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Geometry of Integrable Hierarchies and Their Dispersionless Limits

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Geometry of Integrable Hierarchies and Their Dispersionless Limits

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Geometry of Integrable Hierarchies and Their Dispersionless Limits

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This thesis describes a geometric approach to integrable systems. In the first part we describe the geometry of Drinfeld–Sokolov integrable hierarchies including the corresponding tau-functions. Motivated by a relation between Drinfeld–Sokolov hierarchies and certain physical partition functions, we define a dispersionless limit of Drinfeld–Sokolov systems. We introduce a class of solutions which we call string solutions and prove that the tau-functions of string solutions satisfy Virasoro constraints generalizing those familiar from two-dimensional quantum gravity. In the second part we explain how procedures of Hamiltonian and quasi-Hamiltonian reductions in symplectic geometry arise naturally in the context of shifted symplectic structures. All constructions that appear in quasi-Hamiltonian reduction have a natural interpretation in terms of the classical Chern-Simons theory that we explain. As an application, we construct a prequantization of character stacks purely locally.

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Chapter 1

Introduction

This thesis is divided into two main parts: the first one is concerned with integrable systems of Drinfeld–Sokolov type while the second part describes a new approach to Hamiltonian and quasi-Hamiltonian using derived symplectic geometry.

1.1 Drinfeld–Sokolov hierarchies

Drinfeld–Sokolov hierarchies are generalizations of the so-called Korteweg–de Vries (KdV) hierarchy – that is, a family of nonlinear PDEs – to an arbitrary semisimple Lie group G. For example, the first nontrivial equation of the $G = SL_2$ hierarchy is the well-known KdV equation

$$4u_t = u_{xxx} + 6uu_x$$

for a function $u(t, x)$. The whole KdV hierarchy can be conveniently rephrased in terms of the Schrödinger operator $L = \partial_x^2 + u(x)$ known as the Lax operator in this case. One can similarly obtain the Gelfand–Dikii, or the nKdV hierarchies, by considering an n-th order differential operator $L = \partial_x^n + u_{n-2}(x)\partial_x^{n-2} + ... + u_0(x)$. Drinfeld and Sokolov's idea [DS84] was to replace the differential operator L by a special connection known as a G-oper. Thus, the phase space of Drinfeld–Sokolov hierarchies is the moduli space of G-opers on the formal disk, and one recovers the original nKdV phase space when $G = SL_n$. We will follow a description of *generalized* Drinfeld–Sokolov hierarchies due to Ben-Zvi and Frenkel [BZF99], which starts with a data of a smooth projective curve X, a semisimple group G and a maximal torus $A \subset LG$ of the loop group called a Heisenberg subgroup. The phase space of these hierarchies is the moduli space of affine opers on the formal disk. Considering $X = \mathbf{P}^1$ and A the principal Heisenberg subgroup one recovers the original Drinfeld–Sokolov phase space.

Alternatively, we can also generalize the picture from a different point of view. Using the inverse scattering method one can replace the "coordinate" $u(x)$ on the phase space of the KdV hierarchy by the spectral properties of the corresponding Schrödinger operator L. In the situation we are considering here we get a rank 2 vector bundle on $\mathbf{A}^1$, whose fiber at $\lambda \in \mathbf{A}^1$ is the eigenspace $\mathrm{Ker}(L - \lambda)$. This vector bundle can be extended to the whole $\mathbf{P}^1$ in a canonical way; moreover, the extension carries an extra structure near infinity: the vector bundle comes as a pushforward from the trivial line bundle on a fully-ramified spectral cover $\mathrm{Spec}\, k[[\lambda^{-1/2}]] \to \mathrm{Spec}\, k[[\lambda^{-1}]]$. This picture has an extension to generalized Drinfeld–Sokolov hierarchies by considering the moduli space (known as the abelianized Grassmannian) of G-bundles on X together with a reduction to the Heisenberg subgroup A near a marked point. Ben-Zvi and Frenkel defined a natural isomorphism between the oper description of the Drinfeld–Sokolov hierarchy and the spectral description via bundles on curves. The spectral description of integrable hierarchies has an attractive quality that the time evolution is linearized: namely, the Drinfeld–Sokolov flows simply change the data of a reduction near the marked point.

The abelianized Grassmannian is a scheme of infinite type, so to get a handle on this space we would like to be able to describe certain special points. These points also admit an interesting geometric description we are about to describe.

While studying Gelfand–Dikii hierarchies, Krichever considered special Lax operators L, such that there is another differential operator M, which commutes with L and has a coprime order. He called these Lax operators *algebro-geometric*, since they can be recovered from the data of the spectral curve $\operatorname{Spec} k[L, M]$ together with a line bundle over it, whose fiber over (λ, μ) is the joint eigenspace $\operatorname{Ker}(L - \lambda) \cap \operatorname{Ker}(M - \mu)$. The reader is referred to [Mu94] for a gentle introduction to the classical approach to these solutions.

Instead, we will follow a more general approach to algebro-geometric solutions of generalized Drinfeld–Sokolov hierarchies outlined in [BZF99]. They are described in terms of abstract Higgs bundles of Donagi, such that the cameral cover near the marked point on X is identified with that of the Heisenberg subgroup A. The orbits of Drinfeld–Sokolov flows are finite-dimensional and can be identified with torsors over Picard and Prym varieties associated to the cameral covers. Informally, one can say that algebro-geometric solutions admit a sufficiently big stabilizer under the Drinfeld–Sokolov flows. Essentially, the only known solutions of Drinfeld–Sokolov hierarchies are algebro-geometric.

1.1.1 String solutions

In addition to algebro-geometric solutions we have the so-called *string* solutions, the simplest of which is $u(x, t) = -\frac{2x}{3t}$ in the KdV case. The corresponding Lax operators L admit a differential operator P, such that $[P, L] = 1$ (the string equation). The famous conjecture of Witten, proved by Kontsevich, states that the partition function of 2d quantum gravity (Gromov–Witten potential of a point) satisfies Virasoro constraints and the equations of the KdV hierarchy. In fact, this partition function is the tau-function associated to the solution written above. More generally, partition functions of certain Landau-Ginzburg models parametrized by the ADE groups are known to be string solutions of the corresponding Drinfeld–Sokolov hierarchies [BM12]. Recently this has been extended to include non simply-laced groups; in this case one considers orbifold theories of the ADE Landau-Ginzburg theories.

In this paper we generalize the notion of string solutions from nKdV hierarchies to generalized Drinfeld–Sokolov hierarchies for arbitrary base curves and arbitrary Heisenbergs. Just as algebro-geometric solutions had a sufficiently big stabilizer under the Drinfeld–Sokolov flows, we define a bigger Lie algebra action on the phase space for *varying* curves; the string solutions then possess a big stabilizer under the action of the latter Lie algebra. The goal of the paper is to understand what kind of geometry is behind these solutions and to prove the Virasoro constraints (certain second-order differential equations) on the tau-functions of string solutions.

1.1.2 Dispersionless limit

One can consider the KdV equation with a parameter λ:

$$4u_t = \lambda^2 u_{xxx} + 6uu_x.$$

Similarly, the τ-function will be a function of λ. The introduction of this parameter is very natural from the point of view of the Gromov–Witten potential: the parameter λ appears there as the genus-counting parameter:

$$\tau(\lambda) = \exp\left(\sum_{g=0}^{\infty} F_g \lambda^{2g-2}\right).$$

The term u_{xxx} is responsible for a non-linear dispersion relation for the wave solutions of KdV, so the limit $\lambda \to 0$ is known as the dispersionless limit. In this paper we define the phase spaces in the dispersionless limit. Note, that one cannot just take the limit $\lambda \to 0$ of the phase spaces as the flows become trivial in this limit and one instead has to take a first-order approximation in λ.

We show that an open dense subspace of the phase space of the principal dispersionless Drinfeld–Sokolov hierarchy is isomorphic to $\mathrm{Map}(\mathrm{Spec}\, k[[t]], \mathfrak{h}/W)$. We thus make a connection with Dubrovin's theory of integrable hierarchies of hydrodynamic type, where the phase space is $\mathrm{Map}(\mathrm{Spec}\, k[[t]], F)$ for some Frobenius manifold F. In a future work we plan to make this connection precise by identifying the Poisson structures and the flows on both sides.

1.2 Derived symplectic geometry

Let us recall the definition of the Hitchin integrable system. Given a smooth projective curve X, let $\mathrm{Higgs}_{GL_n}(X)$ be the moduli space of GL_n-Higgs bundles; that is, vector bundles $E \to X$ together with a Higgs field $\phi \in \Gamma(X, \mathrm{End}\, E \otimes K_X)$. Note, that $\mathrm{Higgs}_{GL_n}(X)$ has a natural symplectic structure since it is isomorphic to the cotangent bundle $T^*\mathrm{Bun}_{GL_n}(X)$ of the moduli space of bundles. The Hitchin base $\mathrm{Hitch}_{GL_n}(X)$ is defined to be $\sum_{n=1}^{\infty} \Gamma(X, K_X^{\otimes n})$; this is simply an affine space.

We have the Hitchin morphism $\chi_X : \mathrm{Higgs}_{GL_n}(X) \to \mathrm{Hitch}_{GL_n}(X)$ which sends the matrix ϕ to its characteristic polynomial. Hitchin observed that χ_X is a proper Lagrangian fibration thus making $\mathrm{Higgs}_{GL_n}(X)$ into an integrable system. One can generalize this picture to arbitrary reductive groups G, so that $\chi_X : \mathrm{Higgs}_G(X) \to \mathrm{Hitch}_G(X)$ is an integrable system.

Observe that one can write $\mathrm{Higgs}_G(X)$ as the space of $\mathbf{G}_m$-equivariant maps

$$\mathrm{Higgs}_G(X) \cong \mathrm{Map}_{\mathbf{G}_m}(T^*X^\times, \mathfrak{g}/G),$$

where $T^*X^\times$ is the cotangent bundle minus the zero section. Similarly, the Hitchin base $\mathrm{Hitch}_G(X)$ can be written as $\mathrm{Map}_{\mathbf{G}_m}(T^*X^\times, \mathfrak{h}/W)$. The Hitchin morphism χ_X is induced from the "characteristic polynomial" morphism $\chi : \mathfrak{g}/G \to \mathfrak{h}/W$. One is thus led to wonder whether the map χ makes $\mathfrak{g}/G$ into an integrable system in a certain sense. One could then try to deduce properties of the Hitchin system from the corresponding properties of a simpler integrable system on $\mathfrak{g}/G$.

It turns out that $\mathfrak{g}/G$ is not symplectic, but it has what is known as a 1-shifted symplectic structure [PTVV11] that we now explain. Using this definition, χ becomes a Lagrangian morphism thus making $\mathfrak{g}/G$ into an integrable system.

1.2.1 Shifted symplectic structures

Recall that a symplectic structure on a space X is an isomorphism $\omega_X : \mathbb{T}_X \to \mathbb{L}_X$ from the tangent bundle to the cotangent bundle satisfying a certain integrability condition, that can be expressed by saying that the underlying two-form ω_X is closed. Working in derived algebraic geometry the tangent and cotangent bundles $\mathbb{T}_X, \mathbb{L}_X$ are now complexes of vector bundles. Therefore, one can consider shifted symplectic structures. An n-shifted symplectic structure is a quasi-isomorphism $\omega_X : \mathbb{T}_X \to \mathbb{L}_X[n]$, such that the underlying two-form (which has degree n) is closed. We give a precise definition in the main body of the dissertation.

Like in ordinary symplectic geometry, one can introduce the notion of isotropic and Lagrangian morphisms $L \to X$. Note, that we no longer have to assume that $L \to X$ is an embedding. We say that $f : L \to X$ is isotropic if we are given a null-homotopy of $f^*\omega_X$. The morphism is Lagrangian if this null-homotopy satisfies a non-degeneracy condition. If we restrict our attention to ordinary symplectic manifolds, these notions become the classical notions of isotropic and Lagrangian submanifolds.

One way to get new symplectic spaces out of old ones is to consider intersections of Lagrangians: given two Lagrangians $L_1, L_2 \to X$ in an n-shifted symplectic stack X, their derived intersection $L_1 \times_X L_2$ carries a natural $(n-1)$-shifted symplectic structure. We give a generalization of this theorem to Lagrangian correspondences, see Theorem 3.1.2.

1.2.2 Hamiltonian and quasi-Hamiltonian reductions

In classical symplectic geometry there is another way to obtain new symplectic manifolds from symplectic manifolds with a G-action via a procedure of Hamiltonian reduction. Let (M, ω) be a symplectic manifold with a G-action preserving the symplectic form and a moment map $\mu : M \to \mathfrak{g}^*$, which is a G-equivariant morphism satisfying the moment map equation.

A classic theorem of Marsden and Weinstein states that the reduced space

$$M_{red} := \mu^{-1}(0)/G$$

is again a symplectic manifold. Let us show that it can be phrased as an intersection of Lagrangians. We can rewrite

$$M_{red} \cong (M \times^*_{\mathfrak{g}} \mathrm{pt})/G \cong M/G \times_{\mathfrak{g}^*/G} \mathrm{pt}/G.$$

Since $\mathfrak{g}^*/G \cong T^*[1]BG$ is a shifted cotangent bundle, it has a natural 1-shifted symplectic structure. Given any G-equivariant map $\mu : M \to \mathfrak{g}^*$ we prove that the quotient $M/G \to \mathfrak{g}^*/G$ is Lagrangian iff M has a G-invariant symplectic structure and μ satisfies the moment map equation. In other words, given a Hamiltonian space M, the reduced space M_{red} is an intersection of two Lagrangians M/G and pt/G inside of a 1-shifted symplectic stack $\mathfrak{g}^*/G$, hence is symplectic. We thus recover the Marsden-Weinstein theorem in a completely new way [PTVV11], [Ca13].

There is a variant of Hamiltonian reduction known as the quasi-Hamiltonian reduction due to Alekseev, Malkin and Meinrenken [AMM97]. It replaces moment maps $\mu : M \to \mathfrak{g}^*$ into the dual Lie algebra by moment maps $\mu : M \to G$ into the group G. There are two immediate complications. First, the two-form on M is no longer symplectic: it is neither closed nor non-degenerate. Moreover, the moment map equation has to be modified a bit to include the data of a nondegenerate pairing on $\mathfrak{g}$.

One can write down a natural 1-shifted symplectic structure on G/G similar to one on $\mathfrak{g}^*/G$ and then wonder what kind of structure one should put on M for $\mu : M/G \to G/G$ to be Lagrangian. It turns out that in this way one recovers the quasi-Hamiltonian moment map equation of Alekseev, Malkin and Meinrenken (a similar result was previously obtained by Calaque [Ca13]).

1.2.3 AKSZ field theory

The symplectic structure on $\mathfrak{g}^*/G$ was fairly natural as it was the symplectic structure on the cotangent bundle $T^*[1]BG$. We explain that, similarly, the symplectic structure on G/G is natural in the sense that it is a symplectic structure on the mapping space

$\mathrm{Map}(S^1_B, \mathrm{B}G) \cong G/G$.

Given a compact oriented manifold M of dimension d and an n-shifted symplectic stack X, the AKSZ construction produces an $(n-d)$-shifted symplectic structure on the mapping stack $\mathrm{Map}(M_B, X)$. For instance, one can show that the space of 2-shifted symplectic structures on the classifying stack $\mathrm{B}G$ is isomorphic to the space of G-invariant non-degenerate quadratic forms on $\mathfrak{g}$. Therefore, given such a form, we have a 1-shifted symplectic structure on $\mathrm{Map}(S^1_B, \mathrm{B}G)$. One of our main results is the computation of this symplectic structure. Namely, it coincides with the 1-shifted symplectic structure on G/G that appear in quasi-Hamiltonian reduction.

Let us explain the relevance of this result. Given an n-shifted symplectic stack X, Calaque defined an $(n+1)$-dimensional topological field theory $M \mapsto \mathrm{Map}(M_B, X)$ taking value in the category of Lagrangian correspondences. That is, to a point we attach the n-shifted symplectic stack X. To a closed 1-manifold M we attach the $(n-1)$-shifted symplectic stack $\mathrm{Map}(M_B, X)$. To a d-dimensional cobordism M from N_1 to N_2 we attach $\mathrm{Map}(M_B, X)$, a Lagrangian correspondence from $\mathrm{Map}((N_1)_B, X)$ to $\mathrm{Map}((N_2)_B, X)$.

For the 2-shifted symplectic stack $X = \mathrm{B}G$ we get a 3-dimensional topological field theory, which is just the classical Chern–Simons theory. To the circle S^1 it associates G/G, while to a surface it associates the character stack, the phase space of the Chern–Simons theory.

Let us interpret the story of quasi-Hamitlonian reduction in terms of the classical Chern–Simons theory. Let Σ be a compact oriented surface obtained by gluing a disk into a non-compact surface Σ':

$$\Sigma = \Sigma' \cup_{S^1} D.$$

We get

$$\mathrm{Map}(\Sigma_B, \mathrm{B}G) \cong \mathrm{Map}(\Sigma'_B, \mathrm{B}G) \times_{G/G} \mathrm{pt}/G.$$

In other words, $\mathrm{Map}(\Sigma_B, \mathrm{B}G)$, the character stack of a surface, is presented as a quasi-Hamiltonian reduction. This recovers a result of Alekseev, Malkin and Meinrenken on symplectic structures on character varieties.

More generally, one can interpret a quasi-Hamiltonian space as a defect in this field theory with boundary the circle S^1. Then the quasi-Hamiltonian reduction corresponds to gluing in a disk. Given two such defects, i.e. two quasi-Hamiltonian spaces, we can glue in a pair of pants. In other words, given two Lagrangians in G/G, we can perform an integral transform along the correspondence

$$G/G \times G/G \leftarrow (G \times G)/G \rightarrow G/G$$

given by the pair of pants. In this way we obtain a new Lagrangian in G/G, i.e. another quasi-Hamiltonian space. We prove that this procedure coincides with the *fusion* of quasi-Hamiltonian spaces that appears in the quasi-Hamiltonian literature.

1.2.4 Prequantization

Given a symplectic manifold (X, ω_X), a prequantization of X is a choice of a line bundle with a connection whose curvature coincides with the symplectic form ω_X. One is often interested in prequantizations of symplectic manifolds as it is the first step in the procedure of geometric quantization.

As it is well-known, character stacks do not admit prequantization in general: for instance, for the torus the GL_1-character variety is $GL_1 \times GL_1$ with the symplectic structure $\mathrm{d}_{\mathrm{dR}} \log x \wedge \mathrm{d}_{\mathrm{dR}} \log y$. This space does not admit nontrivial line bundles, so the curvature of any connection is exact. However, $\mathrm{d}_{\mathrm{dR}} \log x \wedge \mathrm{d}_{\mathrm{dR}} \log y$ represents a nontrivial de Rham cohomology, so the space does not admit a prequantization.

Thus we are led to a modified notion of a prequantization as a lift of an element $\omega_X \in \Gamma(X, \Omega^{2,cl})$ to an element $\omega_{\mathcal{A}} \in \Gamma(X, \mathcal{A})$ for some abelian group $\mathcal{A}$ with a map to $\Omega^{2,cl}$. For instance, for $\mathcal{A} = (\mathcal{O}^\times \to \Omega^1)[1]$ one recovers the usual notion of prequantization.

Having made this definition, one can immediately generalize it to symplectic structures of nonzero degrees by shifting the complex $\mathcal{A}$. We show that the universal prequantization of $\mathrm{B}G$ is given by a certain $\mathcal{K}_2$-gerbe. Similarly, the universal prequantization of G/G compatible with the procedure of fusion is a certain $\mathcal{K}_2$-torsor on G/G.

Furthermore, we show that given the $\mathcal{K}_2$-prequantization of $\mathrm{B}G$ one can construct a $\mathcal{K}_2$-prequantization of the character stack. That is, we construct a class $\omega_K \in K_2(\mathrm{Map}(M_B, \mathrm{B}G))$, the second algebraic K-theory, which maps to the symplectic structure on the character stack under the differential symbol. This is done by prequantizing the whole Chern–Simons theory and running the machine of cobordism hypothesis, which allows us to integrate the prequantization of $\mathrm{B}G$ to a prequantization of character stacks. Working complex-analytically, we show that applying the Beilinson regulator one obtains a line bundle with a connection, i.e. an analytic prequantization of the character stack.

1.3 Organization

The structure of the dissertation is as follows. We have divided the thesis into two main chapters which can be read independently.

1.3.1 Chapter 2

In section 2.1 we review background material on Heisenberg subgroups and abstract Higgs bundles. We then proceed to define the phase space of generalized Drinfeld–Sokolov hierarchies together with Drinfeld–Sokolov flows for a fixed curve X and in moduli. A geometric definition of tau-functions for generalized Drinfeld–Sokolov hierarchies was missing in the literature, so we give their brief description in **??** relating, in particular, to a more algebraic definition in the case $X = \mathbf{P}^1$ that can be found in [Wu12]. Note, that in this paper we only consider a linear action of the Virasoro algebra on the tau-function.

In section 2.3 we define the dispersionless phase space and show how it arises in the limit $\lambda \to 0$ of the original Drinfeld–Sokolov phase space. In subsection 2.4.1 we discuss algebro-geometric solutions for generalized Drinfeld–Sokolov hierarchies in a way that will be easily generalized to string solutions. We also discuss the linear differential equations the algebro-geometric tau-functions satisfy.

The first main contribution of this dissertation is subsection 2.4.2. We define string solutions as points having a big stabilizer under the Heisenberg-Virasoro Lie algebra acting on the phase space. We then give a geometric description of string solutions (Theorem 2.4.4) as spectral bundles with a connection, which has a standard structure near the marked point. Generalizing a theorem of F. Plaza Martín [PM11, Theorem 3.1] from the case of nKdV hierarchies to generalized Drinfeld–Sokolov hierarchies, we prove that algebro-geometric and string conditions are mutually exclusive if the base curve X has genus 0 (Theorem 2.4.5). To obtain Virasoro constraints on string tau-functions, we first define the Sugawara embedding for arbitrary Heisenbergs. Using the Sugawara currents we define an action of the negative part of the Virasoro algebra $\Gamma(X\backslash\infty, T_X)$ on the space of tau-functions. When G is simply-laced, the space of tau-functions is one-dimensional. Moreover, all finite-dimensional representations of the negative part of the Virasoro algebra in this case are trivial since it is simple, thus we get Virasoro constraints, i.e. second-order differential equations on string tau-functions. This is the content of Theorem 2.4.10. Note, that we prove the Virasoro constraints in the greatest possible generality: for all simply-laced groups, all Heisenbergs and all curves.

1.3.2 Chapter 3

In section 3.1 we define the notion of shifted symplectic structures and Lagrangian morphisms. We show that one can compose Lagrangian correspondences using pullbacks, which gives rise to the composition in the category $\mathrm{LagrCorr}_n$. section 3.2 is devoted to explicit calculations showing how ordinary symplectic reduction and quasi-Hamiltonian reduction fit into the framework of shifted symplectic structures. In section 3.3 we give interpretations of the constructions in quasi-Hamiltonian reduction in terms of the AKSZ topological field theory.

The second main contribution of this dissertation is section 3.4. It is devoted to computations of the AKSZ symplectic forms on character stacks and their interpretation in terms of a multiplicative $\Omega^{2,cl}$-torsor on G coming from the $\Omega^{2,cl}$-gerbe on $\mathrm{B}G$. Finally, section 3.5 is devoted to the theory of prequantizations. We start with a basic case of universal central extensions of Lie algebras. We show that there is a universal L_∞ extension of an L_∞ algebra by its complex of Chevalley-Eilenberg chains shifted by -2. In particular, it gives a central extension of $\mathfrak{sl}_n$ by the first cyclic homology. We conclude with the construction of the universal central extension in the group case and the construction of the canonical element of $\mathrm{K}_2(\mathrm{Loc}_{GL_n}(M))$, the second K-group of the GL_n-character variety of a surface M.

Chapter 2

Drinfeld–Sokolov hierarchies

2.1 Loop groups and principal bundles on curves

2.1.1 Principal bundles

2.1.1.1 Basic definitions

Recall that a G-*torsor* on X is a morphism $P \to X$ together with a right action of G on P, such that for some étale cover $U \to X$ the pullback is isomorphic to G compatibly with the action map.

Clearly, given any sheaf of groups $\underline{G}$ over X, we can define a notion of a $\underline{G}$-torsor on X.

Let BG be the classifying stack of G, i.e. the stack whose S-points parametrize G-torsors on S. Its tangent complex is $T_{BG} = \mathfrak{g}[1]$ thought of as a G-representation.

2.1.1.2 Connections

Let G be an affine reductive group and $p : P \to X$ a G-torsor on X. We have an exact sequence

$$T_{P/X} \to T_P \to p^*T_X,$$

where we can identify $T_{P/X} \cong \mathfrak{g} \otimes \mathcal{O}_P$ using the right action. Pushing forward to X and taking G-invariants, we get a sequence

$$(p_* T_{P/X})^G \to (p_* T_P)^G \to T_X,$$

which we rename as

$$\operatorname{ad} P \to \mathcal{A}_P \to T_X,$$

which is known as the Atiyah sequence (the sheaf $\mathcal{A}_P$ is known as the Atiyah bundle). For a vector bundle $\mathcal{A}_P$ is the bundle of differential operators on P of order at most 1.

Definition. A *connection* ∇ on P is a splitting $T_X \to \mathcal{A}_P$ of the Atiyah sequence.

We can modify the definition a bit to arrive at a notion of a λ-connection. Let $p_* : \mathcal{A}_P \to T_X$ be the pushforward morphism.

Definition. A *λ-connection* ∇ on P is a map $\nabla : T_X \to \mathcal{A}_P$, such that $p_* \circ \nabla = \lambda \cdot \mathrm{id}_{T_X}$.

Clearly, for $\lambda \neq 0$ the ratio ∇/λ defines an ordinary connection, while for $\lambda = 0$ one simply gets a section of $\Omega^1 \otimes \operatorname{ad} P$.

2.1.1.3 Reductions

Let $K \subset G$ be a subgroup and P a G-bundle on a space X.

Definition. A K-bundle P_K is a *K-reduction* of P if we are given an isomorphism $P \cong P_K \times_K G$ of G-bundles.

Let ∇ be a connection on P. We say that the reduction P_K is *flat* if the connection ∇ is induced from a connection on P_K.

We can measure the failure of ∇ to preserve P_K using the notion of relative position. Let $\mathbf{O} \subset \mathfrak{g}/\mathfrak{k}$ be a K-orbit.

Definition. The reduction P_K has *relative position* $\mathbf{O}$ with respect to ∇ if

$$\nabla(v) - \nabla_K(v) \in \mathbf{O} \times_K P_K \subset \mathfrak{g}/\mathfrak{k} \times_K P_K$$

for any vector field $v \in T_X$ and a connection ∇_K on P_K.

Clearly, this notion makes sense only if the orbit $\mathbf{O}$ is $\mathbf{G}_m$-invariant. For orbits $\mathbf{O}$ which are not $\mathbf{G}_m$-invariant, we will say that P_K has relative position $\mathbf{O}$ with respect to ∇_v if the condition is satisfied for the vector field v.

2.1.1.4 Atiyah morphism

Given an exact sequence

$$\operatorname{ad} P \to \mathcal{A}_P \to T_X,$$

we can rotate it to obtain the sequence

$$\mathcal{A}_P \to T_X \xrightarrow{\text{at}} \operatorname{ad} P[1].$$

The same sequence can also be obtained from the following construction. Let $f : X \to BG$ be the classifying map of the bundle P. Then we get a short exact sequence

$$T_{X/BG} \to T_X \to f^* T_{BG}.$$

One has $T_{BG} \cong \mathfrak{g}[1]$ as G-representations, so $f^* T_{BG}$ is simply $\operatorname{ad} P[1]$. In other words, the exact sequence associated to the relative tangent bundle of $X \to BG$ is the same as the rotated Atiyah sequence. The role of the Atiyah morphism $\mathrm{at} : T_X \to \mathrm{ad}\, P[1]$ is played by the pushforward map $T_X \to f^*T_{BG}$.

Let us explicitly describe the morphism $\mathrm{at} : T_X \to \mathrm{ad}\, P[1]$. First, recall that the data of a *sequence*

$$\mathrm{ad}\, P \xrightarrow{i} \mathcal{A}_P \xrightarrow{p_*} T_X$$

is the null-homotopy $p_* i \sim 0$. In other words, it is a map $h : \mathrm{ad}\, P \to T_X[-1]$, such that $dh + hd = p_* i$. The sequence is *exact* if the morphism

$$\overline{h} : \mathrm{ad}\, P \to \mathcal{A}_P \oplus T_X[-1]$$

defined by $\overline{h}(a) = (i(a), h(a))$ is a quasi-isomorphism. Note, that the complex on the right is the cone of p_*.

To compute $\mathrm{at} : T_X \to \mathrm{ad}\, P[1]$, resolve all bundles, so that there is a splitting $s : T_X \to \mathcal{A}_P$ (as graded vector spaces, not as complexes). For instance, one can use the Čech resolution using the trivializing cover of P. Alternatively, for complex manifolds one could instead use the Dolbeault resolution. Then $(ds(v), v)$ is a closed element of $\mathcal{A}_P \oplus T_X[-1]$ for every $v \in T_X$. Then one can define $\mathrm{at}(v) = \overline{h}^{-1}(ds(v), v)$.

Using the morphism at, one can say that the closed elements of $\mathcal{A}_P$ are the same as elements $v \in T_X, g \in \mathrm{ad}\, P$, such that $dv = 0$ and $dg = \mathrm{at}(v)$.

Proposition 2.1.1. *A connection ∇ on a principal G-bundle $P \to X$ is the same as the data of elements $g_v \in \mathrm{ad}\, P$ for every closed element $v \in T_X$ satisfying $dg = \mathrm{at}(v)$.*

2.1.1.5 Drinfeld-Simpson theorem

Let $\mathrm{Bun}_G(X)$ be the moduli stack of G-bundles on a projective curve X. It is well-known that it is an Artin stack locally of finite type. We recall a theorem due to Drinfeld and Simpson that gives a concrete, albeit infinite-type, presentation of this stack.

Let D be a disk and $D^\times$ a punctured disk. Non-canonically we can identify

$$D \cong \operatorname{Spec} k[[z]], \quad D^\times \cong \operatorname{Spec} k((z)).$$

We define the loop group LG and its positive part $LG_+ \subset LG$ to be

$$LG := \mathrm{Map}(D^\times, G), \quad LG_+ := \mathrm{Map}(D, G).$$

One can show that LG is representable by an ind-scheme, while LG_+ is representable by a scheme of infinite type.

Given a curve X together with a smooth point $x \in X$ we denote by

$$LG_- = \mathrm{Map}(X \backslash x, G)$$

the negative part of the loop group.

Theorem 2.1.2 (Drinfeld–Simpson)**.** *Suppose G is a semisimple group. Then we have an isomorphism* $\mathrm{Bun}_G(X) \cong LG_- \backslash LG / LG_+$.

More explicitly, Drinfeld and Simpson show that any G-bundle on an affine curve, such as $X \backslash x$, is trivial, so a G-bundle is simply given by a transition function on the punctured disk around x.

2.1.2 Heisenberg subgroups

2.1.2.1 General definitions

Let G again be a connected reductive group over k. Recall that since k is algebraically closed, any two maximal tori are conjugate. This implies that the variety of all maximal tori in G is isomorphic to $G/N(H)$ for any maximal torus H. We denote by $\mathcal{H}$ the universal abelian group scheme over $G/N(H)$.

Definition. A *Heisenberg subgroup* $A \subset LG$ is a maximal torus in LG.

