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Some Inequalities in Fourier Analysis and Applications

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Some Inequalities in Fourier Analysis and Applications

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Some Inequalities in Fourier Analysis and Applications

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Abstract. We prove several inequalities involving the Fourier transform of functions which are compactly supported. The constraint that the functions have compact support is a simplifying feature which is desirable in applications, but there is a tradeoff in control of other relevant quantities– such as the mass of the function. With applications in mind, we prove inequalities which quantify these types of trade-offs.

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Chapter 1

Introduction

This thesis consists of four main parts:

(A) A variant of the Beurling-Selberg problem,

(B) An extremal problem for convex bodies,

(C) Uniform dilations in higher himensions, and

(D) Separated nets arising from linear toral flows.

The common theme between these works is the study and use of functions which, in a precise sense, contain only low frequency waves. Such functions are known as band-limited functions, or entire functions of exponential type. In parts (A) and (B) we introduce new kinds of approximation problems and report some first steps towards their resolution. Parts (C) and (D) are papers in which well known constructions in trigonometric approximation are used to obtain effective estimates in dynamical systems. (C) is joint work with Lê Thái Hoàng, and (D) is joint work with Alan Haynes and Barak Weiss. These parts are self contained and contain their own introductions.

1.1 Selberg's Functions and Modifications

In the 1970's A. Selberg constructed two functions $C(t)$ and $c(t)$ with the following properties

(i) $C(t)$ and $c(t)$ are integrable,

(ii) $c(t) \leq \chi_{[a,b]}(t) \leq C(t)$ for every $t \in \mathbb{R}$,

(iii) $\hat{C}(\xi) = \hat{c}(\xi) = 0$ if $|\xi| > \delta$, and

(iv) $\displaystyle\int_{-\infty}^{\infty} \left\{C(t) - \chi_{[a,b]}(t)\right\} dt = \int_{-\infty}^{\infty} \left\{\chi_{[a,b]}(t) - c(t)\right\} dt = \delta^{-1}.$

He went on to show that the integrals appearing in (iv) are minimized among all functions satisfying conditions (i)-(iii) if, and only if, $\delta(b-a)$ is an integer. The case when $\delta(b-a)$ is not an integer has been carried out in recent work of Friedrich Littmann [49].

Properties (i) and (iii) guarantee that $C(z)$ is actually (almost everywhere) the restriction to the real axis of an entire function. Indeed, by the Fourier inversion formula, the function

$$z \mapsto \int_{-\delta}^{\delta} \hat{C}(\xi) e(z\xi) d\xi$$

is equal to $C(t)$ for real t, where $e(t) = e^{2\pi i t}$. It follows from Morera's theorem and Fubini's theorem that the above function is in fact entire. By an abuse of notation, let $C(z)$ denote the extension of $C(t)$ to $\mathbb{C}$.

Theorem 2.2. *Suppose $\delta(b-a) < 1$. Then $C(z)$ is zero free in a ball of radius $(100\delta)^{-1}$ centered at $(b+a)/2$.*

In applications one may wish to take the parameter δ to be quite small, or even to take it to zero. E. Bombieri has asked [private communication] if given a fixed α in the upper half plane, how to construct an analgoue of $C(z)$ which in addition to (i)-(iii) above, also vanishes at α. In view of the above theorem, if δ is sufficiently small, then the condition that $C(\alpha) = 0$ is a non-trivial one.

Problem 2.1. *Let $\chi(t)$ denote the characteristic function of the interval $[a, b]$, $\delta > 0$, and $\alpha \in \mathbb{C}$ be in the upper half plane. Define*

$$\rho(a, b; \alpha, \delta) = \inf \int_{-\infty}^{\infty} F(t)dt - (b - a) \tag{2.0.1}$$

where the infimum is taken over entire functions $F : \mathbb{C} \to \mathbb{C}$ with the following properties:

(i) *$F(t)$ is integrable on the real axis,*

(ii) *$F(t) \geq \chi(t)$ for each real t,*

(iii) *$\hat{F}(\xi) = 0$ if $|\xi| > \delta$, and*

(iv) *$F(\alpha) = 0$.*

Determine the value of $\rho(a, b; \alpha, \delta)$ and if possible give a formula for an extremal.

This question forms the basis of our investigation in Chapter 1. We are able to show

$$\rho(a, b; \alpha, \delta) \approx \delta^{-1} \quad \text{as} \quad \delta \to \infty$$

and

$$\delta^{-2} \ll \rho(a,b;\alpha,\delta) \ll \delta^{-3} \quad \text{as} \quad \delta \to 0$$

where the implied constants depend on a, b, and α. Our methods allow us to explicitly produce such modifications of Selberg's functions and to even create majorants which vanish at any finite number of points off of the real axis. This work is detailed in Chapter 1. [1] Our construction is based upon the following extremal problem which we solve.

Theorem 2.4. *Let $\delta > 0$, $\alpha \in \mathscr{U}$, and $\beta \in \mathbb{C}$. If $F(z)$ is a real entire function of exponential type at most $2\pi\delta$ that is non-negative on the real axis and $F(\alpha) = \beta$, then*

$$\frac{|\beta|K(\alpha,\alpha) - \delta \mathrm{Re}(\beta)}{K(\alpha,\alpha)^2 - \delta^2} \leq \frac{1}{2}\int_{-\infty}^{\infty} F(x)dx. \tag{2.3.1}$$

where $K(\omega, z)$ is defined by (2.1.4). There is equality in (2.3.1) if and only if $F(z) = U(z)U^*(z)$ where

$$U(z) = \lambda_1 K(\alpha, z) + \lambda_2 K(\overline{\alpha}, z),$$

and λ_1 and λ_2 can be given explicitly.

1.2 An Extremal Problem for Convex Bodies and the Fourier Transform

A *convex body* K is a compact, convex, symmetric subset of $\mathbb{R}^N$ with non-empty interior. Convex bodies arise naturally as the unit balls of norms

[1] Littmann and Spanier have recently communicated to me that they have solved a similar problem for the signum function, when α is purely imaginary.

on $\mathbb{R}^N$, and any convex body is the unit ball of some norm on $\mathbb{R}^N$. In this paper we study an extremal problem at the interface of Fourier analysis and the geometry of convex bodies. For an integrable function $F(\mathbf{x})$ the Fourier transform $\hat{F}(\boldsymbol{\xi})$ is defined by

$$\hat{F}(\boldsymbol{\xi}) = \int_{\mathbb{R}^N} e(-\mathbf{x} \cdot \boldsymbol{\xi}) F(\mathbf{x}) d\mathbf{x}$$

where $e(\theta) = e^{2\pi i \theta}$. We are interested in the following problem.

Problem 3.1. Given a convex body K define

$$\eta(K) = \inf \int_{\mathbb{R}^N} F(\mathbf{x}) d\mathbf{x} \tag{3.1.1}$$

where the infimum is taken over non-zero integrable functions $F(\mathbf{x})$ which satisfy

(i) $F(\mathbf{x}) \geq 0$ for each $\mathbf{x} \in \mathbb{R}^N$,

(ii) $F(\mathbf{0}) \geq 1$, and

(iii) $\hat{F}(\boldsymbol{\xi}) = 0$ if $\boldsymbol{\xi} \notin K$.

Determine the value of $\eta(K)$.

It is easy to show that

$$1 \leq \text{vol}_N(K)\eta(K) \leq 2^N \tag{3.1.2}$$

where $\text{vol}_N(K)$ is the Lebesgue measure on $\mathbb{R}^N$. The rightmost inequality is obtained by considering the function

$$F(\mathbf{x}) = \left| \fint_{1/2K} e(\mathbf{x} \cdot \boldsymbol{\xi}) d\boldsymbol{\xi} \right|^2 \tag{3.1.2}$$

and the left most inequality follows from some basic Fourier analysis.

To each convex body K there is an associated convex body K^*, called the *dual body* or *polar body* of K, defined by

$$K^* = \left\{ \boldsymbol{\xi} \in \mathbb{R}^N : \mathbf{x} \cdot \boldsymbol{\xi} \le 1 \ \text{ for each } \mathbf{x} \in K \right\}. \tag{3.1.3}$$

We will call K^* the dual body of K. The following is the main result of chapter 3.

Theorem 3.1. Let K be a symmetric body and let $\eta(K)$ be defined by (3.1.1). Then

$$\frac{\eta(K)}{\mathrm{vol}_N(K^*)} \ge \frac{\eta(B)}{\mathrm{vol}_N(B)} = \frac{2^N}{\mathrm{vol}_N(B)^2} \tag{3.1.4}$$

where B is the unit ball of $\mathbb{R}^N$ and K^* is the dual body of K defined by (3.1.3).

An immediate consequence of Theorem 3.1 and (3.1.2) is the following classical inequality of Santaló.

Corollary (Santaló's inequality). *Let K be a convex body in $\mathbb{R}^N$, K^* be its dual body. Then*

$$\mathrm{vol}_N(K)\mathrm{vol}_N(K^*) \le \Big(\mathrm{vol}_N(B)\Big)^2$$

where B is the unit ball of $\mathbb{R}^N$.

Let $F(\mathbf{x})$ satisfy conditions (i)-(iii) in Problem 3.1, let Λ be a unimodular lattice and Λ^* be its dual lattice. Then by the Poisson summation formula[2]

[2]Poisson summation holds pointwise for $F(\mathbf{x})$ because it is non-negative and the series converges absolutely.

we have

$$1 \leq \sum_{\mathbf{u}\in\Lambda^*} F(\mathbf{u}) = \sum_{\mathbf{v}\in\Lambda} \hat{F}(\mathbf{v}) \leq \hat{F}(0)\#\text{interior}(K)\cap\Lambda$$

which implies $\eta(K)^{-1} \leq \#\text{interior}(K)\cap\Lambda$. Combining this with (3.1.2) gives the following form of Minkowski's convex body theorem.

Corollary 3.1 (Minkowski's convex body theorem)**.** Let K be a convex body in and Λ be a unimodular lattice in $\mathbb{R}^N$. Then

$$\frac{\text{vol}_N(K)}{2^N} \leq \eta(K)^{-1} \leq \#\text{interior}(K)\cap\Lambda.$$

Indeed the admissible function (3.1.2) constructed to achieve the upper bound $2^N/\text{vol}_N(K)$ for $\eta(K)$ is essentially the function used by Siegel [66] in his proof of the convex body theorem. It would be interesting to know if there is a convex body K for which $\eta(K) < 2^N/\text{vol}_N(K)$. If there is such a body, then the above inequality is a strengthening of Minkowski's theorem.

If $Q = [-1,1]^N$ and $\Lambda = \mathbb{Z}^N$, then $\#\text{interior}(Q)\cap\Lambda = 2^{-N}\text{vol}_N(K) = 1$. By Corollary 3.1 it follows that $\eta(Q) = 1 = 2^N/\text{vol}_N(Q)$ and in this case (3.1.2) is just the suitably normalized Fejér kernel for $\mathbb{R}^N$. The solution to Problem 3.1 when $K = B$ is more difficult and is the focus of Section 3.3. Although it is not stated explicitly, the solution to Problem 3.1 in this case is implicit in the work of Holt and Vaaler [39].

Lemma 3.1. Let B be the $N-$dimensional Euclidean unit ball. Then

$$\eta(B) = \frac{2^N}{\text{vol}_N(B)}.$$

We finally remark that if T is a non-singular linear transformation on $\mathbb{R}^N$, then a simple change of variables shows that $|\det(T)|\eta(TK) = \eta(K)$. From our previous remarks it follows that

$$\eta(K) = \frac{2^N}{\mathrm{vol}_N(K)}$$

whenever K is an ellipsoid or a parallelotope. Similarly this show that the inequality appearing in Theorem 3.1 is equality when K is an ellipsoid. In view of these observations, Vaaler and I have formulated the following:

Conjecture 3.1. For any convex body K in $\mathbb{R}^N$,

$$\eta(K) = \frac{2^N}{\mathrm{vol}_N(K)}. \tag{3.1.5}$$

This conjecture, if true, would imply that every convex body has an associated extremal function $F(\mathbf{z})$ which factors as a square, i.e. there exist entire functions $U(\mathbf{z})$ and $V(\mathbf{z})$ such that $F(\mathbf{z}) = U(\mathbf{z})U^*(\mathbf{z}) + V(\mathbf{z})$ where $\int V(\mathbf{t})d\mathbf{t} = 0$. The failure of the conjecture would be striking and lead to further questions: *what is the mechanism which allows the mass to escape?*. It would also imply that for some K, one cannot find extremals for $\eta(K)$ which factor. We note that if K is a parallelotope, then it is easy to obtain such a factorization using known interpolation formulas.

We also investigate the following problem (the reader may wish to set $\nu = (N-2)/2$ upon the first reading which simplifies the expressions):

Problem 3.2. Let $\nu > -1$, K be a convex body and S be the boundary of a star body in $\mathbb{R}^N$, and define

$$\mu_\nu(K,S) = \inf \int_{\mathbb{R}^N} F(\mathbf{x})\|\mathbf{x}\|^{2\nu+2-N} d\mathbf{x} \tag{3.1.6}$$

where the infimum is taken over all integrable functions $F : \mathbb{R}^N \to \mathbb{R}$ which satisfy

(i) $F(\mathbf{x}) \geq 0$ for every $\mathbf{x} \in \mathbb{R}^N$,

(ii) $F(\mathbf{x}) \geq 1$ for every $\mathbf{x} \in S$, and

(iii) $\hat{F}(\boldsymbol{\xi}) = 0$ if $\boldsymbol{\xi} \notin K$.

Determine the value of $\mu_\nu(K,S)$.

The quantity $\mu_\nu(\pi^{-1}B, \xi\mathrm{S}^{N-1})$ is related to $H_\nu^{(N)}(\xi, \pi^{-1})$ from the paper [39] of Holt and Vaaler by

$$\mu_\nu(\pi^{-1}B, \xi\mathrm{S}^{N-1}) \leq H_\nu^{(N)}(\xi, \pi^{-1}) \leq \omega_{N-1} u_\nu(\xi, \pi^{-1}).$$

Lemma 3.2. Let $\delta, \kappa, \xi > 0$, $\nu > -1$, and B be the Euclidean unit ball. Then

$$\mu_\nu(\delta B, \xi\mathrm{S}^{N-1}) = \kappa^{2\nu+2}\mu_\nu(\delta\kappa B, \xi\kappa^{-1}\mathrm{S}^{N-1}) \tag{3.1.7}$$

and

$$\mu_\nu(\pi^{-1}B, \xi\mathrm{S}^{N-1}) = \left(\frac{1}{\omega_{N-1} u_\nu(\xi, \pi^{-1})} + \left| \frac{J_\nu(\xi) J_{\nu+1}(\xi)}{2\omega_{N-1}\xi^{2\nu+1}} \right| \right)^{-1} \tag{3.1.8}$$

where $J_\nu(\xi)$ is the Bessel function and $u_\nu(\xi, \pi^{-1})$ is defined by

$$u_\nu(\xi, \pi^{-1})^{-1} = \frac{\xi J_\nu(\xi)^2 + \xi J_{\nu+1}(\xi)^2 - (2\nu+1) J_\nu(\xi) J_{\nu+1}(\xi)}{2\xi^{2\nu+1}}. \tag{3.1.9}$$

We conjecture that $\mu_\nu(\pi^{-1}B, \xi\mathrm{S}^{N-1}) \leq H_\nu^{(N)}(\xi, \pi^{-1})$. This has recently been shown in the case $\nu = -1/2$ by Littmann [49].

1.3 Notation

We let $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the set of integers, rational numbers, real numbers and complex numbers respectively. n, m, N, M typically represent integers, x, y real numbers, and ω, z denote complex numbers. If N is a positive integer we let $\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}$ usually denote vectors in $\mathbb{R}^N$. We let $\mathscr{U}$ denote the open upper half plane of $\mathbb{C}$, i.e. $\mathscr{U} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ where $\mathrm{Im}(z)$ and $\mathrm{Re}(z)$ denote the real and imaginary parts of $z \in \mathbb{C}$. $B_r(\mathbf{x}) = \left\{\mathbf{y} \in \mathbb{R}^N : |\mathbf{x} - \mathbf{y}| < r\right\}$ denotes the open ball of radius r in $\mathbb{R}^N$. If A is a measurable subset of $\mathbb{R}^N$ we let $|A|$ or $\mathrm{vol}_N(A)$ denote the Lebesgue measure of A. A function $F : \mathbb{C}^N \to \mathbb{C}$ is an entire function if it is entire in each coordinate separately. If $F(\mathbf{z})$ is an entire function, the complex conjugate $F^*(\mathbf{z})$ of $F(\mathbf{z})$ defined by $F^*(\mathbf{z}) = \overline{F(\overline{\mathbf{z}})}$ is also an entire function.

Chapter 2

A variant of the Beurling-Selberg Problem

The goal of this chapter is to gain some non-trivial solutions to the following variant of the Beurling-Selberg problem:

Problem 2.1. *Let* $\chi(t)$ *denote the characteristic function of the interval* $[a, b]$, $\delta > 0$, *and* $\alpha \in \mathbb{C}$ *be in the upper half plane. Define*

$$\rho(a, b; \alpha, \delta) = \inf \int_{-\infty}^{\infty} F(t)dt - (b - a) \tag{2.0.1}$$

where the infimum is taken over entire functions $F : \mathbb{C} \to \mathbb{C}$ *with the following properties:*

(i) $F(t)$ *is integrable on the real axis,*

(ii) $F(t) \geq \chi(t)$ *for each real* t,

(iii) $\hat{F}(\xi) = 0$ *if* $|\xi| > \delta$, *and*

(iv) $F(\alpha) = 0$.

Determine the value of $\rho(a, b; \alpha, \delta)$ *and if possible give a formula for an extremal.*

2.1 Background and Notation

In this chapter x, y will always be real numbers and $z = x + iy$ will be complex. $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ will denote the real and complex parts of z, $\overline{z}$ the complex conjugate of z, if $F(z)$ is an entire function then its complex conjugate is given by $F^*(z) = \overline{F(\overline{z})}$, $\mathscr{U}$ the open upper half plane of $\mathbb{C}$, and α will always be an element of $\mathscr{U}$. If $F : \mathbb{R} \to \mathbb{R}$ is integrable, we define the Fourier transform $\hat{F}(\xi)$ of $F(x)$ by

$$\hat{F}(\xi) = \int_{-\infty}^{\infty} e(-x\xi)F(x)dx$$

where $e(t) = e^{2\pi it}$. We extend the Fourier transform in the usual way to L^2, and when it is convenient we will use the alternate notation $\mathscr{F}(F)(\xi)$ to denote the Fourier transform of $F(x)$.

Suppose that $\hat{F}(\xi) = 0$ whenever $|\xi| > \delta$ for some $\delta > 0$. Then $F(x)$ can be recovered from $\hat{F}(\xi)$ by

$$F(x) = \int_{-\delta}^{\delta} e(x\xi)\hat{F}(\xi)d\xi. \tag{2.1.1}$$

for a.e. $x \in \mathbb{R}$. Using this representation we can easily show that $F(x)$ has an analytic continuation to $\mathbb{C}$ as an entire function. Suppose γ is a simple closed curve in $\mathbb{C}$, then by Fubini's theorem[1]

$$\int_\gamma \int_{-\delta}^{\delta} e(z\xi)\hat{F}(\xi)d\xi dz = \int_{-\delta}^{\delta} \int_\gamma e(z\xi)dz\ \hat{F}(\xi)d\xi = 0.$$

Since γ was arbitrary, it follows from Morera's theorem that, except possibly on a set of measure zero, $F(x)$ has an extension to $\mathbb{C}$ as an entire function.

[1] $\hat{F}(\xi)$ is continuous because we have assumed $F(x)$ is integrable.

By an abuse of notation, let $F(z)$ denote this extension. It follows easily from (2.1.1) that

$$|F(z)| \ll_\epsilon e^{(2\pi\delta+\epsilon)|z|} \tag{2.1.2}$$

for each $\epsilon > 0$. An entire function $F(z)$ which satisfies the growth estimate (2.1.2) is said to be *of exponential type at most* $2\pi\delta$. We have demonstrated that any integrable function $F(x)$ whose Fourier transform $\hat{F}(\xi)$ is supported in $[-\delta, \delta]$ extends to an entire function of exponential type at most $2\pi\delta$. To complete this picture we recall the following classical theorem of Paley and Wiener [71].

Theorem 2.1 (Paley-Wiener)**.** *If $F(z)$ is an entire function of exponential type $2\pi\delta$ that is square-integrable on the real axis, then $\hat{F}(\xi) = 0$ if $|\xi| > \delta$. Conversely, any function $F(x)$ that is square-integrable on $\mathbb{R}$ and that satisfies $\hat{F}(\xi) = 0$ if $|\xi| > \delta$ is a.e. equal to the restriction to the real axis of an entire function of exponential type $2\pi\delta$.*

Let

$$\mathbf{H}_\delta = \Big\{ F \in L^2(\mathbb{R}) \ : \ \hat{F}(\xi) = 0 \ \text{ if } \ |\xi| > \delta/2 \Big\}$$

and identify elements of this space with their extensions to entire functions. $\mathbf{H}_\delta$ is a Hilbert space with respect to the L^2 inner product $\langle \cdot, \cdot \rangle$ with the property that

$$|F(\omega)| \leq C_\omega \|F\|_2$$

for every $F \in \mathbf{H}_\delta$ and $\omega \in \mathbb{C}$. That is, point evaluation is continuous. It follows from the Riesz representation theorem that evaluation at the point ω

is given by the inner product with an element of the space

$$F(\omega) = \langle F, K(\omega, \cdot)\rangle \tag{2.1.3}$$

for every $F \in \mathbf{H}_\delta$ and $\omega \in \mathbb{C}$ where

$$K(\omega, z) = \frac{\sin \pi\delta(z - \overline{\omega})}{\pi(z - \overline{\omega})}. \tag{2.1.4}$$

Throughout our investigation there are many instances where we require an entire function $F(z)$ to be real valued and non-negative on the real axis. We conclude this section with the following factorization lemma for such functions in $\mathbf{H}_\delta$.

Lemma 2.1. *Suppose $F(z)$ is not identically zero, real valued and non-negative on the real axis, and that $F(z) \in \mathbf{H}_\delta$ for some $\delta > 0$. Then there exists an entire function $U(z) \in \mathbf{H}_{\delta/2}$ such that $U(z)$ is zero-free in $\mathscr{U}$ and $F(z) = U(z)U^*(z)$.*

Proof. Let $\{\omega_n : n = 1, 2...\}$ be the zeros of $F(z)$, listed with appropriate multiplicity, in the upper half plane and let

$$B_N(z) = \prod_{n=1}^{N} \frac{1 - z/\overline{\omega_n}}{1 - z/\omega_n}.$$

We define a sequence of entire functions $F_N(z)$ by $F_N(z) = B_N(z)F(z)$. Each of the functions $F_N(z)$ is in $\mathbf{H}_\delta$ by the Paley-Wiener theorem. Since $\|F\| = \|F_N\|$ for each N, it follows that a subsequence of F_N converges weakly to some $G(z)$ in $\mathbf{H}_\delta$. Since the space is a reproducing kernel space, it follows that $F_N(z) \to G(z)$ pointwise for a subsequence. Since $|B_N(z)| \geq 1$ if $z \in \mathscr{U}$ with equality when z is real, it follows that $G(z)$ is zero free in $\mathscr{U}$ and that $|G(t)| = |F(t)|$ for real t. This shows that $F(z)^2 = F(z)F^*(z) = G(z)G^*(z)$. In particular the non-real zeros of $G(z)$ occur with even multiplicity.

Since $F(z)$ is real valued and non-negative on $\mathbb{R}$, the zeros of $G(z)$ occur with even multiplicity and so there is an entire function $U(z)$ for which $G(z) = U(z)^2$. Then $F(z)^2 = \{U(z)U^*(z)\}^2$ and since $F(z)$ is real valued and non-negative on $\mathbb{R}$ it follows that $F(z) = U(z)U^*(z)$. □

2.2 Selberg's Functions - Not yet modified

Let $\chi(x)$ denote the characteristic function of the interval $[-r, r]$ and $\delta > 0$. In the late 1970's Selberg [63, 72] constructed two real entire functions $C(z)$ and $c(z)$ of exponential type $2\pi\delta$ which satisfy

$$c(x) \leq \chi(x) \leq C(x) \quad \text{for each } x \in \mathbb{R},$$

and

$$\int_{-\infty}^{\infty} \{C(x) - \chi(x)\}\, dx = \int_{-\infty}^{\infty} \{\chi(x) - c(x)\}\, dx = \delta^{-1}.$$

In this section we study the behavior of Selberg's functions in a disc of radius $\sim \delta^{-1}$. Our first result is about a zero-free region of $C(z)$.

Theorem 2.2. *Suppose $2r\delta < 1$. Then $C(z) \neq 0$ whenever $|z| \ll \delta^{-1}$.*

It will follow from the proof below that the expression $|z| \ll \delta^{-1}$ can be replaced by $|z| \leq (100\delta)^{-1}$. In the proof of Theorem 2.2 we will invoke the following propositions. The following can be found in [10, p. 83].

Proposition 2.1. *If $f(z)$ is a real entire function of exponential type at most $2\pi\delta$ that is bounded on the real axis, then*

$$|f(x+iy)| \leq \|f\|_\infty \cosh 2\pi\delta y \tag{2.2.1}$$

for all real numbers x and y, where $\|f\|_\infty = \sup_{-\infty<x<\infty} |f(x)|$.

Proposition 2.2. *If $F(z)$ is a real entire function of exponential type at most $2\pi\delta$ that is integrable and nonnegative on the real axis, then for any $p \geq 1$*

$$\left\{\int_{-\infty}^{\infty} |F(x)|^p dx\right\}^{1/p} \leq \delta^{1-1/p} \int_{-\infty}^{\infty} F(x)dx. \tag{2.2.2}$$

Proof. Indeed if $F(z)$ satisfies the condition of the theorem, then by Propositions 2.1 there exists an entire function $U(z)$ of exponential type at most $\pi\delta$ such that $F(z) = U(z)U^*(z)$. It is obvious that

$$\int_{-\infty}^{\infty} |F(x)|^p dx \leq \sup_{-\infty<t<\infty} |F(t)|^{p-1} \int_{-\infty}^{\infty} F(x)dx. \tag{2.2.3}$$

And by the Cauchy-Schwarz inequality

$$\sup_{-\infty<t<\infty} |F(t)| = \sup_{-\infty<t<\infty} |U(t)|^2 \leq \|U\|_2^2 K(t,t) = \delta\|F\|_1.$$

□

Proof of Theorem 2.2. Suppose $C(\omega) = 0$. We will show that $|\omega| \gg \delta^{-1}$. By the mean value theorem and the fact that $C(0) \geq 1$ we have

$$|C'(b\omega)| = \frac{|C(\omega) - C(0)|}{|\omega - 0|} \geq \frac{1}{|\omega|} \tag{2.2.4}$$

for some $0 < b < 1$. We provide an upper bound for $|C'(b\omega)|$ in the following way. By writing $\|C\|_\infty = \lim_{p\to\infty} \|C\|_p$ we find from Proposition 2.2 that $\|C\|_\infty \le \delta(2r + \delta^{-1})$. Therefore Bernstein's inequality and Lemma 2.1 yield

$$|C'(r\omega)| \le 2\pi\delta^2 \cosh(2\pi\delta \mathrm{Im}(\omega))(2r + \delta^{-1}). \tag{2.2.5}$$

Combining (2.2.4) and (2.2.5) yields

$$\frac{1}{2r\delta + 1} \le 2\pi\delta|\omega| \cosh(2\pi\delta \mathrm{Im}(\omega))$$

But since $2r\delta < 1$ it follows that $1/2 \le 2\pi\delta|\omega|\cosh(2\pi\delta\mathrm{Im}(\omega))$. This clearly implies that $|\omega| \gg \delta^{-1}$. □

It is therefore unnatural for extremal majorants of $\chi(t)$ to vanish in a disc around the center of mass of $\chi(t)$.

We will now consider how well Selberg type functions, which have the property that they vanishing at a prescribed point in the upper half plane, can approximate $\chi(t)$. If δ is small enough, then we know this condition is unnatural and Selberg's function is not admissible. Recall $\rho(a, b; \alpha, \delta)$ is the quantity defined in Problem 2.1.

Theorem 2.3. *$\rho(a, b; \alpha, \delta) \gg_{a,b,\alpha} \delta^{-2}$ as $\delta \to 0$.*

Proof. Assume that $a = -b = r$. Following the proof of Theorem 2.2 we find that

$$\frac{1}{|\alpha|} \le \frac{|f_\delta(\alpha) - f_\delta(0)|}{|\alpha|} \le 2\pi\delta^2 \cosh(2\pi\delta\mathrm{Im}(\alpha)) \{2r + \rho(\delta)\}. \tag{2.2.6}$$

But since $\cosh(2\pi\delta\text{Im}(\alpha)) \sim 1$ when $\delta \ll 1$ we have

$$\delta^{-2} \ll_{\alpha,r} 1 + \rho(\delta) \tag{2.2.7}$$

when $\delta \ll 1$. □

In view of these results, another natural question is to ask of the local behavior of $C(\alpha)$ as $\delta \to 0$.

Proposition 2.3. *For each $\alpha \in \mathscr{U}$, $|C(\alpha)| - \text{Re}(C(\alpha)) \ll_{a,b,\alpha} \delta$ as $\delta \to 0$.*

Proof. By following the above proofs we have

$$|C(\alpha) - C(0)| \le 2\pi\delta^2|\alpha|\cosh(2\pi\delta\text{Im}(\alpha))\left\{b - a + \delta^{-1}\right\} \ll \delta. \tag{2.2.8}$$

But upon writing $|C(\alpha) - C(0)| = \left\{[\text{Re}(C(\alpha)) - C(0)]^2 + [\text{Im}(C(\alpha))]^2\right\}^{1/2}$, we find that $|\text{Re}(C(\alpha)) - C(0)| \ll \delta$. By the triangle inequality we have $|\text{Im}(C(\alpha))| = |C(\alpha) - \text{Re}(C(\alpha))| \ll \delta$. Again by the triangle inequality we have $|C(\alpha)| \le |\text{Re}(C(\alpha))| + |\text{Im}(C(\alpha))|$. So $|C(\alpha)| - \text{Re}(C(\alpha)) \le |\text{Im}(C(\alpha))| \ll \delta$. □

2.3 Functions which Interpolate off the Real Axis

Theorem 2.4. *Let $\delta > 0$, $\alpha \in \mathscr{U}$, and $\beta \in \mathbb{C}$. If $F(z)$ is a real entire function of exponential type at most $2\pi\delta$ that is non-negative on the real axis and $F(\alpha) = \beta$, then*

$$\frac{|\beta|K(\alpha,\alpha) - \delta\text{Re}(\beta)}{K(\alpha,\alpha)^2 - \delta^2} \le \frac{1}{2}\int_{-\infty}^{\infty} F(x)dx. \tag{2.3.1}$$

where $K(\omega, z)$ *is defined by (2.1.4). There is equality in (2.3.1) if and only if* $F(z) = U(z)U^*(z)$ *where*

$$U(z) = \lambda_1 K(\alpha, z) + \lambda_2 K(\overline{\alpha}, z),$$

and λ_1 *and* λ_2 *are given by*

$$\lambda_1 = \frac{\beta K(\alpha,\alpha) - \delta}{K(\alpha,\alpha)^2 - \delta^2} \quad \text{and} \quad \lambda_2 = \frac{K(\alpha,\alpha) - \beta\delta}{K(\alpha,\alpha)^2 - \delta^2}. \tag{2.3.2}$$

The proof of this theorem is based on the principal from Hilbert space:

Theorem 2.5. *Let* $\mathbf{H}$ *be a complex vector space with inner product* $\langle \cdot, \cdot \rangle$, $\beta \in \mathbb{C}$, $\mathbf{u}, \mathbf{v} \in \mathbf{H}$ *be linearly independent,* $\eta = \|\mathbf{u}\|\|\mathbf{v}\|$, *and* $\nu = \langle \mathbf{u}, \mathbf{v} \rangle$. *If*

$$\langle \mathbf{h}, \mathbf{u} \rangle \, \overline{\langle \mathbf{h}, \mathbf{v} \rangle} = \beta \tag{2.3.3}$$

then

$$\|\mathbf{h}\|^2 \geq 2\frac{|\beta|\eta - \operatorname{Re}(\beta\nu)}{\eta^2 - |\nu|^2}. \tag{2.3.4}$$

If $\mathbf{h}$ *satisfies (2.3.3), then* $\mathbf{h}$ *achieves equality in (2.3.4) if and only if*

$$\omega \mathbf{h} = \lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} \tag{2.3.5}$$

for some $|\omega| = 1$, *where*

$$\lambda_1 = \frac{\gamma\beta\eta - |\nu|}{\eta^2 - |\nu|^2} \quad \text{and} \quad \lambda_2 = \frac{\eta - \beta\nu}{\eta^2 - |\nu|^2}.$$

and $\gamma = \nu/|\nu|$.

Proof. By scaling considerations it suffices to prove the claim when $|\beta| = \|\mathbf{u}\| = \|\mathbf{v}\| = 1$ and $\nu = \overline{\nu}$.

