almost surely constant. Future discussions of such dimensions should be understood to refer to this constant, and thus may only hold almost surely. In [3](Proposition 5.2) it was shown that in the one-sided case, the Hausdorff dimensions of $T_j^{+}(K)$ and $T_j^{-}(K)$ do not depend on $j \geq 0$. So:

$$\begin{aligned}
\dim_H(T_0^{+}(K)) &= \dim_H(T_{\infty}^{+}(K))(= \dim_H(T_j^{+}(K)) \text{ for all } j), \\
\dim_H(T_0^{-}(K)) &= \dim_H(T_{\infty}^{-}(K))(= \dim_H(T_j^{-}(K)) \text{ for all } j).
\end{aligned}$$

Modifying their argument, we obtain:

Proposition 4. *For $T_j^{\pm}(K)$ as defined above, we have:*

$$\sup_{K'<K} \dim_H(T_{\infty}^{\pm}(K')) \leq \dim_H(T_1^{\pm}(K)) \leq \dim_H(T_{\infty}^{\pm}(K)) \leq \inf_{K''>K} \dim_H(T_1^{\pm}(K'')).$$

The reason we take $j = 1$ instead of $j = 0$ is to prevent the first step of the walk from pushing $|S_0^\tau|$ past $j + K\sqrt{t}$ when $K < 1$. If we are only interested in $K \geq 1$ we can take $j = 0$ and obtain analogous bounds involving $T_0^\pm(K)$. Notice that $T_1^\pm(K)$ and $T_\infty^\pm(K)$ are increasing functions of K, and thus must be continuous for all but countably many K. So for all but countably many K, the inequalities from Proposition 4 collapse into equalities, and the dimensions will not depend on j. We would conjecture that at least the center inequality should be an equality for all K, giving j-independence as in the one-sided case, but we are unable to prove this.

The proof of Proposition 4 is motivated by the proof of Proposition 5.2 from [3]. The second inequality is trivial; the first and third follow from the same argument. To see this, pick any $K_1 < K_2, j \geq 1$ and notice that:

$$\{\tau : |S_0^\tau(t)| \leq 1 \text{ for all } t \in [0, 2n]\} \cap \{\tau : |S_{(0,2n)}^\tau(t)| \leq j + K_1\sqrt{t - 2n} \text{ for all } t \geq 2n\}$$

is contained in $T_1^\pm(K_2)$ for n sufficiently large. This is because $S_0^\tau(2n) = 0$ on the first set, and $j + K_1\sqrt{t - 2n} \leq 1 + K_2\sqrt{t}$ for large n (this fails for $K_1 = K_2$, which is why we don't get full j-independence). The second set is just a translated version of $T_j^\pm(K_1)$, and thus has the same Hausdorff dimension. The first set consists of τ at which an independent (of $S_{(0,2n)}^\tau$) event of positive probability occurs. Thus, by the same ergodicity arguments used in Proposition 5.2 from [3], intersection with the first set does not decrease the dimension. So:

$$\dim_H(T_j^\pm(K_1)) \leq \dim_H(T_1^\pm(K_2)) \text{ for all } j \geq 1, K_1 < K_2,$$

which proves both the first and third inequalities.

Now we focus on comparing the Hausdorff dimensions of $T^{\pm}_{\infty}(K)$ and other, related sets of exceptional times. We may drop j from the notation, and when j is not specified it should be understood that we are discussing $T^{\pm}_{\infty}(K)$ (i.e., $T^{\pm}(K) := T^{\pm}_{\infty}(K)$). Proposition 4 allows us to translate the coming bounds into bounds for $\dim_H(T^{\pm}_1(K))$ (or $\dim_H(T^{\pm}_0(K))$ for $K \geq 1$).

We now consider dynamical times at which $S^{\tau}_0(t)$ displays exceptional behaviour as t goes to ∞. That is, we look at times at which S^{τ}_0 *is* K*-subdiffusive in the limit:*

$$\tilde{T}^{\pm}(K) := \{\tau \in [0,\infty) : \limsup_{t\to\infty} |S^{\tau}_0(t)|/\sqrt{t} \leq K\}.$$

We'd like to relate this set to $T^{\pm}(K)$. Notice that:

$$\begin{aligned} T^{\pm}(K) :=& \{\tau \in [0,\infty) : \exists j \text{ s.t. } |S^{\tau}_0(t)| \leq j + K\sqrt{t} \quad \forall t\} \\ =& \{\tau \in [0,\infty) : \exists N, j \text{ s.t. } |S^{\tau}_0(t)| \leq j + K\sqrt{t} \quad \forall t \geq N\}. \end{aligned} \tag{7.4}$$

This is because the first set is clearly contained in the second, and for any τ in the second set we can simply choose a larger value for j to make the inequality $|S^{\tau}_0(t)| \leq j + K\sqrt{t}$ hold for all t. This implies:

$$\tilde{T}^{\pm}(K) = \bigcap_{K'>K} T^{\pm}(K'),$$

so:

$$\dim_H(\tilde{T}^{\pm}(K)) = \inf_{K'>K} \dim_H(T^{\pm}(K')). \tag{7.5}$$

As in the discussion following Proposition 4, monotonicity in K implies that (7.5) will also equal $\dim_H(T^{\pm}(K))$ except for at most countably many K. It may be that there is equality for all K, but this does not follow from our arguments.