The loop group LG is an algebraic group over $k((t))$, which is no longer algebraically closed: for instance, the polynomial $x^2 - t$ does not have any roots in $k((t))$. Therefore, not any two Heisenbergs are LG-conjugate. However, we can use the fact that $LG \to \operatorname{Spec} k((t))$ is obtained as a base change from $G \to \operatorname{Spec} k$ to understand the maximal tori better. Indeed, a Heisenberg A is then the same as a map

$$C_A : D^\times \to G/N(H).$$

Then Heisenberg A thought of as a group scheme over $D^\times$ is obtained as a pullback $A = C_A^* \mathcal{H}$.

The scheme $G/N(H)$ is not complete, so we do not expect the map C_A to extend to a map from the disk D. However, there is a canonical completion $\widetilde{G/N(H)} \supset G/N(H)$. A maximal torus is a centralizer of a regular semisimple element. We denote by $\widetilde{G/N(H)}$ the variety of all centralizers of regular elements. Let $\mathcal{H}$ be the universal abelian group scheme over $\widetilde{G/N(H)}$.

Proposition 2.1.3 (Donagi-Gaitsgory [DG00]). *The variety $\widetilde{G/N(H)}$ of regular centralizers of G is complete.*

By the valuative criterion of properness, a map $D^\times \to G/N(H)$ uniquely extends to a map $D \to \widetilde{G/N(H)}$. Note, however, that this statement is not true in families.

Let $A \subset LG$ be a Heisenberg with a classifying map $C_A : D^\times \to G/N(H)$. Then the unique extension $C_{A_+} : D \to \widetilde{G/N(H)}$ classifies the subgroup $A_+ = A \cap LG_+$, i.e. $A_+ = C_{A_+}^* \mathcal{H}$.

Recall that the Weyl group W is defined to be $W = N(H)/H$. We have a W-torsor $G/H \to G/N(H)$ over the variety of maximal tori. It extends to a ramified W-cover $\widetilde{G/H} \to \widetilde{G/N(H)}$ over the variety of regular centralizers. The space G/H parametrizes maximal tori $H' \subset G$ together with an isomorphism $H' \cong H$ given by conjugation.

2.1.2.2 Example

Let $G = SL_2$ with the maximal torus

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbf{G}_m \right\}.$$

Its normalizer $N(H)$ consists of matrices in H and matrices of the form

$$\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}.$$

The space G/H is naturally embedded into $\mathbf{P}^1 \times \mathbf{P}^1$ via the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ([a:c],[b:d]).$$

This allows one to identify

$$G/H \cong (\mathbf{P}^1 \times \mathbf{P}^1) \backslash \Delta,$$

where $\Delta \subset \mathbf{P}^1 \times \mathbf{P}^1$ is the diagonal. The left action of G on G/H becomes the diagonal action of G on $\mathbf{P}^1 \times \mathbf{P}^1$ via fractional linear transformations.

Similarly, we can embed $G/N(H)$ into $\mathbf{P}^2$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [ab : (ad+bc) : cd].$$

This allows one to identify

$$G/N(H) \cong \mathbf{P}^2 \backslash Q,$$

where Q is the quadric given in homogeneous coordinates $[z_0 : z_1 : z_2]$ by $z_1^2 - 4z_0z_2 = 0$. The left action of G on $G/N(H)$ is given by the action of $G = SL_2$ on the projectivization of its 3-dimensional irreducible representation.

Given an element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G/N(H),$$

the corresponding maximal torus is

$$\left\{ \begin{pmatrix} adx - bcx^{-1} & ab(x^{-1} - x) \\ cd(x - x^{-1}) & adx^{-1} - bcx \end{pmatrix} \middle| \; x \in \mathbf{G}_m \right\}.$$

Note, that $G/H \to G/N(H)$ coincides with the symmetric square morphism

$$\mathbf{P}^1 \times \mathbf{P}^1 \to \mathrm{Sym}^2(\mathbf{P}^1) \cong \mathbf{P}^2$$

given by

$$([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0y_0 : (x_0y_1 + x_1y_0) : x_1y_1].$$

It is a ramified double cover with Q the ramification locus.

Now let's describe the compactifications $\widetilde{G/N(H)}$ and $\widetilde{G/H}$. Every regular non-semisimple element can be conjugated to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Its centralizer is

$$\left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \middle| \; y \in \mathbf{G}_a \right\}.$$

Therefore, we get a family

$$\left\{ \begin{pmatrix} 1 - a_0 a_1 y & a_0^2 y \\ -a_1^2 y & 1 + a_0 a_1 y \end{pmatrix} \middle| \; y \in \mathbf{G}_a \right\}$$

of regular centralizers parameterized by $[a_0 : a_1] \in \mathbf{P}^1 \cong Q$, which are not maximal tori.

Let us explain how a family of maximal tori degenerates to a regular centralizer once we approach the divisor Q. Consider the family of maximal tori given by

$$\left\{ \begin{pmatrix} \frac{\epsilon+2\lambda}{2\epsilon} x - \frac{2\lambda-\epsilon}{2\epsilon} x^{-1} & \frac{x^{-1}-x}{\epsilon} \\ \frac{4\lambda^2-\epsilon^2}{4\epsilon}(x - x^{-1}) & \frac{\epsilon+2\lambda}{2\epsilon} x^{-1} - \frac{2\lambda-\epsilon}{2\epsilon} x \end{pmatrix} \middle| \; x \in \mathbf{G}_m \right\},$$

where λ, ϵ are parameters.

Let us substitute $y = \frac{x^{-1}-x}{\epsilon}$. We have a unique solution $x = x(y, \epsilon)$ satisfying $x(y, 0) = 1$ once we work with formal ϵ, i.e. over $k((\epsilon))$. Then the maximal torus becomes

$$\left\{ \begin{pmatrix} \frac{x+x^{-1}}{2} - \lambda y & y \\ \frac{\epsilon^2-4\lambda^2}{4} y & \frac{x+x^{-1}}{2} + \lambda y \end{pmatrix} \right\}.$$

For $\epsilon \to 0$ we obtain the family

$$\left\{ \begin{pmatrix} 1 - \lambda y & y \\ -\lambda^2 y & 1 + \lambda y \end{pmatrix} \middle| \; y \in \mathbf{G}_a \right\}$$

of regular centralizers.

2.1.2.3 Higgs bundles and spectral curves

We refer the reader to [DG00] for a comprehensive treatment of abstract Higgs bundles, here we will only sketch the necessary basic facts.

Definition. A *regular Higgs bundle* (P, σ) on X is a G-torsor $P \to X$ together with a G-equivariant map $\sigma : P \to \widetilde{G/N(H)}$.

In other words, a regular Higgs bundle is a subbundle $c \subset \operatorname{ad} P$ of regular centralizers. Given a Higgs field σ, $c = (\sigma^* \mathcal{H})/G$ is the corresponding subbundle of regular centralizers on X.

Remark. If we are given a Higgs field $\phi \in \Gamma(X, \operatorname{ad} P \otimes \Omega^1_X)$ which is regular at every point, we can form a subbundle $c \subset \operatorname{ad} P$ consisting of centralizers of ϕ in $\operatorname{ad} P$ at every point. In the case $G = GL_n$ the subbundle c encodes eigenspaces of the Higgs field ϕ.

For a Higgs field σ we can define a ramified W-cover $X[\sigma]$ of X called the *cameral cover* in the following way. If we pull back the ramified W-cover $\widetilde{G/H} \to \widetilde{G/N(H)}$ to P along σ, we obtain a G-equivariant W-cover, which descends to X.

We call a Higgs field σ *unramified* if the corresponding W-cover $X[\sigma]$ is unramified. Equivalently, a Higgs field $\sigma : P \to \widetilde{G/N(H)}$ is unramified if it factors through $G/N(H) \subset \widetilde{G/N(H)}$.

2.1.2.4 Torsors over Heisenbergs

A Heisenberg A is given by the classifying map

$$C_A : D^\times \to G/N(H),$$

which is the same as an unramified Higgs field on the trivial bundle $\underline{G}$ over the punctured disk $D^\times$. In particular, to a Heisenberg we can associate a cameral cover $D^\times[A]$.

Let E be an A-torsor on $D^\times$. Then the induced G-bundle $E \times_A LG$ over $D^\times$ carries a subbundle of regular centralizers $\operatorname{ad} E \subset \operatorname{ad} P$. In fact, we have the following theorem:

Proposition 2.1.4. *Let P be a G-bundle on $D^\times$. The following three categories are equivalent:*

1. *The category of reductions of P to A.*
2. *The category of unramified Higgs fields σ on P together with an isomorphism of cameral covers $D^\times[\sigma] \cong D^\times[A]$.*
3. *The category of reductions $P_{N(H)}$ of P to $N(H)$ together with an isomorphism of the induced W-torsors $P_{N(H)} \times_{N(H)} W \cong D^\times[A]$.*

2.1.2.5 Monodromy of Heisenbergs

Recall that the set of Heisenbergs is isomorphic to the space of maps $\operatorname{Map}(D^\times, G/N(H))$. Therefore, the set of $LG = \operatorname{Map}(D^\times, G)$-conjugacy classes of Heisenbergs is isomorphic to $\operatorname{Map}(D^\times, BN(H))$, i.e. the set of $N(H)$-bundles on $D^\times$.

From the exact sequence

$$1 \to H \to N(H) \to W \to 1$$

one obtains an $\operatorname{Map}(D^\times, BH)$-principal bundle $\operatorname{Map}(D^\times, BN(H)) \to \operatorname{Map}(D^\times, BW)$. From Noether's version of Hilbert's theorem 90, one concludes that every H-bundle on the punctured disk $D^\times$ is trivial. We obtain the following proposition ([KL88, Lemma 2]):

Proposition 2.1.5. *The set of LG-conjugacy classes of Heisenbergs is in bijection with the set of W-bundles on $D^\times$.*

Now, the set of W-bundles on $D^\times$ is isomorphic to

$$\mathrm{Hom}(\pi_1(D^\times), W)/W \cong \mathrm{Hom}(\hat{\mathbf{Z}}, W)/W \cong W/W,$$

the set of conjugacy classes in W. For a Heisenberg A, we call the corresponding conjugacy class $[w] \in W/W$ the monodromy of the Heisenberg A.

The simplest example of a Heisenberg is the homogeneous Heisenberg LH. Its monodromy is trivial.

A general Heisenberg A has fibers non-canonically isomorphic to H. To make them canonically isomorphic to H, consider the pullback p^*A of the Heisenberg to its cameral cover $p : D^\times[A] \to D^\times$. Recall that the universal cameral cover $G/H \to G/N(H)$ parametrizes maximal tori H' together with an isomorphism $H' \cong H$. Therefore, the pullback of the universal group scheme $\mathcal{H}$ to G/H is isomorphic to the constant group scheme with a fiber H. We conclude that

$$p^*A \cong LH.$$

In particular, we can identify

$$A \cong \mathrm{Map}_W(D^\times[A], H).$$

If $[w]$ is the conjugacy class of the monodromy in the Weyl group, then the cameral cover $D^\times[A]$ splits as a disjoint union of $h:1$ fully-ramified covers of $D^\times$, where h is the order of w. In particular, we can construct a Heisenberg from w in the following way. Let

t be a formal coordinate. Then the action of w extends from $\mathfrak{h}$ to $\mathfrak{h}((t))$ by the following formula:

$$w.(at^m) = (w.a)e^{2\pi i m/h}t^m, \quad a \in \mathfrak{h}.$$

The Heisenberg $\mathfrak{a}$ is then obtained as the subspace w-fixed vectors in $\mathfrak{h}((t))$.

Let $E \to D^\times$ be an $\underline{A}$-torsor. One can view it as a bundle with a varying structure group non-canonically isomorphic to a fixed maximal torus H. As before, the pullback $E[A]$ of E to its own cameral cover $D^\times[E] \cong D^\times[A]$ is naturally an H-bundle.

Being a pullback to the W-cover, the underlying space $E[A]$ is W-equivariant. However, the H-torsor structure is only $N(H)$-shifted W-equivariant [DG00]. Concretely, this means that

$$w.(hx) = (whw^{-1})(w.x), \quad w \in W, h \in H, x \in E[A].$$

2.1.2.6 Atiyah bundle for Heisenbergs

Given an A-torsor E over $D^\times$, we can define the bundle of vertical fields $\operatorname{ad} E$ as $(T_{E/X})^A$. As A is non-constant group scheme over X, one can't define the Atiyah bundle in the same way.

Recall that an A-torsor E is the same as an $N(H)$-torsor $E_{N(H)}$ with an identification of the cameral covers. Since the quotient $N(H)/H \cong W$ is discrete, we have an identification $\operatorname{ad} E \cong \operatorname{ad} E_{N(H)}$. Hence we can define the Atiyah bundle for E to be simply the Atiyah bundle for $N(H)$:

$$\mathcal{A}_E := \mathcal{A}_{E_{N(H)}}.$$

We get the Atiyah sequence

$$0 \to \operatorname{ad} E \to \mathcal{A}_E \to T_{D^\times} \to 0.$$

It induces a canonical connection on $\operatorname{ad} E$ in the following way. Given a vector field $v \in T_{D^\times}$, consider an arbitrary lift $\tilde{v} \in \mathcal{A}_E$. We define the connection to be

$$\nabla_v(-) = [\tilde{v}, -].$$

It is independent of the lift since the Lie algebra $\operatorname{ad} E$ is abelian.

Therefore, we get an action of the Witt algebra $\Gamma(D^\times, T_{D^\times})$ on the Heisenberg $\mathfrak{a} \cong \Gamma(D^\times, \operatorname{ad} E)$. Let z be a coordinate on $D^\times$. Then the action of the vector field $L_0 = z\frac{\partial}{\partial z}$ gives a grading on $\mathfrak{a}$.

Definition. A conjugacy class $[w] \in W/W$ is called *elliptic* if the action of w on the Lie algebra of the torus $\mathfrak{h}$ has no fixed points.

An important class of Heisenbergs are those having an elliptic monodromy. An important property of such Heisenbergs is given by the following proposition.

Proposition 2.1.6. *The degree 0 subspace of a Heisenberg $\mathfrak{a}$ is coincides with the space of w-fixed vectors in $\mathfrak{h}$, where w is the monodromy of the Heisenberg.*

Proof. Suppose $\nabla_{L_0} v = 0$ for some $v \in \mathfrak{a} \cong \operatorname{Map}_W(D^\times[A], \mathfrak{h})$. The tangent bundle to $D^\times[A]$ is trivial, so the map v is annihilated by any vector field.

Therefore, v is a locally-constant map $D^\times[A] \to \mathfrak{h}$. Moreover, the image is fixed by the monodromy. We see that the degree 0 subspace of the Heisenberg coincides with the space of fixed vectors by the monodromy of A, which concludes the proof of the theorem. □

Corollary 2.1.7. *The degree 0 subspace of a Heisenberg* $\mathfrak{a}$ *is trivial iff its monodromy is elliptic.*

Here is another version of the same property.

Proposition 2.1.8. *The intersection* $\mathfrak{g} \cap \mathfrak{a}$ *is trivial if the monodromy of the Heisenberg* A *is elliptic.*

2.1.2.7 Examples of elliptic conjugacy classes

In type A an elliptic conjugacy class is the conjugacy class of Coxeter elements. Indeed, suppose $W = S_n$. Then its conjugacy classes are parametrized by partitions of n. The Cartan subalgebra $\mathfrak{h} \subset k^n$ is the subset of vectors $(x_1, ..., x_n)$, such that $\sum_i x_i = 0$. Suppose $n = n_1 + ... + n_k$. If $k > 1$, then the corresponding elements have fixed vectors. For instance, for $n = 4$ the element $(12)(34)$ fixes the vector $(x, x, -x, -x)$. Therefore, an elliptic conjugacy class is the conjugacy class of n-cycles, whose elements are Coxeter elements.

All other types have elliptic conjugacy classes which do not contain Coxeter elements. Let us illustrate this in three examples.

Consider the root system of type $B_2 = C_2$. It has simple roots α, β, such that $||\beta||^2 = 2||\alpha||^2$. We will use the standard normalization where $||\beta||^2 = 2$ (β is the longest root). Then we have $\alpha \cdot \beta = -1$.

The Coxeter element is given by the product $c = s_\alpha s_\beta$, where s_i is the reflection along root i. It has order 4 equal to the Coxeter number. We have another element $w = s_\beta s_{\beta+2\alpha}$ (note that $\beta + 2\alpha$ is the highest root). Since β and $\beta + 2\alpha$ are orthogonal, the corresponding reflections commute, hence w has order 2. w is clearly elliptic as can be seen by decomposing

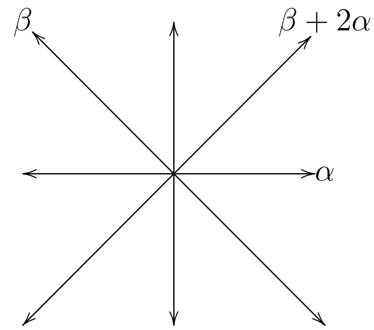


Figure 2.1: The C_2 root system

any vector into a linear combination of β and $\beta + 2\alpha$. w is not conjugate to the Coxeter element c since they have different orders.

Now consider the root system of type G_2. It has simple roots α, β with $||\beta||^2 = 3||\alpha||^2 = 2$. The scalar product between the simple roots is $\alpha \cdot \beta = -1$.

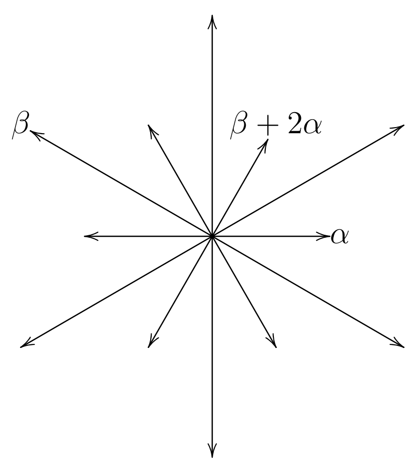


Figure 2.2: The G_2 root system

The Coxeter element is given by the product $c = s_\alpha s_\beta$ and it has order 6. Again we have an elliptic element $w = s_\beta s_{\beta+2\alpha}$ of order 2.

Finally, consider the root system of type D_4 with Dynkin diagram

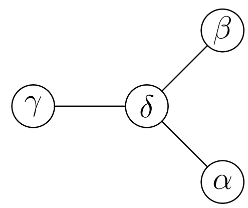


Figure 2.3: The D_4 Dynkin diagram

If we normalize all roots to have length $\sqrt{2}$, the scalar products between them are as follows:

$$\alpha \cdot \delta = \beta \cdot \delta = \gamma \cdot \delta = -1$$
$$\alpha \cdot \beta = \alpha \cdot \gamma = \beta \cdot \gamma = 0.$$

The Coxeter element $c = s_\alpha s_\beta s_\gamma s_\delta$ has order 6. The highest root is $\alpha + \beta + \gamma + 2\delta$, which is orthogonal to α, β and γ. Therefore, the element $w = s_\alpha s_\beta s_\gamma s_{\alpha+\beta+\gamma+2\delta}$ has order 2 and is elliptic.

2.1.2.8 Examples of Heisenbergs

1. $A = LH$, the homogeneous Heisenberg. Its monodromy is trivial and $\nabla = d$, so the grading is the homogeneous one. That is, elements of the form $xt^n \in \mathfrak{a}$ for $x \in \mathfrak{g}$ have degree n.

2. Choose Chevalley generators $e_1, ..., e_n, f_1, ..., f_n, h_1, ..., h_n$ of $\mathfrak{g}$. If $\theta \in \mathfrak{h}^\vee$ is the maximal root, then $e_0 = f_\theta \otimes z$, $f_0 = e_\theta \otimes z^{-1}$ and $\{e_i, f_i, h_i\}_{i=1}^n$ are the Chevalley generators of $L\mathfrak{g}$.

Let $p_{-1} = \sum_{i=0}^n f_i \in L\mathfrak{g}$. It is a regular semisimple element of $L\mathfrak{g}$. The principal Heisenberg A by definition is the centralizer of p_{-1} in LG. Its monodromy is the Coxeter element, whose order is h_{Cox}, the Coxeter number.

The principal gradation on $L\mathfrak{g}$ is given by assigning degree $1/h_{Cox}$ to e_i, degree 0 to h_i and degree $-1/h_{Cox}$ to f_i. For example, $p_{-1} \in \mathfrak{a}$ has degree $-1/h_{Cox}$ and multiplication by z increases the degree by 1.

One has an explicit action of the Witt algebra given by

$$\nabla_{L_0} = z\frac{\partial}{\partial z} + \sum_{j=1}^n c_j h_j,$$

where c_j are solutions to $\sum_{j=1}^n c_j a_{ij} = 1/h_{Cox}$ for $\{a_{ij}\}$ the Cartan matrix and h_{Cox} the Coxeter number.

The monodromy of the cameral cover is a Coxeter element, so the Heisenberg has no degree 0 vectors. Indeed, suppose that $p \in \mathfrak{a}$ is an element of degree 0. Then $p = \sum_{j=1}^n c_j h_j$ for some numbers c_j. Since p centralizes p_{-1},

$$-\sum_{i=1,j=1}^n c_j a_{ij} f_i + \sum_{j=1}^n c_j \theta(h_j) e_\theta = 0.$$

Since $\mathfrak{g}$ is semisimple, the Cartan matrix a_{ij} is nondegenerate. So, $c_j = 0$, i.e. $p = 0$.

2.1.3 Borel-Weil theorem for loop groups

In this section $\mathcal{L}$ will denote the basic ample line bundle on $\mathrm{Bun}_G(X)$. Let $\rho \in \mathfrak{h}^*$ be the half-sum of positive roots and $h^\vee$ the dual Coxeter number.

We consider the following two Grassmannians

$$\mathrm{Gr}_X(A) = LG_-\backslash LG/A_+, \quad \widehat{\mathrm{Gr}}_X = LG_-\backslash LG.$$

These will be phase spaces of Drinfeld–Sokolov hierarchies as explained in the next section.

Recall a Borel-Weil theorem for loop groups [Te98, Theorem 4]:

Theorem 2.1.9 (Teleman)**.** *There is an isomorphism of $\widehat{\mathfrak{g}}$-representations*

$$\Gamma(\widehat{\mathrm{Gr}}_X, \mathcal{L}) \cong \bigoplus_\lambda L(\lambda) \otimes \Gamma(\mathrm{Bun}_G(X), \mathcal{L} \otimes \mathcal{V}_\lambda),$$

where $L(\lambda)$ is a level 1 irreducible highest-weight representation with highest weight $\lambda \in \mathfrak{h}^*$, $\mathcal{V}_\lambda$ is the evaluation bundle corresponding to the $\mathfrak{g}$-representation with highest weight λ and the summation goes over the weights λ, such that $\lambda + \rho$ is inside the positive alcove at level $1 + h^\vee$.

From now on we assume that G is simply-laced. Let $\alpha_i \in \mathfrak{h}^*$ be the simple roots of $\mathfrak{g}$. Then $\lambda + \rho$ is in the positive alcove if

$$(\theta, \lambda + \rho) < 1 + h^\vee,$$
$$(\alpha_i, \lambda + \rho) > 0.$$

Since $(\theta, \rho) = h^\vee - 1$ and $(\alpha_i, \rho) = 1$, we see that the inequalities can be rewritten in the form

$$(\theta, \lambda) \leq 1,$$
$$(\alpha_i, \lambda) \geq 0.$$

In other words, λ is an integral dominant weight at level 1. We see that $\Gamma(\widehat{\mathrm{Gr}}_X, \mathcal{L})$ contains only level 1 highest-weight representations with integral dominant highest weights.

Kac and Peterson [KP85, p. 288] prove that the space of invariants $L(\lambda)^{A_+}$ is finite-dimensional for every integral dominant weight λ at level 1. Finally, since X is projective, the spaces of conformal blocks $\Gamma(\mathrm{Bun}_G(X), \mathcal{L} \otimes \mathcal{V}_\lambda)$ are finite-dimensional.

Consider the space $\Gamma(\mathrm{Gr}_X(A), \mathcal{L})$, which we will call the space of tau-functions (see next section). It can be written as

$$\Gamma(\mathrm{Gr}_X(A), \mathcal{L}) \cong \Gamma(\widehat{\mathrm{Gr}}_X, \mathcal{L})^{A_+} \cong \bigoplus_\lambda L(\lambda)^{A_+} \otimes \Gamma(\mathrm{Bun}_G(X), \mathcal{L} \otimes \mathcal{V}_\lambda).$$

There are finitely-many summands on the right-hand side and each of them is finite-dimensional. Therefore, we conclude:

Proposition 2.1.10. *Let G be a simply-laced group and $\mathcal{L}$ the basic ample line bundle on $\mathrm{Bun}_G(X)$. Then the space of tau-functions $\Gamma(\mathrm{Gr}_X(A), \mathcal{L})$ is finite-dimensional.*

2.1.3.1 Counter-example

Let us show that the conditions of the theorem (G being simply-laced and $\mathcal{L}$ having level 1) are necessary.

Consider $A = LH$, the homogeneous Heisenberg. The space $\Gamma(\mathrm{Gr}_X(A), \mathcal{L})$ has two actions of the Virasoro algebra. One of them comes from the Sugawara construction L_n^H for H; it is an action of central charge $c_H = \mathrm{rk}(\mathfrak{g})$. The other one comes from the Sugawara construction L_n^G for G; it is an action of central charge

$$c_G = \frac{\dim(\mathfrak{g})k}{k + h^\vee} = \frac{\mathrm{rk}(\mathfrak{g})(h+1)k}{k + h^\vee}.$$

The action of both L_n^H and L_n^G on $L\mathfrak{h}$ is the same. Therefore, $L_n^{GH} = L_n^G - L_n^H$

commutes with $L\mathfrak{h}$. Moreover, the elements L_n^{GH} define a Virasoro algebra of central charge

$$c_{GH} = c_G - c_H = \frac{\mathrm{rk}(\mathfrak{g})(h+1)k}{k+h^\vee} - \mathrm{rk}(\mathfrak{g}) = \mathrm{rk}(\mathfrak{g})\frac{kh-h^\vee}{k+h^\vee}.$$

Therefore, we obtain an action of the Virasoro algebra of central charge c_{GH} on $\Gamma(\mathrm{Gr}_X(A), \mathcal{L})$.

Proposition 2.1.11. *A representation of the Virasoro algebra is finite-dimensional only if it has central charge 0.*

Proof. The commutation relation for the Virasoro algebra is

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}c.$$

In particular, we have

$$[L_1, L_{-1}] = 2L_0$$
$$[L_2, L_{-2}] = 4L_0 + \frac{c}{2}.$$

Suppose V is a representation of the Virasoro algebra of dimension d.

Taking traces of both sides, we obtain

$$0 = 2\mathrm{tr}(L_0)$$
$$0 = 4\mathrm{tr}(L_0) + \frac{cd}{2}.$$

Therefore, $\mathrm{tr}(L_0) = 0$ and $c = 0$. □

We see that $\Gamma(\mathrm{Gr}_X(A), \mathcal{L})$ is infinite-dimensional unless $c_{GH} = 0$, i.e. $kh = h^\vee$. From a list of Coxeter numbers for simple groups one can easily see that $kh = h^\vee$ only if the group is simply-laced ($h^\vee = h$) and the level k is 1.

2.1.3.2 Example

In this section G is still a simply-laced group. We have

$$\begin{aligned}\Gamma(\mathrm{Bun}_G(X), \mathcal{L} \otimes \mathcal{V}_\lambda) &\cong (\Gamma(LG/LG_+, \mathcal{L}) \otimes V_\lambda)^{L\mathfrak{g}_-} \\ &= (L(0)^* \otimes V_\lambda)^{L\mathfrak{g}_-},\end{aligned}$$

where $L(0) = \Gamma(LG/LG_+, \mathcal{L})^*$ is the irreducible level 1 representation with the zero highest weight [Ku87, Proposition 2.11]. We can write $L(0)$ as a quotient of the vacuum Verma module $V(0)$

$$L(0) = V(0)/I.$$

Decompose

$$\widehat{L\mathfrak{g}} \cong \mathfrak{g}[[z]] \oplus z^{-1}\mathfrak{g}[z^{-1}] \oplus k.$$

Then the vacuum module is

$$V(0) = \mathrm{Ind}_{L\mathfrak{g}_+ \oplus k}^{\widehat{L\mathfrak{g}}}(kv_0) \cong U(z^{-1}\mathfrak{g}[z^{-1}])v_0.$$

Let $X = \mathbf{P}^1$. Consider the evaluation module V_λ at the origin $z^{-1} = 0$ and pick an element $s \in \Gamma(\mathrm{Bun}_G(X), \mathcal{L} \otimes \mathcal{V}_\lambda)$. We can split $L\mathfrak{g}_- \cong \mathfrak{g} \oplus z^{-1}\mathfrak{g}[z^{-1}]$. The Lie algebra $z^{-1}\mathfrak{g}[z^{-1}]$ annihilates V_λ, so from the invariance of s under $z^{-1}\mathfrak{g}[z^{-1}]$ we see that $s \in L(0)^* \otimes V_\lambda$ is uniquely determined by its value $s(v_0) \in V_\lambda$. The invariance of s under $\mathfrak{g}$ is equivalent to the statement that $s(v_0)$ is $\mathfrak{g}$-invariant. But V_λ is irreducible, hence $s \neq 0$ only if $V_\lambda = k$, i.e. $\lambda = 0$.

We see that $\Gamma(\mathrm{Bun}_G(\mathbf{P}^1), \mathcal{L} \otimes \mathcal{V}_\lambda) = 0$ unless $\lambda = 0$. Therefore, the space of tau-functions is

$$\Gamma(\mathrm{Gr}_X(A), \mathcal{L}) \cong L(0)^{A_+} \otimes \Gamma(\mathrm{Bun}_G(\mathbf{P}^1), \mathcal{L}) \cong L(0)^{A_+}.$$

Kac and Peterson computed the dimension of $L(0)^{A_+}$ in terms of the so-called *defect* of the monodromy of the Heisenberg A. Let us briefly recall what it is.

Let $Q \subset \mathfrak{h}^*$ be the root lattice (where $\mathfrak{h}$ is a $\mathbf{Q}$-vector space). We normalize the inner product, so that the length of every root is $\sqrt{2}$. Then the inner product is even on Q. For an element $w \in W$ define the set $M_w \subset \widehat{\mathfrak{g}}^*$ to consist of those elements $\alpha \in \widehat{\mathfrak{g}}^*$, such that $\alpha - w(\alpha)$ is in Q.

We define a bilinear form

$$\psi : M_w \times M_w \to \mathbf{Q}/\mathbf{Z}$$

by

$$\psi(\alpha, \beta) = (\alpha, \beta - w(\beta)).$$

The form ψ is antisymmetric. To show that it is enough to show that $\psi(\alpha, \alpha) = 0$. Indeed,

$$||\alpha - w(\alpha)||^2 = (\alpha, \alpha - w(\alpha)) - (w(\alpha), \alpha - w(\alpha)) = (\alpha, \alpha - w(\alpha)) + (\alpha, \alpha - w(\alpha)).$$

In the last equality we used the fact that the inner product is W-invariant. Since $\alpha \in M_w$, we have $\alpha - w(\alpha) \in Q$ where the inner product is even. Therefore, $(\alpha, \alpha - w(\alpha))$ is an integer and hence $\psi(\alpha, \alpha) = 0$.

Let $M_w' \subset M_w$ be the radical of ψ, i.e. the subgroup of elements α, such that $\psi(\alpha, \beta) = 0$ for all $\beta \in M_w$. The quotient M_w/M_w' is a finite abelian group with a nondegenerate bilinear form, hence its order is a square c_w^2, where c_w is the defect of w.