For any c_1 and c_2

$$\|\mathbf{h} - (c_1\mathbf{u} + c_2\mathbf{v})\|^2 \geq 0. \tag{2.3.6}$$

Expanding this out gives

$$2\mathrm{Re}\Big\{ \langle \mathbf{h}, c_1\mathbf{u} + c_2\mathbf{v}\rangle \Big\} - \|c_1\mathbf{u} + c_2\mathbf{v}\|^2 \leq \|\mathbf{h}\|^2. \tag{2.3.7}$$

Equality occurs in (2.3.7) if and only if $\mathbf{h} = c_1\mathbf{u} + c_2\mathbf{v}$. We let $\mathbf{h}$ satisfy $\langle \mathbf{h}, \mathbf{u}\rangle\,\overline{\langle \mathbf{h}, \mathbf{v}\rangle} = \beta$ and set

$$c_1 = \frac{\beta - \nu}{1 - \nu^2}$$

and

$$c_2 = \frac{1 - \beta\nu}{1 - \nu^2}.$$

It can be checked that

$$\|c_1\mathbf{u} + c_2\mathbf{v}\|^2 = \frac{2 - 2\mathrm{Re}\,(\beta)\,\nu}{1 - \nu^2} = 2\mathrm{Re}(c_2). \tag{2.3.8}$$

Scaling $\mathbf{h}$ by a suitable constant of absolute value 1 we find that

$$\langle \mathbf{h}, \mathbf{u}\rangle = \beta/r \quad \text{and} \quad \langle \mathbf{h}, \mathbf{v}\rangle = r$$

for some $r > 0$. Thus

$$2\mathrm{Re}\Big\{ \langle \mathbf{h}, c_1\mathbf{u} + c_2\mathbf{v}\rangle \Big\} = 2\left\{ \frac{\mathrm{Re}(c_1\overline{\beta})}{r} + \mathrm{Re}(c_2)r \right\}$$

Seeing that $\mathrm{Re}(c_1\overline{\beta}) = \mathrm{Re}(c_2)$, we have

$$\left(\frac{1}{r} + r - 1\right) 2\mathrm{Re}(c_2) \leq \|\mathbf{h}\|^2. \tag{2.3.9}$$

This completes the proof. □

Proof of Theorem 2.4. Suppose the integral of $F(x)$ is finite. Then by Lemma 2.1 there is a function $U(z) \in \mathbf{H}_{\delta/2}$ such that $F(z) = U(z)U^*(z)$. In view of (2.1.3), the condition that $F(\alpha) = \beta$ can be rewritten as

$$\langle U, \mathbf{u}\rangle \, \overline{\langle U, \mathbf{v}\rangle} = \beta$$

where $\mathbf{u} = K(\alpha, z)$ and $\mathbf{v} = K(\overline{\alpha}, z)$. We notice that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent because the determinant of their inner-product matrix is given by

$$K(\alpha,\alpha)^2 - \delta^2 = \frac{\sinh 2\pi\delta \mathrm{Im}(\alpha)}{2\pi \mathrm{Im}(\alpha)} - \delta^2 > 0.$$

The result now follows from Lemma 2.5. □

It may be useful to have knowledge of the Fourier transform of the extremal function $F(z)$ coming from Corollary 2.4.

Theorem 2.6. *Let $K(\omega, z)$ be given by (2.1.4), $\mathscr{F}$ denote the Fourier transform, and*

$$G_{\alpha,\omega}(t) = \mathscr{F}\big(K(\alpha,\cdot)K(\omega,\cdot)\big)(t). \tag{2.3.10}$$

Then

$$G_{\alpha,\omega}(t) = e^{-\pi i t(\overline{\omega}+\overline{\alpha})} \frac{\sin\left\{\pi(\overline{\omega}-\overline{\alpha})(\delta - |t|)_+\right\}}{\pi(\overline{\omega}-\overline{\alpha})}$$

Let $F(z)$ is the extremal function identified in Lemma 2.4, then

$$\mathscr{F}\big(F\big)(t) = (|\lambda_1|^2+|\lambda_2|^2)G_{\alpha,\overline{\alpha}}(t)+(\overline{\lambda_1}\lambda_2 e(-\alpha)+\lambda_1\overline{\lambda_2}e(-\overline{\alpha}))(\delta-|t|)_+, \tag{2.3.11}$$

where λ_1 and λ_2 are given by (2.3.2), and $(x)_+ = \max\{0, x\}$.

Proof. If $a \leq b$ and $\xi \in \mathbb{C}$, Then

$$\int_a^b e^{2\pi i s\xi} ds = \frac{\sin(\pi\xi(b-a))}{\pi\xi} e^{\pi i(b+a)\xi}.$$

Now if $\xi \in \mathbb{C}$ and $\delta > 0$, then

$$\int_{-\infty}^{\infty} e^{2\pi i s\xi} \chi_{[-\delta/2,\delta/2]}(s)\chi_{[-\delta/2,\delta/2]}(t-s) ds = \frac{\sin \pi\xi(\delta - |t|)_+}{\pi\xi} e^{\pi i t\xi}$$

This is seen by observing that

$$\chi_{[-\delta/2,\delta/2]}(s)\chi_{[-\delta/2,\delta/2]}(t-s) \quad = \quad \chi_{[-\delta/2,\delta/2]\cap[t-\delta/2,t+\delta/2]}(s)$$

Hence

$$\int_{-\infty}^{\infty} e^{2\pi i s\xi} \chi_{[-\delta/2,\delta/2]}(s)\chi_{[-\delta/2,\delta/2]}(t-s) ds = \begin{cases} \displaystyle\int_{t-\delta/2}^{\delta/2} e^{2\pi i s\xi} ds & \text{if } t \geq 0 \\ \displaystyle\int_{-\delta/2}^{t+\delta/2} e^{2\pi i s\xi} ds & \text{if } t < 0 \end{cases}$$

But when $t \geq 0$

$$\int_{t-\delta/2}^{\delta/2} e^{2\pi i s\xi} ds \quad = \quad \frac{\sin \pi\xi(\delta - t)}{\pi\xi} e^{\pi i t\xi}$$

and when $t < 0$

$$\int_{-\delta/2}^{t+\delta/2} e^{2\pi i s\xi} ds \quad = \quad \frac{\sin \pi\xi(\delta - |t|)}{\pi\xi} e^{\pi i t\xi}.$$

Finally if $|t| < \delta$, then

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-2\pi i s t} K_\alpha(s) K_\omega(s) ds &= e^{-2\pi i \overline{\omega} t} \int_{-\infty}^{\infty} e^{2\pi i(\overline{\omega}-\overline{\alpha})s} \chi_{[-\delta/2,\delta/2]}(s)\chi_{[-\delta/2,\delta/2]}(t-s) ds \\ &= e^{-2\pi i \overline{\omega} t} e^{\pi i t(\overline{\omega}-\overline{\alpha})} \frac{\sin \pi(\overline{\omega}-\overline{\alpha})(\delta - |t|)}{\pi(\overline{\omega}-\overline{\alpha})} \\ &= e^{-\pi i t(\overline{\omega}+\overline{\alpha})} \frac{\sin \pi(\overline{\omega}-\overline{\alpha})(\delta - |t|)}{\pi(\overline{\omega}-\overline{\alpha})}. \end{aligned}$$

The rest of the proof is straightforward. ☐

If we let $F(z;\alpha,\beta)$ be the unique function determined in Theorem 2.4, then $F(z;\alpha,\beta)$ converges uniformly to zero on $\mathbb{R}$ rapidly when δ goes to infinity. Notice that $K(\alpha,\overline{\alpha}) = \delta$ and $K(\alpha,\alpha) = \sinh 2\pi\delta\mathrm{Im}(\alpha)/2\pi\mathrm{Im}(\alpha)$. Let

$$u(\alpha,\beta,\delta) = 2\frac{|\beta|K(\alpha,\alpha) - \delta\mathrm{Re}(\beta)}{K(\alpha,\alpha)^2 - \delta^2}. \tag{2.3.12}$$

Proposition 2.4. *Then for all real x*

$$|F(x;\alpha,\beta)| \le \delta u(\alpha,\beta,\delta). \tag{2.3.13}$$

In particular,

$$|F(x;\alpha,\beta)| \ll_{\alpha,\beta} \delta e^{-2\pi\mathrm{Im}(\alpha)\delta}. \tag{2.3.14}$$

Proof. By the Cauchy-Schwarz inequality we have

$$|F(x;\alpha,\beta)| = |U(x;\alpha,\beta)|^2 \le \delta\|U(\cdot;\alpha,\beta)\|_2^2 = \delta u(\alpha,\beta,\delta).$$

But since $K(\alpha,\overline{\alpha}) = \delta$ we have

$$u(\alpha,\beta,\delta) = O_{\alpha,\beta}(K(\alpha,\alpha)^{-1}) = O_{\alpha,\beta}(e^{-2\pi\mathrm{Im}(\alpha)\delta}). \tag{2.3.15}$$

□

2.4 Modifications of Selberg's Functions

In this section we show how to modify Selberg's functions so that they vanish at a prescribed point in the upperhalf plane. Let $\delta > 0$ and $F(z;\alpha,\beta)$ be the extremal function described in Corollary 2.4. Notice that we have supressed the dependence of $F(z;\alpha,\beta)$ on δ.

Given N points $\alpha_1,...,\alpha_N \in \mathscr{U}$ represented by $\boldsymbol{\alpha}$ define

$$G^-_{\boldsymbol{\alpha}}(z) = \prod_{n=1}^{N} \left(1 + F(z/N;\alpha_n/N,-1)\right). \tag{2.4.1}$$

It is easy to see that $G^-_{\boldsymbol{\alpha}}(z)$ is an entire function of exponential type $2\pi\delta$ such that $G^-_{\boldsymbol{\alpha}}(x) \geq 1$ on the real axis and $G^-_{\boldsymbol{\alpha}}(\alpha_n) = 0$ for each $n = 1,...,N$. Now observe that the following modification of Selberg's majorant

$$C_{\boldsymbol{\alpha}}(z) = C(z)G^-_{\boldsymbol{\alpha}}(z) \tag{2.4.2}$$

has exponential type $4\pi\delta$, $C_{\boldsymbol{\alpha}}(x) \geq C(x)$ for all real x, and $C_{\boldsymbol{\alpha}}(\alpha_n) = 0$ for each $n = 1,...,N$.

Proposition 2.5.

$$\int_{-\infty}^{\infty} C_{\boldsymbol{\alpha}}(x) - \chi(x)dx \ll \delta^{-2N-1} \quad \text{as } \delta \to 0,$$

and

$$\int_{-\infty}^{\infty} C_{\boldsymbol{\alpha}}(x) - \chi(x)dx \ll \delta^{-1} \quad \text{as } \delta \to \infty,$$

where the implied constants depends on a, b, and $\boldsymbol{\alpha}$.

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} C_{\boldsymbol{\alpha}}(x)dx &= \int_{-\infty}^{\infty} C(x)dx \\ &\quad + \sum_{i_1<\cdots<i_k} \int_{-\infty}^{\infty} C(x)F(x/N;\alpha_{i_1}/N,-1)\cdots F(x/N;\alpha_{i_k}/N,-1)dx \\ &\leq b - a + \delta^{-1} + \|C\|_\infty \sum_{i_1<\cdots<i_k} \prod_{\ell=1}^{k} \|F(x/N;\alpha_{i_\ell}/N,-1)\|_k.\end{aligned} \tag{2.4.3}$$

Proposition 2.2 gives

$$\|F(x/N;\alpha_{i_\ell}/N,-1)\|_k \;=\; N^{1/k}\|F(x;\alpha_{i_\ell},-1)\|_k \le N^{1/k}\frac{2\delta^{1-1/k}}{K(\alpha_\ell,\alpha_\ell)-\delta}$$

and

$$\|C\|_\infty \le \delta(b-a)+1$$

which combined with (2.4.4) gives

$$\int_{-\infty}^{\infty} C_{\boldsymbol{\alpha}}(x)dx \le b-a+\delta^{-1}+N(\delta(b-a)+1)\sum_{i_1<\cdots<i_k} 2^k\delta^{k-1}\prod_{\ell=1}^{k}\frac{1}{K(\alpha_\ell,\alpha_\ell)-\delta} \tag{2.4.4}$$

and by writing

$$K(\alpha_\ell,\alpha_\ell)=\delta+\sum_{n=2}^{\infty}\frac{(2\pi\mathrm{Im}(\alpha_\ell))^{2n-2}}{(2n-2)!}\delta^{2n-1}$$

we find that

$$\int_{-\infty}^{\infty} C_{\boldsymbol{\alpha}}(x)dx \le b-a+\delta^{-1}+N\delta^{-2N-1}(\delta(b-a)+1)\prod_{n=1}^{N}\frac{4}{\min\{1,2\pi\mathrm{Im}(\alpha_n)\}}.$$

This plainly shows that

$$\int_{-\infty}^{\infty} C_{\boldsymbol{\alpha}}(x)-\chi(x)dx \ll \delta^{-2N-1} \quad \text{as } \delta\to 0$$

where the implied constant depends on a, b, and $\boldsymbol{\alpha}$. The last estimate holds by combining (2.4.4) with the estimate $K(\alpha_\ell,\alpha_\ell)\gg e^{2\pi\mathrm{Im}(\alpha_\ell)\delta}$ for large δ. $\square$

2.5 Concluding Remarks (and generalizations)

2.5.1 Minimization in a de Branges space

In this section we show how Corollary 2.4 can be generalized so that the minimization occurs in a fairly general *de Branges space*. A Hilbert space

H which is nontrivial and whose elements are *entire functions* is called a *de Branges space* if (i) $F(z) \in H$ and ω is a non-real zero of $F(z)$, then $(z - \overline{\omega})F(z)/(z-\omega) \in H$ and has the same norm as $F(z)$, (ii) $F(z) \in H$ implies $F^*(z) \in H$ and has the same norm as $F(z)$, and (iii) for every $\omega \in \mathbb{C}$, then functional $F \mapsto F(\omega)$ is continuous. It is a fundamental theorem of de Branges [23] that to each space H one can find an entire function $E(z)$ satisfying the elementary inequality

$$|E(\overline{z})| < |E(z)| \qquad \text{for each } z \in \mathscr{U} \tag{2.5.1}$$

such that the Hilbert space whose elements come from H but whose inner product is given by

$$\langle F, G\rangle_E = \int_{-\infty}^{\infty} F(t)\overline{G(t)}\frac{dt}{|E(t)|^2},$$

with induced norm $\|\cdot\|_E$, is isometric to H. Condition (iii) implies that a de Branges space is a reproducing kernel Hilbert space. We will let $K_E(\omega, z)$ denote the corresponding reproducing kernel.

Conversely, given an entire function which satisfies (2.5.1), there exists a de Branges space H_E which consists of entire functions $F(z)$ which satisfy (i) $\|F\|_E < \infty$, and (ii) $F(z)/E(z)$ and $F^*(z)/E(z)$ are of *bounded type* and non-positive *mean type* in $\mathscr{U}$. A function $g(z)$ which is analytic in $\mathscr{U}$ is said to be of *bounded type* in $\mathscr{U}$ if it can be expressed as the quotient of bounded analytic functions in $\mathscr{U}$. The *mean type* of a function $g(z)$ of bounded type in $\mathscr{U}$ is the number

$$\nu(g) = \limsup_{y\to\infty} y^{-1}\log|g(iy)|$$

if $g(z)$ is not identically zero, and $-\infty$ if $g \equiv 0$. We can now formulate a generalization of Corollary 2.4 for de Branges spaces.

Corollary 2.1. *Let $\alpha \in \mathscr{U}$, $\beta \in \mathbb{C}$, and $E(z)$ be an entire function that satisfies (2.5.1) and that is of bounded type in $\mathscr{U}$. Assume in addition that $K_E(\alpha, z)$ and $K_E(\overline{\alpha}, z)$ are linearly independent. If $F(z)$ is an entire function of exponential type at most $2\tau(E)$ satisfying*

1. *$F(x) \geq 0$ for real x, and*

2. *$F(\alpha) = \beta$,*

then

$$\frac{|\beta| K_E(\alpha,\alpha) - \mathrm{Re}(\beta K_E(\alpha,\overline{\alpha}))}{K_E(\alpha,\alpha)^2 - |K_E(\alpha,\overline{\alpha})|^2} \leq \frac{1}{2}\int_{-\infty}^{\infty} F(x)|E(x)|^{-2}dx. \tag{2.5.2}$$

Equality occurs in (2.5.2) if and only if $F(z) = U(z)U^*(z)$, where

$$U(z) = \lambda_1 K_E(\alpha, z) + \lambda_2 K_E(\overline{\alpha}, z). \tag{2.5.3}$$

The coefficients λ_1 and λ_2 are given by

$$\lambda_1 = \frac{\gamma\beta K_E(\alpha,\alpha) - |K_E(\alpha,\overline{\alpha})|}{K_E(\alpha,\alpha)^2 - |K_E(\alpha,\overline{\alpha})|^2} \quad \text{and} \quad \lambda_2 = \frac{K_E(\alpha,\alpha) - \beta K_E(\alpha,\overline{\alpha})}{K_E(\alpha,\alpha)^2 - |K_E(\alpha,\overline{\alpha})|^2}.$$

Where $\gamma = K_E(\alpha,\overline{\alpha})|K_E(\alpha,\overline{\alpha})|^{-1}$.

Proof. If we can write $F(z) = U(z)U^*(z)$ for some $U(z) \in H_E$ then the result follows from Theorem 2.5 by taking $\mathbf{u} = K_E(\alpha, z)$ and $\mathbf{v} = K_E(\overline{\alpha}, z)$. That $F(z) = U(z)U^*(z)$ for some $U(z) \in H_E$ follows exactly from the proof of Theorem 15 of [39]. □

The condition that $K_E(\alpha, z)$ and $K_E(\overline{\alpha}, z)$ are linearly independent is necessary because of examples such as $E(z) = z + i$. Notice that $A(z) = (1/2)(E(z) + E^*(z)) = z$ and $B(z) = (i/2)(E(z) - E^*(z)) = -1$. The function $K(\omega, z)$ is given by

$$K(\alpha, z) = \frac{B(z)\overline{A(\alpha)} - A(z)\overline{B(\alpha)}}{\pi(z - \overline{\alpha})} = \frac{z - \overline{\alpha}}{\pi(z - \overline{\alpha})} = \pi^{-1}.$$

It follows that $K(\alpha, z) = K(\beta, z)$ for every $\alpha, \beta \in \mathbb{C}$. But $E(z)$ has bounded type and mean type 0 in $\mathscr{U}$, and

$$|E(x - iy)| = |x + i(1 - y)| < |x + i(1 + y)| = |E(x + iy)|$$

when $y > 0$. In fact $H_E \cong \mathbb{C}$ as Hilbert spaces.

The condition that $K_E(\alpha, z)$ and $K_E(\overline{\alpha}, z)$ are linearly independent is superfluous for a large class of $E(z)$, particularly if $E(z)$ has positive mean type then $K_E(\alpha, z)$ and $K_E(\overline{\alpha}, z)$ are linearly independent for all $\alpha \in \mathscr{U}$.

Lemma 2.2. *Let $E(z)$ be an entire function that is of bounded type in $\mathscr{U}$ and $|E^*(z)| < |E(z)|$ for every $z \in \mathscr{U}$. Then given $\alpha \in \mathscr{U}$, the functions $K_E(\alpha, z)$ and $K_E(\overline{\alpha}, z)$ are linearly independent in H_E if either of the following conditions hold:*

1. *$E(z)$ has positive mean type, $\nu(E) > 0$;*

2. *$E(z)$ has more than one non-real zero.*

Proof. We will begin by showing that it suffices to prove the lemma when $E(z)$ has no real zeros. Suppose $E(z)$ has real zeros, then $E(z)$ can be written as $E(z) = E_0(z)S(z)$ where $S(z) = S^*(z)$ and $E_0(z)$ has no real zeros. But since $|E^*(z)| < |E(z)|$ for all $z \in \mathscr{U}$ it follows that $S(z)$ has only real zeros and that $|E_0^*(z)| < |E_0(z)|$ for all $z \in \mathscr{U}$. Following problem 44 from [23] we find that $F(z) \mapsto F(z)S(z)$ is a linear isometry from H_{E_0} onto H_E. Now we will show that if there does not exist a nonzero constant $c \in \mathbb{C}$ such that $F_0(\alpha) = cF_0(\overline{\alpha})$ for all $F_0 \in H_{E_0}$, then there does not exist a non-zero constant $\tilde{c} \in \mathbb{C}$ such that $F(\alpha) = \tilde{c}F(\overline{\alpha})$ for all $F \in H_E$. Suppose there does not exist a non-zero constant $c \in \mathbb{C}$ such that $F_0(\alpha) = cF(\overline{\alpha})$ for all $F_0 \in H_{E_0}$. But $F(z) = F_0(z)S(z)$ is in H_E for every $F_0 \in H_{E_0}$. Suppose, by way of contradiction, that there was a non-zero constant $c \in \mathbb{C}$ such that $F(\alpha) = cF(\overline{\alpha})$ for every $F \in H_E$. Then $F_0(\alpha)S(\alpha) = cF_0(\overline{\alpha})S(\overline{\alpha})$ for every $F_0 \in H_{E_0}$. But seeing that $S(\alpha) = S^*(\alpha) \neq 0$, the number $\tilde{c} = cS(\overline{\alpha})/S(\alpha)$ is non-zero, and $F_0(\alpha) = \tilde{c}F_0(\overline{\alpha})$ for all $F_0 \in H(E_0)$, a contradiction. Therefore it suffices to prove the lemma when $E(z)$ has no real zeros.

(1) Suppose $\nu(E) = \pi\delta > 0$. Let us first deal with the case when $E(z)$ has no zeros. Then up to scaling $E(z) = e^{-i\pi\delta z}$, by Nevanlinna's factorization or the product representation for Pólya class. The reproducing kernel for H_E is then given by

$$K(\omega, z) = \frac{\sin \pi\delta(z - \overline{\omega})}{\pi(z - \overline{\omega})}. \tag{2.5.4}$$

And $K(\alpha, z)$ and $K(\overline{\alpha}, z)$ are linearly independent in H_E if and only if $K(\alpha, \alpha)^2 - |K(\alpha, \overline{\alpha})|^2 \neq 0$. But

$$K(\alpha, \alpha)^2 - |K(\alpha, \overline{\alpha})|^2 = \left\{ \frac{\sinh 2\pi\delta \mathrm{Im}(\alpha)}{2\pi \mathrm{Im}(\alpha)} \right\}^2 - \delta^2.$$

So the condition that $K(\alpha,\alpha)^2 - |K(\alpha,\overline{\alpha})|^2 \neq 0$ is equivalent to $\sinh 2\pi\delta \mathrm{Im}(\alpha) \neq 2\pi\delta\mathrm{Im}(\alpha)$, which is true when $\mathrm{Im}(\alpha) > 0$.

Now suppose that $E(z)$ has one zero ω in the open lower half plane. Then up to scaling $E(z) = e^{-i\pi\delta z}(z-\omega)$. The function $G(z) = 1$ is then in H_E since $\nu(1/E) = 0 - \nu(E) \leq 0$ and $|t-\omega|^{-2}$ is integrable. Similarly the function $H(z) = e^{-i\pi\delta z}$ is in H_E. Now suppose, by way of contradiction, that there is a non-zero constant $c \in \mathbb{C}$ such that $F(\alpha) = cF(\overline{\alpha})$ for all $F \in H_E$. Then $c = 1$ since $1 = G(\alpha) = cG(\overline{\alpha}) = c$. But $|H(\alpha)| \neq |H(\overline{\alpha})|$ so it is impossible for $H(\alpha) = H(\overline{\alpha})$, a contradiction.

(2) Now we need not suppose that $\nu(E) > 0$. Let us first treat the case when $E(z)$ has $N > 1$ zeros $\omega_1, ..., \omega_N$ all of which are in the open lower half plane. By a similar argument as above, the function $G(z) = 1$ is in H_E. And the function $L(z) = z$ is also in H_E since $\nu(z/E) = \nu(z) - \nu(E) = 0 - \nu(E) \leq 0$ and $t^2|t-\omega_1|^{-2}\cdots|t-\omega_N|^2$ is integrable. Now suppose by way of contradiction that there exists a non-zero constant $c \in \mathbb{C}$ such that $F(\alpha) = cF(\overline{\alpha})$ for all $F \in H_E$. Then $c = 1$ since $G \in H_E$, and $L(\alpha) = cL(\overline{\alpha})$ implies $\alpha = \overline{\alpha}$, a contradiction.

Now suppose $E(z)$ has infinitely many zeros $\omega_1, \omega_2, ...$ in the lower half plane, ordered in such a way that $|\omega_k| \leq |\omega_{k+1}|$. The condition $|E^*(z)| < |E(z)|$ for all $z \in \mathscr{U}$ guarantees that $E(z) \not\equiv 0$ and so $|\omega_k| \to \infty$ as $k \to \infty$. For each positive integer k let $F_k(z) = E(z)(z - \omega_k)^{-1}$. Observe that $\nu(F_k/E) = 0$ and $\nu(F_k^*/E) = \nu(E^*) - \nu(E)$. But seeing that $|E^*(iy)| < |E(iy)|$ for $y > 0$ we find that $\nu(E^*) \leq \nu(E)$ from which it follows that $\nu(F_k^*/E) \leq 0$. It is now clear that $F_k(z)$ is in H_E for all k.

Suppose, by way of contradiction, that there exists a non-zero constant $c \in \mathbb{C}$ such that $F(\alpha) = cF(\overline{\alpha})$ for all $F(z)$ in H_E. If $E(\overline{\alpha}) \neq 0$ we have

$$\begin{aligned} F_k(\alpha) &= E(\alpha)(\alpha - \omega_k)^{-1} \\ &= cE(\overline{\alpha})(\overline{\alpha} - \omega_k)^{-1} \\ &= cF_k(\overline{\alpha}) \end{aligned}$$

which implies

$$c\frac{E(\overline{\alpha})}{E(\alpha)} = \frac{\alpha - \omega_k}{\overline{\alpha} - \omega_k}.$$

Taking limits as $k \to \infty$ gives $|cE(\overline{\alpha})| = |E(\alpha)|$ which implies $|c| > 1$. But seeing that $F_k^*(z)$ is in H_E we have $F_k^*(\alpha) = cF_k^*(\overline{\alpha}) = c\overline{F_k(\alpha)} = c\overline{c}F_k^*(\alpha)$ which implies $|c| = 1$, a contradiction.

Now suppose $E(\overline{\alpha}) = 0$ and recall that $E(\alpha) \neq 0$. Then $F_k(\overline{\alpha}) \neq 0$ for sufficiently large k but $F_k(\alpha) = 0$ for all k. Therefore there cannot exist a non-zero constant $c \in \mathbb{C}$ such that $F_k(\alpha) = cF_k(\overline{\alpha})$ for all k.

$\square$

2.5.2 Trigonometric Polynomials

Let $N \geq 1$ and $\mathscr{P}_N \subset \mathbb{C}[z]$ be the complex vector space of polynomials of degree at most N. Define an inner product $\langle \cdot, \cdot \rangle$ on $\mathscr{P}_N$ by

$$\langle p, q \rangle = \int_{S^1} p(\theta)\overline{q(\theta)} d\sigma(\theta) \tag{2.5.5}$$

where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and σ is the Haar probability measure on S^1. Let $\| \cdot \|$ be the norm induced by $\langle \cdot, \cdot \rangle$ and $e(t) = e^{2\pi i t}$.

Proposition 2.6. *The polynomials $1, z, z^2, ..., z^N$ form an orthonormal basis for $\mathscr{P}_N$ and the space $\mathscr{P}_N$ is complete with respect to $\| \cdot \|$.*

Proof. These polynomials clearly span $\mathscr{P}_N$, so we only need to show that they are orthonormal. Observe

$$\begin{aligned} \langle z^n, z^m \rangle &= \int_{S^1} \theta^n \overline{\theta}^m d\sigma(\theta) \\ &= \int_0^1 e((n-m)t) dt \\ &= \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases} \end{aligned}$$

Let $p_1, p_2, ...$ is a Cauchy sequence in $\mathscr{P}_N$. Then there exist complex numbers $a_{k,n}$ for $k = 1, 2, ...$ and $n = 0, 1, ..., N$ such that

$$p_k(z) = a_{k,0} + a_{k,1} z + \cdots + a_{k,N} z^N.$$

Clearly $\|p_k - p_\ell\|^2 = \sum_{n=0}^{N} |a_{k,n} - a_{k,\ell}|^2$ which implies $a_{1,n}, a_{2,n}, ...$ is a Cauchy sequence for each $n = 1, ..., N$. Completeness follows from the completeness of $\mathbb{C}$. $\square$

Lemma 2.3. *The space* $(\mathscr{P}_N, \langle\cdot,\cdot\rangle)$ *is a reproducing kernel Hilbert space with reproducing kernel* $K : \mathbb{C}\times\mathbb{C}\to\mathbb{C}$ *given by*

$$K(\omega, z) = \sum_{n=0}^{N} z^n \overline{\omega}^n. \tag{2.5.6}$$

Furthermore, if $\alpha \neq 0$*, the functions* $z \mapsto K(\alpha, z)$ *and* $z \mapsto K(1/\overline{\alpha}, z)$ *are linearly independent if and only if* $\alpha \notin S^1$*.*

Proof. The form of the reproducing kernel is a standard fact, but we note that it doesn't depend on the choice of orthonormal basis. We need only verify the latter statement. The functions in question are linearly independent if and only if

$$K(\alpha,\alpha)K(1/\overline{\alpha},1/\overline{\alpha}) - |K(1/\overline{\alpha},\alpha)|^2 \neq 0. \tag{2.5.7}$$

But $K(\alpha,\alpha) = \sum_{n=0}^{N} |\alpha|^n$, $K(\alpha, 1/\overline{\alpha}) = N+1$ and $K(1/\overline{\alpha},1/\overline{\alpha}) = \sum_{n=0}^{N} |\alpha|^{-n}$. So (2.5.7) can be rewritten as

$$\left\{\sum_{n=0}^{N} |\alpha|^n\right\}\left\{\sum_{n=0}^{N} |\alpha|^{-n}\right\} - (N+1)^2 \neq 0. \tag{2.5.8}$$

But by the arithmetic-geometric mean inequality we have

$$(N+1)^{-2}\left\{\sum_{n=0}^{N} |\alpha|^n\right\}\left\{\sum_{n=0}^{N} |\alpha|^{-n}\right\} \geq \left\{\prod_{n=0}^{N} |\alpha|^n\right\}^{1/(N+1)}\left\{\prod_{m=0}^{N} |\alpha|^{-m}\right\}^{1/(N+1)} = 1 \tag{2.5.9}$$

and equality holds if and only if $|\alpha|^m = |\alpha|^n$ for $n, m = 1, 2, ..., N$ which implies $|\alpha| = 1$. □

To state the next theorem, we will intoduce the notation $[\omega; M]$ where $\omega \in \mathbb{C}$ and $M \in \mathbb{Z}^+$ by

$$[\omega; M] = \sum_{m=0}^{M} |\omega|^m. \tag{2.5.10}$$

Corollary 2.2. *Suppose $\alpha \neq 0$, $\alpha \notin S^1$, $\beta \in \mathbb{C}$, and $N \geq 1$. If $F(z)$ is a Laurent polynomial of degree at most N,*

1. *$F(\theta) \geq 0$ for $\theta \in S^1$, and*

2. *$F(\alpha) = \beta$,*

then

$$\frac{|\beta|[\alpha; N]^{1/2}[\alpha^{-1}; N]^{1/2} - (N+1)\mathrm{Re}(\beta)}{[\alpha; N][\alpha^{-1}; N] - (N+1)^2} \leq \frac{1}{2}\int_{S^1} F(\theta) d\sigma(\theta). \tag{2.5.11}$$

Equality occurs above if and only if $F(z) = p(z)p^*(z)$, where

$$p(z) = \lambda_1 K(\alpha, z) + \lambda_2 K(1/\overline{\alpha}, z). \tag{2.5.12}$$

The coefficients λ_1 and λ_2 can be explicitly computed in terms of K, α, β and N.

Proof. Let $p^*(z) = \overline{p(1/\overline{z})}$. Then by Fejer's theorem $F(z) = p(z)p^*(z)$ for some $p \in \mathscr{P}_N$. The Corollary is proved upon appealing to Theorem 2.5 when $\mathbf{H} = \mathscr{P}_N$. □

2.5.3 Another Uncertainty type principle

Corollary 2.3. *Let G be a locally compact Abelian group with Haar measure μ and $\hat{G}$ be its unitary dual with measure ν. Select the measures in such a way that makes the scaling constant in the Fourier inversion formula equal to 1. If U is a Borel subset of G and V is a Borel subset of $\hat{G}$, both of finite measure, then let*

$$\rho(U,V) = \int_U \int_V \overline{\chi(g)} d\nu(\chi) d\mu(g). \tag{2.5.13}$$

Then given $\beta \in \mathbb{C}$, if $f \in L^2(G,\mu)$ satisfies

$$\int_U f(g) d\mu(g) \int_V \hat{f}(\chi) d\nu(\chi) = \beta, \tag{2.5.14}$$

then

$$\frac{1}{2} \|f\|^2_{L^2(G,\mu)} \geq \frac{|\beta| \mu(U)^{1/2} \nu(V)^{1/2} - \operatorname{Re}\big\{\rho(U,V)\beta\big\}}{\mu(U)\nu(V) - |\rho(U,V)|^2}. \tag{2.5.15}$$

Proof. We wish to apply Theorem 2.5 when $\mathbf{H} = L^2(G,\mu)$. We let $\mathbf{u} = \Delta_U$ and $\mathbf{v} = \check{\Delta}_V$, where

$$\Delta_A(a) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{otherwise.} \end{cases}$$

The Fourier transform of Δ_V is given by

$$\begin{aligned} \check{\Delta}_V(g) &= \int_{\hat{G}} \chi(g) \Delta_V(\chi) d\nu(\chi) \\ &= \int_V \chi(g) d\nu(\chi). \end{aligned}$$

Hence

$$\left\langle \Delta_U, \check{\Delta}_V \right\rangle_{L^2(G,\mu)} = \int_G \Delta_U(g)\overline{\check{\Delta}_V(g)}d\mu(g)$$
$$= \int_U \overline{\int_V \chi(g)d\nu(\chi)}d\mu(g)$$
$$= \rho(U,V).$$

Now if $\langle f, \Delta_U \rangle \left\langle f, \check{\Delta}_V \right\rangle = \beta$, then there is a $|\omega| = 1$ such that

$$\langle \omega f, \Delta_U \rangle \overline{\left\langle \omega \hat{f}, \Delta_V \right\rangle} = \beta.$$

And $\|\omega f\| = \|f\|$. □

Chapter 3

Extremal Problems for Convex Bodies and the Fourier Transform

3.2 Preliminaries

To each convex body K we associate the dual norm (extended to $\mathbb{C}^N$) by

$$\|\mathbf{z}\|_K^* = \sup\{|\mathbf{z}\cdot\mathbf{y}| : \mathbf{y}\in K\}. \tag{3.2.1}$$

This norm on $\mathbb{R}^N$ is also called the *support function* of K. The dual norm will be important to us in two ways. (1) For growth estimates for certain entire functions, and (2) the following volume formula for K^*.

