The Dissertation Committee for Samuel Joseph Taylor
certifies that this is the approved version of the following dissertation:

Geometric properties of outer automorphism groups of free groups

Committee:

Alan W. Reid, Supervisor

Cameron Gordon

John Luecke

Daniel Allcock

Hossein Namazi

Jeffrey Brock

Geometric properties of outer automorphism groups of free groups

by

Samuel Joseph Taylor, B.A.

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2014

Dedicated to my wife, Marygrace.

Acknowledgments

I first thank my advisor, Alan Reid, for his continuous support, encouragement, and advice throughout my time at the University of Texas. I thank Hossein Namazi for introducing me to surface topology and for conversations that inspired much of this work. I have also learned much from my friends and colleagues during the past five years, especially Sam Ballas, Chris White, Nick Zufelt, Matt Pancia, Aaron Royer, Jeff Danciger, and Alex Zupan.

During my time as an undergraduate, the faculty at The College of New Jersey introduced me to mathematics and devoted much time and energy to my education. Conversations with my closest friends, Jeff Hatley, Chris Geddis, and Chris Westenberger, were particularly important during this time.

Finally, any success achieved on my part is only possible with the love and support of my wonderful family, who are far too numerous to name. I only mention my dad, whose work ethic I've attempted to emulate; my mom, whose support I can always depend on; and my grandparents, for everything they've done for my brother and I. Most importantly, I thank my wife Marygrace. She has shown me endless love and confidence throughout each stage of my education, and I can't imagine facing any of the challenges along the way without her by my side.

Geometric properties of outer automorphism groups of free groups

Samuel Joseph Taylor, Ph.D.
The University of Texas at Austin, 2014

Supervisor: Alan W. Reid

This thesis examines geometric aspects of the outer automorphism group of a finitely generate free group. Using recent advances made in understanding mapping class groups as our primary motivation, we refine methods to understand the structure of $\mathrm{Out}(F_n)$ via its action on free factors of F_n. Our investigation has a number of applications: First, we give a natural notion of projection between free factors, extending a construction of Bestvina-Feighn. Second, we provide a new method to produce fully irreducible automorphisms of F_n using combinations of automorphism supported on free factors. Finally, we use these results to give a general construction of quasi-isometric embeddings from right-angled Artin groups into $\mathrm{Out}(F_n)$.

Table of Contents

List of Figures

Chapter 1

Introduction

Significant progress in understanding the outer automorphism group of a free group has come from adapting ideas and techniques that were originally developed to study the mapping class group of a surface. The analogy between outer automorphism groups and mapping class groups is derived from several sources. For example, the Dehn-Nielsen-Baer Theorem identifies the extended mapping class group $\mathrm{Mod}^{\pm}(S)$ of a closed, orientable surface S with $\mathrm{Out}(\pi_1(S))$, the outer automorphism group of $\pi_1(S)$, by associating to each mapping class its action on $\pi_1(S)$. When S is a surface with punctures, $\mathrm{Mod}^{\pm}(S)$ is precisely the subgroup of $\mathrm{Out}(\pi_1(S))$ that preserves the set of conjugacy classes that correspond to the punctures of S. In the case when S has punctures, $\pi_1(S)$ is a finitely generated free group and we have the well known fact that $\mathrm{Mod}(S) \leq \mathrm{Out}(F_n)$ for $n = 1 - \chi(S)$.

The philosophy that knowledge of mapping class groups should serve as a guide to study $\mathrm{Out}(F_n)$ is prevalent in the literature. In [16], Culler and Vogtmann introduced a space of marked metric graphs, known as Outer space $\mathcal{X}_n$, that has a discontinuous action of $\mathrm{Out}(F_n)$, and they used this space to study the cohomology of $\mathrm{Out}(F_n)$. In many respects, $\mathcal{X}_n$ plays the role for

Out(F_n) that Teichmüller spaces plays for the mapping class group. Since its introduction $\mathcal{X}_n$ has become a central tool to understand the geometry of Out(F_n). In particular, $\mathcal{X}_n$ deformation retracts to a simplicial complex that is quasi-isometric to Out(F_n) [16, 26]. More recently, Francaviglia and Martino introduced an Out(F_n)-invariant asymmetric metric on $\mathcal{X}_n$ [18]. This metric, called the Lipschitz metric, is defined similarly to Thurston's metric on Teichmüller space and has been used to study different features of both Out(F_n) and $\mathcal{X}_n$. For example, Bestvina used the Lipschitz metric on $\mathcal{X}_n$ to classify elements of Out(F_n) via the dynamics of their action, providing a new proof of the train track existence theorem of [10]. Bestvina's argument closely follows Bers' proof of the classification of mapping classes (originally due to Thurston). Other applications of the Lipschitz metric on $\mathcal{X}_n$ appear in [1] and [9].

The work contained in this thesis studies the geometry of Out(F_n) following the approach to study mapping class groups initiated by Masur and Minsky [39, 40]. A summary of this approach is given below. In short, Masur and Minsky introduce the curve graph $\mathcal{C}(S)$ of the surface S as a primary tool to understand the geometry of Mod(S). They first proved that $\mathcal{C}(S)$ is hyperbolic (in the sense of Gromov) and then studied the connection between the geometry of $\mathcal{C}(S)$ and that of Mod(S). This approach has recently been adapted to study Out(F_n). Two graphs that are naturally associated to F_n (and serve as an analog of the curve graph) are the free splitting graph $\mathcal{S}_n$ and free factor graph $\mathcal{F}_n$. Both of these graphs admit an isometric action of

$\mathrm{Out}(F_n)$ and both graphs are now known to be hyperbolic. This was first proven for $\mathcal{F}_n$ by Bestvina-Feighn in [8], using the Lipschitz metric on $\mathcal{X}_n$, and for $\mathcal{S}_n$ by Handel-Mosher in [23]. New proofs of these facts appear in [31, 29, 9]. Many of the results in this thesis are concerned with developing tools to connect the hyperbolic geometry of these graphs back to the structure of $\mathrm{Out}(F_n)$, which itself is not hyperbolic.

We warn the reader that although this brief literature review seems to suggest that the analogy between $\mathrm{Out}(F_n)$ and the mapping class group is strong, we remark that there are aspects of $\mathrm{Out}(F_n)$ that do not appear when studying mapping class groups. Besides the positive results in this thesis, we provide a number of examples that demonstrate where analogies between $\mathrm{Mod}(S)$ and $\mathrm{Out}(F_n)$ fail. The source of many of these differences is easy to describe: For a surface S, any essential subsurface Y of S determines a cyclic splitting of $\pi_1(S)$ simply by taking the graph-of-groups decomposition induced by the boundary ∂Y. However, for the free group F_n there is no corresponding association between a free factor A of F_n and a free splitting of F_n with A as a vertex stabilizer. In other words, there is no natural way to complement a free factor; in the expression $F_n = A * B$, the factor B is highly non-unique. This simple difference between free groups and surface groups translates into major difficulties when studying $\mathrm{Out}(F_n)$ and will be discussed in detail throughout this thesis.

1.1 Motivation from Mod(S)

To explain our motivation for the results in this thesis, we briefly review the Masur-Minsky approach for understanding geometric aspects of the mapping class group. Let $S = S_{g,p}$ be an orientable surface of genus $g \geq 0$ with $p \geq 0$ punctures and let $\omega(S) = 3g - 3 + p \geq 1$. We call $\omega(S)$ the complexity of S; it is equal to the maximum number of essential, simple closed curves (or simply *curves*) that can be pairwise disjointly embedded in S where no two of which are isotopic. As is common in the literature, we blur the distinction between a curve and its isotopy class.

Let Mod(S) denote the mapping class group of S and $\mathcal{C}(S)$ denote its curve graph. This is the graph whose vertices correspond to isotopy class of curves on S and whose edges join vertices that correspond to curves that can be disjointly realized on S. There is an obvious action of Mod(S) on $\mathcal{C}(S)$ and this action can be used to understand Mod(S) from a geometric group theory point of view. First, Masur and Minsky proved that when $\mathcal{C}(S)$ is connected (that is, when $\omega(S) \geq 2$), $\mathcal{C}(S)$ is δ-hyperbolic [39]. This hyperbolicity makes the global geometry of $\mathcal{C}(S)$ tractable, despite the fact that $\mathcal{C}(S)$ is locally infinite. In [40], Masur and Minsky provide a strong connection between the geometry of Mod(S) and the collection of curve graphs associated to subsurface of S. Thus, although the mapping class group is not hyperbolic, it can be understood by using (infinitely many) of these hyperbolic graphs. (See [7] for a precise statement.) To do this, Masur and Minsky construct special paths between mapping classes, called *hierarchies*, that are built out of geodesics in various curve graphs of subsurfaces. *Subsurface projections* are a fundamental tool to understand how hierarchies organized these geodesics. Informally, let $\alpha \in \mathcal{C}^0(S)$ be a curve and fix an essential subsurface $Y \subset S$. The projection of α to the curve graph of Y, denoted $\pi_Y(\alpha) \subset \mathcal{C}(Y)$, is defined to be the collection of curves that one obtains by surgering arcs in the collection $\alpha \cap Y$ (once we have isotoped α to intersect ∂Y minimally).

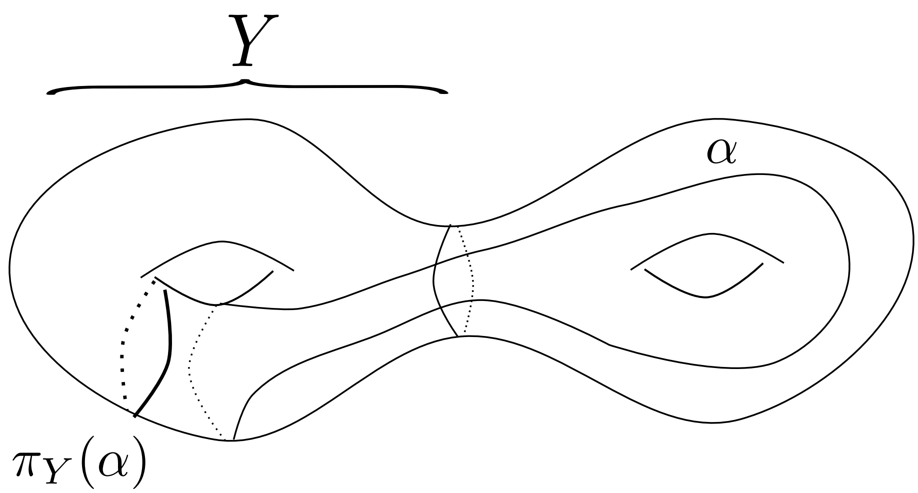

Figure 1.1: Subsurface projection of α to Y.

Hence, the projection is empty if and only if α can be isotoped to miss Y. For essential (non-annular) subsurfaces $X, Y \subset S$, set $\pi_Y(X) = \pi_Y(\partial X)$. These subsurface projections have a number of properties well suited for controlling distance in $\mathrm{Mod}(S)$. In particular, they satisfy the following:

Theorem 1.1.1 ([39, 3]). *Let X and Y be subsurfaces of S as above. Then either*

1. $Y \subset X$*, up to isotopy,*

2. Y *and* X *are disjoint, or*

3. $\pi_Y(X)$ *is nonempty with diameter less than or equal to* 3.

Moreover, there is an $M \geq 0$ *so that if* $\pi_Y(X), \pi_X(Y) \neq \emptyset$ *then for any curve* α *that intersects* X *and* Y

$$\min\{d_X(\alpha, Y), d_Y(\alpha, X)\} \leq M.$$

By combining these properties of subsurface projections with a detailed study of hierarchies, Masur and Minsky produce distance formulas for $\mathrm{Mod}(S)$. Specifically, if one fixes a word metric on $\mathrm{Mod}(S)$, which is well known to be finitely generated, the word norm $|f|$ of a mapping class $f \in \mathrm{Mod}(S)$ can be calculated directly in terms of subsurface projections (up to bounded multiplicative and additive error depending on the choice of $|\cdot|$). See [40] for the details and a precise statement.

From the Masur-Minsky distance formulas, a number of major implication for $\mathrm{Mod}(S)$ follow (with varying degrees of difficultly): Masur and Minsky proved the linear bounded conjugator property for psuedo-Anosov elements of $\mathrm{Mod}(S)$ in [40]. In [5], Behrstock and Minsky proved the rank conjecture of Brock and Farb [12]; that $\mathbb{Z}^n$ quasi-isometrically embeds into $\mathrm{Mod}(S)$ if and only if $n \leq \omega(S)$. Behrstock, Kleiner, Minsky and Mosher [4], and independently Hamenstädt [21], proved that $\mathrm{Mod}(S)$ is quasi-isometrically rigid, and Bestvina-Bromberg-Fujiwara [7] proved that the asymptotic dimension of the mapping class group is finite. In the proofs of each of these theorems, subsurface projections, and in particular the distance formulas, play an essential role. Subsurface projections are also used to study the structure of geodesics in Teichmüller space [42] and to organize the construction of model manifolds in the proof of the Ending Lamination Conjecture [41].

We remark on one further application of subsurface projections, which will be generalized to $\mathrm{Out}(F_n)$ in this thesis. In [14], Clay, Leininger, and Mangahas provide a flexible method to quasi-isometrically embed right-angled Artin groups into $\mathrm{Mod}(S)$. To state their result, say that two subsurface $X, Y \subset S$ *overlap* if they are not disjoint and one is not contained in the other, up to isotopy.

Theorem 1.1.2 (Theorem 1.1 of [14])**.** *Suppose that* $f_1, \ldots, f_n \in \mathrm{Mod}(S)$ *are fully supported on disjoint or overlapping non-annular subsurfaces. Then after raising to sufficiently high powers, the elements generate a quasi-isometrically embedded right-angled Artin subgroup of* $\mathrm{Mod}(S)$*. Furthermore, the orbit map to Teichmüller space is a quasi-isometric embedding.*

They also prove:

Corollary 1.1.3 (Corollary 1.2 of [14])**.** *Any right-angled Artin group admits a homomorphism to some mapping class group which is a quasi-isometric embedding, and for which the orbit map to Teichmüller space is a quasi-isometric embedding.*

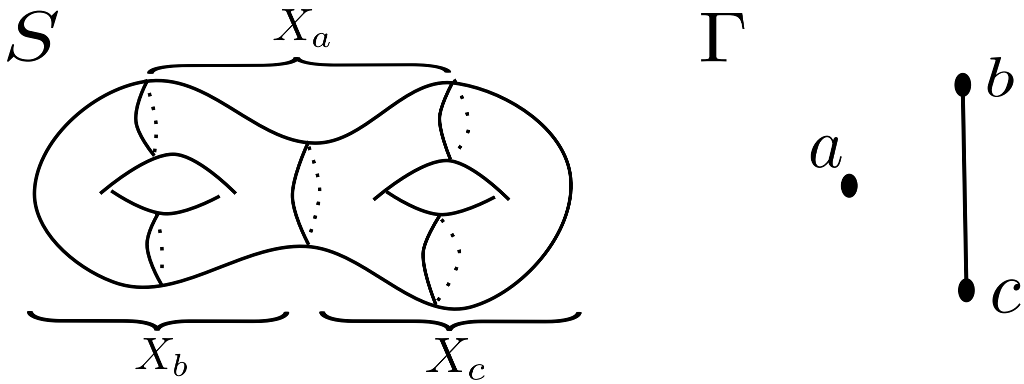

Figure 1.2: Quasi-isometrically embedded $A(\Gamma)$ in $\mathrm{Mod}(S)$.

In [38], Mangahas and the author of this thesis use Theorem 1.1.2 to explicitly construct convex cocompact subgroups of mapping class groups. Convex cocompact subgroups of mapping class groups are those subgroups that induce hyperbolic extensions of surface groups [17, 20], and the main result of [38] allows one to build such subgroups of $\mathrm{Mod}(S)$ using the cubical geometry associated to right-angled Artin groups.

1.2 Summary of results

This thesis contains three main results. The first provides an analog of the construction of Clay, Leininger, and Mangahas in [14]. Specifically, we prove the following: (See Chapter 4 for definitions and a more precise statement.)

Theorem 1.2.1. *Suppose that $f_1, \ldots, f_n \in \mathrm{Out}(F_n)$ are fully supported on an admissible collection of free factors. Then after raising to sufficiently high pow-*

ers, the elements generate a quasi-isometrically embedded right-angled Artin subgroup of $\mathrm{Out}(F_n)$.

The *admissible collection* condition on the set of free factors in Theorem 1.2.1 is meant to mimic the situation in Theorem 1.1.2, where the subsurfaces considered are either disjoint or overlapping. We note that if Γ is the "coincidence graph" for the involved free factors, then the right-angled Artin group generated in Theorem 1.2.1 is $A(\Gamma)$. This is made precise in Section 4.2. We also obtain

Corollary 1.2.2. *Any right-angled Artin group admits a homomorphism to* $\mathrm{Out}(F_n)$, *for some* n, *which is a quasi-isometric embedding.*

The proof of Theorem 1.2.1 involves an ad hoc notion of projection that was developed specifically for this purpose. There are two other notions of projections for $\mathrm{Out}(F_n)$ (modeled on subsurface projections) that appear in the literature, and these are discussed in detail in Chapter 4.

The Bestvina-Feighn subfactor projections, introduced in [9], have properties most similar to subsurface projection. Bestvina and Feighn show that for free factors A and B of F_n, one may project B to the splitting complex of A if either A and B have distance at least 5 in the free factor complex of F_n or if they have the same color in a specific finite coloring of the factor complex. In Chapter 5, we show that if one considers projections to the *free factor complex of a free factor* $\mathcal{F}(A)$, simpler and more natural conditions can be given. Say that free factors A and B are *disjoint* if they are nonconjugate vertex stabilizers of a splitting of F_n, or equivalently, if they can be represented by disjoint subgraphs of a marked graph G. Our second main result of this thesis is the following:

Theorem 1.2.3. *There is a constant D depending only on $n = \mathrm{rank}(F_n)$ so that if A and B are free factors of F_n with $\mathrm{rank}(A) \geq 2$, then either*

1. *$A \subset B$, up to conjugation,*
2. *A and B are disjoint, or*
3. *$\pi_A(B) \subset \mathcal{F}(A)$ is nonempty and has diameter $\leq D$.*

Moreover, these projections are equivariant with respect to the action of $\mathrm{Out}(F_n)$ on conjugacy classes of free factors and they satisfy the following: There is an $M \geq 0$ so that if $A, B < F_n$ are free factors with $\pi_A(B), \pi_B(A) \neq \emptyset$ then for any marked graph G

$$\min\{d_A(B, G), d_B(A, G)\} \leq M.$$

Hence, we show that the situation for subfactor projections to the free factor complex is exactly the same as the situation given in Theorem 1.1.1 for subsurface projections to the curve graph. Of course, one would ultimately like to produce distance formulas for $\mathrm{Out}(F_n)$, as for $\mathrm{Mod}(S)$, but Theorem 1.2.3 represents progress towards this goal. We also have the following strengthening of the Bounded Geodesic Image Theorem of [9]. For subsurface projections, this was first shown in [40].

Theorem 1.2.4. *For $n \geq 3$, there is $M \geq 0$ so that if A is a free factor of F_n with $\operatorname{rank}(A) \geq 2$ and γ is a geodesic of $\mathcal{F}_n$ with each vertex of γ meeting A (i.e. having well-defined projection to $\mathcal{F}(A)$) then $\operatorname{diam}(\pi_A(\gamma)) \leq M$.*

Finally, as an application of subfactor projections we give a construction of fully irreducible automorphism similar to Proposition 3.3 of [37], where pseudo-Anosov mapping classes are constructed. Here, free factors A and B *fill* F_n if no free factor C is disjoint from both A and B. This is our third main result of this thesis.