In [3] it was shown that almost surely, for all τ, all walks in the DyDW are recurrent, and all pairs of walks coalesce (see Theorem 2.1, Remark 2.3 from [3]). This implies:

$$\begin{aligned} T^{\pm}(K) &= \{\tau \in [0,\infty) : \exists N, j \text{ s.t. } |S_0^\tau(t)| \le j + K\sqrt{t} \quad \forall t \ge N\} \quad \text{(by (7.4))} \\ &\overset{\text{a.s.}}{=} \bigcup_{n\ge 0} \{\tau \in [0,\infty) : \exists j \text{ s.t. } |S_{(0,2n)}^\tau(t)| \le j + K\sqrt{t} \quad \forall t \ge 2n\}. \end{aligned} \tag{7.6}$$

To see this, notice that on the second set, S_0^τ will a.s. eventually coalesce with $S_{(0,2n)}^\tau$, so for t large we will have $|S_0^\tau(t)| = |S_{(0,2n)}^\tau(t)| \le j + K\sqrt{t}$. For τ in the first set, let N^* be the first time S_0^τ returns to zero after N. Then $|S_{(0,N^*)}(t)| \le j + K\sqrt{t} \quad \forall t \ge N^*$. This proves (7.6).

Using (7.6) and the recurrence of all paths, we also get:

$$T^{\pm}(K) \overset{\text{a.s.}}{=} \bigcap_{m\ge 0} \bigcup_{n\ge m} \{\tau \in [0,\infty) : \exists j \text{ s.t. } |S_{(0,2n)}^\tau(t)| \le j + K\sqrt{t} \quad \forall t \ge 2n\},$$

which is a tail random variable with respect to the underlying $\xi_{(x,t)}^\tau$ processes. Similar reasoning also applies to $\tilde{T}^{\pm}(K)$. These observations imply:

$$\begin{aligned} &\mathbb{P}(T^{\pm}(K) \cap [0,\epsilon] = \varnothing) = 0 \text{ or } 1 \qquad \text{for all } K > 0, \epsilon \ge 0, \\ &\mathbb{P}(\tilde{T}^{\pm}(K) \cap [0,\epsilon] = \varnothing) = 0 \text{ or } 1 \qquad \text{for all } K > 0, \epsilon \ge 0. \end{aligned} \tag{7.7}$$

An easy consequence of (7.7) is given by the following proposition:

Proposition 5. *Almost surely, for every $K > 0$, $\tilde{T}^{\pm}(K)$, $T^{\pm}(K)$ will each be either empty, or dense in* $[0, \infty)$.

Now we prove a lower bound for the Hausdorff dimension of $\tilde{T}^{\pm}(K)$. The above results (Proposition 4, (7.5)) allow us to translate the following bound into lower bounds for $\dim_H(T^{\pm}(K))$ and $\dim_H(T_j^{\pm}(K))$. In fact, our lower bound is continuous in K, so we get the same bound for all these sets of exceptional times. Now, let $\tilde{\gamma}(K) := \sqrt{K^2 + 1}$, so that:

$$\limsup_{t \to \infty} \frac{\sigma_{\tilde{\gamma}(K)}(t)}{\sqrt{t}} \leq K \tag{7.8}$$

(see Proposition 3). Given γ, let $C_\infty(\gamma)$ be the corresponding rectangle event for Brownian motions; that is, the event that two independent Brownian motions started at $\pm\gamma^{-1}$ stay within $[-1, 1]$ for $0 \leq t \leq 1$. Then we have:

Proposition 6.

$$\dim_H(\tilde{T}^{\pm}(K)) \geq 1 - \frac{\log \mathbb{P}(C_\infty(\tilde{\gamma}(K)))^{-1}}{\log \tilde{\gamma}(K)} =: 1 - b_\infty(K). \tag{7.9}$$