Proposition 3.1. *Let K be a symmetric body and K^* be its dual body. Then*

$$\mathrm{vol}_N(K^*) = \frac{1}{N}\int_{\mathrm{S}^{N-1}} \left\{\frac{1}{\|\boldsymbol{\theta}\|_K^*}\right\}^N d\sigma(\boldsymbol{\theta}) \tag{3.2.2}$$

where $\mathbf{d}\sigma$ *is the usual surface measure on the unit sphere* S^{N-1} *in* $\mathbb{R}^N$.

Proof. For $\boldsymbol{\theta}\in\mathrm{S}^{N-1}$ let

$$\rho(\boldsymbol{\theta}) = \sup\{r>0 : r\boldsymbol{\theta}\in K^*\}.$$

By writing the volume of K^* in spherical coordinates we have

$$\begin{aligned}\mathrm{vol}_N(K^*) &= \int_{\mathrm{S}^{N-1}} \int_0^{\rho(\boldsymbol{\theta})} r^{N-1} dr d\sigma(\boldsymbol{\theta}) \\ &= \frac{1}{N} \int_{\mathrm{S}^{N-1}} \{\rho(\boldsymbol{\theta})\}^N d\sigma(\boldsymbol{\theta})\end{aligned}$$

It is not difficult to check that $\rho(\boldsymbol{\theta}) = 1/\|\boldsymbol{\theta}\|_K^*$, see [62, Theorem 1.7.6]. □

An integrable function $F(\mathbf{x})$ will be called *admissible* if $F(\mathbf{x})$ satisfies conditions (i)-(iii) in Problem 3.1. $F(\mathbf{x})$ will be called *extremal* if it is admissible and its integral is equal to $\eta(K)$. Suppose $F(\mathbf{x})$ is an admissible function for Problem 3.1. The function

$$\mathbf{z} \mapsto \int_K e(\mathbf{z} \cdot \boldsymbol{\xi}) \hat{F}(\boldsymbol{\xi}) d\boldsymbol{\xi} \tag{3.2.3}$$

is equal to $F(\mathbf{x})$ for almost every $\mathbf{x}$ in $\mathbb{R}^N$ and for each closed curve γ in $\mathbb{C}$ we have for each $\ell = 1, ..., N$

$$\int_\gamma \int_K e(\mathbf{z} \cdot \boldsymbol{\xi}) \hat{F}(\boldsymbol{\xi}) d\boldsymbol{\xi} dz_\ell = \int_K \int_\gamma e(\mathbf{z} \cdot \boldsymbol{\xi}) dz_\ell \hat{F}(\boldsymbol{\xi}) d\boldsymbol{\xi} = 0$$

where we have used Fubini's theorem to interchange the order of integration. Since the curve γ was arbitrary (3.2.3) defines an entire function by Morera's theorem. Consequentially, except for possibly on a set of measure zero, $F(\mathbf{x})$ is the restriction to $\mathbb{R}^N$ of an entire function. We will always identify admissible functions with their extensions to entire functions. The representation (3.2.3) shows that admissible functions satisfy the following growth estimate in $\mathbb{C}^N$

$$|F(\mathbf{z})| \ll_\epsilon e^{2\pi(1+\epsilon)\|\mathbf{z}\|_K^*}. \tag{3.2.4}$$

for each $\epsilon > 0$. In the single variable case, entire functions which satisfy an estimate of the same type as (3.2.4) are called *entire functions of exponential type.* An entire function $F(z)$ is said to be of exponential type $2\pi\tau > 0$ if

$$|F(z)| \ll_\epsilon e^{2\pi\tau(1+\epsilon)|z|}$$

for each $\epsilon > 0$. The Paley-Wiener theorem gives two equivalent ways of looking at entire functions of exponential type which are square-integrable on the real axis.

3.3 Discussion and Proof that $\eta(B) = 2^N/\mathrm{vol}_N(B)$

As the title suggests, our main goal of this section is to prove

$$\eta(B) = \frac{2^N}{\mathrm{vol}_N(B)}$$

where B is the Euclidean unit ball in $\mathbb{R}^N$. This result is implicit in the work Holt and Vaaler [39]. And since the proof of this result does not require the full force of the Holt-Vaaler machinery we will provide a self contained proof here.

Suppose $F(\mathbf{z})$ is an admissible function for this problem. By averaging over $SO(N)$ we find that

$$\int_{\mathbb{R}^N} F(x)d\mathbf{x} = \int_{\mathbb{R}^N}\int_{SO(N)} F(\mathbf{gx})d\mu(\mathbf{g})d\mathbf{x}$$

where μ is the normalized Haar measure on $SO(N)$, and that the function

$$\mathbf{x} \mapsto \int_{SO(N)} F(\mathbf{gx})d\mu(\mathbf{g})$$

is admissible. In view of this observation we can safely limit our search to extremal functions which are *radial*. We will see momentarily that the extremal function we find can be factored as $F(\mathbf{z}) = U(\mathbf{z})U^*(\mathbf{z})$ where $U(\mathbf{z})$ is square integrable and radial on $\mathbb{R}^N$ and $\hat{U}(\boldsymbol{\xi})$ is supported in $1/2B$. This allows us to recast the extremal problem as a minimization problem in the Hilbert space

$$\mathbf{H} = \left\{ U(\mathbf{x}) \in L^2(\mathbb{R}^N) \ : \ U(\mathbf{x}) \text{ is continuous and } \operatorname{supp}(\hat{U}(\boldsymbol{\xi})) \subset 1/2B \right\}.$$

$\mathbf{H}$ is a Hilbert space with respect to the L^2-inner product $\langle \cdot, \cdot \rangle$ with the property that for every $\mathbf{z} \in \mathbb{C}^N$ and $f \in H$

$$f(\mathbf{z}) = \langle f, K(\mathbf{z}, \cdot) \rangle \tag{3.3.1}$$

where

$$K(\boldsymbol{\omega}, \mathbf{z}) = \int_{1/2B} e\big(-(\mathbf{z} - \overline{\boldsymbol{\omega}}) \cdot \boldsymbol{\xi} \big) d\boldsymbol{\xi}. \tag{3.3.2}$$

We identify the elements of $\mathbf{H}$ with their entire extensions to $\mathbb{C}^N$. Let $\mathbf{H}_1$ be the $N = 1$ case of $\mathbf{H}$. Functions in $\mathbf{H_1}$ which are real-valued and non-negative on the real axis enjoy a factorization akin to that for non-negative trigonometric polynomials given by the Fejér-Riesz theorem . The following is an extension of Lemma 2.1.

Proposition 3.2. *Suppose $F(z) \in \mathbf{H}_1$ is real valued and non-negative on the real axis and that $F(z)$ is not identically zero. Then there exists an entire function $U(z) \in \mathbf{H}_1$ such that $U(z)$ is zero-free in $\mathscr{U}$ and $F(z) = U(z)U^*(z)$. If $F(z)$ is also even, then $F(z)$ admits the factorization*

$$F(z) = z^{2k} Q(z) V(z) V^*(z)$$

where k is the multiplicity of the possible zero at $z = 0$, $Q(z)$ has only purely imaginary zeros, and $V(z)$ is even.

Proof. By Lemma 2.1 we have the factorization $F(z) = U(z)U^*(z)$.

If $F(z)$ is even write $U(z) = z^k p(z)R(z)R^*(-z)$ where $R(z)$ contains the zeros of $U(z)$ which have strictly positive real part, $p(z)$ contains only purely imaginary zeros, and k is the multiplicity of the zero at 0. Let $V(z) = R(z)R(-z)$ and $Q(z) = p(z)p^*(z)$. □

We now introduce a notation for restrictions and extensions for dealing with radial functions. If $G(\mathbf{z})$ is a radial function, that is the restriction of $G(\mathbf{z})$ to $\mathbb{R}^N$ is radial, we let $g(z)$ denote its restriction to a line, say one of the coordinate axes. Similarly if $g(z)$ is an even entire function, we may extend $g(z)$ to a radial function $G(\mathbf{z})$ on $\mathbb{C}^N$ by

$$G(\mathbf{z}) = \sum_{\ell=0}^{\infty} \frac{g^{(2\ell)}(0)}{(2\ell)!} \{z_1^2 + \cdots + z_N^2\}^{\ell}.$$

Let $F(\mathbf{z})$ be an admissible function for our problem and assume that $F(\mathbf{z})$ is radial. Then the corresponding restriction $f(z)$ is an even function in $\mathbf{H_1}$ that is real-valued and non-negative on the real axis. Therefore $f(z)$ admits the representation

$$f(z) = q(z)v(z)v^*(z)$$

where $q(z)$ and $v(z)$ are even entire functions and $q(z)$ has only purely imaginary zeros. We choose the functions in such a way that $|v(0)|^2 = q(0) = 1$.

Seeing that $q(z)$ and $v(z)$ are even, we extend them to $\mathbb{C}^N$ to obtain the following factorization for $F(\mathbf{z})$

$$F(\mathbf{z}) = Q(\mathbf{z})V(\mathbf{z})V^*(\mathbf{z}).$$

The integral of $F(\mathbf{x})$ now has the form

$$\int_{\mathbb{R}^N} F(\mathbf{x})d\mathbf{x} = \int_{\mathbb{R}^N} Q(\mathbf{x})|V(\mathbf{x})|^2 d\mathbf{x}$$

But if $F(\mathbf{x})$ is extremal, then $q(z)$ is zero free. Suppose, by way of contradiction, that $q(z)$ has a zero at say iy for $y > 0$. Then

$$q(z) = \left(1 + \frac{z^2}{y^2}\right)\tilde{q}(z)$$

for some even entire function $\tilde{q}(z)$ such that $\tilde{q}(0) = 1$, and $\tilde{q}(x) \geq 0$ for real x. In particular, $\tilde{q}(x) < q(x)$ for all non-zero real numbers x. This plainly shows that the admissible function $\tilde{F}(\mathbf{z}) = \tilde{Q}(\mathbf{z})V(\mathbf{z})V^*(\mathbf{z})$ has smaller L^1-norm than $F(\mathbf{z})$. Therefore we may assume

$$F(\mathbf{z}) = V(\mathbf{z})V^*(\mathbf{z})$$

where $V(\mathbf{x}) \in H$. But by the Cauchy-Schwarz inequality and (3.3.1)

$$1 \leq F(\mathbf{0}) = |V(\mathbf{0})|^2 \leq K(\mathbf{0},\mathbf{0})\|V\|_2^2 = \mathrm{vol}_N(1/2B)\|V\|_2^2 \tag{3.3.3}$$

where equality occurs if and only if $F(\mathbf{0}) = 1$ and $V(z)$ is a scalar multiple of $K(\mathbf{0},\mathbf{z})$. But

$$\|V\|_2^2 = \int_{\mathbb{R}^N} F(\mathbf{x})d\mathbf{x}.$$

Therefore

$$\eta(B) = \frac{2^N}{\mathrm{vol}_N(B)}.$$

3.4 Proof of Theorem 3.1

Let $F(\mathbf{x})$ be an admissible function for Problem 3.1. We may assume that $F(\mathbf{x})$ is even, because $1/2(F(\mathbf{x})+F(-\mathbf{x}))$ is admissible and it has the same integral as $F(\mathbf{x})$. For each $\boldsymbol{\theta} \in \mathrm{S}^{N-1}$ the function

$$z \mapsto F(z\boldsymbol{\theta})$$

is entire of exponential type at most $2\pi\|\boldsymbol{\theta}\|_K^*$ by (3.2.4). Let $F(\mathbf{z};\boldsymbol{\theta})$ be the radial extension of this function. Observe

$$\begin{aligned}
\int_{\mathbb{R}^N} F(\mathbf{x})d\mathbf{x} &= \int_{\mathrm{S}^{N-1}} \int_0^\infty F(r\boldsymbol{\theta})|r|^{N-1} dr d\sigma(\boldsymbol{\theta}) \\
&= \frac{1}{\omega_{N-1}} \int_{\mathrm{S}^{N-1}} \int_{\mathbb{R}^N} F(\mathbf{x};\boldsymbol{\theta}) d\mathbf{x} d\sigma(\boldsymbol{\theta}) \\
&\geq \fint_{\mathrm{S}^{N-1}} \eta(\|\boldsymbol{\theta}\|_K^* B) d\sigma(\boldsymbol{\theta}) \\
&= \frac{2^N}{\mathrm{vol}_N(B)} \fint_{\mathrm{S}^{N-1}} \left\{ \frac{1}{\|\boldsymbol{\theta}\|_K^*} \right\}^N d\sigma(\boldsymbol{\theta}) \qquad (3.4.1)
\end{aligned}$$

where ω_{N-1} is the surface area of S^{N-1}, and we have used Lemma 3.1 in the last line. But by Proposition 3.1

$$\frac{1}{N} \int_{\mathrm{S}^{N-1}} \left\{ \frac{1}{\|\boldsymbol{\theta}\|_K^*} \right\}^N d\sigma(\boldsymbol{\theta}) = \mathrm{vol}_N(K^*)$$

Seeing that $\omega_{N-1} = N\mathrm{vol}_N(B)$ it follows that

$$\frac{\mathrm{vol}_N(K^*)}{\mathrm{vol}_N(B)} = \fint_{\mathrm{S}^{N-1}} \left\{ \frac{1}{\|\boldsymbol{\theta}\|_K^*} \right\}^N \mathbf{d}\sigma(\boldsymbol{\theta}). \qquad (3.4.2)$$

By substituting (3.4.2) into (3.4.1), we obtain the desired result.

3.5 Discussion and Proof of Lemma 3.2

Throughout this section suppose *a priori* that $F : \mathbb{R}^N \to \mathbb{R}$ is admissible and extremal for Lemma 3.2. Let

$$\mathbf{H}_\nu = \left\{ U(\mathbf{x}) \in L^2(\mathbb{R}^N; \|\mathbf{x}\|^{2\nu+2-N} d\mathbf{x}) \ : \ \hat{U}(\boldsymbol{\xi}) = 0 \text{ if } \boldsymbol{\xi} \notin (1/2\pi)B \right\}.$$

By our previous discussion we may assume F is radial. Positivity and extremality give that there exists a radial entire function $U(z)$ in $\mathbf{H}_\nu$ such that

$$|U(\mathbf{x})| \geq 1 \quad \text{whenever} \quad \mathbf{x} \in S \quad \text{and} \quad F(\mathbf{z}) = U(\mathbf{z})U^*(\mathbf{z}).$$

Now for each $\nu > -1$ there is an entire function $E_\nu(z)$, the de Branges structure function, associated to a de Branges homogeneous space $\mathscr{H}(E_\nu)$ which has the property that

$$c_\nu \int_{-\infty}^{\infty} |f(t)|^2 |t|^{2\nu+1} dt = \int_{-\infty}^{\infty} \left| \frac{f(t)}{E_\nu(t)} \right|^2 dt$$

for each $f \in \mathscr{H}(E_\nu)$ where $c_\nu = \pi 2^{-2\nu-1}\Gamma(\nu+1)^{-2}$. It follows from the work of Holt and Vaaler that $z \mapsto U(z\boldsymbol{\theta})$ is in $\mathscr{H}(E_\nu)$ for any $\boldsymbol{\theta} \in \mathrm{S}^{N-1}$. We will call such a selection $U(z)$. Therefore

$$\begin{aligned}
\mu_\nu(B, \xi \mathrm{S}^{N-1}) &= \int_{\mathbb{R}^N} F(\mathbf{x}) \|\mathbf{x}\|^{2\nu+2-N} d\mathbf{x} && (3.5.1)\\
&= \frac{\omega_{N-1}}{2} \int_{-\infty}^{\infty} |U(t)|^2 |t|^{2\nu+1} dt && (3.5.2)\\
&= \frac{\omega_{N-1}}{2c_\nu} \int_{-\infty}^{\infty} \left| \frac{U(t)}{E_\nu(t)} \right|^2 dt && (3.5.3)\\
&= \frac{\omega_{N-1}}{2c_\nu} \|U\|_{E_\nu}^2 && (3.5.4)
\end{aligned}$$

where $\|\cdot\|_{E_\nu}$ is the de Branges space norm and we will let $\langle \cdot, \cdot \rangle_{E_\nu}$ is the associated inner product. Evidently $U(z)$ satisfies the following relations

$$| \langle U, K_\nu(-\xi, \cdot) \rangle_{E_\nu} | \geq 1 \quad \text{and} \quad | \langle U, K_\nu(\xi, \cdot) \rangle_{E_\nu} | \geq 1$$

where $K_\nu(\omega, z)$ is the reproducing kernel for $\mathscr{H}(E_\nu)$.

The crucial inequality giving extremality in the previous discussion is the use of the Cauchy-Schwarz inequality in (3.3.3) in conjunction with the reproducing property. In this section we will need the following *2 point* version of this inequality:

Lemma 3.3. *Let* $(\mathbf{H}, \langle \cdot, \cdot \rangle)$ *be a complex inner product space, and let* $\mathbf{h}_1, \mathbf{h}_2 \in \mathbf{H}$ *be linearly independent and satisfy*

$$\|\mathbf{h}_1\| = \|\mathbf{h}_2\| \quad \textit{and} \quad \langle \mathbf{h}_1, \mathbf{h}_2 \rangle = \langle \mathbf{h}_2, \mathbf{h}_1 \rangle .$$

If $\mathbf{h} \in \mathbf{H}$ *satisfies*

$$| \langle \mathbf{h}, \mathbf{h}_i \rangle | \geq 1 \quad \textit{for } i = 1, 2,$$

then

$$\frac{1}{\|\mathbf{h}_1\|\|\mathbf{h}_2\| + | \langle \mathbf{h}_1, \mathbf{h}_2 \rangle |} \leq \frac{1}{2}\|\mathbf{h}\|^2.$$

We will use the following elementary proposition to establish this lemma:

Proposition 3.3. *Suppose* $y_1, y_2 \in \mathbf{H}$ *are linearly independent and* $\|y_1\| = \|y_2\|$*. Then*

$$\min \left\{ \|s_1 y_1 + s_2 y_2\|^2 : |s_1| \geq 1, \quad |s_2| \geq 1 \right\} = 2\|y_1\|\|y_2\| - 2| \langle y_1, y_2 \rangle |. \quad (3.5.5)$$

Proof. We may assume without loss of generality that $\|y_1\| = \|y_2\| = 1$ and that y_1 and y_2 are not orthogonal. Then

$$\|s_1 y_1 + s_2 y_2\|^2 = |s_1|^2 + |s_2|^2 + 2\rho \text{Re}(s_1 \overline{s_2})$$

where $\rho = \langle y_1, y_2 \rangle$. We may also assume $s_2 = 1$ and write $s_1 = re^{i\theta}$ where $r \geq 1$.

$$\|s_1 y_1 + s_2 y_2\|^2 = r^2 + 2\rho r \cos\theta + 1.$$

But $r^2 + 2\rho r \cos\theta + 1 \geq r^2 - 2|\rho| r + 1$ so we choose θ such that $\cos\theta = -\text{sgn}(\rho)$. But the minimum of the function $r \mapsto r^2 - 2|\rho| r + 1$ occurs at $r = |\rho| < 1$ and so we must take $r = 1$ and so

$$\|s_1 y_1 + s_2 y_2\|^2 \geq 2 - 2|\langle y_1, y_2 \rangle|$$

and equality is achieved if $s_1 = -\text{sgn}(\rho)$ and $s_2 = 1$. □

Proof. (of Lemma 3.3) By a basic projection argument, we see that if $\mathbf{h}$ is extremal, then there are complex numbers λ_1 and λ_2 such that

$$\mathbf{h} = \lambda_1 \mathbf{h}_1 + \lambda_2 \mathbf{h}_2.$$

There are also complex numbers s_1, s_2 with $|s_1| \geq 1$ and $|s_2| \geq 1$ that satisfy

$$\langle \mathbf{h}, \mathbf{h}_1 \rangle = s_1 \qquad \langle \mathbf{h}, \mathbf{h}_2 \rangle = s_2.$$

Notice that the choices of λ_1, λ_2 and s_1, s_2 are in a one-to-one correspondence. This is best seen by considering the following linear system.

$$\begin{aligned} s_1 &= \langle \mathbf{h}, \mathbf{h}_1 \rangle = \lambda_1 + \lambda_2 \langle \mathbf{h}_1, \mathbf{h}_2 \rangle \\ s_2 &= \langle \mathbf{h}, \mathbf{h}_2 \rangle = \lambda_1 \langle \mathbf{h}_1, \mathbf{h}_2 \rangle + \lambda_2 \end{aligned}$$

which we rewrite in matrix form

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 1 & \nu \\ \nu & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{3.5.6}$$

where $\nu = \langle \mathbf{h}_1, \mathbf{h}_2 \rangle$. The matrix $G = \begin{pmatrix} 1 & \nu \\ \nu & 1 \end{pmatrix}$ is the Gram matrix for $\mathbf{h}_1$ and $\mathbf{h}_2$, and since $\mathbf{h}_1$ and $\mathbf{h}_2$ are linearly independent it follows that G is nonsingular. Therefore

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{1-\nu^2} \begin{pmatrix} 1 & -\nu \\ -\nu & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}. \tag{3.5.7}$$

This now allows us to write $\mathbf{h}$ is terms of s_1 and s_2

$$\mathbf{h} = \frac{1}{1-\nu^2}\Big((s_1 - s_2\nu)\mathbf{h}_1 + (s_2 - \nu s_1)\mathbf{h}_2\Big). \tag{3.5.8}$$

We would now like to apply Proposition 3.3 to (3.5.8). We set $y_1 = \mathbf{h}_1 - \nu \mathbf{h}_2$ and $y_2 = \mathbf{h}_2 - \nu \mathbf{h}_1$. But $\nu \neq 1$ since $\|\mathbf{h}_1\| = \|\mathbf{h}_2\| = 1$ and $\mathbf{h}_1$ and $\mathbf{h}_2$ are linearly independent. So y_1 and y_2 are linearly independent and

$$\|y_1\|^2 = \|y_2\|^2 = (1-\nu)(1+\nu), \quad \langle y_1, y_2 \rangle = \nu(\nu-1)(\nu+1).$$

Therefore by Proposition 3.3 and (3.5.8) we have for $\|\mathbf{h}_1\| = \|\mathbf{h}_2\| = 1$

$$\|\mathbf{h}\|^2 = \frac{2(1-\nu)(1+\nu) - 2|\nu(\nu-1)(\nu+1)|}{(1-\nu^2)^2} = \frac{2(1-|\nu|)}{1-\nu^2} \tag{3.5.9}$$

and by scaling we have for $\|\mathbf{h}_1\| = \|\mathbf{h}_2\|$

$$\|\mathbf{h}\|^2 = \frac{2}{\|\mathbf{h}_1\|\|\mathbf{h}_2\| + |\langle \mathbf{h}_1, \mathbf{h}_2 \rangle|}.$$

□

Now the proof of Theorem 3.2 is nearly complete. We take $\mathbf{h}_1 = K_\nu(-\xi, z)$ and $\mathbf{h}_2 = K_\nu(\xi, z)$ in the lemma to obtain

$$\begin{aligned}\mu_\nu(\pi^{-1}B, \xi \mathrm{S}^{N-1}) &= \frac{\omega_{N-1}}{c_\nu}\left\{\|K_\nu(\xi,\cdot)\|_{E_\nu}^2 + |\langle K_\nu(\xi,\cdot), K_\nu(-\xi,\cdot)\rangle|\right\}^{-1} \\ &= \frac{\omega_{N-1}}{c_\nu}\left\{K_\nu(\xi,\xi) + |K_\nu(-\xi,\xi)|\right\}^{-1}\end{aligned}$$

But

$$\begin{aligned}K_\nu(-\xi,\xi) &= \frac{A_\nu(-\xi)B_\nu(\xi) - A_\nu(\xi)B_\nu(-\xi)}{2\pi\xi} \\ &= \frac{A_\nu(\xi)B_\nu(\xi)}{\pi\xi} \\ &= \frac{\Gamma(\nu+1)^2\,(\xi/2)^{-2\nu}\,J_\nu(\xi)J_{\nu+1}(\xi)}{\pi\xi} \\ &= \frac{J_\nu(\xi)J_{\nu+1}(\xi)}{2c_\nu\xi^{2\nu+1}}\end{aligned}$$

where $A_\nu(z)$ and $B_\nu(z)$ are defined in Vaaler and Holt. But $c_\nu K_\nu(\xi,\xi) = u_\nu(\xi,\pi^{-1})^{-1}$ as defined in Vaaler and Holt by

$$u_\nu(\xi,\pi^{-1})^{-1} = \frac{\xi J_\nu(\xi)^2 + \xi J_{\nu+1}(\xi)^2 - (2\nu+1)J_\nu(\xi)J_{\nu+1}(\xi)}{2\xi^{2\nu+1}}.$$

So

$$K_\nu(\xi,\xi) = \frac{\xi J_\nu(\xi)^2 + \xi J_{\nu+1}(\xi)^2 - (2\nu+1)J_\nu(\xi)J_{\nu+1}(\xi)}{2c_\nu\xi^{2\nu+1}}.$$

This shows that

$$\mu_\nu(\pi^{-1}B, \xi\mathrm{S}^{N-1}) = \left(\frac{1}{\omega_{N-1}u_\nu(\xi,\pi^{-1})} + \left|\frac{J_\nu(\xi)J_{\nu+1}(\xi)}{2\omega_{N-1}\xi^{2\nu+1}}\right|\right)^{-1}.$$

3.6 Concluding Remarks

Remark 3.1. Bourgain and Milman [12] have shown that there is an absolute constant $c > 0$ such that $c^N \mathrm{vol}_N(B)^2 \le \mathrm{vol}_N(K)\mathrm{vol}_N(K^*)$. This in conjunction with (3.1.2) implies that

$$\eta(K) \le c^{-N} \frac{2^N \mathrm{vol}_N(K^*)}{(\mathrm{vol}_N(B))^2}.$$

Conversely, the existence of an absolute constant in the above inequality would imply Bourgain and Milman's *reverse Santaló inequality* above. In view of this observation we pose the following problem which is equivalent to the reverse Santaló inequality.

Problem 3.3. *Show that there exists an absolute constant* $C > 0$ *such that*

$$\int_{\mathbb{R}^N} \left| \fint_K e(\mathbf{x} \cdot \boldsymbol{\xi}) d\boldsymbol{\xi} \right|^2 d\mathbf{x} \le C^N \frac{\mathrm{vol}_N(K^*)}{(\mathrm{vol}_N(B))^2}$$

using Fourier analysis.

It has been brought to my attention that Nazarov has shown this in [56] using results at the intersection of Fourier analysis and several complex variables. Our arguments and ideas are very similar to those of Nazaraov.

Remark 3.2. Problem 3.1 can be regarded as the simplest problem in a larger program aimed at tackling the Beurling-Selberg problem in several variables with fairly general Fourier support. Indeed this is the author's motivation for studying Problem 3.1.

Remark 3.3. In the previous section, regarding the 2 point lemma. If $\mathbf{H} = \mathcal{PW}$ of type $\pi\delta$, and t_1, t_2 are distinct real numbers, $\mathbf{h}_1 = K(t_1, z)$ and $\mathbf{h}_2 = K(t_2, z)$, then $\|\mathbf{h}_1\|^2 = \|\mathbf{h}_2\|^2 = \delta$ and $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle = (\pi|t_1 - t_2|)^{-1} \sin \pi\delta|t_1 - t_2|$. In this case

$$\Delta = \frac{1}{2}\int_{-\infty}^{\infty} F(t)dt = \left\{\delta + \left|\frac{\sin \pi\delta|t_1 - t_2|}{\pi|t_1 - t_2|}\right|\right\}^{-1} \tag{3.6.1}$$

where $F(t)$ is the extremal majorant of type $2\pi\delta$ of the characteristic function of the points t_1 and t_2. If $\delta|t_1 - t_2| = L \in \mathbb{Z}$ then $\Delta = \dfrac{|t_1 - t_2|}{L}$

Remark 3.4. It is natural to recast Problem 3.1 for a more generic set in place of K. For instance one could select K to be a set of positive and finite Lebesgue measure, a compact set, a *star body*, etc. Regardless of the choice of set, we remark that one may as well choose the set to be symmetric since $F(\mathbf{x})$ being real valued implies that $\hat{F}(\boldsymbol{\xi}) = 0$ if and only if $\hat{F}(-\boldsymbol{\xi}) = 0$, and so the support of $\hat{F}(\boldsymbol{\xi})$ for any real valued function $F(\mathbf{x})$ is naturally symmetric.

Chapter 4

Uniform Dilations in Higher Dimensions

This chapter is a joint paper written with Thái Hoàng Lê [42].

4.1 Introduction

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. A subset $X \subset \mathbb{T}$ is called ϵ-*dense* in $\mathbb{T}$ if it intersects every interval of length 2ϵ in $\mathbb{T}$. A *dilation* of X is a set of the form $nX = \{nx : x \in X\} \subset \mathbb{T}$. The following theorem of Glasner [31] is the basis for our investigation.

Theorem I (Glasner)**.** *Let X be an infinite subset of $\mathbb{T}$ and $\epsilon > 0$, then there exists a positive integer n such that the dilation nX is ϵ-dense in $\mathbb{T}$.*

Theorem I can be made effective in the sense that every sufficiently large subset X has an ϵ-dense dilation of the form nX for some positive integer n, and 'sufficiently large' can be quantified. The first result in this direction was obtained by Berend and Peres in [9]. Given $\epsilon > 0$, let $k(\epsilon)$ be the minimal integer k such that for any set $X \subset \mathbb{T}$ of cardinality at least k, some dilation nX is ϵ-dense in $\mathbb{T}$. Berend and Peres showed that

$$c/\epsilon^2 \leq k(\epsilon) \leq (c_1/\epsilon)^{c_2/\epsilon} \tag{4.1.1}$$

where c, c_1, c_2 are absolute constants.

The question of determining the correct order of magnitude of $k(\epsilon)$ was further studied in depth by Alon and Peres [2], who gave the bound

$$k(\epsilon) \ll_\delta \left(\frac{1}{\epsilon}\right)^{2+\delta} \tag{4.1.2}$$

for any $\delta > 0$. This is almost best possible in view of (4.1.1). Actually, they gave a more precise bound

$$k(\epsilon) \ll \left(\frac{1}{\epsilon}\right)^{2+\frac{3}{\log\log(1/\epsilon)}}. \tag{4.1.3}$$

In [2], Alon and Peres provided two different approaches to this problem. On the one hand, the probabilistic approach gives more information about the dilation, such as its discrepancy. On the other hand, the second approach, using harmonic analysis, is particular suited when one is interested in dilating the set X by a sequence of arithmetic nature, such as the primes or the squares. They proved

Theorem II (Alon-Peres). *(i) For any $\delta > 0$, every set X in $\mathbb{T}$ of cardinality*

$$k \gg_\delta \frac{1}{\epsilon^{2+\delta}},$$

has an ϵ-dense dilation pX with p prime.

(ii) Let f be a polynomial of degree $L > 1$ with integer coefficients and let $\delta > 0$. Then any set X in $\mathbb{T}$ of cardinality

$$k \gg_{\delta,f} \left(\frac{1}{\epsilon}\right)^{2L+\delta},$$

has an ϵ-dense dilation of the form $f(n)X$, for some $n \in \mathbb{Z}$.

It is shown in [55] that in part (ii) of the above theorem there is an ϵ-dense dilation of the form $f(p)X$ where p is a prime number.

In this paper we investigate high dimensional analogues of Glasner's theorem and the above results of Alon and Peres using Alon-Peres' harmonic analysis approach. One problem that comes to mind is that of determining the natural analogue of "dilating by n" in the one-dimensional case. Any continuous endomorphism of $\mathbb{T}$ is represented this way, so we may regard the dilation as the action by a continuous endomorphism. When considering higher dimensional generalizations of the above theorems we need not restrict ourselves from maps of a torus into itself. We will instead consider maps between tori of possibly different dimension. A continuous homomorphism between $\mathbb{T}^N$ and $\mathbb{T}^L$ is represented by left multiplication of an $L \times N$ matrix with entries in $\mathbb{Z}$. This will be our analogue of dilation. We say that a subset of $\mathbb{T}^L$ is *ϵ-dense* in $\mathbb{T}^L$ if it intersects any box of side length 2ϵ.

Our first theorem is a high dimensional analogue of Glasner's theorem.

Theorem 4.1. *For any $\epsilon > 0$ and any infinite subset $X \subset \mathbb{T}^N$ there exists a continuous homomorphism $T : \mathbb{T}^N \to \mathbb{T}^L$ such that TX is ϵ-dense in $\mathbb{T}^L$.*

The proof of this result is similar to the proof of (4.1.2). Our main investigation, however, is an analogue of the fact that if $X \subset \mathbb{T}$ is infinite, then there is a dilation of the form $f(n)X$ that is ϵ-dense, where $f(x)$ is a non-constant polynomial with integral coefficients. Let us introduce the set-up to this problem and lay out some of the complications that arise when moving to high dimensions. In this paper, a *subtorus* of $\mathbb{T}^N$ is defined to be a non-trivial closed and connected Lie subgroup.