Theorem 1.2.5. *Let A and B be rank ≥ 2 free factors of F_n that fill and let $f, g \in \operatorname{Out}(F_n)$ satisfy the following:*

1. *$f(A) = A$ and $f|_A \in \operatorname{Out}(A)$ is fully irreducible, and*
2. *$g(B) = B$ and $g|_B \in \operatorname{Out}(B)$ is fully irreducible.*

Then there is an $N \geq 0$ so that the subgroup $\langle f^N, g^N \rangle \leq \operatorname{Out}(F_n)$ is free of rank 2 and any nontrivial automorphism in $\langle f^N, g^N \rangle$ that is not conjugate to a power of f or g is fully irreducible.

See Section 5.6 for a stronger statement. Theorem 1.2.5 adds a new construction of fully irreducible automorphisms to the methods of Clay-Pettet [13], where they arise as compositions of Dehn twists, and of Kapovich-Lustig [30], where they are compositions of powers of other fully irreducible automorphisms.

We also give examples to show that the hypothesis that A and B fill cannot be weakened. Interestingly, this hypothesis can be removed if we place additional hypotheses on the automorphisms f and g. This phenomenon, which does not happen in the mapping class group case, is explained in detail in Chapter 5.

Chapter 2

Background

In this chapter, we review the background material that will be needed throughout this thesis. We begin by reviewing basic notions in coarse geometry and then give a quick introduction to $\mathrm{Out}(F_n)$.

2.1 Quasi-isometries

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is a (K, L) - *quasi-isometric embedding* if for all $x_1, x_2 \in X$

$$\frac{1}{K} d_X(x_1, x_2) - L \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + L.$$

If, in addition, every point of Y is within distance L from the image $f(X)$, then f is a *quasi-isometry* and X and Y are said to be *quasi-isometric*. In this thesis, the metric spaces of interest arise from finite dimensional simplicial complexes. For a particular complex, the metric is induced by giving each simplex the structure of a standard Euclidean simplex. Recall that if K is a finite dimensional simplicial complex, then this piecewise Euclidean metric on K is quasi-isometric to K^1, the 1-skeleton of K, with its standard graph metric (see [11] for details). Since we are interested in the coarse geometry of such complexes, i.e. their metric structure up to quasi-isometry, this justifies our convention of when working with a complex K to consider only the graph metric on K^1. Here, and below, a *graph* is a 1-dimensional CW complex and a simply connected graph is a *tree*.

2.2 Out(F) basics

Fix $n \geq 2$ and let F_n denote the free group of rank n with outer automorphism group $\mathrm{Out}(F_n)$. When it is clear from context, the subscript n will be dropped from the notation. First, a *splitting* of F is a minimal, simplicial action $F \curvearrowright T$ on a non-trivial simplicial tree. The action is determined by a homomorphism $\psi : F \to \mathrm{Aut}(T)$ into the simplicial automorphisms of T. An action on a tree is *minimal* if there is no proper invariant subtree. By a *free splitting*, we mean a splitting with trivial edge stabilizers and refer to a k*-edge splitting* as a free splitting with k natural edge orbits. Here, natural edges are the edges of the cell structure on T whose vertices all have valence ≥ 3. From Bass-Serre theory, k-edge splittings correspond to graph of groups decompositions of F with k edges, each edge with trivial edge group. Two actions $F \curvearrowright T$ and $F \curvearrowright T'$ are *conjugate* if there is a F-equivariant homeomorphism $\chi : T \to T'$, and the conjugacy class of an action is denoted by $[F \curvearrowright T]$. We will usually drop the action symbol from the notation and refer to the splitting by T. Finally, an equivariant surjection $c : T \to T'$ between F-trees is a *collapse map* if all point preimages are connected. In this case, T is said to be a refinement of T'.

The *free splitting complex* $\mathcal{S}_n$ of the free group F_n is the simplicial complex defined as follows (see [23] for details): The vertex set $\mathcal{S}_n^0$ is the set of conjugacy classes of 1-edge splittings of F_n, and $k+1$ vertices $[T_0], \ldots, [T_k]$ determine a k-simplex of $\mathcal{S}_n$ if there is a $(k+1)$-edge splitting T and collapse maps $c_i : T \to T_i$, for each $i = 0, \ldots, k$. That is, a collection of vertices span a simplex in $\mathcal{S}_n$ if they have a common refinement. We will mostly work with the barycentric subdivision of the free splitting complex, denoted by $\mathcal{S}_n'$. The vertices of $\mathcal{S}_n'$ are conjugacy classes of free splittings of F_n and two vertices are joined by an edge if, up to conjugacy, one refines the other. Higher dimensional simplicies are determined similarly.

For $n \geq 3$, the *free factor complex* $\mathcal{F}_n$ of F_n is the simplicial complex defined as follows (see [25] or [8] for details): The vertices are conjugacy classes of free factors of F_n and $k+1$ conjugacy classes $[A_0], \ldots, [A_k]$ span a k-simplex if there are representative free factors in these conjugacy classes with $A_0 \subset A_1 \subset \ldots \subset A_k$. When $n = 2$, the definition is modified so that $\mathcal{F}_2$ is the standard Farey graph. In this case, vertices of $\mathcal{F}_2$ are conjugacy classes of rank 1 free factors and two vertices are jointed by an edge if there are representatives in these conjugacy classes that form a basis for F_2.

$\mathrm{Out}(F_n)$ acts simplicially on these complexes. For $\mathcal{F}$, if $f \in \mathrm{Out}(F)$ is represented by an automorphism ϕ, we define $f[A] = [\phi A]$. It is clear that this is independent of choice of ϕ and that the action extends to a simplicial action on all of $\mathcal{F}$. For $\mathcal{S}$ the action is defined as follows: with f and ϕ as above and $[T] \in \mathcal{S}^0$, suppose that the action on T is given by the homomorphism $\psi : F \to \mathrm{Aut}(T)$. Then $f[T]$ is the conjugacy class of F-tree determined by

$\psi \circ \phi^{-1} : F \to \mathrm{Aut}(T)$. That is, the underlying tree is unchanged and the action is precomposed with the inverse of a representative automorphism for f. Again, checking that this is a well-defined action that extends to all of $\mathcal{S}$ (or $\mathcal{S}'$) is an easy exercise. These definitions have the convenient property that if $[T]$ is a conjugacy class of free splitting with vertex stabilizers $[A_1], \ldots, [A_l]$, then $f[T]$ has vertex stabilizers $f[A_1], \ldots, f[A_l]$, for any $f \in \mathrm{Out}(F)$.

There is a natural, coarsely defined map $\pi : \mathcal{S}' \to \mathcal{F}$. For $T \in (\mathcal{S}')^0$, we set $\pi(T)$ equal to the set of free factors that arise as a vertex group of a 1-edge collapse of T. That is, $A \in \pi(T)$ if and only if there is a tree $T_0 \in \mathcal{S}^0$, T refines T_0, and A is a vertex group of T_0. Letting $d_{\mathcal{F}}$ denote distance in $\mathcal{F}$ and setting $d_{\mathcal{F}}(\pi(T), \pi(T')) = \mathrm{diam}_{\mathcal{F}}(\pi(T) \cup \pi(T'))$, it is easily verified that π is coarsely 4-Lipschitz [8]. Note that here, and throughout the thesis, the brackets that denote conjugacy classes of trees and free factors will often be suppressed when it should cause no confusion to do so.

Recent efforts to understand the free splitting and free factor complex have focused on their metric properties along with their similarity to the curve graph of a surface. In particular, both complexes are now known to be Gromov-hyperbolic. Hyperbolicity of the free factor complex was proven by Bestvina-Feighn in [8], and hyperbolicity of the free splitting complex was proven by Handel-Mosher in [23]. See [31], [29], and [9] for alternative proofs and perspectives.

We remark that the action $\mathrm{Out}(F_n) \curvearrowright \mathcal{S}_n$ is far from proper; all vertices have infinite stabilizers. There is, however, an invariant subcomplex of $\mathcal{S}_n'$ that is locally finite, and the inherited action is proper. This is the spine of Outer space and we refer the reader to [16] or [22] for details beyond what is discussed here. Also, see [24] or [2] for an alternative perspective.

The *spine of Outer space* $\mathcal{K}_n$ is the subcomplex of $\mathcal{S}'_n$ spanned by vertices that correspond to proper splittings of F_n. Recall that a splitting T is *proper* if no element of F_n fixes a vertex in T. Hence, $T \in \mathcal{S}'_n$ is proper if and only if T/F_n is a graph with fundamental group isomorphic to F_n. Observe that since $\mathrm{Out}(F_n)$ preserves the vertices of $\mathcal{S}'_n$ corresponding to proper splittings there is an induced simplicial action $\mathrm{Out}(F_n) \curvearrowright \mathcal{K}_n$. The spine of Outer space is a deformation retract of Outer space $\mathcal{X}_n$, which is discussed further in Section 2.4.

It is well-known that $\mathcal{K}_n$ is a locally finite, connected complex and that the action $\mathrm{Out}(F_n) \curvearrowright \mathcal{K}_n$ is proper and cocompact (see [16]). Hence, for any tree $T \in \mathcal{K}_n^0$, the orbit map $g \mapsto gT$ defines a quasi-isometry from $\mathrm{Out}(F_n)$ to $\mathcal{K}_n$ by the Švarc-Milnor lemma [11]. As remarked above, the metric considered here is the standard graph metric on $\mathcal{K}_n^1$, the 1-skeleton of the spine of Outer space. This metric on $\mathcal{K}_n^1$ will serve as our geometric model for $\mathrm{Out}(F_n)$.

2.3 The sphere complex

We recall the $\mathrm{Out}(F_n)$-equivalent identification between the free splitting complex and the sphere complex. See [2] for details. Take $M_n = \#_n(S^1 \times S^2)$, or equivalently, the double of the handlebody of genus n. Let $M_{n,s}$ be M_n with s open 3-balls removed. Note that $\pi_1 M_n$ is isomorphic to F_n and, once and for all, fix such an isomorphism. A sphere S in $M_{n,s}$ is *essential* if it is not boundary parallel and does not bound a 3-ball. A collection of disjoint, essential, pairwise non-isotopic spheres in M_n is called a *sphere system.* By [35], spheres S_1 and S_2 are homotopic in M_n if and only if they are isotopic.

The *sphere complex* $\mathcal{S}(M_n)$ is the simplicial complex whose vertices are isotopy classes of essential spheres and vertices $[S_0], \ldots, [S_k]$ span a k-simplex if there are representatives in these isotopy classes that are pairwise disjoint in M_n. It is a theorem of Laudenbach [35] that with $\mathrm{Mod}(M_n) = \pi_0(\mathrm{Diff} M_n)$ there is an exact sequence

$$1 \to K \to \mathrm{Mod}(M_n) \to \mathrm{Out}(F_n) \to 1,$$

where K is a finite group generated by "Dehn twists" about essential spheres. Since elements of K act trivially on $\mathcal{S}(M_n)$, we have a well-defined action $\mathrm{Out}(F_n) \curvearrowright \mathcal{S}(M_n)$. The following proposition of Aramayona and Souto identifies $\mathcal{S}_n$ and $\mathcal{S}(M_n)$. See Section 4.3.1 for how one constructs splittings from essential spheres.

Proposition 2.3.1 ([2])**.** *For $n \geq 2$, $\mathcal{S}_n$ and $\mathcal{S}(M_n)$ are $\mathrm{Out}(F_n)$-equivariantly isomorphic.*

2.4 Outer space and folding paths

A finite graph is a *core* graph if all its vertices have valence at least 2. Any connected graph with finitely generated, nontrivial fundamental group has a unique core subgraph that carries its fundamental group. A core graph has a unique CW structure, or *triangulation*, where each vertex has valence at least 3 and we refer to vertices and edges in this triangulation as *natural*. If the modifier natural is omitted then we are referring to the graph with its given triangulation.

To study $\mathrm{Out}(F_n)$, Culler and Vogtmann introduce Outer space $\mathcal{X}_n$, the space of metric graphs marked by F_n, or equivalently, the space of minimal, proper actions of F_n on simplicial $\mathbb{R}$-trees [16]. Recall that a *marking* of the graph G is a homotopy equivalence $\phi : R_n \to G$, where R_n is the rose with n petals whose fundamental group has been identified with F_n. A metric $l : E(G) \to \mathbb{R}_+$ on the marked graph G is an assignment of a positive real number, or length, to each edge of G and a marked metric graph is the ordered triple (G, ϕ, l), which we usually simplify to G. The volume of G is the sum of the lengths of the edges of G. *Outer space* $\mathcal{X}_n$ is defined to be the space of marked metric core graphs of volume one, up to equivalence. Here, (G, ϕ, l) and (G', ϕ', l') are equivalent if there is an isometry $i : G \to G'$ that is homotopic to $\phi' \circ \phi^{-1} : G \to G'$. In general, any map $h : G \to G'$ homotopic to $\phi' \circ \phi^{-1}$ is called a *change of marking*. For $G \in \mathcal{X}_n$ and α a conjugacy class of F_n, let $l_G(\alpha)$ denote the length of the immersed loop in G that correspond to α through the marking for G. We use the notation $\hat{\mathcal{X}_n}$ to denote *unprojectivized Outer space*, where the requirement that graphs have volume one is dropped.

We consider $\mathcal{X}_n$ with its Lipschitz metric defined by

$$d_{\mathcal{X}}(G, G') = \inf\{\log L(h) : h \simeq \phi' \circ \phi^{-1}\},$$

where $L(h)$ is the Lipschitz constant for the change of marking h and $\phi : R_n \to G$ and $\phi' : R_n \to G'$ are the corresponding markings. We remark that this (asymmetric) metric induces the standard topology on $\mathcal{X}_n$ that is got by considering lengths of immersed loops representing conjugacy classes in F_n [18]. Also, viewing $\mathcal{X}_n$ as the space of minimal, proper F_n-actions on simplicial $\mathbb{R}$-trees, we have the map $\pi : \mathcal{X}_n \to \mathcal{S}_n \to \mathcal{F}_n$, as described above. Note that free factors in the image $\pi(G)$ of $G \in \mathcal{X}_n$ are represented by embedded subgraphs of G.

It is well known that the infimum in the definition of the Lipschitz metric is realized by some (non-unique) *optimal* map [18, 8]. We briefly describe the folding path induced by an optimal $f : G \to G'$ and refer to [8] for more details. First, an *illegal turn structure* on G is an equivalence relation on the set of directions at each vertex of G; the equivalence classes are called *gates*. Here, a *turn* is a unordered pair of distinct directions at a vertex and a turn is *illegal* if both directions are contained in the same gate and is *legal* otherwise. An illegal turn structure is a *train track structure* if, in addition, every vertex has at least 2 gates. For marked graphs $G, G' \in \mathcal{X}_n$ any change of marking map $h : G \to G'$ that is linear on edges induces an illegal turn structure on G whose gates are the directions at each vertex that are identified by h. In fact, there is always a change of marking $f : G \to G'$, called an *optimal map*, that is constant slope (i.e. stretch) on each edge of G with the property that the subgraph $\triangle(f) \subset G$ consisting of edges of maximal slope, $L(f)$, is a core subgraph and that the illegal turn structure on G induced by f restricts to a train track structure on $\triangle(f)$ [18, 6]. From these properties, it follows that f has minimal Lipschitz constant over all change of markings $G \to G'$. If, in addition, $\triangle(f) = G$, i.e. every edge is stretched by $L(f)$, then there is an induced *folding path* $t \mapsto G_t$ joining G and G' in $\mathcal{X}_n$. Such a path is locally obtained by folding all illegal turns at unit speed and then rescaling to maintain volume one. For each $a \leq b$, there is an induced optimal map $f_{ab} : G_a \to G_b$. These folding maps compose naturally and send legal segments to legal segments, where a *legal segment* of G_a is an immersed path that makes only legal turns. See [8] for a detailed construction. Arbitrary points $G, G' \in \mathcal{X}_n$ are joined by a geodesic path that first rescales edge lengths of G and is then followed by a folding path. For a folding path G_t in $\mathcal{X}_n$, a family of subgraphs $H_t \subset G_t$ is called *forward invariant* if for all $a \leq b$, H_a maps into H_b under the folding map $f_{ab} : G_a \to G_b$.

2.5 Translation length in $\mathcal{F}_n$

An outer automorphism $f \in \mathrm{Out}(F_n)$ is *fully irreducible* if no positive power of f fixes a conjugacy class of a free factor. That is, for any $A \in \mathcal{F}_n^0$, $f^n(A) = A$ implies that $n = 0$. Recall that the *(stable) translation length* of an outer automorphism $f \in \mathrm{Out}(F_n)$ on $\mathcal{F}_n$ is defined as

$$\ell_{\mathcal{F}}(f) = \lim_{k\to\infty} \frac{d_{\mathcal{F}}(A, f^k A)}{k}$$

where $A \in \mathcal{F}_n^0$. It is not difficult to verify that $\ell_{\mathcal{F}}(f)$ is well-defined and independent of $A \in \mathcal{F}_n^0$. It also satisfies the property $\ell_{\mathcal{F}}(f^n) = n \cdot \ell_{\mathcal{F}}(f)$ for

$n \geq 0$. Further, $\ell_{\mathcal{F}}(f) \geq c$ if and only if for all $A \in \mathcal{F}_n^0$, $d_{\mathcal{F}}(A, f^n A) \geq c|n|$. Throughout this thesis, when we refer to the translation length of $f \in \mathrm{Out}(F_n)$ we will always mean the translation length of f on the free factor complex $\mathcal{F}_n$. The following proposition characterizes those outer automorphisms with positive translation length on $\mathcal{F}_n$.

Proposition 2.5.1 ([8])**.** *Let $f \in \mathrm{Out}(F_n)$, f is fully irreducible if and only if $\ell_{\mathcal{F}}(f) > 0$.*

It appears to be an open question whether there is a uniform lower bound on translation length for fully irreducible outer automorphisms of $\mathrm{Out}(F_n)$. In the mapping class group situation, this is indeed the case. That is, for a fixed surface S there is an $\eta > 0$ so that if $f \in \mathrm{Mod}(S)$ is pseudo-Anosov then the curve complex translation length of f is greater than or equal to η [39]. It is worthwhile to note that when $n = 2$, Proposition 2.5.1 reduces to the following statement: if an outer automorphism is infinite order and does not fix a conjugacy class of a primitive element in F_2, then it acts with positive translation length on $\mathcal{F}_2$, which as noted above is the Farey graph.

Chapter 3

Projections for $\mathrm{Out}(F_n)$

For a finitely generated subgroup $H \leq F$, let $\mathcal{S}(H)$ and $\mathcal{F}(H)$ denote the free splitting complex and free factor complex of H, respectively. A subgroup H is *self-normalizing* if $N(H) = H$, where $N(H)$ is the normalizer of H in F. When H is self-normalizing the complexes $\mathcal{S}(H)$ and $\mathcal{F}(H)$ depend only on the conjugacy class of H in F. More precisely, if $H' = gHg^{-1}$ for $g \in F$, then g induces an isomorphism between $\mathcal{S}(H)$ and $\mathcal{S}(H')$ (and between $\mathcal{F}(H)$ and $\mathcal{F}(H')$) via conjugation. For any other $x \in F$ with $H' = xHx^{-1}$ we see that $x^{-1}g$ normalizes H and so $x^{-1}g \in H$. In this case, $gH = xH$ and it is easily verified that g and x induce identical isomorphisms between $\mathcal{S}(H)$ and $\mathcal{S}(H')$. Hence, when H is self-normalizing we obtain a canonical identification between the free splitting complex of H and the free splitting complex of each of its conjugates. The same holds for the free factor complex of H. This allows us to unambiguously refer to the free splitting complex or free factor complex for the conjugacy class $[H]$. Finally, recall that a subgroup $C \leq F$ is *malnormal* if $xCx^{-1} \cap C \neq \{1\}$ implies that $x \in C$. For example, free factors of F are malnormal and malnormal subgroups are self-normalizing.