As an immediate consequence of this we have $\dim_H(\tilde{T}^{\pm}(K)) \to 1$ as $K \to \infty$. Proposition 6 is established by a modification of the arguments used in Proposition 5.3 from [3]. We will drop the K dependence from the notation for the moment. First we define a family of random measures, $\sigma_{n,m}$, that play the role of the σ_n from their proof. As above, we take $C_k := C_k^0$. Given a Borel set E in $[0, 1]$, $n \geq m$, we define:

$$\sigma_{n,m}(E) := \int_E \prod_{k=m}^{n} \frac{\mathbb{1}_{C_k^\tau}}{\mathbb{P}(C_k)} d\tau,$$

and notice that $\sigma_{n,m}$ is supported on $\bar{E}_{n,m}$, the closure of:

$$E_{n,m} := \left\{ \tau \in [0,1] : \bigcap_{k=m}^{n} C_k^\tau \text{ occurs} \right\}. \tag{7.10}$$

Now, reasoning as in [3], we would like to show $\mathbb{P}(\sigma_{n,m}([0,1]) > 1/2) > c'$ and bound the expectation of the α-energy of $\sigma_{n,m}$, defined to be:

$$\int_0^1 \int_0^1 \frac{1}{|\tau - \tau'|^\alpha} d\sigma_{n,m}(\tau) d\sigma_{n,m}(\tau').$$

Reasoning as in Lemma 4, we have for all $n \geq m$:

$$\begin{aligned}\mathbb{E}[\sigma_{n,m}([0,1])^2] &= \int_0^1 \int_0^1 \prod_{k=m}^{n} \frac{\mathbb{P}(C_k^\tau \cap C_k^{\tau'})}{\mathbb{P}(C_k)^2} d\tau d\tau' \\ &\leq c \int_0^1 \int_0^1 \frac{1}{|\tau - \tau|^{b_m}} d\tau d\tau',\end{aligned}$$

with:

$$b_m = \log[\sup_{k \geq m} (\mathbb{P}(C_k)^{-1})] / \log \tilde{\gamma}.$$

For $b_m < 1$, an application of the Cauchy-Schwartz inequality as in (4.3)-(4.5) gives:

$$\begin{aligned}\mathbb{P}\left[\sigma_{n,m}([0,1]) > 1/2\right] \geq & \frac{\left(\mathbb{E}\left[\sigma_{n,m}([0,1])\mathbb{1}_{\sigma_{n,m}([0,1])>1/2}\right]\right)^2}{\mathbb{E}\left[\sigma_{n,m}([0,1])^2\right]} \\ \geq & \frac{\left(\mathbb{E}\left[\sigma_{n,m}([0,1])\right] - 1/2\right)^2}{\mathbb{E}\left[\sigma_{n,m}([0,1])^2\right]} \\ = & \frac{(1-1/2)^2}{\mathbb{E}\left[\sigma_{n,m}([0,1])^2\right]} \\ \geq & c' \left(\int_0^1 \int_0^1 \frac{1}{|\tau - \tau|^{b_m}} d\tau d\tau'\right)^{-1}.\end{aligned}$$

So $\mathbb{P}(\sigma_{n,m}([0,1]) > 1/2) > c''$, where c'' doesn't depend on n.

We assume that K is large enough to make the right-hand side of (7.9) positive (otherwise there is nothing to prove). Then $b_\infty < 1$ and by the diffusive scaling of the events C_k we have $b_m \to b_\infty < 1$ as $m \to \infty$. So given any $\alpha < 1 - b_\infty$, there exists m large enough such that $\alpha + b_m < 1$. Now, arguing as in Proposition 5.3 of [3], this gives a uniform bound the expectation of the α-energy of $\sigma_{n,m}$. Then we can use the extension of Frostman's lemma from [15] to conclude that:

$$\dim_H\left(\bigcap_{n\geq m} \bar{E}_{n,m}\right) \geq \alpha \text{ with positive probability.} \tag{7.11}$$

Now, for our chosen K, $\bigcap_{n\geq m} E_{n,m} \subset \tilde{T}^{\pm}(K)$ for all m (using (7.8) and the a.s. coalescence of all paths). We've shown that given any $\alpha < 1 - b_\infty$, (7.11) holds for some sufficiently large m. Also, $\bigcap_{n\geq m} E_{n,m} = \bigcap_{n\geq m} \bar{E}_{n,m}$ almost surely, by the same argument used to establish (4.7). Combining these observations, we have:

$$\dim_H(\tilde{T}^{\pm}(K)) \geq 1 - b_\infty(K),$$

(almost surely by ergodicity in τ of the DyDW). This proves Proposition 6.

Remark 2. *One may wish to consider "asymmetrical" exceptional times. That is, exceptional times where the K of $\tilde{T}^{\pm}(K)$, $T^{\pm}(K)$, $T_j^{\pm}(K)$, etc. is replaced by two constants, K_L, and K_R, giving different bounds on the left and right sides. One can obtain an analogous lower bound for the dimension of these asymmetrical exceptional times using the "skewed rectangle" construction described in Remark 1.*

Now we look at upper bounds for the Hausdorff dimension of the sets of two-sided exceptional times. This is a straightforward extension of the results in Chapter 5.2 of [3]. Following [3], we state the results for the asymmetrical case. So we give an upper bound for $\dim_H(T_1^-(K_L) \cap T_1^+(K_R))$. Recall:

$$T_1^-(K_L) \cap T_1^+(K_R) = \{\tau \in [0, \infty) : -1 - K_L\sqrt{t} \leq S_0^\tau(t) \leq 1 + K_R\sqrt{t} \text{ for all } t\}, \tag{7.12}$$

using the definitions given earlier in this chapter.