Proposition 2.1.12 (Kac-Peterson)**.** *The space $L(0)^{A_+}$ of A_+-invariants has dimension equal to c_w, the defect of the monodromy of the Heisenberg A.*

For example, if A is the homogeneous Heisenberg LH, $w = 1$. Therefore, $M_w = \mathfrak{h}^*$ and $\psi = 1$. Hence $c_w = 1$.

For an elliptic element w one can compute the defect using the formula

$$c_w^2 = \frac{\det(1-w)}{\det A},$$

where A is the Cartan matrix. For instance, for w being the Coxeter element, $\det(1-w) = \det A$, so $c_w = 1$.

On the other hand, we have exhibited an elliptic element $w \in W$ for the D_4 root system of order 2. Its eigenvalues must all be -1. Hence

$$\det(1-w) = \prod_i (1-\lambda_i) = 2^4.$$

Therefore, the defect $c_w = 2$. Moreover, one can extract from [Ree11] that for E_8 one has elliptic elements with defect equal to any of

$$1, 2, 3, 4, 5, 6, 8, 9, 16.$$

Finally, in type A defects of all Heisenbergs are $c_w = 1$.

We deduce that the space of tau-functions $\Gamma(\mathrm{Gr}_X(A), \mathcal{L})$ in genus 0 is one-dimensional for the principal and homogeneous Heisenbergs.

2.2 Drinfeld–Sokolov hierarchies

2.2.1 Principal case

In this section we define the phase space of a Drinfeld–Sokolov hierarchy for A being the principal Heisenberg. In this case the phase space has an explicit parametrization in terms of differential operators (for $G = SL_n$) or opers (general G). Phase spaces for more general Heisenbergs will be defined in the next section.

2.2.1.1 Opers

Let us split a simple Lie algebra $\mathfrak{g}$ according to the action of the Cartan subalgebra:

$$\mathfrak{g} \cong \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi_+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in\Phi_+} \mathfrak{g}_{-\alpha} = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{n}_-.$$

Let $\chi : \mathfrak{n} \to k$ be a Lie algebra character. That is, it is a map so that $\chi([\mathfrak{n}, \mathfrak{n}]) = 0$. In particular, it induces a map $\chi : \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] \to k$. The action of $\mathfrak{h}$ on the quotient $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ is decomposed into lines $\mathfrak{g}_\alpha$ parametrized by *simple* roots $\alpha \in \Delta$.

Definition. A character $\chi : \mathfrak{n} \to k$ is *nondegenerate* if $\chi : \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] \to k$ is nonzero on each line $\mathfrak{g}_\alpha$ for $\alpha \in \Delta$.

Now suppose we have a G-invariant identification $\mathfrak{g} \cong \mathfrak{g}^*$ given by a bilinear symmetric form $(-, -)$ on $\mathfrak{g}$. Then we can identify

$$\mathfrak{n}^* \cong \mathfrak{g}/\mathfrak{b}.$$

Let us see which elements of $\mathfrak{g}/\mathfrak{b}$ correspond to characters $\mathfrak{n} \to k$ under the identification $\mathfrak{g} \cong \mathfrak{g}^*$. For $p \in \mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}^*$ to be a character, we must have

$$(p, [x, y]) = 0, \quad x, y \in \mathfrak{n}.$$

Since the pairing is G-invariant, we must have $[x, p] \in \mathfrak{b}$ for all $x \in \mathfrak{n}$. If we identify $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_-$ then p must have components in $\mathfrak{g}_{-\alpha}$ only if α is a simple root. Similarly, the character corresponding to p is nondegenerate if elements in $\mathfrak{g}_{-\alpha}$ are nonzero for all simple roots $\alpha \in \Delta$.

Proposition 2.2.1. *Under the identification $\mathfrak{n}^* \cong \mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_-$, nondegenerate characters correspond to principal nilpotent elements.*

Given a principal nilpotent element $p \in \mathfrak{g}/\mathfrak{b}$, let $\mathbf{O} \subset \mathfrak{g}/\mathfrak{b}$ be its B-orbit.

Remark. The orbit $\mathbf{O}$ is independent of the choice of the principal nilpotent element. For instance, for $\mathfrak{sl}_n$ it consists of matrices of the form

$$\begin{pmatrix} * & * & * \\ + & * & * \\ 0 & + & * \end{pmatrix},$$

where $*$ is an arbitrary entry and $+$ is any nonzero entry.

Definition. An *oper* on a curve X is a G-bundle P_G on X with a B-reduction P_B and a connection ∇ on P_G, such that P_B has relative position $\mathbf{O}$ with respect to ∇.

We denote by $\mathrm{Op}_G(X)$ the moduli space of G-opers on X.

2.2.1.2 Poisson structure

The space of opers on the formal disk $X = D^\times$ has a Poisson structure, which can be explicitly described using Poisson reduction.

Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra corresponding to $\mathfrak{g}$ and $\widehat{\mathfrak{g}}^*$ its dual space. We denote by $\widehat{\mathfrak{g}}^*_\lambda$ the hyperplane of functionals $\widehat{\mathfrak{g}} \to k$ whose value on the central charge c is λ.

Lemma 2.2.2. *The space $\widehat{\mathfrak{g}}^*_\lambda$ can be identified with the space of λ-connections on the trivial bundle in a way compatible with the Poisson structures and the LG-action.*

The moment map for the coadjoint action of LG on $\widehat{\mathfrak{g}}^*_\lambda$ is given by the composite

$$\mu : \widehat{\mathfrak{g}}^*_\lambda \hookrightarrow \widehat{\mathfrak{g}}^* \twoheadrightarrow L\mathfrak{g}^*.$$

We will consider the restriction of the Hamiltonian LG-action to an LN-action with the moment map $\mu_N : \widehat{\mathfrak{g}}^* \to L\mathfrak{g}^* \to L\mathfrak{n}^*$ and similarly for the LB-moment map μ_B.

Consider the Poisson reduction

$$\widehat{\mathfrak{g}}^*_1 //_\chi LN := \mu_N^{-1}(\chi)/LN.$$

We have an isomorphism $\mu_N^{-1}(\chi)/LN \cong \mu_B^{-1}(\mathbf{O})/LB$.

The space $\mu_B^{-1}(\mathbf{O})$ parametrizes the following data:

- A G-bundle $P \to D^\times$ with a connection ∇
- A trivial B-reduction $P_B \to D^\times$ in relative position $\mathbf{O}$ with respect to ∇.

Modding out by LB corresponds to forgetting the trivialization of the B-reduction, so we reach the following conclusion:

Theorem 2.2.3. *The Poisson reduction $\widehat{\mathfrak{g}}^*_1 //_\chi LN$ is isomorphic to the space of G-opers on the formal disk.*

This construction shows that the space of G-opers has a natural Poisson structure.

2.2.1.3 The case $G = SL_n$

When $G = GL_n$, we can reformulate the definition of opers using vector bundles:

Definition. A GL_n-oper on a curve X is a rank n vector bundle $E \to X$ with a connection ∇ and a complete flag $0 \subset E_1 \subset E_2 \subset ... \subset E_n = E$, satisfying

1. $\nabla(E_i) \subset E_{i+1} \otimes \Omega^1$
2. $\nabla : E_i/E_{i-1} \to E_{i+1}/E_i \otimes \Omega^1$ is an isomorphism.

Now let's describe explicitly SL_n-opers. Consider a second order differential operator $L = \partial_t^2 + u(t)$. This is a self-adjoint operator with respect to the inner product

$$(f, g) = \int dt f(t)g(t)$$

with the principal symbol 1.

The differential equation $L\psi = 0$ can be rewritten in a matrix form as follows:

$$\left(\partial_t + \begin{pmatrix} 0 & u \\ -1 & 0 \end{pmatrix}\right)\begin{pmatrix} \psi' \\ \psi \end{pmatrix} = 0.$$

The connection matrix is upper triangular up to an element

$$p = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

We see that to any such second-order differential operator one can associate a PGL_2-oper.

Given any differential operator $L : \mathcal{L}_1 \to \mathcal{L}_2$ between two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over X, we define the adjoint operator $L^\dagger : \mathcal{L}_2^* \otimes \Omega^1 \to \mathcal{L}_1^* \otimes \Omega^1$ to be the unique such differential operator satisfying

$$\langle Ls_1, s_2 \rangle = \langle s_1, L^\dagger s_2 \rangle, \quad \forall s_1 \in \mathcal{L}_1, s_2 \in \mathcal{L}_2^* \otimes \Omega^1,$$

where the equality is understood as taking place in $\Omega^1/d\mathcal{O}$. One can show that $L^\dagger$ and L have the same order.

Recall that to a differential operator $L : \mathcal{L}_1 \to \mathcal{L}_2$ we can associate its principal symbol $\sigma(L) \in \mathcal{L}_2 \otimes \mathcal{L}_1^* \otimes K^{-n}$. One has the following description of PGL_2-opers [BD05, Section 2.6.1] on a curve.

Theorem 2.2.4. *PGL_2-opers on a curve X are in a 1-1 correspondence with second-order self-adjoint differential operators $L : K^{-1/2} \to K^{3/2}$ with principal symbol 1 for any choice of $K^{1/2}$.*

To describe SL_n-opers, we first need the following definition.

Definition. An n-th order differential operator $L : \mathcal{L} \to \mathcal{L}^* \otimes \Omega^1$ is said to have *subprincipal symbol 0* if $L - (-1)^n L^\dagger$ has order less than $n-1$.

Note, that for $n = 2$ an operator L is self-adjoint iff it has subprincipal symbol 0.

Theorem 2.2.5. *SL_n-opers on a curve X are in a 1-1 correspondence with line bundles $\mathcal{L}$, such that $\mathcal{L}^n \cong K^{-n(n-1)/2}$ and n-th order differential operators $L : \mathcal{L} \to \mathcal{L} \otimes K^n$ with principal symbol 1 and subprincipal symbol 0.*

Note, that the only difference between SL_2-opers and PGL_2-opers is that in the SL_2 case the choice of the square root $K^{1/2}$ is an additional datum.

For other matrix groups opers can also be identified with differential operators of a certain form. Thus, the notion of a G-oper is a natural generalization of differential operators to more general groups.

2.2.1.4 Affine opers

Just like opers generalize differential equations of the form $L\psi = 0$. We will also be interested in the eigenvalue problems of the form $(L - z^{-1})\psi = 0$. In that case a generalization is known as an affine oper. The reader is referred to [BZF99] for an introduction and proofs of the theorems.

Recall that $LG^+ \subset LG_+$ is the Iwahori subgroup. We also denote by

$$LG_- = \mathrm{Map}(k[z^{-1}], G)$$

the negative part of the loop group. Given a principal nilpotent element $p \in \mathfrak{n}_-$ we let $p_{-1} = p + f_0$, a regular semisimple element. Its LG^+-orbit in $L\mathfrak{g}/L\mathfrak{g}^+$ is denoted by $\mathbf{O}_{aff}$. Note, that $\mathbf{O}_{aff}$ is no longer $\mathbf{G}_m$-invariant, so it makes no sense to require relative position of a reduction with respect to a connection.

Definition. An *affine oper* on the formal disk D is an LG-bundle $P_{LG} \to X$ together with a connection ∇, a flat LG_--reduction P_{LG_-} and an LG^+-reduction P_{LG^+} in relative position $\mathbf{O}_{aff}$ with respect to $\nabla_{\partial/\partial t}$.

An LG-bundle on D with a reduction to LG^+ and LG_- is the same as a G-bundle on $\mathbf{P}^1 \times D$ with a reduction to the Borel subgroup $B \subset G$ along $\infty \times D$. Then an affine oper is a connection along the D direction with a certain asymptotic condition near $\infty \times D$.

Let $\mathrm{AOp}_G(D)$ be the moduli space of affine G-opers. We say that an affine oper is *generic* if the corresponding G-bundle on $\mathbf{P}^1 \times D$ is trivializable.

We have the following theorem relating generic affine opers and ordinary opers [BZF99, Proposition 7.2.5].

Theorem 2.2.6. *There is a natural inclusion* $\mathrm{Op}_G(D) \subset \mathrm{AOp}_G(D)$*, which is an isomorphism onto the subspace of generic affine opers.*

2.2.1.5 Tautological relative position

Recall that the centralizer of p_{-1} in LG is the so-called principal Heisenberg A. Let $A_{-1} \subset A$ be the formal group integrating the one-dimensional Lie algebra spanned by p_{-1}. A_+, the positive part of A, is defined as $LG_+ \cap A$.

Let X be a curve with a smooth point $\infty \in X$. Let D_∞ be the completion of X along ∞ and $X^o = X - \infty$ the complement. Denote by $\mathcal{M}_G(X, p_{-1})$ the space parametrizing the following data:

- A G-bundle $P \to X \times A_{-1}$.
- A reduction P_{A_+} of the LG_+-bundle $P|_{D_\infty \times A_{-1}} \to A_{-1}$ to A_+.
- A relative connection ∇ on P along the A_{-1} factor regular away from $\infty \times A_{-1}$. We require that the reduction P_{A_+} has a *tautological relative position* with respect to ∇. That is, for an invariant vector field $v \in \mathfrak{a} \subset T_{A_{-1}}$ the reduction P_{A_+} has a relative position $v \in \mathfrak{a}/\mathfrak{a}_+ \subset L\mathfrak{g}/\mathfrak{a}_+$ with respect to ∇_v.

One can induce an A_+-bundle to an $LG^+ \supset A_+$-bundle; one can check that it gives a map $\mathcal{M}_G(\mathbf{P}^1, p_{-1}) \to \mathrm{AOp}_G(A_{-1})$ [BZF99, Proposition 7.3.10].

Theorem 2.2.7. *The map* $\mathcal{M}_G(\mathbf{P}^1, p_{-1}) \to \mathrm{AOp}_G(A_{-1})$ *is an isomorphism.*

2.2.1.6 Lie algebra equivariance

Consider two spaces X and Y with an action of a Lie algebra $\mathfrak{g}$. That is, we have morphisms of Lie algebras $a_X : \mathfrak{g} \to \Gamma(X, T_X)$ and $a_Y : \mathfrak{g} \to \Gamma(Y, T_Y)$.

Definition. The datum for a morphism $f : X \to Y$ to be $\mathfrak{g}$-*equivariant* is a null-homotopy of the morphism $\mathfrak{g} \to \Gamma(X, f^* T_Y)$ given by $v \mapsto a_Y(v) - f_* a_X(v)$.

For instance, when X and Y are schemes, the tangent complex is concentrated in non-negative degrees, so in this case equivariance is a condition on a morphism. We will denote by $\mathrm{Map}_{\mathfrak{g}}(X, Y)$ the space of $\mathfrak{g}$-equivariant morphisms from X to Y.

Recall the definition of the Grassmannian $\mathrm{Gr}_X(A)$ parametrizing G-bundles on X with a reduction to A_+ near the marked point. The tangent complex to $\mathrm{Gr}_X(A)$ is given by

$$T_{\mathrm{Gr}_X(A)} \cong T_{LG} \oplus (\mathfrak{a}_+ \oplus L\mathfrak{g}_-) \otimes \mathcal{O}_{LG}[1]$$

with a differential given by the difference of inclusions $\mathfrak{a}_+ \subset L\mathfrak{g}$ and $L\mathfrak{g}_- \subset L\mathfrak{g}$.

Let $\mathfrak{a}_{-1}$ be the Lie algebra spanned by p_{-1} and A_{-1} the corresponding formal group. We would like to describe the stack $\mathrm{Map}_{\mathfrak{a}_{-1}}(A_{-1}, \mathrm{Gr}_X(A))$.

The stack $\mathrm{Map}_{\mathfrak{a}_{-1}}(A_{-1}, \mathrm{Gr}_X(A))$ parametrizes LG-bundles $P_{LG} \to A_{-1}$ together with reductions $P_{LG_-}, P_{A_+} \to A_{-1}$ to LG_- and A_+. Given a map $f \in \mathrm{Map}(A_{-1}, \mathrm{Gr}_X(A))$, the pushforward $f_* : T_{A_{-1}} \to T_{\mathrm{Gr}_X(A)}$ is given by the difference of the Atiyah maps for LG_- and A_+. The datum of $\mathfrak{a}_{-1}$-equivariance is a collection of two elements $h_{LG_-} \in \Gamma(A_{-1}, \mathrm{ad}\, P_{LG_-})$ and $h_{A_+} \in \Gamma(A_{-1}, \mathrm{ad}\, P_{A_+})$ satisfying the following relations taking place in $\mathrm{ad}\, P_{LG}$, $\mathrm{ad}\, P_{LG_-}$

and ad P_{A_+} respectively:

$$p_{-1} = h_{A_+} - h_{LG_-}$$
$$\mathrm{at}_{A_+}(p_{-1}) = dh_{A_+}$$
$$\mathrm{at}_{LG_-}(p_{-1}) = dh_{LG_-}.$$

Here the element p_{-1} is thought of as a section of ad P_{LG} using the embedding

$$p_{-1} \times_{A_+} P_{A_+} \subset \mathfrak{a} \times_{A_+} P_{A_+} \subset \mathrm{ad}\, P_{LG}.$$

We can rewrite these relations as

$$\mathrm{at}_{LG_-}(p_{-1}) = dh_{LG_-}$$
$$\mathrm{at}_{A_+}(p_{-1}) = d(h_{LG_-} + p_{-1}).$$

Using Proposition 2.1.1 we see that this is the same as a connection ∇ on P_{LG}, such that the reduction P_{LG_-} is flat, while P_{A_+} is in tautological relative position.

We summarize this discussion in the following theorem.

Theorem 2.2.8. *We have an isomorphism of stacks* $\mathrm{Map}_{\mathfrak{a}_{-1}}(A_{-1}, \mathrm{Gr}_X(A)) \cong \mathcal{M}_G(X, p_{-1})$.

Similarly to the definition of $\mathfrak{a}_{-1}$-equivariance, we have a definition of A_{-1}-equivariance of morphisms. Since A_{-1} is a formal group, we also have

$$\mathrm{Map}_{\mathfrak{a}_{-1}}(A_{-1}, \mathrm{Gr}_X(A)) \cong \mathrm{Map}_{A_{-1}}(A_{-1}, \mathrm{Gr}_X(A)) \cong \mathrm{Gr}_X(A).$$

We conclude:

Corollary 2.2.9. *We have an isomorphism of stacks $\mathcal{M}_G(X, p_{-1}) \cong \mathrm{Gr}_X(A)$.*

An alternative proof of the corollary are given in [BZF99, Proposition 2.3.10].

The considerations of this subsection can be summarized in the following diagram:

$$
\begin{array}{ccccc}
 & & \mathcal{M}_G(\mathbf{P}^1, p_{-1}) & & \\
 & \overset{\sim}{\swarrow} & & \overset{\sim}{\searrow} & \\
\mathrm{Op}_G(A_{-1}) \hookrightarrow \mathrm{AOp}_G(A_{-1}) & & & & \mathrm{Gr}_{\mathbf{P}^1}(A)
\end{array}
$$

2.2.2 General phase space

The input data for a generalized Drinfeld–Sokolov hierarchy consists of:

- A projective curve X with a smooth point $x \in X$.
- A reductive group G.
- A choice of a Heisenberg subgroup $A \subset LG$.

We define the phase space of the Drinfeld–Sokolov hierarchy to be $\mathcal{M}_G(X, p_{-1}) \cong \mathrm{Gr}_X(A)$, where A it the centralizer of p_{-1}. For $X = \mathbf{P}^1$ we have an open dense subspace $\mathrm{Gr}_X^0(A) \subset \mathrm{Gr}_X(A)$ consisting of trivializable G-bundles. For semisimple groups one can present the big cell as

$$\mathrm{Gr}_X^0(A) \cong G \backslash LG_+ / A_+.$$

The space $\mathrm{Map}(A_{-1}, \mathrm{Gr}_X(A))$ carries an action of $\mathrm{Map}(A_{-1}, A)$ given by the right A-action on $\mathrm{Gr}_X(A)$. This action preserves the $\mathfrak{a}_{-1}$-equivariance data if the map $f \in \mathrm{Map}(A_{-1}, A)$ is constant, i.e. we simply have an action of A on the phase space. The corresponding vector fields constitute what's known as the *Drinfeld–Sokolov flows*. Clearly, the action factors through the quotient $A \to A/A_+$.

We can also define a version of the Drinfeld–Sokolov phase space for varying curves. The Drinfeld–Sokolov Grassmannian $\widehat{\mathrm{Gr}}_g$ is the moduli stack whose S-points parametrize the following data:

- A family of genus g smooth projective curves $X \to S$ together with a marked point $\infty : S \to X$. We denote by $X_0 = X \backslash \infty$ the affine part and X_∞ the spectrum of the completed local ring at infinity.
- A local coordinate $z : X_\infty \xrightarrow{\sim} D \times S$.
- A G-torsor $P \to X$ together with a reduction of $P|_{X_\infty}$ to an $z^*\underline{A}_+$-torsor $E \to X_\infty$.
- A trivialization $E \xrightarrow{\sim} z^*\underline{A}_+$.

To not obscure the notation, the local coordinate z will be implicit from now on.

Note, that a trivialization of E induces a trivialization of $P|_{X_\infty}$, hence the Grassmannian $\widehat{\mathrm{Gr}}_g$ is independent of the choice of a Heisenberg.

Let $\mathrm{Gr}_g(A)$ be the moduli space obtained from $\widehat{\mathrm{Gr}}_g$ by forgetting the trivialization of E. We have the maps

$$\widehat{\mathrm{Gr}}_g \to \mathrm{Gr}_g(A) \to \widehat{\mathcal{M}}_{g,1},$$

where $\widehat{\mathcal{M}}_{g,1}$ is the moduli space of genus g curves with a marked point together with a choice of a local coordinate. We denote by $\widehat{\mathrm{Gr}}_X$ and $\mathrm{Gr}_X(A)$ the fibers of $\widehat{\mathrm{Gr}}_g \to \widehat{\mathcal{M}}_{g,1}$ and $\mathrm{Gr}_g(A) \to \widehat{\mathcal{M}}_{g,1}$ respectively over a curve $X \in \widehat{\mathcal{M}}_{g,1}$.

For G semisimple one can explicitly realize $\widehat{\mathrm{Gr}}_X$ and $\mathrm{Gr}_X(A)$ as the spaces of cosets

$$\widehat{\mathrm{Gr}}_X = LG_-\backslash LG, \quad \mathrm{Gr}_X(A) = LG_-\backslash LG/A_+.$$

Remark. If two Heisenbergs A and A' are LG_+-conjugate, the corresponding Drinfeld–Sokolov phase spaces are canonically isomorphic and the isomorphism intertwines the flows. However, as noted in [BZF99], there are continuous families of LG_+-conjugacy classes of Heisenbergs in the same LG-conjugacy class.

2.2.2.1 Examples

All known examples of Drinfeld–Sokolov hierarchies start with a genus 0 curve, so let $X = \mathbf{P}^1$. In this case we have an open dense subset $\mathrm{Gr}^0_X(A) \subset \mathrm{Gr}_X(A)$ known as the big cell consisting of trivializable G-torsors P.

We have the following standard choices of Heisenbergs:

1. $A = LH$ is the homogeneous Heisenberg. For $G = SL_2$ one gets the non-linear Schrödinger hierarchy.

2. A is the principal Heisenberg. Then the big cell $\mathrm{Gr}^0_X(A)$ parametrizes G-opers on the disk and we recover the original description of Drinfeld and Sokolov. For example, the case $G = SL_2$ corresponds to the KdV hierarchy and $G = SL_n$ to its generalizations known as Gelfand–Dikii or nKdV hierarchies.

The reader is referred to [Kr01] for explicit coordinates on Drinfeld–Sokolov phase spaces.

2.2.3 Phase space for $G = GL_n$

Let us describe explicitly the phase space for the following triple:

- The curve $X = \mathbf{P}^1$
- The group $G = GL_n$
- The Heisenberg A is the principal one

Let us start with an explicit description of the Heisenberg A. Recall that it is defined to be the centralizer of

$$p_{-1} = \sum_{i=0}^{n} f_i.$$

One can show that the Heisenberg $\mathfrak{a}$ is spanned by p_{-1}^k and p_1^k, where

$$p_1 = \sum_{i=0}^{n} e_i.$$

Let's prove it for $n = 2$. We have

$$p_{-1} = \begin{pmatrix} 0 & z^{-1} \\ 1 & 0 \end{pmatrix}.$$

The commutator between $x \in L\mathfrak{gl}_2$ and p_{-1} is

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & z^{-1} \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} b - cz^{-1} & (a-d)z^{-1} \\ d - a & cz^{-1} - b \end{pmatrix}.$$

Therefore, we must have

$$a = d, \quad b = cz^{-1}.$$

So, the centralizer of p_{-1} consists of elements $a + cp_{-1}$, where $a, b \in k((z))$. Note, that we have $p_{-1}^2 = z^{-1}$, hence we obtain the claim.

For SL_n the principal Heisenberg $\mathfrak{a}$ is spanned by p_{-1}^k and p_1^k for k not divisible by n.

Let's identify the Heisenberg as $\mathfrak{a} \cong k((t))$ and $\mathfrak{a}_+ \cong k[[t]]$, where $t^n = z$. The trivial rank n bundle on the disk $k[[z]]^{\oplus n}$ has an action of the Heisenberg if we identify

$$k[[z]]^{\oplus n} \cong k[[t]].$$

Recall that the Grassmannian $\widehat{\mathrm{Gr}}_X$ parametrizes rank n bundles on $X = \mathbf{P}^1$ together with a trivialization on the disk D. Therefore, it can be parametrized by a $k[z^{-1}]$-module W together with an identification $W \otimes_{k[z^{-1}]} k((z)) \cong k((z))^{\oplus n}$.

The cohomology of the bundle can be computed by the complex

$$0 \to W \oplus k[[z]]^{\oplus n} \to k((z))^{\oplus n} \to 0,$$

which has to be perfect. In other words, the map

$$W \subset k((z))^{\oplus n} \twoheadrightarrow k((z))^{\oplus n}/k[[z]]^{\oplus n}$$

must be Fredholm. Its index coincides with the Euler characteristic of the corresponding bundle.

To recapitulate, the Grassmannian $\widehat{\mathrm{Gr}}_X$ parametrizes subspaces $W \subset k((t))$, such that

1. $t^{-n} W \subset W$.

2. The composition

$$W \subset k((t)) \twoheadrightarrow k((t))/k[[t]]$$

is Fredholm.

Note, that the second condition implies the isomorphism $W \otimes_{k[z^{-1}]} k((z)) \cong k((z))^{\oplus n}$.

It is a standard computation that the tangent bundle $T_W\widehat{Gr}_X$ is isomorphic to $k((t))/W$. An action of $f(t) \in \mathfrak{a} \cong k((t))$ on the Grassmannian, i.e. the map $\mathfrak{a} \to T_W\widehat{\mathrm{Gr}}_X$, is given by the composite

$$f(t)W \subset k((t)) \twoheadrightarrow k((t))/W.$$

The connected components of $\mathrm{Gr}_X(A)$ are parametrized by the degree of the bundle; alternatively, one can parametrize them by the Euler characteristic due to the Riemann-Roch theorem:

$$\chi(V) = \deg(V) + n(1-g) = \deg(V) + n.$$

The component of the GL_n Grassmannian that corresponds to the SL_n Grassmannian has $\deg(V) = 0$. This is equivalent to saying that the index of the Fredholm operator

$$W \subset k((t)) \twoheadrightarrow k((t))/k[[t]]$$

is n.

The big cell $\mathrm{Gr}^0_X(A) \subset \mathrm{Gr}_X(A)$ is an open dense subset of the SL_n Grassmannian which consists of trivializable bundles. In terms of subspaces, this means the kernel K of $W \to k((z))^{\oplus n}/k[[z]]^{\oplus n}$ has dimension n and $k((z))K = k((z))^{\oplus n}$. Here is an easier way to describe the big cell.

The big cell $\widehat{\mathrm{Gr}}_X^0$ consists of subspaces $W \subset k((t))$, such that

$$W \hookrightarrow k((t)) \twoheadrightarrow \frac{k((t))}{t^n k[[t]]}$$

is an isomorphism.

Proposition 2.2.10. *For $G = SL_n$ a point $W \in \mathrm{Gr}_X(A)$ is fixed by the action of $p_{-1} \in \mathfrak{a}$ only if W is in the big cell.*

Proof. Suppose a point $W \in \mathrm{Gr}_X(A)$ is fixed by p_{-1}. Then there is a power series $f(z) \in k[[z]]$, such that

$$(z^{-1} + f(z))W \subset W.$$

Let's denote $p(z) = z^{-1} + f(z)$ and we denote the map $W \to k((t))/t^n k[[t]]$ by π. GIven an element $f(z) \in k((z))$ of the form $f(z) = a_{-n}z^{-n} + a_{-n+1}z^{-n+1} + ...$ with a_{-n} we say that f has order n.

Since the kernel of π is finite-dimensional, there is an element w of the maximal order. Let W_0 be the span of $p^n w$ for all n. Suppose $W_0 \subsetneq W$ and consider an element $v \in W \backslash W_0$. Then $p^n v$ do not lie in W_0 and have the same degree as some elements in W_0. We obtain that the cokernel of $W \to k((t))/k[[t]]$ is infinite in this case. Hence, $W_0 = W$.

If $\deg w < n$, the kernel of π is trivial, so the index of π is negative, which contradicts the fact that W corresponds to an SL_n-bundle. Therefore, $\deg w \geq n$. Since the index of π is zero we conclude that π is an isomorphism, i.e. W is in the big cell. $\square$

2.2.4 Line bundles

2.2.4.1

Let $\langle , \rangle$ be a G-invariant bilinear form on $\mathfrak{g}$ taking even values on the coroots. It defines a central extension

$$1 \to \mathbf{G}_m \to \widehat{LG} \to LG \to 1.$$

On the level of Lie algebras, one has $\widehat{L\mathfrak{g}} \cong L\mathfrak{g} \oplus k$ as vector spaces together with the Lie bracket

$$[(a, \alpha), (b, \beta)] = [a, b] + \mathrm{Res}_{z=0}\langle a, db\rangle.$$

In particular, we see that the central extension is trivial when restricted to LG_-, i.e. we have an embedding $LG_- \subset \widehat{LG}$. Therefore, we get a line bundle

$$LG_-\backslash\widehat{LG} \to LG_-\backslash LG.$$

Since the central extension restricted to LG_+ is also split, thus defined line bundle is LG_+-equivariant, i.e. it descends to a line bundle $\mathcal{L}$ on the moduli space of bundles $\mathrm{Bun}_G(X) \cong LG_-\backslash LG/LG_+$. Similarly, it descends to the abelianized Grassmannian $\mathrm{Gr}_X(A)$.

Let us also explain how to extend this line bundle to $\mathrm{Gr}_g(A)$. The following construction can be found in [BK, Section 5]. The bilinear form $\langle , \rangle$ defines an element $\omega_K \in H^2(BG, \mathcal{K}_2)$, where $\mathcal{K}_2$ is the sheafification of the K-theory sheaf. The line bundle $\mathcal{L}$ on $\mathrm{Bun}_G(X)$ is obtained by an integral transform of ω_K along the correspondence

$$\begin{array}{ccc} & \mathrm{Bun}_G(X) \times X & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle \mathrm{ev}} \\ \mathrm{Bun}_G(X) & & BG \end{array}$$

The pullback $\mathrm{ev}^*\omega_K$ defines a $\mathcal{K}_2$-gerbe on $\mathrm{Bun}_G(X) \times X$, which can then be pushed forward to a $\mathbf{G}_m$-torsor on $\mathrm{Bun}_G(X)$. The corresponding pushforward

$$H^2(\mathrm{Bun}_G(X) \times X, \mathcal{K}_2) \to H^1(\mathrm{Bun}_G(X), \mathcal{O}^*)$$

is the pushforward in the Chow groups. Such a pushforward can also be done in families, which gives the line bundle $\mathcal{L}$ on $\mathrm{Gr}_g(A)$.