3.1 Projecting trees

Given a free splitting $T \in \mathcal{S}'$ and a finitely generated subgroup $H \leq F$ denote by T^H the *minimal H-subtree* of T. This is the unique minimal H-invariant subtree of the restricted action $H \curvearrowright T$. For any such H, T^H is either trivial, in which case H fixes a unique vertex in T, or T^H is the union of axes of elements in H that act hyperbolically on T. When T^H is not trivial, we define the *projection* of T to the free splitting complex of H as $\pi_{\mathcal{S}(H)}(T) = [H \curvearrowright T^H]$, where the brackets denote conjugacy of H-trees. Note that this projection is a well-defined vertex of $\mathcal{S}'(H)$ and it depends only on the conjugacy class of T. To see this, note that any conjugacy between F-trees will induce a conjugacy between their minimal H-subtrees. Further define the projection to the free factor complex of H to be the composition $\pi_H(T) = \pi(\pi_{\mathcal{S}(H)}(T))$, where $\pi : \mathcal{S}(H) \to \mathcal{F}(H)$ is the 4-Lipschitz map defined in Section 2.2. Hence, $\pi_H(T) \subset \mathcal{F}(H)$ is the collection of free factors of H that arise as a vertex group of a one-edge collapse of the splitting $H \curvearrowright T^H$. When H is also self-normalizing, e.g. a free factor, these projections are independent of the choice of H within its conjugacy class. The following lemma verifies that such projections are coarsely Lipschitz.

Lemma 3.1.1. *Let $F_n \curvearrowright T$ be a free splitting and $H \leq F_n$ a finitely generated subgroup with T^H non-trivial. Let T_0 be a refinement of T with equivalent collapse map $c : T_0 \to T$. Then there is an induced collapse map $c_H : T_0^H \to T^H$. Hence, T_0^H is a refinement of T^H.*

Proof. Since $c(T_0^H) \subset T$ is an invariant H-tree, it contains T^H. Also, the axis in T_0 of any hyperbolic $h \in H$ is mapped by c to either h's axis in T or a singe vertex stabilized by h; each of which is contained in T^H. Since T_0^H is the union of such axes, we see that $c(T_0^H) = T^H$. Hence the map c_H described in the lemma is given by restriction. It remains to show that c_H is a collapse map. This is the case since for any $p \in T^H$,

$$c_H^{-1}(p) = T_0^H \cap c^{-1}(p)$$

is the intersection of two subtrees of T_0 and is, therefore, connected. □

For a free factor A of F we use the symbol d_A to denote distance in $\mathcal{F}(A)$ and for F_n-trees T_1, T_2 we use the shorthand

$$d_A(T_1, T_2) := d_A(\pi_A(T_1), \pi_A(T_2)) = \mathrm{diam}_A(\pi_A(T_1) \cup \pi_A(T_2))$$

when both projections are defined. The following proposition follows immediately from the definitions in this section and Lemma 3.1.1.

Proposition 3.1.2 (Basic properties I)**.** *Let T_1, T_2 be adjacent vertices in $\mathcal{K}_n$, $A \in \mathcal{F}_n$, and H a finitely generated and self-normalizing subgroup of F_n containing A, up to conjugacy. Then we have the following:*

1. $\mathrm{diam}_{\mathcal{F}(A)}(\pi_A(T)) \leq 4$,

2. $d_A(T_1, T_2) \leq 4$,

3. $\pi_A(T_1) = \pi_A(\pi_{\mathcal{S}(H)}(T_1))$ *and so* $d_A(T_1, T_2) = d_A(\pi_{\mathcal{S}(H)}(T_1), \pi_{\mathcal{S}(H)}(T_2))$.

3.2 Projecting factors

Let A and B be rank ≥ 2 free factors of F_n. Define A and B to be *disjoint* if they are nonconjugate vertex groups of a free splitting of F_n. Disjoint free factors are those that will support commuting outer automorphisms in our construction. Define A and B to *meet* if there exist representatives in their conjugacy classes whose intersection is nontrivial and proper in each factor. In this section, we show that this intersection provides a well-defined projection of $[B]$ to $\mathcal{F}(A)$, the free factor complex of A. Note that if A and B meet, then $d_{\mathcal{F}}([A],[B]) = 2$.

Fix free factors A and B in F_n. Define the projection of B into $\mathcal{F}(A)$ to be

$$\pi_A(B) = \{[A \cap gBg^{-1}] : g \in F_n\} \setminus \{[1],[A]\},$$

where conjugacy is taken in A. Observe that A and B meet exactly when $\pi_A(B) \neq \emptyset \neq \pi_B(A)$. We show that members of $\pi_A(B)$ are vertex groups of a single (non-unique) free splitting of A and so $\pi_A(B)$ has diameter less than or equal to 4 in $\mathcal{F}(A)$. Since the projection is independent of the conjugacy class of B, this provides the desired projection from $[B]$ to the free factor complex of A.

Lemma 3.2.1. *Suppose the free factors A and B meet. Then* $\operatorname{diam}_{\mathcal{F}(A)}\pi_A(B) \leq 4$.

Proof. First, observe that g uniquely determines the class $[A \cap gBg^{-1}] \in \pi_A(B)$ up to double coset in F. Precisely, $[A \cap gBg^{-1}] = [A \cap hBh^{-1}] \neq 1$ if and only if $AgB = AhB$; this follows from the fact that free factors are malnormal. Now choose any marked graph G which contains a subgraph G^B whose fundamental group represents B up to conjugacy. Let $p_A : \tilde{G}_A \to G$ be the cover of G corresponding to free factor A and let G_A denote the core of $\tilde{G}_A$. By covering space theory, the components of $p^{-1}(G^B)$ are in bijective correspondence with the double cosets $\{AgB : g \in F\}$. Also, the fundamental group of the component corresponding to AgB is $A \cap gBg^{-1}$. Since the core carries the fundamental group of $\tilde{G}_A$, all nontrivial subgroups $A \cap gBg^{-1}$ correspond to double cosets representing components of $p^{-1}(G^B)$ in the core G_A. Hence, G_A is a marked A-graph that contains disjoint subgraphs whose fundamental groups (up to conjugacy in A) are the subgroups of $\pi_A(B)$. This completes the proof. □

If $A \in \mathcal{F}_n^0$ and $f \in \mathrm{Out}(F_n)$ stabilizes A, then f induces an outer automorphism of A, denoted $f|_A \in \mathrm{Out}(A)$. In this case, let $\ell_A(f)$ represent the translation length of $f|_A$ on $\mathcal{F}(A)$. By Proposition 2.5.1, if $f|_A$ is fully irreducible in $\mathrm{Out}(A)$, then $\ell_A(f) > 0$. The following proposition provides the additional properties of the projections that will be needed in the next chapter. Its proof is a straightforward exercise in working through the definitions of this section.

Proposition 3.2.2 (Basic Properties II)**.** *Let $A, B, C \in \mathcal{F}_n^0$ so that A and B meet and A and C are disjoint. Let $c \in \mathrm{Out}(F)$ stabilize the free factors A and C with $c|_A = 1$ in $\mathrm{Out}(A)$. Finally, let $T \in \mathcal{K}^0$ and $f \in \mathrm{Out}(F)$ be arbitrary. Then f induces an isomorphism $f : \mathcal{F}(A) \to \mathcal{F}(fA)$ and we have the following:*

1. *f(A) and f(B) meet and* $\pi_{fA}(fB) = f(\pi_A(B)) \subset \mathcal{F}(fA)$.

2. $\pi_{fA}(fT) = f(\pi_A(T)) \subset \mathcal{F}(fA)$.

3. $\pi_A(cB) = \pi_A(B) \subset \mathcal{F}(A)$.

4. $\pi_A(cT) = \pi_A(T) \subset \mathcal{F}(A)$.

For the applications in the next chapter, a slightly stronger condition than meeting is necessary on the free factors A and B. In particular, we need their meeting representatives to generate the "correct" subgroup of F. More precisely, say that two free factors A and B of F *overlap* if there are representatives in their conjugacy classes, still denoted A and B, so that $A \cap B = x \neq \{1\}$ is proper in both A and B and the subgroup generated by these representatives $\langle A, B\rangle \leq F$ is isomorphic to $A *_x B$. Note that the first condition here is exactly that A and B meet.

Example 1. Here is an example of free factors that meet but do not overlap. Let $F_6 = \langle a, b, c, d, e, f\rangle$ and consider the free factors $A = \langle a, b, c, d, f\rangle$ and $B = \langle aec, bed, f\rangle$. It is quickly verified that $A \cap B = \langle f\rangle$, so A and B meet. However, $\langle A, B\rangle = F_6$ is not isomorphic to $A *_{\langle f\rangle} B = \langle a, b, c, d\rangle * B$, which has rank 7.

Remark 3.2.3. Suppose the free factors $[A], [B] \in \mathcal{F}$ overlap and select representatives in their conjugacy classes so that $A \cap B = x$ is nontrivial and proper in both A and B. Note that as in Lemma 3.2.1 the free factor x is not necessarily unique up to conjugacy, but once the conjugacy class of x is fixed the subgroup $H = \langle A, B \rangle$ generated by these conjugacy class representatives is itself determined up to conjugacy in F. Since A and B overlap, x can be chosen so that $H \cong A *_x B$ and it is not difficult to verify that H is finitely generated and self-normalizing. So, for example, if $T \in S'$, then $\pi_A(T) = \pi_A(\pi_{\mathcal{S}(H)})(T)$ by Lemma 3.1.2. Projections of meetings factors, however, may slightly change. In particular, A and B are free factors of H that overlap but, now as subgroups of H, x is their unique intersection up to conjugacy. In general, we use the notation $\pi_A(B \leq H)$ to denote the projection of B to the free factor complex of A when B is considered as a free factor of H. Note that in this case $\pi_A(B \leq H) = \{[x]\} \subset \pi_A(B) \subset \mathcal{F}(A)$ and so although the choice of x and, hence, H is not uniquely determined by the overlapping free factors A and B, this ambiguity is not significant when considering projections.

Remark 3.2.4. The definitions in this section, including *meeting* and *overlapping*, were made to fit the applications in Chapter 4. However, in Chapter 5 we greatly extend these notions by combining our projections with those considered by Bestvina-Feighn [9].

Chapter 4

Right-angled Artin groups in $\mathrm{Out}(F_n)$

4.1 RAAGs as subgroups

For a finite simplicial graph Γ with vertex set Γ^0, the right-angled Artin group $A(\Gamma)$ is the group presented with generators $s_i \in \Gamma^0$ and relators $[s_i, s_j] = 1$ whenever s_i and s_j are joined by an edge in Γ. Although they are simple to define, right-angled Artin groups have been at the center of recent developments in geometric group theory and low-dimensional topology. This interest is, in part, because many geometrically significant groups contain right-angled Artin subgroups. For example, Wang constructed injective homomorphisms from certain right-angled Artin groups into $\mathrm{SL}_n(\mathbb{Z})$, for $n \geq 5$ [48]. In [32], Kapovich proved that for any finite simplicial graph Γ and any symplectic manifold (M, ω), $A(\Gamma)$ embeds into the group of Hamiltonian symplectomorphisms of (M, ω). Turning our attention to the mapping class group of a surface, Koberda showed that under general conditions the subgroup generated by sufficiently high powers of finitely many mapping classes is a right-angled Artin subgroup of $\mathrm{Mod}(S)$ [34]. In [14], Clay, Leininger, and Mangahas constructed quasi-isometric embeddings of right-angled Artin groups into mapping class groups using partial pseudo-Anosov mapping classes. Their theorem is stated in Chapter 1 as Theorem 1.1.2.

In this chapter, we develop the theory necessary to quasi-isometrically embed right-angled Artin groups into $\mathrm{Out}(F_n)$. Here, we show the following (see Section 4.2 for definitions and a more general statement):

Theorem 4.1.1. *Suppose that $f_1, \dots, f_n \in \mathrm{Out}(F_n)$ are fully supported on an admissible collection of free factors. Then after raising to sufficiently high powers, the elements generate a quasi-isometrically embedded right-angled Artin subgroup of $\mathrm{Out}(F_n)$.*

The *admissible collection* condition on the set of free factors in Theorem 4.1.1 is meant to mimic the situation in Theorem 1.1.2, where the subsurfaces considered are either disjoint or overlapping. We note that if Γ is the "coincidence graph" for the involved free factors, then the right-angled Artin group generated in Theorem 4.1.1 is $A(\Gamma)$. This is made precise in Section 4.2. We also obtain

Corollary 4.1.2. *Any right-angled Artin group admits a homomorphism to $\mathrm{Out}(F_n)$, for some n, which is a quasi-isometric embedding.*

We remark that although much of the inspiration for this chapter is drawn from [14], there are several significant points of departure. First, the methods of Clay, Leininger, and Mangahas rely heavily on subsurface projections for $\mathrm{Mod}(S)$, as discussed in Chapter 1. When working in $\mathrm{Out}(F_n)$, however, there are different possible projections that one could employ. In [9], Bestvina and Feighn begin with free factors A and B of F_n that are in "general position," and they define the projection of A to the free splitting complex of B. These projections, though powerful in other settings, are not delicate enough for our application. In particular, the presence of commuting outer automorphisms in our construction precludes the free factors from satisfying the conditions for finite diameter Bestvina-Feighn projections. See [46] for recent work that extends the Bestvina-Feighn projections to a larger class of free factors. In [43], a different sort of projection is developed. Sabalka and Savchuk consider a topologically defined projection using sphere systems in M_n, the double of the handlebody of genus n. Although these projections are interesting in their own right, they do not always give free splittings of free factors and so they cannot be used in this chapter. These difficulties are discussed in detail in Section 4.3.3. To resolve these issues, we develop our own projections which are tailored for the applications in this chapter. In the process, we demonstrate the relationship between the projections of [9] and [43], answering a question that appears in both papers.

Second, the authors of [14] use the Masur-Minsky distance formulas for $\mathrm{Mod}(S)$ to verify that the homomorphisms they construct are quasi-isometric embeddings. For $\mathrm{Out}(F_n)$, however, there are no general distance formulas available. Instead, in Section 4.8 we address this issue by using the partial ordering on the syllables of $g \in A(\Gamma)$. This partial ordering allows us to control distance in $\mathrm{Out}(F_n)$ by using the projections that are defined in Section 3.2. This suffices for proving the lower bounds on $\mathrm{Out}(F_n)$-distance that is needed in our main theorem.

Finally, we note that there is another method to construct quasi-isometrically embedded right-angled Artin subgroups of $\mathrm{Out}(F_{2g})$. One could start with a once-punctured genus g surface $\dot{S}$ and use the methods of [14] to build a quasi-isometric embedding from $A(\Gamma)$ into $\mathrm{Mod}(\dot{S})$. In [19], the authors show that the injective homomorphism $\mathrm{Mod}(\dot{S}) \to \mathrm{Out}(F_{2g})$ induced by the action of $\mathrm{Mod}(\dot{S})$ on $\pi_1(\dot{S}) = F_{2g}$, is itself a quasi-isometric embedding. Composing two such maps then gives a quasi-isometric embedding from $A(\Gamma)$ into $\mathrm{Out}(F_{2g})$. These homomorphisms have the property that they factor through mapping class groups and, hence, fix the conjugacy class in F_{2g} corresponding to the puncture. In our approach, however, homomorphisms into $\mathrm{Out}(F_n)$ do not factor through mapping class groups.

4.2 The homomorphisms $A(\Gamma) \to \mathrm{Out}(F_n)$

In this section, we present the most general version of our theorem. Technical conditions are unavoidable since, unlike the surface case, free factors do not uniquely determine splittings. Also, some care must be taken when defining the support of an outer automorphism. After presenting the general conditions, we also give a specific construction for applying the main theorem. The idea is to replace the surface in the mapping class group situation with a graph of groups decomposition of F.

4.2.1 Admissible systems

Let $\mathcal{A} = \{A_1, \ldots, A_n\} \subset \mathcal{F}^0$ be a collection of (conjugacy classes of) rank ≥ 2 free factors of F such that for $i \neq j$ either

1. A_i and A_j are *disjoint*, that is they are vertex groups of a common splitting, or
2. A_i and A_j *overlap*, so in particular $\pi_{A_i}(A_j) \neq \emptyset \neq \pi_{A_j}(A_i)$.

Then we say that $\mathcal{A}$ is an *admissible collection* of free factors of F. Let $\Gamma = \Gamma_{\mathcal{A}}$ be the coincidence graph for $\mathcal{A}$. This is the graph with a vertex v_i for each A_i and an edge connecting v_i and v_j whenever the free factors A_i and A_j are disjoint.

An outer automorphism $f_i \in \mathrm{Out}(F)$ is said to be *supported* on the factor A_i if $f_i(A_j) = A_j$ for each v_j in the star of $v_i \in \Gamma^0$ and $f_i|_{A_j} = 1 \in \mathrm{Out}(A_j)$ for each v_j in the link of $v_i \in \Gamma^0$. Informally, f_i is required to stabilize and act trivially on each free factor in $\mathcal{A}$ that is disjoint from A_i as well as stabilize A_i itself. We say that f_i is *fully supported* on A_i if, in addition, $f_i|_{A_i} \in \mathrm{Out}(A_i)$ is fully irreducible. Finally, we call the pair $\mathcal{S} = (\mathcal{A}, \{f_i\})$ an *admissible system* if the f_i are fully supported on the collection of free factors $\mathcal{A}$ and for each v_i, v_j joined by an edge in Γ, f_i and f_j commute in $\mathrm{Out}(F)$ (this condition is made unnecessary in the construction of the next section).

Given an admissible system $\mathcal{S} = (\mathcal{A}, \{f_i\})$, we have the induced homomorphism

$$\phi = \phi_{\mathcal{S}} : A(\Gamma) \to \mathrm{Out}(F_n)$$

defined by mapping $v_i \mapsto f_i$. Our main theorem is the following:

Theorem 4.2.1. *Given an admissible collection $\mathcal{A}$ of free factors for F with coincidence graph Γ there is a $C \geq 0$ so that if outer automorphisms $\{f_i\}$ are chosen to make $\mathcal{S} = (\mathcal{A}, \{f_i\})$ an admissible system with $\ell_{A_i}(f_i) \geq C$ then the induced homomorphism $\phi = \phi_{\mathcal{S}} : A(\Gamma) \to \mathrm{Out}(F)$ is a quasi-isometric embedding.*

It is worth noting that since right-angled Artin groups are torsion-free, homomorphisms from $A(\Gamma)$ that are quasi-isometric embeddings are injective.

The majority of this chapter is devoted to proving Theorem 4.2.1. Before completing its proof in Section 4.9, we first prove a version of Behrstock's inequality in Section 4.4. This will be central to our proofs of Theorem 4.7.1 and Theorem 4.8.2, which are the main ingredients in the proof of Theorem 4.2.1.