Proposition 5.5 from [3] gives the bound $\dim_H(T_1^-(K)) \leq 1 - p(K)$, where $p(K) \in (0, 1)$ is the solution to:

$$f(p, K) := \frac{\sin(\pi p/2)\Gamma(1 + p/2)}{\pi} \sum_{n=1}^{\infty} \frac{(\sqrt{2}K)^n}{n!} \Gamma((n - p)/2) = 1. \tag{7.13}$$

They also prove that $T_1^-(K_L) \cap T_1^+(K_R)$ is empty when $p(K_L) + p(K_R) > 1$ (see [3], Proposition 5.8). The function $p(K)$ comes from [13], where it is shown that $p(K)$ is continuous and decreasing on $(0, \infty)$, tending to 0 as K goes ∞, tending to 1 as K goes to 0.

The upper bound from [3] is established by partitioning $[0, 1]$ into intervals of equal length, and estimating the number of these needed to cover $T_1^-(K)$. An application of the FKG inequality, as in Proposition 5.8 of [3], extends the bound to the two-sided case, giving:

Proposition 7.

$$\dim_H(T_1^-(K_L) \cap T_1^+(K_R)) \leq 1 - p(K_L) - p(K_R),$$

so:

$$\dim_H(T_1^{\pm}(K)) \leq 1 - 2p(K).$$

Note that, as with the lower bound, continuity of the bounding function combined with our previous results gives an identical bound for $\dim_H(T^{\pm}(K))$, $\dim_H(T_j^{\pm}(K))$, $\dim_H(\tilde{T}^{\pm}(K))$, and their asymmetrical analogues.

7.2 Superdiffusive Exceptional Times

In this chapter we will derive a lower bound for the Hausdorff dimension of the superdiffusive exceptional times. We consider the rectangle events $\hat{A}_k^{\tau}$ from Chapter 5. The proof relies on the same techniques as in Proposition 5.3 from [3] and Proposition 6 above. As in Chapter 5 of this thesis, the proof is complicated by the fact that $\mathbb{P}(\hat{A}_k) \to 0$. We will give a bound for the set of τ such that $\hat{A}_k^\tau$ occurs for all $k \geq 0$. This set is a subset of the set of the superdiffusive exceptional times, so a lower bound on its dimension gives the bound we need. To obtain an upper bound on the dimension of the superdiffusive exceptional times we would instead need to consider a (possibly) larger set. Attempts in this direction have not been succesful, so we will only give a lower bound.

Let $\hat{T}^+(C), \hat{T}^-(C)$ denote the sets of C-superdiffusive times, i.e.:

$$\hat{T}^+(C) = \{\tau \in [0,\infty) : \limsup_{t\to\infty} S_0^\tau(t)/\sqrt{t\log(t)} \geq C\}$$
$$\hat{T}^-(C) = \{\tau \in [0,\infty) : \liminf_{t\to\infty} S_0^\tau(t)/\sqrt{t\log(t)} \leq -C\}.$$

These two sets will have the same Hausdorff dimension due to symmetry. So we will focus on $\hat{T}^+(C)$. Taking a' to be the value given by Lemma 5 we have the following proposition:

Proposition 8. *For γ, C such that $a'(1-\gamma^{-3/2}) - 2C^2 > 0$ we have:*

$$\dim_H(\hat{T}^+(C)) \geq 1 - \frac{2\gamma^2 C^2}{(\gamma-1)}. \tag{7.14}$$

As an immediate consequence of Proposition 8 we see that $\dim_H(\hat{T}^+(C)) \to 1$ as $C \to 0$. The remainder of this chapter will be devoted to the proof of Proposition 8.