Let $\mathbf{A}(x) \in \mathrm{M}_{L\times N}(\mathbb{Z}[x])$ be non-constant and let D be the positive integer representing the largest of the degrees of the entries of $\mathbf{A}(x)$. Then there are $A_0, ..., A_D \in \mathrm{M}_{L\times N}(\mathbb{Z})$ such that

$$\mathbf{A}(x) = A_0 + xA_1 + \cdots + x^D A_D = A_0 + \mathbf{A}_*(x)$$

where $\mathbf{A}_*(x)$ is the *non-constant part* of $\mathbf{A}(x)$. We wish to consider dilations of subsets $X \subset \mathbb{T}^N$ of the form $\mathbf{A}(n)X$.

Simple examples show that, unlike Theorem 4.1, there are configurations of $\mathbf{A}(x)$ and X for which $\mathbf{A}(n)X$ is never ϵ-dense in the full torus. Take, for instance, $\mathbf{A}(n) = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$ and X to live in a proper subtorus, then $A(n)X$ is also in the same subtorus, for every n. Furthermore, if we take X to be in a translate of a subtorus, then $A(n)X$ is also in a translate of a subtorus (where the translate depends of n). So the best one can hope for in this situation is to achieve an ϵ-dense dilation in a *translate of a subtorus*. Before stating our results, we give some examples to show that even this restriction is not always achieved.

Example 4.1. If $\mathbf{A}(n) = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}$ and $X = \{(0, x) : |x| \leq 1/4\}$, then there is no value of n such that $A(n)X$ is $1/4$-dense in a translate of a subtorus. Basically, this is because the matrix $\mathbf{A}_*$ is degenerate in a sense so that $\mathbf{A}(n)X$ doesn't "move X around."

Example 4.2. If $\mathbf{A}(n) = \begin{pmatrix} n & 0 \\ 0 & n+1 \end{pmatrix}$ and $X = \{(1/j, 1/j) : j = 1, 2, \ldots\}$, then clearly $\mathbf{A}(n)X$ is not $1/4$-dense in any translate of the diagonal. On the other hand, one can show that for any n, for any subtorus HT of $\mathbb{T}^2$ that is different from the diagonal, $\mathbf{A}(n)X$ is not ϵ-dense in any translate of $\mathcal{T}$ (since the set of dot products of elements of $\mathbf{A}(n)X$ with $(-1 \quad 1)$ has only one accumulation point). The reason of such a failure can be attributed to the lack of a compromise between the constant part and the non-constant part of $\mathbf{A}$.

Our main result says that the only obstructions to ϵ-dense dilations are the ones described in Examples 4.1 and 4.2.

Theorem 4.2. *Let $\mathbf{A}(x) \in \mathrm{M}_{L\times N}(\mathbb{Z}[x])$. The following are equivalent:*

1. *For any infinite subset $X \subset \mathbb{T}^N$ there exists a subtorus $\mathcal{T} = \mathcal{T}(X, \mathbf{A})$ of $\mathbb{T}^L$ such that for any $\epsilon > 0$ there exists an integer n such that $\mathbf{A}(n)X = \{\mathbf{A}(n)\mathbf{x} : \mathbf{x} \in X\}$ is ϵ-dense in a translate of $\mathcal{T}$.*
2. *(a) The columns of $\mathbf{A}_*(x)$ are $\mathbb{Q}$-linearly independent, and*

   (b) If there are $\mathbf{v} \in \mathbb{Q}^L$ and $\mathbf{w} \in \mathbb{Q}^N$ satisfying

$$\mathbf{v} \cdot A_d\mathbf{w} = 0 \quad \textit{for each } \ d = 1, ..., D, \tag{4.1.4}$$

   then $\mathbf{v} \cdot A_0\mathbf{w} = 0$.

Remarks 4.1.

- Theorem 4.2 shows one how to construct matrices $\mathbf{A}(n)$ such that the conclusion (1) holds. The condition (2a) tells us how to choose the non-constant part $\mathbf{A}_*(n)$, and the condition (2b) tells us that the constant part A_0 has to behave accordingly.

- In the case $N = L = 1$, (2) is automatically satisfied if $\mathbf{A}$ is not constant, which explains why in Theorem II (ii) we can take f to be any non-constant polynomial.

- If we replace $\mathbb{Q}$ with $\mathbb{C}$ in (2b), then by Hilbert's Nullstellensatz, it would imply that A_0 is a linear combination of $A_1, \ldots, A_D$. It would be interesting to construct examples of $\mathbf{A}$ satisfying (2b) without A_0 being a linear combination of $A_1, \ldots, A_D$.

We also prove an effective form of this result. Define $k(\epsilon; L, N, \mathbf{A})$ to be the largest integer k such that there exist k distinct points $X = \{\mathbf{x}_1, ..., \mathbf{x}_k\} \subset \mathbb{T}^N$ such that $\mathbf{A}(n)X = \{\mathbf{A}(n)\mathbf{x}_1, ..., \mathbf{A}(n)\mathbf{x}_k\}$ is not ϵ-dense in any translate of any subtorus for any $n = 1, 2, 3, ...$

Theorem 4.3. *Let $\mathbf{A}(x)$ be of degree at most D and satisfy (2a) and (2b) from Theorem 4.2. Then there are constants $c_1(N, L, D)$ and $c_2(N, L, D)$ such that*

$$k(\epsilon; L, N, \mathbf{A}) \ll_{N,L,D} \|\mathbf{A}_*\|_\infty^{c_1(N,L,D)} \left(\frac{1}{\epsilon}\right)^{c_2(N,L,D)}. \qquad (4.1.5)$$

where $\|\mathbf{A}_*\|_\infty$ is the max of the heights[1] *of the entries of $\mathbf{A}_*$.*

[1] Recall that the height of a polynomial is the maximum of the absolute values of its coefficients.

Remark 4.1. Theorem 4.2 would be a mere consequence of Theorem 4.3, if not for the fact that the subtorus $\mathcal{T}$ is independent of ϵ in the conclusion of Theorem 4.2.

The exponents c_1 and c_2 can be given explicitly. We do not try to find the best possible exponents, since these are not known even in the case $N = L = 1$, though our values can certainly be improved. Finally, we remark that it is straightforward to prove a version of Theorem 4.3 in the spirit of [55], with bounds of the same quality, for dilations of the form $\mathbf{A}(p)X$ where p is prime. Indeed, the proof would proceed exactly the same way, albeit with an appropriate modification of Lemma 4.2. We leave the details to the interested reader.

The paper is organized as follows. In Section 4.2 we gather some useful facts that we need in our proofs, including Alon-Peres' machinery. In Section 4.3 we prove Theorem 4.2, and in Section 4.4 we prove Theorem 4.3. In Section 4.5 we prove (a variant of) a quantitative version of Theorem 4.1. Finally, in Section 4.6 we discuss some applications of our results.

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4.2 Notation and preliminaries

4.2.1 Notation

Throughout this paper, we will use Vinogradov's symbols $\ll$ and $\gg$. For two quantities A, B, we write $A \ll B$, or $B \gg A$ if there is a positive constant c such that $|A| \leq cB$. If the constant c depends on another quantity t, then we indicate this dependence as $A \ll_t B$. The numbers N, L, D are fixed throughout this paper, so dependence on these quantities is implicitly understood.

Given a vector $\mathbf{v}$, we denote by $\|\mathbf{v}\|_\infty$ its usual sup norm. Given a matrix A, let us denote by $\|A\|_\infty$ the maximal of the absolute values of its entries. Finally, for a matrix $\mathbf{A}(x) = A_0 + xA_1 + \cdots + x^D A_D$ whose entries are a polynomials in x, we define $\|\mathbf{A}\|_\infty = \max\{\|A_d\|_\infty : d = 0, 1, \ldots, D\}$. While we use the same symbol for slightly different objects, the use should be clear from the context.

For $x \in \mathbb{R}$, we denote by $\|x\|$ the distance from x to the nearest integer. For $\mathbf{x} = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$, let $\|\mathbf{x}\| = \max_{i=1,\ldots,\ell} \|x_i\|$. In other words, $\|\mathbf{x}\|$ denotes the distance from $\mathbf{x}$ to the nearest integer lattice point under $\|\cdot\|_\infty$.

Throughout the paper, we always identify a point in a torus $\mathbb{T}^\ell$ with its unique representative in $[0, 1)^\ell$. This point of view is important, since it enables us to define subtori in terms of equations.

4.2.2 Preliminaries

Let $\{x_1, ..., x_k\}$ be a set of k distinct numbers in $\mathbb{T}$. Define

$$h_m = \#\{(i,j) : 1 \le i, j \le k \ \text{ and } m(x_i - x_j) \in \mathbb{Z}\} \tag{4.2.1}$$

and $H_m = h_1 + \cdots + h_m$. The quantities h_i, H_m certainly depend on the sequence $\{x_1, ..., x_k\}$, but we always specify the sequence we are working with. The numbers h_m and H_m appear in several of the arguments in [2] and they will make an appearance in the proof of our main results. We will need the following simple estimate:

Proposition 4.1. $H_m \le km^2$.

Proof. Observe that for fixed i and m, there are at most m values of j such that $m(x_i - x_j) \in \mathbb{Z}$. Thus for fixed i, the number of couples (j, m) such that $m(x_i - x_j) \in \mathbb{Z}$ is at most $1 + \cdots + M \le M^2$. Summing this up over all i gives the desired estimate. □

Remark 4.2. Since we are not concerned with optimal exponents, this estimate will suffice for our purposes, but we note that it is shown in [2] that the (essentially sharp) bound $H_m \ll_\gamma (mk)^{1+\gamma}$ holds for any $\gamma > 0$.

Corollary 4.1. *If* $\mathtt{s}_2, \mathtt{s}_3, ...$ *is a sequence of positive integers such that* $\mathtt{S}_b = \mathtt{s}_2 + \cdots + \mathtt{s}_b \le H_b$ *and* $\mathtt{S}_b \le k^2$*, then*

$$\sum_{b=2}^{\infty} \mathtt{s}_b b^{-1/D} \ll_D k^{2-1/(2D)}. \tag{4.2.2}$$

Proof. We follow the proof of a similar estimate in [2]. For $b \geq \sqrt{k}$ use the bound $\mathtt{S}_b \leq k^2$ and if $b < \sqrt{k}$ use $\mathtt{S}_b \leq H_b \ll kb^2$ so we have by summation by parts

$$\sum_{b=2}^{\infty} \mathtt{S}_b \Big(b^{-1/D} - (b+1)^{-1/D} \Big) \ll k^2 k^{-1/(2D)} + k \sum_{b=2}^{\sqrt{k}} b^2 b^{-1/D-1}.$$

But

$$\sum_{b=2}^{\sqrt{k}} b^{1-1/D} \ll_D k^{1-1/(2D)}.$$

□

The following Lemma is a high dimensional analogue of an inequality used in the several of the results in [2]. It may be regarded as a general principle which connects the lack of ϵ-denseness to exponential sums.

Proposition 4.2. *Let $A(1), A(2), \ldots$ be a sequence of linear transformations taking $\mathbb{T}^N$ to $\mathbb{T}^\ell$ and assume $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a subset of $\mathbb{T}^N$ of cardinality k such that $\mathbf{A}(n)X$ is not ϵ-dense in $\mathbb{T}^\ell$ for any $n \in \mathbb{Z}$. Then for any $\epsilon > 0$ there is an integer $0 \leq M \ll_\ell \epsilon^{-1}$ such that*

$$k^2 \ll_\ell \frac{1}{\epsilon^\ell} \sum_{\substack{0 < \|\mathbf{m}\|_\infty \leq M \\ \mathbf{m} \in \mathbb{Z}^\ell}} \sum_{i=1}^{k} \sum_{j=1}^{k} \lim_{R \to \infty} \frac{1}{R} \sum_{r=1}^{R} e_{\mathbf{m}} \Big(\mathbf{A}(r)(\mathbf{x}_i - \mathbf{x}_j) \Big) \tag{4.2.3}$$

where $e_{\mathbf{m}}(\mathbf{t}) = \exp(2\pi i \mathbf{m} \cdot \mathbf{t})$.

Alon-Peres proved the one-dimensional version of Proposition 4.2 using a classical result of Denjoy and Carleman, and obtained the same inequality with $M \ll (1/\epsilon) \log^2(1/\epsilon)$. Their method can be extended in a straightforward manner to higher dimensions. As pointed out to us by Vaaler, one could as well use the machinery developed by Barton-Montgomery-Vaaler [8] to improve this to $M \ll 1/\epsilon$. We will follow the latter approach in our proof of Proposition 4.2 since it gives us a cleaner value for M, though this is inconsequential. Indeed, even in the case $N = L = 1$, this improved value of M does not lead to any improvement on Alon-Peres' bound (4.1.3).

We first recall the following consequence of [8, Corollary 2]:

Lemma 4.1. *Let $0 < \epsilon \leq 1/2$. Let $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k \in \mathbb{R}^\ell$ be such that $\|\boldsymbol{\xi}_i\| \geq \epsilon$ for any $i = 1, \ldots, \ell$. Then we have*

$$\frac{k}{3} \leq \sum_{\substack{\mathbf{m} \in \mathbb{Z}^\ell \\ 0 < \|\mathbf{m}\|_\infty \leq \left[\frac{\ell}{\epsilon}\right]}} \left| \sum_{i=1}^{k} e_{\mathbf{m}}(\boldsymbol{\xi}_i) \right|$$

Proof of Lemma 4.2. For any r, since $\mathbf{A}(r)X$ is not ϵ-dense in $\mathbb{T}^\ell$, there exists $\boldsymbol{\alpha}_r \in \mathbb{R}^\ell$ such that $\|\boldsymbol{\alpha}_r - \mathbf{A}(r)\mathbf{x}_i\| \geq \epsilon$ for any $i = 1, \ldots, k$. Let $M = \left[\frac{\ell}{\epsilon}\right]$. By Lemma 4.1, we have

$$\frac{k}{3} \leq \sum_{\substack{\mathbf{m} \in \mathbb{Z}^\ell \\ 0 < \|\mathbf{m}\|_\infty \leq M}} \left| \sum_{i=1}^{k} e_{\mathbf{m}}(\boldsymbol{\alpha}_r - \mathbf{A}(r)\mathbf{x}_i) \right|$$

By Cauchy-Schwarz, we have

$$\begin{aligned} k^2 &\ll_\ell M^\ell \sum_{\substack{\mathbf{m} \in \mathbb{Z}^\ell \\ 0 < \|\mathbf{m}\|_\infty \leq M}} \left| \sum_{i=1}^{k} e_{\mathbf{m}}(\boldsymbol{\alpha}_r - \mathbf{A}(r)\mathbf{x}_i) \right|^2 \\ &\ll_\ell \frac{1}{\epsilon^\ell} \sum_{\substack{\mathbf{m} \in \mathbb{Z}^\ell \\ 0 < \|\mathbf{m}\|_\infty \leq M}} \sum_{i=1}^{k} \sum_{i=1}^{k} e_{\mathbf{m}}(\mathbf{A}(r)(\mathbf{x}_i - \mathbf{x}_j)) \end{aligned}$$

This is true for any r so by taking the average of the right hand side over $1 \leq r \leq R$, we have

$$k^2 \ll_\ell \frac{1}{\epsilon^\ell} \sum_{\substack{0<\|\mathbf{m}\|_\infty \leq M \\ \mathbf{m}\in\mathbb{Z}^\ell}} \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{1}{R} \sum_{r=1}^{R} e_{\mathbf{m}}\Big(\mathbf{A}(r)(\mathbf{x}_i - \mathbf{x}_j)\Big)$$

Letting $R \to \infty$ we have the desired inequality. □

We also recall the following classical estimate due to Hua [21, 57]:

Lemma 4.2 (Hua)**.** *Suppose* $f(x) = a_d x^d + \cdots a_1 x + a_0 \in \mathbb{Z}[x]$ *and* q *is a positive integer such that* $\gcd(a_1, ..., a_d, q) = 1$. *Then*

$$\left| \sum_{r=1}^{q} e^{2\pi i f(r)/q} \right| \ll_d q^{1-1/d}.$$

4.3 The infinite version

Of the two implications, the implication (1) ⇒ (2) is the more difficult so let us begin by quickly proving the implication (2) ⇒ (1). We will need the following lemma in the proof of the necessity of (2b). The assertion of the lemma is that by taking the dot product with a vector $\mathbf{v}$, an ϵ-dense subset of a torus becomes an $\tilde{\epsilon}$-dense set in $\mathbb{T}$ where $\tilde{\epsilon}$ is comparable to ϵ, as long as $\mathbf{v}$ is not orthogonal to the original torus.

Lemma 4.3. *Let* $\epsilon > 0$, $\mathbf{b} \in \mathbb{R}^L$, V *a proper subspace of* $\mathbb{R}^L$, $\mathbf{v} \in \mathbb{Z}^L, \mathbf{v} \notin V^\perp$, *and*

$$X \subset S = \left\{\mathbf{b} + \mathbf{x} + \mathbb{Z}^L : \mathbf{x} \in V\right\} \subset \mathbb{T}^L.$$

If X *is* ϵ*-dense in* S, *then* $\{\mathbf{v} \cdot \mathbf{x} + \mathbb{Z} : \mathbf{x} \in X\}$ *is* $L\|\mathbf{v}\|_\infty\epsilon$*-dense in* $\mathbb{T}$.

Proof. Let $t \in \mathbb{T}$. We want to find a $\mathbf{x} \in X$ such that $\mathbf{v} \cdot \mathbf{x}$ is contained in an interval of length $2L\|\mathbf{v}\|_\infty \epsilon$ in $\mathbb{T}$ centered at t. That is we wish to show the existence of an $\mathbf{x} \in X$ such that $\|\mathbf{v} \cdot \mathbf{x} - t\| \leq L\|\mathbf{v}\|_\infty \epsilon$.

Since $\mathbf{v} \notin V^\perp$ we may write $t = \mathbf{v} \cdot \boldsymbol{\alpha}$ for some $\boldsymbol{\alpha} \in V$. And since X is ϵ-dense in S there exists an $\mathbf{x} \in X \cap S$ and a $\mathbf{w} \in \mathbb{Z}^L$ such that $\|\mathbf{x} - \boldsymbol{\alpha} - \mathbf{w}\|_\infty \leq \epsilon$. But since $\mathbf{v} \cdot \mathbf{w} \in \mathbb{Z}$ we have

$$\|\mathbf{v} \cdot \mathbf{x} - t\| = \|\mathbf{v} \cdot (\mathbf{x} - \boldsymbol{\alpha} - \mathbf{w})\| \leq |\mathbf{v} \cdot (\mathbf{x} - \boldsymbol{\alpha} - \mathbf{w})| \leq L\|\mathbf{v}\|_\infty \epsilon.$$

□

Proof of necessity of (2a). Suppose, by way of contradiction, that the columns of $\mathbf{A}_*(x)$ are not $\mathbb{Q}-$linearly independent. Then there is a nonzero $\mathbf{m} \in \mathbb{Q}^N$ such that

$$\mathbf{A}_* \mathbf{m} = 0.$$

If

$$X = \{\mathbf{m}/j : j = 1, 2, ...\},$$

then $\mathbf{A}(n)X = A_0 X = \{\mathbf{x}_j = A_0 \mathbf{m}/j : j = 1, 2, ...\}$ which is not ϵ-dense in a translate of a subtorus for any sufficiently small $\epsilon > 0$. □

Proof of necessity of (2b). Suppose that there are vectors $\mathbf{v} \in \mathbb{Z}^L$ and $\mathbf{w} \in \mathbb{Z}^N$ such that

$$\mathbf{v} \cdot A_d \mathbf{w} = 0 \quad \text{for each } d = 1, ..., D$$

but $\mathbf{v} \cdot A_0 \mathbf{w} = t \neq 0$. In particular $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$. Let $X = \{\mathbf{w}/j : j = 1, 2, ...\} \subset \mathbb{T}^N$. Note that X is an infinite set. It then follows that

$$\mathbf{v} \cdot \mathbf{A}(n)\mathbf{x}_j = t/j \searrow 0 \quad \text{for each } n = 1, 2, ... \tag{4.3.1}$$

Suppose for a contradiction that there is a translate of a subtorus S in $\mathbb{T}^L$ such that for any $\epsilon > 0$, there exists n such that $\mathbf{A}(n)X$ is dense in a translate of S. Suppose S is given by $S = \left\{\mathbf{b} + \boldsymbol{\alpha} + \mathbb{Z}^L : \boldsymbol{\alpha} \in V\right\}$ where V is a proper subspace of $\mathbb{R}^L$ and $\mathbf{b} \in \mathbb{R}^L$. Let $\epsilon > 0$ be sufficiently small and suppose there is a subset $Y \subset X$, an integer n such that $\mathbf{A}(n)Y$ is ϵ-dense in S. We have two possibilities:

- If $\mathbf{v} \in V^{\perp}$, then $\mathbf{v} \cdot \mathbf{A}(n)\mathbf{y}$ is a constant (namely $\mathbf{v} \cdot \mathbf{b}$) for any $\mathbf{y} \in Y$, which is not true in view of (4.3.1).
- If $\mathbf{v} \notin V^{\perp}$, then by Lemma 4.3 we have $\mathbf{v} \cdot \mathbf{A}(n)Y$ is $L\|\mathbf{v}\|_{\infty}\epsilon$-dense in $\mathbb{T}$. Again, in view of (4.3.1), this is impossible if $\epsilon > 0$ is sufficiently small.

□

In the remainder of the paper we will say the *rank* (*corank*) of $\mathbf{A}(x)$ is the rank of the $\mathbb{Z}$-module generated by the rows (columns) of $\mathbf{A}(x)$. First we describe briefly the ideas of the proof of the implication (2) $\Rightarrow$ (1). Observe that we can't expect $\mathbf{A}(n)X$ to be ϵ-dense in the whole of $\mathbb{T}^L$ since there may be some linear dependencies between the rows of $\mathbf{A}$. If $\mathbf{A}(n)X$ fails to be ϵ-dense in the "natural" subtorus defined by these linear dependencies for every

n, then we use Proposition 4.2 to conclude that X has *structure*, in the sense that it has an infinite intersection with a translate of a subtorus of $\mathbb{T}^N$. This enables us to perform induction on N. Let us now introduce some preparatory lemmas.

Lemma 4.4. *Let $\mathbf{A}(x) \in \mathrm{M}_{L\times N}(\mathbb{Z}[x])$ satisfy condition (2b) from Theorem 4.2 and suppose $\mathbf{A}_*(x)$ is of rank ℓ. Then there exist matrices $T \in \mathrm{M}_{L\times\ell}(\mathbb{Q})$, $\mathbf{B}(x) \in \mathrm{M}_{\ell\times L}(\mathbb{Z}[x])$ such that*

(i) $\mathbf{A}(x) = T\mathbf{B}(x)$,

(ii) $\mathbf{B}_*(x)$ *has full rank, and*

(iii) There is a positive integer q such that qT is integral and $\|qT\|_\infty \ll_\ell \|\mathbf{A}_*\|_\infty^\ell$.

Proof. Without loss of generality we may assume the first ℓ rows of $\mathbf{A}_*(x)$ are $\mathbb{Q}$-linearly independent. Then there is an $L \times \ell$ matrix T with entries in $\mathbb{Q}$ such that $\mathbf{A}_* = T\mathbf{B}_*$ where $\mathbf{B}_* = \mathbf{B}_*(x) \in \mathrm{M}_{\ell\times N}(\mathbb{Z}[x])$ is the block of the first ℓ rows of $\mathbf{A}_*(x)$. We claim that condition (b) guarantees that $A_0 = TB_0$ for some $\ell \times N$ integral matrix B_0. First we show $\ker(T^t) \subset \ker(A_0^t)$.

Suppose $\mathbf{v} \in \ker(T^t)$. Then $\mathbf{A}_*^t\mathbf{v} = \mathbf{B}_*^t T^t\mathbf{v} = 0$, which implies $\mathbf{v}\cdot\mathbf{A}_*\mathbf{w} = 0$ for any $\mathbf{w} \in \mathbb{Q}^{\mathbb{N}}$. But by condition (b) this implies that $\mathbf{v}\cdot A_0\mathbf{w} = 0$ for each $\mathbf{w} \in \mathbb{Q}^{\mathbb{N}}$, which implies $A_0^t\mathbf{v} = 0$. That is, $\mathbf{v} \in \ker(A_0^t)$.

Therefore there exists $B_0 \in M_{\ell\times N}(\mathbb{Q})$ such that $A_0 = TB_0$. But the uppermost $\ell \times \ell$ block of T is the identity. Thus B_0 is none other than the uppermost $\ell \times N$ block of A_0, and consequently B_0 is integral. Upon putting $\mathbf{B} = \mathbf{B}_* + B_0$, we have $\mathbf{B}$ is integral and $\mathbf{A} = T\mathbf{B}$.

Let A be the $L \times DN$ matrix given by $A = [A_1 \cdots A_D]$ and B be the $\ell \times DN$ matrix given by $B = [B_1 \cdots B_D]$. Since $A_d = TB_d$ for each $d = 1, ..., D$, we have $A = TB$. B must have rank ℓ since $\mathbf{B}_*(x)$ does, so there is an invertible $\ell \times \ell$ minor B' of B. Let A' be the corresponding minor of A and observe we have the equality $A'(B')^{-1} = T$. Let $q = \det B' \neq 0$ and $C = q^{-1}(B')^{-1}$ be the adjugate of B'. We then have the inequality

$$\|qT\|_\infty = \|A'C\|_\infty \ll_\ell \|A'\|_\infty \|C\|_\infty \ll_\ell \|\mathbf{A}_*\|_\infty^\ell$$

as required. Clearly we may assume q to be positive. $\square$

Our crucial tool is the following consequence of Proposition 4.2. We regard it as some sort of *inverse result* since it tells about the structure of X if dilations of X fail to be ϵ-dense. In this respect our use of Proposition 4.2 is rather different from Alon-Peres. It is perhaps no surprise that our proof of Proposition 4.3 involves Ramsey's theorem.

Proposition 4.3. *Suppose $\epsilon > 0$, X is an infinite subset of $\mathbb{T}^N$, and $\mathbf{B}(x) \in M_{\ell \times N}(\mathbb{Z}[x])$ such that $\mathbf{B}_*(x)$ has full rank. If $\mathbf{B}(r)X$ is not ϵ-dense in $\mathbb{T}^\ell$ for any $r \in \mathbb{Z}$, then there exists a point $\mathbf{y}_0 \in X$, an integer J, and nonzero $\mathbf{w} \in \mathbb{Z}^N$ such that $\mathbf{w} \cdot (\mathbf{y} - \mathbf{y}_0) = J$ for infinitely many $\mathbf{y} \in X$.*

Note that the last equation is an equality in $\mathbb{R}$ rather than in $\mathbb{T}$, by our identification of points in $\mathbb{T}^N$ with their representatives in $[0, 1)^N$.

Proof. We create a complete graph whose vertex set is X and whose edges $(\mathbf{x}, \mathbf{y})$ are colored $\mathbf{w} \in \mathbb{Z}^N$ $(0 < \|\mathbf{w}\|_\infty \leq M\ell\|\mathbf{B}_*\|_\infty)$ if $\mathbf{w} \cdot (\mathbf{x} - \mathbf{y}) \in \mathbb{Z}$ and[2] colored $\boldsymbol{\omega}$ otherwise. By the infinite version of Ramsey's theorem there exists an infinite complete monochromatic subgraph whose vertex set is $Y \subset X$. We now would like to show that this graph cannot be $\boldsymbol{\omega}$-colored.

Suppose, by way of contradiction, that the graph is $\boldsymbol{\omega}-$colored. For any distinct $\mathbf{x}_1, ..., \mathbf{x}_k$ in Y and $R > 0$ we have, by Proposition 4.2:

$$\begin{aligned} k^2 &\ll_\ell \frac{1}{\epsilon^\ell} \sum_{\substack{0<\|\mathbf{m}\|_\infty \leq M \\ \mathbf{m}\in\mathbb{Z}^\ell}} \sum_{i=1}^k \sum_{j=1}^k \lim_{R\to\infty} \frac{1}{R} \sum_{r=1}^R e_{\mathbf{m}}\big(\mathbf{B}(r)(\mathbf{x}_i - \mathbf{x}_j)\big) \\ &= \frac{1}{\epsilon^\ell} \sum_{\substack{0<\|\mathbf{m}\|_\infty \leq M \\ \mathbf{m}\in\mathbb{Z}^\ell}} \sum_{i=1}^k \sum_{j=1}^k \lim_{R\to\infty} \frac{1}{R} \sum_{r=1}^R e_{\mathbf{m}}\left(\sum_{d=0}^D r^d B_d(\mathbf{x}_i - \mathbf{x}_j)\right) \\ &= \frac{1}{\epsilon^\ell} \sum_{\substack{0<\|\mathbf{m}\|_\infty \leq M \\ \mathbf{m}\in\mathbb{Z}^\ell}} \sum_{i=1}^k \sum_{j=1}^k \lim_{R\to\infty} \frac{1}{R} \sum_{r=1}^R e\left(\sum_{d=1}^D r^d B_d^t\mathbf{m} \cdot (\mathbf{x}_i - \mathbf{x}_j)\right) \\ &\ll_\ell \frac{M^\ell}{\epsilon^\ell} \sum_{i=1}^k \sum_{j=1}^k \lim_{R\to\infty} \frac{1}{R} \sum_{r=1}^R e\left(\sum_{d=1}^D r^d B_d^t\mathbf{m} \cdot (\mathbf{x}_i - \mathbf{x}_j)\right) \qquad (4.3.2) \end{aligned}$$

where $\mathbf{m}$ is the lattice point which maximizes the last sum. Let $\tilde{d}$ be the largest index such that $B_{\tilde{d}}^t\mathbf{m} \neq \mathbf{0}$. Then $\tilde{d} > 0$ because $\mathbf{B}_*^t(x)$ has $\mathbb{Q}$-linearly independent columns, which implies $B_d^t\mathbf{m}$ is not zero for some $d = 1, \ldots, D$. For any $i \neq j$, since $(\mathbf{x}_i, \mathbf{x}_j)$ is $\boldsymbol{\omega}$-colored under our coloring and $\|B_{\tilde{d}}^t\mathbf{m}\| \leq M\ell\|\mathbf{B}_*\|_\infty$, we have

$$B_{\tilde{d}}^t\mathbf{m} \cdot (\mathbf{x}_i - \mathbf{x}_j) \neq 0 \qquad (4.3.3)$$

[2]Observe we are allowing multiple colors per edge.

Therefore, if $i \neq j$, the polynomial

$$\Phi_{ij}(r) = \mathbf{m} \cdot \mathbf{B}_*(r)(\mathbf{x}_i - \mathbf{x}_j) = \sum_{d=1}^{D} r^d B_d^t \mathbf{m} \cdot (\mathbf{x}_i - \mathbf{x}_j)$$

has degree $\tilde{d}$. By Weyl's equidistribution theorem and Hua's bound (Lemma 4.2), we have:

$$\lim_{R\to\infty} \frac{1}{R} \sum_{r=1}^{R} e\left(\Phi_{ij}(r)\right) = \begin{cases} 0, & \text{if } \Phi_{ij} \text{ has at least one irrational coefficient} \\ \ll_D b^{-1/\tilde{d}} \leq b^{-1/D}, & \text{if } \Phi_{ij}(x) \in \mathbb{Q}[x], \end{cases}$$

where in the second case $b = b(i,j)$ is the least positive integer such that $b(\mathbf{m} \cdot \mathbf{B}_*(x)(\mathbf{x}_i - \mathbf{x}_j)) \in \mathbb{Z}[x]$.

For each $b > 1$ we define

$$S_b = \Big\{(i,j) \ : \ 1 \leq i,j \leq k, \ \ b \text{ is the smallest positive integer such that } \ b(\mathbf{m} \cdot \mathbf{B}_*(x)(\mathbf{x}_i - \mathbf{x}_j)) \in \mathbb{Z}[x]\Big\}.$$

Let $\mathtt{s}_b = \#S_b$ and $\mathtt{S}_b = \mathtt{s}_2 + \cdots + \mathtt{s}_b$. Let $x_i = B_{\tilde{d}}^t \mathbf{m} \cdot \mathbf{x}_i$ for any $i = 1, \ldots, k$, then the x_i are distinct in $\mathbb{T}$ in view of (4.3.3). We notice that if $(i,j) \in S_b$ then $b(x_i - x_j) \in \mathbb{Z}$. Consequently, $\mathtt{S}_b \leq H_b$ where $H_b = h_1 + \cdots + h_b$ and h_m is the quantity defined by (4.2.1) for the sequence $x_1, \ldots, x_k$. We also have the trivial bound $\mathtt{S}_b \leq k^2$ for any b, since for each couple (i,j) we associate at most one b. Therefore

$$k^2 \ll_{l,D} \frac{M^\ell}{\epsilon^\ell} \left(k + \sum_{b=2}^{\infty} \mathtt{s}_b b^{-1/D} \right)$$

Combining this with Corollary 4.1 we have

$$k^2 \ll_{D,\epsilon,\ell} k^{2-1/(2D)} \tag{4.3.4}$$

which is a contradiction.

Therefore there is an infinite complete monochromatic subgraph whose color is $\mathbf{w}$ for some $\mathbf{w} \in \mathbb{Z}^N$ and $0 < \|\mathbf{w}\|_\infty \leq M\ell\|B_*\|_\infty$. More specifically we find that there is an infinite subset $Y \subset X$ such that $\mathbf{w} \cdot (\mathbf{y} - \mathbf{y}') \in \mathbb{Z}$ for any $\mathbf{y}, \mathbf{y}' \in Y$. Now fix an element $\mathbf{y}_0 \in Y$. Upon noticing that the map $\mathbf{y} \mapsto \mathbf{w} \cdot (\mathbf{y} - \mathbf{y}_0)$ has a finite image (since $\mathbf{y}, \mathbf{y}_0 \in [0,1)^N$) and Y is infinite, there exists an integer J such that $\mathbf{w} \cdot (\mathbf{y} - \mathbf{y}_0) = J$ for infinitely many $\mathbf{y} \in Y$. $\square$

We are now in a position to finish the proof of Theorem 4.2.