4.2.2 Splitting construction

Here we present a particular type of graph of groups decomposition of F that allows for easy applications of Theorem 4.2.1. Let $\mathcal{G}$ be a free splitting of F along with a family of collapse maps

$$p_i : \mathcal{G} \to \mathcal{G}_i$$

to splittings $\mathcal{G}_i$, satisfying the following conditions:

1. Each splitting $\mathcal{G}_i$ has a preferred vertex $v_i \in \mathcal{G}_i$ so that all edges of $\mathcal{G}_i$ are incident to v_i.

2. Setting $G_i = p^{-1}(v_i) \subset \mathcal{G}$ we require that for $i \neq j$ one of the two following conditions hold: either (i) G_i and G_j are *disjoint*, meaning that $G_i \cap G_j = \emptyset$, or (ii) $G_i \cap G_j$ is a subgraph whose induced subgroup is nontrivial and proper in each of the subgroups induced by G_i and G_j. In the latter case, we say the subgraphs *overlap*.

We call the splitting $\mathcal{G}$ satisfying these conditions a *support graph*, and we note that the above data is determined by the collection of subgraphs G_i. For such a splitting of F, we set $A_i = \pi_1(G_i) = (\mathcal{G}_i)_{v_i} \in \mathcal{FF}$. This is the vertex groups of the vertex v_i in $\mathcal{G}_i$. It is clear from the above conditions that such a collection of free factors forms an admissible collection $\mathcal{A}(\mathcal{G})$ and that $\Gamma_{\mathcal{A}(\mathcal{G})}$ is precisely the coincidence graph of the G_i in $\mathcal{G}$.

Next, we consider the outer automorphisms that will generate the image of our homomorphism. For each i, chose an $f_i \in \mathrm{Out}(F_n)$ which preserves the splitting $\mathcal{G}_i$, induces the identity automorphism on the underlying graph of $\mathcal{G}_i$, and restricts to the identity on the complement of v_i in $\mathcal{G}_i$. In this case, we say that f_i is *supported* on G_i (or v_i), and if the restriction of f_i to the free factor A_i is fully irreducible, we say that f_i is *fully supported* on G_i (or v_i). With these choices, the pair $\mathcal{S}(\mathcal{G}) = (\mathcal{A}(\mathcal{G}), \{f_i\})$ is an admissible system. Indeed, the only condition to check is that if v_i and v_j represent disjoint free factors, then the outer automorphisms f_i and f_j commute. Observe that since G_i and G_j are disjoint subgraphs of $\mathcal{G}$ we may collapse each to a vertex to obtain a common refinement $\mathcal{G}_{ij}$ of $\mathcal{G}_i$ and $\mathcal{G}_j$, which has vertices with associated groups $(\mathcal{G}_i)_{v_i}$ and $(\mathcal{G}_j)_{v_j}$. Label these vertices of $\mathcal{G}_{ij}$ v_i and v_j corresponding to the subgraphs G_i and G_j of $\mathcal{G}$. From the fact that f_i and f_j are supported on G_i and G_j, respectively, it follows that they both stabilize the common refinement $\mathcal{G}_{ij}$ and are each supported on distinct vertices, namely v_i and v_j. This implies that f_i and f_j commute in $\mathrm{Out}(F)$. Hence, $\mathcal{S}(\mathcal{G}) = (\mathcal{A}(\mathcal{G}), \{f_i\})$ is an admissible system inducing a homomorphism

$$\phi_{\mathcal{S}(\mathcal{G})} : A(\Gamma_{\mathcal{G}}) \to \mathrm{Out}(F)$$

given by

$$v_i \mapsto f_i$$

as before. With this setup, our main result can be restated as follows:

Corollary 4.2.2. *Suppose $\mathcal{G}$ is a free splitting of F that is a support graph with subgraphs G_i for $1 \leq i \leq k$. Let Γ be the coincidence graph for these subgraphs. There is a $C \geq 0$ so that if for each i, $f_i \in \mathrm{Out}(F)$ is fully supported on G_i with $\ell_{A_i}(f_i) \geq C$, then the induced homomorphism $\phi_{\mathcal{S}(\mathcal{G})} : A(\Gamma) \to \mathrm{Out}(F)$ is a quasi-isometric embedding.*

We remark that once a support graph $\mathcal{G}$ is constructed with $\pi_1\mathcal{G} = F_n$, there is no obstruction to finding f_i fully supported on G_i with large translation length on $\mathcal{F}(A_i)$. Corollary 4.2.2 then implies that there exist homomorphisms $\phi_{\mathcal{S}(\mathcal{G})} : A(\Gamma) \to \mathrm{Out}(F_n)$ which are quasi-isometric embeddings.

4.2.3 Constructions and applications

We use our main theorem to construct quasi-isometric homomorphisms into $\mathrm{Out}(F_n)$ beginning with an arbitrary right-angled Artin group $A(\Gamma)$. We provide a bound on n given a measurement of complexity of Γ.

First, it is easy to use the splitting construction of Section 4.2.2 to start with a graph Γ and find a quasi-isometric embedding $A(\Gamma) \to \mathrm{Out}(F_n)$, with n depending on Γ. We illustrate this with an example and then give a general procedure. Note that although using the splitting construction is simple, it will always require that n is rather large compared to Γ. As demonstrated in Example 3, more creative choices of admissible systems can be used to reduce n.

Example 2. Let $\Gamma = \Gamma_5$ be the pentagon graph with vertices labeled counterclockwise v_0, v_2, v_4, v_1, v_3 as in Figure 4.1, and let Γ^c be the same graph with vertices labeled cyclically $v_0, \ldots, v_4$. Take $\mathcal{G}$ to be the graph of groups with underlying graph $(\Gamma^c)'$, the barycentric subdivision of Γ^c, with trivial vertex group labels on the vertices of Γ^c and infinite cyclic group labels on the subdivision vertices. Note that $\pi_1\mathcal{G} = F_6$. Set G_i $(1 \leq i \leq 4)$ equal to the subgraph of $\mathcal{G}$ consisting of the vertex labeled v_i, its two adjacent subdivision vertices, and the edges joining these vertices to v_i. Observe that G_i and G_j have empty intersection if and only if v_i and v_j are joined by an edge in Γ. Also, if G_i and G_j intersect then their intersection is a vertex with nontrivial vertex group. Hence, $\mathcal{G}$ is a support graph with subgraphs G_i whose coincidence graph is Γ. By Corollary 4.2.2 there is a constant C such that choosing any collection of outer automorphisms f_i fully supported on the collection G_i with $\ell_{A_i}(f_i) \geq C$ determines a homomorphism $A(\Gamma_5) \to \mathrm{Out}(F_6)$ that is a quasi-isometric embedding. In Example 3, we improve this construction by modifying $\mathcal{G}$.

Now fix any simplicial graph Γ with n vertices labeled $v_1, \ldots, v_n$. We give a general procedure for producing a support graph $\mathcal{G}$ with subgraphs G_i whose coincidence graph is Γ. By Corollary 4.2.2, this provides examples of homomorphisms $A(\Gamma) \to \mathrm{Out}(\pi_1(\mathcal{G}))$ that are quasi-isometric embeddings. First, assume that the *complement graph* Γ^c is connected. Recall that Γ^c is the subgraph of the complete graph on Γ^0 whose edge set is the complement of the edge set of Γ. Let $(\Gamma^c)'$ be the barycentric subdivision of Γ^c. We reserve labels v_i for the vertices of $(\Gamma^c)'$ that are vertices of Γ^c and label the vertex of $(\Gamma^c)'$ corresponding to the edge (v_i, v_j) of Γ^c by v_{ij}. Hence, in $(\Gamma^c)'$ the vertex v_{ij} is valence two and is connected by an edge to both v_i and v_j. Set G_i equal to the star of the vertex v_i in $(\Gamma^c)'$, i.e. G_i is the union of edges incident to v_i together with their vertices. Now take $\mathcal{G}$ to be the graph of groups with underlying graph $(\Gamma^c)'$ and infinite cyclic vertex group labels for each vertex v_{ij}, $i \neq j$. For vertices v_i there are two cases for vertex groups: If v_i has valence one in $\mathcal{G}$ then we label it with an infinite cyclic vertex group and otherwise we give it a trivial vertex group.

With these vertex groups, $\mathcal{G}$ becomes of graph of groups decomposition for F_n. Moreover, $\mathcal{G}$ is a support graph for the collection of subgraphs G_i with coincidence graph Γ. Indeed, G_i and G_j have nonempty intersection in $\mathcal{G}$ if and only if v_i and v_j are joined by an edge in Γ^c. When this is the case, their intersection is a single vertex with infinite cyclic vertex group and this vertex group is proper in each of the groups induced by G_i and G_j. We can also calculate the rank of $\pi_1\mathcal{G}$. By construction, the rank of $\pi_1\mathcal{G}$ is equal to the rank of the fundamental group of the underlying graph plus the number of nontrivial vertex groups on $\mathcal{G}$. Since there is a nontrivial vertex group for each edge of Γ^c and each vertex of Γ^c of valence one, the rank of $\pi_1\mathcal{G}$ equals

$$1 + 2|\mathrm{E}(\Gamma^c)| - |\mathrm{V}(\Gamma^c)| + |\text{valence 1 vertices of } \Gamma^c|.$$

Translating this into a function of Γ, we see that the rank of $\pi_1\mathcal{G}$ is

$$1 + |\mathrm{V}(\Gamma)| \cdot (|\mathrm{V}(\Gamma)| - 2) - |\mathrm{E}(\Gamma)| + |\text{valence } n-2 \text{ vertices of } \Gamma|,$$

and we refer to this quantity as the complexity of Γ, denoted $c(\Gamma)$.

When Γ^c is not connected it decomposes into components $\Gamma^c = \sqcup_{i=1}^{l} \Delta_i$ and it is not difficult to show that $A(\Gamma) = A(\Delta_1^c) \times \ldots \times A(\Delta_l^c)$. In this case, we set $c(\Gamma) = \sum_i c(\Delta_i^c)$ and the corresponding supported graph is constructed as follows: Let $\mathcal{G}(\Delta_i^c)$ be the support graph constructed as above for the graph Δ_i^c. Let $\mathcal{G}$ to be the support graph built by taking the wedge of l intervals (at one endpoint of each) and attaching the other endpoint of the ith interval to an arbitrary vertex of $\mathcal{G}(\Delta_i^c)$. The graph of groups structure on $\mathcal{G}$ is induced by that of $\mathcal{G}(\Delta_i^c)$ along with a trivial group label at the wedge vertex. Then $\mathcal{G}$ is a support graph with coincidence graph Γ and complexity $c(\Gamma)$. As noted above, the existence of a support graph with coincidence graph Γ implies the following:

Corollary 4.2.3. *For any simplicial graph* Γ*,* $A(\Gamma)$ *admits a homomorphism into* $\mathrm{Out}(F_n)$*, with* $n \leq c(\Gamma)$*, which is a quasi-isometric embedding.*

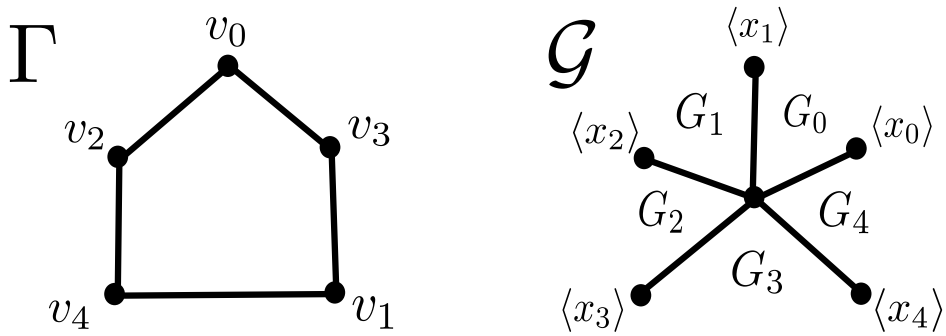

Figure 4.1: $F_5 = \pi_1(\mathcal{G})$

The next example shows how Theorem 5.1.1 can be used to give quasi-isometric embeddings into $\mathrm{Out}(F_n)$ for smaller n than by using support graphs.

Example 3. Again, let $\Gamma = \Gamma_5$ be the pentagon graph with vertices labeled counter-clockwise v_0, v_2, v_4, v_1, v_3 as in Figure 4.1. Take $\mathcal{G}$ as in Figure 4.1. This is a graph of groups decomposition for F_5; the central vertex has trivial vertex group and the 5 valence one vertices joined to the central vertex each have infinite cyclic vertex group, with generators labeled $x_0 \dots, x_4$. $\mathcal{G}$ can be thought of as a "folded" version of the support graph that appears in Example 2. For $0 \leq i \leq 4$, let G_i be the smallest connected subgraph containing the vertices labeled x_i and x_{i+1}, with indices taken mod 5. Note that $\mathcal{G}$ together with the subgraphs G_i is *not* a support graph; for example G_0 and G_2 intersect in a vertex with trivial vertex group. Despite this, for $i = 0, \dots, 4$, $A_i = \pi_1 G_i = \langle x_i, x_{i+1}\rangle$ does form an admissible collection of free factors with coincidence graph Γ_5. Hence, by Theorem 4.2.1, there exists a $C \geq 0$ so that if there are outer automorphisms $f_i \in \mathrm{Out}(F_5)$ making

$(\{A_i\}, \{f_i\})$ an admissible system with $\ell_{A_i}(f_i) \geq C$ then the induced homomorphism $\phi : A(\Gamma_5) \to \mathrm{Out}(F_5)$ is a quasi-isometric embedding. Choosing such a collection in this case is straightforward. Specifically, let $B_i = \langle x_{i+2}, x_{i+3}, x_{i+4} \rangle$ and choose $f_i \in \mathrm{Out}(F_5)$ for $i = 0, \ldots, 4$ so that

1. $f_i(A_i) = A_i$ and $f_i(B_i) = B_i$,
2. the restriction $f_i|A_i \in \mathrm{Out}(A_i)$ is fully irreducible with $\ell_{A_i}(f_i) \geq C$, and
3. the restriction $f_i|B_i = 1 \in \mathrm{Aut}(B_i)$.

With these choices, it is clear that each f_i is fully supported on A_i and that f_i and f_j commute if and only if v_i and v_j are joined by an edge of Γ. This makes $\mathcal{S} = (\{A_i\}, \{f_i\})$ into an admissible system with $\ell_{A_i}(f_i) \geq C$ and so the induced homomorphism

$$\phi_{\mathcal{S}} : A(\Gamma_5) \to \mathrm{Out}(F_5)$$

is a quasi-isometric embedding. In fact, as we shall see in the proof of the main theorem, the required translation length is simple to determine. Further, as each of free factors A_i in the admissible system is rank 2, the free factor complex $\mathcal{F}(A_i)$ is the Farey graph where translation lengths can be computed.

For an application, recall that $A(\Gamma_5)$ contains quasi-isometrically embedded copies of $\pi_1(\Sigma_2)$, the fundamental group of the closed genus 2 surface (see [15]). Restricting the homomorphism constructed above to such a subgroup, we obtain quasi-isometric embeddings

$$\pi_1(\Sigma_2) \to \mathrm{Out}(F_5).$$

4.3 Splittings and submanifolds

We need a topological interpretation of our projections in order to prove the version of Behrstock's inequality that appears in the next section. We first review some facts about embedded surfaces in 3-manifolds and the splittings they induce.

4.3.1 Surfaces and splittings

It is well-known that codimension 1 submanifolds induce splittings of the ambient manifold group [44]. We review some details here, focusing on the case when then inclusion map is not necessarily π_1-injective.

For our application, begin with an orientable, connected 3-manifold X possibly with boundary and a property embedded, orientable surface F. We do not require that F is connected or that each component of F is π_1-injective. Working, for example, in the smooth setting, choose a tubular neighborhood $N \cong F \times I$ of F in X whose restriction $N \cap \partial X$ is a tubular neighborhood of the boundary of F in ∂X. Let G denote the graph *dual* to F in X. This is the graph with a vertex for each component of $X \setminus \mathrm{int}(N)$ and an edge e_f, for each component $f \subset F$, that joins the vertices corresponding to the (not necessarily distinct) components on either side of f. We may consider G as embedded in X and, after choosing an appropriate embedding, G is easily seen to be a retract of X. The retraction is obtained by collapsing each complementary component of N to its corresponding vertex and projecting $f \times I$ to I for each component f of F. Here, I is the closed interval $[-1, 1]$ and $f \times \{0\}$

corresponds under the identification $N \cong F \times I$ to $f \subset N$.

Let $\tilde{X}$ denote the universal cover of X and let $\tilde{N}$ and $\tilde{F}$ denote the complete preimage of N and F, respectively. Let T_F denote the graph dual to $\tilde{F}$ in $\tilde{X}$. Since T_F is a retract of the connected, simply connected space $\tilde{X}$, T_F is a tree. We call T_F the *dual tree* to the surface F in X. As $\tilde{F}$ and $\tilde{X} \setminus \tilde{N}$ are permuted by the action of $\pi_1(X)$, we obtain a simplicial action $\pi_1(X) \curvearrowright T_F$, up to the usual ambiguity of choosing basepoints. The following is an exercise in covering space theory; it appears in [44].

Proposition 4.3.1. *With the above notation, let v be a vertex of T_F corresponding to a lift of a component $C \subset X \setminus N$ and e an edge of T_F corresponding to a lift of a component $f \subset F$. Then*

1. *stab*$(v) = \mathrm{im}(\pi_1 C \to \pi_1 X)$
2. *stab*$(e) = \mathrm{im}(\pi_1 f \to \pi_1 X)$

where both equalities are up to conjugation in $\pi_1 X$.

The action $\pi_1 X \curvearrowright T_F$ provides a splitting of $\pi_1 X$ via Basse-Serre theory. The corresponding graph of groups decomposition of $\pi_1 X$ has underlying graph $G = T_F / \pi_1 X$ with vertex and edge groups as given in Proposition 4.3.1. A subgraph $G' \subset G$ *carries* a subgroup $H \leq \pi_1 X$ if the subgroup induced by G' contains H, up to conjugacy.

We now specialize to the situation where the action $\pi_1 X \curvearrowright T_F$ has trivial edge stabilizers. The following proposition determines when the dual tree to a surface is minimal. First, say that a connected component $f \subset F$ is *superfluous* if f separates X and to one side bounds a relatively simply connected submanifold, i.e. $X \setminus f = X_1 \sqcup X_2$ and $\operatorname{im}(\pi_1(X_1) \to \pi_1(X)) = 1$. A component of F that is not superfluous is said to *split* X. Also, use the notation $T^{\min}$ to denote the unique minimal subtree associated to an action on the tree T, see Section 2.2.

Proposition 4.3.2. *Let F be an orientable, properly embedded surface in the orientable 3-manifold X with $\operatorname{im}(\pi_1 f \to \pi_1 X) = 1$ for each component f of F. Then the edge $e_f \subset T$ corresponding to a lift of the component $f \subset F$ is contained in the minimal subtree $T_F^{\min}$ if and only if f splits X.*

Proof. First suppose that the edge e_f whose orbit corresponds to the lifts of f is not in the minimal subtree $T^{\min}$. Setting $G = T_F/\pi_1 X$ and $G^{\min} = T_F^{\min}/\pi_1 X$, the image of e_f in G does not lie in $G^{\min}$. Since $G^{\min}$ carries the fundamental group of X, the image of e_f in G must separate and the component of its complement not containing $G^{\min}$ has all trivial vertex groups. In X, this implies that the component $f \subset F$ separates X and to one side bounds a component whose fundamental group, when included into $\pi_1 X$, is trivial. Hence, f is superfluous.