Remark. In types A and C the line bundle $\mathcal{L}$ can be obtained as the determinant line bundle of a fundamental representation.

2.2.5 Lie algebra actions

Recall the definition of the Atiyah bundle $\mathcal{A}_E$ of an $\underline{A}$-torsor $E \to D^\times$. The Atiyah sequence

$$0 \to \mathrm{ad}\, E \to \mathcal{A}_E \to T_D \to 0$$

is non-canonically split, so we have an identification $\mathcal{A} := \Gamma(D^\times, \mathcal{A}_E) \cong \Gamma(D^\times, T_{D^\times}) \ltimes \mathfrak{a}$.

There is an action of the Lie algebra $\mathcal{A}$ on the Grassmannian $\widehat{\mathrm{Gr}}_g$ given by deforming both the curve X and the bundle E. Note, that the action of $\mathcal{A}$ does not descend to the Grassmannian $\mathrm{Gr}_g(A)$ since $\mathcal{A}$ and $\mathfrak{a}_+$ do not commute.

The subalgebra $\mathcal{A}^0 = \Gamma(D^\times, \mathrm{ad}\, E) \cong \mathfrak{a}$ of vertical vector fields acts along the fibers of $\widehat{\mathrm{Gr}}_g \to \widehat{\mathcal{M}}_{g,1}$, i.e. it preserves $\widehat{\mathrm{Gr}}_X$. Moreover, the action descends to the abelianized Grassmannian $\mathrm{Gr}_X(A)$.

This action of $\mathcal{A}^0$ generates the Drinfeld–Sokolov flows, while $\mathcal{A}$ represents the so-called Orlov-Shulman extended symmetries.

As line bundles $\mathcal{L}$ on $\widehat{\mathrm{Gr}}_g$ correspond to central extensions of LG, the action of $\mathcal{A}^0 \cong \mathfrak{a}$ lifts canonically to the line bundle by restricting the central extension to the Heisenberg. Similarly, we have a central extension $\widehat{\mathcal{A}}$, which lifts the action of $\mathcal{A}$ to the line bundle.

2.2.6 Tau-function

The Lie algebra $\mathcal{A}$ exponentiates to an ind-group $K = \mathrm{Aut}(D^\times) \ltimes A$. Similarly, there is a central extension $\widehat{K}$ which lifts the action of K to the line bundle $\mathcal{L}$ on $\widehat{\mathrm{Gr}}_g$.

Consider the action and projection maps

$$\widehat{\mathrm{Gr}}_g \times \widehat{K} \underset{p}{\overset{a}{\rightrightarrows}} \widehat{\mathrm{Gr}}_g.$$

Then the action of $\widehat{K}$ on $\mathcal{L}$ can be expressed as the data of an isomorphism $p^*\mathcal{L} \cong a^*\mathcal{L}$.

Fix a nonzero section $\sigma \in \Gamma(\mathrm{Gr}_g(A), \mathcal{L}) \cong \Gamma(\widehat{\mathrm{Gr}}_g, \mathcal{L})^{A_+}$. Then one defines a rational function on $\widehat{\mathrm{Gr}}_g \times \widehat{K}$ called the *extended tau-function* $\overline{\tau}$ as

$$\overline{\tau} = \frac{p^*\sigma}{a^*\sigma}.$$

Explicitly, one has

$$\overline{\tau}(P, g) = \frac{\sigma(g^{-1}P)}{g^{-1}\sigma(P)}.$$

Thus, it measures the failure of the section σ to be $\widehat{K}$-invariant. Since σ is A_+-invariant, the extended tau-function is a function on $\widehat{\mathrm{Gr}}_g \times \widehat{K}/A_+$.

Fixing a point $P \in \widehat{\mathrm{Gr}}_g$ we define the *tau-function* of P as the restriction of the extended tau-function $\overline{\tau}$ to the slice $\{P\} \times \widehat{A}/A_+$. It is a rational function on $\widehat{A}/A_+$.

Although A is abelian, its central extension $\widehat{A}$ is not. Its failure is measured by the commutator pairing $c : A \times A \to \mathbf{G}_m$ given by arbitrarily lifting the elements of A to the central extension and computing the commutator.

For example, suppose $a \in A_+$. Then

$$\tau_{aP}(g) = \frac{\sigma(g^{-1}aP)}{g^{-1}\sigma(aP)} = \frac{a\sigma(g^{-1}P)}{g^{-1}a\sigma(P)} = c(a, g^{-1})\frac{a\sigma(g^{-1}P)}{ag^{-1}\sigma(P)} = c(a, g^{-1})\tau_P(g).$$

In particular, the tau-function is not well-defined as a function on $\mathrm{Gr}_g(A)$ and is instead a section of an associated line bundle to $\widehat{\mathrm{Gr}}_g \to \mathrm{Gr}_g(A)$.

There is an equivalent, more algebraic, way to write down the tau-function. Fix a point $P \in \widehat{\mathrm{Gr}}_X$ and pull back the line bundle $\mathcal{L}$ to the orbit $\widehat{A}$ of the Drinfeld–Sokolov flows. σ defines a section of $\mathcal{L}$; in particular, it gives a connection $d\log\sigma$ with regular singularities at the zeros of σ.

We define a meromorphic 1-form $d\log\tau$ on $\widehat{A}$ by

$$d\log\tau[a] = d\log\sigma[a] - v_a,$$

where $a \in \widehat{\mathfrak{a}}$ and $v_a \in \mathcal{A}_{\mathcal{L}}$ is the vector field exhibiting the action of $\widehat{A}$ on $\mathcal{L}$.

The connection $d\log\sigma$ splits the sequence

$$0 \to \mathcal{O} \to \mathcal{A}_{\mathcal{L}} \to T_{\widehat{A}} \to 0$$

whenever σ does not vanish, which happens generically when X has genus 0. Then $d\log\tau[a]$ is simply the image of $-v_a$ under the splitting $\mathcal{A}_{\mathcal{L}} \to \mathcal{O}$. This is the definition of tau-functions, which can be found in [Wu12].

2.3 Dispersionless limit

2.3.1 Dispersion in the KdV equation

Consider the KdV equation and replace all times t_n by λt_n. For instance, the first KdV equation becomes

$$4u_t = \lambda^2 u_{xxx} + 6uu_x.$$

Let us look for solutions of the form $u(x,t) = f(x+vt)$ with $s = x+vt$. Then the KdV equation becomes

$$4vf' = \lambda^2 f''' + 6ff'.$$

Integrating with respect to s, we get

$$4vf = \lambda^2 f'' + 3f^2 + c_1.$$

Multiplying this equation by f' and integrating again, we get

$$2vf^2 = \frac{\lambda^2 f'^2}{2} + f^3 + c_1 s + c_2. \tag{2.1}$$

Assuming $\lambda \neq 0$ and $c_1 = 0$ we get the Weierstrass differential equation, whose solution is the Weierstrass $\wp$-function. For instance, for $c_2 = 0$ the solution is

$$u(x,t) = \frac{2v}{\cosh^2\left(\frac{\sqrt{v}}{\lambda}(x+vt)\right)}.$$

On the other hand, in the limit $\lambda \to 0$, known as the dispersionless limit, the equation (2.1) becomes *algebraic*. In other words, in the dispersionless limit the integrable system greatly simplifies.

2.3.2 Dispersionless Drinfeld–Sokolov hierarchies

One can obtain the dependence on λ of the KdV equation by replacing the Sturm–Liouville operator $L = \partial_t^2 + u(t)$ by $L = \lambda^2\partial_t^2 + u(t)$. In the interpretation of Sturm–Liouville operators as opers, the parameter λ appears if we instead consider λ-opers: these are G-bundles with a λ-connection ∇ and a B-reduction in a certain relative position with respect to ∇. Let us denote the space of λ G-opers by $\mathrm{Op}_G^\lambda(D)$. One similarly introduces the space of affine λ G-opers by replacing connections by λ-connections; this space is denoted by $\mathrm{AOp}_G^\lambda(D)$.

Finally, the space $\mathcal{M}_G(X, p_{-1}) = \mathrm{Map}_{\mathfrak{a}_{-1}}(A_{-1}, \mathrm{Gr}_X(A))$ also admits a λ-version.

Definition. The datum for a morphism $f : X \to Y$ to be λ $\mathfrak{g}$-*equivariant* is a null-homotopy of the morphism $\mathfrak{g} \to \Gamma(X, f^*T_Y)$ given by $v \mapsto a_Y(v) - \lambda f_* a_X(v)$.

We denote by $\mathrm{Map}_{\mathfrak{g}}^\lambda(X, Y)$ the space of λ $\mathfrak{g}$-equivariant morphisms $X \to Y$. Note, that for $\lambda = 0$ we recover the mapping space $\mathrm{Map}(X, Y^{\mathfrak{g}})$ from X to the stack of $\mathfrak{g}$-fixed points in Y. As before, we have the following maps:

$$\mathrm{Op}_G^\lambda(A_{-1}) \subset \mathrm{AOp}_G^\lambda(A_{-1}) \xleftarrow{\sim} \mathcal{M}_G^\lambda(\mathbf{P}^1, p_{-1}).$$

However, $\mathcal{M}_G^\lambda(X, p_{-1})$ is isomorphic to $\mathrm{Gr}_X(A)$ only if $\lambda \neq 0$. In other words, we do not have a spectral description of the integrable hierarchy in the dispersionless limit.

2.3.2.1 Dispersionless phase spaces

In the limit $\lambda \to 0$ the $\mathfrak{a}$-action on $\mathcal{M}_G^\lambda(X, p_{-1})$ becomes trivial. Therefore, we have to consider the first-order jet of flows at $\lambda = 0$ to get an interesting family of flows.

We define the dispersionless Drinfeld–Sokolov phase space $\tilde{\mathcal{M}}^0_G(X, p_{-1})$ to be the moduli space parametrizing the following data:

- An LG-bundle $P_{LG} \to A_{-1}$ together with reductions P_{LG_-} and P_{A_+}
- A Higgs field $\phi \in \Gamma(A_{-1}, \operatorname{ad} P_{LG_-} \otimes \Omega^1)$, such that P_{A_+} has a tautological relative position with respect to ϕ
- A connection ∇ on P_{LG}, such that the reductions P_{LG_-} and P_{A_+} are both flat.

The condition of tautological relative position is understood in the following way: the Higgs field $\phi\Gamma(A_{-1}, \operatorname{ad} P_{LG} \otimes \Omega^1)$ can be projected to $\Gamma(A_{-1}, \operatorname{ad} P_{LG}/\operatorname{ad} P_{A_+} \otimes \Omega^1)$. An element $v \in \mathfrak{a}_{-1}$ gives rise to a vector field $v \in T_{A_{-1}}$. The reduction P_{A_+} has tautological relative position with respect to ϕ if $v \times_{A_+} P_{A_+} \subset \operatorname{ad} P_{LG}/\operatorname{ad} P_{A_+}$ coincides with the image of $\phi(v)$ in $\operatorname{ad} P_{LG}/\operatorname{ad} P_{A_+}$.

Note, that the space $\mathcal{M}^0_G(X, p_{-1})$ parametrizes the same data as $\tilde{\mathcal{M}}^0_G(X, p_{-1})$ except the connection. Thus, we have a map $\tilde{\mathcal{M}}^0_G(X, p_{-1}) \to \mathcal{M}^0_G(X, p_{-1})$ given by forgetting the connection.

2.3.2.2 Zero-curvature representation

The Drinfeld–Sokolov phase space $\operatorname{Gr}_X(A)$ can be presented in many ways as the space of equivariant morphisms: for any subgroup K of A we have

$$\operatorname{Gr}_X(A) \cong \operatorname{Map}_{\mathfrak{k}}(K, \operatorname{Gr}_X(A)).$$

The space $\operatorname{Map}_{\mathfrak{k}}(K, \operatorname{Gr}_X(A))$ parametrizes the following data:

- An LG-bundle $P_{LG} \to K$ together with reductions P_{LG_-} and P_{A_+}
- A flat connection ∇ on P_{LG}, such that P_{LG_-} is a flat reduction while P_{A_+} has tautological relative position with respect to ∇.

Once one constructs such a connection, the action of K on $\mathrm{Gr}_X(A)$ is simply the translation on the base. In the literature this is known as the zero-curvature presentation of an integrable hierarchy.

Motivated by this and the semiclassical definition of the dispersionless phase space, the zero-curvature representation in the dispersionless limit consists of the following data:

- An LG-bundle $P_{LG} \to K$ together with reductions P_{LG_-} and P_{A_+}
- An integrable Higgs field ϕ on P_{LG}, such that P_{LG_-} is a flat reduction while P_{A_+} has tautological relative position with respect to ϕ
- A flat connection ∇ on P_{LG}, such that both P_{LG_-} and P_{A_+} are flat reductions. Moreover, the Higgs field ϕ is flat with respect to ∇.

Using the connection ∇ we can trivialize all bundles. Therefore, the zero-curvature representation consists of writing down a Higgs field $\phi \in \Gamma(K, \Omega^1_K) \otimes \mathrm{ad}\, P_{LG}$ satisfying the following two equations:

$$\phi \wedge \phi = 0$$

$$d\phi = 0.$$

2.3.2.3 Dispersionless opers

Recall that in the case $X = \mathbf{P}^1$ we have the big cell $\mathrm{Gr}^0_X(A) \subset \mathrm{Gr}_X(A)$. Therefore, we can define an open dense subset $\tilde{\mathcal{M}}^{generic}_G(X, p_{-1}) \subset \tilde{\mathcal{M}}^0_G(X, p_{-1})$ whose underlying G-bundles on $\mathbf{P}^1$ are trivializable.

Note, that Proposition 2.2.10 implies that for $G = SL_n$ and A the principal Heisenberg, the inclusion $\tilde{\mathcal{M}}^{generic}_G(X, p_{-1}) \subset \tilde{\mathcal{M}}^0_G(X, p_{-1})$ is an isomorphism.

We have a forgetful map $\tilde{\mathcal{M}}^0_G(X, p_{-1}) \to \mathcal{M}^0_G(X, p_{-1})$. It is clearly surjective since we always have a flat connection on the formal disk. The fiber over a tuple $(P_{LG}, P_{LG_-}, P_{A_+}, \phi)$ consists of Higgs fields $\psi \in \Gamma(A_{-1}, \mathrm{ad}\, P_{LG} \otimes \Omega^1)$, such that both reductions P_{LG_-} and P_{A_+} are flat with respect to ψ. If the triple $(P_{LG}, P_{LG_-}, P_{A_+})$ is classified by a map $f : A_{-1} \to \mathrm{Gr}_X(A)$, then using the description of the tangent complex of $\mathrm{Gr}_X(A)$ we can think of ψ as an element of $H^{-1}(A_{-1}, f^* T_{\mathrm{Gr}_X(A)})$.

For generic elements of the dispersionless phase space the map f maps into the big cell $\mathrm{Gr}^0_X(A) \cong G \backslash LG_+ / A_+$. Therefore, to show that ψ is necessarily trivial we just have to show that the tangent complex of $\mathrm{Gr}^0_X(A)$ has no cohomology in degree -1. Indeed, suppose A has an elliptic monodromy. Then we have to show that none of the LG_+-conjugates of $\mathfrak{a}_+$ intersect $\mathfrak{g}$. Since the monodromy does not change under LG_+-conjugacy, it follows from Proposition 2.1.8 that this is indeed true. Therefore, we conclude that the forgetful map $\tilde{\mathcal{M}}^0_G(X, p_{-1}) \to \mathcal{M}^0_G(X, p_{-1})$ is an isomorphism over the big cell.

For A the principal Heisenberg, we have an identification between the big cell of $\mathcal{M}^0_G(X, p_{-1})$ and the space of dispersionless opers $\mathrm{Op}^0_G(A_{-1})$. Finally, one has the characteristic polynomial map $\chi : \mathrm{Op}^0_G(A_{-1}) \to \mathrm{Map}_K(A_{-1}, \mathfrak{h}/W)$ given by taking the characteristic polynomial of the Higgs field. Here $\mathrm{Map}_K(A_{-1}, \mathfrak{h}/W)$ is the space of maps $A_{-1} \to \mathfrak{h}/W$ twisted by the canonical bundle of A_{-1}: let K^0 be the total space of the canonical bundle of A_{-1} minus the zero section; then $\mathrm{Map}_K(A_{-1}, \mathfrak{h}/W)$ is the space of $\mathbf{G}_m$-equivariant maps $K^0 \to \mathfrak{h}/W$.

Proposition 2.3.1 (Kostant, Beilinson–Drinfeld)**.** *The map*

$$\chi : \mathrm{Op}^0_G(A_{-1}) \to \mathrm{Map}_K(A_{-1}, \mathfrak{h}/W)$$

is an isomorphism.

Corollary 2.3.2. *Assume $X = \mathbf{P}^1$ and A is the principal Heisenberg. Then an open dense subset of the phase space of the dispersionless Drinfeld–Sokolov hierarchy $\tilde{\mathcal{M}}^0_G(\mathbf{P}^1, p_{-1})$ is isomorphic to* $\mathrm{Map}_K(A_{-1}, \mathfrak{h}/W)$*. For $G = SL_n$ this is the full phase space.*

This corollary gives a relation between our definition of the dispersionless phase space and Dubrovin's definition of an integrable hierarchy on the mapping space $\mathrm{Map}(D, F)$ into a Frobenius manifold F. The Frobenius manifold structure on $\mathfrak{h}/W$ is discussed in [Du96, Lecture 4].

2.4 $\mathcal{W}$-constraints

2.4.1 Algebro-geometric solutions

2.4.1.1 Geometric descrpition

We have an action of A/A_+ on $\mathrm{Gr}_X(A)$ on the right. A point $P \in \mathrm{Gr}_X(A)$ is algebro-geometric if its stabilizer is big enough. Let us make it precise.

Proposition 2.4.1. *The stabilizer of the $\mathfrak{a}/\mathfrak{a}_+$-action on $\mathrm{Gr}_X(A)$ is isomorphic to*

$$\mathcal{A}^0_{stab} \cong \mathfrak{a} \cap \Gamma(X_0, \mathrm{ad}\, P).$$

Proof. We have $\mathrm{Gr}_X(A)/(A/A_+) \cong LG_-\backslash LG/A$. Its tangent complex is

$$\Gamma(X_0, \mathrm{ad}\, P) \oplus \mathfrak{a} \to \Gamma(D^\times, \mathrm{ad}\, P).$$

The stabilizer of the $\mathfrak{a}/\mathfrak{a}_+$-action is the cohomology of the tangent complex in degree -1, which is isomorphic to $\mathcal{A}^0_{stab}$. □

$\mathcal{A}^0_{stab}$ has a natural structure of a torsion-free finitely-generated $\mathcal{O}(X_0)$-module. If X_0 is smooth, $\mathcal{A}^0_{stab}$ is in fact projective. So, we can localize $\mathcal{A}^0_{stab}$ to a vector bundle c on X_0.

We have an inclusion

$$\mathcal{A}^0_{stab} \otimes_{\mathcal{O}(X_0)} \mathcal{O}(D^\times) \hookrightarrow \mathfrak{a}$$

of projective $\mathcal{O}(D^\times)$-modules. The module $\mathfrak{a}$ has rank $\mathrm{rk}\, G$, so the rank of $\mathcal{A}^0_{stab}$ is at most $\mathrm{rk}\, G$.

Definition. A point $P \in \mathrm{Gr}_X(A)$ is *algebro-geometric* if the stabilizer $\mathcal{A}^0_{stab}$ has rank $\mathrm{rk}\, G$.

In this case the map $\mathcal{A}^0_{stab} \otimes_{\mathcal{O}(X_0)} \mathcal{O}(D^\times) \to \mathfrak{a}$ is an isomorphism of $\mathcal{O}(D^\times)$-modules. In particular, we can extend c to a vector bundle on X by gluing $\mathrm{ad}\, E$ on D, where E is the A_+-torsor, to c on X_0.

Proposition 2.4.2. *Let $P \in \mathrm{Gr}_X(A)$ be a point. The following conditions are equivalent:*

1. *P is algebro-geometric.*

2. *The map* $\mathcal{A}^0_{stab} \otimes_{\mathcal{O}(X_0)} \mathcal{O}(D^\times) \to \mathfrak{a}$ *is an isomorphism.*

3. *The subbundle* $\mathrm{ad}\, E \subset \mathrm{ad}\, P$ *of regular centralizers on* D *extends to a regular Higgs field on* X*.*

Proof. The equivalence of the first two conditions is clear. Now suppose $c' \subset \mathrm{ad}\, P$ is a subbundle of regular centralizers extending $\mathrm{ad}\, E$ on D. Then

$$\Gamma(X_0, c') \subset \mathfrak{a} \cap \Gamma(X_0, \mathrm{ad}\, P) \cong \mathcal{A}^0_{stab}.$$

In particular,

$$c'|_{X_0} \subset c.$$

Since the rank of c' is $\mathrm{rk}\, G$, the rank of c is at least $\mathrm{rk}\, G$. Hence it it exactly $\mathrm{rk}\, G$, i.e. the point is algebro-geometric. □

We see that algebro-geometric points in $\mathrm{Gr}_X(A)$ can be reconstructed from the following data:

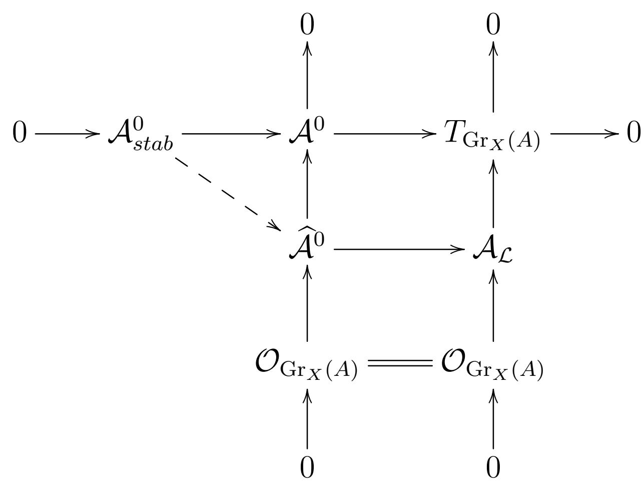
- A G-torsor $P \to X$.
- A regular Higgs field $c \subset \mathrm{ad}\, P$.
- An isomorphism of cameral covers $D[c] \cong D[A]$.

Consider two points $P_1, P_2 \in \mathrm{Gr}_X(A)$ which differ by an action of A/A_+. The corresponding Higgs fields c_{P_1} and c_{P_2} are isomorphic away from infinity, so we have an isomorphism of cameral covers $X_0[c_{P_1}] \cong X_0[c_{P_2}]$. Assume the complements of $X_0[c_{P_i}]$ in $X[c_{P_i}]$

consist of smooth points. Since both $X[c_{P_1}]$ and $X[c_{P_2}]$ are projective curves, this gives an isomorphism $X[c_{P_1}] \cong X[c_{P_2}]$. Using the results of [DG00] we see that an algebro-geometric orbit of A is a torsor over the corresponding Prym variety. In particular, it is finite-dimensional.

2.4.1.2 Tau-function

Consider the diagram



where the middle column is a pullback of the rightmost column along $\mathcal{A}^0 \to T_{\mathrm{Gr}_X(A)}$.

There is a unique lift $\mathcal{A}^0_{stab} \to \widehat{\mathcal{A}}^0$, such that the composite $\mathcal{A}^0_{stab} \to \widehat{\mathcal{A}}^0 \to \mathcal{A}_{\mathcal{L}}$ is zero.

Let $v \in \mathcal{A}^0_{stab}(P)$ be any vector stabilizing a point $P \in \mathrm{Gr}_X(A)$. Since $\mathcal{A}^0_{stab} \to \mathcal{A}_{\mathcal{L}}$ is the zero map, v preserves the fiber of $\mathcal{L}$ at P. Therefore, the tau-function

$$\tau_P(g) = \frac{\sigma(g^{-1}P)}{g^{-1}\sigma(P)}$$

obeys a first-order differential equation

$$v\tau_P = 0$$

for any $v \in \mathcal{A}^0_{stab}(P)$. This simply means that the tau-function descends to a well-defined function on the orbit.

2.4.1.3 Differential side

Recall that an algebro-geometric points $P \in \mathrm{Gr}_X(A)$ possess unique extensions of the abstract Higgs field of A on the disk D to the whole curve X, i.e. we have a bundle of regular centralizers $c \subset \operatorname{ad} P$. Suppose, moreover, that we picked a concrete Higgs field $\phi \in \Gamma(X, \operatorname{ad} P \otimes \Omega^1)$, such that its bundle of centralizers coincides with c.

Under the isomorphism $\mathrm{Gr}_X(A) \cong \mathcal{M}_G(X, p_{-1})$, the algebro-geometric points have the following description. Recall that $\mathcal{M}_G(X, p_{-1})$ parametrizes a G-bundle on $X \times A_{-1}$ with an A_+-reduction on $D \times A_{-1}$ and a connection ∇ along the A_{-1}-factor with a certain asymptotic condition (expressed as a relative position condition) near the marked point.

Proposition 2.4.3. *Algebro-geometric points $(P, \nabla) \in \mathcal{M}_G(X, p_{-1})$ correspond to those connections ∇, which extend as an integrable Higgs field ϕ satisfying the following condition: the centralizer of ϕ in $\operatorname{ad} P$ on the formal disk D coincides with the abstract Higgs field $\operatorname{ad} E \subset \operatorname{ad} P$.*

2.4.2 String solutions

2.4.2.1 Geometric description

Algebro-geometric points were characterized by the property that enough elements of the Heisenberg $\mathfrak{a}$ stabilize a point. Similarly, one can consider the action of the Virasoro-

Heisenberg algebra $\mathcal{A}$, which can be thought of as the algebra of first-order differential operators with coefficients in $\mathfrak{a}$.

Just as $\mathcal{A}^0_{stab} = \mathcal{A}^0 \cap \Gamma(X_0, \operatorname{ad} P)$, we also have $\mathcal{A}^{stab} = \mathcal{A} \cap \Gamma(X_0, \mathcal{A}_P)$. In fact, we have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(X_0, \operatorname{ad} P) & \longrightarrow & \Gamma(X_0, \mathcal{A}_P) & \longrightarrow & \Gamma(X_0, T_X) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathcal{A}^0_{stab} & \longrightarrow & \mathcal{A}_{stab} & \longrightarrow & \Gamma(X_0, T_X) & &
\end{array}$$

Note, that there is no reason to expect in general that the map $\mathcal{A}_{stab} \to \Gamma(X_0, T_X)$ is surjective.

Definition. A point $P \in \widehat{\mathrm{Gr}}_X$ is *string* if the map $\mathcal{A}_{stab} \to \Gamma(X_0, T_X)$ is surjective.

Note, that in the string case the sequence of Lie algebras

$$0 \to \mathcal{A}^0_{stab} \to \mathcal{A}_{stab} \to \Gamma(X_0, T_X) \to 0$$

is non-canonically split since T_X is locally-free.

Let us pick a splitting $\Gamma(X_0, T_X) \to \mathcal{A}^{stab}_0$ and consider the composite

$$\Gamma(X_0, T_X) \to \mathcal{A}_{stab} \hookrightarrow \Gamma(X_0, \mathcal{A}_P).$$

By definition, this means that P carries a connection on the affine part X_0. Moreover, the connection on $D^\times$ preserves the Heisenberg in the sense that for any vectors $v \in T_{D^\times}$ and $a \in \mathfrak{a}$ we have $[\nabla_v, a] \in \mathfrak{a}$. We get the following statement:

Theorem 2.4.4. *A point $P \in \widehat{\mathrm{Gr}}_X$ is string iff the torsor P has a connection ∇ on the affine part X_0, such that $\nabla_{z\partial/\partial z}$ induces the canonical grading on the Heisenberg $\mathfrak{a}$.*

Proof. Suppose that we have a connection ∇, such that $\nabla_{z\partial/\partial z}$ induces the canonical grading. Since the module of vector fields is free over $\mathcal{O}(D^\times)$, we see that the whole action of the Witt algebra on the Heisenberg $\mathfrak{a}$ coincides with the canonical action. We want to prove that it implies that the image of ∇ lands $\Gamma(D^\times, \mathcal{A}_E) \subset \Gamma(D^\times, \mathcal{A}_P)$.

Pick a splitting $s : \Gamma(D^\times, T_X) \to \Gamma(D^\times, \mathcal{A}_E)$. Since ∇ gives the canonical action of the Witt algebra on the Heisenberg, the difference $\nabla_v - s_v$ is an element $g(v)$ of $\Gamma(D^\times, \operatorname{ad} P)$ for any $v \in \Gamma(D^\times, T_X)$. Moreover, as both ∇ and s preserve the Heisenberg, $g(v)$ commutes with $\mathfrak{a}$. But then $g(v) \in \mathfrak{a}$ and hence ∇ lands in $\Gamma(D^\times, \mathcal{A}_E)$.

We see that there is a map $\Gamma(X_0, T_X) \to \Gamma(D^\times, \mathcal{A}_E) \cap \Gamma(X_0, \mathcal{A}_P)$ and hence the point is string. □

We get the following geometric structure on string solutions:

- A G-torsor $P \to X$ together with an $\underline{A}_+$-reduction $E \to D$.
- A subbundle $c \subset \operatorname{ad} P|_{X_0}$ of abelian Lie algebras, such that $c|_{D^\times} \subset \operatorname{ad} E|_{D^\times}$.
- A connection ∇ on the affine part X_0 which preserves c and gives the canonical gradation on the Heisenberg $\Gamma(D^\times, \operatorname{ad} E)$.

As before, c is the localization of $\mathcal{A}^0_{stab}$.

If the genus of X is zero, we can try to exploit the action of $\mathfrak{sl}_2 \cong \Gamma(X, T_X)$ on c.

Theorem 2.4.5. *Suppose X has genus 0 and the Heisenberg A has elliptic monodromy. Then string points have $c = 0$.*

Proof. Let $v \in \Gamma(X, T_X)$ be a regular vector field with a zero of order 2 at ∞. In local coordinates $v = z^2 \frac{\partial}{\partial z}$. The corresponding derivation of $\Gamma(D^\times, \mathcal{O}_X)$ has order 1. Therefore, ∇_v also has order 1 on the Heisenberg $\mathfrak{a}$.

Take a nonzero element $e \in \mathcal{A}^0_{stab}$. ∇_v raises its order, so for some n we have $(\nabla_v)^n e \in \Gamma(D, \operatorname{ad} E)$. On the other hand,

$$\Gamma(D, \operatorname{ad} E) \cap \Gamma(X_0, \operatorname{ad} P) \subset \Gamma(X, \operatorname{ad} P)$$

is finite-dimensional, so, possibly increasing n, we can assume that $(\nabla_v)^n e = 0$. Let n be the minimal such exponent, so that $s = (\nabla_v)^{n-1} e \neq 0$. Then s is annihilated by ∇, i.e. it is a flat section. But it cannot happen by assumption on the Heisenberg and Corollary 2.1.7. Therefore, $\mathcal{A}^0_{stab} = 0$. □

In particular, since $c = 0$, we see that string points cannot be algebro-geometric.

A similar statement was obtained previously by F. Plaza Martín [PM11, Theorem 3.1] when $G = SL_n$ and A is the principal Heisenberg.

2.4.2.2 Sugawara construction for twisted Heisenbergs

Any vector $a \in \mathcal{A}_{stab}$ annihilates the extended tau-function

$$a\overline{\tau} = 0$$

since it preserves the section σ. However, unless $a \in \mathcal{A}^0_{stab}$, such a constraint does not make sense on the ordinary tau-function since it has components in the Virasoro direction.