Similar to Proposition 6 above, we consider the measures:

$$\hat{\sigma}_{n,m}(E) := \int_E \prod_{k=m}^{n} \frac{\mathbb{1}_{\hat{A}_k^\tau}}{\mathbb{P}(\hat{A}_k)} d\tau,$$

which are supported on $\bar{\hat{E}}_{n,m}$, the closure of:

$$\hat{E}_{n,m} := \left\{ \tau \in [0,1] : \bigcap_{k=m}^{n} \hat{A}_k^\tau \text{ occurs} \right\}. \tag{7.15}$$

For the sake of simplicity we will only consider the case $m = 0$ and let $\hat{\sigma}_n = \hat{\sigma}_{n,0}$, $\hat{E}_n = \hat{E}_{n,0}$. For $n \geq 0$ we want to bound $\mathbb{E}[\hat{\sigma}_n([0,1])^2]$ and the expectation of the α-energy of $\hat{\sigma}_n$:

$$\mathbb{E}\left[\int_0^1 \int_0^1 \frac{1}{|\tau - \tau'|^\alpha} d\hat{\sigma}_n(\tau) d\hat{\sigma}_n(\tau')\right] = \int_0^1 \int_0^1 \frac{1}{|\tau - \tau'|^\alpha} \prod_{k=0}^{n} \frac{\mathbb{P}(\hat{A}_k^\tau \cap \hat{A}_k^{\tau'})}{\mathbb{P}(\hat{A}_k)^2} d\tau d\tau'. \tag{7.16}$$

$\mathbb{E}[\hat{\sigma}_n([0,1])^2]$ corresponds to $\alpha = 0$ in (7.16), so if we can obtain an integrable bound for (7.16) with $\alpha > 0$ we will have $\dim_H(\hat{T}(C)) > \alpha$. This follows from Cauchy-Schwarz and Frostman's lemma, as in Proposition 5.3 of [3] and Proposition 6 above. We will use Lemma 5 from Chapter 5 to bound the product on the right side of (7.16). Lemma 5 gives $c', a' \in (0, \infty)$ independent of k, τ and τ' such that:

$$\frac{\mathbb{P}(\hat{A}_k^\tau \cap \hat{A}_k^{\tau'})}{\mathbb{P}(\hat{A}_k)^2} \leq 1 + \frac{c'}{\mathbb{P}(\hat{A}_k)^2} \left(\frac{1}{\gamma^{\gamma^k} |\tau - \tau'|} \right)^{a'}.$$

Let $N_0 = \lfloor \log_\gamma(2 \log_\gamma(1/|\tau - \tau'|)) \rfloor + 1$, so that $\gamma^{\gamma^{N_0}/2} |\tau - \tau'| \geq 1$. We will split the product at N_0 to obtain our bound. First consider $k > N_0$:

$$\begin{aligned}
\prod_{k=N_0+1}^{n} \frac{\mathbb{P}(\hat{A}_k^\tau \cap \hat{A}_k^{\tau'})}{\mathbb{P}(\hat{A}_k)^2} &\leq \prod_{k=N_0+1}^{\infty} 1 + \frac{c'}{\mathbb{P}(\hat{A}_k)^2}\left(\frac{1}{\gamma^{\gamma^k}|\tau-\tau'|}\right)^{a'} \\
&= \prod_{k=N_0+1}^{\infty} 1 + \frac{c'}{\mathbb{P}(\hat{A}_k)^2}\left(\frac{1}{\gamma^{\gamma^k-\gamma^{N_0/2}}(\gamma^{\gamma^{N_0/2}}|\tau-\tau'|)}\right)^{a'} \\
&\leq \prod_{k=N_0+1}^{\infty} 1 + \frac{c'}{\mathbb{P}(\hat{A}_k)^2}\left(\frac{1}{\gamma^{\gamma^k-\gamma^{N_0/2}}}\right)^{a'} \\
&\leq \prod_{k=N_0+1}^{\infty} 1 + \frac{c'}{\mathbb{P}(\hat{A}_k)^2}\left(\frac{1}{\gamma^{\gamma^k(1-\gamma^{-3/2})}}\right)^{a'} \\
&\leq \prod_{k=1}^{\infty} 1 + \frac{c'}{\mathbb{P}(\hat{A}_k)^2}\left(\frac{1}{\gamma^{\gamma^k(1-\gamma^{-3/2})}}\right)^{a'} \qquad (7.17)
\end{aligned}$$

In the second to last step we used $N_0 \geq 1$, $k \geq N_0 + 1$. In Chapter 5 (see (5.9)) we saw that:

$$\mathbb{P}(\hat{A}_k) \geq K''(\gamma^{\gamma^k})^{-(1+\epsilon_k'')^2 C^2},$$

where $\epsilon_k'' \to 0$. So provided that $a'(1-\gamma^{-3/2}) - 2C^2 > 0$ (7.17) gives:

$$\prod_{k=N_0+1}^{n} \frac{\mathbb{P}(\hat{A}_k^\tau \cap \hat{A}_k^{\tau'})}{\mathbb{P}(\hat{A}_k)^2} \leq K(C). \qquad (7.18)$$