Now suppose that f is a component of F that is superfluous. Then f corresponds to a separating edge e in $G = T_F/\pi_1 X$, with lift $e_f \subset T_F$, whose complement in G contains a component with trivial induced subgroup. Hence, this component of $G \setminus e$ is a tree with trivial vertex groups. Set G' equal to the other component of the complement of e in G. Then G' carries all of $\pi_1 X$ and so its complete preimage in T_F is connected, $\pi_1 X$ invariant, and does not contain the edge e_f. Hence, e_f is not in $T^{\min}$. $\square$

We will use the above proposition in the following manner: If $f \subset F$ splits X then T_f is a 1-edge collapse of T_F corresponding to a 1-edge splitting of $\pi_1 X$.

4.3.2 Topological projections

The purpose of this section is to give a topological description of the projection $\pi_A(T)$ in terms of submanifolds of the manifold M_n. As discussed below, these are similar to the submanifold projections of [43], and this section serves to explain the connection between these projections and the projections of [9]. To verify that our description is accurate, we rely on Hatcher's normal position for spheres in $M = M_n$ and its generalization in [27]. Let $\tilde{M}$ denote the universal covers of M. We say that essential sphere systems S_1 and S_2 in M are in *normal position* if for $\tilde{S}_1$ and $\tilde{S}_2$, the complete preimage of S_1 and S_2 in $\tilde{M}$, any spheres $s_1 \in \tilde{S}_1$ and $s_2 \in \tilde{S}_2$ satisfy each of the following:

1. s_1 and s_2 intersect in at most one component and
2. no component of $s_1 \setminus s_2$ is a disk that is isotopic relative its boundary to a disk in s_2.

This definition is easily seen to be equivalent to Hatcher's original notion of normal position in the case where one of the sphere systems is maximal [24].

In particular, the authors of [27] use Hatcher's original proof of existence and uniqueness of normal position to show the following:

Lemma 4.3.3. *Any two essential sphere systems S_0 and S can be isotoped to be in normal position. Also, normal position is unique in the following sense: Let S_0 be a sphere system of M_n, and let S, S' be two isotopic spheres in M_n which are in normal position with respect to S_0. Then there is a homotopy between S and S' which restricts to an isotopy on S_0.*

Fix sphere systems S and S_A and a preferred component $C_A \subset M \setminus S_A$. In what follows we assume that $S_A = \partial C_A$. When this is the case, we say C_A is a *splitting component* and observe that C_A is homeomorphic to $M_{k,s}$, as defined in Section 2.3. Let A be the (conjugacy class of) free factor $\pi_1(C_A)$ and let $T = T_S$ be the free splitting of F determined by the sphere system S. Since we are interested the projection of the splitting $F \curvearrowright T$ to the free splitting complex of A, our aim is a topological interpretation of the projection $\pi_A(T) = [A \curvearrowright T^A]$.

Put S and S_A in normal position and consider the collection of connected components of the surface $F = S \cap C_A$. This family of surfaces is well-defined up to homotopy in C_A that restricts to isotopy on S_A, by Lemma 4.3.3. Consider the graph of spaces decomposition of C_A given by F with dual tree T_F, see Section 4.3.1. Recall that a connected component $f \subset F$ is *superfluous* if f separates C_A and to one side bounds a relatively simply connected submanifold, that is $C_A \setminus f = C_1 \sqcup C_2$ and $\mathrm{im}(\pi_1(C_1) \to \pi_1(C_A)) = 1$. A

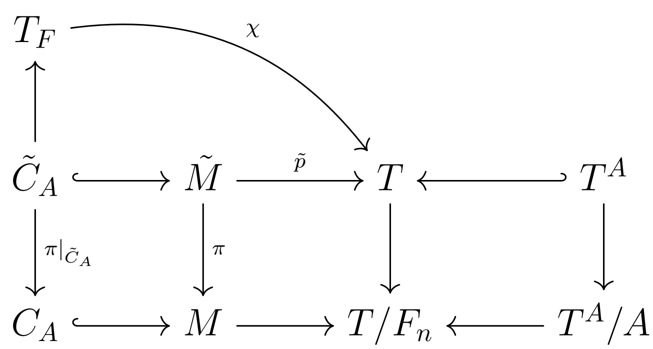

Figure 4.2: Defining the map $\chi : T_F \to T$

component of F that is not superfluous is said to *split* C_A. Set $\bar{F}$ equal to F minus its superfluous components and let $T_{\bar{F}}$ be its dual tree (that is, the tree dual to the complete preimage of $\bar{F}$ in the universal cover of C_A).

We claim the following about the associated splitting of $\pi_1 C_A = A$:

1. there is an A-equivariant simplicial embedding $\chi : T_F \to T$ whose image contains T^A,

2. an edge e of T_F maps to an edge in T^A if and only if e corresponds to the lift of a component of $f \subset F$ that splits C_A, and

3. the projection $\pi_A(T) = A \curvearrowright T^A$ is conjugate to the A-tree $T_{\bar{F}}$.

To prove the above claim we refer to Figure 4.2, where as above the free splitting $F \curvearrowright T$ corresponds to the sphere system $S \subset M$. Let $\pi : \tilde{M} \to M$ be the universal cover of M and let $\tilde{S}$ be the complete preimage of S in $\tilde{M}$. The map labeled $\tilde{p}$ is the equivariant map from $\tilde{M}$ to the tree T obtained by retracting $\tilde{M}$ to the tree dual to $\tilde{S} \subset \tilde{M}$, as explained in Section 4.3.1. Hence, if we let m denote the set of midpoints of edges of T then $\tilde{S} = \tilde{p}^{-1}(m)$. Setting $F = S \cap C_A$ as above, we note that if $\tilde{C}_A$ is a fixed component of the preimage of C_A in $\tilde{M}$ then $\pi|_{\tilde{C}_A} : \tilde{C}_A \to C_A$ is the universal cover and $\tilde{F} = (\pi|_{\tilde{C}_A})^{-1}(F) = \tilde{S} \cap \tilde{C}_A$. Hence, by definition of the dual tree to F in C_A, T_F is precisely the tree dual to $\tilde{S} \cap \tilde{C}_A$ in $\tilde{C}_A$.

Because $\tilde{p}$ is F-equivariant, $T' = \tilde{p}(\tilde{C}_A)$ is an A-invariant subtree of T and so it contains T^A, the minimal A-subtree of T. Note that by carefully choosing the projection $\tilde{p}$, we may assume that T' is a subcomplex of T. We first show that the A-tree T' is conjugate to the A-tree T_F. Since T is dual to $\tilde{S}$ in $\tilde{M}$ and T_F is dual to $\tilde{F} = \tilde{S} \cap \tilde{C}_A$ in $\tilde{C}_A \subset \tilde{M}$ each complementary component of $\tilde{F}$ in $\tilde{C}_A$ corresponds to a complementary component of $\tilde{S}$ in $\tilde{M}$. This induces a map from the vertices of T_F to those of T. As components of $\tilde{F}$ are contained in components of $\tilde{S}$, this map extends to a simplicial map of A-trees $\chi : T_F \to T$ with image T'. We show that this map does not fold edges and is, therefore, an immersion. This suffices to prove that $\chi : T_F \to T'$ is an A-conjugacy.

To see that χ does not fold edge, suppose to the contrary that two edges e_1 and e_2 with common initial vertex v are identified by χ (Figure 4.3). Then the edge e_i is dual to a component $f_i \subset \tilde{F}$ in $\tilde{C}_A$ and these components are disjoint. Since e_1 and e_2 are folded by χ, their common image e in T corresponds to a sphere $s \subset \tilde{S}$ which must contain f_1 and f_2 as subsurfaces. Let τ be an arc in s that connects the interiors of f_1 and f_2 and intersects only the

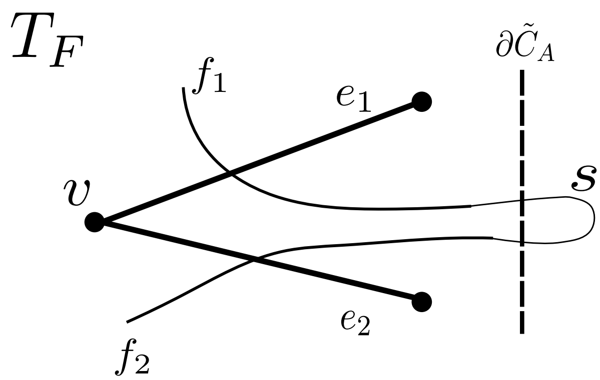

Figure 4.3: Folding edges

components of $s \cap \partial\tilde{C}_A$ that separate f_1 and f_2. Since each component of $\partial\tilde{C}_A$ separates $\tilde{M}$, as do all essential spheres in $\tilde{M}$, the first and last components of $\partial\tilde{C}_A$ intersected by τ must be the same. This implies that f_1 and f_2 each have a boundary component on the same component of $\partial\tilde{C}_A$. Hence, the sphere s intersects the same component of $\partial\tilde{C}_A$ in at least 2 circles. This, however, contradicts normal position of the sphere systems S and ∂C_A. We conclude that the A-trees T_F and T' are simplicially conjugate. This proves claim (1) and justifies identifying T_F and T' through χ. Observe that since T' contains T^A, we get an induced A-conjugacy $\chi : T_F^A \to T^A$ on minimal subtrees.

It remains to show that $T_{\bar{F}} = T_F^A$, as this identifies the edges of T_F that correspond to components of F that split C_A with those contained in T^A. Since the components of $\bar{F}$ are precisely those that split C_A, Proposition 4.3.2 implies that the minimal A-subtree of T_F is $T_{\bar{F}}$ and so $T_{\bar{F}} = T_F^A$, as required. This completes the proofs of claims (2) and (3).

To summarize the above discussion:

Proposition 4.3.4. *Let $T \in \mathcal{S}'_n$ be a free splitting of F_n corresponding to the sphere system $S \subset M_n$. Fix a submanifold C_A, as above, with $\pi_1(C_A) = A$ and ∂C_A and S in normal position. If $\bar{F}$ is the surface obtained from $F = S \cap C_A$ by removing the components that separate and bound relatively simply connected components, then $\bar{F}$ is nonempty if and only if T^A is nontrivial. When this is the case, the resulting splitting $T_{\bar{F}}$ is conjugate as an A-tree to $\pi_{\mathcal{S}(A)}(T)$.*

4.3.3 Relations between the various projections

In [43], Sabalka and Savchuk define projections from the sphere complex $\mathcal{S}(M_n)$ to the sphere and disk complex of certain submanifolds of M_n. Their projections can be interpreted within the framework developed in this section, providing a simple relationship to $\pi_{\mathcal{S}(A)}(T)$. This answers a question asked in [43, 9]. However, it is important to note that, as demonstrated below, it is possible for each of the projections to be defined in situations when the other is not. Also, it is not clear whether the distances in the target complexes of the two projections are comparable. This section is not necessary for the rest of the chapter.

Let $X \subset M_n$ denote a component of the complement of some sphere system. In [43], such X are referred to as *submanifolds*. Note that X is homeomorphic to $M_{k,s}$ for some $k < n$ and $s > 0$. The disk and sphere complex of X, denoted $\mathcal{DS}(X)$, is defined to be the simplicial complex whose vertices are isotopy classes of essential spheres and essential properly embedded disks in X with $k+1$ vertices spanning a k-simplex whenever the disks and spheres representing these vertices can be realized disjointly in X. Sabalka and Savchuk define their projections as follows: Let S be an essential sphere system in M_n. Put S and ∂X in normal position and set $F = S \cap X$. The projection $\pi_X^{\mathrm{SS}}([S]) \subset \mathcal{DS}(X)$ is then defined to be the components of F which are either spheres or disks. If there are no such components of F, then the projection is left undefined.

Fix a submanifold X with $A = \pi_1 X$ a rank ≥ 2 free factor of $F_n = \pi_1 M_n$. There is a *partially* defined map $\Phi : \mathcal{DS}^0(X) \to \mathcal{S}^0(A)$ given by taking $D \in \mathcal{DS}^0(X)$ and mapping it to the A-tree T_D if D splits X. If D does not split X, then $\Phi(D)$ is left undefined. Recall that as in Section 4.3.1, T_D is the dual tree to $D \subset X$. Note that this map will be defined on all vertices of $\mathcal{DS}(X)$ *only* when X is homeomorphic to $M_{k,1}$. When D and D' are adjacent in $\mathcal{DS}(X)$ and both $\Phi(D)$ and $\Phi(D')$ are defined, then it is clear that $d_{\mathcal{S}(A)}(\Phi(D), \Phi(D')) \leq 1$. With this setup, we can show the following:

Proposition 4.3.5. *Let T be a free splitting of F_n and S its corresponding sphere system in M_n. Let X be a submanifold of M_n with $\pi_1 X = A \neq 1$. If the composition $\Phi \circ \pi_X^{\mathrm{SS}}(S)$ is defined, then it is a free splitting of A that has $\pi_{\mathcal{S}(A)}(T)$ as a refinement.*

Proof. By Proposition 4.3.4, if S and ∂X are in normal position and $F = S \cap X$ then T^A is conjugate to $T_{\bar{F}}$ where $\bar{F}$ is the union of connected components of F that split X. By definition, $\Phi \circ \pi_X^{\mathrm{SS}}(S)$ is the tree dual to the collection of disks and spheres $D \subset F$ that split X, which is nonempty by assumption. Since $D \subset \bar{F}$, the induced map $T_{\bar{F}} \to T_D$ is a collapse map. Hence, $T_{\bar{F}}$ refines $\Phi \circ \pi_X^{\mathrm{SS}}(S)$. □

This proposition also gives the connection between the projections of [43] and those of [9]. Recall that the projection $\pi^{\mathrm{BF}}_{\mathcal{S}(A)}(B)$ is well-defined, i.e. has bounded diameter image, when either (1) A and B have the same color in a specific finite coloring of the vertices of $\mathcal{F}_n$ or (2) $d_{\mathcal{F}}(A, B) > 4$. See [9] for the definition of the coloring and further details.

Corollary 4.3.6. *Let A, B be free factors of F_n satisfying one of the above conditions so that the projection $\pi^{\mathrm{BF}}_{\mathcal{S}(A)}(B)$ is well-defined. Let X be a submanifold of M with $\pi_1 X = A$ and let S be any sphere system that contains a sphere system $S' \subset S$ whose dual tree $T_{S'}$ has B as a vertex stabilizer. If the composition $\Phi \circ \pi_X^{\mathrm{SS}}(S)$ is defined, then it has bounded distance from $\pi^{\mathrm{BF}}_{\mathcal{S}(A)}(B)$ in $\mathcal{S}(A)$, where the bound depends only on n.*

It is important to note that whether $\Phi \circ \pi_X^{\mathrm{SS}}(S)$ is defined is highly dependent on the choice of X and S that represent the free factors A and B in Corollary 4.3.6. This is demonstrated in the examples below.

Proof. By definition, we may take $\pi^{\mathrm{BF}}_{\mathcal{S}(A)}(B) = \pi_{\mathcal{S}(A)}(T_{S'})$. By Lemma 3.1.1, this is refined by the projection $\pi_{\mathcal{S}(A)}(T_S)$, and by Proposition 4.3.5, $\pi_{\mathcal{S}(A)}(T_S)$ also refines $\Phi \circ \pi_X^{\mathrm{SS}}(S)$. This completes the proof since we may take as our bound the diameter of the Bestvina-Feighn projection plus 2. □

We end this section with some examples that illustrate cases when one of the projections is defined and the other is not. The general idea is that while the Bestvina-Feighn projections are robust, i.e. they do not depend on how a factor is complemented, the Sabalka and Savchuk projections are highly sensitive to the submanifold that is chosen to represent a free factor.

Example 4. Take $M = M_4$ and $S = S_1 \cup S_2$ to be a union of two essential spheres so that $X = M \setminus S$ connected with $\pi_1 X = A$. Let $f \in \mathrm{Out}(F_n)$ with $f(A) = A$ but f has no power that fixes S in $\mathcal{S}(M)$. Then $\pi_{\mathcal{S}(A)}(f^n T_S) = \pi_{\mathcal{S}(A)}(T_S)$ is undefined, as A fixes a vertex of T_S, but $\pi_X^{\mathrm{SS}}(f^n S)$ is defined for all $n \geq 1$ by construction. Hence, it must be the case that each disk of $\pi_X^{\mathrm{SS}}(f^n S)$ is superfluous in X. Informally, each disk of $\pi_X^{\mathrm{SS}}(f^n S)$ $(n \geq 1)$ simply encloses some boundary components of X without splitting $\pi_1 X$.

Example 5. Take M, X, A as above and refer to Figure 4.4 where M is drawn as a handlebody and spheres are drawn as properly embedded disks; doubling the picture gives an illustration of what is described. Let S_3 be any sphere that separates M into two components, one of which contains $S = \partial X$ and the other, denoted Y, has $\pi_1 Y = A$. Let R be the essential sphere shown in Figure 4.4 with dual tree T_R; R is in normal position with S_3. Note that R splits Y with non-trivial projection $\pi_{\mathcal{S}(A)}(T_R)$. However, $Y \cap R$ has no disks of intersection and so $\pi_Y^{\mathrm{SS}}(R)$ is undefined. If instead we use the submanifold X to represent the free factor A, we see that $\pi_X^{\mathrm{SS}}(R)$ is the sphere $R \subset X$ and $\Phi \circ \pi_X^{\mathrm{SS}}(R) = \pi_{\mathcal{S}(A)}(T_R)$.

Even if we only use the submanifold X, which *exhausts* M in the terminology of [43], to represent the free factor A, the question of whether the composition $\Phi \circ \pi_X^{\mathrm{SS}}$ is defined still depends on the choice of sphere that is projected. This is because the existence of a disk in $\pi_X^{\mathrm{SS}}(R)$ that splits X is highly depended on R itself. In fact, it is not difficult to show the following: for any nonseperating sphere $R \subset X$ there is a $f \in \mathrm{Out}(F_4)$ with $f(A) = A$ and $f|_A = 1 \in \mathrm{Out}(A)$, so in particular $\pi_A(fT_R) = \pi_A(T_R) = \Phi \circ \pi_X^{\mathrm{SS}}(R)$, but $\Phi \circ \pi_X^{\mathrm{SS}}(fR)$ is undefined. This implies that all disks of $\pi_X^{\mathrm{SS}}(fR)$ are superfluous even though $\pi_A(fT_R) = \pi_A(T_R)$.

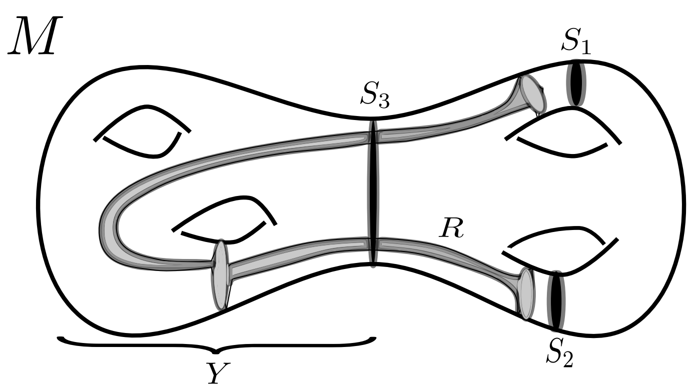

Figure 4.4: Projecting R to Y

4.4 Behrstock's Inequality

We now introduce an analog of Behrstock's inequality for projections to the free factor complex of a free factor. For the original statement and proof in the case of subsurface projections from the curve complex, see [3]. The proof of the free group version given in Proposition 4.4.1 is similar in spirit to the proof of the original version of Behrstock's inequality that is recorded in [36], where it is attributed to Chris Leininger. Both proofs investigate intersections of submanifolds and give explicit bounds on the distances of the projections that are considered.