Proof of sufficiency of (2a) and (2b). First we will provide a proof when $N = 1$ and then proceed by induction on N.

Let $X \subset \mathbb{T}$ be an infinite subset, $0 < \ell \leq L$ be the rank of $\mathbf{A}_*(x)$, and $\mathbf{B}(x)$ and T be given by Lemma 4.4. We claim that for any $\epsilon > 0$ there is an integer n such that $\mathbf{B}(n)X$ is ϵ-dense in $\mathbb{T}^\ell$. Assume, by way of contradiction, that there exists an $\epsilon_0 > 0$ such that $\mathbf{B}(n)X$ is not ϵ_0-dense in $\mathbb{T}^\ell$ for any $n \in \mathbb{Z}$. By Proposition 4.3 there exists an integer $m \neq 0$, a point $y_0 \in X$, an integer J such that $m(y - y_0) = J$ for infinitely many $y \in X$. This is clearly impossible (recall that this is an equality in $\mathbb{R}$). Therefore for every $\epsilon > 0$ there exists an integer n such that $\mathbf{B}(n)X$ is ϵ-dense in $\mathbb{T}^\ell$. Let $\mathcal{T} = \mathrm{Im}(T)/\mathbb{Z}^L$ where $\mathrm{Im}(T) \subset \mathbb{R}^L$ is the image of T. Let q be given by Lemma 4.4. Then qT is integral and well-defined when considered as a map from $\mathbb{T}^\ell$ to $\mathcal{T}$. Letting $X/q = \left\{\mathbf{x}/q : \mathbf{x} \in [0,1)^N \text{ and } \mathbf{x} \in X\right\}$ we find that $\mathbf{A}(n)X = (qT)\mathbf{B}(n)(X/q)$.

Therefore for any $\epsilon > 0$ there exists an integer n such that $\mathbf{A}(n)X$ is ϵ-dense in $\mathcal{T}$.

Now we assume the theorem holds for each integer up to $N-1$. Again, by Lemma 4.4 there exist an $L \times \ell$ matrix T with entries in $\mathbb{Q}$, an $\ell \times N$ matrix $\mathbf{B} = \mathbf{B}(x)$ with entries in $\mathbb{Z}[x]$, a positive integer such that

$$\mathbf{A} = T\mathbf{B}$$

and the rows of $\mathbf{B}_*$ are $\mathbb{Q}$-linearly independent. Define

$$X/q = \left\{\mathbf{x}/q : \mathbf{x} \in [0,1)^N \text{ and } \mathbf{x} \in X\right\}.$$

and $\mathcal{T} = \operatorname{Im}(T)/\mathbb{Z}^L$, so that qT is integral and well-defined as a map from $\mathbb{T}^{N-1}$ to $\mathcal{T}$. We have two possibilities:

(i) Either for every $\epsilon > 0$ there exists an integer n such that $\mathbf{B}(n)(X/q)$ is ϵ-dense in $\mathbb{T}^\ell$. This implies that $\mathbf{A}(n)X = (qT)\mathbf{B}(n)(X/q)$ is $\tilde{\epsilon}$-dense in $\mathcal{T} \subset \mathbb{T}^L$, where $\tilde{\epsilon} \ll \epsilon \|qT\|_\infty$.

(ii) Or there exists an $\epsilon_0 > 0$ such that $\mathbf{B}(n)(X/q)$ is not ϵ_0-dense in $\mathbb{T}^\ell$ for any $n \in \mathbb{Z}$.

If we are in the first case, then we are done. We suppose (ii), and rename X/q as X. Proposition 4.3 tells us that there is a nonzero $\mathbf{w} \in \mathbb{Z}^N$ and an infinite subset $Y \subset X$ such that $\mathbf{y} \mapsto \mathbf{w} \cdot \mathbf{y}$ is constant on Y. We can assume $\mathbf{w} \cdot \mathbf{y} = 0$ for each $\mathbf{y} \in Y$ since this amounts to translating X by a fixed $\boldsymbol{\theta} \in \mathbb{T}^N$. Let the subtorus $\mathcal{T}$ of $\mathbb{T}^N$ be defined by $\mathcal{T} = \{\mathbf{t} \in [0,1)^N : \mathbf{w} \cdot \mathbf{t} = 0\}$. Then there is an $N \times (N-1)$ matrix H with full rank and integral entries such that

$$\mathrm{Im}(H)/\mathbb{Z}^N = \mathcal{T} \tag{4.3.5}$$

Since the mapping $\mathbf{t} \mapsto H\mathbf{t} + \mathbb{Z}^N \in \mathcal{T}$ is surjective, there is an infinite subset $Z \subset \mathbb{T}^{N-1}$ such that $HZ = Y$.

Let $\mathbf{C}(x) = \mathbf{A}(x)H$, then $\mathbf{C}$ is an $\ell \times (N-1)$ matrix. Let us verify that $\mathbf{C}$ satisfies conditions (2a) and (2b). Suppose there is $\mathbf{q} \in \mathbb{Q}^{N-1}$ such that $\mathbf{C}_*\mathbf{q} = \mathbf{0}$. Then $\mathbf{A}_*(x)H\mathbf{q} = \mathbf{0}$. Since $\mathbf{A}$ satisfies (2a), it follows that $H\mathbf{q} = \mathbf{0}$. Since H has a trivial kernel, this implies that $\mathbf{q} = \mathbf{0}$ and $\mathbf{C}$ satisfies condition (2a). To see that $\mathbf{C}$ satisfies condition (2b), let vectors $\mathbf{v} \in \mathbb{Q}^\ell$ and $\mathbf{w} \in \mathbb{Q}^{N-1}$ be such that $\mathbf{v} \cdot \mathbf{C}_*(x)\mathbf{w} = 0$ identically. Upon setting $\tilde{\mathbf{w}} = H\mathbf{w} \in \mathbb{Q}^N$, we find that $\mathbf{v} \cdot \mathbf{A}_*(x)\tilde{\mathbf{w}} = 0$ is the zero polynomial. Since $\mathbf{A}(x)$ satisfies condition (2b), it follows that $0 = \mathbf{v} \cdot A_0\tilde{\mathbf{w}} = \mathbf{v} \cdot A_0H\mathbf{w} = \mathbf{v} \cdot \mathbf{C}(0)\mathbf{w}$.

Let us now invoke the inductive hypothesis for $\mathbf{C}$. It follows that there is a subtorus $\mathcal{T}$ such that for every $\epsilon > 0$ there exists n such that $\mathbf{C}(n)Z$ is ϵ-dense in a translate of $\mathcal{T}$. But $\mathbf{A}(n)Y = \mathbf{C}(n)Z$, so we are done. □

Remarks 4.2. It may not be clear from the proof why conditions (2a), (2b) are the correct ones. At first sight, it would seem that the only conditions we need in order to make the proof work are the weaker ones:

- $T \neq 0$, which is equivalent to $\mathbf{A} \neq 0$.
- $\mathrm{Ker}(T^t) \subset \mathrm{Ker}(A_0^t)$, which is equivalent to $\mathrm{Ker}(\mathbf{A}_*^t) \subset \mathrm{Ker}(A_0^t)$.

But we want to maintain these requirements throughout our inductive process. Recall that our matrix $\mathbf{A}$ is changed after each step, so keeping these requirements at each step ultimately leads to conditions (2a) and (2b).

4.4 The finite version

In order to make the proof of Theorem 4.2 effective, we need to keep track of all the quantities involved when we move from one dimension to the next. The main obstacle in the proof of Theorem 4.3 is finding an effective version of Proposition 4.3. One could use the finite version of Ramsey's theorem, but currently we don't have a sensible bound for Ramsey numbers which involve more than two colors. We can get past this, by noticing that the graph we used in Proposition 4.3 is a very special graph. The following lemma is an effective form of Proposition 4.3.

Proposition 4.4. *Let* $\mathbf{B}(x) \in \mathrm{M}_{\ell\times N}(\mathbb{Z}[x])$ *have full rank and let* $X = \{\mathbf{x}_1, ..., \mathbf{x}_k\} \subset \mathbb{T}^N$ *be a set of* k *distinct points. If* $\mathbf{B}(n)X$ *is not* ϵ*-dense in* $\mathbb{T}^\ell$ *for any* $n = 1, 2, ...$ *then there exists a subset* $Y \subset X$*,* $\mathbf{y}_0 \in X$*,* $\mathbf{w} \in \mathbb{Z}^N$*, and* $J \in \mathbb{Z}$ *such that*

$$\mathbf{w}\cdot(\mathbf{y}-\mathbf{y}_0) = J \quad \text{for each } \mathbf{y} \in Y, \tag{4.4.1}$$

$$\|\mathbf{w}\|_\infty \ll_{\ell,N} \|\mathbf{B}_*\|_\infty \epsilon^{-1}, \text{ and} \tag{4.4.2}$$

$$\epsilon^{\ell+1}k^{1/4D}\|\mathbf{B}_*\|_\infty^{-1} \ll_{\ell,N,D} |Y|. \tag{4.4.3}$$

Note that again, (4.4.1) is an equality in $\mathbb{R}$.

Proof. By Proposition 4.2 we have a constant $M \ll_\ell \epsilon^{-1}$ such that

$$k^2 \ll_\ell \frac{1}{\epsilon^\ell} \sum_{\substack{0<\|\mathbf{m}\|_\infty \le M \\ \mathbf{m}\in\mathbb{Z}^\ell}} \sum_{\mathbf{x}\in X} \sum_{\mathbf{y}\in X} \lim_{R\to\infty} \frac{1}{R} \sum_{r=1}^{R} e\left(\sum_{d=0}^{D} r^d B_d^t \mathbf{m} \cdot (\mathbf{x}-\mathbf{y}) \right) \tag{4.4.4}$$

where $e(t) = \exp(2\pi i t)$ and $M \ll_\ell \epsilon^{-2}$. By an abuse of notation, let $\mathbf{m} \in \mathbb{Z}^\ell$ (with $0 < \|\mathbf{m}\|_\infty \le M$) be the lattice point which maximizes the first sum. Then

$$k^2 \ll_\ell \frac{M^\ell}{\epsilon^\ell} \sum_{\mathbf{x}\in X} \sum_{\mathbf{y}\in X} \omega(\mathbf{x},\mathbf{y}) \tag{4.4.5}$$

where $\omega(\mathbf{x},\mathbf{y})$ is the weight given by

$$\omega(\mathbf{x},\mathbf{y}) = \left| \lim_{R\to\infty} \frac{1}{R} \sum_{r=1}^{R} e\left(\sum_{d=1}^{D} r^d B_d^t \mathbf{m}(\mathbf{x}-\mathbf{y}) \right) \right|$$

Let d be the largest integer such that $B_d^t\mathbf{m} \neq 0$, then $d > 1$ since $\mathbf{B}_*$ has full rank. We partition X into equivalence classes $R_1, ..., R_s$, with $|R_i| = c_i$, where $\mathbf{x} \sim \mathbf{y}$ if $B_d^t\mathbf{m} \cdot (\mathbf{x}-\mathbf{y}) \in \mathbb{Z}$.

Define

$$\Phi_{i,j}(r) = \mathbf{m} \cdot \mathbf{B}_*(r)(\mathbf{x}_i - \mathbf{x}_j) = \sum_{d=1}^{D} r^d B_d^t \mathbf{m}(\mathbf{x}-\mathbf{y})$$

then Φ has degree d. We use Weyl's equidistribution theorem and Hua's bound to obtain

$$\omega(\mathbf{x}_i, \mathbf{x}_j) \le \begin{cases} 1 & \text{if } \mathbf{x} \sim \mathbf{y} \\ b^{-1/d} & \text{if } \mathbf{x} \not\sim \mathbf{y} \quad \text{and } \Phi_{ij}(x) \in \mathbb{Q}[x] \\ 0 & \text{if } \Phi_{ij} \text{ has at least one irrational coefficient.} \end{cases} \tag{4.4.6}$$

where in the second case $b = b(i,j)$ is the smallest positive integer such that $b\Phi_{ij}(x) \in \mathbb{Z}[x]$.

Let $y_1, ..., y_s \in \mathbb{T}$ be given by $y_i = B_d^t \mathbf{m} \cdot \mathbf{x}_i$ for some $\mathbf{x}_i \in R_i$. Then by the way we define equivalence classes, $y_1, ..., y_s$ are distinct in $\mathbb{T}$. By substituting the bound (4.4.6) into (4.4.5), we have:

$$\begin{aligned} k^2 &\ll_\ell \left(\frac{M}{\epsilon}\right)^\ell \sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{\mathbf{x}_i \in R_i} \sum_{\mathbf{x}_j \in R_j} \omega(\mathbf{x}_i, \mathbf{x}_j) \\ &\leq \left(\frac{M}{\epsilon}\right)^\ell \left\{ \sum_{i=1}^{s} c_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^{s} \sum_{j=1}^{s} \sum_{\mathbf{x}_i \in R_i} \sum_{\mathbf{x}_j \in R_j} \omega(\mathbf{x}_i, \mathbf{x}_j) \right\} \\ &\leq \left(\frac{M}{\epsilon}\right)^\ell \left\{ \sum_{i=1}^{s} c_i^2 + c^2 \sum_{b=2}^{\infty} \mathbf{s}_b b^{-1/d} \right\} \end{aligned}$$

where

$$\mathbf{s}_b = \#\{(i,j) \ \ : \ \ 1 \leq i,j \leq s, \ \ b \text{ is the smallest positive integer such that } \ b\Phi_{ij}(x) \in \mathbb{Z}[x]\}$$

and $c = \max\{c_1, ..., c_s\}$. Clearly the sequence $\mathbf{s}_b$ satisfies the conditions of Corollary 4.1. Upon writing $c_1 + \cdots + c_s = k$ and noticing $s \leq k$, we have

$$k^2 \ll_{D,\ell} \left(\frac{1}{\epsilon}\right)^\ell \left\{kc + c^2 s^{2-1/(2D)}\right\} \ll_{D,\ell} \epsilon^{-2\ell} c^2 k^{2-1/(2D)}.$$

That is,

$$\epsilon^\ell k^{1/4D} \ll_{\ell,D} c.$$

Now let Y' be equal to one of the equivalence classes $R_1, ..., R_s$ whose cardinality is c, and $\mathbf{w} = B_d^t \mathbf{m}$. Then $\mathbf{w} \cdot (\mathbf{x} - \mathbf{y}) \in \mathbb{Z}$ for each $\mathbf{x}, \mathbf{y} \in Y'$. But seeing that $|\mathbf{w}\cdot(\mathbf{x}-\mathbf{y})| \leq N\|\mathbf{w}\|_\infty$, we are guaranteed the existence of an integer $|J| \leq N\|\mathbf{w}\|_\infty$ and $\mathbf{y}_0 \in Y'$ such that $\mathbf{w}\cdot(\mathbf{y}-\mathbf{y}_0) = J$ for at least $c/N\|\mathbf{w}\|_\infty$ elements $\mathbf{y}$ of Y'. But

$$\|\mathbf{w}\|_\infty \ll_{N,\ell} \|\mathbf{B}_*\|_\infty M \ll_{N,\ell} \|\mathbf{B}_*\|_\infty \epsilon^{-1}$$

Combining this with the above we have the existence of a subset $Y \subset Y' \subset X$ such that

$$\epsilon^{\ell+1} k^{1/4D} \|\mathbf{B}_*\|_\infty^{-1} \ll_{\ell,N,D} |Y|$$

as desired. □

We also need to estimate the entries of the matrix H introduced in (4.3.5).

Lemma 4.5. *Let* $\mathbf{w} \in \mathbb{Z}^N$ *be nonzero and* $\mathbf{w}^\perp = \left\{\mathbf{v} \in \mathbb{R}^N : \mathbf{v}\cdot\mathbf{w} = 0\right\}$. *There exists an* $(N-1)\times N$ *integral matrix* H *whose image is* $\mathbf{w}^\perp$ *and* $\|H\|_\infty = \|\mathbf{w}\|_\infty$.

Proof. Since $\mathbf{w} = (w_1, ..., w_N)$ is nonzero we may assume without loss of generality that $w_N \neq 0$. Let

$$\mathbf{v}_j = w_N \mathbf{e}_j - w_j \mathbf{e}_N.$$

where $(\mathbf{e}_1, \ldots, \mathbf{e}_N)$ is the standard basis of $\mathbb{R}^N$. Then $\mathbf{v}_j \in \mathbf{w}^\perp$ because

$$\mathbf{v}_j \cdot \mathbf{w} = w_N \mathbf{e}_j \cdot \mathbf{w} - w_j \mathbf{e}_N \cdot \mathbf{w} = 0.$$

Clearly $\mathbf{v}_1, ..., \mathbf{v}_{N-1}$ are linearly independent and therefore form a basis for $\mathbf{w}^\perp$. Letting H be the $N \times (N-1)$ matrix whose columns are $\mathbf{v}_1, ..., \mathbf{v}_{N-1}$ gives the result. □

We are now in a position to prove Theorem 4.3.

Proof of Theorem 4.3. Let us proceed by induction.

Base case: Let $N = 1$ and $\mathbf{A}(x)$ be an $L \times 1$ matrix with entries in $\mathbb{Z}[x]$, such that $\mathbf{A}_*(x)$ has rank ℓ, degree at most D, and such that $\mathbf{A}(x)$ satisfies conditions (2a) and (2b) of Theorem 4.2. Let $X = \{x_1, ..., x_k\}$ be a set of k distinct points in $\mathbb{T}$ such that there does not exist a subtorus $\mathcal{T}$ such that $\mathbf{A}(n)X$ is not ϵ-dense in a translate of $\mathcal{T}$ for any $n = 1, 2, \ldots$.

By Lemma 4.4, there exist an $\ell \times N$ matrix $\mathbf{B}(x)$ whose rows are rows of $\mathbf{A}(x)$, an $L \times \ell$ matrix T with entries in $\mathbb{Q}$ such that $\mathbf{B}_*(x)$ has full rank and $\mathbf{A}(x) = T\mathbf{B}(x)$. Furthermore, there is a positive integer q such that qT is integral and $\|qT\|_\infty \ll_\ell \|\mathbf{A}_*\|_\infty^\ell$. Define

$$X/q = \{x/q + \mathbb{Z} : x \in [0, 1) \text{ and } x \in X\}$$

then X/q also has cardinality k, and $(qT)\mathbf{B}(n)(X/q) = \mathbf{A}(n)X$ is not ϵ-dense in any translate of $\mathcal{T} = \mathrm{Im}(T)/\mathbb{Z}^L$. This implies that $\mathbf{B}(n)(X/q)$ is not ϵ_1 dense in $\mathbb{T}^\ell$ for any $n = 1, 2, \ldots$, where $\epsilon_1 \gg_L \epsilon/\|qT\|_\infty$. Therefore by Proposition 4.4, there exists a subset $Y \subset X/q$, $y_0 \in \mathbb{T}$, integers J and w such that

$$w(y - y_0) = J \quad \text{for each } y \in Y, \tag{4.4.7}$$

$$\epsilon_1^{\ell+1} k^{1/4D} \|\mathbf{B}_*\|_\infty^{-1} \ll_{L,D} |Y|, \tag{4.4.8}$$

But (4.4.7) cannot happen for more than one value of y (recall that it's an equality in $\mathbb{R}$), Combining this with (4.4.8), we have

$$k \ll_{L,D} \|\mathbf{B}_*\|_\infty^{4D} \left(\frac{1}{\epsilon_1}\right)^{4D(\ell+1)} \leq \|\mathbf{B}_*\|_\infty^{4D} \left(\frac{1}{\epsilon_1}\right)^{4D(L+1)} \tag{4.4.9}$$

Recall that $\epsilon_1 \gg_L \epsilon/\|qT\| \gg_L \epsilon\|\mathbf{A}_*\|_\infty^{-\ell} \geq \epsilon\|\mathbf{A}_*\|_\infty^{-L}$. We also trivially have $\|\mathbf{B}_*\|_\infty \leq \|\mathbf{A}_*\|_\infty$ (since the rows of $\mathbf{B}$ are the rows of $\mathbf{A}$ by construction) so

$$k \ll_{L,D} \|\mathbf{A}_*\|_\infty^{4D(L(L+1)+1)} \left(\frac{1}{\epsilon}\right)^{4D(L+1)} \tag{4.4.10}$$

which shows that $k(\epsilon; L, 1, \mathbf{A})$ exists and can be bounded by the right hand side.

Inductive step. Now we assume that for each $\mathbf{C} \in \mathrm{M}_{L\times n}(\mathbb{Z}[x])$ having degree D and that satisfies conditions (2a) and (2b) of Theorem 4.2, there exist constants $c_1(n, L, D)$ and $c_2(n, L, D)$ such that

$$k(\epsilon; L, n, \mathbf{C}) \ll_{N,L,D} \|\mathbf{C}_*\|_\infty^{c_1(n,L,D)} \left(\frac{1}{\epsilon}\right)^{c_2(n,L,D)}. \tag{4.4.11}$$

for $n = 1, 2, ..., N-1$.

Let $\mathbf{A}(x) \in \mathrm{M}_{L\times N}(\mathbb{Z}[x])$ have degree at most D and satisfy conditions (2a) and (2b) from Theorem 4.2. Suppose that $X = \{\mathbf{x}_1, ..., \mathbf{x}_k\}$ is a set of k distinct points in $\mathbb{T}^N$ such that there does not exist a subtorus $\mathcal{T}$ of $\mathbb{T}^L$ such that $\mathbf{A}(n)X$ is ϵ-dense in a translate of $\mathcal{T}$ for any $n = 1, 2, ...$ Suppose $\mathbf{A}_*(x)$ has rank ℓ. Again, let $\mathbf{B}(x) \in \mathrm{M}_{\ell\times N}(\mathbb{Z}[x])$, $T \in \mathrm{M}_{L\times\ell}(\mathbb{Q})$ and $q \in \mathbb{Z}$ be given by Proposition 4.4, and let $X/q = \left\{\mathbf{x}/q : \mathbf{x} \in [0,1)^N \text{ and } x \in X\right\}$. As before we see that $\mathbf{B}(n)(X/q)$ cannot be $\epsilon_1 \gg \epsilon/\|qT\|-$dense in $\mathbb{T}^L$ for any $n = 1, 2, \ldots$. Therefore by Lemma 4.4 then there exists a subset $Y \subset X/q$, $\mathbf{y}_0 \in \mathbb{T}^N$, $J \in \mathbb{Z}$ and a $\mathbf{w} \in \mathbb{Z}^N$ such that

$$\mathbf{w} \cdot (\mathbf{y} - \mathbf{y}_0) = J \quad \text{for each } \mathbf{y} \in Y, \tag{4.4.12}$$

$$\epsilon_1^{\ell+1} k^{1/(4D)} \|\mathbf{B}_*\|_\infty^{-1} \ll_{N,L,D} |Y|, \text{ and} \tag{4.4.13}$$

$$0 < \|\mathbf{w}\|_\infty \ll_{L,N} \|\mathbf{B}_*\|_\infty \epsilon_1^{-1}. \tag{4.4.14}$$

Clearly Y lies in a translate of the torus $\mathcal{T} = \left\{\mathbf{x} + \mathbb{Z}^N : \mathbf{x} \in [0,1)^N, \mathbf{x} \cdot \mathbf{w} = 0\right\} \subset \mathbb{T}^N$. By Lemma 4.5, there is a matrix $H \in \mathrm{M}_{N \times (N-1)}(\mathbb{Z})$ of rank $N-1$ such that the range of H is $\mathbf{w}^\perp$ and $\|H\|_\infty = \|\mathbf{w}\|_\infty$. H is surjective as a map from $\mathbb{T}^{N-1}$ to $\mathcal{T}$ so there is a set Z of cardinality $|Z| = |Y|$ points in $\mathbb{T}^{N-1}$ such that $HZ = Y$. By the definition of the function $k(\epsilon; L, N, \mathbf{A})$, we have that

$$|Y| = |Z| \le k(\epsilon_1; L, N-1, \mathbf{A}H). \tag{4.4.15}$$

Note that the degree of $\mathbf{A}H$ is at most D, so by the inductive hypothesis and (4.4.13) we have

$$\epsilon_1^{L+1} k^{1/(4D)} \|\mathbf{B}_*\|_\infty^{-1} \ll_{N,L,D} \|(\mathbf{A}H)_*\|_\infty^{c_1(N-1,L,D)} \left(\frac{1}{\epsilon_1}\right)^{c_2(N-1,L,D)} \tag{4.4.16}$$

But

$$\|(\mathbf{A}H)_*\|_\infty \ll_{N,L} \|\mathbf{A}_*\|_\infty \|H\|_\infty = \|\mathbf{A}_*\|_\infty \|\mathbf{w}\|_\infty \ll \|\mathbf{A}_*\|_\infty \epsilon_1^{-1}$$

and $\|\mathbf{B}_*\|_\infty \le \|\mathbf{A}_*\|_\infty$. Therefore,

$$k^{1/(4D)} \ll_{N,L,D} \|\mathbf{A}_*\|_\infty^{1+c_1(N-1,L,D)} \left(\frac{1}{\epsilon_1}\right)^{c_1(N-1,L,D)+c_2(N-1,L,D)+L+1}$$

Recalling that $\epsilon_1 \gg_{N,L} \epsilon \|qT\|_\infty^{-1} \gg \epsilon \|\mathbf{A}_*\|_\infty^{-L}$, we have

$$k \ll_{N,L,D} \|\mathbf{A}_*\|_\infty^{c_1(N,L,D)} \left(\frac{1}{\epsilon}\right)^{c_2(N,L,D)} \tag{4.4.17}$$

where

$$c_2(N,L,D) = 4D\Big(c_1(N-1,L,D) + c_2(N-1,L,D) + L + 1\Big)$$

and

$$c_1(N,L,D) = Lc_2(N,L,D) + 4D\Big(1 + c_1(N-1,L,D)\Big)$$

This shows that $k(\epsilon; L, N, \mathbf{A})$ exists, and establishes a bound of the desired form for $k(\epsilon; L, n, \mathbf{A})$. □

Remark 4.3. As we noted in the introduction, we do not attempt to find the optimal values of the exponents c_1 and c_2 and the values that we achieve can be improved. We found in the base step that $c_1(1,L,D) = 4D(L(L+1)+1)$ and $c_2(1,L,D) = 4D(L+1)$. It is not difficult to show that $c_1(N,L,D) \leq (CD)^N L^{N+1}$ and $c_2(N,L,D) \leq (CDL)^N$ for $N, D, L \geq 1$, and C is a positive constant with $C \leq 20$. It would be interesting to know the true order of magnitude for the optimal exponents, even for fixed values of N, L, and D. When $N \geq L$ and $X = X_m^N$ where X_m is the Farey sequence of order $m = 2/\epsilon$, no dilation $n\mathbb{P}X$, where $\mathbb{P}$ is projection onto the first L components, contains a point in the cube $(0,\epsilon)^L$. But $\#X = \Omega(\epsilon^{-2N})$ which implies that the optimal choice for $c_2(N,L,1)$ is at least $2N$ when $N \geq L$. This is how the lower bound for k is obtained in [9] when $N = L = 1$ and it is *nearly* sharp in this case.

4.5 The High Dimensional Glasner Theorem

In this section we prove a stronger result than Theorem 4.1. The proof of Theorem 4.1 follows along the same lines of the proof of [2, Proposition 6.1]. Without any extra effort effort, we can add the extra requirement that the entries of T be relatively prime. This is reminiscent of Theorem II (i) though perhaps any resemblance stops here. We have the following:

Theorem 4.4. *For any $\epsilon > 0$ and any subset $X \subset \mathbb{T}^N$ of cardinality at least $k \gg_L \epsilon^{-3LN}$ there exists a matrix $T \in \mathrm{M}_{L\times N}(\mathbb{Z})$ with relatively prime entries such that TX is ϵ-dense in $\mathbb{T}^L$.*

We note that the exponents we obtain can be easily improved, but we opt for cruder bounds for the sake of brevity.

Proof. Let $\epsilon > 0$ and let $X \subset \mathbb{T}^N$ have cardinality k and let $X_j \subset \mathbb{T}$ be the projection of X onto the j^{th} coordinate axis for $j = 1, 2, ..., N$. The projection homomorphism $\mathbb{P}_j$ is represented by inner product with the vector $(0, ..., 1, ..., 0)$ where the 1 is in the j^{th} entry. Clearly

$$k = \#X \leq \prod_{j=1}^{N} \#X_j. \tag{4.5.1}$$

Consequently there is a projection X_i for which $\#X_i \geq k^{1/N}$. Let Y be a subset of X such that its projection on the i^{th} coordinate $Y_i \subset \mathbb{T}$ has cardinality at least $K = \lceil k^{1/N} \rceil$. Now if we can find a primitive vector $\boldsymbol{\alpha} \in \mathbb{Z}^L$ such that $\boldsymbol{\alpha} Y_i$ is ϵ-dense in $\mathbb{T}^L$ we are done once setting T equal to the composition of

$\mathbb{P}_i$ and the homomorphism induced by multiplication by $\boldsymbol{\alpha}$. We will show that we can choose $\boldsymbol{\alpha}$ to be of the following form

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}(n) = (q_1 n, q_2 n + 1, q_3 n, ..., q_L n)$$

where we choose $q_\ell = (M+1)^{\ell-1}$ for $n \geq 1$ where $M = [L/\epsilon]$. Note that $\boldsymbol{\alpha}$ is primitive since $(n, q_2 n + 1) = 1$.

Suppose, by way of contradiction, that there is no n for which $\boldsymbol{\alpha} Y = \boldsymbol{\alpha}(n) Y$ is ϵ-dense in $\mathbb{T}^L$. Then we have by Proposition 4.2

$$K^2 \ll_L \frac{1}{\epsilon^L} \sum_{\substack{0<\|\mathbf{m}\|_\infty \leq M \\ \mathbf{m}\in\mathbb{Z}^L}} \sum_{x\in Y_i} \sum_{y \in Y_i} \lim_{R\to\infty} \frac{1}{R} \sum_{r=1}^{R} e\Big(\mathbf{m}\cdot\boldsymbol{\alpha}(r)(x-y)\Big). \tag{4.5.2}$$

By abuse of notation, let $\mathbf{m}$ be the lattice point which maximizes the first sum. Then

$$K^2 \ll_L \frac{M^L}{\epsilon^L} \sum_{x\in Y_i} \sum_{y \in Y_i} \lim_{R\to\infty} \frac{1}{R} \sum_{r=1}^{R} e\Big(\mathbf{m}\cdot\boldsymbol{\alpha}(r)(x-y)\Big).$$

But

$$\begin{aligned} \lim_{R\to\infty} \frac{1}{R} \sum_{r=1}^{R} e\Big(\mathbf{m}\cdot\boldsymbol{\alpha}(r)(x-y)\Big) &= \lim_{R\to\infty} \frac{1}{R} \sum_{r=1}^{R} e\left(r(x-y)\sum_{\ell=1}^{L} m_\ell q_\ell\right) \\ &= \begin{cases} 1 & \text{if } (x-y)\sum_{\ell=1}^{L} m_\ell q_\ell \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence,

$$K^2 \ll_L \epsilon^{-2L} \#\{(x,y) : x, y \in Y_i,\ Q(x-y) \in \mathbb{Z}\}$$

where $Q = \sum_{\ell=1}^{L} m_\ell q_\ell$. Our choices of $q_1, ..., q_L$ guarantee that Q is non-zero. The right hand side of the above inequality can be trivially be bounded (by the same reasoning as in Proposition 4.1) by

$$\epsilon^{-2L} KQ \ll \epsilon^{-2L} KM^L \ll \epsilon^{-3L} K$$

Recalling $K = \lceil k^{1/N} \rceil$ gives

$$k \ll_L \epsilon^{-3LN}.$$

□

4.6 Concluding Remarks

We conclude with a few remarks concerning our main results. For example, it is obvious by Theorem 4.1 that if $X \subset \mathbb{T}^N$ is an infinite subset then the union $\cup_T TX$ over all $T \in \mathrm{M}_{L\times N}(\mathbb{Z})$ is dense in $\mathbb{T}^L$. Moreover, if X is invariant under the action of $\mathrm{M}_{L\times N}(\mathbb{Z})$, then X is dense in $\mathbb{T}^L$. Similarly, a simple compactness argument implies the following corollary Theorem 4.2.

Corollary 4.2. *Let $\mathbf{A}(x) \in \mathrm{M}_{L\times N}(\mathbb{Z}[x])$ satisfy conditions (2a) and (2b) of Theorem 4.2. If $X \subset \mathbb{T}^N$ is an infinite subset, then the closure of $\cup_n \mathbf{A}(n)X$ contains a translate of a subtorus $\mathcal{T}$.*

In particular, if X is infinite and $X \subset \mathbf{A}(n)X$ for each n, then the closure of X contains a translate of a subtorus $\mathcal{T}$.

It would be interesting to see what kind of generalizations can be made of Theorem 4.1. That is, what conditions on an infinite topological group G_1 and a metric group G_2 guarantee that for any infinite subset $X \subset G_1$, and $\epsilon > 0$, there exists a continuous homomorphism $\varphi : G_1 \to G_2$ such that $\varphi(X)$ is ϵ-dense in G_2? An interesting special case of this question occurs when G_1 is a compact (or locally compact) Abelian group and $G_2 = U(1) = \{z \in \mathbb{C} : |z| = 1\}$, the problem is to find a unitary character φ of G_1 which distributes a prescribed set of points evenly throughout $U(1)$.