To get rid of these components, we would like to know how the tau-function changes under the Virasoro action. In this section we define an embedding of the Virasoro algebra into the universal enveloping algebra of a Heisenberg called the Sugawara construction.

Recall from subsection 2.1.2 that a Heisenberg $\mathfrak{a}$ can be obtained as the subspace of s-invariants in $\mathfrak{h}((t))$, where s acts on $\mathfrak{h}((t))$ by

$$s.(at^m) = (s.a)e^{2\pi i m/h}t^m, \quad a \in \mathfrak{h}.$$

Similarly, one can obtain the central extension $\widehat{\mathfrak{a}}$ as the subspace of s-invariants in $\widehat{\mathfrak{h}((t))}$, where s acts trivially on the central element.

Consider a decomposition

$$\mathfrak{h} = \bigoplus_{m=0}^{h-1} \mathfrak{h}_m$$

into eigenspaces of s, where s acts on $\mathfrak{h}_m$ by $\exp(2\pi i m/h)$. We denote the dimension $d_\alpha = \dim \mathfrak{h}_{\alpha \bmod h}$.

Since s is orthogonal with respect to the inner product $\langle , \rangle$ on $\mathfrak{h}$, the spaces $\mathfrak{h}_m$ and $\mathfrak{h}_{h-m}$ are naturally paired. Let $\{a^i_m\}_i$ be a basis of $t^m\mathfrak{h}_m$ and let $\{a^{\bar{i}}_{-m}\}_i$ be the dual basis of $t^{-m}\mathfrak{h}_{h-m}$.

Let us define the universal enveloping algebra $U_k(\widehat{\mathfrak{a}})$ as the quotient $U(\widehat{\mathfrak{a}})/(\mathbf{K} - k \cdot 1)$, where $\mathbf{K}$ is the central element of $\widehat{\mathfrak{a}}$. We can similarly define the universal enveloping algebras $U_c(\mathrm{Vir})$ and $U_{k,c}(\mathrm{Vir} \ltimes \widehat{\mathfrak{a}})$.

The Sugawara currents we are about to define will involve infinite expressions in the elements a^i_α, so they belong to a completion of the universal enveloping algebra $U_k(\widehat{\mathfrak{a}})$ in the "positive" direction:

$$\widehat{U}_k(\widehat{\mathfrak{a}}) = \varprojlim_n U_k(\widehat{\mathfrak{a}})/(\widehat{\mathfrak{a}}_n U_k(\widehat{\mathfrak{a}})),$$

where we denote $\widehat{\mathfrak{a}}_n = t^{hn}\widehat{\mathfrak{a}}_+$. Similarly, we can define a completion $\widehat{U}_{k,c}(\mathrm{Vir} \ltimes \widehat{\mathfrak{a}})$.

Proposition 2.4.6. *The elements of* $\widehat{U}_k(\widehat{\mathfrak{a}})$

$$L_n^S = \frac{1}{kh} \sum_{\alpha<nh/2,i} a^i_{hn-\alpha} a^{\bar{i}}_{\alpha} + \frac{1}{2kh} \sum_i a^i_{hn/2} a^{\bar{i}}_{hn/2} - \delta_{n,0} \frac{1}{4h^2} \sum_{0<l<h} d_l l(h-l)$$

obey the Virasoro commutation relations

$$[L_n^S, L_m^S] = (n-m)L^S_{n+m} - \delta_{n,-m} \frac{\dim \mathfrak{h}}{12}(n^3 - n).$$

Proof. Let us assume that h, n and m are all odd for simplicity. Moreover, assume $m \geq 0$. Other cases are treated similarly. We denote $C^{ij} = \langle a^i, a^j \rangle$.

Then we get

$$\begin{aligned}
(kh)^2[L_n^S, L_m^S] &= \sum_{\alpha<hn/2,\beta<hm/2,i,j} [a^i_{hn-\alpha} a^{\bar{i}}_{\alpha}, a^j_{hm-\beta} a^{\bar{j}}_{\beta}] \\
&= k \sum_{\alpha,\beta,i,j} a^i_{hn-\alpha} a^{\bar{j}}_{\beta} \alpha \delta_{\alpha,\beta-hm} \delta_{ij} + a^{\bar{i}}_{\alpha} a^{\bar{j}}_{\beta} (hn-\alpha) \delta_{hn-\alpha,\beta-hm} C^{ij} \\
&+ a^j_{hm-\beta} a^{\bar{i}}_{\alpha} (hn-\alpha) \delta_{hn-\alpha,-\beta} \delta_{ij} + a^j_{hm-\beta} a^i_{hn-\alpha} \alpha \delta_{\alpha,-\beta} C^{\bar{i}\bar{j}} \\
&= k \sum_{\alpha<hn/2,i} a^i_{hn-\alpha} a^{\bar{i}}_{hm+\alpha} \alpha \delta_{\alpha<-hm/2} + a^{\bar{i}}_{\alpha} a^i_{hn+hm-\alpha} (hn-\alpha) \delta_{\alpha>hn+hm/2} \\
&+ a^i_{hm+hn-\alpha} a^{\bar{i}}_{\alpha} (hn-\alpha) \delta_{\alpha<hn+hm/2} + a^{\bar{i}}_{hm+\alpha} a^i_{hn-\alpha} \alpha \delta_{\alpha>-hm/2}
\end{aligned}$$

We see that one may combine the terms to get

$$k \sum_{\alpha<hn/2,i} a^i_{hn-\alpha} a^{\bar{i}}_{hm+\alpha} \alpha + a^i_{hm+hn-\alpha} a^{\bar{i}}_{\alpha} (hn-\alpha).$$

Let's shift the index of summation in the first term:

$$k \sum_{\alpha<hn/2+hm,i} a^i_{hn+hm-\alpha} a^{\bar{i}}_\alpha (\alpha - hm) + k \sum_{\alpha<hn/2,i} a^i_{hm+hn-\alpha} a^{\bar{i}}_\alpha (hn - \alpha).$$

We want to make both sum to go up to $\alpha = h(n+m)/2$, so split off the necessary part in the first sum:

$$k \sum_{\alpha<h(n+m)/2,i} a^i_{hn+hm-\alpha} a^{\bar{i}}_\alpha (\alpha - hm) + k \sum_{h(n+m)/2\leq\alpha<hn/2+hm,i} a^i_{hn+hm-\alpha} a^{\bar{i}}_\alpha (\alpha - hm)$$
$$+ k \sum_{\alpha<hn/2,i} a^i_{hm+hn-\alpha} a^{\bar{i}}_\alpha (hn - \alpha).$$

The extra piece of the sum combines with the last term to finally give

$$k \sum_{\alpha<h(n+m)/2,i} a^i_{hn+hm-\alpha} a^{\bar{i}}_\alpha (hn - hm) + k \sum_i a^i_{h(n+m)/2} a^{\bar{i}}_{h(n+m)/2} h(n-m)/2$$
$$- k^2 \delta_{n,-m} \sum_{0<\alpha<hm/2,i} \alpha(\alpha - hm).$$

The first two terms combine into $(kh)^2(n-m)L^S_{n+m}$ (for $n \neq -m$), while the last sum is

$$S := \sum_{0<\alpha<hm/2} \alpha(\alpha - hm) d_\alpha.$$

Since d_α depends on α only modulo h, let us substitute $\alpha = hb + l$. Then we get

$$\begin{aligned}
S &= \sum_{-h/2<l<h/2} d_l \sum_{-l/h<b<m/2-l/h} (hb+l)(hb-hm+l) \\
&= \sum_{-h/2<l<h/2} d_l \sum_{-l/h<b<m/2-l/h} h^2b^2 + l(l-hm) + 2hbl - h^2mb \\
&= \sum_{0<l<h/2} d_l l(l-hm) + \sum_{-h/2<l<h/2} d_l \sum_{0<b<m/2-l/h} h^2b^2 + l(l-hm) + 2hbl - h^2mb \\
&= \sum_{0<l<h/2} d_l l(l-hm) + \sum_{-h/2<l<h/2} d_l \left(h^2 \frac{m^2-1}{24} m + l(l-hm)\frac{m-1}{2} + \frac{m^2-1}{8} h(2l-hm) \right)
\end{aligned}$$

Since $d_l = d_{-l}$, only even powers of l survive in the last sum. Moreover, $\sum_l d_l = \dim \mathfrak{h}$. Therefore, we get

$$S = m \sum_{0<l<h/2} d_l l(l-h) - h^2 \dim \mathfrak{h} \frac{m^2-1}{12} m.$$

The second term gives the usual Virasoro cocycle, while the first term can be absorbed into a redefinition of L_0^S. □

Remark. Note the unusual ordering in the Sugawara currents. We use this particular ordering as we will be interested in representations like $\mathcal{O}(A/A_+)$, which are not highest-weight, but instead dual to the highest-weight ones (namely, the vacuum representation $U\mathfrak{a}/U\mathfrak{a}_+$). This is also reflected in the minus sign in front of the central charge.

Remark. The Sugawara operators do not change if we multiply the bilinear form $\langle , \rangle$ on $\mathfrak{h}$ by a number.

We get an embedding of the Virasoro algebra into $\widehat{U}_k(\widehat{a})$. Now we want to see that the action of this copy of Virasoro coincides with the canonical Virasoro action on the Heisenberg.

Proposition 2.4.7. *The Sugawara currents L_n^S of Proposition 2.4.6 have the following commutation relations with elements of $\mathfrak{a}$:*

$$[L_n^S, a_\alpha^i] = -\frac{\alpha}{h} a^i_{\alpha+hn}.$$

Proof. For simplicity, we again assume that both h and n are odd. Then the commutator is

$$\begin{aligned}
[L_n^S, a_\alpha^i] &= \frac{1}{kh} \sum_{\beta<nh/2, j} [a^j_{hn-\beta} a^{\bar{j}}_\beta, a^i_\alpha] \\
&= \frac{1}{h} \sum_{\beta<nh/2, j} (hn-\beta)\delta_{hn-\beta,-\alpha} C^{ji} a^{\bar{j}}_\beta + a^j_{hn-\beta} \beta \delta_{\beta,-\alpha} \delta_{ij} \\
&= -\frac{\alpha}{h} a^i_{\alpha+hn}.
\end{aligned}$$

$\square$

2.4.2.3

Let $L_n \in \mathrm{Vir}$ be the standard generators of the Virasoro algebra. The last two propositions give an embedding

$$S : U_{c'}(\mathrm{Vir}) \to \widehat{U}_{k,c}(\mathrm{Vir} \ltimes \widehat{\mathfrak{a}})$$

given by

$$S(L_n) = L_n - L_n^S.$$

Indeed,

$$\begin{aligned}
[S(L_n), S(L_m)] &= [L_n, L_m] + [L_n^S, L_m^S] - [L_n, L_m^S] - [L_n^S, L_m] \\
&= [L_n, L_m] - [L_n^S, L_m^S] \\
&= (n-m)S(L_{n+m}) + \delta_{n,-m}\frac{c + \dim \mathfrak{h}}{12}(n^3 - n).
\end{aligned}$$

Therefore, the central charge is

$$c' = c + \dim \mathfrak{h}.$$

Furthermore, the image of S commutes with $U_k(\widehat{\mathfrak{a}}) \subset \widehat{U}_{k,c}(\mathrm{Vir} \ltimes \widehat{\mathfrak{a}})$.

2.4.2.4 Virasoro constraints

Any vector $v \in \Gamma(X_0, T_X)$ acts by endomorphisms on $\Gamma(\widehat{\mathrm{Gr}}_X, \mathcal{L})$ and hence we have an action of $S(v)$.

Since the operators $S(v)$ commute with A_+, the action preserves the subspace of invariants $\Gamma(\mathrm{Gr}_X(A), \mathcal{L}) \subset \Gamma(\widehat{\mathrm{Gr}}_X, \mathcal{L})$. In other words, we have an action of $\Gamma(X_0, T_X)$ on the space of tau-functions $\Gamma(\mathrm{Gr}_X(A), \mathcal{L})$.

We have the following lemma [BFM, Lemma 2.5.1]:

Lemma 2.4.8 (Beilinson-Feigin-Mazur)**.** *The Lie algebra $\Gamma(X_0, T_X)$ of vector fields on a curve is simple. In particular, since it is infinite-dimensional, it has no nontrivial finite-dimensional representations.*

Combining this lemma with Proposition 2.1.10 we get the following proposition:

Proposition 2.4.9. *Any vector field $v \in \Gamma(X_0, T_X)$ preserves the section σ:*

$$\sigma(S(v)P) = S(v)\sigma(P).$$

This implies that the tau-function

$$\tau_P(g) = \frac{\sigma(g^{-1}P)}{g^{-1}\sigma(P)}$$

is invariant under the Sugawara currents on the right. But since $g \in A$ and the Sugawara currents commute with A, it is also invariant under the Sugawara currents on the left.

Suppose $a \in A_+$. Then $\tau_P(ag) = c(a, g)\tau_P(g)$. Therefore, A_+ acts on $\tau_P(\cdot g) \in \mathcal{O}(A/A_+)$ via multiplication by a function in $\mathcal{O}(A/A_+)$. In contrast, elements of A/A_+ act by translations. Infinitesimally, it means that $\mathfrak{a}/\mathfrak{a}_+$ acts on $\tau_P(\cdot g)$ by vector fields, i.e. first-order differential operators.

Denote the projection $p : \mathcal{A}_{stab} \to \Gamma(X_0, T_X)$. Combining the invariance of the tau-function under the Sugawara currents and vectors $a \in \mathcal{A}_{stab}$ we get

Theorem 2.4.10. *For any vector $a \in \mathcal{A}_{stab}$ we have a second-order differential equation on the tau-function:*

$$(a - S(p(a)))\tau = 0.$$

These are the famous Virasoro constrains of two-dimensional quantum gravity [DVV91]. For example, pick a global coordinate z^{-1} on $\mathbf{P}^1$ vanishing to the first order at ∞. Let $\{t^i_\alpha\}_{\alpha>0,i} \in \mathcal{O}(A/A_+)$ be the time coordinates, so that a^i_α acts by $\partial/\partial t^i_{-\alpha}$ for $\alpha \leq 0$, and it acts by $(-k\alpha)t^{\bar{i}}_\alpha$ for $\alpha > 0$.

Suppose $p(g) = z^2\frac{\partial}{\partial z}$ for some $g \in \mathcal{A}_{stab}$. Let's consider the case $h = 2$. Then the operator

$$S(z^2\partial/\partial z) - g = L_1 - L_1^S - g = \frac{1}{2}\sum_{\alpha\geq 0,i}(\alpha+2)t^i_{2+\alpha}\frac{\partial}{\partial t^i_\alpha} + \frac{1}{4}\sum_i (t^i_1)^2 - \sum_{i,\alpha} g_{i,\alpha}a^i_\alpha$$

is known as the string operator, where $g_{i,\alpha}$ are some coefficients involved in the definition of g. Similarly,

$$S(z\partial/\partial z) - z^{-1}g = L_0 - L_0^S - z^{-1}g = \frac{1}{2}\sum_{\alpha>0,i}\alpha t^i_\alpha\frac{\partial}{\partial t^i_\alpha} + \frac{1}{4}\sum_i\left(\frac{\partial}{\partial t^i_0}\right)^2 + \frac{d_1}{16} - \sum_{i,\alpha} g_{i,\alpha}a^i_{\alpha-2},$$

which is related to the dilaton operator.

2.4.2.5 Differential side

Suppose that $P \in \mathrm{Gr}_X(A)$ is a string point. Then there is a connection ∇_{string} on $P \to X_0$, which preserves the reduction to A near infinity. Hence, the oper connection ∇_{oper} is flat with respect to ∇_{string}, i.e. together they form an absolute flat connection on $X_0 \times A_{-1}$. Combining this with Theorem 2.4.4 we get a converse statement.

Theorem 2.4.11. *An affine oper* (P, P_B, ∇) *is string iff the relative oper connection* ∇_{oper} *on* $X_0 \times A_{-1}$ *extends to an absolute flat connection preserving the A-reduction near infinity.*

Chapter 3

Derived symplectic geometry

3.1 Derived symplectic geometry

3.1.1 Symplectic structures

Let us remind basic notions of differential forms in derived algebraic geometry [PTVV11].

Let $X = \operatorname{Spec} A$ be a derived affine scheme for $A \in \mathrm{cdga}^{\leq 0}$ a non-positively graded commutative differential graded algebra. Recall that we have the cotangent complex $\mathbb{L}_A$, which is an A-module, and the complex of differential forms $\Omega(X) := \mathrm{Sym}_A(\mathbb{L}_A[1])$. It has a $\mathbf{G}_m$ action given by scaling the cotangent complex and we denote by $\Omega^p(X)[p]$ the weight p piece. Define the complex $\Omega^p(X, n)$ of p-forms of degree n to be $\Omega^p(X)[n]$.

The space $\mathrm{H}^0(\Omega^p(X, n))$ of p-forms of degree n is an algebraic analog of the space $\mathrm{H}^{p,n}(X)$ of (p, n)-forms in complex analytic geometry.

The de Rham differential is a morphism $\mathrm{d}_{\mathrm{dR}} \colon \Omega(X) \to \Omega(X)$ of degree -1 and weight 1, which squares to zero. We define the complex of closed forms to be

$$\Omega^{cl}(X) := (\mathrm{Sym}_A(\mathbb{L}_A[1]) \otimes_k k[[u]], d + u\mathrm{d}_{\mathrm{dR}}),$$

where d is the differential on $\Omega(X)$ and u has degree 2 and weight -1. Let $\Omega^{p,cl}(X)[p]$ be the weight p piece of $\Omega^{cl}(X)$. The complex $\Omega^{p,cl}(X, n)$ of closed p-forms of degree n is the weight p piece of $\Omega^{cl}(X)[n-p]$. We have a map $\Omega^{p,cl}(X, n) \to \Omega^p(X, n)$ given by evaluation at $u = 0$.

Geometrically, closed forms can be interpreted as S^1-equivariant functions on the free loop space [TV09], [BZN10]. This explains the action of $k[[u]] \cong \mathcal{O}(BS^1)$ on the complex of closed forms.

Explicitly, an element $\omega \in \mathrm{H}^0(\Omega^{p,cl}(X,n))$ is a collection of differential forms ω_i for $i = 0, 1, ...,$ such that ω_i has weight $p + i$ and degree $n - 2i$ and the following equations are satisfied:

$$d\omega_0 = 0$$

$$d\omega_{i+1} + \mathrm{d}_{\mathrm{dR}}\omega_i = 0.$$

In other words, ω_{i+1} expresses closedness of the form ω_i.

Both prestacks $\Omega^p(-,n)$ and $\Omega^{p,cl}(-,n)$ satisfy étale descent, so we can define the complexes of forms for a general derived stack as mapping stacks

$$\Omega^p(X,n) = \mathrm{Map}_{\mathrm{dSt}}(X, \Omega^p(-,n)), \quad \Omega^{p,cl}(X,n) = \mathrm{Map}_{\mathrm{dSt}}(X, \Omega^{p,cl}(-,n)).$$

For an Artin stack the complex of p-forms $\Omega^p(X,n)$ is the space of sections

$$\Omega^p(X,n) \cong \Gamma(X, \mathrm{Sym}^p_{\mathcal{O}_X}(\mathbb{L}_X[1])[n-p]).$$

A two-form $\omega \in \Omega^2(X,n)$ defines a morphism $\mathbb{T}_X \to \mathbb{L}_X[n]$.

Definition. A two-form $\omega \in \Omega^2(X,n)$ is *nondegenerate* if $\mathbb{T}_X \to \mathbb{L}_X[n]$ is an isomorphism.

We denote $\mathcal{A}^{nd}(X,n) \subset |\Omega^2(X,n)|$ the subspace of nondegenerate forms, where $|\Omega^2(X,n)|$ is the simplicial set corresponding to the complex $\Omega^2(X,n)$ under the Dold-Kan correspondence (as the complex $\Omega^2(X, n)$ is not connective in general, we consider its truncation $\tau_{\leq 0}$). We define the space $\mathrm{Symp}(X, n)$ of n-shifted symplectic forms to be the pullback

$$\begin{array}{ccc} \mathrm{Symp}(X,n) & \longrightarrow & \mathcal{A}^{nd}(X,n) \\ \downarrow & & \downarrow \\ |\Omega^{2,cl}(X,n)| & \longrightarrow & |\Omega^2(X,n)|. \end{array}$$

3.1.1.1 Example

The main example of a symplectic stack relevant for this paper is the classifying stack $X = \mathrm{B}G$ of an affine algebraic group G. See [TV02, Section 3.4] for a definition of G-torsors over derived affine schemes. The category of quasi-coherent sheaves $\mathrm{QC}(\mathrm{B}G)$ is naturally identified with the category of comodules over $\mathcal{O}(G)$. The cotangent complex of $\mathrm{B}G$ is $\mathbb{L}_{\mathrm{B}G} \cong \mathfrak{g}^*[-1] \in \mathrm{QC}(\mathrm{B}G)$, where $\mathfrak{g}^*$ is the coadjoint representation of G. If G is reductive, the functor of G-invariants is exact, so $\Omega^2(\mathrm{B}G)$ is concentrated in degree 2 and we have $\mathrm{H}^0(\Omega^2(\mathrm{B}G, 2)) \cong \mathrm{Sym}^2(\mathfrak{g}^*)^G$. One similarly has $\mathrm{H}^0(\Omega^{2,cl}(\mathrm{B}G, 2)) \cong \mathrm{Sym}^2(\mathfrak{g}^*)^G$ since $\mathrm{d}_{\mathrm{dR}} = 0$. A class $\omega \in \mathrm{Sym}^2(\mathfrak{g}^*)^G$ is nondegenerate if the induced G-equivariant map $\mathfrak{g} \to \mathfrak{g}^*$ is an isomorphism.

3.1.2 Lagrangian structures

An n-shifted symplectic form $\omega \in \mathrm{Symp}(X, n)$ can be viewed as an element of $\mathrm{H}^n(X, \bigwedge^2 \mathbb{L}_X)$. For example, 1-shifted symplectic structures can be thought of as defining torsors over $\bigwedge^2 \mathbb{L}_X$ together with a trivialization of its de Rham differential and higher closedness conditions (which are void if $\mathbb{L}_X$ is concentrated in nonnegative degrees). We will take up this point of view in the future sections.

Let (X, ω) be an n-shifted symplectic stack with $\omega \in \Omega^{2,cl}(X, n)$.

Definition. An *isotropic structure* on $f\colon L \to X$ is a homotopy from $f^*\omega$ to 0 in $\Omega^{2,cl}(L, n)$.

In other words, it is an element $h \in \Omega^{2,cl}(L, n-1)$, such that $(d + u\mathrm{d}_{\mathrm{dR}})h = f^*\omega$.

Explicitly, we have a collection of differential forms h_i satisfying the conditions

$$dh_0 = f^*\omega_0$$

$$dh_{i+1} + \mathrm{d}_{\mathrm{dR}} h_i = f^*\omega_{i+1}.$$

The form h_0 defines a map $\mathbb{T}_L \to \mathbb{L}_L[n-1]$, which is not a chain map in general since h_0 is not closed. Consider instead the relative tangent bundle

$$\mathbb{T}_f = f^*\mathbb{T}_X[-1] \oplus \mathbb{T}_L$$

with the differential given by the map $\mathbb{T}_L \to f^*\mathbb{T}_X$. We have a chain map $\mathbb{T}_f \to \mathbb{L}_L[n-1]$ defined to be $f^*\omega_0$ on the first summand and h_0 on the second summand.

Definition. An isotropic structure $f\colon L \to X$ is *Lagrangian* if $\mathbb{T}_f \to \mathbb{L}_L[n-1]$ is an isomorphism.

Here is a way to unpack this definition (see [Ca13]). An isotropic structure on $f\colon L \to X$ is a commutativity data of the diagram

$$\begin{array}{ccc} \mathbb{T}_L & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ f^*\mathbb{T}_X & \longrightarrow & \mathbb{L}_L[n] \end{array}$$

The isotropic structure is Lagrangian if the diagram is a pullback. In other words,

$$\mathbb{T}_L \to f^*\mathbb{T}_X \to \mathbb{L}_L[n]$$

is an exact sequence.

Theorem 3.1.1 ([PTVV11])**.** *Let (X, ω) be an n-shifted symplectic stack together with two Lagrangians $L_1 \to X$ and $L_2 \to X$. Then their intersection $L_1 \times_X L_2$ carries a natural $(n-1)$-shifted symplectic structure.*

Let us prove a generalization of this theorem, which can also be found in [Ca13, Theorem 4.4]. Let X and Y be n-shifted symplectic stacks.

Definition. A *Lagrangian correspondence* is a correspondence

$$\begin{array}{ccc} & L & \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

together with a Lagrangian structure on the map $L \to X \times \overline{Y}$.

Here $\overline{Y}$ is Y with the opposite symplectic structure.

The following theorem allows us to compose Lagrangian correspondences, which will be used in section 3.3 to describe the AKSZ topological field theory.

Theorem 3.1.2. *Let (X, ω_X), (Y, ω_Y) and (Z, ω_Z) be n-shifted symplectic stacks and*

$$\begin{array}{ccccccc} & L_1 & & & & L_2 & \\ {\scriptstyle p_X}\swarrow & & \searrow{\scriptstyle p_Y^1} & & {\scriptstyle p_Y^2}\swarrow & & \searrow{\scriptstyle p_Z} \\ X & & & Y & & & Z \end{array}$$

are Lagrangian correspondences. Then the pullback $L_1 \times_Y L_2$ *is a Lagrangian correspondence between* Z *and* X*.*

Proof. Suppose that the Lagrangian structures on L_1 and L_2 are given by the forms h_1 and h_2 respectively, i.e.

$$p_X^*\omega_X - (p_Y^1)^*\omega_Y = (d + u\mathrm{d_{dR}})h_1, \quad (p_Y^2)^*\omega_Y - p_Z^*\omega_Z = (d + u\mathrm{d_{dR}})h_2.$$

Denote $L = L_1 \times_Y L_2$ and let $\pi_i \colon L \to L_i$ be the projections.

Then

$$\pi_1^* p_X^*\omega_X - \pi_1^*(p_Y^1)^*\omega_Y + \pi_2^*(p_Y^2)^*\omega_Y - \pi_2^* p_Z^*\omega_Z = (d + u\mathrm{d_{dR}})\pi_1^* h_1 + (d + u\mathrm{d_{dR}})\pi_2^* h_2.$$

Therefore,

$$\pi_1^* p_X^*\omega_X - \pi_2^* p_Z^*\omega_Z = (d + u\mathrm{d_{dR}})(\pi_1^* h_1 + \pi_2^* h_2),$$

i.e. $\pi_1^* h_1 + \pi_2^* h_2$ defines an isotropic structure on $L \to X \times \overline{Z}$. Let us check that it is in fact Lagrangian.

$L_1 \to X \times \overline{Y}$ and $L_2 \to Y \times \overline{Z}$ are Lagrangian, so we have the following pullback squares

$$\begin{array}{ccc} \mathbb{T}_{L_1} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ p_X^*\mathbb{T}_X \oplus (p_Y^1)^*\mathbb{T}_Y & \longrightarrow & \mathbb{L}_{L_1}[n] \end{array} \qquad \begin{array}{ccc} \mathbb{T}_{L_2} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ (p_Y^2)^*\mathbb{T}_Y \oplus p_Z^*\mathbb{T}_Z & \longrightarrow & \mathbb{L}_{L_2}[n] \end{array}$$

Pulling them back to L and adding together we get a pullback square

$$\begin{array}{ccc} \pi_1^*\mathbb{T}_{L_1} \oplus \pi_2^*\mathbb{T}_{L_2} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \pi_1^* p_X^*\mathbb{T}_X \oplus \pi_1^*(p_Y^1)^*\mathbb{T}_Y \oplus \pi_2^*(p_Y^2)^*\mathbb{T}_Y \oplus \pi_2^* p_Z^*\mathbb{T}_Z & \longrightarrow & \pi_1^*\mathbb{L}_{L_1}[n] \oplus \pi_2^*\mathbb{L}_{L_2}[n] \end{array}$$

We can split off two summands of $\pi_1^*(p_Y^1)^*\mathbb{T}_Y$ into the diagonal and antidiagonal parts obtaining the pullback of the form

$$\begin{array}{ccc} \pi^*\mathbb{T}_L \oplus g^*\mathbb{T}_Z \oplus \pi_1^* p_1^* \mathbb{T}_Y[-1] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \pi_1^* p_X^* \mathbb{T}_X \oplus \pi_2^* p_Z^* \mathbb{T}_Z & \longrightarrow & \pi_1^*\mathbb{L}_{L_1}[n] \oplus \pi_2^*\mathbb{L}_{L_2}[n] \oplus \pi_1^* p_1^* \mathbb{T}_Y[1] \end{array}$$

with the obvious differentials in the top-left and bottom-right corners. Finally, using the identification in the bottom-right corner $\mathbb{T}_Y \cong \mathbb{L}_Y[n]$ given by the symplectic form ω_Y we get a pullback

$$\begin{array}{ccc} \mathbb{T}_L & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \pi_1^* p_X^* \mathbb{T}_X \oplus \pi_2^* p_Z^* \mathbb{T}_Z & \longrightarrow & \mathbb{L}_L[n] \end{array}$$

In other words, $L \to X \times \overline{Z}$ is Lagrangian as claimed. $\square$

In the case $X = Z = \mathrm{pt}$ Theorem 3.1.2 reduces to Theorem 3.1.1. Indeed, a Lagrangian $L \to \mathrm{pt}$ is the same as an $(n-1)$-shifted symplectic stack.

More generally, suppose $f\colon X \to Y$ is a symplectic morphism. Then the graph

$$\Gamma_f = X \times_Y Y \to X \times \overline{Y}$$

carries an isotropic structure. We say that the morphism f is *nondegenerate* if its graph is Lagrangian. In this case we can pullback Lagrangians: let $L_2 \to Y$ be a Lagrangian and $L_1 = \Gamma_f$. Then the theorem gives a natural Lagrangian structure on the pullback $L_2 \times_Y X$. One can view Theorem 3.1.2 for Z being a point as a way to perform integral transforms for Lagrangians.

3.2 Symplectic reduction

In this and future sections G will denote a reductive group of finite type over k.

3.2.1 General definition

The general procedure for a symplectic reduction starts with a 1-shifted symplectic stack X together with a choice of a Lagrangian $L \to X$. Then the data of a symplectic reduction consists of:

1. A stack M with a G-action.
2. A moment map $\mu\colon M/G \to X$ together with a Lagrangian structure.

The isotropic conditions $dh_0 = f^*\omega_0, dh_1 + \mathrm{d}_{\mathrm{dR}} h_0 = f^*\omega_1, \ldots$ will be called the *moment map equations*. We will see that these equations coincide with the usual moment map equations familiar from the theory of symplectic reduction.

By definition the *reduced space* is $M/G \times_X L$. Theorem 3.1.1 gives a natural symplectic structure on the reduced space.

3.2.2 Ordinary Hamiltonian reduction

Let $X = \mathfrak{g}^*/G$. The category of quasi-coherent sheaves $\mathrm{QC}(X)$ is the category of G-equivariant sheaves on $\mathfrak{g}^*$. The tangent complex $\mathbb{T}_X \in \mathrm{QC}(X)$ is

$$\mathbb{T}_X = \mathfrak{g} \otimes_k \mathcal{O}_{\mathfrak{g}^*}[1] \oplus \mathfrak{g}^* \otimes_k \mathcal{O}_{\mathfrak{g}^*},$$

with the differential given by the coadjoint action.