Now we consider the $k \leq N_0$ terms. Given any fixed $\epsilon'' > 0$ we can make $\epsilon_k'' < \epsilon''$ for all k by decreasing K'', since $\epsilon_k'' \to 0$. Similarly to (5.7)-(5.9) above we will absorb lower order terms into K'' and ϵ''. To keep the notation simple we will continue to use K'', ϵ'' to denote these updated values. So for $n \le N_0$, we have:

$$\begin{aligned}
\prod_{k=0}^{n} \frac{\mathbb{P}(\hat{A}_k^{\tau} \cap \hat{A}_k^{\tau'})}{\mathbb{P}(\hat{A}_k)^2} &\le \prod_{k=0}^{N_0} \frac{1}{\mathbb{P}(\hat{A}_k)} \\
&\le \prod_{k=0}^{N_0} \frac{1}{K''(\gamma^{\gamma^k})^{-(1+\epsilon_k'')^2 C^2}} \\
&\le \frac{1}{(K'')^{N_0+1} \gamma^{-(1+\epsilon'')^2 C^2 \sum_{k=0}^{N_0} \gamma^k}} \\
&\le \frac{(\gamma^{\gamma^{N_0+1}})^{(1+\epsilon'')^2 C^2/(\gamma-1)}}{(K'')^{N_0+1}} \\
&\le K'' \left(\frac{1}{|\tau - \tau'|} \right)^{\frac{2\gamma^2 (1+\epsilon'')^2 C^2}{(\gamma-1)}} .
\end{aligned}$$

Combined with (7.18) this gives the bound needed in (7.16) which completes the proof.

One could derive a similar lower bound for the dimension of the two-sided superdiffusive times, $\hat{T}^{\pm}(C) = \hat{T}^{+}(C) \cap \hat{T}^{-}(C)$, using the same techniques combined with the ideas of Chapter 5.1. We will not present a proof of such a result.

Chapter 8

Zero Temperature Dynamical Ising Models

In this chapter we describe some recent joint work with Charles Newman and Daniel Stein on zero temperature dynamical Ising models with nearest neighbor interactions. Given a graph, $G = (V, E)$, we consider an Ising model with Hamiltonian:

$$H = -\sum_{x,y \in E} J_{x,y} \sigma_x \sigma_y. \tag{8.1}$$

We are interested in the zero temperature limit of Glauber dynamics for this model, with the initial spin values given by independent $p = 1/2$, ± 1-valued Bernoulli random variables. This will be a stochastic process $\sigma^t(\omega)$ taking values in $\{-1, +1\}^V$. Physically, this corresponds to an instantaneous quench from infinite to zero temperature.

We'll begin by describing the models of interest in more detail. A magnetic system consists of a collection of particles, each with a corresponding "spin". We will denote the spin at a site $x \in V$ by σ_x, and for our purposes spins will only take two values, denoted $+1$ and -1. Pairs of nearby particles interact with each other, exerting influence on each other's spin values. Ising models (with nearest neighbor interactions) are meant to provide a simple model of magnetism, in which particles are placed at each vertex of a graph and only nearest neighbors interact with each other. In a ferromagnetic system neighboring spins tend to take matching values, while in an antiferromagnetic system neighboring spins tend to take opposite values. In a "spin glass" there are both ferromagnetic and antiferromagnetic interactions.

An Ising model can be defined on any graph. Common choices include the d-dimensional integer lattice, $\mathbb{Z}^d$, or the complete graph on N points, K_N. Given our graph, $G = (V, E)$, we define a probability measure on the space of spin configurations, $\{-1, +1\}^V$, such that the probability of a given configuration is proportional to $\exp(-\beta H)$. Here H denotes the Hamiltonian or "energy" of the system, which is given by (8.1), and β denotes the inverse temperature. Such a measure is called a Gibbs distribution. With this choice of measure the system will favor lower energy configurations, and this tendency will increase for larger values of β (i.e. lower temperature). The choice of the $J_{x,y}$ in (8.1) will determine whether the model describes a ferromagnet, an antiferromagnet or a spin glass. We can see that if $J_{x,y} \geq 0$ for all x, y, then the energy will be lower if neighboring spins agree, giving a ferromagnet. Typical choices are to take $J_{x,y} = 1$ for all x, y, or to draw the $J_{x,y}$ independently from a probability distribution supported on $[0, \infty)$. Similarly, if $J_{x,y} \leq 0$ for all

x, y this will give an antiferromagnet. In a spin glass model, the $J_{x,y}$ will take on both positive and negative values. Typical choices for a spin glass are to draw the $J_{x,y}$'s from i.i.d. standard normal distributions, or from i.i.d. ± 1-valued symmetric Bernoulli distributions.