One necessary condition on G_1 is that for each $\epsilon > 0$ there must exist a characters φ for which $\varphi(G_1)$ is ϵ-dense in $U(1)$. Even though this condition is inherently necessary, it cannot be dismissed as a triviality. For instance, if $G_1 = \mathbf{F}_2^\infty$ with the metric $d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$, then the group of all (continuous) characters of G_1 is $\mathbf{F}_2^\omega = \{x = (x_1, x_2 \ldots) : x_i \neq 0 \text{ for finitely many } i\}$ via $x(y) = (-1)^{x \cdot y}$ for all $x \in \mathbf{F}_2^\omega, y \in \mathbf{F}_2^\infty$ (note that the dot product is well defined). But the image of the whole of G_1 under any x is the set $\{-1, 1\}$ and can't be ϵ-dense.

As noted in the introduction, Alon and Peres are able to estimate the discrepancy of dilations of the form nX using the probabilistic method (see Theorem 1.2 from [2]). It would be interesting to see an analogous result in higher dimensions.

Baker [6] has proven a quantitative lemma about dilations of the form nX where $X \subset \mathbb{T}^N$, though his hypotheses and conclusion differ from our results. His proof makes use of Lemma 4.1 as well.

Chapter 5

Equivalence Relations on Separated Nets Arising from Linear Toral Flows

5.1 Introduction

A *separated net* in $\mathbb{R}^d$ is a subset Y for which there are $0 < r < R$ such that any two distinct points of Y are at least a distance r apart, and any ball of radius R in $\mathbb{R}^d$ contains a point of Y. Separated nets are sometimes referred to as *Delone sets*. The simplest example of a separated net is a lattice in $\mathbb{R}^d$, and it is natural to inquire to what extent a given separated net resembles a lattice. To this end we define equivalence relations on separated nets: we say that Y_1, Y_2 are *bi-Lipschitz equivalent*, or *BL*, if there is a bijection $f : Y_1 \to Y_2$ which is bi-Lipschitz, i.e. for some $C > 0$,

$$\frac{1}{C}\|x - y\| \leq \|f(x) - f(y)\| \leq C\|x - y\|$$

for all $x, y \in Y_1$; we say they are *bounded displacement*, or *BD*, if there is a bijection $f : Y_1 \to Y_2$ for which

$$\sup_{y \in Y} \|f(y) - y\| < \infty; \tag{5.1.1}$$

finally, we say they are *bounded displacement after dilation*, or *BDD*, if there is a $\lambda > 0$ such that Y_1 and λY_2 (the dilation of Y_2 by a factor λ) are BD. Clearly

BD implies BDD, and it is not hard to show that for separated nets, BDD implies BL. Moreover it follows from the Hall marriage lemma (see Proposition 5.1) that all lattices are in the same BDD (hence BL) class, and in the same BD class if they have the same covolume. A fundamental result in this context was the discovery in 1998 (by Burago-Kleiner [13] and McMullen [50]) that there are separated nets which are not BL to a lattice. In fact their arguments showed that there are uncountably many BL-inequivalent separated nets.

A simple way to construct separated nets is via an $\mathbb{R}^d$-action. Namely, suppose X is a compact space, equipped with a continuous action of $\mathbb{R}^d$. We denote the action by $\mathbb{R}^d \times X \ni (v, x) \mapsto v.x \in X$. Now given $x \in X$ and a subset $\mathcal{S} \subset X$, we can define the 'visit set'

$$Y = Y_{\mathcal{S},x} \stackrel{\text{def}}{=} \{v \in \mathbb{R}^d : v.x \in \mathcal{S}\}. \tag{5.1.2}$$

It is quite easy (see §5.2.2) to impose conditions on $\mathcal{S}$ guaranteeing that Y is a separated net for all x. For example, this will hold if X is a k-dimensional manifold, $\mathcal{S}$ is a *Poincaré section* (i.e., an embedded submanifold of dimension $k - d$ everywhere transverse to orbits) and the $\mathbb{R}^d$-action is *minimal* (i.e. all orbits are dense).

The net Y obviously depends on the dynamical system X chosen. We will focus on what is perhaps the simplest nontrivial case, labelly when $X = \mathbb{T}^k \stackrel{\text{def}}{=} \mathbb{R}^k/\mathbb{Z}^k$ is the standard k-torus, and $\mathbb{R}^k$ acts linearly. That is, denoting $\pi : \mathbb{R}^k \to \mathbb{T}^k$ the standard projection, and letting $V \cong \mathbb{R}^d$ be a d-dimensional linear subspace of $\mathbb{R}^k$, the action is given by

$$v.\pi(\mathbf{x}) = \pi(v + \mathbf{x}). \tag{5.1.3}$$

In this context we will say that Y is a *toral dynamics separated net*, with *associated dimensions* (d, k). We will say that a section $\mathcal{S} \subset \mathbb{T}^k$ is *linear* if it is the image under π of a bounded subset of a $(k-d)$-dimensional plane transverse to V. We remark that the toral dynamics separated nets are intimately connected to the well-studied *cut-and-project* constructions of separated nets. We briefly discuss this connection in §5.2.3, and refer the reader to [4, 5, 52, 65] for more information.

Note that the separated net Y depends nontrivially on the choices of the subspace V, the section $\mathcal{S}$, and the orbit $V.x$. We will be interested in *typical* toral dynamical nets; e.g. this might mean randomly choosing the acting subspace V in the relevant Grassmannian variety, and/or the section $\mathcal{S}$ in a finite dimensional set of shapes such as parallelotopes, etc. We remark (see §5.2.2) that different choices of x do not have a significant effect on the properties of Y.

The constructions of [13, 50] were rather indirect, and left open the question of whether any of the nets constructed via toral dynamics is equivalent (in the sense of either BL or BDD) to a lattice. In [14], Burago and Kleiner addressed this issue, and showed that a typical toral dynamics separated net with associated dimensions (2,3) is BL to a lattice. We analyze the situations in arbitrary dimensions (d, k). Our first result shows that being BL to a lattice is quite common for toral dynamics nets:

Theorem 5.1. *For a.e. d-dimensional subspace $V \subset \mathbb{R}^k$, for any $x \in \mathbb{T}^k$, and any linear section $\mathcal{S}$ which is $k-d$ dimensionally open and bounded, and satisfies $\dim_M \partial\mathcal{S} < k-d$, the corresponding separated net is BL to a lattice.*

The assumptions on the section appearing in the statement are explained in §5.2.2. The notation $\dim_M$ signifies the upper Minkowski dimension, a notion we recall in §5.4. It would be interesting to know whether there is a toral dynamics separated net which is *not* BL to a lattice.

Our second result deals with the equivalence relation BDD. Here the situation is more delicate, and we have the following:

Theorem 5.2. *Consider toral dynamics nets with associated dimensions (d, k).*

1. *If $(k+1)/2 < d \leq k$, then for almost every V, any $x \in \mathbb{T}^k$, and linear section $\mathcal{S}$ which is $k-d$ dimensionally open and bounded, and satisfies $\dim_M \partial\mathcal{S} = k-d-1$, the coresponding separated net is BDD to a lattice.*

2. *For any $2 \leq d \leq k$, for almost every V, for any $x \in \mathbb{T}^k$ and any linear section $\mathcal{S}$ which is a box with sides parallel to $k-d$ of the coordinate axes, the corresponding net is BDD to a lattice.*

3. *For almost every linear section $\mathcal{S} \subset B$ which is a parallelotope, there is a residual set of subspaces V for which the corresponding net is not BDD to a lattice.*

Our strategy of proof is inspired by [14, 25, 68]. We use work of Burago-Kleiner [14] and Laczkovich [43] to relate the notions of BL and BDD to rates of convergence of some ergodic averages for our toral $\mathbb{R}^d$-action. This rate of convergence is studied via harmonic analysis on $\mathbb{T}^k$, and leads to the study of Diophantine properties of the acting subspace V. The connection between Diophantine properties of V and rates of convergence of ergodic averages on $\mathbb{T}^k$ is standard and well-studied in the literature on discrepancy, see e.g. [28]. However none of the existing results in the literature supplied the estimates we needed. Before stating our results in this direction, we introduce some notation.

We will use boldface letters such as $\mathbf{v}, \mathbf{x}$ to denote vectors in $\mathbb{R}^k$, and denote their inner product by $\mathbf{v} \cdot \mathbf{x}$. Let $V = \mathrm{span}\,(\mathbf{v}_1, \dots, \mathbf{v}_d)$. For $T > 0$ we set

$$B_T \stackrel{\text{def}}{=} \left\{ \sum a_i \mathbf{v}_i : \max_i |a_i| \leq T \right\}. \tag{5.1.4}$$

The notation $|A|$ denotes the Lebesgue measure of a measurable set A in $\mathbb{R}^k$ or $\mathbb{T}^k$. Given $U \subset \mathbb{T}^k$, $T \geq 0$ and $\mathbf{x} \in \mathbb{R}^k$ we set

$$N_T(U, \mathbf{x}) \stackrel{\text{def}}{=} \int_{B_T} \chi_U \left(\pi(\mathbf{x} + \mathbf{t}) \right) \, d\mathbf{t}.$$

The reader should note that this notation suppresses the dependence of $N_T(U, \mathbf{x})$ on the choice of the subspace V as well as the basis $\mathbf{v}_1, \dots, \mathbf{v}_d$. We will denote by $\|\mathbf{m}\|$ the sup-norm of a vector $\mathbf{m} \in \mathbb{R}^k$, and say that $\mathbf{v}$ is *Diophantine* if there are positive constants c, s such that

$$|\mathbf{m} \cdot \mathbf{v}| \geq \frac{c}{\|\mathbf{m}\|^s}\ , \quad \text{for all nonzero } \mathbf{m} \in \mathbb{Z}^k. \tag{5.1.5}$$

We will say that V is *Diophantine* if it contains a Diophantine vector.

By an *aligned box* in $\mathbb{T}^k$ we mean the image, under π, of a set of the form $[a_1, b_1] \times \cdots \times [a_k, b_k]$ (a box with sides parallel to the coordinate axes), where $b_i - a_i < 1$ for all i (so that π is injective on the box).

Theorem 5.3. *Suppose V is Diophantine. Then there are constants C and $\delta > 0$ such that for any $\mathbf{x} \in \mathbb{R}^k$, any $T > 1$, and any aligned box $U \subset \mathbb{T}^k$,*

$$\left| N_T(U, \mathbf{x}) - |U||B_T| \right| \le CT^{d-\delta}. \tag{5.1.6}$$

We remark that under a stronger Diophantine assumption, which still holds for almost every subspace V, conclusion (5.1.6) can be strengthened, replacing $T^{d-\delta}$ with $(\log T)^{k+2d+\delta}$. See Proposition 5.10.

Given a basis $\mathcal{T} = (\mathbf{t}_1, \dots, \mathbf{t}_k)$ of $\mathbb{R}^k$, we denote

$$r_{\mathcal{T}}(\mathbf{m}) \stackrel{\text{def}}{=} \prod_{i=1}^{k} \min\left(1, \frac{1}{|\mathbf{t}_i \cdot \mathbf{m}|}\right), \tag{5.1.7}$$

and say that $\mathbf{v}_1, \dots, \mathbf{v}_d$ are *strongly Diophantine (with respect to $\mathcal{T}$)* if for any $\varepsilon > 0$ there is $C > 0$ such that for any $M > 0$,

$$\sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|\mathbf{m}\| \le M}} r_{\mathcal{T}}(\mathbf{m}) \prod_{i=1}^{d} \frac{1}{|\mathbf{m} \cdot \mathbf{v}_i|} \le CM^{\varepsilon}. \tag{5.1.8}$$

We say that $U \subset \mathbb{T}^k$ is *a parallelotope aligned with $\mathcal{T}$* if there are positive $b_1, \dots, b_k$, and $\mathbf{x} \in \mathbb{R}^k$, such that $U = \pi(\widetilde{U} + \mathbf{x})$, where

$$\widetilde{U} \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{d} a_i \mathbf{t}_i : \forall i,\ a_i \in [0, b_i] \right\},$$

and π is injective on $\widetilde{U} + \mathbf{x}$. Let $\mathbf{e}_1, \dots, \mathbf{e}_k$ be the standard basis for $\mathbb{R}^k$.

Theorem 5.4. *Suppose $\mathcal{T} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is a basis for $\mathbb{R}^k$ such that $\mathbf{v}_i \in \{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ for each $i = d+1, \ldots, k$, and $\mathbf{v}_1, \ldots, \mathbf{v}_d$ is strongly Diophantine with respect to $\mathcal{T}$. Then for any $\delta > 0$ there is $C > 0$ such that for all $\mathbf{x} \in \mathbb{T}^k$, and any U which is a parallelotope aligned with $\mathcal{T}$, with sidelengths bounded above by η, we have*

$$\left| N_T(U, \mathbf{x}) - |U||B_T| \right| \leq C(1+\eta)^k T^\delta. \tag{5.1.9}$$

As above, we will show in Proposition 5.10 that there is a stronger Diophantine hypothesis, which still holds for almost every V, under which T^δ in (5.1.9) can be replaced by $(\log T)^{k+2d+\delta}$.

Justifying the terminology, we will see in §5.7 that a subspace with a strongly Diophantine basis is Diophantine. We will also see that almost every choice of V (respectively $\mathcal{T}$) satisfies the Diophantine properties which are the hypotheses of Theorem 5.3 (resp., Theorem 5.4). The conclusions of Theorems 5.1 and 5.2 hold for these choices.

Besides the cut-and-project method, another well-studied construction of a separated net is the *substitution system* construction, and results analogous to ours have appeared for separated nets arising via substitution systems in recent work of Solomon [67, 68] and Aliste-Prieto, Coronel and Gambaudo [1]. Briefly, it was shown in these papers that all substitution system separated nets are BL to lattices and many but not all are BDD to lattices. A particular case of interest is the *Penrose net* obtained by placing one point in each tile of the Penrose aperiodic tiling of the plane. As shown by de Bruijn [24], the Penrose net admits alternate descriptions via both the cut-and-project and substitution system constructions. Using the latter approach, Solomon [68] showed that that the Penrose net is BDD to a lattice.

5.1.1 Organization of the paper

In §5.2 we review basic material relating sections for minimal flows and separated nets, and the relation to cut-and-project constructions. In §5.3 we state the results of Burago-Kleiner and Laczkovich, and use these to connect the properties of the separated net to quantitative equidistribution statements for flows. In §5.4 we discuss Minkowski dimension and show how to approximate a section by aligned boxes if the Minkowski dimension of the boundary is strictly smaller than d. The main result of §5.5 is Theorem 5.8, which provides good approximations to the indicator function of parallelotopes in $\mathbb{T}^k$ by trigonometric polynomials. We believe this result will be helpful for other problems in Diophantine approximation and ergodic theory of linear toral flows. In §5.6 we deduce an Erdős-Turán type inequality from Theorem 5.8 and apply it to prove Theorems 5.3 and 5.4. In §5.7 we adapt arguments of W. Schmidt to show that our Diophantine conditions are satisfied almost surely, and deduce Theorem 5.1 and parts (1) and (2) of Theorem 5.2 in §5.8. In §5.9 we prove Theorem 5.2(3).

5.2 Basics

5.2.1 Bounded displacement

We first recall the following well-known facts.

Proposition 5.1. *Any two lattices of the same covolume are BD to each other, and any two lattices are BDD to each other. Moreover, if* $Y \subset \mathbb{R}^d$ *is BDD to a lattice, and* $T : \mathbb{R}^d \to \mathbb{R}^d$ *is a linear isomorphism, then* $T(Y)$ *is also BDD to a lattice.*

Proof. Suppose L_1 and L_2 are lattices of the same covolume λ, and define a bipartite graph G whose vertices are the points of $L_1 \cup L_2$, and $x_1 \in L_1$, $x_2 \in L_2$ are joined by an edge if $\|x_1 - x_2\| \le r_1 + r_2$, where r_i is the diameter of a compact fundamental domain for L_i. To verify the conditions of the Hall marriage lemma [35], let D_2 be a fundamental domain for L_2, so that $\mathbb{R}^d = \bigsqcup_{y \in L_2} y + D_2$, and let $A \subset L_1$ with $N \stackrel{\mathrm{def}}{=} \# A$. Let F denote the set of points in $\mathbb{R}^d$ which are within a distance r_1 from points of A. Then F contains at least N copies of a fundamental domain for L_1 so has volume at least $N\lambda$. Therefore F intersects at least N of the sets $\{y + D_2 : y \in L_2\}$. By the definition of G, if F intersects $y + D_2$ then y is connected to an element of A by an edge. This implies that the number of neighbors of A is at least N. By the marriage lemma there is a perfect matching in G, which gives our required bijection.

If the covolumes of L_1, L_2 are not the same, first apply a homothety to one of them to reduce to the previous case. Now suppose L is a lattice in $\mathbb{R}^d$ and $\phi : Y \to L$ is a bijection moving points a uniformly bounded amount, then $T \circ \phi \circ T^{-1}$ is a bijection $T(Y) \to T(L)$ and it moves points a bounded amount because T is Lipschitz. This proves the second assertion. $\square$

5.2.2 Sections and minimal actions

A standard technique for studying flows was introduced by Poincaré. Suppose X is a manifold with a flow, i.e. an action of $\mathbb{R}$. Given an embedded submanifold $\mathcal{S}$ transverse to the orbits, we can study the return map to $\mathcal{S}$ along orbits, and in this way reduce the study of the $\mathbb{R}$-action to the study of a $\mathbb{Z}$-action. We will be interested in a similar construction for the case of an $\mathbb{R}^d$-action, $d > 1$. Namely, given a space X equipped with an $\mathbb{R}^d$-action, we say that $\mathcal{S} \subset X$ is a *good section* if there are bounded neighborhoods $\mathcal{U}_1, \mathcal{U}_2$ of 0 in $\mathbb{R}^d$, such that for any $x \in X$:

(i) there is at most one $u \in \mathcal{U}_1$ such that $u.x \in \mathcal{S}$.

(ii) there is at least one $u \in \mathcal{U}_2$ such that $u.x \in \mathcal{S}$.

These conditions immediately imply that the set $Y = Y_{\mathcal{S},x}$ of visit times defined in (5.1.2) is a separated net; moreover the parameters r, R appearing in the definition of a separated net may be taken to be the same for all $x \in X$, since they depend only on $\mathcal{U}_1, \mathcal{U}_2$ respectively.

The action is called *minimal* if there are no proper invariant closed subsets of X, or equivalently, if all orbits are dense. The following proposition shows that good sections always exist for minimal actions on manifolds:

Proposition 5.2. *Suppose X is a compact k-dimensional manifold equipped with a minimal $\mathbb{R}^d$-action, and suppose $\mathcal{S} \subset X$ is the image of an open bounded $\mathcal{O} \subset \mathbb{R}^{k-d}$ under a smooth injective map which is everywhere transverse to the orbits and extends to the closure of $\mathcal{O}$. Then $\mathcal{S}$ is a good section.*

Proof. Since $\mathcal{S}$ is transverse to orbits, for every $x \in \overline{\mathcal{S}}$ there is a bounded neighborhood $U = U_x$ of identity in $\mathbb{R}^d$ so that for $u \in U \smallsetminus \{0\}$, $u.x \notin \overline{\mathcal{S}}$. Since $\mathcal{O}$ is bounded, a compactness argument shows that U may be taken to be independent of x, and we can take $\mathcal{U}_1$ so that $\mathcal{U}_1 - \mathcal{U}_1 = \{x - y : x, y \in \mathcal{U}_1\} \subset U$, which immediately implies (i). Let

$$\widehat{\mathcal{S}} \stackrel{\mathrm{def}}{=} \{u.s : u \in U, s \in \mathcal{S}\}.$$

Then $\widehat{\mathcal{S}}$ is open in X. By a standard fact from topological dynamics (see e.g. [3]), the set of return times

$$\{u \in \mathbb{R}^d : u.x \in \widehat{\mathcal{S}}\}$$

is *syndetic*, i.e. there is a bounded set K such that for any $w \in \mathbb{R}^d$, there is $k \in K$ with $(w + k).x \in \widehat{\mathcal{S}}$. By minimality this implies that for any $x \in X$ there is $k \in K$ such that $k.x \in \widehat{\mathcal{S}}$. Taking $\mathcal{U}_2 = K - U$ we obtain (ii). □

If X is not minimal, there will be some x and $\mathcal{S}$ for which Y is not syndetic. However good sections exist for any action:

Proposition 5.3. *For any action of $\mathbb{R}^d$ on a compact manifold, there are good sections.*

Proof. Fix a bounded symmetric neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^d$. We can assume that $\mathcal{U}$ is sufficiently small, so that for each $x \in X$ there is an embedded submanifold $\mathcal{S}_x$ of dimension $k-d$ such that the map

$$\mathcal{U} \times \mathcal{S}_x \to X, \quad (u, x) \mapsto u.x$$

is a diffeomorphism onto a neighborhood $\mathcal{O}_x$ of x. By compactness we can choose $x_1, \ldots, x_r$ so that the sets $\mathcal{O}_j = \mathcal{O}_{x_j}$ are a cover of X. By a small perturbation we can ensure that the closures of the $\mathcal{S}_j = \mathcal{S}_{x_j}$ are disjoint. Let $\mathcal{S} = \bigcup_j \mathcal{S}_j$, then it is clear by construction that (ii) holds for $\mathcal{U}_2 = \mathcal{U}$. Since the $\mathcal{S}_j$ are disjoint, a compactness argument shows (i). □

The following will be useful when we want to go from a section to a smaller one.

Proposition 5.4. *Suppose $\mathcal{S}$ is a section for an $\mathbb{R}^d$ action on a space X, $x \in X$, and $\mathcal{S} = \bigsqcup_{i=1}^r \mathcal{S}_i$ is a partition into subsets. Suppose that for $i = 1, \ldots, r$, each $Y_i \stackrel{\text{def}}{=} Y_{\mathcal{S}_i,x}$ is BD to a fixed lattice L. Then $Y_{\mathcal{S},x}$ is BDD to a lattice.*

Proof. Clearly $Y_{\mathcal{S},x} = \bigsqcup_1^r Y_i$, and by assumption, for each i there is a bijection $f_i : Y_i \to L$ moving points a bounded distance. Let $\hat{L}$ be a lattice containing L as a subgroup of index r and let $v_1, \ldots, v_r$ be coset representatives for $\hat{L}/L$. Then

$$f(y) = f_i(y) + v_i \quad \text{for } y \in Y_i$$

is the required bijection between Y and $\hat{L}$. □

Proposition 5.5. *Suppose $\mathcal{S}_1, B$ are two good sections for an $\mathbb{R}^d$-action on a space X. Let $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_1', \mathcal{U}_2'$ be the corresponding sets as in (i) and (ii), for $\mathcal{S}_1$ and B respectively, and assume that*

$$\mathcal{U}_2' - \mathcal{U}_2' \subset \mathcal{U}_1. \tag{5.2.1}$$

Then there is $\mathcal{S}_2 \subset B$, a good section for the action, such that for each $x \in X$, the nets $Y_i = Y_{\mathcal{S}_i,x}$ as in (5.1.2) *$(i = 1, 2)$ are BD to each other.*

Proof. For each $x \in X$, let $u_x \in \mathcal{U}_2'$ be such that $u_x.x \in B$. Let $\mathcal{S}_2 \stackrel{\text{def}}{=} \{u_x.x : x \in \mathcal{S}_1\}$. First note that $\mathcal{S}_2$ is a good section: $\mathcal{U}_2'' \stackrel{\text{def}}{=} \mathcal{U}_2 + \{u_x : x \in \mathcal{S}_1\}$ satisfies (ii) for $\mathcal{S}_2$. Since $\mathcal{U}_1'$ satisfies (i) for B, it also satisfies (i) for $\mathcal{S}_2$.

Let $Y_i = Y_{\mathcal{S}_i,x}$ $(i = 1, 2)$. It remains to show that the Y_i are BD. For each $u \in Y_1$ we have $z = u.x \in \mathcal{S}_1$ so that $F(u).x \in \mathcal{S}_2$, where $F(u) = u + u_z$ and $u_z \in \mathcal{U}_2'$. Clearly F moves all points a bounded distance, and maps Y_1 to Y_2. We need to show that it is a bijection. If $u' \in Y_2$ then $u'.x = s_2 \in \mathcal{S}_2$, which implies that there is $s_1 \in \mathcal{S}_1$ with $s_2 = u_{s_1}.s_1$. This implies that $s_1 = (u' - u_{s_1}).x$ so that $u' - u_{s_1} \in Y_1$ satisfies $F(u' - u_{s_1}) = u' - u_{s_1} + u_{s_1} = u'$. Thus F is surjective. Now suppose $u_1, u_2 \in Y_1$ such that

$$u_1 + u_{z_1} = F(u_1) = F(u_2) = u_2 + u_{z_2},$$

where $z_i = u_i x \in \mathcal{S}_1$. Then by (5.2.1), $u_2 - u_1 = u_{z_1} - u_{z_2} \in \mathcal{U}_1$, so by (i) with $x = s_1$, we conclude that $u_2 = u_1$. $\square$

The nets $Y_{\mathcal{S},x}$ depend on the choice of x and $\mathcal{S}$. As Theorem 5.2 shows, different choices of $\mathcal{S}$ will lead to very different separated nets. However, as the following result shows, for much of our discussion the choice of x is immaterial.

Proposition 5.6. *Suppose X is a minimal dynamical system and $\mathcal{S}$ is a good section which is $(k-d)$-dimensionally open. If there is $x_0 \in X$ for which the separated net $Y_{\mathcal{S},x_0}$ is BD (resp. BDD, BL) to a lattice, then for every $x \in X$, the net $Y_{\mathcal{S},x}$ is also BD (resp. BDD, BL) to a lattice.*

Proof. We will prove the statement for the case of the BD equivalence relation, leaving the other cases to the reader.

Write $Y_0 \stackrel{\text{def}}{=} Y_{\mathcal{S},x_0}$ and $Y \stackrel{\text{def}}{=} Y_{\mathcal{S},x}$. Let $\mathcal{L} \subset \mathbb{R}^d$ be a lattice and let $f : Y_0 \to \mathcal{L}$ be a bijection satisfying

$$K \stackrel{\text{def}}{=} \sup_{y \in Y} \|y - f(y)\| < \infty.$$

Let Ω be a compact fundamental domain for the action of $\mathcal{L}$ on $\mathbb{R}^d$, that is for each $z \in \mathbb{R}^d$ there are unique $\ell = \ell(z) \in \mathcal{L}$, $\omega = \omega(z) \in \Omega$ with $z = \ell + \omega$. Let $x \in X$ and let $u_n \in \mathbb{R}^d$ such that $u_n.x_0 \to x$. Using the continuity of the action on X, and the assumption that $\mathcal{S}$ is $(k-d)$-dimensionally open, it is easy to see that the translated nets $Y_0 - u_n$ converge to Y in the following sense. Let $B(x,T)$ denote the Euclidean open ball of radius T around x. For any $T > 0$ for which there is no element of Y of norm T, and any $\varepsilon > 0$ there is n_0 such that for any $n > n_0$, there is a bijection between $B(0,T) \cap Y$ and $B(0,T) \cap (Y_0 - u_n)$ moving points at most a distance ε.

Now for each k we take $n = n(k)$ large enough so that for each $y \in B(0,k) \cap Y$, there is $x = x(y) \in Y_0 - u_n$ with $\|y - x\| < 1$. Define $f_k : B(0,k) \cap Y \to \mathcal{L}$ by

$$f_k(y) \stackrel{\text{def}}{=} f(x(y) + u_n) - \ell(u_n).$$

Then for each $k \geq k_0 > 0$, and each $y \in B(0,k) \cap Y$,

$$\begin{aligned} \|y - f_k(y)\| &\leq \|y - x(y)\| + \|x(y) + u_n - f(x(y) + u_n)\| + \|u_n - \ell(u_n)\| \\ &\leq 1 + K + \operatorname{diam}(\Omega); \end{aligned}$$

that is, points in $B(0,k_0) \cap Y$ are moved a uniformly bounded distance by the maps f_k, $k \geq k_0$. In particular the set of possible values of the maps $f_k(y)$, $k \geq k_0$ is finite. Thus by a diagonalization procedure we may choose a subset of the f_k so that for each $y \in Y$, $f_k(y)$ is eventually constant. We denote this constant by $\hat{f}(y)$. Now it is easy to check that $\hat{f}$ is a bijection satisfying (5.1.1). □

We now specialize to linear actions on tori. It is known that a linear action of a d-dimensional subspace $V \subset \mathbb{R}^k$ on $\mathbb{T}^k$ as in (5.1.3) is minimal if and only if V is *totally irrational*, i.e., not contained in a proper $\mathbb{Q}$-linear subspace of $\mathbb{R}^k$. Suppose V is totally irrational and of dimension d, so that the action of V on $\mathbb{T}^k$ is minimal. Note that when using this action to define separated nets via (5.1.2), one needs to fix an identification of V with $\mathbb{R}^d$; however, in light of Proposition 5.1, for the questions we will be considering, this choice will be immaterial.

Let W be a subspace of dimension $k-d$, such that $\mathbb{R}^k = V \oplus W$. For any bounded open subset B' in W, such that $\pi|_{\bar{B}'}$ is injective, $B \stackrel{\text{def}}{=} \pi(B')$ is a good section, in view of Proposition 5.2. We do not assume that W is totally irrational, so that π need not be globally injective on W. We remind the reader that such sections will be called *linear* sections.

When discussing sections, there is no loss of generality in considering linear sections:

Corollary 5.1. *Let V and W be as above, and assume W is totally irrational. Then for any section $\mathcal{S}$ for the linear action of V on $\mathbb{T}^k$, there is a linear section $\mathcal{S}' \subset \pi(W)$ such that for any $x \in \mathbb{T}^k$, $Y_{\mathcal{S},x}$ and $Y_{\mathcal{S}',x}$ are BDD.*

Proof. Since W also acts minimally, for any $\varepsilon > 0$, there is a sufficiently large ball $B' \subset W$ such that $B = \pi(B')$ is ε-dense in $\mathbb{T}^k$. That is, we can make the neighborhood $\mathcal{U}_2$ appearing in (ii) as small as we wish. Thus, given any section $\mathcal{S}$ for the action of V, we can make B' large enough so that (5.2.1) holds. So the claim follows from Proposition 5.5. $\square$

When we say that the section $\mathcal{S}$ is $k-d$ dimensionally open, bounded, is a parallelotope, etc., we mean that $\mathcal{S} = \pi(\mathcal{S}')$ where $\mathcal{S}' \subset W$ has the corresponding properties as a subset of $W \cong \mathbb{R}^{k-d}$.

5.2.3 Cut and project nets

Fix a direct sum decomposition $\mathbb{R}^k = V \oplus W$ into $V \cong \mathbb{R}^d$, $W \cong \mathbb{R}^{d-k}$. Let $\pi_V : \mathbb{R}^k \to V$ and $\pi_W : \mathbb{R}^k \to W$ be the projections associated with this direct sum decomposition. Suppose $L \subset \mathbb{R}^k$ is a lattice, and $K \subset W$ is a non-empty bounded open set. The *cut-and-project construction* associated to this data is

$$\mathcal{N} = \mathcal{N}_{L,K,V,W} \stackrel{\text{def}}{=} \{x \in V : \exists y \in L, \pi_V(y) = x, \pi_W(y) \in K\}.$$

The set $\mathcal{N}$ is always a separated net in $V \cong \mathbb{R}^d$, and under suitable assumptions, is aperiodic (e.g. is not a finite union of lattices). This is a particular case of a family of more general constructions involving locally compact abelian groups. We refer to [4, 5, 65] for more details.

Unsurprisingly, the construction above may be seen as a toral dynamics separated net. Since we will not be using it, we leave the proof of the following to the reader:

Proposition 5.7. *Given L, $\mathbb{R}^k = V \oplus W$ and $K \subset W$ as above, there is a linear subspace $V' \subset \mathbb{R}^k$, a section $\mathcal{S} \subset \mathbb{T}^k$, and $x \in \mathbb{T}^k$, such that $\mathcal{N}_{L,K,V,W} = Y_{\mathcal{S},x}$, where $Y_{\mathcal{S},x}$ is as in* (5.1.2) *for the action* (5.1.3)*.*

□

5.3 Results of Burago-Kleiner and Laczkovich, and their dynamical interpretation

Let Y be a separated net. The question of whether Y is BL or BDD to a lattice is related to the number of points of Y in large sets in $\mathbb{R}^d$. More precisely, fix a positive number λ, which should be thought of as the asymptotic density of Y, and for $E \subset \mathbb{R}^d$, define

$$\mathrm{disc}_Y(E, \lambda) \stackrel{\mathrm{def}}{=} \Big| \# Y \cap E - \lambda |E| \Big|,$$

where $|E|$ denotes the d-dimensional Lebesgue measure of E ('disc' stands for *discrepancy*). If Y is a lattice, and E is sufficiently regular (e.g. a large ball), then one has precise estimates showing that $\mathrm{disc}_Y(E, \lambda)$ is small, relative to the measure of E. In this section we present some results which show that for arbitrary Y, bounds on $\mathrm{disc}_Y(E, \lambda)$ are sufficient to ensure that Y is BL or BDD to a lattice.

For each $\rho \in \mathbb{N}$ and $\lambda > 0$, let

$$D_Y(\rho, \lambda) \stackrel{\mathrm{def}}{=} \sup_B \frac{\mathrm{disc}_Y(B, \lambda)}{\lambda |B|},$$

where the supremum is taken over all cubes $B \subset \mathbb{R}^d$ of the form

$$B = [a_1\rho, (a_1+1)\rho] \times \cdots \times [a_d\rho, (a_d+1)\rho], \quad \text{with } a_1, \ldots, a_d \in \mathbb{Z}.$$

Theorem 5.5 (Burago-Kleiner)**.** *If there is $\lambda > 0$ for which*

$$\sum_\rho D_Y(2^\rho, \lambda) < \infty \tag{5.3.1}$$

then Y is BL to a lattice.