On $\mathfrak{g}^*$ we have a canonical "Maurer–Cartan" form $\omega_0 \in \Omega^1(\mathfrak{g}^*) \otimes_k \mathfrak{g}^*$ given by the identity map $T_x\mathfrak{g}^* = \mathfrak{g}^* \to \mathfrak{g}^*$. It defines a two-form $\omega_0 \in \Omega^2(\mathfrak{g}^*/G, 1)$ of degree 1. It is closed: $(d + u\mathrm{d}_{\mathrm{dR}})\omega_0 = 0$, where $\mathrm{d}_{\mathrm{dR}}\omega_0 = 0$ follows from the fact that ω_0 does not depend on the point $x \in \mathfrak{g}^*$ and $d\omega_0 = 0$ follows from the equivariance of ω_0 with respect to the coadjoint action.

It gives a morphism $\omega_0 \colon \mathbb{T}_X \to \mathbb{L}_X[1]$

$$\begin{array}{ccc} \mathfrak{g} \otimes_k \mathcal{O}_{\mathfrak{g}^*} & \longrightarrow & \mathfrak{g}^* \otimes_k \mathcal{O}_{\mathfrak{g}^*} \\ \downarrow \sim & & \downarrow \sim \\ \mathfrak{g} \otimes_k \mathcal{O}_{\mathfrak{g}^*} & \longrightarrow & \mathfrak{g}^* \otimes_k \mathcal{O}_{\mathfrak{g}^*} \end{array}$$

which is clearly an isomorphism.

We have $\mathfrak{g}^*/G = T^*[1]\mathrm{B}G$. Therefore, the symplectic structure can alternatively be defined using a canonical one-form θ of degree 1. It is given by the identity function $\mathrm{id} \in \mathcal{O}_{\mathfrak{g}^*} \otimes_k \mathfrak{g}^*$. Then $\omega_0 = \mathrm{d}_{\mathrm{dR}}\theta$.

An isotropic structure on $M/G \to X$ is a closed two-form h of degree 0 on M, which is G-equivariant. Moreover, the condition $dh_0 = f^*\omega_0$ is equivalent to

$$-\iota_{a(v)} h_0 = d\mu(v)$$

for $v \in \mathfrak{g}$ and $a \colon \mathfrak{g} \to \Gamma(M, \mathbb{T}_M)$ the action map. Lagrangian condition translates into the fact that h_0 has to be nondegenerate.

For example, the map $L = \mathrm{pt}/G \to \mathfrak{g}^*/G$ induced from the inclusion of the origin is Lagrangian.

3.2.2.1 Example

Let M be a stack with a G action. We define a moment map $\mu\colon T^*M \to \mathfrak{g}^*$ as follows. The action map $a\colon \mathfrak{g} \to \Gamma(M, \mathbb{T}_M)$ gives an element of

$$\mathfrak{g}^* \otimes_k \Gamma(M, \mathbb{T}_M) \subset \mathfrak{g}^* \otimes_k \Gamma(M, \mathrm{Sym}_{\mathcal{O}_M} \mathbb{T}_M) \cong \Gamma(T^*M, \mathfrak{g}^* \otimes_k \mathcal{O}_{T^*M}).$$

Recall that the canonical one-form θ on T^*M is defined to be the composite

$$\mathbb{T}_{T^*M} \to p^*\mathbb{T}_M \to \mathcal{O}_{T^*M},$$

where $p\colon T^*M \to M$ is the projection and $p^*\mathbb{T}_M \to \mathcal{O}_{T^*M}$ is adjoint to

$$\mathbb{T}_M \to p_*\mathcal{O}_{T^*M} \cong \mathrm{Sym}_{\mathcal{O}_M} \mathbb{T}_M.$$

Observe that

$$\iota_{a(v)}\theta = \mu(v).$$

We define $h_0 = \mathrm{d}_{\mathrm{dR}}\theta$. The moment map equation follows from the following calculation:

$$-\iota_{a(v)}\mathrm{d}_{\mathrm{dR}}\theta = \mathrm{d}_{\mathrm{dR}}\iota_{a(v)}\theta = \mathrm{d}_{\mathrm{dR}}\mu(v),$$

where we used G-invariance of θ in the second equality.

The moment map equation follows by observing that

$$\iota_v\theta = \mu(v)$$

for $v \in \mathfrak{g}$.

The symplectic reduction

$$T^*M/G \times_{\mathfrak{g}^*/G} \mathrm{pt}/G$$

is isomorphic to $T^*(M/G)$.

3.2.3 Quasi-Hamiltonian reduction

Consider the right action of G on itself by conjugation: $a \mapsto g^{-1}ag =: \mathrm{Ad}_g(a)$ and let $X = G/G$. The tangent complex is

$$\mathfrak{g} \otimes_k \mathcal{O}_G \to \mathbb{T}_G$$

in degrees -1 and 0 with the differential $\mathfrak{g} \to \Gamma(G, \mathbb{T}_G)$ given by the adjoint action

$$x \in \mathfrak{g} \mapsto x^R - x^L,$$

where x^L and x^R are vector fields generating the left and right actions of G on itself. The cotangent complex is

$$\mathbb{L}_G \to \mathfrak{g}^* \otimes_k \mathcal{O}_G$$

in degrees 0 and 1 with the differential d given by

$$(d\phi)(x) = -\iota_{(x^R - x^L)}\phi$$

for $\phi \in \mathbb{L}_G$. At any point $a \in G$ we have $x^L = \mathrm{Ad}_a x^R$.

Recall the left and right Maurer–Cartan forms $\theta, \overline{\theta} \in \Omega^1(G) \otimes_k \mathfrak{g}$ defined by

$$\iota_v\theta = (a \in G \mapsto (L_{a^{-1}})_* v_a), \quad \iota_v\overline{\theta} = (a \in G \mapsto (R_{a^{-1}})_* v_a)$$

for a vector field $v \in \Gamma(G, \mathbb{T}_G)$. For any point $a \in G$ we have

$$\theta = \mathrm{Ad}_a\overline{\theta}.$$

The contraction of the Maurer–Cartan forms with the invariant vector fields are as follows:

$$\iota_{(x^L)}\theta = \mathrm{Ad}_a(x), \iota_{(x^R)}\theta = x, \iota_{(x^L)}\overline{\theta} = x, \iota_{(x^R)}\overline{\theta} = \mathrm{Ad}_{a^{-1}}(x).$$

Furthermore, we have the Maurer–Cartan equations

$$\mathrm{d}_{\mathrm{dR}}\theta + \frac{1}{2}[\theta,\theta] = 0, \quad \mathrm{d}_{\mathrm{dR}}\overline{\theta} - \frac{1}{2}[\overline{\theta},\overline{\theta}] = 0.$$

The sheaf of two-forms on G/G is

$$\bigwedge^2 \mathbb{L}_G \oplus \mathbb{L}_G \otimes_k \mathfrak{g}^*[-1] \oplus \mathcal{O}_G \otimes_k \mathrm{Sym}^2(\mathfrak{g}^*)[-2].$$

Let $(-,-)\colon \mathfrak{g}\otimes_k \mathfrak{g} \to k$ be a G-invariant nondegenerate symmetric bilinear form. Then we can define a two-form ω_0 of degree 1 by

$$\omega_0(y) = -\frac{1}{2}(\theta + \overline{\theta}, y) \tag{3.1}$$

for any $y \in \mathfrak{g}$.

Lemma 3.2.1. *ω_0 is d-closed.*

Proof. If we view $d\omega_0$ as an element of $\mathfrak{g}^* \otimes_k \mathfrak{g}^*$, we have to prove that it is antisymmetric.

$$\begin{aligned} d\omega_0(x,y) &= \frac{1}{2}(\iota_{(x^R - x^L)}\theta + \iota_{(x^R-x^L)}\overline{\theta}, y) \\ &= \frac{1}{2}(x - \mathrm{Ad}_a(x) + \mathrm{Ad}_{a^{-1}}(x) - x, y) \\ &= \frac{1}{2}(\mathrm{Ad}_{a^{-1}}(x), y) - \frac{1}{2}(x, \mathrm{Ad}_{a^{-1}}(y)). \end{aligned}$$

□

Although ω_0 is not d_{dR}-closed, it is homotopically d_{dR}-closed: there is a differential form ω_1, such that $\mathrm{d}_{\mathrm{dR}}\omega_0 + d\omega_1 = 0$. Indeed, define a three-form ω_1 of degree 0 by

$$\omega_1 = \frac{1}{12}(\theta, [\theta,\theta]). \tag{3.2}$$

Lemma 3.2.2. *The equation* $\mathrm{d_{dR}}\omega_0 + d\omega_1 = 0$ *is satisfied.*

Proof. For $x \in \mathfrak{g}$ we must prove

$$-\mathrm{d_{dR}}\frac{1}{2}(\theta + \overline{\theta}, x) - \iota_{(x^R - x^L)}\omega_1 = 0.$$

Let us split

$$\omega_1 = \frac{1}{24}(\theta, [\theta, \theta]) + \frac{1}{24}(\overline{\theta}, [\overline{\theta}, \overline{\theta}]).$$

Then we have to prove

$$\frac{1}{4}([\theta, \theta], x) - \frac{1}{4}([\overline{\theta}, \overline{\theta}], x) - \frac{1}{8}(\iota_{(x^R - x^L)}\theta, [\theta, \theta]) - \frac{1}{8}(\iota_{(x^R - x^L)}\overline{\theta}, [\overline{\theta}, \overline{\theta}]) = 0.$$

This is equivalent to

$$2([\theta, \theta], x) - 2([\overline{\theta}, \overline{\theta}], x) - (x - \mathrm{Ad}_a(x), [\theta, \theta]) - (\mathrm{Ad}_{a^{-1}}(x) - x, [\overline{\theta}, \overline{\theta}]) = 0.$$

The claim follows from the invariance of the bilinear form under conjugation. □

Lemma 3.2.3. *The form* ω_1 *is* $\mathrm{d_{dR}}$*-closed.*

Proof. From the Maurer–Cartan equation we see that $[\theta, \theta]$ is $\mathrm{d_{dR}}$-closed. Then

$$\mathrm{d_{dR}}\omega_1 = \frac{1}{12}(\mathrm{d_{dR}}\theta, [\theta, \theta]) = -\frac{1}{12}([\theta, \mathrm{d_{dR}}\theta], \theta) = \frac{1}{12}(\mathrm{d_{dR}}[\theta, \theta], \theta) = 0,$$

where we used invariance of the bilinear form in the second equality. □

The previous three lemmas prove that $\omega_0 + u\omega_1$ is a closed two-form. To see that it is symplectic, we have to check that it is nondegenerate.

Lemma 3.2.4. *The two-form* $\omega_0\colon \mathbb{T}_{G/G} \to \mathbb{L}_{G/G}[1]$ *is nondegenerate.*

Proof. ω_0 gives the following chain map:

$$\begin{array}{ccc} \mathfrak{g} \otimes_k \mathcal{O}_G & \longrightarrow & \mathbb{T}_G \\ \downarrow & & \downarrow \\ \mathbb{L}_G & \longrightarrow & \mathfrak{g}^* \otimes_k \mathcal{O}_G, \end{array}$$

where the vertical maps are dual to each other. As the vertical maps are morphisms of vector bundles of the same rank, we just have to check that one of them (say, the left one) is injective on cohomology.

Consider a point $a \in G$ and a closed degree 0 element $v \in \mathfrak{g}$ of $\mathbb{T}_{G/G}[-1]$. Closedness of v is equivalent to the equation $\mathrm{Ad}_{a^{-1}} v = v$.

Its image under ω_0 is

$$-\frac{1}{2}(\theta + \overline{\theta}, v).$$

If this form is zero, its contraction with every vector field of the form x^L is zero as well. That is,

$$-\frac{1}{2}(\mathrm{Ad}_a(x) + x, v) = 0.$$

However,

$$-\frac{1}{2}(\mathrm{Ad}_a(x) + x, v) = -\frac{1}{2}(x, \mathrm{Ad}_{a^{-1}}(v) + v) = -(x, v).$$

It is zero for all $x \in \mathfrak{g}$ if and only if $v = 0$, i.e. the left vertical map is injective. □

Consider a G-equivariant map $\mu\colon M \to G$ of *right* G-spaces. It induces an isotropic map $M/G \to G/G$ if we are given a two-form h_0 of degree 0 on M/G, such that

$$\mu^*\omega_0 = dh_0, \quad \mu^*\omega_1 = \mathrm{d_{dR}} h_0.$$

Substituting the expressions for ω_0 and ω_1 we get

$$\begin{aligned} \iota_{a(v)} h_0 &= \frac{1}{2}\mu^*(\theta + \overline{\theta}, v) \\ \mathrm{d_{dR}} h_0 &= \frac{1}{12}\mu^*(\theta, [\theta, \theta]). \end{aligned}$$

These are precisely the moment map equations for the quasi-Hamiltonian reduction. One sees that the equations coincide with [AMM97, Definition 2.2] up to a sign in the second equation since [AMM97] consider left G-actions. In the future we will call Lagrangians $X \to G/G$ *quasi-Hamiltonian spaces.*

3.2.3.1 Example

This example is due to Alekseev, Malkin and Meinrenken [AMM97, Section 9].

Let M be a closed oriented surface together with a point $x \in M$. Let $\mathrm{Loc}_G(M)$ be the moduli space of local systems on M also known as the *character stack*. The moment map

$$\mu\colon \mathrm{Loc}_G(M\backslash x) \to G/G$$

is given by the monodromy around the puncture x. This gives $\mathrm{Loc}_G(M\backslash x)$ the structure of a quasi-Hamiltonian space. The symplectic reduction of $\mathrm{Loc}_G(M\backslash x)$ is $\mathrm{Loc}_G(M)$, which inherits a symplectic form.

For instance, let $M = T^2$ be the 2-torus. More general character varieties can be obtained by fusion (see the next section). We have a moment map

$$\mu\colon G \times G \to G$$

given by the commutator $a, b \mapsto aba^{-1}b^{-1}$. The two-form h_0 on $G \times G$ is given by

$$h_0 = \frac{1}{2}(p_1^*\theta, p_2^*\overline{\theta}) + \frac{1}{2}(p_2^*\theta, p_1^*\overline{\theta}) + \frac{1}{2}(m^*\theta, i^*m^*\overline{\theta}), \tag{3.3}$$

where $m\colon G \times G \to G$ is the multiplication, $p_i\colon G \times G \to G$ are the projection and $i\colon G \times G \to G \to G$ is the inversion on each factor.

The geometric meaning of the form h_0 will be explained in section 3.4.

3.3 AKSZ topological field theory

3.3.1 Lagrangian correspondences

Consider the symmetric monoidal 1-category $\mathrm{LagrCorr}_n$ whose objects are $(n-1)$-shifted symplectic stacks and morphisms $X \to Y$ are Lagrangian correspondences $X \leftarrow L \to Y$. Theorem 3.1.2 defines a composition on this category. To make the composition well-defined, we consider Lagrangian correspondences only up to an isomorphism. Let us spell out the notion of isomorphisms explicitly.

Two Lagrangians $f_i\colon L_i \to X \times \overline{Y}$ are isomorphic if we have an isomorphism of stacks $g\colon L_1 \to L_2$ together with a commutative diagram

$$\begin{array}{ccc} L_1 & \xrightarrow{f_1} & X \times \overline{Y} \\ {\scriptstyle g}\downarrow & \nearrow_{f_2} & \\ L_2 & & \end{array}$$

We have a loop at the origin in $\Omega^{2,cl}(L_2, n)$ given by

$$0 \sim f_2^*\omega_X - f_2^*\omega_Y \sim g^* f_1^*\omega_X - g^* f_1^*\omega_Y \sim 0$$

which we require to be contractible.

The symmetric monoidal structure on $\mathrm{LagrCorr}_n$ is given by the Cartesian product of symplectic stacks.

Recall also the symmetric monoidal 1-category of cobordisms Cob_n^{or} whose objects are closed oriented $(n-1)$-manifolds and morphisms are oriented cobordisms between them.

Given a topological space M we can assign to it a constant stack M_{B}. Let us recall the following two theorems ([PTVV11, Theorem 2.5] and [Ca13, Section 3.1.2]).

Theorem 3.3.1. *Let M be a closed oriented $(n-1)$-manifold and X an m-shifted symplectic stack. Then the derived mapping stack $\mathrm{Map}_{\mathrm{dSt}}(M_{\mathrm{B}}, X)$ carries a natural $(m-n+1)$-shifted symplectic structure.*

Theorem 3.3.2. *Let M be a compact oriented n-manifold. Then the restriction map*

$$\mathrm{Map}_{\mathrm{dSt}}(M_{\mathrm{B}}, X) \to \mathrm{Map}_{\mathrm{dSt}}((\partial M)_{\mathrm{B}}, X)$$

carries a natural Lagrangian structure.

One can recover the previous theorem since $\partial M \cong \varnothing$ and Lagrangian maps from the stack $\mathrm{Map}_{\mathrm{dSt}}(M_{\mathrm{B}}, X)$ into the point equipped with a unique $(m-n+1)$-shifted symplectic structure are the same as $(m-n)$-shifted symplectic structures on $\mathrm{Map}_{\mathrm{dSt}}(M_{\mathrm{B}}, X)$.

Following [Ca13] we define the AKSZ topological field theory $Z_X\colon \mathrm{Cob}_n^{or} \to \mathrm{LagrCorr}_{m-n+2}$ whose value on any manifold M is given by the derived mapping stack

$$Z_X(M) = \mathrm{Map}_{\mathrm{dSt}}(M_{\mathrm{B}}, X).$$

See *loc. cit* for more details.

3.3.2 Classical Chern–Simons theory

We would like to interpret objects appearing in [AMM97] from the point of view of the AKSZ topological field theory.

The classifying stack $\mathrm{B}G$ carries a 2-shifted symplectic structure constructed from a nondegenerate G-invariant quadratic form $q \in \mathrm{Sym}^2(\mathfrak{g}^*)^G$. The field theory

$$Z_{\mathrm{B}G}\colon \mathrm{Cob}_2^{or} \to \mathrm{LagrCorr}_2$$

is the classical Chern–Simons theory truncated to dimensions 1 and 2. Let's consider some simple cobordisms.

- $M = S^1$. $Z_{\mathrm{B}G}(S^1) = G/G$ and it carries a 1-shifted symplectic structure.
- M is the disk. Then $Z_{\mathrm{B}G}(M) = \mathrm{pt}/G$ which carries a Lagrangian map $\mathrm{pt}/G \to G/G$.
- $M = S^1 \times I$ viewed as a cobordism from pt to $S^1 \sqcup S^1$. We call

  $$D(G) := Z_{\mathrm{B}G}(S^1 \times I) = G/G$$

  the double of G. The map $G/G \to G/G \times G/G$ given by $a \mapsto (a, a^{-1})$ is Lagrangian.

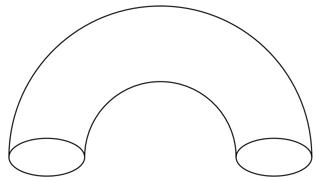


Figure 3.1: The double $D(G)$.

- M is a pair of pants viewed as a cobordism from $S^1 \sqcup S^1$ to S^1. Then

$$Z_{\mathrm{B}G}(M) = (G \times G)/G.$$

The AKSZ field theory then gives a Lagrangian correspondence

$$G/G \longleftarrow (G \times G)/G \longrightarrow G/G \times G/G$$

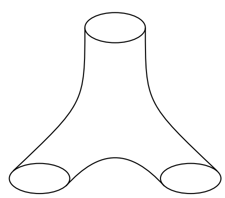


Figure 3.2: Fusion

For example, for any Lagrangian $L \to G/G \times G/G$ we get a Lagrangian

$$L \times_{(G \times G)/G} (G/G \times G/G) \to G/G$$

which is called the *internal fusion* of L.

- M is a 2-torus with a disk removed. We can view M as a composition of the cylinder with a pair of pants, so $Z_{\mathrm{B}G}(M)$ is the fusion of the double $D(G)$. Explicitly,

$$Z_{\mathrm{B}G}(M) = (G \times G)/G$$

with a Lagrangian morphism $Z_{\mathrm{B}G}(M) \to G/G$ given by $(a, b) \mapsto aba^{-1}b^{-1}$.

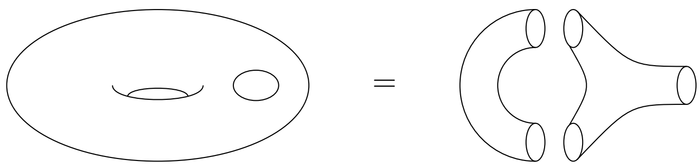
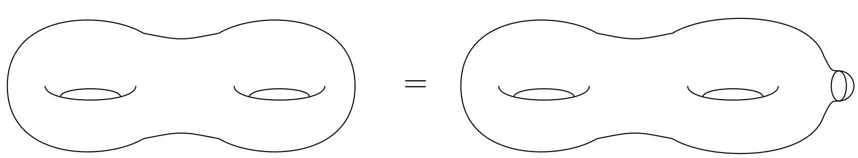


Figure 3.3: Punctured torus as a fusion of $D(G)$

- M is a closed oriented surface of genus g. We can split it as a composition of M with a disk removed M' and a disk. This allows us to compute

$$Z_{\mathrm{B}G}(M) = Z_{\mathrm{B}G}(M') \times_{G/G} \mathrm{pt}/G.$$

In other words, the character variety of M with its symplectic structure obtained by the AKSZ construction can be also obtained from a quasi-Hamiltonian reduction of $Z_{\mathrm{B}G}(M') = (\underbrace{G \times ... \times G}_{2g \text{ times}})/G$.



M $\qquad$ M'

3.4 Multiplicative torsor on the group

3.4.1 Multiplicative structures

3.4.1.1

In this section we will show that there is a multiplicative $\Omega^{2,cl}$-torsor on G, which gives rise to the 1-shifted symplectic form on G/G described previously.

Let $\mathcal{A}$ be a natural system of sheaves of abelian groups on stacks. That is, it is a collection of sheaves $\mathcal{A}_X$ for every stack X together with compatible maps $f^{-1}\mathcal{A}_Y \to \mathcal{A}_X$ for every morphism $f\colon X \to Y$. Given an $\mathcal{A}_Y$-torsor $\mathcal{T}$ on Y, we define the pullback $\mathcal{A}$-torsor $f^*\mathcal{T}$ to be

$$f^*\mathcal{T} = f^{-1}\mathcal{T} \times_{f^{-1}\mathcal{A}_Y} \mathcal{A}_X.$$

Definition. A *multiplicative $\mathcal{A}$-torsor* $\mathcal{T}$ on G is an $\mathcal{A}$-torsor $\mathcal{T}$ together with the data of an isomorphism

$$\phi\colon m^*\mathcal{T} \cong p_1^*\mathcal{T} \times_{\mathcal{A}} p_2^*\mathcal{T} =: \mathcal{T} \boxtimes \mathcal{T}$$

satisfying the following pentagon diagram expressing associativity:

$$\begin{array}{ccccc}
m_{12}^* m^*\mathcal{T} & & \xrightarrow{\sim} & & m_{23}^* m^*\mathcal{T} \\
\downarrow{\scriptstyle m_{12}^*\phi} & & & & \downarrow{\scriptstyle m_{23}^*\phi} \\
m_{12}^*(\mathcal{T} \boxtimes \mathcal{T}) & & & & m_{23}^*(\mathcal{T} \boxtimes \mathcal{T}) \\
& \searrow{\scriptstyle \phi\boxtimes\mathrm{id}} & & \swarrow{\scriptstyle \mathrm{id}\boxtimes\phi} & \\
& & \mathcal{T} \boxtimes \mathcal{T} \boxtimes \mathcal{T} & &
\end{array}$$

The maps $m_{12}, m_{23}\colon G \times G \times G \to G \times G$ are multiplications of the first two and the last two factors respectively.

Let BG be the classifying stack of a group G and let the simplicial scheme $B^\bullet G$ be the nerve of the map pt $\to G$ classifying the trivial torsor. The simplicial scheme $B^\bullet G$ is G^n in degree n with the degeneracy maps coming from the multiplication of adjacent elements.

Suppose that all $\mathcal{A}$-gerbes on a point admit a trivialization. Then a multiplicative torsor is just an element of

$$\operatorname{Tot}\Gamma(B^\bullet G, \mathcal{A}[2]).$$

Indeed, an element $\mathcal{T} \in \operatorname{Tot}\Gamma(B^\bullet G, \mathcal{A}[2])$ is an $\mathcal{A}$-gerbe on a point which we assume to be trivial together with an isomorphism between two pullbacks $G \rightrightarrows \mathrm{pt}$ given by an $\mathcal{A}$-torsor $\mathcal{T}$ on G. Finally, we have a trivialization of $p_2^*\mathcal{T} \times_{\mathcal{A}} m^*\mathcal{T}^{-1} \times_{\mathcal{A}} p_1^*\mathcal{T}$ on $G \times G$, i.e. an identification $m^*\mathcal{T} \cong p_2^*\mathcal{T} \times_{\mathcal{A}} p_1^*\mathcal{T}$ satisfying the associativity condition written above.

More generally, given a complex of sheaves of abelian groups $\mathcal{A}$, we define a multiplicative torsor over $\mathcal{A}$ to be an element of $\operatorname{Tot}\Gamma(B^\bullet G, \mathcal{A}[2])$.

By the universal property of totalization we have a natural map

$$\Gamma(\mathrm{B}G, \mathcal{A}[2]) \to \operatorname{Tot}\Gamma(B^\bullet G, \mathcal{A}[2]).$$

If $\mathcal{A}$ satisfies descent with respect to the smooth topology, this map is an isomorphism. Hence, in this case a multiplicative $\mathcal{A}$-torsor on G is the same as an $\mathcal{A}$-gerbe on BG.

Given a multiplicative torsor over $\mathcal{A}$, we can descend it to an $\mathcal{A}$-torsor on the adjoint quotient G/G. Indeed, let $f\colon G \to G \times G$ be the map that sends g to (g^{-1}, g). Then fm is the constant map that sends $g \mapsto e$. Therefore, we have a trivialization

$$\mathcal{A} \cong f^*m^*\mathcal{T} \overset{\phi}{\cong} f^*(\mathcal{T} \boxtimes \mathcal{T}).$$

Consider the composite $\mathrm{Ad}\colon G \times G \xrightarrow{p_2 \times f_{13}} G \times G \times G \xrightarrow{m} G$ given by

$$a, g \mapsto (g^{-1}, a, g) \mapsto g^{-1} a g.$$

Then the pullback of $\mathcal{T}$ along Ad is isomorphic to $\mathcal{T} \boxtimes \mathcal{A}$.

A section $s \in \mathrm{H}^0(G, \mathcal{T})$ is G-invariant, i.e. is a pullback of a section over G/G, if $\mathrm{Ad}^* s \in \mathrm{H}^0(G \times G, \mathrm{Ad}^* \mathcal{T})$ coincides with $p_1^* s \in \mathrm{H}^0(G \times G, \mathcal{T} \boxtimes \mathcal{A})$ under the isomorphism $\mathrm{Ad}^* \mathcal{T} \cong \mathcal{T} \boxtimes \mathcal{A}$.

3.4.1.2 Multiplicative torsors over $\mathcal{A} = \Omega^2$

As G is affine, $\Gamma(B^\bullet G, \Omega^2[2])$ is concentrated in degree 2. Therefore, an element $\mathcal{T} \in \mathrm{H}^0(\mathrm{Tot}\,\Gamma(B^\bullet G, \Omega^2[2]))$ boils down to a two-form $\phi \in \Omega^2(G \times G)$ satisfying the associativity condition

$$m_{23}^* \phi + p_{23}^* \phi = m_{12}^* \phi + p_{12}^* \phi. \tag{3.4}$$

Let us fix this form to be

$$\phi = -\frac{1}{2}(p_1^* \theta, p_2^* \overline{\theta}). \tag{3.5}$$

To check associativity, let us write down the pullbacks of the Maurer–Cartan forms under multiplication:

$$m^* \theta = \mathrm{Ad}_b p_1^* \theta + p_2^* \theta, \quad m^* \overline{\theta} = p_1^* \overline{\theta} + \mathrm{Ad}_{a^{-1}} p_2^* \overline{\theta},$$

where a and b are coordinates on the two factors of G. Hence, the associativity condition becomes

$$(p_1^* \theta, p_2^* \overline{\theta} + \mathrm{Ad}_{b^{-1}} p_3^* \overline{\theta}) + (p_2^* \theta, p_3^* \overline{\theta}) = (\mathrm{Ad}_b p_1^* \theta + p_2^* \theta, p_3^* \overline{\theta}) + (p_1^* \theta, p_2^* \overline{\theta}).$$

3.4.1.3

Finally, let us work out what it means for a section $s \in \mathrm{H}^0(G, \mathcal{T}) \cong \mathrm{H}^0(G, \Omega^2)$ to be invariant under conjugation. As before, denote by $f\colon G \to G \times G$ the map $g \mapsto (g^{-1}, g)$ and $p_2 \times f_{13}\colon G \times G \to G \times G \times G$ the map $(a, g) \mapsto (g^{-1}, a, g)$. The section s is G-invariant if

$$\mathrm{Ad}^* s - f_{13}^* m_{12}^* \phi - f_{13}^* p_{12}^* \phi + f_{13}^* p_{13}^* \phi = p_1^* s.$$

The term $f_{13}^* p_{13}^* \phi$ vanishes, since it is equal to $\frac{1}{2}(p_2^*\overline{\theta}, p_2^*\overline{\theta}) = 0$. The other two terms containing ϕ are

$$\begin{aligned}\frac{1}{2} f_{13}^*(\mathrm{Ad}_g p_1^*\theta + p_2^*\theta, p_3^*\overline{\theta}) + \frac{1}{2} f_{13}^*(p_1^*\theta, p_2^*\overline{\theta}) &= \frac{1}{2}(-\mathrm{Ad}_a p_2^*\overline{\theta} + p_1^*\theta, p_2^*\overline{\theta}) - \frac{1}{2}(p_2^*\overline{\theta}, p_1^*\overline{\theta}) \\ &= \frac{1}{2}(p_1^*\theta - \mathrm{Ad}_a p_2^*\overline{\theta} + p_1^*\overline{\theta}, p_2^*\overline{\theta}).\end{aligned}$$

Therefore, a section s is G-invariant if

$$\mathrm{Ad}^* s = p_1^* s - \frac{1}{2}(p_1^*\theta - \mathrm{Ad}_a p_2^*\overline{\theta} + p_1^*\overline{\theta}, p_2^*\overline{\theta}). \tag{3.6}$$

Picking out different components of this equation, we get the following consequences:

1. Restricting to $G \times \{g\} \subset G \times G$, we get

$$\mathrm{Ad}_g^* s = s,$$

   i.e. the form s has to be invariant under the adjoint action.

2. Contracting the equation with a vector field v^L generating a left action on the second factor of G and then restricting it to the first factor of G, we get

$$\iota_{(v^R - v^L)} s = \frac{1}{2}(\theta + \overline{\theta}, v). \tag{3.7}$$

3. Finally, contracting the equation with v^L and w^L along the second G factor, we get

$$\iota_{(w^R - w^L)}\iota_{(v^R - v^L)} s = \frac{1}{2}(\mathrm{Ad}_a(v), w) - \frac{1}{2}(\mathrm{Ad}_a(w), v) = \frac{1}{2}(\mathrm{Ad}_{a^{-1}}(w) - \mathrm{Ad}_a(w), v),$$

which follows from equation (3.7) since $\iota_{(w^R - w^L)}(\theta + \overline{\theta}) = \mathrm{Ad}_{a^{-1}}(w) - \mathrm{Ad}_a(w)$.

On G/G we can write the equation (3.7) as

$$ds(x) = -\frac{1}{2}(\theta + \overline{\theta}, x) = \omega_0(x).$$

In other words, s is a section of the Ω^2-torsor with class $\omega_0 \in \mathrm{H}^1(G/G, \Omega^2)$.