We can introduce dynamics to the Ising model in a similar manner to the dynamical discrete web or dynamical percolation. The first step is to assign independent, rate one Poisson clocks to each site, x. When the clock at x rings, the spin at x is updated. The spin update will depend on the local field at x, which we define as:

$$Z_x^t = \sum_{y:y\sim x} J_{x,y}\sigma_y^t.$$

A natural rule for updating spins is to choose the new spin in a way which preserves the Gibbs distribution described above. To accomplish this, the distribution of the new spin value can be taken to be the conditional distribution of that spin under the Gibbs measure, conditioned on the current values of all other spins. So if the clock at a site, x, rings at time t, then the new spin value, $\sigma_x^{t^+}$, is chosen according to:

$$\mathbb{P}(\sigma_x^{t^+} = +1) = \frac{\exp\left[\beta Z_x^t\right]}{\exp\left[\beta Z_x^t\right] + \exp\left[-\beta Z_x^t\right]}, \tag{8.2}$$

$$\mathbb{P}(\sigma_x^{t^+} = -1) = \frac{\exp\left[-\beta Z_x^t\right]}{\exp\left[\beta Z_x^t\right] + \exp\left[-\beta Z_x^t\right]}. \tag{8.3}$$

To see that these values are equal the conditional probabilities discussed above, recall that the probability of a configuration is proportional to $\exp(-\beta H)$. In (8.2), for example, start with the probability that the spin at x is $+1$ in the current configuration, and divide by the sum of the probabilities that x is $+1$ or -1 in the current configuration. Any pair in the sum for H (see (8.1)) which does not include the site x will lead to common terms which can be factored from all the exponentials. After canceling these common terms we are left with the equation in (8.2).

The dynamics described above are referred to as Glauber dynamics; they were introduced by Glauber in [6]. We are interested in zero temperature Glauber dynamics, corresponding to the limit as $\beta \to \infty$. If we let $\beta \to \infty$ in (8.2),(8.3) we see that $\mathbb{P}(\sigma_x^{t^+} = \text{sgn}(Z_x^t)) \to 1$ for $Z_x^t \neq 0$ and $\mathbb{P}(\sigma_x^{t^+} = +1) = \mathbb{P}(\sigma_x^{t^+} = -1) = 1/2$ for $Z_x^t = 0$. In other words, when the clock at x rings, the new spin value, $\sigma_x^{t^+}$, will be equal to $\text{sgn}(Z_x^t)$ if $Z_x^t \neq 0$ and will be equal to ± 1 with equal probability if $Z_x^t = 0$. These zero temperature dynamics define the update rules we will use. Notice that a spin flip at site x under these dynamics will cause H to decrease by $2|Z_x^t|$. As a consequence, H will be non-increasing as a function of t. Our initial, $t = 0$, spin values will be given by independent, $p = 1/2$, ± 1-valued Bernoulli random variables. These initial conditions correspond to the infinite temperature, or $\beta \to 0$, limit of the Gibbs measure described above.

Notice that we have taken infinite temperature initial conditions, but zero temperature dynamics. This choice corresponds to instantaneously lowering from high (infinite) temperature to very low (zero) temperature. One might expect the spin distribution to approach the zero temperature Gibbs distribution, but this is not the case. The zero temperature Gibbs distribution concentrates on spin configurations with globally minimized energy. However, the zero-temperature dynamics never allow a spin flip which increases the energy, so the system can be caught in local energy minima. In other words, our spin system gets stuck in 1-spin-flip-stable configurations, which are configurations whose energy, H, can not be reduced by flipping any single spin. It has been shown by Newman and Stein that there are a large number of these 1-spin-flip-stable configurations, and they have energy which is strictly greater than the energy of the ground states [12].

To construct the models described above, we just need to choose a specific graph, $G = (V, E)$, and distribution for the couplings, $J_{x,y}$. For the sake of clarity and brevity we will focus on the Sherrington-Kirkpatrick (SK) model [16], where $G = K_N$, the complete graph on N vertices, and the $J_{x,y}$'s are distributed as i.i.d. mean 0, variance $1/N$ Gaussian random variables. The $1/N$ normalization ensures that the Z_x^t's will stay $O(1)$ as $N \to \infty$. One advantage of this choice is the graph, K_N, is finite, so the Hamiltonian, H, is well defined and the above construction is rigorous. There is a natural extension of this analysis to infinite graphs; one common choice is the integer lattice, $\mathbb{Z}^d$, known as the Edwards-Anderson model. However, we will focus on the finite case to keep things simple. We will rely on the fact that H is well-defined in the following discussion, though a similar analysis should be possible for infinite graphs by considering the mean energy per site. Another important consequence of this choice is that the $J_{x,y}$'s come from a continuous distribution, so $Z_x^t \neq 0$ almost surely for all x, t and spin flips will always strictly decrease H.