Proof. The theorem was proved in case $d = 2$ in [14], and in [1] for general d. □

Using this we state a dynamical sufficient condition guaranteeing that a dynamical separated net is BL to a lattice. We will denote the Lebesgue measure of $B \subset \mathbb{R}^d$ by $|B|$ and write the Lebesgue measure element as dt. Let $\mathbf{v}_1, \ldots, \mathbf{v}_d$ be a basis of $\mathbb{R}^d$ and define B_T via (5.1.4). Note that $|B_T| = CT^d$ for some $C > 0$. For $W \subset X$ and $x \in X$, denote

$$N_T(W, x) \stackrel{\text{def}}{=} \int_{B_T} \chi_W(\mathbf{t}.x)\, d\mathbf{t},$$

where χ_W is the indicator function of W. The asymptotic behavior of such *Birkhoff integrals* as $T \to \infty$ is a well-studied topic in ergodic theory. The action of $\mathbb{R}^d$ on X is said to be *uniquely ergodic* if there is a measure μ on X such that for any continuous function f on X, and any $x \in X$,

$$\left| \int_{B_T} f(\mathbf{t}.x)\, d\mathbf{t} - |B_T| \int_X f\, d\mu \right| = o(|B_T|).$$

We now show that a related quantitative estimate implies that certain dynamical nets are BL to a lattice.

Corollary 5.2. *Suppose $\mathbb{R}^d$ acts on X and $\mathcal{S}$ is a good section for the action. Let $\mathcal{U}_1$ be a neighborhood of identity in $\mathbb{R}^d$ satisfying (i) of §5.2.2, and let*

$$W \stackrel{\text{def}}{=} \{u.x : u \in \mathcal{U}_1,\, x \in \mathcal{S}\} \subset X. \tag{5.3.2}$$

Suppose there are positive constants a, C, δ such that for all $x \in X$ and $T > 1$,

$$\Big| N_T(W, x) - a|B_T| \Big| < C\, T^{d-\delta}. \tag{5.3.3}$$

Then for any $x \in X$, the net $Y_{\mathcal{S},x}$ as in (5.1.2) *is BL to a lattice.*

Proof. Let $x \in X$, $Y = Y_{\mathcal{S},x}$ and let $B = x' + B_T \subset \mathbb{R}^d$, i.e. B is a cube of side length $2T$, with sides parallel to the coordinate hyperplanes, and center at x'.

We want to bound $\# Y \cap B$ in terms of $N_T(W, x')$. Let r denote the diameter of $\mathcal{U}_1$, and let $b = |\mathcal{U}_1|$. If $y \in Y \cap B$ then $y.x \in \mathcal{S}$ and hence $(y+u).x \in W$ for any $u \in \mathcal{U}_1$. This implies that

$$N_{T+r}(W, x) \geq (\# Y \cap B)\, b.$$

Similarly, if $\chi_W(y.x) = 1$ then there is $y' \in Y$ with $\|y' - y\| \leq r$, which implies that

$$N_{T-r}(W, x) \leq (\# Y \cap B)\, b.$$

Applying (5.3.3) we find that

$$\frac{a}{b}|B_{T-r}| - \frac{C}{b}(T-r)^{d-\delta} \leq \# Y \cap B \leq \frac{a}{b}|B_{T+r}| + \frac{C}{b}(T+r)^{d-\delta}.$$

So for any $\delta' < \delta$ there is T_0 such that for $T > T_0$, setting $\lambda = a/b$ gives

$$\mathrm{disc}_Y(B_T, \lambda) \leq T^{d-\delta'}.$$

Since $|B_T| = cT^d$ for some $c > 0$, we find that $D_Y(T, \lambda) = O(T^{-\delta'})$. From this (5.3.1) follows. □

We now turn to analogous results for the relation BDD. Our results in this regard rely on work of Laczkovich. We first introduce some notation. For a measurable $B \subset \mathbb{R}^d$, we denote by $|B|$ the Lebesgue measure of B, by ∂B the boundary of B, and by $|\partial B|_{d-1}$ the $(d-1)$-dimensional volume of ∂B. By a *unit cube* (respectively, *dyadic cube*) we mean a cube of the form

$$[a_1, b_1) \times \cdots \times [a_k, b_k),$$

where for $i = 1, \ldots, k$ we have $a_i \in \mathbb{Z}$ and $b_i - a_i = 1$ (respectively, $b_i - a_i = 2^j$ for a non-negative integer j independent of i).

Theorem 5.6 ([43], Theorem 1.1)**.** *For a separated net $Y \subset \mathbb{R}^d$, and $\lambda > 0$, the following are equivalent:*

1. *Y is BD to a lattice of covolume λ^{-1}.*

2. *There is $c > 0$ such that for every finite union of unit cubes $\mathcal{C} \subset \mathbb{R}^d$,*

$$\mathrm{disc}_Y(\mathcal{C}, \lambda) \leq c\, |\partial\, \mathcal{C}|_{d-1}.$$

3. *There is $c > 0$ such that for any measurable A,*

$$\mathrm{disc}_Y(A, \lambda) \leq c\, \left|(\partial\, A)^{(1)}\right|,$$

where $(\partial\, A)^{(1)}$ denotes the set of points at distance 1 from the boundary of A.

When applying this result, another result of Laczkovich is very useful. For sets $\mathcal{C}, Q_1, \ldots, Q_n$, we say that $\mathcal{C} \in S(Q_1, \ldots, Q_n)$ if $\mathcal{C}$ can be presented using $Q_1, \ldots, Q_n$ and the operations of disjoint union and set difference, with each Q_i appearing at most once. Then we have:

Theorem 5.7 ([43], Theorem 1.3)**.** *There is a constant κ, depending only on d, such that if $\mathcal{C}$ is a finite union of unit cubes in $\mathbb{R}^d$, then there are dyadic cubes $Q_1, \ldots, Q_n$, such that $\mathcal{C} \in S(Q_1, \ldots, Q_n)$ and for each j,*

$$\#\left\{i : Q_i \text{ has sidelength } 2^j\right\} \leq \kappa\, \frac{|\partial\, \mathcal{C}|_{d-1}}{2^{j(d-1)}}. \tag{5.3.4}$$

Corollary 5.3. *Suppose $\mathbb{R}^d$ acts on X and $\mathcal{S}$ is a good section for the action. Let $\mathcal{U}_1$ be a neighborhood of identity in $\mathbb{R}^d$ satisfying (i) of §5.2.2, and let W be as in* (5.3.2). *Suppose there are positive constants a, C, δ such that for all $x \in X$ and $T > 1$,*

$$\left| N_T(W, x) - a|B_T| \right| < C\, T^{d-1-\delta}. \tag{5.3.5}$$

Then for any $x \in X$, the net $Y_{\mathcal{S},x}$ as in (5.1.2) *is BDD to a lattice.*

Proof. Let $b = |\mathcal{U}_1|$ and let $\lambda = a/b$. We verify condition (2) of Theorem 5.6. Arguing as in the proof of Corollary 5.2, we deduce from (5.3.5) that there are $0 < \delta' < \delta$, T_0 and C' such that for every cube Q of sidelength $T \geq T_0$ in $\mathbb{R}^d$,

$$\mathrm{disc}_Y(Q, \lambda) \leq C'\, T^{d-1-\delta'}. \tag{5.3.6}$$

By enlarging C' we can assume (5.3.6) holds for every $T \geq 1$. Given a finite union of unit cubes $\mathcal{C}$, let $Q_1, \ldots, Q_n$ be as in Theorem 5.7. Then we have:

$$\begin{aligned} \mathrm{disc}_Y(\mathcal{C}, \lambda) &\leq \sum_{i=1}^{n} \mathrm{disc}_Y(Q_i, \lambda) \\ &\overset{(5.3.4),(5.3.6)}{\leq} C'\kappa \sum_j \frac{|\partial\, \mathcal{C}|_{d-1}}{2^{j(d-1)}} 2^{j(d-1-\delta')} \\ &\leq \frac{C'\kappa}{1-2^{-\delta'}}\, |\partial \mathcal{C}|_{d-1}, \end{aligned}$$

as required. □

5.4 Minkowski dimension and approximation

Let $A \subset \mathbb{R}^k$ be bounded and let $r > 0$. We denote by $N(A, r)$ the minimal number of balls of radius r needed to cover A, and

$$\dim_M A \stackrel{\text{def}}{=} \limsup_{r \to 0} \frac{\log N(A, r)}{-\log r}.$$

Equivalently (see e.g. [30, Chap. 3]), for $r > 0$ let $\mathcal{B}$ be the collection of boxes $[a_1, a_1 + r] \times \cdots \times [a_k, a_k + r]$ where the a_i are integer multiples of r, and let $S(A, r)$ denote the number of elements of $\mathcal{B}$ which intersect A. Then

$$\dim_M A = \limsup_{r \to 0} \frac{\log S(A, r)}{-\log r}.$$

From Theorem 5.3 we derive:

Corollary 5.4. *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{R}^k$ *be such that* $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_d)$ *is Diophantine, and suppose* U *is a bounded closed set in* $\mathbb{T}^k$*, such that* $\dim_M \partial U < k$*. Then there are constants* C *and* $\delta > 0$ *such that for any* $\mathbf{x} \in \mathbb{R}^k$ *and any* $T > 1$*,*

$$\Big| N_T(U, \mathbf{x}) - |U||B_T| \Big| \leq C\, T^{d-\delta}.$$

Proof (assuming Theorem 5.3). Let K be a positive integer and for each $\mathbf{m} \in \mathbb{Z}^k$ let

$$C(\mathbf{m}) = \left[\frac{m_1}{K}, \frac{m_1 + 1}{K}\right] \times \cdots \times \left[\frac{m_k}{K}, \frac{m_k + 1}{K}\right].$$

Define $A_1, A_2 \subset \mathbb{R}^k$ by

$$A_1 = \bigcup_{\substack{\mathbf{m} \in \mathbb{Z}^k \\ C(\mathbf{m}) \subset U}} C(\mathbf{m}) \quad \text{and} \quad A_2 = \bigcup_{\substack{\mathbf{m} \in \mathbb{Z}^k \\ C(\mathbf{m}) \cap U \neq \emptyset}} C(\mathbf{m}).$$

Clearly $N_T(A_1, \mathbf{x}) \leq N_T(U, \mathbf{x}) \leq N_T(A_2, \mathbf{x})$, so that

$$\Big| N_T(U, \mathbf{x}) - |U||B_T| \Big| \leq \max_{i=1,2} \Big| N_T(A_i, \mathbf{x}) - |U||B_T| \Big|. \tag{5.4.1}$$

Now by the triangle inequality

$$\Big| N_T(A_1, \mathbf{x}) - |U||B_T| \Big| \leq \Big| N_T(A_1, \mathbf{x}) - |A_1||B_T| \Big| + |B_T| \Big| |A_1| - |U| \Big|. \tag{5.4.2}$$

The number of $\mathbf{m} \in \mathbb{Z}^k$ with $C(\mathbf{m}) \subset U$ is bounded above by a constant times M^k, so applying Theorem 5.3 to each of the aligned boxes $C(\mathbf{m})$ gives

$$\Big| N_T(A_1, \mathbf{x}) - |A_1||B_T| \Big| \leq c_1 T^{d-\delta_0} K^k,$$

where c_1 and δ_0 are positive constants that are independent of K. Now our hypothesis on the dimension of the boundary guarantees that there is an $\varepsilon > 0$ such that the number of $\mathbf{m} \in \mathbb{Z}^k$ for which $C(\mathbf{m})$ intersects ∂U is bounded above by a constant times $K^{k-\varepsilon}$. Each of these boxes has volume K^{-k} and thus we have that

$$|B_T| \Big| |A_1| - |U| \Big| \leq c_2 \, |B_T| \frac{K^{k-\varepsilon}}{K^k} \leq c_3 \, T^d K^{-\varepsilon},$$

with $0 < c_2 < c_3$ independent of K. Now we return to (5.4.2) and set $K = \lfloor T^{\delta_0/(k+\varepsilon)} \rfloor$ to obtain the bound

$$\Big| N_T(A_1, \mathbf{x}) - |U||B_T| \Big| \leq c_1 T^{d-\delta_0} K^k + c_3 T^d K^{-\varepsilon} \leq (c_1 + c_3) T^{d - \delta_0 \varepsilon/(k+\varepsilon)}.$$

Setting $C = c_1 + c_3$, $\delta = \dfrac{\delta_0 \varepsilon}{k + \varepsilon}$ and applying the same argument to A_2 finishes the proof via (5.4.1). □

We now give a similar argument for bounded displacement.

Corollary 5.5. *Suppose* $d > (k+1)/2$ *and* $\mathcal{T} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ *is a basis of* $\mathbb{R}^k$ *satisfying the conditions of Theorem 5.4. Let* $\mathcal{S}$ *be a good section lying in a translate of* $\mathrm{span}(\mathbf{v}_{d+1}, \ldots, \mathbf{v}_k)$*, which is closed in this affine subspace, and satisfies* $\dim_M \partial\mathcal{S} = k - d - 1$*. Then we can choose* $\mathcal{U}_1$ *satisfying (i) of §5.2.2 so that, for the set* W *defined as in* (5.3.2)*, there are constants* C *and* $\delta > 0$ *such that for any* $\mathbf{x} \in \mathbb{R}^k$ *and any* $T > 1$*,*

$$\Big| N_T(W, \mathbf{x}) - |W||B_T| \Big| \leq C\, T^{d-1-\delta}.$$

Proof (assuming Theorem 5.4). Much of this proof is analogous to the previous one, so to simplify the exposition we omit some of the notational details. We begin by covering the set $\mathcal{S}$ by $(k-d)$-dimensional boxes which are translates of aligned boxes in $\mathrm{span}(\mathbf{v}_{d+1}, \ldots, \mathbf{v}_k)$ of sidelength $\eta = 1/K,\ K \geq 1$,. As before we construct disjoint unions A_1, A_2 of such boxes with the property that $A_1 \subset \mathcal{S} \subset A_2$, and we have that

$$\Big| N_T(W, \mathbf{x}) - |W||B_T| \Big| \leq \max_{i=1,2} \Big| N_T(A_i', \mathbf{x}) - |W||B_T| \Big|,$$

with

$$A_i' \stackrel{\mathrm{def}}{=} \{u.x : u \in \mathcal{U}_1,\ x \in A_i\}.$$

We choose $\mathcal{U}_1$ to be any parallelotope in $\mathbb{R}^d$ which satisfies (i) of §5.2.2, and which has sides parallel to $\mathbf{v}_1, \ldots, \mathbf{v}_d$. This is clearly possible since we can always replace our original choice of this set by any sub-neighborhood of the origin. With this choice of $\mathcal{U}_1$ our sets A_i' are unions of parallelotopes aligned with $\mathcal{T}$, with a uniform bound on their sidelengths. That is, parallelotopes to which Theorem 5.4 applies. The number of parallelotopes in A_1' is bounded above by a constant times K^{k-d}, so Theorem 5.4 tells us that for any $\delta_0 > 0$ there is a $c_1 > 0$ (which is independent of K) for which

$$\left| N_T(A_1', \mathbf{x}) - |A_1'||B_T| \right| \leq c_1 T^{\delta_0} K^{k-d}.$$

Our hypothesis that $\dim_M \partial\mathcal{S} = k - d - 1$ leads to the inequality

$$|B_T| \left| |A_1'| - |W| \right| \leq c_2 \, |B_T| \frac{K^{k-d-1}}{K^{k-d}} \leq \frac{c_3 T^d}{K},$$

and using the triangle inequality as in (5.4.2) we have that

$$\left| N_T(A_1', \mathbf{x}) - |W||B_T| \right| \leq c_1 T^{\delta_0} K^{k-d} + \frac{c_3 T^d}{K}.$$

Now using the hypothesis that $d > (k+1)/2$, we may assume that δ_0 has been chosen small enough so that there is a $\delta > \delta_0$ with $(1+\delta)(k-d) < (d-1-2\delta)$. Then setting $K = \lfloor T^{1+\delta} \rfloor$ we have that

$$\left| N_T(A_1', \mathbf{x}) - |W||B_T| \right| \leq c_4 T^{d-1-\delta}.$$

Since the same analysis holds for A_2', the proof is complete. □

5.5 Trigonometric polynomials approximating aligned parallelotopes

The proofs of Theorems 5.3 and 5.4 proceed with two major steps. The first step to prove an Erdős-Turán type inequality for Birkhoff integrals, and the second is to use Diophantine properties of the acting subspace to produce a further estimate on the error terms coming from the Erdős-Turán type inequality. Our goal in this section is to build up the necessary machinery to complete the first step.

Our approach to proving the Erdős-Turán type inequality requires approximations of the indicator function of an aligned parallelotope by trigonometric polynomials which *majorize* and *minorize* it. To obtain the quality of estimates that we need, we require the trigonometric polynomials to be close to the indicator function of the parallelotope in L^1-norm and to have suitably fast decay in their Fourier coefficients. The following theorem is the main result of this section, the Fourier analysis notation will be explained shortly.

Theorem 5.8. *Suppose that* $\mathcal{T} = (\mathbf{t}_1, ..., \mathbf{t}_k)$ *is a basis for* $\mathbb{R}^k$ *and that* L *is the linear isomorphism mapping* $\mathbf{e}_i$ *to* $\mathbf{t}_i$*. Suppose* $U \subset \mathbb{R}^k$ *is a parallelotope, aligned with* $\mathcal{T}$*, given by* $U = LB$ *for a box*

$$B = \prod_{\ell=1}^{k} [-b_\ell, b_\ell]$$

such that $\pi|_U$ *is injective. Let* $\chi_U^{\mathbb{T}} : \mathbb{T}^k \to \mathbb{R}$ *denote the indicator function of* $\pi(U)$*. Then for each* $M \in \mathbb{N}$ *there are trigonometric polynomials* $\varphi_U(\mathbf{x})$ *and* $\psi_U(\mathbf{x})$ *whose Fourier coefficients are supported in* $\{\mathbf{m} \in \mathbb{Z}^k : \|L^t\mathbf{m}\| \leq M\}$*, where* L^t *denotes the transpose of* L*, and*

$$\varphi_U(\mathbf{x}) \leq \chi_U^{\mathbb{T}}(\mathbf{x}) \leq \psi_U(\mathbf{x}) \tag{5.5.1}$$

for each $\mathbf{x} \in \mathbb{T}^k$*. Moreover, there exists a constant* $C > 0$*, depending only on* k*, such that*

$$\max\left\{|U| - \hat{\varphi}_U(\mathbf{0}), \hat{\psi}_U(\mathbf{0}) - |U|\right\} \leq \frac{Cb^{k-1}|\det L|}{M}, \tag{5.5.2}$$

and the Fourier coefficients of $\varphi_U(\mathbf{x})$ *and* $\psi_U(\mathbf{x})$ *satisfy*

$$\max\left\{\hat{\varphi}_U(\mathbf{m}), \hat{\psi}_U(\mathbf{m})\right\} \leq k2^{k+1}(1+2b)^k |\det L| r_{\mathcal{T}}(\mathbf{m}) \tag{5.5.3}$$

for all nonzero $\mathbf{m} \in \mathbb{Z}^k$, *where* $r_{\mathcal{T}}(\mathbf{m})$ *is defined by* (5.1.7) *and* $b = \max_{\ell}\{b_\ell\}$.

We note that some form of this result is alluded to in [37, Proof of Theorem 5.25], and since we could not find a suitable reference we will give the full details here. There are, however, known constructions which handle the case when U is rectangular [8, 22, 28, 36]. Our proof requires the well known construction of Selberg regarding extremal approximations of the indicator functions of intervals by integrable functions with compactly supported Fourier transforms, which we will recall below. To move to several variables we bootstrap from the single variable theory using another construction due to Selberg, who never published his results. A similar construction can be found in [22, 36].

Let $e(x) \overset{\text{def}}{=} \exp(2\pi i x)$. We will use the same notation for the Fourier transform of a function $F \in L^1(\mathbb{R}^N)$ and for a function $f : \mathbb{R}^N \to \mathbb{R}$ which is periodic with respect to $\mathbb{Z}^N$. That is

$$\hat{F}(\mathbf{t}) \overset{\text{def}}{=} \int_{\mathbb{R}^N} F(\mathbf{x}) e(-\mathbf{t}\cdot\mathbf{x})\, d\mathbf{x},\ \mathbf{t} \in \mathbb{R}^N;\ \ \hat{f}(\mathbf{m}) \overset{\text{def}}{=} \int_{[0,1)^N} f(\boldsymbol{\theta}) e(-\mathbf{m}\cdot\boldsymbol{\theta})\, d\boldsymbol{\theta},\ \mathbf{m} \in \mathbb{Z}^N.$$

The reader should have no difficulty disambiguating these two uses.

If $I \subset \mathbb{R}$ is an interval, let $\chi(t) = \chi_I(t)$ be its characteristic function. The following lemma is due to Selberg.

Lemma 5.5.1. For each positive integer M there exist integrable functions $C_I, c_I : \mathbb{R} \to \mathbb{R}$ such that

1. $c_I(t) \leq \chi(t) \leq C_I(t)$ for each $t \in \mathbb{R}$;

2. $\hat{C}_I(\xi) = \hat{c}_I(\xi) = 0$ whenever $|\xi| \geq M$,

3. $$\|C_I - \chi\|_{L^1(\mathbb{R})} = \|\chi - c_I\|_{L^1(\mathbb{R})} = \frac{1}{M}, \text{ and} \tag{5.5.4}$$

4. $$\max\left\{\left|\hat{C}_I(\xi)\right|, |\hat{c}_I(\xi)|\right\} \leq \min\left\{1 + |I|, \frac{2}{|\xi|}\right\}.$$

   for each $\xi \in \mathbb{R}$.

Proof. We will only verify the estimates on the Fourier coefficients appearing in (4) above. The other properties are well known and can be found in [72] or in [64]. From (5.5.4) we have

$$\sup_{\xi\in\mathbb{R}}\left|\hat{C}_I(\xi) - \hat{\chi}(\xi)\right| \leq \|C_I - \chi\|_{L^1(\mathbb{R})} = \frac{1}{M}.$$

In particular for any fixed ξ we have

$$\left|\hat{C}_I(\xi)\right| \leq |\hat{\chi}(\xi)| + \frac{1}{M}. \tag{5.5.5}$$

For any $1 < |\xi| < M$ we have $|\hat{\chi}(\xi)| = |\sin(\pi\xi|I|)/\pi\xi| \leq |\pi\xi|^{-1}$, hence

$$|\hat{\chi}(\xi)| + \frac{1}{M} < |\pi\xi|^{-1} + |\xi|^{-1} < \frac{2}{|\xi|},$$

therefore

$$|\hat{C}_I(\xi)| \leq \frac{2}{|\xi|} \quad \text{for} \;\; 1 < |\xi| < M.$$

Recall that $\hat{C}_I(\xi) = 0$ if $|\xi| \geq M$ so it remains to show $|\hat{C}_I(\xi)| \leq 1 + |I|$ when $|\xi| < 1$. But by (5.5.5) we have

$$|\hat{C}_I(\xi)| \leq \sup_{|\xi|<1} \left| \frac{\sin(\pi\xi|I|)}{\pi\xi} \right| + \frac{1}{M} \leq |I| + 1.$$

This concludes the proof for $\hat{C}_I$, and the proof for $\hat{c}_I$ is nearly identical. □

5.5.1 Majorizing and minorizing a rectangle in $\mathbb{R}^k$

From here on out we will use the notation $C_i(x) = C_{[-b_i,b_i]}(x)$. For any $M \in \mathbb{N}$ the indicator function χ_B of $B \subset \mathbb{R}^k$ is clearly majorized by the function

$$G_B(\mathbf{x}) \stackrel{\text{def}}{=} \prod_{j=1}^{k} C_j(x_j). \tag{5.5.6}$$

Minorizing χ_B requires a little more effort. For $i = 1, 2, \ldots, k$ define

$$L_B(\mathbf{x}; i) \stackrel{\text{def}}{=} c_i(x_i) \prod_{\substack{j=1 \\ j \neq i}}^{k} C_j(x_j),$$

and then set

$$g_B(\mathbf{x}) \stackrel{\text{def}}{=} -(k-1)G_B(\mathbf{x}) + \sum_{i=1}^{k} L_B(\mathbf{x}; i).$$

We claim that

$$g_B(\mathbf{x}) \leq \chi_B(\mathbf{x}) \quad \text{for every } \mathbf{x} \in \mathbb{R}^k. \tag{5.5.7}$$

To establish this we use the following elementary inequality, which can be proved by induction on k:

$$\text{For any } \beta_1 \geq 1, \ldots, \beta_k \geq 1, \quad \sum_{i=1}^{k} \prod_{\substack{j=1 \\ j \neq i}}^{k} \beta_j \leq 1 + (k-1) \prod_{j=1}^{k} \beta_j. \tag{5.5.8}$$

To verify the inequality (5.5.7), first suppose that $\mathbf{x} \notin B$. Then there is an $1 \leq i \leq k$ with $|x_i| > b_i$. Since $L_B(\mathbf{x}; i) \leq 0$ and $L_B(\mathbf{x}; j) \leq G_B(\mathbf{x})$ for all $j \neq i$, we have that

$$\sum_{i=1}^{k} L_B(\mathbf{x}; j) \leq (k-1) G_B(\mathbf{x}),$$

which implies $g_B(\mathbf{x}) \leq 0$. On the other hand if $\mathbf{x} \in B$ then we have that

$$c_j(x_j) \leq 1 \leq C_j(x_j).$$

Then by (5.5.8) we have that

$$\sum_{i=1}^{k} L_B(\mathbf{x}; i) \quad \leq \quad \sum_{i=1}^{k} \prod_{\substack{j=1 \\ j \neq i}}^{k} C_j(x_j) \leq 1 + (k-1) G_B(\mathbf{x}),$$

and this together with the definition of g_B establishes (5.5.7).

5.5.2 Proof of Theorem 5.8

Define

$$\mathscr{G}_U(\mathbf{x}) \stackrel{\text{def}}{=} G_B \circ L^{-1}(\mathbf{x}) \text{ and } \mathscr{F}_U(\mathbf{x}) \stackrel{\text{def}}{=} g_B \circ L^{-1}(\mathbf{x}).$$

The results of §5.5.1 show that

$$\mathscr{F}_U(\mathbf{x}) \leq \chi_U(\mathbf{x}) \leq \mathscr{G}_U(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^k.$$

For the majorants and minorants of $\chi_U^{\mathbb{T}}$ define

$$\varphi_U(\mathbf{x}) \stackrel{\text{def}}{=} \sum_{\mathbf{m}\in\mathbb{Z}^k} \mathscr{F}_U(\mathbf{x}+\mathbf{m}) \quad \text{and} \quad \psi_U(\mathbf{x}) \stackrel{\text{def}}{=} \sum_{\mathbf{m}\in\mathbb{Z}^k} \mathscr{G}_U(\mathbf{x}+\mathbf{m}).$$

These functions are $\mathbb{Z}^k$ invariant, so we can view them as functions on $\mathbb{T}^k$, and since $\pi|_U$ is injective we have

$$\varphi_U(\mathbf{x}) \le \chi_U^{\mathbb{T}}(\mathbf{x}) \le \psi_U(\mathbf{x}) \;\text{ for all }\; \mathbf{x} \in \mathbb{T}^k. \tag{5.5.9}$$

To determine the Fourier transform of $\mathscr{G}_U$ and $\mathscr{F}_U$, observe that if $f : \mathbb{R}^k \to \mathbb{C}$ is an integrable function then $f \circ L^{-1}$ is also integrable and

$$\widehat{f \circ L^{-1}}(\boldsymbol{\xi}) = |\det L| \hat{f}(L^t \boldsymbol{\xi}). \tag{5.5.10}$$

Since $\hat{G}_B(\boldsymbol{\xi}) = 0$ and $\hat{g}_B(\boldsymbol{\xi}) = 0$ when $\|\boldsymbol{\xi}\| \ge M$, both $\hat{\mathscr{F}}_U$ and $\hat{\mathscr{G}}_U$ are supported on $\{\boldsymbol{\xi} \in \mathbb{R}^k : \|L^t\boldsymbol{\xi}\| \le M\}$. Thus, by the Poisson summation formula and a classical theorem of Pólya and Plancherel [58], we have the following *pointwise* identities

$$\psi_U(\mathbf{x}) = \sum_{\substack{\mathbf{m}\in\mathbb{Z}^k \\ \|L^t\mathbf{m}\|\le M}} \hat{\mathscr{G}}_U(\mathbf{m}) e(\mathbf{m}\cdot\mathbf{x}) \tag{5.5.11}$$

and

$$\varphi_U(\mathbf{x}) = \sum_{\substack{\mathbf{m}\in\mathbb{Z}^k \\ \|L^t\mathbf{m}\|\le M}} \hat{\mathscr{F}}_U(\mathbf{m}) e(\mathbf{m}\cdot\mathbf{x}). \tag{5.5.12}$$

We will need the following formulas for the Fourier coefficients of ψ_U and φ_U:

$$\hat{\psi}_U(\mathbf{m}) = |\det L| \prod_{i=1}^{k} \hat{C}_i(\mathbf{t}_i \cdot \mathbf{m}) \tag{5.5.13}$$

and

$$\hat{\varphi}_U(\mathbf{m}) = |\det L| \left(-(k-1)\prod_{i=1}^{k}\hat{C}_i(\mathbf{t}_i\cdot\mathbf{m}) + \sum_{i=1}^{k}\hat{c}_i(\mathbf{t}_i\cdot\mathbf{m})\prod_{\substack{j=1\\ j\neq i}}^{k}\hat{C}_j(\mathbf{t}_j\cdot\mathbf{m}) \right). \tag{5.5.14}$$

To see (5.5.13), first observe that $\hat{\psi}_U(\mathbf{m}) = \hat{\mathscr{G}}_U(\mathbf{m})$ then by (5.5.10) and basic properties of the Fourier transform we have that

$$\begin{aligned}\hat{\mathscr{G}}_U(\mathbf{m}) = \widehat{G_B \circ L^{-1}}(\mathbf{m}) &= |\det L|\hat{G}_B(L^t\mathbf{m}) \\ &= |\det L|\prod_{i=1}^{k}\hat{C}_i(\mathbf{t}_i\cdot\mathbf{m}).\end{aligned}$$

The proof of (5.5.14) is similar. By (5.5.4) we see that

$$\hat{C}_i(0) = 2b_i + \frac{1}{M} \quad\text{and}\quad \hat{c}_i(0) = 2b_i - \frac{1}{M}. \tag{5.5.15}$$

Now by using (5.5.13) and (5.5.14), together with (5.5.15) we find that

$$\hat{\psi}_U(\mathbf{0}) = |\det L|\prod_{i=1}^{k}\left(2b_i + M^{-1}\right)$$

and

$$\begin{aligned}\hat{\varphi}_U(\mathbf{0}) &= |\det L|\left(-(k-1)\prod_{i=1}^{k}(2b_i+M^{-1}) + \sum_{j=1}^{k}(2b_j - M^{-1})\prod_{\substack{i=1\\ i\neq j}}^{k}(2b_i+M^{-1}) \right) \\ &= 2^k b_1\cdots b_k|\det L| + |\det L| O(b^{k-1}/M).\end{aligned}$$

The bounds (5.5.2) follow upon recalling that $|U| = 2^k b_1\cdots b_k|\det L|$. For the other Fourier coefficients we use (4) from Lemma 5.5.1 to obtain the inequalities

$$|\hat{\psi}_U(\mathbf{m})| = |\det L|\prod_{i=1}^{k}\left|\hat{C}_i(\mathbf{t}_i\cdot\mathbf{m})\right| \leq 2^k(1+2b)^k|\det L|\prod_{i=1}^{k}\min\left\{1, \frac{1}{|\mathbf{t}_i\cdot\mathbf{m}|}\right\}$$

and

$$
\begin{aligned}
|\hat{\varphi}_M(\mathbf{m})| &\leq |\det L| \left(\left| (k-1) \prod_{i=1}^{k} \hat{C}_i(\mathbf{t}_i \cdot \mathbf{m}) \right| + \sum_{j=1}^{k} \left| \hat{c}_j(\mathbf{t}_j \cdot \mathbf{m}) \prod_{\substack{i=1 \\ i \neq j}}^{k} \hat{C}_i(\mathbf{t}_i \cdot \mathbf{m}) \right| \right) \\
&\leq 2^k (2k-1)(1+2b)^k |\det L| \prod_{i=1}^{k} \min \left\{ 1, \frac{1}{|\mathbf{t}_i \cdot \mathbf{m}|} \right\}.
\end{aligned}
$$

Combining these estimates with (5.5.9), (5.5.11), and (5.5.12) finishes our proof. □

5.6 An Erdős-Turán type inequality for Birkhoff integrals

From Theorem 5.8 we deduce:

Theorem 5.9. *For any positive integer $k \geq 2$ there is a constant $C > 0$ such that the following holds. Suppose that $d < k$ is a positive integer and that $V \subset \mathbb{R}^k$ is a subspace of dimension d spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$. Let $\tilde{L} : \mathbb{R}^k \to \mathbb{R}^k$ be an affine isomorphism such that π is injective on the parallelotope $U = \tilde{L}B$ where B and b are as in Theorem 5.8. Let $\mathcal{T}$ denote the basis $L(\mathbf{e}_i), i = 1, \ldots, k$, where L is the linear part of $\tilde{L}$. Then for any $M \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^k$ we have*

$$
\Big| N_T(U, \mathbf{x}) - |U||B_T| \Big| \leq C(1+2b)^k |\det L| \left(\frac{|B_T|}{M} + \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|L^t \mathbf{m}\| \leq M}} r_{\mathcal{T}}(\mathbf{m}) \left| \int_{B_T} e(\mathbf{m} \cdot \mathbf{s}) \, d\mathbf{s} \right| \right). \tag{5.6.1}
$$

Proof. If $\tilde{L}(\mathbf{y}) = L(\mathbf{y}) + \mathbf{y}_0$, we may replace $\mathbf{x}$ with $\mathbf{x} - \mathbf{y}_0$ to assume that $\tilde{L} = L$, so that Theorem 5.8 applies. For $M \geq 1$ we have from Theorem 5.8

$$\chi_U^{\mathbb{T}}(\mathbf{x}) - |U| \leq \psi_U(\mathbf{x}) - |U| \leq \frac{C' b^{k-1} |\det L|}{M} + \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|L^t \mathbf{m}\| \leq M}} \hat{\psi}_U(\mathbf{m}) e(\mathbf{m} \cdot \mathbf{x}), \tag{5.6.2}$$

for some constant C' which depends only on k. By integrating both sides of (5.6.2) over $B_T - \mathbf{x}$ we find that

$$\begin{aligned}
N_T(U, \mathbf{x}) - |U||B_T| &\leq \left| \frac{C' b^{k-1} |\det L|\, |B_T|}{M} + \int_{B_T} \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|L^t \mathbf{m}\| \leq M}} \hat{\psi}_U(\mathbf{m}) e(\mathbf{m} \cdot \mathbf{s}) d\mathbf{s} \right| \\
&\leq \frac{C' b^{k-1} |\det L|\, |B_T|}{M} + \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|L^t \mathbf{m}\| \leq M}} \left| \hat{\psi}_U(\mathbf{m}) \right| \cdot \left| \int_{B_T} e(\mathbf{m} \cdot \mathbf{s})\ d\mathbf{s} \right| \\
&\overset{(5.5.3)}{\leq} C(1+2b)^k |\det L| \left(\frac{|B_T|}{M} + \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|L^t \mathbf{m}\| \leq M}} r_{\mathcal{T}}(\mathbf{m}) \cdot \left| \int_{B_T} e(\mathbf{m} \cdot \mathbf{s})\ d\mathbf{s} \right| \right),
\end{aligned}$$

where $C \stackrel{\text{def}}{=} \max(C', k2^{k+2})$. For a lower bound on $N_T(U, \mathbf{x}) - |U||B_T|$ we use $\varphi_U(\mathbf{x}) \leq \chi_U^{\mathbb{T}}(\mathbf{x})$ in a similar way. □

Specializing to aligned boxes we obtain a generalization of the Erdős-Turán inequality.