3.4.2 Relation to the AKSZ construction

The aim of this section is to compute the 1-shifted symplectic form on G/G via the AKSZ construction and compare it to the form $\omega_0 + u\omega_1$ defined previously.

3.4.2.1

Let us describe the isomorphism $\Gamma(\mathrm{B}G, \Omega^n) \to \mathrm{Tot}\,\Gamma(\mathrm{B}^\bullet G, \Omega^n)$. This can be thought of as the Dolbeault version of the Chern-Weil homomorphism $\mathrm{Sym}^n(\mathfrak{g}^*)^G \to \mathrm{H}^{n,n}(\mathrm{B}^\bullet G)$.

Consider a G-torsor $f\colon P \to X$. On the one hand, the complex of one-forms on X fits into an exact sequence

$$f^*\Omega^1_X \to \Omega^1_P \to \Omega^1_{P/X}.$$

Moreover, using the G-action on P we can identify $\Omega^1_{P/X} \cong \mathfrak{g}^* \otimes_k \mathcal{O}_P$. Therefore, global one-forms on X are represented by the complex

$$\Gamma(X, \Omega^1) \cong (\Omega^1(P)^G \to (\mathfrak{g}^* \otimes_k \mathcal{O}(P))^G).$$

The differential $\Omega^1(P)^G \to (\mathfrak{g}^* \otimes_k \mathcal{O}(P))^G$ is $\omega \mapsto (v \mapsto -\iota_{a(v)}\omega)$, where $v \in \mathfrak{g}$ and $a(v)$ is the vector field generating the right action of G on P.

On the other hand, we can compute the complex of global one-forms on X using the descent along $P \to X$. Let $P^\bullet$ be the Čech nerve of $P \to X$. We will use isomorphisms

$$P^{n-1} = P \times_X P \times_X ... \times_X P \cong P \times G \times ... \times G$$

given by

$$(p_1, p_2, ...) \mapsto (p_1, p_1^{-1}p_2, ..., p_{n-1}^{-1}p_n).$$

We have a quasi-isomorphism $\Gamma(X, \Omega^1) \cong \operatorname{Tot}\Gamma(P^\bullet, \Omega^1)$, which has the following description. We have the evaluation morphisms $ev_n\colon P^n \times \Delta^n \to X$. Given a one-form $\omega \in \Gamma(X, \Omega^1)$, we get one-forms $\omega_n \in \Gamma(P^n, \Omega^1)$ given by $(ev_n^*\omega)[\Delta^n]$. The forms ω_n are not closed since Δ^n has a boundary, but together they combine into a closed element $\omega_0 + \omega_1 + ...$ of $\operatorname{Tot}\Gamma(P^\bullet, \Omega^1)$.

This gives a presentation of the complex of global one-forms as

$$\Gamma(X, \Omega^1) \cong (\Omega^1(P) \to \Omega^1(P \times G) \to \Omega^1(P \times G \times G) \to ...)$$

The differential $\Omega^1(P) \to \Omega^1(P \times G)$ is $\omega \mapsto a^*\omega - p_1^*\omega$ and $\Omega^1(P \times G) \to \Omega^1(P \times G \times G)$ is $\omega \mapsto a^*\omega - m_{23}^*\omega + p_{12}^*\omega$, where $a\colon P \times G \to P$ is the action map, p_{ij} are the projection maps and $m\colon G \times G \to G$ the multiplication.

Descent gives a canonical chain map

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^1(P)^G & \longrightarrow & (\mathfrak{g}^* \otimes_k \mathcal{O}(P))^G & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(P) & \longrightarrow & \Omega^1(P \times G) & \longrightarrow & \Omega^1(P \times G \times G) & \longrightarrow & ...
\end{array}$$

The map $\Omega^1(P)^G \to \Omega^1(P)$ is the standard inclusion. Let us work out the map

$$(\mathfrak{g}^* \otimes_k \mathcal{O}(P))^G \to \Omega^1(P \times G).$$

Note, that a form $\omega \in \Omega^1(P)$ is G-invariant if $a(g)^*\omega = \omega$ for every $g \in G$.

The composite $\Omega^1(P)^G \to \Omega^1(P) \to \Omega^1(P \times G)$ is $\omega \mapsto a^*\omega - p_1^*\omega$. Let $g\colon P \to P \times G$ be the slice (p, g). Then $g^*(a^*\omega - p_1^*\omega) = a(g)^*\omega - \omega = 0$. In other words, the form $a^*\omega - p_1^*\omega$ has components only in the G direction. For $v \in \mathfrak{g}$ let v^L be the vector field on $P \times G$ generating the left translation on the G factor. Then

$$\iota_{v^L}(a^*\omega - p_1^*\omega) = \iota_{v^L} a^*\omega = \iota_{a(v)}\omega,$$

where we used G-invariance of ω in the last equality. Let $\sum_i e^i \otimes_k e_i \in \mathfrak{g}^* \otimes_k \mathfrak{g}$ be the image of the identity morphism under $\mathrm{End}(\mathfrak{g}) \cong \mathfrak{g}^* \otimes_k \mathfrak{g}$. Since $\iota_{a(v)}\omega$ is constant on the G factor, we get

$$a^*\omega - p_1^*\omega = \sum_i \iota_{a(e_i)}\omega \cdot e^i(p_2^*\overline{\theta}).$$

Therefore, we are forced to let $(\mathfrak{g}^* \otimes_k \mathcal{O}(P))^G \to \Omega^1(P \times G)$ be $t \mapsto -t(p_2^*\overline{\theta})$ for every $t \in (\mathfrak{g}^* \otimes_k \mathcal{O}(P))^G$.

Finally, we have to check that $-t(p_2^*\overline{\theta})$ is a cocycle for every t. Indeed, its differential is

$$-\sum_i (a^* - m_{23}^* + p_{12}^*)t(\overline{\theta}) = -a^*t(\overline{\theta}) + t(p_2^*\overline{\theta} + \mathrm{Ad}_{g_1} p_3^*\overline{\theta}) - t(p_2^*\overline{\theta}) = 0.$$

We can repeat the process for forms of higher degree, we will just write down the resulting differentials for $n = 2$. The chain map

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^2(P)^G & \longrightarrow & (\mathfrak{g}^* \otimes_k \Omega^1(P))^G & \longrightarrow & (\mathrm{Sym}^2(\mathfrak{g}^*) \otimes_k \mathcal{O}(P))^G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^2(P) & \longrightarrow & \Omega^2(P \times G) & \longrightarrow & \Omega^2(P \times G \times G) & \longrightarrow & \ldots \end{array}$$

has the following components

- The map $\Omega^2(P)^G \to \Omega^2(P)$ is the standard inclusion.
- The map $(\mathfrak{g}^* \otimes_k \Omega^1(P))^G \to \Omega^2(P \times G)$ is given by

$$t \mapsto \sum_i t(e_i) \wedge e^i(p_2^*\overline{\theta}) - \frac{1}{2} \sum_{i,j} \iota_{a(e_j)} t(e_i) e^i(p_2^*\overline{\theta}) \wedge e^j(p_2^*\overline{\theta}), \tag{3.8}$$

  where $t \in (\mathfrak{g}^* \otimes_k \Omega^1(P))^G$.

- The map $(\mathrm{Sym}^2(\mathfrak{g}^*) \otimes_k \mathcal{O}(P))^G \to \Omega^2(P \times G \times G)$ is

$$(-,-) \mapsto -\frac{1}{2}(p_2^*\overline{\theta}, \mathrm{Ad}_{g_1} p_3^*\overline{\theta}),$$

  where $(-,-)$ is the symmetric bilinear form associated to the quadratic form on Lie algebra $q(-) \in \mathrm{Sym}^2(\mathfrak{g}^*) \otimes_k \mathcal{O}(P)$; in particular, we have $(v, v) = 2q(v)$ for $v \in \mathfrak{g}$.

Let us apply the considerations above to the universal G-torsor $\mathrm{pt} \to \mathrm{B}G$. Then we see that the form

$$\phi = -\frac{1}{2}(p_2^*\theta, p_3^*\overline{\theta}) \in \Omega^2(G \times G)$$

representing a degree 2 two-form on $\mathrm{B}G$ in the Čech description comes from the quadratic form $v \mapsto (v, v)/2$ on $\mathfrak{g}$.

3.4.2.2

Given the explicit description of the isomorphism $\Gamma(\mathrm{B}G, \Omega^2) \cong \mathrm{Tot}\,\Gamma(\mathrm{B}^\bullet G, \Omega^2)$, let us now compute the integral transform of the symplectic structure on $\mathrm{B}G$ along

$$\begin{array}{ccccc} & & G/G \times S^1 & & \\ & \swarrow^{p} & & \searrow^{ev} & \\ G/G & & & & \mathrm{B}G. \end{array}$$

We can think of the map ev as the self-homotopy h of the map $G/G \to \mathrm{B}G$ classifying the G-torsor $G \to G/G$. The self-homotopy h induces a chain map $\overline{h}\colon \Omega^2(\mathrm{B}G, 2) \to \Omega^2(G/G, 1)$, which coincides with $p_* ev^*$.

If we write $S^1 = \mathrm{B}\mathbf{Z}$, then a map $G/G \times S^1 \to \mathrm{B}G$ is the same as a G-torsor P on G/G together with an automorphism $P \to P$ of G-torsors on G/G. To describe the torsor with the automorphism corresponding to the map ev, first we have to better understand the isomorphism $G/G \cong \mathcal{L}\mathrm{B}G$.

For an derived affine scheme S, $\mathcal{L}\mathrm{B}G(S)$ is the simplicial set of G-torsors $P \to S$ together with an automorphism $a\colon P \to P$. Similarly, $G/G(S)$ is the simplicial set of G-torsors $P \to S$ together with a G-equivariant map $P \to G$. Given an automorphism $a\colon P \to P$, the isomorphism $\mathcal{L}\mathrm{B}G \cong G/G$ sends it to the map $p \in P \mapsto p^{-1}a(p)$. From this description we see that the G-torsor on G/G corresponding to ev is the projection map $G \to G/G$, where $G \cong \Omega\mathrm{B}G$ is identified with the stack of pointed G-torsors with an automorphism. The automorphism

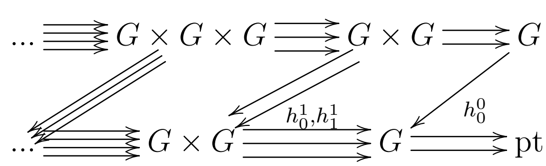
$$\begin{array}{ccc} G & \longrightarrow & G \\ & \searrow \quad \swarrow & \\ & G/G & \end{array}$$

sends the pointed torsor (P, p) with a map $a\colon P \to G$ to $(P, pa(p))$ with the same map f. Since $a(pa(p)) = a(p)^{-1}a(p)a(p) = a(p)$, the map is the identity on G. The data of the homotopy commutativity captures the automorphism of the torsor P that sends the basepoint p to $f(p)$. More explicitly, the self-homotopy of $G \to G/G$ is an element $h(g)$, such that $h(g)^{-1}gh(g) = g$. In our case, $h(g)$ is simply g.

Now we write down the map ev on the level of Čech nerves $A^\bullet$ for $G \to G/G$ and $\mathrm{B}^\bullet G$ for $\mathrm{pt} \to \mathrm{B}G$. It will consist of a map $f\colon A^\bullet \to \mathrm{B}^\bullet G$ and a self-homotopy h. The map f in low degrees looks as follows: $f^0\colon A^0 \cong G \to \mathrm{pt} \cong \mathrm{B}^0 G$ is trivial; $f^1\colon A^1 \cong G \times G \to G \cong \mathrm{B}^1 G$ is the projection to the second factor.

Recall that a self-homotopy of a map $f\colon A^\bullet \to \mathrm{B}^\bullet G$ between two simplicial sets is a collection of maps $h^n_i\colon A^n \to \mathrm{B}^{n+1}G$ for $0 \le i \le n$ satisfying certain relations with respect to boundary and degeneracy maps [GJ09, Lemma III.2.13]. In our case we get the relations

$$
\begin{aligned}
p_2 h^1_0 &= f^1 \\
p_1 h^1_0 &= h^0_0 p_1 \\
p_1 h^1_1 &= f^1 \\
p_2 h^1_1 &= h^0_0 a.
\end{aligned}
$$

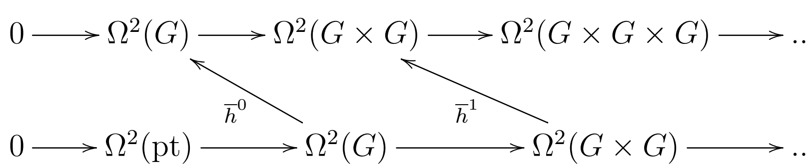


From the explicit description of the map $ev\colon G/G \times S^1 \to \mathrm{B}G$, we see that the map h^0_0 is the identity map $G \to G$. Since both G/G and $\mathrm{B}G$ are homotopy 1-types, one expects that the homotopy, i.e. the maps h_i^p, is uniquely determined by the map h_0^0. It turns out to be indeed the case.

The equation $p_2 h_0^1 = f^1$ implies that $h_0^1(a,g) = (?, g)$. Similarly, $p_1 h_0^1 = h_0^0 p_1$ implies that $h_0^1(a,g) = (a,g)$.

The equation $p_1 h_1^1 = f^1$ implies that $h_1^1(a,g) = (g, ?)$. Finally, $p_2 h_1^1 = h_0^0 a$ implies that $h_1^1(a,g) = (g, g^{-1}ag)$.

The self-homotopy $h\colon f \Rightarrow f$ induces a self-homotopy $\overline{h}\colon f^* \Rightarrow f^*$ between the complexes of differential forms:



Explicitly, we have

$$\overline{h}^0 = (h_0^0)^*, \quad \overline{h}^1 = (h_0^1)^* - (h_1^1)^*, ...$$

We are ready to compute the image of ϕ under $\overline{h}\colon \Omega^2(\mathrm{B}G, 2) \to \Omega^2(G/G, 1)$. It is

$$(h_0^1)^*\phi - (h_1^1)^*\phi = -\frac{1}{2}(p_1^*\theta, p_2^*\overline{\theta}) + \frac{1}{2}(p_2^*\theta, \mathrm{Ad}^*\overline{\theta}).$$

We can decompose $\mathrm{Ad}^* = f_{13}^* m_{12}^* m^*$, where $f_{13}\colon G\times G \to G\times G\times G$ is $(a,g) \mapsto (g^{-1}, a, g)$. Let us compute $\mathrm{Ad}^*\overline{\theta}$ step by step:

$$m^*\overline{\theta} = p_1^*\overline{\theta} + \mathrm{Ad}_{a^{-1}} p_2^*\overline{\theta}$$
$$m_{12}^* m^*\overline{\theta} = p_1^*\overline{\theta} + \mathrm{Ad}_{a^{-1}} p_2^*\overline{\theta} + \mathrm{Ad}_{g^{-1}a^{-1}} p_3^*\overline{\theta}$$
$$f_{13}^* m_{12}^* m^*\overline{\theta} = -p_2^*\theta + \mathrm{Ad}_g p_1^*\overline{\theta} + \mathrm{Ad}_{a^{-1}g} p_2^*\overline{\theta}.$$

Therefore,

$$\begin{aligned}\overline{h}(\phi) &= -\frac{1}{2}(p_1^*\theta, p_2^*\overline{\theta}) + \frac{1}{2}(p_2^*\theta, -p_2^*\theta + \mathrm{Ad}_g p_1^*\overline{\theta} + \mathrm{Ad}_{a^{-1}g} p_2^*\overline{\theta}) \\ &= -\frac{1}{2}(p_1^*\theta + p_1^*\overline{\theta} + \mathrm{Ad}_{a^{-1}} p_2^*\overline{\theta}, p_2^*\overline{\theta}) \\ &= -\frac{1}{2}(p_1^*\theta + p_1^*\overline{\theta} - \mathrm{Ad}_a p_2^*\overline{\theta}, p_2^*\overline{\theta}).\end{aligned}$$

This is exactly the form appearing in equation (3.6).

So far we have computed the image of ϕ in $\Omega^2(G/G, 1)$ using the Čech presentation. Now we show that the Čech cocycle comes from the degree 1 two-form ω_0 under the map (3.8). Indeed, the image of ω_0 under (3.8) is

$$\begin{aligned}&-\frac{1}{2}\sum_i (p_1^*\theta + p_1^*\overline{\theta}, e_i) \wedge p_2^* e^i(\overline{\theta}) + \frac{1}{4}\sum_{i,j} \iota_{e_j^R - e_j^L}(\theta + \overline{\theta}, e_i) \cdot e^i(p_2^*\overline{\theta}) \wedge e^j(p_2^*\overline{\theta}) \\ &= -\frac{1}{2}(p_1^*\theta + p_1^*\overline{\theta}, p_2^*\overline{\theta}) + \frac{1}{4}\sum_j (\mathrm{Ad}_{a^{-1}}(e_j) - \mathrm{Ad}_a(e_j), p_2^*\overline{\theta}) \wedge e^j(p_2^*\overline{\theta}) \\ &= -\frac{1}{2}(p_1^*\theta + p_1^*\overline{\theta}, p_2^*\overline{\theta}) + \frac{1}{2}(\mathrm{Ad}_a p_2^*\overline{\theta}, p_2^*\overline{\theta}).\end{aligned}$$

All the calculations in this section are summarized in the following theorem.

Theorem 3.4.1. *The integral transform of the quadratic form* $q \in \mathrm{Sym}^2(\mathfrak{g}^*)^G \cong \Omega^2(\mathrm{B}G, 2)$ *under*

$$\begin{array}{ccccc} & & G/G \times S^1 & & \\ & \overset{p}{\swarrow} & & \overset{ev}{\searrow} & \\ G/G & & & & \mathrm{B}G \end{array}$$

is equal to

$$\omega_0 = -\frac{1}{2}(\theta + \overline{\theta}, -) \in \Omega^2(G/G, 1).$$

3.4.3 Multiplicative torsors over $\mathcal{A} = \Omega^{2,cl}$

Since G is affine, the fibers of the forgetful map

$$\{\text{multiplicative } \Omega^{2,cl}\text{-torsors } \mathcal{T}\} \to \{\text{multiplicative } \Omega^{2}\text{-torsors } \mathcal{T}\}$$

consist of multiplicative sections of the induced Ω^3-torsor $\mathrm{d_{dR}}\mathcal{T}$. Explicitly, these are 3-forms s, such that

$$m^* s + \mathrm{d_{dR}}\phi = p_1^* s + p_2^* s.$$

Lemma 3.4.2. *The three-form $\omega_1 = \frac{1}{12}(\theta, [\theta, \theta])$ is a multiplicative section of the multiplicative Ω^2-torsor defined by the two-form $-\phi$.*

Proof. Recall that $m^*\theta = \mathrm{Ad}_{h_2^{-1}} p_1^*\theta + p_2^*\theta$. Therefore,

$$\begin{aligned} m^*\omega_1 &= \frac{1}{12}(m^*\theta, [m^*\theta, m^*\theta]) \\ &= \frac{1}{12}(\mathrm{Ad}_{h_2^{-1}} p_1^*\theta + p_2^*\theta, [\mathrm{Ad}_{h_2^{-1}} p_1^*\theta + p_2^*\theta, \mathrm{Ad}_{h_2^{-1}} p_1^*\theta + p_2^*\theta]). \end{aligned}$$

We also have

$$\begin{aligned} \mathrm{d_{dR}}\phi &= -\frac{1}{2}(p_1^*\mathrm{d_{dR}}\theta, p_2^*\overline{\theta}) + \frac{1}{2}(p_1^*\theta, p_2^*\mathrm{d_{dR}}\overline{\theta}) \\ &= \frac{1}{4}(p_1^*[\theta,\theta], p_2^*\overline{\theta}) + \frac{1}{4}(p_1^*\theta, p_2^*[\overline{\theta}, \overline{\theta}]) \end{aligned}$$

There are eight terms in $m^*\omega_1$. Two of them are just $p_1^*\omega_1 + p_2^*\omega_1$. Another six terms break into two triples:

$$\begin{aligned} &\frac{1}{12}(\mathrm{Ad}_{h_2^{-1}} p_1^*\theta, [p_2^*\theta, p_2^*\theta]) + \frac{1}{12}(p_2^*\theta, [\mathrm{Ad}_{h_2^{-1}} p_1^*\theta, p_2^*\theta]) \\ &\frac{1}{12}(p_2^*\theta, [p_2^*\theta, \mathrm{Ad}_{h_2^{-1}} p_1^*\theta]) = \frac{1}{4}(p_1^*\theta, [p_2^*\overline{\theta}, p_2^*\overline{\theta}]) \end{aligned}$$

and similarly for the other triple. We see that these six terms cancel with the terms in $\mathrm{d}_{\mathrm{dR}}\phi$. $\square$

To summarize, we have constructed a multiplicative torsor over $\Omega^{2,cl}$ on G, such that the induced $\Omega^{2,cl}$ torsor on G/G is represented by the differential forms (ω_0, ω_1).

Note, that ω_1 is uniquely determined by ω_0. Indeed, difference between any two choices of ω_1 defines a multiplicative Ω^3-torsor on G. However, the space of multiplicative Ω^3-torsors on G is $\Omega^3(\mathrm{B}G, 2)$, which is contractible.

Theorem 3.4.3. *The integral transform of the quadratic form* $q \in \mathrm{Sym}^2(\mathfrak{g}^*)^G \cong \Omega^{2,cl}(\mathrm{B}G, 2)$ *under*

$$G/G \xleftarrow{\;p\;} G/G \times S^1 \xrightarrow{\;ev\;} \mathrm{B}G$$

is equal to

$$\omega_0 + u\omega_1 = -\frac{1}{2}(\theta + \overline{\theta}, -) + \frac{u}{12}(\theta, [\theta, \theta]) \in \Omega^{2,cl}(G/G, 1).$$

3.4.4 Fusion

A geometric way to think of the isomorphism $m^*\mathcal{T} \cong \mathcal{T} \boxtimes \mathcal{T}$ is as the data of an isotropic correspondence

$$G/G \longleftarrow (G \times G)/G \longrightarrow G/G \times G/G$$

In other words, $(m, p_1, p_2)\colon (G\times G)/G \to G/G \times G/G \times G/G$ is isotropic, where the homotopy is given by $-\phi$, a two-form on $(G \times G)/G$. This is exactly the correspondence given in the classical Chern–Simons theory by a pair of pants. This is not a coincidence. Indeed, in subsection 3.4.2 we computed the pullback of $q \in \mathrm{Sym}^2(\mathfrak{g}^*)^G \cong \Omega^2(\mathrm{B}G, 2)$ under $ev_2\colon \mathrm{B}^2G \times \Delta^2 \to \mathrm{B}G$, where $\mathrm{B}^2G \cong G \times G$. We obtained

$$\phi = (ev_2^* q)[\Delta^2].$$

Consider the pair $(\Delta^2, p_0 \sqcup p_1 \sqcup p_2)$, where p_i are the vertices of Δ^2. Then

$$\mathrm{B}^2G = \mathrm{Map}_*(\Delta^2, \mathrm{B}G),$$

where p_i are sent to the basepoint of $\mathrm{B}G$.

However, we can also consider a "framed" pair of pants (M, N) relative to the subspace N as illustrated in the picture. Then the space of pointed maps $\mathrm{Map}_*(M, \mathrm{B}G) \cong G\times G$ is isomorphic to $\mathrm{Map}_*(\Delta^2, \mathrm{B}G)$. Indeed, both spaces M/N and $\Delta^2/(p_0 \sqcup p_1 \sqcup p_2)$ are homeomorphic.

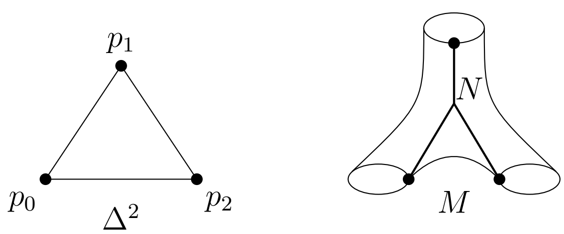


Figure 3.4: A framed 2-simplex and a framed pair of pants

Corollary 3.4.4. $(m, p_1, p_2)\colon G/G \leftarrow (G\times G)/G \to G/G \times G/G$ *is a Lagrangian correspondence with the isotropic structure* $-\phi$*.*

Suppose $\mu = (\mu_1, \mu_2)\colon M \to G \times G$ is a $G \times G$-equivariant map. Then $\tilde{\mu}\colon M \to G \times G \overset{m}{\to} G$ is G-equivariant for the diagonal action of G.

If $M/(G \times G) \to G/G \times G/G$ is Lagrangian, we have a section $\omega \in \mathrm{H}^0(M, \mu^*(\mathcal{T} \boxtimes \mathcal{T}))$. Using the multiplicative structure on $\mathcal{T}$ we get a section $\tilde{\omega} \in \mathrm{H}^0(M, \mu^* m^* \mathcal{T})$; in fact, since $\mathcal{T}$ is trivial, we can write it as

$$\tilde{\omega} = \omega - \mu^* \phi = \omega + \frac{1}{2}(\mu_1^* \theta, \mu_2^* \overline{\theta}).$$

We see that M/G with the moment map coming from the product $\mu_1 \mu_2$ is the internal fusion of $M/(G \times G)$.

3.4.5 Punctured torus

3.4.5.1

If

$$1 \to H \to \tilde{G} \to G \to 1$$

is a central extension of G by H, we can canonically lift commutators $aba^{-1}b^{-1}$ to the central extension: pick any lifts $\tilde{a}, \tilde{b}$ of a and b. Then $\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1}$ is a lift of $aba^{-1}b^{-1}$. It is easy to see that the lift of the commutator does not depend on the individual lifts.

One can formulate the same result in the language of multiplicative torsors. Consider $G \times G$ with the moment map $\mu\colon G \times G \to G$ given by the commutator. Let us use the notation $f(g) = (g, g^{-1})$ and $\overline{f}(a, b) = (a, b, a^{-1}, b^{-1})$. Then the canonical section of $\mu^* \mathcal{T}$ over $G \times G$ obtained as $(\tilde{a}\tilde{b})(\tilde{a}^{-1}\tilde{b}^{-1})$ is

$$h_0 = -\overline{f}^* m_{12}^* m_{23}^* \phi - \overline{f}^* m_{12}^* p_{23}^* \phi - \overline{f}^* p_{12}^* \phi + p_1^* f^* \phi + p_2^* f^* \phi.$$

For our choice of the multiplicative structure $f^*\phi = 0$. So, we get

$$h_0 = \frac{1}{2}((ab)^*\theta, (a^{-1}b^{-1})^*\overline{\theta}) + \frac{1}{2}((a^{-1})^*\theta, (b^{-1})^*\overline{\theta}) + \frac{1}{2}(a^*\theta, b^*\overline{\theta}).$$

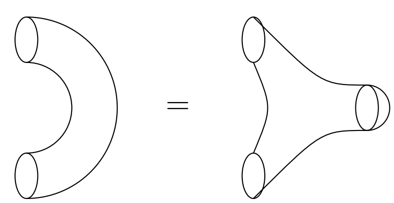
We can simplify it further using $(a^{-1})^*\theta = -a^*\overline{\theta}$; we obtain

$$h_0 = \frac{1}{2}(a^*\theta, b^*\overline{\theta}) + \frac{1}{2}(a^*\overline{\theta}, b^*\theta) + \frac{1}{2}((ab)^*\theta, (a^{-1}b^{-1})^*\overline{\theta}). \tag{3.9}$$

3.4.5.2

Let us now compute the Lagrangian structure h_0 on the character stack of the punctured torus in the AKSZ formalism.

First, we can represent the double $D(G)$ as a capped pair of pants:



So, we can compute the Lagrangian structure on $D(G)$ by representing it as

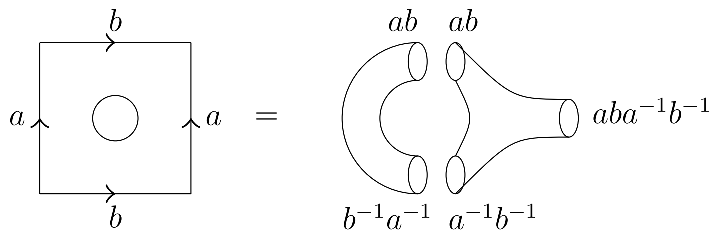
$$G/G \cong (G \times G)/G \times_{G/G} \mathrm{pt}/G.$$

Let $f\colon G \to G \times G$ be $g \mapsto (g, g^{-1})$, then the G-equivariant form on G that equips G/G with a Lagrangian structure is

$$f^*\phi = -f^*\frac{1}{2}(p_1^*\theta, p_2^*\overline{\theta}) = \frac{1}{2}(\theta, \theta) = 0.$$

This is not surprising since the double $D(G)$ comes from the cylinder representing the diagonal Lagrangian $G/G \to G/G \times G/G$, where the Lagrangian structure is trivial.

To compute the Lagrangian structure on the character stack of the punctured torus $(G \times G)/G$, let us represent it as a fusion of the double $D(G)$:



This gives a pullback diagram

$$\begin{array}{ccc} (G \times G)/G & \xrightarrow{f_1} & (G \times G)/G \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} \\ G/G & \xrightarrow{f_2} & G/G \times G/G, \end{array}$$

where the maps are

$$f_1(a, b) = (ab, a^{-1}b^{-1})$$

$$g_1(a, b) = a$$

$$f_2(a) = (a, a^{-1})$$

$$g_2(a, b) = (a, b).$$

Note, that the diagram has a nontrivial homotopy commutativity data

$$h\colon (G \times G)/G \times I \to G/G \times G/G$$

given by the path $(ab, b^{-1}a^{-1}) \sim (ab, a^{-1}b^{-1})$. On the level of differential forms, h induces a homotopy $\overline{h}\colon g_1^* f_2^* \Rightarrow f_1^* g_2^*$, i.e. we have

$$d\overline{h} + \overline{h}d = g_1^* f_2^* - f_1^* g_2^*.$$

Consider the chain complex $g_1^*\Omega^2_{G/G} \oplus f_1^*\Omega^2_{(G\times G)/G} \oplus f_1^* g_2^* \Omega^2_{G/G\times G/G}[1]$ with the differential

$$f_1^* g_2^* \Omega^2_{G/G\times G/G}[1] \to g_1^*\Omega^2_{G/G} \oplus f_1^*\Omega^2_{(G\times G)/G}$$

given by

$$\gamma \mapsto (f_2^*\gamma, -g_2^*\gamma).$$

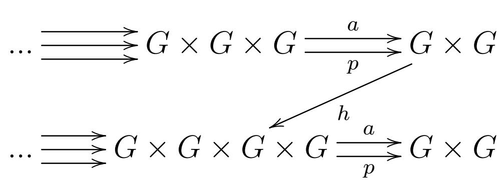
The Lagrangian structure on $(G\times G)/G$ is given by the image of $(0, -\phi, p_1^*\omega_0 + p_2^*\omega_0)$ under the map

$$g_1^*\Omega^2_{G/G} \oplus f_1^*\Omega^2_{(G\times G)/G} \oplus f_1^* g_2^* \Omega^2_{G/G\times G/G}[1] \to \Omega^2_{(G\times G)/G}$$

given by

$$(\alpha, \beta, \gamma) \mapsto g_1^*\alpha + f_1^*\beta - \overline{h}\gamma.$$

To compute $\overline{h}\colon \Omega^2(G/G\times G/G, 1) \to \Omega^2((G\times G)/G, 0)$, we will use the Čech presentation of differential forms on $G/G \times G/G$:



The groupoid maps for $(G\times G)/G$ are

$$a(g, a, b) = (g^{-1}ag, g^{-1}bg)$$
$$p(g, a, b) = (a, b).$$

The groupoid maps for $G/G \times G/G$ are

$$a(g_1, g_2, a, b) = (g_1^{-1} a g_1, g_2^{-1} b g_2)$$
$$p(g_1, g_2, a, b) = (a, b).$$

The homotopy h is given by

$$h(a, b) = (e, b^{-1}, ab, b^{-1}a^{-1}).$$

The Lagrangian structure on the double $D(G)$ is thus given by the two-form

$$h_0 = \frac{1}{2} f_1^*(p_1^*\theta, p_2^*\overline{\theta}) + \frac{1}{2}\overline{h}(p_3^*\theta + p_3^*\overline{\theta} + \mathrm{Ad}_a p_1^*\overline{\theta}, p_1^*\overline{\theta}) + \frac{1}{2}\overline{h}(p_4^*\theta + p_4^*\overline{\theta} - \mathrm{Ad}_b p_2^*\overline{\theta}, p_2^*\overline{\theta}).$$

The second summand is zero since $\overline{h}(p_1^*\overline{\theta}) = 0$. We get

$$\begin{aligned} h_0 &= \frac{1}{2}((ab)^*\theta, (a^{-1}b^{-1})^*\overline{\theta}) + \frac{1}{2}((ab)^*\theta + (ab)^*\overline{\theta} - \mathrm{Ad}_{b^{-1}a^{-1}} b^*\theta, b^*\theta) \\ &= \frac{1}{2}((ab)^*\theta, (a^{-1}b^{-1})^*\overline{\theta}) + \frac{1}{2}(\mathrm{Ad}_b a^*\theta + b^*\theta + a^*\overline{\theta} + \mathrm{Ad}_{a^{-1}} b^*\overline{\theta} - \mathrm{Ad}_{a^{-1}} b^*\overline{\theta}, b^*\theta) \\ &= \frac{1}{2}((ab)^*\theta, (a^{-1}b^{-1})^*\overline{\theta}) + \frac{1}{2}(a^*\theta, b^*\overline{\theta}) + \frac{1}{2}(a^*\overline{\theta}, b^*\theta), \end{aligned}$$

which coincides with the previously obtained form h_0 (3.9).