Proposition 4.4.1. *There is an $M \geq 0$ so that if A and B are free factors of F of rank ≥ 2 that overlap, then for any $T \in \mathcal{S}'$ with $\pi_A(T) \neq \emptyset \neq \pi_B(T)$ we have*

$$\min\{d_A(B,T), d_B(A,T)\} \leq M.$$

Proof. Fix $T \in \mathcal{S}'$ that has nontrivial projection to both the free factors complex of A and the free factor complex of B. Since A and B overlap we may, as in Section 3.2, choose conjugates (still denoted A and B) so that $A \cap B = x$, where $x \neq \{1\}$ is a proper free factor of A and B. Write $A = A' * x$ and $B = B' * x$ so that

$$H = \langle A, B \rangle \cong A *_x B \cong A' * x * B'.$$

Since $\pi_A(B \leq H) = \{[x]\} \subset \pi_A(B)$ in $\mathcal{F}(A)$ and $\pi_B(A \leq H) = \{[x]\} \subset \pi_B(A)$ in $\mathcal{F}(B)$ and by Lemma 3.2.2, $\pi_A(\pi_{\mathcal{S}(H)}(T)) = \pi_A(T)$ and $\pi_B(\pi_{\mathcal{S}(H)}(T)) = \pi_B(T)$, we have

$$\begin{aligned} d_A(B,T) &\leq d_A(\pi_A(B \leq H), \pi_{\mathcal{S}(H)}(T)) + \mathrm{diam}_A(\pi_A(B)) \\ &\leq d_A(x, \pi_{\mathcal{S}(H)}(T)) + 4 \end{aligned}$$

and similarly

$$\begin{aligned} d_B(A,T) &\leq d_B(\pi_B(A \leq H), \pi_{\mathcal{S}(H)}(T)) + \mathrm{diam}_B(\pi_B(A)) \\ &\leq d_B(x, \pi_{\mathcal{S}(H)}(T)) + 4. \end{aligned}$$

Hence, it suffices to show that for $T \in \mathcal{S}'$ with $\pi_A(T) \neq \emptyset \neq \pi_B(T)$

$$\min\{d_A(x, \pi_{\mathcal{S}(H)}(T)), d_B(x, \pi_{\mathcal{S}(H)}(T))\} \leq M - 4,$$

where H is fixed as above.

To transition to the topological picture, suppose that $\mathrm{rank}(H) = k$ and set $M = M_k$ with a fixed identification $\pi_1 M = H$. Let S_A, S_B be two disjoint spheres in M that correspond to the splitting $H = A' * x * B'$ via Proposition 2.3.1. Take C_A to be the submanifold with boundary S_A and $\pi_1 C_A = A$ and take C_B to be the submanifold with boundary S_B and $\pi_1 C_B = B$. By construction $S_A \subset C_B$ and $S_B \subset C_A$ and so, in particular, ∂C_A induces a splitting of $B = \pi_1 C_B$ whose projection to $\mathcal{F}(B)$ contains $\pi_B(A \leq H) = \{[x]\}$. Similarly, ∂C_B induces a splitting of $A = \pi_1 C_A$ whose projection to $\mathcal{F}(A)$ contains $\pi_A(B \leq H) = \{[x]\}$.

Now choose any tree $T \in \mathcal{S}'(H)$ with nontrivial projections to $\mathcal{F}(A)$ and $\mathcal{F}(B)$ and let S be the corresponding sphere system in M. Put S and $\partial C_A \cup \partial C_B$ in normal position and recall that by Proposition 4.3.4, $\pi_{\mathcal{S}(A)}(T)$ is given by the collection of components of $C_A \cap S$ that split C_A. With this set-up, we show that

$$\min\{d_A(\partial C_B, S), d_B(\partial C_A, S)\} \leq 12$$

where for any sphere system R in M, $\pi_A(R)$ denotes $\pi_A(T_R)$.

Suppose, toward a contraction, that both $d_B(\partial C_A, S)$ and $d_A(\partial C_B, S)$ are greater than 12 and consider the forest G on S that is dual to the circles of intersection $\partial C_A \cap S$ and $\partial C_B \cap S$. We label the edges of G dual to circles of $\partial C_A \cap S$ with "a" and those dual to $\partial C_B \cap S$ with "b". Label the vertices of G that represent components $S \setminus (\partial C_A \cup \partial C_B)$ contained in $C_A \cap C_B$ with "AB", those in C_A but not C_B with "A", and those in C_B but not C_A with "B".

Call a subtree of G *terminal* if it has a unique vertex that separates it from its complement in G. We say a subtree is an a- tree (or b- tree) if all of its edges are labeled a (or b).

Claim 1. *No AB-vertex which is the boundary of both an a-edge and a b-edge is a vertex for either a terminal a-tree or a terminal b-tree.*

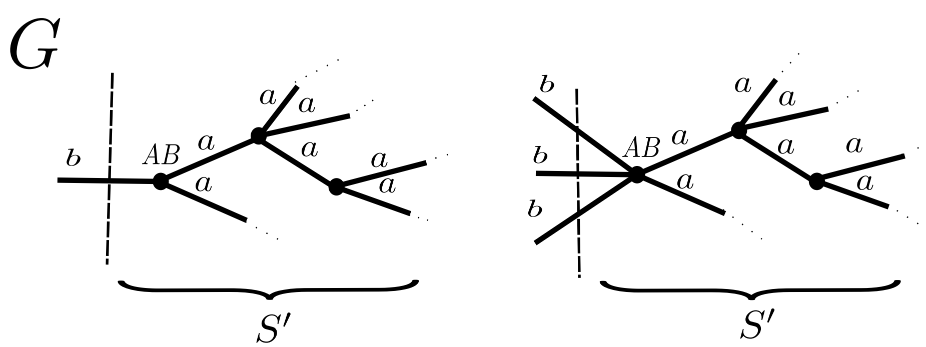

Figure 4.5: Two cases for S'

Proof of claim 1. We prove the claim for terminal a-trees. The proof for b-trees is obtained by switching the symbols a and b.

Suppose that there is an AB-vertex v of G which bounds both an a-edge and a b-edge and is the vertex for a terminal a-tree. Observe that the component S' of $S \cap C_B$ that corresponds to the union of b-edges at v (as in Figure 4.5) splits C_B and so it can be used for the projection $\pi_B(S)$ (see the remark following Proposition 4.3.2). To see that S' splits C_B, recall that if this were not the case then $C_B \setminus S' = C_1 \cup C_2$, where C_1 is relatively simply connected in M. As S' contains a disk of intersection with either $C_B \cap C_A$ or $C_B \setminus C_A$ coming from a valence one vertex of the terminal a-tree, this disk cobounds a region R contained in C_1 with a disk of ∂C_A. This shows that R is relatively simply connected with sphere boundary and basic combinatorial topology implies that R is simply connected in M. This implies that R is a 3-ball and so it can be used to reduce the number of intersections of S and ∂C_A, contradicting normal position. Hence, S' splits C_B.

Now there are two cases (see Figure 4.5). Suppose first that v is an endpoint for at least two b-edges. This implies that S' has at least two boundary components; each of which is contained in ∂C_B. Since these edges share the endpoint v, these boundary components co-bound the same component of $S' \cap C_A$. Let d_1, d_2 be two boundary components of S' which are not separated by another such boundary component of S' in ∂C_B. Let α be an arc between d_1 and d_2 in ∂C_B which intersects no other boundary component of S' and let β be an arc in S' joining d_1 and d_2 with $\partial\beta = \partial\alpha$ that does not intersect ∂C_A.

Since S and ∂C_B are in normal position, β is not homotopic relative endpoints into ∂C_B and so $\gamma = \alpha * \beta$ is an essential loop in $C_A \cap C_B$ which is disjoint from ∂C_A and can be homotoped not to intersect S'. Hence, if $[\gamma]$ denotes the conjugacy class of the smallest free factor containing $\langle \gamma \rangle$ then

$$\begin{aligned} d_B(\partial C_A, S) &\leq \operatorname{diam}_{\mathcal{F}(B)}(S) + d_B(\partial C_A, S') \\ &\leq \operatorname{diam}_{\mathcal{F}(B)}(S) + d_B(\partial C_A, [\gamma]) + d_B([\gamma], S') \\ &\leq 4 + 4 + 4 = 12, \end{aligned}$$

a contradiction.

If v is the endpoint of only one b-edge, this argument does not work. In this case, S' is a disk and we argue as follows: first, any disk component of $S' \cap C_A$ splits C_A and is disjoint from ∂C_B providing the bound $d_A(\partial C_B, S) \leq 4$, a contradiction. So assume that each components of $S' \cap C_A$ has at least two boundary components on ∂C_A, except possibly the unique component with a boundary component on ∂C_B. Among all components of $S' \cap C_A$ choose the component S'' which has two boundary components d_1, d_2 on ∂C_A which are least separated by other components of $S' \cap \partial C_A$. Let α be an arc in ∂C_A between d_1 and d_2 that intersects only the circles of $S' \cap \partial C_A$ that separate d_1 from d_2 in ∂C_A. Note that by our choice of S'' each circle of $S' \cap \partial C_A$ that is crossed by α bounds a distinct component of $S' \cap C_A$. Let β be an arc in S'' joining these boundary components with $\partial \beta = \partial \alpha$. As before, $\gamma = \alpha * \beta$ is an essential loop in $C_A \cap C_B$ and we obtain a similar contradiction as above if γ intersects S' at most once; so suppose that this is not the case. Since intersections between S' and γ must occur along α we conclude that there is a component C of $S' \cap C_A$ which does not have boundary on ∂C_B and intersects γ exactly once. This implies that C is nonseparating in C_A. Hence, C splits C_A and is disjoint from ∂C_B. This provides the upper bound on distance

$$\begin{aligned} d_A(S, \partial C_B) &\leq \mathrm{diam}_A(S) + d_A(C, \partial C_B) \\ &\leq 4 + 4 = 8, \end{aligned}$$

a contradiction. □

Claim 2. *There exists an AB-vertex of G that has both an a-edge and a b-edge.*

Proof of claim 2. Assume to the contrary; that is assume that no component of $S \cap C_A \cap C_B$ has its boundary on both ∂C_A and ∂C_B. Let $s \in S$ be a sphere of S that splits C_B, this sphere exists by assumption. If s intersection ∂C_B then it does not meet ∂C_A and so $d_B(s, \partial C_A) \leq 4$, a contraction. Hence, $s \subset C_B$. If s also splits C_A, i.e. if some component of $s \cap C_A$ splits C_A, then we conclude $d_A(s, \partial C_B) \leq 4$; so it must be the case that every component of $s \cap C_A$ is superfluous, that is, it separates C_A and bounds to one side a component that is relatively simply connected. Note this implies in particular that no component of $s \cap C_A$ is a disk. We show that this also leads to a contraction. The argument is similar to that of the second part of Claim 1.

Among all components of $s \cap C_A$ choose the one with boundary components on ∂C_A that are least separated by circles of $s \cap \partial C_A$, call this component

s''. As in the poof of Claim 2, let α be an arc in ∂C_A between these boundary components of s'' that intersects only the circles of $s \cap \partial C_A$ that separate these boundary components. Note that by our choice of s'' each circle of $s \cap \partial C_A$ that is crossed by α bounds a distinct component of $s \cap C_A$. Let β be an arc in s'' joining these boundary components, with the same endpoints as α. By normal position, $\gamma = \alpha * \beta$ is an essential loop in $C_A \cap C_B$ that can be homotoped to miss s'' and we obtain a similar contradiction as above if γ does not intersect any other components of $s \cap C_A$; so assume that this is not the case. Since additional intersections with s must occur along α we conclude that there is a component C of $s \cap C_A$ that intersects γ exactly once. This implies that C is nonseparating in C_A and contradicts the statement that all components of $s \cap C_A$ are superfluous. $\square$

To conclude the proof of the proposition, first locate an AB-vertex v that has both an a-edge and a b-edge. The existence of v is guaranteed by Claim 2. By the Claim 1, the b-edges at v are not contained in a terminal b-tree. Hence, there is an a-edge adjacent to this b-tree in the complement of the initial vertex; the adjacency necessarily occurring at an AB-vertex. At this new vertex, Claim 1 now implies that the a-edges are not contained in a terminal a-tree. Hence we may repeat the process and find a new AB-vertex to which we may again apply Claim 1. Since G is a forest, these AB vertices are distinct and we conclude that G is infinite. This contradicts that fact that edges of G correspond to components of the intersection of transverse sphere systems S and $S_A \cup S_B$ in M_k and must, therefore, be finite.

□

4.5 Order on overlapping factors

For trees $T, T' \in \mathcal{K}^0$ and $K \geq 2M+1$, define $\Omega(K, T, T')$ to be the set of (conjugacy classes of) free factors with the property that $A \in \Omega(K, T, T')$ if and only if $d_A(T, T') \geq K$. This definition is analogous to [14], where the authors put a partial ordering on the set of subsurfaces with large projection distance between two fixed markings. See [40] and [4] for details on this partial ordering on subsurfaces. Defining a partial ordering on $\Omega(K, T, T')$, however, requires a more general notion of projection than is available in our situation. We resolve this issue by defining a relation that is not necessarily transitive. Lemma 4.8.1 will then compensate for this lack of transitivity.

For $A, B \in \Omega(K, T, T')$ that *overlap* we define $A \prec B$ to mean that $d_A(T, B) \geq M+1$, where M is as in Proposition 4.4.1. As noted above, this does not define a partial order. In particular, if $A \prec B$ and $B \prec C$ there is no reason to expect that A and C will meet as free factors. We do, however, have the following version of Proposition 3.6 from [14].

Proposition 4.5.1. *Let $K \geq 2M+1$ and choose $A, B \in \Omega(K, T, T')$ that overlap. Then A and B are ordered and the following are equivalent*

1. $A \prec B$
2. $d_A(T, B) \geq M+1$
3. $d_B(T, A) \leq M$
4. $d_B(T', A) \geq M+1$
5. $d_A(T', B) \leq M$

Proof. (1) implies (2) is by definition, (2) implies (3) is Proposition 4.4.1, (3) implies (4) is the observation that

$$d_B(T', A) \geq d_B(T, T') - d_B(T, A) \geq 2M + 1 - M = M + 1,$$

and the proofs of the remaining implications are similar. To show that $A, B \in \Omega(K, T, T')$ which overlap are ordered, note that by the equivalence of the above conditions if $A \not\prec B$ then $d_A(T, B) \leq M$ and if $B \not\prec A$, switching the roles of A and B, $d_A(T', B) \leq M$ so that

$$d_A(T, T') \leq d_A(T, B) + d_A(B, T') \leq 2M \leq K,$$

a contradiction. □

4.6 Normal forms in $A(\Gamma)$

Let Γ be a simplicial graph with vertex set $V(\Gamma) = \{s_1, \ldots, s_n\}$ and edge set $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$. The right-angled Artin group, $A(\Gamma)$, associated to Γ is the group presented by

$$\langle s_i \in V(\Gamma) : [s_i, s_j] = 1 \iff (s_i, s_j) \in E(\Gamma) \rangle.$$

We refer to $s_1, \ldots, s_n$ as the *standard generators* of $A(\Gamma)$.

4.6.1 The Clay-Leininger-Mangahas partial order

In this section, we briefly recall a normal form for elements of a right-angled Artin group. For details see Section 4 of [14] and the references provided there. Fix a word $w = x_1^{e_1} \dots x_k^{e_k}$ in the vertex generators of $A(\Gamma)$, with $x_i \in \{s_1, \dots, s_n\}$ for each $i = 1, \dots, k$. Each $x_i^{e_i}$ together with its index, which serves to distinguish between duplicate occurrences of the same generator, is a *syllable* of the word w. Let $\mathrm{syl}(w)$ denote the set of syllables for the word w. We consider the following 3 moves that can be applied to w without altering the element in $A(\Gamma)$ it represents:

1. If $e_i = 0$, then remove the syllable $x_i^{e_i}$.
2. If $x_i = x_{i+1}$ as vertex generators, then replace $x_i^{e_i} x_{i+1}^{e_{i+1}}$ with $x_i^{e_i + e_{i+1}}$.
3. If the vertex generators x_i and x_{i+1} commute, then replace $x_i^{e_i} x_{i+1}^{e_{i+1}}$ with $x_{i+1}^{e_{i+1}} x_i^{e_i}$.

For $g \in A(\Gamma)$, set $\mathrm{Min}(g)$ equal to the set of words in the standard generators of $A(\Gamma)$ that have the fewest syllables among words representing g. We refers to words in $\mathrm{Min}(g)$ as the *normal form* representatives of g. Hermiller and Meier showed in [28] that any word representing g can be brought to any word in $\mathrm{Min}(g)$ by applications of the three moves above. Since these moves do not increase the word (or syllable) length, we see that words in $\mathrm{Min}(g)$ are also minimal length with respect to the standard generators and that any two words in $\mathrm{Min}(g)$ differ by repeated application of move (3) only. It is verified in [14] that for any $g \in A(\Gamma)$ and $w, w' \in \mathrm{Min}(g)$ there is a natural bijection between $\mathrm{syl}(w)$ and $\mathrm{syl}(w')$. Because of this, for $g \in A(\Gamma)$ we can define $\mathrm{syl}(g) = \mathrm{syl}(w)$ for $w \in \mathrm{Min}(g)$. For each $g \in A(\Gamma)$, this permits a strict partial order $\prec$ on the set $\mathrm{syl}(g)$ by setting $x_i^{e_i} \prec x_j^{e_j}$ if and only if for every $w \in \mathrm{Min}(g)$ the syllable $x_i^{e_i}$ precedes $x_j^{e_j}$ in the spelling of w.

4.6.2 Order on meeting syllables

By analogy with the weaker notion of order on free factors, for $g \in A(\Gamma)$ let $\overset{m}{\prec}$ be the relation on $\mathrm{syl}(g)$ defined as follows: $x_i^{e_i} \overset{m}{\prec} x_j^{e_j}$ if and only if $x_i^{e_i} \prec x_j^{e_j}$ and there is a normal form $w \in \mathrm{Min}(g)$ where $x_i^{e_i}$ and $x_j^{e_j}$ are adjacent. The following observation will be important in proving the lower bound on distance in our main theorem.