Corollary 5.6. *Let the notation be as in Theorem 5.3. Suppose $U \subset \mathbb{R}^k$ is an aligned box. Then there is a positive constant C (depending only on k) such* *that for any $M \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^k$ we have that*

$$\left| N_T(U, \mathbf{x}) - |U||B_T| \right| \leq C \left(\frac{|B_T|}{M} + \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|\mathbf{m}\| \leq M}} r(\mathbf{m}) \left| \int_{B_T} e(\mathbf{m} \cdot \mathbf{t}) \, d\mathbf{t} \right| \right), \tag{5.6.3}$$

where

$$r(\mathbf{m}) \stackrel{\text{def}}{=} \prod_{i=1}^{k} \min\left(1, \frac{1}{|m_i|}\right). \tag{5.6.4}$$

Proof. In this case L is the identity matrix, so that $B = U$ and $b \leq 1/2$. □

We will need the following estimate for the integrals appearing on the right-hand-side of (5.6.1):

Proposition 5.8. *There is a constant C (depending only on d, k and the choice of Lebesgue measure on V) such that*

$$\left| \int_{B_T} e(\mathbf{m} \cdot \mathbf{s}) \, d\mathbf{s} \right| \leq C \prod_{i=1}^{d} \frac{1}{|\mathbf{m} \cdot \mathbf{v}_i|}. \tag{5.6.5}$$

Proof. For a constant C_1 depending on the choice of Lebesgue measure on V, we have:

$$\begin{aligned} \left| \int_{B_T} e(\mathbf{m} \cdot \mathbf{s}) \, d\mathbf{s} \right| &= C_1 \prod_{i=1}^{d} \left| \int_{-T}^{T} e((\mathbf{m} \cdot \mathbf{v}_i) s_i) \, ds_i \right| \\ &= C_1 \prod_{i=1}^{d} \frac{|\sin(2\pi(\mathbf{m} \cdot \mathbf{v}_i)T)|}{\pi |\mathbf{m} \cdot \mathbf{v}_i|} \\ &\leq \frac{C_1}{\pi^d} \prod_{i=1}^{d} \frac{1}{|\mathbf{m} \cdot \mathbf{v}_i|}. \end{aligned}$$

□

Proof of Theorem 5.3. Let $\mathbf{v}$ be a Diophantine vector in the subspace spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_d$, and write $\mathbf{v} = \sum_{i=1}^{d} x_i \mathbf{v_i}$. Fix c, s as in (5.1.5), and let $s' > s$. If $\mathbf{m} \in \mathbb{Z}^k$ satisfies

$$\max_{1 \le i \le d} |\mathbf{m} \cdot \mathbf{v}_i| \le \|\mathbf{m}\|^{-s'}$$

then, for all but finitely many $\mathbf{m}$,

$$|\mathbf{m} \cdot \mathbf{v}| \le \left(\sum_{i=1}^{d} |x_i| \right) \|\mathbf{m}\|^{-s'} \le c\|\mathbf{m}\|^{s}.$$

Thus for some $c_1 > 0$ we have

$$\max_{1 \le i \le d} |\mathbf{m} \cdot \mathbf{v}_i| \ge c_1 \|\mathbf{m}\|^{-s'} \quad \text{for all} \quad \mathbf{m} \in \mathbb{Z}^k. \tag{5.6.6}$$

We will apply Corollary 5.6 with

$$M = \lfloor T^{\delta} \rfloor, \quad \text{where } \delta = \frac{1}{d + s' + 1}. \tag{5.6.7}$$

Assume that the maximum in (5.6.6) is attained for $i = 1$. It follows that for some $c_4, c_3, c_2 > 0$,

$$\begin{aligned}
\left| \int_{B_T} e(\mathbf{m} \cdot \mathbf{t}) \, d\mathbf{t} \right| &= c_2 \left| \int_{[-T,T]^d} e\left(\mathbf{m} \cdot \left(\sum t_i \mathbf{v}_i \right) \right) \, d\mathbf{t} \right| \\
&= c_2 \prod_{i=1}^{d} \left| \int_{-T}^{T} e((\mathbf{m} \cdot \mathbf{v}_i) t_i) \, dt_i \right| \\
&\le c_2 \, \frac{|\sin(2\pi (\mathbf{m} \cdot \mathbf{v}_1) T)|}{\pi |\mathbf{m} \cdot \mathbf{v}_1|} \, (2T)^{d-1} \\
&\le c_3 \, \frac{T^{d-1}}{|\mathbf{m} \cdot \mathbf{v}_1|} \overset{(5.6.6),(5.6.7)}{\le} c_4 \, T^{d-1+s'\delta}.
\end{aligned}$$

Therefore

$$\sum_{0<\|\mathbf{m}\|\le M} r(\mathbf{m}) \left| \int_{B_T} e(\mathbf{m}\cdot\mathbf{t})\, d\mathbf{t} \right| \le c_4\, T^{d-1+s'\delta} \sum_{0<\|\mathbf{m}\|\le M} r(\mathbf{m})$$
$$\le c_4\, M^d\, T^{d-1+s'\delta} \overset{(5.6.7)}{\le} c_5\, T^{d-\delta}.$$

It is clear that the constants c_5, δ do not depend on U or $\mathbf{x}$. Thus the theorem follows from Corollary 5.6. □

Remark 5.1. The proof shows that if V is Diophantine with corresponding constant s, then δ can be taken to be any number smaller than $\frac{d+1}{d+s+1}$.

Proof of Theorem 5.4. For a fixed δ, let $\varepsilon = \delta/d$ and let C_1, C_2, C_3 be the constants C appearing in (5.1.8), (5.6.1) and (5.6.5) respectively. Let $L : \mathbb{R}^k \to \mathbb{R}^k$ be the linear isomorphism mapping $\mathbf{e}_i$ to $\mathbf{v}_i$, $i = 1, \dots, k$. Any U which is a parallelotope aligned with $\mathcal{T}$ is of the form $U = \pi \circ \widetilde{L}(B)$, where $\widetilde{L}$ is an affine isomorphism whose linear part is L and $B = \prod[-b_i, b_i]$.
Let $\|\cdot\|_2$ be the Euclidean norm on $\mathbb{R}^k$. If the largest side length of U is η, then

$$\eta = 2 \max_i b_i \|\mathbf{v}_i\|_2.$$

In particular $\eta \ge 2b \min \|\mathbf{v}_i\|_2$ where $b = \max b_i$. There is a constant λ, which depends only on the $\mathbf{v}_1, \dots, \mathbf{v}_k$, such that

$$\{\mathbf{m} \in \mathbb{Z}^k : \|L^t \mathbf{m}\| \le M\} \subset \{\mathbf{m} \in \mathbb{Z}^k : \|\mathbf{m}\| \le \lambda M\}.$$

Applying Proposition 5.8, we find that for any $M > 0$, the right hand side of (5.6.1) is bounded above by

$$C_2|\det L|(1+\eta/\min\|\mathbf{v}_i\|_2)^k\left(\frac{|B_T|}{M}+C_3\sum_{\substack{\mathbf{m}\in\mathbb{Z}^k\setminus\{0\}\\\|\mathbf{m}\|\le\lambda M}} r_{\mathcal{T}}(\mathbf{m})\prod_{i=1}^{d}\frac{1}{|\mathbf{m}\cdot\mathbf{v}_i|}\right).$$

Now taking $M = \lfloor T^d \rfloor$, and using our strongly Diophantine hypothesis, gives the required bound, with

$$C = C_1C_2C_3|\det L|\lambda^{\delta/d}\max\{1, 1/\min\|\mathbf{v}_i\|_2\}^k.$$

□

5.7 Diophantine approximation to subspaces

The main result of this section shows that the Diophantine properties stated in the introduction hold almost surely. More precisely, properties of d-tuples of vectors in $\mathbb{R}^k$ hold almost everywhere with respect to Lebesgue measure on $\times_1^d\mathbb{R}^k \cong \mathbb{R}^{kd}$, and properties of vector spaces hold almost everywhere with respect to the smooth measure class on the Grassmannian variety.

The fact that almost every vector is Diophantine is a standard exercise using the Borel-Cantelli Lemma — or see [27] for a stronger statement. For the extension to strongly Diophantine vectors, we employ some ideas of Schmidt [61]:

Proposition 5.9. *Almost every $\mathbf{v}_1,\ldots,\mathbf{v}_d$ is strongly Diophantine with respect to any basis $\mathcal{T} = (\mathbf{t}_1,\ldots,\mathbf{t}_k)$ for $\mathbb{R}^k$ having the property that for each $i \in \{d+1,\ldots,k\}$, there is a j for which $\mathbf{t}_i$ is a multiple of $\mathbf{e}_j$.*

Proof. Fix $\varepsilon > 0$, let $R_1, \ldots, R_d$ be cubes in $\mathbb{R}^k$ of sidelength 1, and for each $1 \le i \le d$ and $\mathbf{m} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$ let

$$I_{R_i}(\mathbf{m}) \stackrel{\text{def}}{=} \int_{R_i} \frac{d\mathbf{v}}{|\mathbf{m}\cdot\mathbf{v}|(-\log\min(1/2, |\mathbf{m}\cdot\mathbf{v}|))^{1+\varepsilon}}.$$

We estimate this integral by using the change of variables $\mathbf{u} = \mathbf{u}(\mathbf{v})$, where

$$u_i = \mathbf{m}\cdot\mathbf{v}, \quad u_j = v_j \text{ for } 1 \le j \le k,\ j \ne i.$$

The Jacobian determinant of this transformation is $1/m_i$. If we write R_i' for the image of R_i in the $\mathbf{u}$ coordinate system then it is clear that for $j \ne i$ the u_j coordinates of two points in R_i' cannot differ by more than 1. Using this fact we have

$$I_{R_i}(\mathbf{m}) \le \frac{2}{m_i}\int_0^{1/2} \frac{du_i}{u_i|\log u_i|^{1+\varepsilon}} + \frac{1}{m_i(\log 2)^{1+\varepsilon}}\int_{1/2}^{1/2+m_i} \frac{du_i}{u_i} \le c_1\frac{\log(m_i)}{m_i},$$

where c_1 depends only on ε. Thus we have

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_d=1}^{\infty} \frac{I_{R_1}(\mathbf{m})\cdots I_{R_d}(\mathbf{m})}{(\log m_1)^{2+\varepsilon}\cdots(\log m_d)^{2+\varepsilon}} \le c_2$$

with c_2 depending on ε but not on $\mathbf{m}$. On interchanging the orders of integration and summation this implies that for almost every $(\mathbf{v}_1, \ldots, \mathbf{v}_d) \in R_1 \times \cdots \times R_d$,

$$S(\mathbf{v}_1, \ldots, \mathbf{v}_d) \stackrel{\text{def}}{=} \sum_{m_1=1}^{\infty}\cdots\sum_{m_d=1}^{\infty}\prod_{i=1}^{d} \frac{1}{|\mathbf{m}\cdot\mathbf{v}_i|(\log m_i)^{2+\varepsilon}(-\log\min(1/2, |\mathbf{m}\cdot\mathbf{v}_i|))^{1+\varepsilon}} \tag{5.7.1}$$

is finite and independent of $m_{d+1}, m_{d+2}, \ldots, m_k \in \mathbb{Z}$. Since the location of the cubes $R_1, \ldots, R_d$ was arbitrary, $S(\mathbf{v}_1, \ldots, \mathbf{v}_d) < \infty$ for almost every

$(\mathbf{v}_1, \ldots, \mathbf{v}_d) \in (\mathbb{R}^k)^d$. By grouping together the choices for $m_{d+1}, \ldots, m_k$, we obtain

$$\sum_{\substack{\mathbf{m}\in\mathbb{Z}^k \\ 0<m_1,\ldots,m_k\le M}} r_{\mathcal{T}}(\mathbf{m}) \prod_{i=1}^{d} \frac{1}{|\mathbf{m}\cdot\mathbf{v}_i|}$$

$$\le C(\log M)^{k-d}(\log M)^{d(2+\varepsilon)}P(M)S(\mathbf{v}_1,\ldots,\mathbf{v}_d), \tag{5.7.2}$$

where

$$P(M) = \prod_{i=1}^{d} \max_{1\le m_1,\ldots,m_k\le M} (-\log\min(1/2, |\mathbf{m}\cdot\mathbf{v}_i|))^{1+\varepsilon}.$$

In the inequality in (5.7.2) we are using the fact that for each $i \in \{d+1, \ldots, k\}$, the quantity $\mathbf{t}_i \cdot \mathbf{m}$ is always a fixed multiple of m_j for some j.

By a standard application of the Borel-Cantelli Lemma, for almost every $\mathbf{v} \in \mathbb{R}^k$ there is a constant $c = c(\mathbf{v}) > 0$ such that

$$|\mathbf{m}\cdot\mathbf{v}| \ge \frac{c}{M^{2k}} \quad \text{for all} \quad \mathbf{m}\in\mathbb{Z}^k \quad \text{with} \quad 0 < \|\mathbf{m}\| \le M.$$

Thus for almost every $\mathbf{v}_1, \ldots, \mathbf{v}_d$ and for any $\delta > 0$ we have that (5.7.2) is bounded above by a constant times $(\log M)^{k+2d+\delta}$.

Finally we can estimate

$$\sum_{0<\|\mathbf{m}\|\le M} r_{\mathcal{T}}(\mathbf{m}) \prod_{i=1}^{d} \frac{1}{|\mathbf{m}\cdot\mathbf{v}_i|}$$

by partitioning the sum into 2^k subsets of points $\mathbf{m}$, according to which components of $\mathbf{m}$ are 0. To each one of these subsets we may then apply the above arguments to obtain the required bound. □

As a corollary of our proof, the conclusions of Theorems 5.3 and 5.4 can be considerably strengthened, as follows.

Proposition 5.10. *For almost every* $\mathbf{v}_1, \ldots, \mathbf{v}_d$*, and any basis* $\mathcal{T}$ *as in Proposition 5.9, for any* $\delta > 0$ *there is* $c > 0$ *so that*

$$\sum_{0<\|\mathbf{m}\|\leq M} r_{\mathcal{T}}(\mathbf{m}) \prod_{i=1}^{d} \frac{1}{|\mathbf{m} \cdot \mathbf{v}_i|} \leq c\,(\log M)^{k+2d+\delta}. \tag{5.7.3}$$

Under this condition, the error terms on the right hand sides of (5.1.6) *and* (5.1.9) *can be replaced by* $C(\log T)^{k+2d+\delta}$.

Proof. The bound (5.7.3) was already proved above. For the rest of the claim, take $M = T^d$ and use (5.7.3) and Proposition 5.8 in Theorem 5.9. □

To conclude this section we mention the following easy fact:

Proposition 5.11. *If* $\mathbf{v}_1, \ldots, \mathbf{v}_d$ *are strongly Diophantine then each* $\mathbf{v}_i$ *is Diophantine.*

Proof. Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_d$ are strongly Diophantine with respect to $\mathcal{T} = (\mathbf{t}_1, \ldots, \mathbf{t}_k)$, let $s > k + d - 1$, and let $i \in \{1, \ldots, d\}$. Suppose by contradiction that there are infinitely many vectors $\mathbf{m} \in \mathbb{Z}^k$ so that $|\mathbf{m} \cdot \mathbf{v}_i| < \frac{1}{\|\mathbf{m}\|^s}$. If $\mathbf{m}$ is one such vector then setting $M = \|\mathbf{m}\|$ and using Cauchy-Schwarz we find, for each $j \neq i$,

$$|\mathbf{m} \cdot \mathbf{v}_j| \leq M\|\mathbf{v}_j\|.$$

Noting that $r_{\mathcal{T}}(\mathbf{m}) \geq \prod_{i=1}^{k} \frac{1}{\|\mathbf{t}_i\| \cdot \|\mathbf{m}\|}$ gives

$$r_{\mathcal{T}}(\mathbf{m}) \prod_{i=1}^{d} \frac{1}{|\mathbf{m} \cdot \mathbf{v}_i|} \geq M^{-k} \left(\prod_{i=1}^{k} \frac{1}{\|\mathbf{t}_i\|} \right) \left(\prod_{j \neq i} \frac{1}{M \|\mathbf{v}_j\|} \right) \|\mathbf{m}\|^s \geq C M^{s-k-d+1}.$$

This holds along a sequence of $M \to \infty$. However for some $\varepsilon > 0$ this contradicts (5.1.8). □

5.8 Proofs of Theorem 5.1 and 5.2(1),(2)

Proof of Theorem 5.1. Let V be a Diophantine subspace, and let $\mathbf{v}_1, \ldots, \mathbf{v}_d$ be a basis for V. Let $\mathcal{S}$ be a linear section which is $(k-d)$-dimensionally open and bounded, with $\dim_M \partial \mathcal{S} < k - d$, let $\mathcal{U}_1$ be a closed ball around 0 in V, satisfying (i) of §5.2.2, and define W via (5.3.2). Then W is bi-Lipschitz equivalent to $\mathcal{U} \times \mathcal{S}$ and hence, by [30, Formulae 7.2 and 7.3], $\dim_M \partial W < k$. Thus the Theorem follows from Corollaries 5.2 and 5.4. □

Proof of Theorem 5.2(1). Let $\mathbf{v}_1, \ldots, \mathbf{v}_d$ satisfy the conclusion of Proposition 5.9, and for $i = d+1, \ldots, k$, let $\mathbf{v}_i \in \{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ such that $\mathcal{T} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is a basis of $\mathbb{R}^k$. Also let $V = \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_d)$. We need to show that for any linear section $\mathcal{S}$ in a space L transverse to V, such that $\dim \partial S = k - d - 1$, and any $\mathbf{x} \in \mathbb{T}^k$, the corresponding is BDD to a lattice. To this end we will apply Corollaries 5.3 and 5.5 . Let B be a ball in L such that π is injective on B, and sets $\mathcal{U}_1$ and $\mathcal{U}_2$ satisfying conditions (i) and (ii) of §5.2.2 for B'. Also let $L' \stackrel{\mathrm{def}}{=} \mathrm{span}(\mathbf{v}_{d+1}, \ldots, \mathbf{v}_k)$, and let B' be a ball in L' such that π is injective on B'. Then B' is a good section, let $\mathcal{U}_1', \mathcal{U}_2'$ be the corresponding sets as in §5.2.2.

Suppose first that B is small enough so that (5.2.1) holds. Then we can assume with no loss of generality that $\mathcal{S}$ is contained in B'. This in turn shows that the hypotheses of Corollaries 5.5 and 5.3 are satisfied, and Y is BDD to a lattice.

Now suppose (5.2.1) does not hold. Then we can partition $\mathcal{S}$ into smaller sets $\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(r)}$ with equal volume and $\dim_M \partial\mathcal{S}^{(i)} = k - d - 1$, such that the corrresponding sets $\mathcal{U}_1^{(i)}$ satisfy (5.2.1). Now repeating the previous argument separately to each $\mathcal{S}^{(i)}$, we see that the corrresponding net is BD to a fixed lattice L. Note that the lattice is the same because each $\mathcal{S}_i$ has the same volume. Now the result follows via Proposition 5.4. □

Proof of Theorem 5.2(2). Suppose $\mathcal{S}$ is a box with sides parallel to the coordinate axes; that is, there is $J \subset \{1, \ldots, k\}$, $|J| = k - d$, such that $\mathcal{S}$ is the projection under π of an aligned box in the space $V_J \stackrel{\text{def}}{=} \text{span}(\mathbf{e}_j : j \in J)$. As above, we can use Proposition 5.4 to assume that π is injective on a subset of V_J covering $\mathcal{S}$. According to Proposition 5.9, for almost every choice of $\mathbf{v}_1, \ldots, \mathbf{v}_d$, the space $V = \text{span}(\mathbf{v}_i)$ is strongly Diophantine with respect to the basis

$$\mathcal{T} \stackrel{\text{def}}{=} \{\mathbf{v}_i : i = 1, \ldots, d\} \cup \{\mathbf{e}_j : j \in J\}.$$

As in the preceding proof, choose a neighborhood $\mathcal{U}_1$ of 0 in V satisfying property (i) of §5.2.2 which is a box. Then the set W defined by (5.3.2) is a parallelotope aligned with $\mathcal{T}$. According to Theorem 5.4, (5.3.5) holds, and we can apply Corollary 5.3. □

5.9 Irregularities of distribution

In this section we will fix $1 < d < k$ and let $\mathcal{G}$ denote the Grassmannian variety of d-dimensional subspaces of $\mathbb{R}^k$. We denote by $\mathcal{G}(\mathbb{Q})$ the subset of rational subspaces. We will fix a totally irrational $k-d$ dimensional subspace $W \subset \mathbb{R}^k$, and let $\mathcal{S}$ be the image under π of a subset of W which is open and bounded. There is a dense G_δ subset of $V \in \mathcal{G}$ for which $\mathcal{S}$ is a good section for the action of V on $\mathbb{T}^k$; indeed, by the discussion of §5.2.2, this holds whenever V and W are transverse to each other and V is totally irrational.

If $Q \in \mathcal{G}(\mathbb{Q})$ then any orbit $Q.\mathbf{x}$ is compact; further if Q is transverse to W then $Q.\mathbf{x} \cap \mathcal{S}$ is a finite set for every $\mathbf{x} \in \mathbb{T}^k$. We say that $\mathcal{S}$ and Q are *not correlated* if there are $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{T}^k$ such that

$$\# \left(Q.\mathbf{x}_1 \cap \mathcal{S}\right) = \# \left(Q.\mathbf{x}_1 \cap \overline{\mathcal{S}}\right) \neq \# \left(Q.\mathbf{x}_2 \cap \mathcal{S}\right) = \# \left(Q.\mathbf{x}_2 \cap \overline{\mathcal{S}}\right) \qquad (5.9.1)$$

(here $\overline{\mathcal{S}}$ denotes the closure of $\mathcal{S}$). We say that $\mathcal{S}$ is *typical* if there is a dense set of $Q \in \mathcal{G}$ for which $\mathcal{S}$ and Q are not correlated.

It is not hard to find typical $\mathcal{S}$:

Proposition 5.12. *Let $r = k-d$ and let W be a totally irrational r-dimensional subspace of $\mathbb{R}^k$. Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis for W and for $\mathbf{a} = (a_1, \ldots, a_r) \in (0,1)^r, \mathbf{b} = (b_1, \ldots, b_r) \in (0,1)^r$ let*

$$P(\mathbf{a}, \mathbf{b}) \stackrel{\text{def}}{=} \pi\left(\left\{\sum_1^r t_i \mathbf{w}_i : t_i \in (a_i, a_i + b_i)\right\}\right).$$

Then the set of $(\mathbf{a}, \mathbf{b})$ for which $P(\mathbf{a}, \mathbf{b})$ is not correlated with any rational subspace, and hence typical, is of full measure and residual in $(0,1)^{2r}$.

Proof. It is enough to show that for a fixed Q, the set of $\mathbf{a}, \mathbf{b}$ for which $P(\mathbf{a}, \mathbf{b})$ is correlated with Q has zero measure and is a submanifold of dimension less than $2r$ in $[0,1]^{2r}$. To see this, define two functions $F, \overline{F}$ on $\mathbb{T}^k$, by

$$F(\mathbf{x}) \stackrel{\text{def}}{=} \#\left(Q.\mathbf{x} \cap \mathcal{S}\right), \quad \overline{F}(\mathbf{x}) \stackrel{\text{def}}{=} \#\left(Q.\mathbf{x} \cap \overline{\mathcal{S}}\right).$$

We always have $F(\mathbf{x}) \leq \overline{F}(\mathbf{x})$, and $F(\mathbf{x}) = \overline{F}(\mathbf{x})$ unless $Q.\mathbf{x}$ intersects the boundary of $\mathcal{S}$. So if (5.9.1) fails then $F(\mathbf{x})$ always has the same value, for the values of $\mathbf{x}$ for which $Q.\mathbf{x} \cap \partial \mathcal{S} = \varnothing$.

Note that the values of $F, \overline{F}$ are constant along orbits of Q. The space of orbits for the Q-action is itself a compact torus Q' of dimension r. Let $\pi' : \mathbb{T}^k \to Q'$ be the projection mapping a point to its orbit. The discussion in the previous paragraph shows that the requirement that $\mathcal{S}$ and Q are correlated is equivalent to the requirement that the interior of $\mathcal{S}$ projects onto a dense open subset of Q' with fibers of constant cardinality. Clearly this property is destroyed if we vary $\mathcal{S}$ slightly in the direction orthogonal to Q. More precisely, for any $\mathbf{a}$ and $\mathbf{b}$, there is a small neighborhood $\mathcal{U}$ such that which the set of $\mathbf{a}', \mathbf{b}'$ in $\mathcal{U}$ for which (5.9.1) fails is a proper submanifold of zero measure. This proves the claim.

□

By similar arguments one can show that almost every ball, ellipsoid, etc., is typical.

Proposition 5.13. *If $\mathcal{S}$ is a bounded open set whose boundary is of zero measure (w.r.t. the Lebesgue measure on the subspace W), and $\mathcal{S}$ is typical,* *then there is a dense G_δ subset of V for which, for every $\mathbf{x} \in \mathbb{T}^k$, the separated net $Y_{\mathcal{S},\mathbf{x}}$ is not BDD to a lattice.*

Proof. Let $Q_1, Q_2, \ldots$ be a list of rational subspaces in $\mathcal{G}(\mathbb{Q})$ such that $\mathcal{S}$ and Q_i are not correlated for each i, and $\{Q_i\}$ is a dense subset of $\mathcal{G}$. For each i let $\mathbf{x}_1^{(i)}, \mathbf{x}_2^{(i)}$ be two points in $\mathbb{T}^k$ for which (5.9.1) holds. Since the linear action of subspaces on $\mathbb{T}^k$ is the restriction of the continuous natural $\mathbb{R}^k$-action, for any $\varepsilon > 0$ and any $T > 0$ we can find a neighborhood of Q_i in $\mathcal{G}$ consisting of subspaces V such that for any $v \in V$ with $\|v\| < T$, and any $\mathbf{x} \in \mathbb{T}^k$, the distance in $\mathbb{T}^k$ between $v.\mathbf{x}$ and $v'.\mathbf{x}$ is less than ε, where v' is the orthogonal projection of v onto Q_i. We will fix below a sequence of bounded sets $M_i \subset Q_i$ and denote by $M_i^{(V)}$ the preimage, under orthogonal projection $V \to Q_i$, of M_i. Using our assumption on $\mathcal{S}$, by perturbing $\mathbf{x}_1^{(i)}, \mathbf{x}_2^{(i)}$ slightly we can assume that $q.\mathbf{x}_1^{(i)}$ and $q.\mathbf{x}_2^{(i)}$ are not in $\partial\mathcal{S}$ when $q \in M_i$. Since $\mathcal{S}$ is relatively open in W, this implies that there is an open subset $\mathcal{V}_i$ of $\mathcal{G}$ containing Q_i, such that for every $V \in \mathcal{V}_i$ and for $\ell = 1, 2$,

$$\#\left\{q \in M_i : q.\mathbf{x}_\ell^{(i)} \in \mathcal{S}\right\} = \#\left\{v \in M_i^{(V)} : v.\mathbf{x}_\ell^{(i)} \in \mathcal{S}\right\}. \tag{5.9.2}$$

Then

$$\mathcal{G}_\infty \stackrel{\text{def}}{=} \bigcap_{i_0} \bigcup_{i \geq i_0} \mathcal{V}_i$$

is clearly a dense G_δ subset of $\mathcal{G}$, and it remains to show that by a judicious choice of the sequence M_i, we can ensure that for any totally irrational $V \in \mathcal{G}_\infty$, for any $\mathbf{x}$, and any positive λ, c, the separated net $Y_{\mathcal{S},\mathbf{x}}$ does not satisfy condition (3) of Theorem 5.6.

For any i let C_i be a parallelotope which is a fundamental domain for the action of the lattice $Q_i \cap \mathbb{Z}^k$ on Q_i. Specifically we let

$$C_i \stackrel{\text{def}}{=} \left\{ \sum_{j=1}^{d} a_j \mathbf{q}_j : \forall j,\, 0 \le a_j < \|\mathbf{q}_j\| \right\},$$

where $\mathbf{q}_1, \ldots, \mathbf{q}_d$ are a basis of $Q_i \cap \mathbb{Z}^k$. We claim that there are positive constants c_1, c_2, C (depending on i) and sets M_i which are finite unions of translates of C_i, of arbitrarily large diameter, such that:

$$|M_i| \ge c_1 \operatorname{diam}(M_i)^d; \tag{5.9.3}$$

$$\left|(\partial M_i)^{(1)}\right| \le C \operatorname{diam}(M_i)^{d-1} \tag{5.9.4}$$

(where, as before, $(\partial M_i)^{(1)}$ is the set of points at distance 1 from ∂M_i). Indeed, we simply take M_i to be dilations by an integer factor, of C_i around its center. Then each M_i is homothetic to C_i and (5.9.3) and (5.9.4) follow. Now let N_i be the number of copies of C_i in M_i. Then

$$\#\{q \in M_i : q.\mathbf{x}_\ell^{(i)} \in \mathcal{S}\} = N_i \cdot \#\{q \in C_i : q.\mathbf{x}_\ell^{(i)} \in \mathcal{S}\} = N_i \cdot \# \left(Q.\mathbf{x}_\ell^{(i)} \cap \mathcal{S}\right)$$

and

$$|M_i| = N_i \cdot |C_i|,$$

which implies via (5.9.3) and (5.9.4) that for some constant c_2,

$$|(\partial M_i)^{(1)}| \le c_2 N_i^{1-1/d}.$$

If we set

$$c_3 \stackrel{\text{def}}{=} \frac{\left| \# \left(Q.\mathbf{x}_2^{(i)} \cap \mathcal{S}\right) - \# \left(Q.\mathbf{x}_1^{(i)} \cap \mathcal{S}\right) \right|}{2},$$

then for any λ, there is $\ell \in \{1,2\}$ such that for $\mathbf{x}' = \mathbf{x}_\ell^{(i)}$ we have

$$|\#(Q.\mathbf{x}' \cap \mathcal{S}) - \lambda |C_i|| \geq c_3,$$

and hence

$$\frac{|\#(M_i.\mathbf{x}' \cap \mathcal{S}) - \lambda |M_i||}{|(\partial M_i)^{(1)}|} \geq \frac{N_i \, |\#(Q\mathbf{x}' \cap \mathcal{S}) - \lambda |C_i||}{c_2 N_i^{1-1/d}} \geq \frac{c_3}{c_2} N_i^{1/d}.$$

So by choosing N_i large enough we can ensure that for any λ, and $\mathbf{x}'$ one of the $\mathbf{x}_\ell^{(i)}$, we have

$$|\#(M_i.\mathbf{x}' \cap \mathcal{S}) - \lambda |M_i|| \geq i \, |(\partial M_i)^{(1)}|. \tag{5.9.5}$$

Now fixing λ and c we choose $i > c$ and choose $\mathbf{x}'$ as above depending on λ. If $V \in \mathcal{V}_i$ is totally irrational then for any $\mathbf{x} \in \mathbb{T}^k$ there is a sequence $v_n \in V$ such that $v_n.\mathbf{x} \to \mathbf{x}'$. So we may replace $\mathbf{x}'$ with $\mathbf{x}$ and M_i with $v_n + M_i$ for sufficiently large n, and (5.9.5) will continue to hold. In light of (5.9.2), if Y is the net corresponding to V, $\mathcal{S}$ and $\mathbf{x}$, and $E \stackrel{\text{def}}{=} v_n + M_i$, then we have shown $\text{disc}_Y(E, \lambda) > c|(\partial E)^{(1)}|$, and we have a contradiction to condition (3) of Theorem 5.6. □

Proof of Theorem 5.2(3). Immediate from Propositions 5.12 and 5.13. □

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