3.5 Prequantization

3.5.1 General definition

3.5.1.1

Classically, a prequantization consists of lifting the symplectic form $\omega \in \mathrm{H}^0(X, \Omega^{2,cl})$ to a line bundle with a connection $L \in \mathrm{H}^1(X, \mathcal{O}^\times \to \Omega^1)$ whose curvature is ω. That is, a prequantization is a lift

$$\begin{array}{ccc} & & X \\ & \overset{L}{\swarrow} & \downarrow \omega \\ (\mathcal{O}^\times \to \Omega^1) & \xrightarrow{d} & \Omega^{2,cl} \end{array}$$

3.5.1.2 Example

We will be interested in constructing prequantizations of character stacks, so consider the simplest case of a GL_1 character stack of a torus $\mathrm{Loc}_{GL_1}(T)$. Removing a disk from the torus and gluing it back in, we obtain a presentation

$$\begin{aligned} \mathrm{Loc}_{GL_1}(T) &\cong (GL_1 \times GL_1)/GL_1 \times_{GL_1/GL_1} \mathrm{pt}\,/GL_1 \\ &\cong ((GL_1 \times GL_1) \times_{GL_1} \mathrm{pt})/GL_1 \\ &\cong (GL_1 \times GL_1) \times (\Omega GL_1 \times \mathrm{B}GL_1). \end{aligned}$$

In other words, the character stack $\mathrm{Loc}_{GL_1}(T)$ is isomorphic to a product of the character *variety* of the torus $GL_1 \times GL_1$ and the character stack of the sphere

$$\mathrm{Loc}_{GL_1}(S^2) \cong \Omega GL_1 \times \mathrm{B}GL_1.$$

Moreover, the symplectic structure is simply the product symplectic structure.

The symplectic structure on the character variety $GL_1 \times GL_1$ can be read off from the formula (3.9). If we denote the coordinates on $GL_1 \times GL_1$ by (a, b), the symplectic structure is

$$\omega = d \log a \wedge d \log b.$$

Every line bundle on $GL_1 \times GL_1$ is trivializable, so the curvature of a line bundle with a connection is necessarily exact. But ω is not exact, so it cannot be prequantized.

Alternatively, one can observe that the weight of ω in the mixed Hodge structure on the character variety is 4, while Chern classes of line bundles have weight 2.

This should be contrasted with the analytic case, where the character variety $GL_1 \times GL_1$ is isomorphic to the moduli space of holomorphic line bundles with a connection $\mathrm{Pic}^\flat(T)$ as a complex manifold once we choose a complex structure on T. The space $\mathrm{Pic}^\flat(T)$, a twisted cotangent bundle to $\mathrm{Pic}(T)$, admits a prequantization, but the prequantum line bundle is not algebraic when pulled back to $GL_1 \times GL_1$.

3.5.1.3

Therefore, we will consider a more general notion of prequantization applicable in the algebraic situation.

An immediate generalization of the notion of a prequantization is to a sheaf of complexes F together with a map $F \to \Omega^{2,cl}(-)$. One can also consider a sheaf of infinite loop spaces F together with a map $F \to |\mathcal{A}^{2,cl}(-)|$. A prequantization is then a lift of the symplectic form $\omega \in \Omega^{2,cl}(X)$ to an element $\tilde{\omega} \in \mathrm{H}^0(X, F)$.

We can mimic the procedure of symplectic reduction to descend prequantizations. Suppose X is a stack with an F-torsor $\mathcal{T} \to X$. We fix a trivialization of the torsor $\mathcal{T}$ on $\mathrm{pt}/G \to X$.

The data of reduction of prequantizations consists of:

1. A stack M with a G-action.

2. A moment map $\mu\colon M/G \to X$ together with a section $\omega \in \mathrm{H}^0(M/G, \mu^*\mathcal{T})$.

Like before, the reduced space is defined to be $M//G := M/G \times_X \mathrm{pt}/G$. Comparing two trivializations of the torsor $\mathcal{T}$, we get an element $\omega \in \mathrm{H}^0(M//G, F)$.

The 1-shifted symplectic structures on $\mathfrak{g}/G$ and G/G come from $\bigwedge^2 \mathbb{L}$-central extensions of $\mathfrak{g}$ and G respectively. In the following sections we will compute the universal such extensions hence universal prequantizations.

3.5.2 Central extensions of Lie algebras

Let V be a vector space over k. Recall that the reduced symmetric coalgebra $\overline{S}V$ is a cocommutative non-counital coalgebra cofreely cogenerated by V, i.e. the right adjoint to the forgetful functor from such coalgebras (which are conilpotent) to vector spaces. Explicitly, it is the subcoalgebra of the reduced tensor coalgebra

$$\overline{T}V = \oplus_{n\geq 1} V^{\otimes n}$$

given by symmetric tensors. Over a field of characteristic zero we can identify it with the reduced symmetric algebra.

Let $\mathfrak{g}$ be a $\mathbf{Z}$-graded vector space.

Definition. An L_∞ *algebra structure* on $\mathfrak{g}$ is the data of a differential on the Chevalley-Eilenberg complex $\overline{S}(\mathfrak{g}[1])$.

Such a differential is uniquely determined by a map $\overline{S}(\mathfrak{g}[1]) \to \mathfrak{g}[2]$, whose components we will write as $l_n \colon \overline{S}^n(\mathfrak{g}[1]) \to \mathfrak{g}[2]$.

An L_∞ morphism $f\colon \mathfrak{g} \to \mathfrak{k}$ between two L_∞ algebras is a morphism of dg coalgebras $f\colon \overline{S}(\mathfrak{g}[1]) \to \overline{S}(\mathfrak{k}[1])$. It is uniquely determined by maps $f_n\colon \overline{S}^n(\mathfrak{g}[1]) \to \mathfrak{k}[1]$ subject to the condition that the induced map on symmetric coalgebras commutes with the differentials.

Let H be a complex. We can consider it as a trivial L_∞ algebra with the maps $l_n = 0$ for $n \geq 2$.

Definition. A *split central extension* of $\mathfrak{g}$ by H is the data of an L_∞ algebra structure on $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus H$, such that the natural maps $H \to \widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}} \to \mathfrak{g}$ are morphisms of L_∞ algebras. Moreover, $H \subset \widehat{\mathfrak{g}}$ has to be central.

Remark. We only require the central extension to be split as an extension of graded vector spaces, it is not split as an extension of L_∞ algebras in general.

Remark. The condition $H \subset \widehat{\mathfrak{g}}$ is central is equivalent to saying that the operations $l_n(x_1, ..., x_n)$ in $\widehat{\mathfrak{g}}$ vanish as soon as one of $x_i \in H$ and $n \geq 2$.

Any split central extension is uniquely determined by the maps $h_n \colon \overline{S}^n(\mathfrak{g})[1] \to H[2]$. Let us see what kind of conditions they satisfy. The Chevalley-Eilenberg complex of $\widehat{\mathfrak{g}}$ is

$$\overline{S}(\widehat{\mathfrak{g}}[1]) \cong \overline{S}(H[1]) \oplus \overline{S}(\mathfrak{g}[1]) \oplus \overline{S}(\mathfrak{g}[1]) \otimes_k \overline{S}(H[1]).$$

Since $H \to \widehat{\mathfrak{g}}$ is an L_∞ morphism, the differential on $\overline{S}(H[1])$ is just the one coming from H. The differential on the last summand will be uniquely determined by the differential on $\overline{S}(\mathfrak{g}[1])$.

Since $\widehat{\mathfrak{g}} \to \mathfrak{g}$ is an L_∞ morphism, the differential $\overline{S}(\mathfrak{g}[1])$ is just the Chevalley-Eilenberg differential coming from the L_∞ structure on $\mathfrak{g}$. Therefore, the data of a split central extension is equivalent to a degree 1 morphism of dg coalgebras $\overline{S}(\mathfrak{g}[1]) \to \overline{S}(\mathfrak{H}[1])$, which is equivalent to a morphism of chain complexes $\overline{S}(\mathfrak{g}[1]) \to H[2]$.

Finally, maps $\overline{S}(\mathfrak{g}[1]) \to H[2]$ are the same as L_∞ morphisms $f \colon \mathfrak{g} \to H[1]$. We will call the corresponding split central extension $\widehat{\mathfrak{g}}$ the homotopy fiber of f.

Theorem 3.5.1. *The operation of taking homotopy fibers establishes an equivalence between the category of L_∞ morphisms $\mathfrak{g} \to H[1]$ and the category of split central extensions.*

Clearly, an initial object in this category is $H = \overline{S}(\mathfrak{g}[1])[-2]$ with the identity map $\overline{S}(\mathfrak{g}[1]) \to H$.

From the previous theorem we conclude:

Theorem 3.5.2. *There is a universal split central extension of L_∞ algebras*

$$\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \overline{S}(\mathfrak{g}[1])[-2]$$

given by the homotopy fiber of the identity map.

Remark. There is a universal central extension of curved L_∞ algebras if one replaces the reduced symmetric coalgebra by the full symmetric coalgebra.

Remark. If we start with a connective L_∞ algebra $\mathfrak{g}$, the universal central extension $\widehat{\mathfrak{g}}$ is connective iff $\mathfrak{g}$ is perfect. We will encounter a similar issue in the next section when we consider central extensions of groups.

3.5.2.1 Example

Let $\mathfrak{g} = \mathfrak{gl}_\infty$. Then the Chevalley-Eilenberg complex $\overline{S}(\mathfrak{gl}_\infty[1])$ is quasiisomorphic to the reduced symmetric coalgebra on the cyclic homology $\overline{S}(\mathrm{HC}[1])$.

For example, consider $\mathfrak{sl}_\infty \subset \mathfrak{gl}_\infty$. The degree 0 part of the universal central extension is an extension

$$0 \to \mathrm{HC}_1 \to \widehat{\mathfrak{sl}}_\infty \to \mathfrak{sl}_\infty \to 0.$$

In particular, the de Rham differential gives a morphism of complexes

$$\mathrm{d}_{\mathrm{dR}} \colon \mathrm{HC}[-1] \to \Omega^{cl}[-2] \to \Omega^{2,cl},$$

where the latter map is the projection to the weight 2 piece. Therefore, we can view this central extension as a prequantization of the associated symplectic structure.

Given a representation of a semisimple Lie algebra $\mathfrak{g}$, we can pullback the universal central extension of $\mathfrak{sl}_\infty$ to obtain a connective central extension of $\mathfrak{g}$.

3.5.3 Central extensions of groups

We would like to repeat the considerations of the previous section for the group G.

Consider the following three categories:

1. The category of 1-connected infinite loop spaces X together with a map $\mathrm{B}G \to X$. Morphisms are maps of pointed spaces $X_1 \to X_2$ commuting with the maps from $\mathrm{B}G$.

2. The category of connected infinite loop spaces Y together with an $\mathbb{E}_1$ map $G \to Y$.

3. The category of central extensions of G, i.e. multiplicative torsors over infinite loop spaces on G.

These three are equivalent. Indeed, 1-connected spaces are equivalent to connected $\mathbb{E}_1$-spaces, which gives an equivalence of the first two categories. Similarly, an $\mathbb{E}_1$ morphism $G \to Y \cong \mathrm{B}\Omega Y$ is the same as a multiplicative ΩY-torsor on G.

For applications we have in mind, we have to relax the connectivity assumption. Therefore, we will consider the following two categories instead:

1. The category of connected infinite loop spaces X together with a map $\mathrm{B}G \to X$.

2. The category of infinite loop spaces Y together with an $\mathbb{E}_1$ map $G \to Y$.

Given an $\mathbb{E}_1$ map $G \to Y$, there are two ways to produce a central extension of G. Picking a base point in Y and considering the homotopy fiber of $G \to Y$ we obtain an ΩY-torsor $\tilde{G} \to G$, but this map is not surjective in general. Instead, we get an exact sequence

$$1 \to \Omega Y \to \tilde{G} \to G \to \pi_0(Y) \to 1.$$

Alternatively, consider the free loop space $\mathcal{L}Y$, which inherits an $\mathbb{E}_1$ structure from Y. We also have a different $\mathbb{E}_1$ structure on $\mathcal{L}Y \to Y$ coming from the composition of loops. Combining these two structures, one can say that $\mathcal{L}_Y$ is a multiplicative torsor over itself. Pulling it back to G gives a different central extension of G.

Consider the case $G = GL$. Every infinite loop space is simple (i.e. the action of π_1 on the higher homotopy groups is trivial) and it has abelian fundamental group. Therefore, any map $\mathrm{B}GL \to X$ uniquely factors as

$$\mathrm{B}GL \to \mathrm{B}GL^+ \to X,$$

where $\mathrm{B}GL^+$ is the plus construction. It can be written in terms of the K-theory space $\mathrm{K} = \mathrm{B}GL^+ \times \mathbf{Z}$ as $\mathrm{B}GL^+ \cong \mathrm{B}\Omega\mathrm{K}$. In particular, it implies that $\mathrm{B}GL^+$ is an infinite loop space, hence it is an initial object in the first category.

Taking the homotopy fiber, we obtain an extension of GL:

$$1 \to \Omega^2\mathrm{K} \to \widehat{GL} \to GL \to \mathrm{K}_1 \to 1.$$

Restricting to the subgroup of elementary matrices E, we get an honest central extension

$$1 \to \Omega^2 \mathrm{K} \to \widehat{E} \to E \to 1,$$

whose sheafification gives a central extension $\widehat{SL}$ of SL. In particular, once we choose a representation of a semisimple group G, we obtain an $\Omega^2\mathrm{K}$-central extension of G.

The Chern character is a map

$$\Omega^2 \mathrm{K} \to \Omega^2 |\Omega^{cl}| \cong |\Omega^{cl}[-2]|,$$

whose weight 2 part gives the $\Omega^{2,cl}$-torsor we described in section 3.4. In this equation we use Ω for both the based loop space and the complex of differential forms; we hope it does not cause too much confusion.

3.5.4 Application of reduction of prequantizations

In this section we will construct a $\mathcal{K}_2$-class on the character varieties of compact surfaces in two ways: by repeating the AKSZ construction for the natural $\mathcal{K}_2$-gerbe on $\mathrm{B}G$ and by a reduction of prequantizations.

3.5.4.1 Integrating the $\mathcal{K}_2$ class

Let $\mathcal{K}_2$, a sheaf of abelian groups, be the sheafification of the functor

$$\mathrm{K}_2 = \pi_2(\mathrm{K}) \colon \mathrm{cdga}^{\leq 0} \to \mathrm{Ab}.$$

Note, that $\mathcal{K}_2$ is homotopy invariant: if X is a contractible topological space and Y any stack, then $\mathrm{H}^0(X_{\mathrm{B}} \times Y, \mathcal{K}_2) \cong \mathrm{H}^0(Y, \mathcal{K}_2)$.

We have a map $\mathrm{B}SL_n \to \mathrm{B}SL \to \mathrm{B}^2\Omega^2\mathrm{K} \to \mathrm{B}^2\mathcal{K}_2$, where $\mathrm{B}SL \to \mathrm{B}^2\Omega^2\mathrm{K}$ is the central extension we have constructed in the previous section.

Let M be a compact oriented manifold and consider the character stack $\mathrm{Loc}_{SL_n}(M)$. We have a diagram

$$\begin{array}{ccc} & \mathrm{Loc}_{SL_n}(M) \times M_{\mathrm{B}} & \\ \swarrow & & \searrow \\ \mathrm{Loc}_{SL_n}(M) & & \mathrm{B}SL_n \longrightarrow \mathrm{B}^2\mathcal{K}_2, \end{array}$$

which gives an element of $\Gamma(\mathrm{Loc}_{SL_n}(M) \times M_{\mathrm{B}}, \mathcal{K}_2[2])$. Pick a cover of M by contractible open sets and let $U^\bullet$ be the Čech nerve of the cover. We have

$$\operatorname{colim} U^\bullet \cong M.$$

Then $\Gamma(\mathrm{Loc}_{SL_n}(M) \times M_{\mathrm{B}}, \mathcal{K}_2)$ can be computed as

$$\begin{aligned} \Gamma(\mathrm{Loc}_{SL_n}(M) \times M_{\mathrm{B}}, \mathcal{K}_2) &\cong \Gamma(\mathrm{Loc}_{SL_n}(M) \times \operatorname{colim} U^\bullet_{\mathrm{B}}, \mathcal{K}_2) \\ &\cong \lim \Gamma(\mathrm{Loc}_{SL_n}(M) \times U^\bullet_{\mathrm{B}}, \mathcal{K}_2) \\ &\cong \lim \Gamma(\mathrm{Loc}_{SL_n}(M), \mathcal{K}_2) \\ &\cong \check{C}^\bullet(M, \mathbf{Z}) \otimes_{\mathbf{Z}} \Gamma(\mathrm{Loc}_{SL_n}(M), \mathcal{K}_2). \end{aligned}$$

We used descent for $\mathcal{K}_2$ in the second line and homotopy invariance in the third line.

Therefore, using the integration map along M, we get a map

$$\Gamma(\mathrm{Loc}_{SL_n}(M) \times M_{\mathrm{B}}, \mathcal{K}_2) \to \Gamma(\mathrm{Loc}_{SL_n}(M), \mathcal{K}_2)[-\dim M].$$

In particular, if M is a compact oriented surface, we get a $\mathcal{K}_2$-class on the character stack of M. A similar construction of the same $\mathcal{K}_2$-class on a character variety is described in the work of Fock and Goncharov [FG03, Section 15].

3.5.4.2 Prequantization of character stacks via field theories

In this section we construct a prequantization of the symplectic form on the character stack using the machinery of topological field theories. The author learned the idea to prequantize field theories from the paper [FRS13].

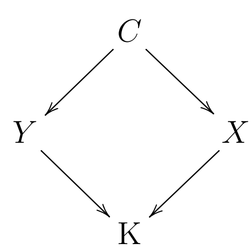
Our main tool is the cobordism hypothesis [Lu09, Theorem 1.2.16]. Given a symmetric monoidal (∞, n)-category $\mathcal{C}$ the functor $Z \mapsto Z(\mathrm{pt})$ gives an equivalence of ∞-groupoids

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \to (\mathcal{C}^{fd})^{\sim}$$

between the category of symmetric monoidal functors Z and the groupoid of fully-dualizable objects in $\mathcal{C}$.

We will be interested in the case when $\mathcal{C}$ is the (∞, n)-category $\mathrm{Corr}_{/\mathrm{K}}$ of correspondences of *underived* stacks together with a map to the presheaf of spaces K. That is,

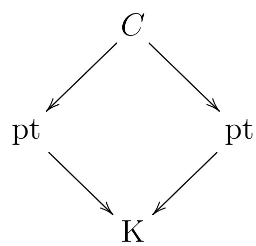
- Objects of $\mathrm{Corr}_{/\mathrm{K}}$ are stacks X with a map of presheaves $X \to \mathrm{K}$.
- A morphism between X and Y is a correspondence



where the map $X \to \mathrm{K}$ is *inverse* to that for X. For instance, if the map $X \to \mathrm{K}$ arises from a virtual vector bundle $E_0 - E_1$, then its inverse arises from the virtual vector bundle $E_1 - E_0$.

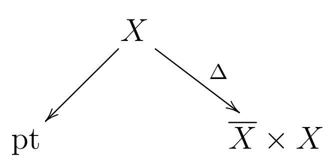
Higher morphisms are similarly correspondences between correspondences. To make $\mathrm{Corr}_{/\mathrm{K}}$ into an (∞, n)-category we only consider invertible k-correspondences whenever $k > n$. The monoidal structure is given by the Cartesian product of stacks.

Consider the symmetric monoidal $(\infty, n-1)$-category $\mathrm{Hom}_{\mathrm{Corr}_{/\mathrm{K}}}(\mathrm{pt}, \mathrm{pt})$ of endomorphisms of the unit object in $\mathrm{Corr}_{/\mathrm{K}}$. Its objects are correspondences

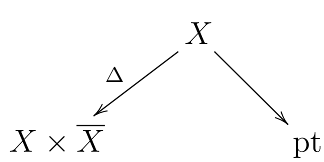


Equivalently, its objects are stacks C with a map to the based loop space $\Omega\mathrm{K}$. Therefore, $\mathrm{Hom}_{\mathrm{Corr}_{/\mathrm{K}}}(\mathrm{pt}, \mathrm{pt}) \cong \mathrm{Corr}_{/\Omega\mathrm{K}}$.

We will be interested in 2-dimensional field theories, but let's begin with an easy case $n = 1$. Any object $X \in \mathrm{Corr}_{/\mathrm{K}}$ is 1-dualizable. Indeed, define its dual to be $\overline{X} \in \mathrm{Corr}_{/\mathrm{K}}$, the same stack as X, but with the inverse map to K. Then the evaluation map is the correspondence



and the coevaluation map is the correspondence



Now we have to check that the snake diagrams commute. We just check one of them, the commutativity of the other diagram is proved analogously.

Consider a pair of correspondences

$$\begin{array}{ccccccccc} & & X \times X & & & & X \times X & & \\ & \overset{p_1}{\swarrow} & & \overset{\mathrm{id}\times\Delta_{23}}{\searrow} & & \overset{\Delta_{12}\times\mathrm{id}}{\swarrow} & & \overset{p_2}{\searrow} & \\ X & & & & X \times \overline{X} \times X & & & & X \end{array}$$

Their composite is

$$\begin{array}{ccccccccc} & & & & X & & & & \\ & & & \overset{\Delta}{\swarrow} & & \overset{\Delta}{\searrow} & & & \\ & & X \times X & & & & X \times X & & \\ & \overset{p_1}{\swarrow} & & \overset{\mathrm{id}\times\Delta_{23}}{\searrow} & & \overset{\Delta_{12}\times\mathrm{id}}{\swarrow} & & \overset{p_2}{\searrow} & \\ X & & & & X \times \overline{X} \times X & & & & X \end{array}$$

which is the identity morphism from X to X as required.

Let us specialize to the case $X = \mathrm{B}G = \mathrm{B}GL_n$ with the map to K given by the universal bundle. Then we obtain a 1-dimensional framed (equivalently, oriented) field theory Z_X^{K}. To understand this field theory, consider a forgetful functor $\mathrm{Corr}_{/\mathrm{K}} \to \mathrm{Corr}$, which forgets the data of the map to K. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^{fr}, \mathrm{Corr}_{/\mathrm{K}}) & \xrightarrow{\sim} & (\mathrm{Corr}_{/\mathrm{K}}^{fd})^{\sim} \\ \downarrow & & \downarrow \\ \mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^{fr}, \mathrm{Corr}) & \underset{\longleftarrow}{\xrightarrow{\sim}} & (\mathrm{Corr}^{fd})^{\sim} \end{array}$$

Here the splitting $(\mathrm{Corr}^{fd})^{\sim} \to \mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^{fr}, \mathrm{Corr})$ is given by assigning to X the field theory Z_X, which to a manifold M assigns to mapping stack $\mathrm{Map}(M_{\mathrm{B}}, X)$. Therefore,

$Z^{\mathrm{K}}_X(M)$ is again the mapping stack $\mathrm{Map}(M_{\mathrm{B}}, X)$ together with a map to K. This is exactly how one we will integrate maps into K using topological field theories.

We get that $Z^{\mathrm{K}}_{\mathrm{B}G}$ is the following field theory:

- To a point it assigns $\mathrm{B}G$ with a map to K given by the universal bundle.
- To the circle S^1 it assigns the adjoint quotient G/G with a map to $\Omega\mathrm{K}$, i.e. it gives a class in $\mathrm{K}_1(G/G)$, the first algebraic K-theory group. We will discuss shadows of this K-theoretic class in the next subsection.

Having warmed up with the 1-dimensional field theories, let us go on an discuss 2-dimensional field theories. Recall [Lu09, Proposition 4.2.3] that a 1-dualizable object $X \in \mathrm{Corr}_{\mathrm{K}}$ is 2-dualizable iff its evaluation map admits right and left adjoints.

We define the right and left adjoints $\mathrm{ev}^\vee$ and ${}^\vee\mathrm{ev}$ of the evaluation to be the coevaluation morphism

$$X \times \overline{X} \xleftarrow{\;\Delta\;} X \longrightarrow \mathrm{pt}$$

To show that they are indeed adjoints, we have to exhibit the maps $u^\vee : \mathrm{id}_{\overline{X}\times X} \to \mathrm{ev}^\vee \circ \mathrm{ev}$ and $v^\vee : \mathrm{ev} \circ \mathrm{ev}^\vee \to \mathrm{id}_{\mathrm{pt}}$ (and similarly for the left adjoint) satisfying two snake diagrams.

The composite $\mathrm{ev} \circ \mathrm{ev}^\vee$ is the free loop space $\mathcal{L}X$ with a map to $\Omega\mathrm{K}$. We let $v^\vee$ be the morphism $X \to \mathcal{L}X$.

The composite $\mathrm{ev}^\vee \circ \mathrm{ev}$ is the correspondence

$$\begin{array}{ccc} & X \times X & \\ {}^{\Delta_1}\swarrow & & \searrow^{\Delta_2} \\ X \times \overline{X} & & X \times \overline{X} \end{array}$$

A map $\mathrm{id}_{X\times\overline{X}} \to\!\!\to \mathrm{ev}^\vee \circ \mathrm{ev}$ is then the same as a map into $(\overline{X}\times X)\times_{\overline{X}\times X\times X\times\overline{X}}(X\times X)$. The latter space is the space of four points (x, y, z, w) on X together with paths $w \sim x$, $y \sim w$, $x \sim z$ and $z \sim y$. But this is again the free loop space $\mathcal{L}X$ and we let $u^\vee$ be the inclusion of constant loops $X \to \mathcal{L}X$. We skip the check that the relevant diagrams commute.

Recall that the Serre morphism $S : X \to X$ is defined to be the composite

$$X \stackrel{\mathrm{ev}^\vee\times\mathrm{id}}{\to} X \times \overline{X} \times X \stackrel{\mathrm{ev}_X\times\mathrm{id}}{\to} X.$$

Since we defined $\mathrm{ev}^\vee$ to be the coevaluation, the Serre morphism S is just the identity. In other words, instead of a framed 2d field theory, we automatically get an oriented field theory. This field theory, however, does not descend to an unoriented field theory: this would force twice the universal bundle on $\mathrm{B}G$ to be trivial in K-theory. This is not the case as can be seen on the level of dimensions.

Thus $Z^{\mathrm{K}}_{\mathrm{B}G}$ extends to an oriented 2d field theory. In particular, to a closed oriented surface Σ it assigns the character stack $\mathrm{Loc}_G(\Sigma)$ with a map to $\Omega^2\mathrm{K}$, i.e. it gives a class in $\mathrm{K}_2(\mathrm{Loc}_G(\Sigma))$ that we will call ω_K.

3.5.4.3 Beilinson regulator

In this section we discuss how the class ω_{K} gives rise to a prequantization of the analytification of the character stack $\mathrm{Loc}_{GL_n}(M)$ in the ordinary sense, i.e. a line bundle with a holomorphic connection whose curvature is the symplectic structure on $\mathrm{Loc}_{GL_n}(M)$.

Recall that to X a complex manifold one can attach the Deligne–Beilinson cohomology groups $\mathrm{H}^m_D(X, \mathbf{Z}(p))$ of degree m and weight p. For instance, $\mathrm{H}^2_D(X, \mathbf{Z}(2))$ parametrizes line bundles with holomorphic connections.

One has the curvature map

$$\mathrm{curv}\colon \mathrm{H}^m_D(X, \mathbf{Z}(p)) \to \Omega^{p,cl}(X, m-p).$$

For instance, for $m = p = 2$ we simply associate the curvature of the corresponding holomorphic connection.

Beilinson [Be85] defined a refinement of the Chern character

$$\mathrm{ch}\colon \mathrm{K}_i(X) \to \oplus_p \Omega^{p,cl}(X, p-i)$$

to the Deligne–Beilinson cohomology

$$\begin{array}{ccc} \mathrm{K}_i(X) & \xrightarrow{\mathrm{ch}} & \oplus_p \Omega^{p,cl}(X, p-i) \\ & {\scriptstyle r}\searrow \quad \nearrow{\scriptstyle \mathrm{curv}} & \\ & \oplus_p \mathrm{H}^{2p-i}_D(X, \mathbf{Z}(p)), & \end{array}$$

where $r\colon \mathrm{K}_i(X) \to \oplus_p \mathrm{H}^{2p-i}_D(X, \mathbf{Z}(p))$ is the so-called Beilinson regulator map.

Using the machinery of topological field theories we have been able to integrate the universal bundle on $\mathrm{B}G$ to a class in $\mathrm{K}_1(G/G)$ and a class in $\mathrm{K}_2(\mathrm{Loc}_G(\Sigma))$ for every closed oriented surface Σ.

The Beilinson regulator gives maps $\mathrm{K}_1(G/G) \to \mathrm{H}^1_D(G/G, \mathbf{Z}(1)) \cong \mathrm{H}^0(G/G, \mathcal{O}^\times)$ and $\mathrm{K}_1(G/G) \to \mathrm{H}^3_D(G/G, \mathbf{Z}(2))$. The first map is simply the determinant map $G/G \to \mathbf{G}_m$.

The second map gives a gerbe on G/G with a connective structure, which is descended from the so-called basic gerbe on the group G.

On the level of surfaces we have the Beilinson regulator map

$$\mathrm{K}_2(\mathrm{Loc}_G(\Sigma)) \to \mathrm{H}^2_D(\mathrm{Loc}_G(\Sigma), \mathbf{Z}(2)).$$

The latter group is identified with the group of holomorphic line bundles with a connection. Moreover, its curvature coincides with the AKSZ symplectic structure on $\mathrm{Loc}_G(\Sigma)$. Therefore, this is a prequantization of the character stack, albeit an analytic one.

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