Lemma 4.6.1. *The strict partial ordering $\prec$ on $\mathrm{syl}(g)$ is the transitive closure of the relation $\overset{m}{\prec}$.*

Proof. From the definition of $\overset{m}{\prec}$ it suffices to show that if $x_i^{e_i} \prec x_j^{e_j}$ in $\mathrm{syl}(w)$ then $x_i^{e_i}$ and $x_j^{e_j}$ cobound a chain of syllables where adjacent terms are ordered by $\overset{m}{\prec}$. To this end, let

$$x_i^{e_i} = a_1 \prec a_2 \prec \ldots \prec a_n = x_j^{e_j}$$

be a chain of maximal length joining $x_i^{e_i}$ and $x_j^{e_j}$ in $\mathrm{syl}(g)$. We show that each pair of consecutive terms in the chain is ordered by $\overset{m}{\prec}$. Take $1 \leq i \leq n$ and consider the $w \in \mathrm{Min}(g)$ for which a_i and a_{i+1} are separated by the least number of syllables in w. If a_i and a_{i+1} are adjacent in w we are done, otherwise write

$$w = w_1 \cdot a_i \cdot s \cdot w_2 \cdot a_{i+1} \cdot w_3$$

where w_1, w_2, w_3 are possibly empty subwords of w and s is a syllable of w. By our choice of w, $a_i \prec s$, for otherwise we could commute s past a_i resulting in a normal form for g with fewer syllables separating a_i and a_{i+1}. Then either $s \prec a_{i+1}$, which contradicts the assumption that the chain is maximal, or s can be commuted past a_{i+1} resulting in a normal form $w' \in \mathrm{Min}(g)$ with

$$w' = w_1 \cdot a_i \cdot w_2' \cdot a_{i+1} \cdot s \cdot w_3'$$

where w_2' is a subword of w_2. This contradicts our choice of w. Hence, a_i and a_j must occur consecutively in w and so $a_i \overset{m}{\prec} a_{i+1}$ as required. □

4.7 Large projection distance

Fix an admissible system $\mathcal{S} = (\mathcal{A}, \{f_i\})$ for F with coincidence graph Γ. This determines a homomorphism $\phi = \phi_{\mathcal{S}} : A(\Gamma) \to \mathrm{Out}(F)$ by mapping the vertex generator s_i to the outer automorphism f_i.

For $g \in A(\Gamma)$ with $w = x_1^{e_1} \dots x_k^{e_k} \in \mathrm{Min}(g)$, let $J : \{1, \dots, k\} \to \{1, \dots, n\}$ be defined so that $x_i = s_{J(i)}$, as generators of $A(\Gamma)$. Hence, $\phi(x_i) = f_{J(i)}$ is supported on $A_{J(i)}$. Write

$$A^w(x_i^{e_i}) = \phi(x_1^{e_1} \dots x_{i-1}^{e_{i-1}})(A_{J(i)})$$

for $i = 2, \dots, k$ and $A^w(x_1^{e_1}) = A_{J(1)}$. This defines a map

$$A^w : \mathrm{syl}(w) \to \mathcal{FF}^0.$$

It is verified in [14] that this map is well-defined for $g \in A(\Gamma)$, independent of the choice of normal form. Then, set $A^g = A^w$ for $w \in \mathrm{Min}(g)$ and set $\textsc{fact}(g)$

equal to the image of the map $A^g : \mathrm{syl}(g) \to \mathcal{FF}^0$. We refer to the free factors in $\textsc{fact}(g)$ as the *active free factors* for $g \in A(\Gamma)$. For notional convenience, set $B_i = A_{J(i)}$ and $g_i = \phi(x_i^{e_i}) = f_{J(i)}^{e_i}$. Note that this notation is for a fixed $w \in \mathrm{Min}(g)$.

Having developed the necessary tools in the free group setting, the proof of the first part of the following theorem is a verification that the arguments of [14] extend to this situation, even with a weaker form of Proposition 4.4.1 . We repeat their argument here for completeness. Let M be the constant determined in Proposition 4.4.1 and let $L = 4$ be the Lipschitz constant for the projection $\pi_A : \mathcal{K} \to \mathcal{F}(A)$, $A \in \mathcal{F}$. In Section **??**, we determine the corresponding Lipschitz constant for $\mathcal{X}^{\geq\epsilon} \to \mathcal{F}(A)$.

Theorem 4.7.1. *Given an admissible collection $\mathcal{A}$ of free factors for F with coincidence graph Γ and $T \in \mathcal{K}^0$, there is a $K \geq 5M + 3L$ so that if outer automorphisms $\{f_i\}$ are chosen to make $(\mathcal{A}, \{f_i\})$ an admissible system with $\ell_{A_i}(f_i) \geq 2K$ then the induced homomorphism $\phi : A(\Gamma) \to \mathrm{Out}(F)$ satisfies the following: For any $g \in A(\Gamma)$ with normal form $w = x_1^{e_1} \dots x_k^{e_k} \in \mathrm{Min}(g)$,*

1. *$d_{A^g(x_i^{e_i})}(T, \phi(g)T) \geq K|e_i|$ for $1 \leq i \leq k$. In particular, $\textsc{fact}(g) \subset \Omega(K, T, \phi(g)T)$.*

2. *If $x_i^{e_i} \overset{m}{\prec} x_j^{e_j}$, then $A^g(x_i^{e_i})$ and $A^g(x_j^{e_j})$ overlap and*

$$A^g(x_i^{e_i}) \prec A^g(x_j^{e_j}).$$

Proof. Set $K = 5M + 3L + 2 \cdot \max\{d_{A_i}(T, A_j)\}$ and observe that this choice of K has the property that if A_i and A_j overlap then $d_{A_i}(T, A_j) \leq K/2 - M$. The proof of (1) is by induction on the syllable length of $w \in \mathrm{Min}(g)$. If w has only one syllable then

$$d_{A_{J(1)}}(T, f_{J(1)}^{e_1} T) \geq \ell_{A_{J(1)}}(f_{J(1)}^{e_1}) \geq 2K|e_1|.$$

Now suppose that (1) has been proven for all elements in $A(\Gamma)$ that have representative with less than or equal to $k-1$ syllables. Take $g \in A(\Gamma)$ with $w = x_1^{e_1} \dots x_k^{e_k}$ a k-syllable normal form representative for g. Using the notation at the beginning of this section, write $\phi(w)$ as $g_1 \dots g_k$ so that for $1 \leq i \leq k$ we must show

$$d_{g_1 \dots g_{i-1} B_i}(T, g_1 \dots g_k T) \geq K|e_i|.$$

With $x_i^{e_i} \in \mathrm{syl}(g)$ fixed and $g_i = \phi(x_i^{e_i})$, we write $\phi(g)$ as abg_ic by choosing a normal form $w \in \mathrm{Min}(g)$ so that

1. $c = g_{i+1} \dots g_k$ and g_i and g_{i+1} do not commute,
2. $a = g_1 \dots g_l$ with l the largest index among $w \in \mathrm{Min}(g)$ so that g_l and g_i do not commute, and
3. $b = g_{l+1} \dots g_{i-1}$, all of which commute with g_i.

Note that we allow a, b or c to be empty.

Using this notation, we show that $d_{abB_i}(T, abg_icT) \geq K|e_i|$. By Lemma 3.2.2 and the triangle inequality,

$$\begin{aligned} d_{abB_i}(T, abg_icT) &= d_{B_i}(b^{-1}a^{-1}T, g_icT) && (4.1)\\ &\geq d_{B_i}(T, g_iT) - d_{B_i}(b^{-1}a^{-1}T, T) - d_{B_i}(g_icT, g_iT). && (4.2) \end{aligned}$$

Since b is written in terms of generators that restrict to the identity outer automorphism on B_i and g_i restricts to an isometry of the free factor complex of B_i, Lemma 3.2.2 implies

$$d_{B_i}(b^{-1}a^{-1}T, T) = d_{B_i}(a^{-1}T, T)$$

and

$$d_{B_i}(g_icT, g_iT) = d_{B_i}(cT, T).$$

This, along with our hypothesis on translation length, allows us to write

$$d_{abB_i}(T, abg_icT) \geq 2K|e_i| - d_{B_i}(a^{-1}T, T) - d_{B_i}(cT, T). \quad (4.3)$$

We use the induction hypotheses to show that both terms subtracted in (3) are $\leq K/2$. This will complete the proof of (1). First, observe that each of $a^{-1} = g_l^{-1} \dots g_1^{-1}$ and $c = g_{i+1} \dots g_k$ is either trivial or is the image of a normal form subword of w with strictly fewer than k syllables and begins with a syllable not commuting with $x_i^{e_i}$. This is all that is needed for the remainder of the proof. We show the inequality $d_{B_i}(a^{-1}T, T) \leq K/2$, the other appears in [14] where the proof follows through without change.

By the induction hypothesis applied to a^{-1},

$$d_{B_l}(T, a^{-1}T) = d_{B_l}(T, g_l^{-1} \dots g_1^{-1}T) \geq K|e_l|,$$

and so since $d_{B_l}(T, B_i) \leq K/2 - M$ by our choice of K, we have $d_{B_l}(B_i, a^{-1}T) \geq K - (K/2 - M) \geq M + 1$. Since B_i and B_l overlap, Proposition 4.4.1 implies that $d_{B_i}(B_l, a^{-1}T) \leq M$, so by another application of $d_{B_i}(T, B_l) \leq K/2 - M$,

$$d_{B_i}(a^{-1}T, T) \leq M + (K/2 - M) \leq K/2,$$

as required. This completes the proof of the first part of the theorem.

The second part of the theorem is also proven by induction on syllable length. If $g \in A(\Gamma)$ has syllable length equal to 1, then there is nothing to prove. Suppose that the ordering statement holds for all g with a minimal syllable representative with less then or equal to $k - 1$ syllables. As in the first part of the proof, take $g \in A(\Gamma)$ with $w = x_1^{e_1} \dots x_k^{e_k}$ a k-syllable normal form representative for g. Write $\phi(w)$ as $g_1 \dots g_k$ and suppose that $x_i^{e_i} \overset{m}{\prec} x_j^{e_j}$ as syllables of g. If $j \leq k - 1$ then we may apply the induction hypothesis to a prefix of w and conclude $A^w(x_i^{e_i}) \prec A^w(x_j^{e_j})$. More precisely, let w' be the word formed by the first $k - 1$ syllables of w; this is a normal form word for some $g' \in A(\Gamma)$. By the induction hypothesis $A^{w'}(x_i^{e_i})$ and $A^{w'}(x_j^{e_j})$ overlap and $A^{w'}(x_i^{e_i}) \prec A^{w'}(x_j^{e_j})$. This suffices since for $l \leq k - 1$ we have $A^w(x_l^{e_l}) = A^{w'}(x_l^{e_l})$, using the obvious identification of the syllables of w' with those of w.

Otherwise, $j = k$ and by definition of $\overset{m}{\prec}$ we may choose $w \in \mathrm{Min}(g)$ so that $w = a x_i^{e_i} x_k^{e_k}$ and so $\phi(w) = \phi(a) g_i g_k$. Since $x_i^{e_i} \overset{m}{\prec} x_k^{e_k}$, B_i and B_k overlap and so $\phi(a)g_iB_i = \phi(a)B_i$ and $\phi(a)g_iB_k$ also overlap. We have

$$\begin{aligned} d_{A^g(x_k^{e_k})}(A^g(x_i^{e_i}), \phi(g)T) &= d_{\phi(a)g_iB_k}(\phi(a)B_i, \phi(a)g_ig_kT) \\ &= d_{B_k}(B_i, g_kT) \\ &\geq d_{B_k}(T, g_kT) - d_{B_k}(B_i, T) \\ &\geq d_{A_{J(k)}}(T, f_{J(k)}^{e_k}T) - d_{A_{J(k)}}(A_{J(i)}, T) \\ &\geq 2K - K \\ &\geq M + 1, \end{aligned}$$

and so since $A^g(x_i^{e_i}), A^g(x_k^{e_k}) \in \Omega(K, T, \phi(g)T)$, by Proposition 4.5.1

$$A^w(x_i^{e_i}) \prec A^w(x_k^{e_k}).$$

□

4.8 The lower bound on distance for admissible systems

Let $\mathcal{A} = (\{A_i\}, \{f_i\})$ be an admissible system satisfying the hypotheses of Theorem 4.7.1 for $T \in \mathcal{K}^0$ and let $K \geq 5M + 3L$ be as in Theorem 4.7.1. For $g \in A(\Gamma)$ and $w \in \mathrm{Min}(g)$ write in normal form

$$w = x_1^{e_1} \ldots x_k^{e_k}.$$

We make use of the notation introduced at the beginning of the previous section.

Set $T' = \phi(g)T$ and choose a geodesic $T = T_0, T_1, \ldots, T_N = T'$ in the 1-skeleton of $\mathcal{K}_n$. Similar to [40], we define the subinterval $I_A = [a_A, b_A] \subset [0, N]$

associated to the free factor $A \in \Omega(K, T, T')$ as follows: Set

$$a_A = \max\{k \in \{0, \ldots, N\} : d_A(T, T_k) \le 2M + L\}$$

and

$$b_A = \min\{k \in \{a_A, \ldots, N\} : d_A(T_k, T') \le 2M + L\}.$$

Since $A \in \Omega(K, T, T')$, $d_A(T, T') \ge K \ge 5M + 3L$ and so both a_A and b_A are well-defined and not equal. Hence, the interval I_A is nonempty and for all $k \in I_A$,

$$d_A(T_k, T) \ge 2M + 1 \quad \text{and} \quad d_A(T_k, T') \ge 2M + 1.$$

This uses that fact that the projection from $\mathcal{K}^0$ to $\mathcal{F}(A)$ is L-Lipschitz. The next lemma shows that if syllables are ordered, then distance in their associated free factors cannot be made simultaneously.

Lemma 4.8.1. *With notation fixed as above, if* $x_i^{e_i}, x_j^{e_j} \in \mathrm{syl}(w)$ *and* $x_i^{e_i} \prec x_j^{e_j}$ *then*

$$I_{A^w(x_i^{e_i})} < I_{A^w(x_j^{e_j})}.$$

That is, the intervals are disjoint and correctly ordered in $[0, N]$.

Proof. We first prove the proposition when $x_i^{e_i} \overset{m}{\prec} x_j^{e_j}$. Recall that since $x_i^{e_i} \overset{m}{\prec} x_j^{e_j}$, Theorem 4.7.1 implies that the free factors $A^w(x_i^{e_i})$ and $A^w(x_j^{e_j})$ overlap and are ordered, $A^w(x_i^{e_i}) \prec A^w(x_j^{e_j})$. If $k \in I_{A^w(x_i^{e_i})}$, then $d_{A^w(x_i^{e_i})}(T_k, T') \ge 2M + 1$ and since $A^w(x_i^{e_i}) \prec A^w(x_j^{e_j})$ we have $d_{A^w(x_i^{e_i})}(A^w(x_j^{e_j}), T') \le M$. The triangle inequality then implies that

$$d_{A^w(x_i^{e_i})}(T_k, A^w(x_j^{e_j})) \ge M + 1.$$

As the free factors $A^w(x_i^{e_i})$ and $A^w(x_j^{e_j})$ overlap, by Proposition 4.4.1 we have

$$d_{A^w(x_j^{e_j})}(T_k, A^w(x_i^{e_i})) \leq M.$$

Combining this with the inequality $d_{A^w(x_j^{e_j})}(A^w(x_i^{e_i}), T) \leq M$, again coming from the ordering, provides

$$d_{A^w(x_j^{e_j})}(T, T_k) \leq 2M.$$

Since this is true for each $k \in I_{A^w(x_i^{e_i})}$ it follows from the definition of $I_{A^w(x_j^{e_j})}$ that $I_{A^w(x_i^{e_i})} \cap I_{A^w(x_j^{e_j})} = \emptyset$. So if there were an index $k \in I_{A^w(x_i^{e_i})}$ with $k > a_{A^w(x_j^{e_j})}$ then by disjointness of the intervals $a_{A^w(x_i^{e_i})} > a_{A^w(x_j^{e_j})}$. This contradiction the choice of $a_{A^w(x_j^{e_j})}$ as the largest index k with $d_{A^w(x_j^{e_j})}(T, T_k) \leq 2M + 1$ and shows that the intervals of interest are disjoint and ordered as $I_{A^w(x_i^{e_i})} < I_{A^w(x_j^{e_j})}$.

Now, if more generally we have that $x_i^{e_i} \prec x_j^{e_j}$, then by Lemma 4.6.1, $x_i^{e_i}$ and $x_j^{e_j}$ can be joined by a chain of syllables

$$x_i^{e_i} = a_0 \overset{m}{\prec} a_1 \overset{m}{\prec} \ldots \overset{m}{\prec} a_l = x_j^{e_j}.$$

Hence, we conclude

$$I_{A^w(x_i^{e_i})} < I_{A^w(a_1)} < \ldots < I_{A^w(x_j^{e_j})},$$

as required. □

Let $s = s(\Gamma)$ be the size of the largest complete subgraph of Γ. This is also the maximal rank of a free abelian subgroup of $A(\Gamma)$. Note by the definition of an admissible system, s is bounded above by a constant depending only on the rank of F. To simplify notations, associated to the free factor $A^g(x_i^{e_i})$ we set $a_i = a_{A^g(x_i^{e_i})}$ and $b_i = b_{A^g(x_i^{e_i})}$.

Lemma 4.8.2 (Lower bound on distance)**.** *With notation fixed as above, K as in Theorem 4.7.1 and $w \in \mathrm{Min}(g)$ in normal form*

$$\sum_{1 \le i \le k} d_{A^g(x_i^{e_i})}(T, \phi(g)T) \le 5sL \cdot d_{\mathcal{K}}(T, \phi(g)T).$$

Proof. Since $A^g(x_i^{e_i}) \in \Omega(K, T, \phi(g)T)$ for all $x_i^{e_i} \in \mathrm{syl}(g)$ by Theorem 4.7.1, we have the collection of nonempty subintervals $\{I_{A^g(x_i^{e_i})} : 1 \le i \le k\}$ of $\{0, 1, \ldots, N\}$. If, for $i \le j$, it is the case that $x_i^{e_i} \prec x_j^{e_j}$ then by Lemma 4.8.1, $I_{A^g(x_i^{e_i})}$ and $I_{A^g(x_j^{e_j})}$ are ordered and, hence, disjoint. Further, any collection of syllables pairwise unordered by $\prec$ has size bounded above by s. This is clear since such a collection of syllables can be commuted to be consecutive in w using move (3) and so correspond to distinct pairwise commuting standard generators. We conclude that for any integer $j \in [0, N]$, j is contained in at most s of the intervals $I_{A^g(x_i^{e_i})}$. Hence,

$$\sum_{1 \le i \le k} |b_i - a_i| \le s \cdot d_{\mathcal{K}}(T, \phi(w)T).$$

Using the Lipschitz condition on the projections and the triangle inequality,

$$\begin{aligned} d_{A^g(x_i^{e_i})}(T, \phi(g)T) &\le d_{A^g(x_i^{e_i})}(T_{a_i}, T_{b_i}) + 4M + 2L \\ &\le L|b_i - a_i| + 4M + 2L. \end{aligned}$$

Since for each $A \in \Omega(K, T, \phi(g)T)$, $d_A(T, \phi(g)T) \geq K \geq 5M + 3L$ we have $|b_A - a_A| \geq \frac{M+L}{L}$. This implies that $d_A(T, \phi(g)T) \leq 5L \cdot |b_A - a_A|$ and so putting this with the inequality above

$$\sum_{1 \leq i \leq k} d_{A^{g(i)}}(T, \phi(g)T) \leq 5sL \cdot d_{\mathcal{K}}(T, \phi(g)T),$$

as required. □

4.9 The quasi-isometric embedding

We can now prove Theorem 4.9.1 using Theorem 4.7.1 and Theorem 4.8.2.