Our work on this subject has focused on the order parameter, q_D, introduced in [11]. This order parameter is defined as $q_D = \lim_{t\to\infty} q^t$, where:

$$q^t = \mathbb{E}_{\vec{J},\vec{\sigma}^0}\left(\langle \sigma_x^t \rangle^2\right).$$

The inner angled brackets denote the expectation over the dynamics, for fixed initial spin value and coupling realizations. The outer expectation is the average over the initial spin values and couplings. This parameter captures the relative importance of the initial conditions and couplings versus the dynamics in determining the spin value at a typical site as $t \to \infty$. For example, $q_D = 0$ would imply $\mathbb{E}\langle|\sigma_x^t|\rangle$ goes to 0 as $t \to \infty$, meaning that the spin values at large times are essentially independent of the initial conditions and couplings. On the other hand, $q_D = 1$ would mean that the large t spin values are essentially determined by the couplings and initial conditions. A numerical analysis of some low dimensional cases was carried out in [17]. We are interested in the behavior of q_D for large dimension, or as $N \to \infty$ in the SK model.

We begin by considering the total energy, or Hamiltonian, as a function of t:

$$H(t) = -\sum_{(x,y):y\sim x} J_{x,y}\sigma_x^t\sigma_y^t = -\frac{1}{2}\sum_x Z_x^t\sigma_x^t.$$

First recall that $H(t)$ is non-increasing in t. Now consider that $H(0)$ is a sum of N^2 terms, each of the form $J_{x,y}\sigma_x^0\sigma_y^0$. The $J_{x,y}$ are i.i.d. Gaussian random variables with variance $1/N$, and the σ_x^0, σ_y^0 are i.i.d., symmetric, ± 1-valued Bernoulli random variables. So each of the terms, $J_{x,y}\sigma_x^0\sigma_y^0$, is itself distributed as a Gaussian random variable with variance $1/N$, and they are independent of each other. This implies that the entire sum, $H(0)$, is Gaussian with variance $N^2(1/N) = N$, so $H(0)$ is typically of order $O(\sqrt{N})$. In addition, it is well-known that the ground state for the SK model has total energy bounded below by $-N$ [1]. The ground state energy is clearly a lower bound for $H(t)$, so the above discussion implies that $H(t) \in [-N, O(\sqrt{N})]$ for all t. Each time a spin flips, say at site x, this causes $H(t)$ to decrease by $2|Z_x^t|$ which is $O(1)$ on average. These observations suggest that there can only be $O(N)$ spin flips in total, or $O(1)$ spin flips per site. This leads to the following conjecture.

Conjecture 1. *There exists a constant C, which is independent of N, such that the expectation of the total number of spin flips at any site in $SK(N)$ is bounded by CN.*

Given this conjecture, we know that as $N \to \infty$ a given site will only flip C times on average. This suggests that knowledge of the initial spin configuration does tell us something about the final spin value, meaning $\lim_{N\to\infty} q_D(N) > 0$. Another way to think about this is to consider Z_x^t as a function of t. Z_x^0 is determined by the initial conditions. As t increases, neighboring spins will flip. This causes Z_x^t to increase or decrease by $2J_{x,y} = O(1/\sqrt{N})$ with each spin flip. In the SK model, all sites are neighbors of x except for x itself. So Conjecture 1 implies that Z_x^t will take CN or fewer steps of size $O(1/\sqrt{N})$ each. This would correspond to a bounded time interval in a diffusive scaling limit. So if we send N to ∞ we might expect Z_x^t to behave (after a time change) as a diffusion process, $\tilde{Z}_x^t$, with bounded volatility run over a finite time interval, $[0, T]$. Since the diffusion runs for a finite amount of time, it should not forget it's initial value. More specifically, the sign of $\tilde{Z}_x^T$ should not be independent of the initial conditions. This suggests $q_D \neq 0$, since σ_x^∞ will take on the sign of $\tilde{Z}_x^T$.

The sketch of a potential proof just before Conjecture 1 has some issues which need to be resolved. The bounds on $H(t)$ are rigorous (see [1], for example); the main weakness is the possibility that $|Z_x^t|$ concentrates near zero at flipping times, which would allow for greater than $O(N)$ flips. This shouldn't happen for small t, but this may be a problem for larger t. Consider a flipping time, t^*, at a site x^*. If t^* is large then with high probability t^* is not the first clock ring. So if σ_{x^*} changes sign at t^* this means that $Z_{x^*}^t$ must have changed its sign between the previous clock ring and t^*. This time interval has length $O(1)$ on average. Initially, we'd expect to see $O(N)$ neighboring spins flip in an $O(1)$ time interval, but this should slow down for large t. As a consequence, we'd see less than $O(1)$ movement in $Z_{x^*}^t$ between the previous flipping time and t^*, meaning $|Z_{x^*}^{t^*}|$ will be less than $O(1)$. In addition to this issue, the connection between Conjecture 1 and q_D needs to be made more precise. One way to validate the discussion after Conjecture 1 would be to show that Z_x^t does converge to a reasonable limit as $N \to \infty$, and this limit satisfies the above assumptions regarding the dependence between the initial and final values. These considerations remain to be investigated.

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