Theorem 4.9.1. *Given an admissible collection $\mathcal{A}$ of free factors for F_n with coincidence graph Γ there is a $C \geq 0$ so that if outer automorphism $\{f_i\}$ are chosen making $\mathcal{S} = (\mathcal{A}, \{f_i\})$ an admissible system with $\ell_{A_i}(f_i) \geq C$ then the induced homomorphism $\phi = \phi_{\mathcal{S}} : A(\Gamma) \to \mathrm{Out}(F_n)$ is a quasi-isometric embedding.*

Proof. Suppose $\mathcal{A}$ is an admissible collection of free factors and $T \in \mathcal{K}^0$. Take $C = 2K$, for K as in Theorem 4.7.1. We show that the orbit map $A(\Gamma) \to \mathcal{K}_n^1$

$$g \mapsto \phi(g)T$$

is a quasi-isometric embedding, where $A(\Gamma)$ is given the word metric in its standard generators. Since $\mathrm{Out}(F_n)$ is quasi-isometric to $\mathcal{K}_n^1$, this suffices to prove the theorem. First, recall that the orbit map is Lipschitz, as is any orbit map induced by an isometric action of a finitely generated group on a metric space. Specifically, $d_{\mathcal{K}}(T, \phi(g)T) \leq A \cdot |g|$, where $A = \max\{d_{\mathcal{K}}(T, \phi(s_i)T : 1 \leq i \leq n\}$ and $s_1, \ldots, s_n$ are the standard generators.

Let $g \in A(\Gamma)$. By Theorem 4.7.1, we know that if $w = x_1^{e_1} \ldots x_k^{e_k} \in \mathrm{Min}(g)$, then

$$d_{A^g(x_i^{e_i})}(T, \phi(g)T) \geq K|e_i|$$

for $1 \leq i \leq k$. Hence, by Lemma 4.8.2

$$\begin{aligned} |g| &= \sum_{1 \leq i \leq k} |e_i| \\ &\leq \frac{1}{K} \sum_{1 \leq i \leq k} d_{A^g(x_i^{e_i})}(T, \phi(g)T) \\ &\leq \frac{5sL}{K} \cdot d_{\mathcal{K}}(T, \phi(g)T). \end{aligned}$$

We conclude that for any $g, h \in A(\Gamma)$

$$\frac{1}{A} d_{\mathcal{K}}(\phi(g)T, \phi(h)T) = \frac{1}{A} d_{\mathcal{K}}(T, \phi(g^{-1}h)T) \leq |g^{-1}h| = d_{A(\Gamma)}(g, h)$$

and

$$d_{A(\Gamma)}(g, h)) \leq \frac{5sL}{K} \cdot d_{\mathcal{K}}(T, \phi(g^{-1}h))T) = \frac{5sL}{K} \cdot d_{\mathcal{K}}(\phi(g)T, \phi(h))T),$$

as required. $\square$

Chapter 5

Subfactor Projections

In this chapter, we extend the Bestvina-Feighn subfactor projections and prove both Theorem 1.2.3 and Theorem 1.2.5. This chapter is independent of Chapter 4 and we use a different set of terminology here. The reason for this change is that the results of this chapter were proven in order to combine the Bestvina-Feighn projections, as introduced in [9], with the projections that were defined in Chapter 4.

5.1 The Bestvina-Feighn subfactor projections

In their recent work on the geometry of $\mathrm{Out}(F_n)$, Mladen Bestvina and Mark Feighn define the projection of a free factor $B < F_n$ to the free splitting complex (or free factor complex) of the free factor A, when the two factors are in "general position." They show that these subfactor projections have properties that are analogous to subsurface projections used to study mapping class groups, and they use their results to show that $\mathrm{Out}(F_n)$ acts on a product of hyperbolic spaces in such a way that exponentially growing automophisms have positive translation length.

Because the authors were primarily interested in projections to the splitting complex of a free factor, relatively strong conditions were necessary in order to guarantee that the projections have uniformly bounded diameter, i.e. that they are well-defined. They show that one may project B to the splitting complex of A if either A and B have distance at least 5 in the free factor complex of F_n or if they have the same color in a specific finite coloring of the factor complex. In this note, we show that if one considers projections to the *free factor complex of a free factor*, simpler and more natural conditions can be given. In particular, we show that for free factors $A, B < F_n$ with $\mathrm{rank}(A) \geq 2$ the projection $\pi_A(B) \subset \mathcal{F}(A)$ into the free factor complex of A is well-defined so long as (1) A is not contained in B, up to conjugation, and (2) A and B are not disjoint. This exactly mimics the case for subsurface projection. Here, free factors A and B are *disjoint* if they are distinct vertex stabilizers of a splitting of F_n, or equivalently, if they can be represented by disjoint subgraphs of a marked graph G. These are also the obvious necessary condition for the projection to be defined. As a consequence of this more inclusive projection, we are able to merge the Bestvina-Feighn projections with those considered in Chapter 4.

The first part of this chapter should be considered as a direct follow-up to the work of Bestvina and Feighn, as our arguments rely on the techniques developed in [9]. Our contribution toward defining subfactor projections is an extension of their results. In summary, we show:

Theorem 5.1.1. *There is a constant D depending only on $n = \mathrm{rank}(F_n)$ so that if A and B are free factors of F_n with $\mathrm{rank}(A) \geq 2$, then either*

1. $A \subset B$, up to conjugation,

2. A and B are disjoint, or

3. $\pi_A(B) \subset \mathcal{F}(A)$ is defined and has diameter $\leq D$.

Moreover, these projections are equivariant with respect to the action of $\mathrm{Out}(F_n)$ *on conjugacy classes of free factors and they satisfy the following: There is an $M \geq 0$ so that if free factors $A, B < F_n$ overlap and G is a marked F_n-graph, then*

$$\min\{d_A(B,G), d_B(A,G)\} \leq M.$$

Here, free factors *overlap* if one is not contained in the other, up to conjugation, and they are not disjoint. Hence, for overlapping free factors both subfactor projections are defined. For subsurface projections, the final property in Theorem 5.1.1 is known as Behrstock's inequality [3]. We also have the following strengthening of the Bounded Geodesic Image Theorem of [9]. For subsurface projections, this was first shown in [40].

Theorem 5.1.2. *For $n \geq 3$, there is $M \geq 0$ so that if A is a free factor of F_n with* $\mathrm{rank}(A) \geq 2$ *and γ is a geodesic of $\mathcal{F}_n$ with each vertex of γ meeting A (i.e. having well-defined projection to $\mathcal{F}(A)$) then* $\mathrm{diam}(\pi_A(\gamma)) \leq M$.

Finally, as an application of subfactor projections we give a construction of fully irreducible automorphism similar to Proposition 3.3 of [37], where pseudo-Anosov mapping classes are constructed. Here, free factors A and B *fill* F_n if no free factor C is disjoint from both A and B.

Theorem 5.1.3. *Let A and B be rank ≥ 2 free factors of F_n that fill and let $f, g \in \mathrm{Out}(F_n)$ satisfy the following:*

1. *$f(A) = A$ and $f|_A \in \mathrm{Out}(A)$ is fully irreducible, and*

2. *$g(B) = B$ and $g|_B \in \mathrm{Out}(B)$ is fully irreducible.*

Then there is an $N \geq 0$ so that the subgroup $\langle f^N, g^N \rangle \leq \mathrm{Out}(F_n)$ is free of rank 2 and any nontrivial automorphism in $\langle f^N, g^N \rangle$ that is not conjugate to a power of f or g is fully irreducible.

See Section 5.6 for a stronger statement. Theorem 5.1.3 adds a new construction of fully irreducible automorphisms to the methods found in [13], where they arise as compositions of Dehn twists, and in [30], where they are compositions of powers of other fully irreducible automorphisms.

5.2 Folding paths and the Bestvina-Feighn projections

First, we recall the projection of a splitting of F_n to the free factor complex of a subfactor. See Chapter 3 for details. For $G \in \mathcal{X}_n$ and a rank ≥ 2 free factor A we can consider the core subgraph of the cover of G corresponding to the conjugacy class of A. We denote this marked A-graph by $A|G$ and the associated immersion by $p : A|G \to G$. Pulling back the metric on G, we obtain $A|G \in \hat{\mathcal{X}}(A)$. Denote by $\pi_A(G) = \pi(A|G) \subset \mathcal{F}(A)$ the projection of $A|G$ to the free factor complex of A. Alternatively, if G corresponds to the action $F_n \curvearrowright T$ (i.e. T is the universal cover of G) with *minimal A-subtree* T^A, then $A \curvearrowright T^A$

represents a point in $\hat{\mathcal{X}}(A)$. The projection $\pi_A(T) = \pi_A(G) \subset \mathcal{F}(A)$ is the set of free factors of A that arise as vertex stabilizers of one-edge collapses of T^A. Note that this projection is defined whenever T is a splitting of F_n where A does not fix a vertex (i.e. where T^A is not trivial).

For a free factor A of F_n, we use the symbol d_A to denote distance in $\mathcal{F}(A)$, the free factor complex of A, and for F_n-trees T_1, T_2 we use the shorthand

$$d_A(T_1, T_2) := d_A(\pi_A(T_1), \pi_A(T_2)) = \mathrm{diam}_A(\pi_A(T_1) \cup \pi_A(T_2)),$$

when both projections are defined.

Let A and B be (conjugacy classes of) free factors of F_n with $\mathrm{rank}(A) \geq 2$. Suppose that A and B are not disjoint and that A is not contained in B, up to conjugation. In this case, we say that B *meets* A. Define *the projection of* B *to the free factor complex of* A to be the following subset of $\mathcal{F}(A)$:

$$\begin{aligned} \pi_A(B) &= \bigcup \{\pi_A(T) : T \text{ is a splitting of } F_n \text{ with vertex stabilizer } B\} \\ &= \bigcup \{\pi_A(G) : G \in \mathcal{X}_n \text{ and } B|G \subset G \text{ is embedded}\}. \end{aligned}$$

In other words, $\pi_A(B)$ is the set of vertex groups of splittings of A that are refined by the splitting $A \curvearrowright T^A$, where T is any free splitting with vertex stabilizer B. For convenience, if $A \subset B$ or A and B are disjoint we define $\pi_A(B)$ to be empty and say that B *misses* A. If A meets B and B meets A, then both projections are nonempty and we say that A and B *overlap.* Note that the conditions for $\pi_A(B)$ to be nonempty are precisely that the tree T^A is non-degenerate for any choice of T with B as a vertex stabilizer. The main result of this note is that $\mathrm{diam}(\pi_A(B))$ is uniformly bounded and, therefore, can be used as a coarse projection. This is shown in [9] in the case that either $d_{\mathcal{F}}(A, B) > 4$ or A and B have the same color in a specific finite coloring of $\mathcal{F}_n$. This, however, excludes cases of interest; for example when the free factors have nontrivial intersection, as in [47]. We note that by "uniformly bounded" we mean bounded by a constant depending only on n, the rank of F_n. Unlike the subsurface case, where the bound is 3, we do not explicitly compute this constant.

We recall some of the technical results from [9] that are needed here. Suppose that G_t is a folding path for $t \in [\alpha, \omega]$ as in Section 2.4 and that A is a free factor. Then for all $t \in [\alpha, \omega]$, we have the immersion $p_t : A|G_t \to G_t$ corresponding to the core of the A-cover of G_t and $A|G_t$ induces a path in $\hat{\mathcal{X}}(A)$. The results of [9] explain the behavior of the path $A|G_t$ and track the progress of $\pi_A(G_t) = \pi(A|G_t)$ in $\mathcal{F}(A)$. Note that $p_t : A|G_t \to G_t$ induces an illegal turn structure on $A|G_t$. Call a valence 2 vertex, i.e. a vertex appearing in the interior of a natural edge, an *interior illegal turn* if it has only one gate.

Lemma 5.2.1 (Lemma 3.1 of [9]). *For a folding path G_t, $t \in [\alpha, \omega]$ and a finitely generated subgroup $A < F_n$, the interval $[\alpha, \omega]$ can be divided into three subintervals $[\alpha, \beta)$, $[\beta, \gamma)$, and $[\gamma, \omega]$ so that the following properties characterize the restriction of $A|G_t$ to the middle interval $[\beta, \gamma)$: all vertices of $A|G_t$ have ≥ 2 gates, there are no interior illegal turns, and all natural edges of $A|G_t$ have length < 2. Moreover, the images of $\{A|G_t : t \in [\alpha, \beta)\}$ and $\{A|G_t : t \in [\gamma, \omega]\}$ in $\mathcal{S}(A)$ (and $\mathcal{F}(A)$) have uniformly bounded diameter.*

From this lemma, it is shown that the projection of the folding path G_t to the free splitting (or free factor) complex of A is an unparameterized quasi-geodesic with uniform constants. We will not need this fact in what follows. Note that for $a, b \in [\beta, \gamma)$, where $[\beta, \gamma)$ is the middle interval given in Lemma 5.2.1, the folding map $f_{ab} : G_a \to G_b$ induces a map $A|G_a \to A|G_b$ between the cores of the A-covers.

For the immersion $p : A|G \to G$, define $\Omega \subset G$ as the set of edge of G that are at least double covered by p and set $\tilde{\Omega} \subset A|G$ to be the subgraph $p^{-1}(\Omega) \subset A|G$. If $\tilde{\Omega} = \emptyset$, then $A|G \to G$ is an embedding and we say that A (or $A|G$) is *embedded* in G. If $\tilde{\Omega}$ is a forest (a disjoint union of trees), then we say that A (or $A|G$) is *nearly embedded.* The following lemma states that if a folding path makes significant progress in $\mathcal{F}(A)$ then A must be nearly embedded along the path.

Lemma 5.2.2. *Let G_t be a folding path for $t \in [\alpha, \omega]$ and let $[\beta, \gamma)$ be the middle interval determined by Lemma 5.2.1. Then after restricting G_t to $t \in [\beta, \gamma)$, the subgraph $\tilde{\Omega}_t \subset A|G_t$ is forward invariant and if for some t_0, $\tilde{\Omega}_{t_0}$ is not a forest (i.e. A is not nearly embedded in G_{t_0}), then $\pi_A(\{G_t : t \geq t_0\})$ has uniformly bounded diameter in $\mathcal{F}(A)$.*

Proof. That $\tilde{\Omega}_t$ is forward invariant on the middle interval is contained in Lemma 4.3 of [9]. The other statement is essentially Lemma 4.4 in [9]. There, it is shown that if $\tilde{\Omega}_{t_1} = A|G$ then $\pi_A(\{G_t : t \geq t_1\})$ is uniformly bounded. Since $\tilde{\Omega}_t$ is forward invariant it suffices to show that progress of $A|G_t$ in $\mathcal{F}(A)$

is bounded so long as $\tilde{\Omega}_t$ is a proper subgraph of $A|G_t$ that is not a forest. Suppose this is the case for $\tilde{\Omega}_{t_0} \subset A|G_{t_0}$ and let x_0 be an immersed loop in $A|G_{t_0}$ that is contained in $\tilde{\Omega}_{t_0}$. Denote by x_t the immersed representative of the image of x_0 through $A|G_{t_0} \to A|G_t$. Since $\tilde{\Omega}_t$ is a proper subgraph, x_t fails to cross some edge of $A|G_t$. This implies that the cyclic free factor represented by x_0 has distance ≤ 5 from $\pi_A(G_t) = \pi(A|G_t)$ in $\mathcal{F}_n$, so long as $\tilde{\Omega}_t$ is a proper subgraph. This completes the proof. □

5.3 Diameter bounds

The following lemmas determine when the projection of a factor B to the free factor complex of the factor A is well-defined. The first provides a criterion for when two free factors can be embedded in a common marked graph and the second shows that the failure of a joint embedding is enough to block progress of subfactor projections along a folding path. We recall that in [9], the authors show that if the finitely generated subgroup $A < F_n$ is nearly embedded in G, then A is a free factor of F_n. Similar arguments are used to prove the following:

Lemma 5.3.1. *Suppose that $p : A|G \to G$ is the canonical immersion and that $B|G \subset G$ is an embedding for free factors A and B of F_n. Let E^B be the collection of edges of $A|G$ that map to edges of $B|G$. If $\tilde{\Omega} \cup E^B \subset A|G$ is a forest, then there is a marked graph G' where A and B are disjointly embedded.*

Proof. Enlarge the forest $\tilde{\Omega} \cup E^B$ to a maximal tree T and let E be the set of edges not contained in T. These edges are in bijective correspondence with the edges of $p(E)$, since they are not in $\tilde{\Omega}$. For $x \in T$, define

$$G' = A|G \vee_{x=p(x)} (G \setminus p(E)).$$

As in [9], we have the morphism (edge isometry) $G' \to G$ induced by $p : A|G \to G$ and the inclusion of $G \setminus p(E)$ into G. Folding the edges of the tree T into $G \setminus p(E)$, we arrive at an intermediate graph G'' with an induced morphism $G'' \to G$. Because T is a tree, such folds do not change the homotopy type of the graph. Further, since no edges outside of T are identified when mapped to G, the morphism $G'' \to G$ is bijective. We conclude that the map $G' \to G$ is a homotopy equivalence and that G' contains disjoint, embedded copies of both $A|G' = A|G$ and $B|G' = B|G \subset G \setminus p(E)$.

□

We show that for any marked graphs G and G' where B is embedded, $d_A(G, G')$ is uniformly bounded. For this, fix a marked graph G_0 that is a rose and for which B is embedded. For any metric graph $G \in \mathcal{X}_n$ with $B|G$ embedded we can choose edge lengths for G_0 so that $G_0 \in \mathcal{X}_n$ and there is an optimal map $f : G_0 \to G$ with $\triangle(f) = G_0$ and $f(B|G_0) \subset B|G$. Then the folding path $\{G_t : t \in [0, T]\}$ induced by f with $G_T = G$ has the property that $B|G_t$ is embedded in G_t for all $t \in [0, T]$, and for all $s \leq t$, $f_{st} : G_s \to G_t$ maps $B|G_s$ into $B|G_t$. Hence, $B|G_t$ is forward invariant. It suffices to show that the image $\pi_A(G_t) \subset \mathcal{F}(A)$ of the folding path is bounded by a constant depending only on n. To do this, first restrict to a subinterval $[a, b] \subset [0, T]$ where

1. for $t \in [a, b]$, the immersion $p_t : A|G_t \to G_t$ induces a train track structure on $A|G_t$, i.e. $A|G_t$ has no interior illegal turns and all vertices have ≥ 2 gates. Also, each natural edge of $A|G_t$ has length < 2,

2. for $t \in [a, b]$, the subgraph $\tilde{\Omega}_t \subset A|G_t$ is a forward invariant forest, and

3. the projections $\pi_A(\{G_t : t \in [0, a]\})$ and $\pi_A(\{G_t : t \in [b, T]\})$ in $\mathcal{F}(A)$ have uniformly bounded diameter.

Note the such an interval exists by Lemma 5.2.1 and Lemma 5.2.2. For $p : A|G_t \to G_t$, let $E_t^B \subset A|G_t$ be the set of edges in the triangulation induced from G_t that project to edges of $B|G_t \subset G_t$, as in Lemma 5.3.1.

Lemma 5.3.2. *With $\{G_t : t \in [a, b]\}$ as above, if there is a $t_0 \in [a, b]$ so that $A|G_{t_0}$ has an embedded loop x_0 all of whose edges are contained in $\tilde{\Omega}_{t_0} \cup E_{t_0}^B$, then the projection $\pi_A(\{A|G_t : t \geq t_0\})$ has uniformly bounded diameter in $\mathcal{F}(A)$.*

Proof. Let x_t be the image of x_0 in $A|G_t$ pulled tight, i.e. its immersed representative. We show that for any edge e of $A|G_t$ not in E_t^B, x_t crosses e a bounded number of times. Since by assumption A is not contained in B such an edge is guaranteed to exist. By Lemma 3.2 of [8], this implies that $\pi_A(G_t) = \pi(A|G_t)$ has bounded distance from the cyclic factor of A represented by x_0, for all $t \geq t_0$.

Suppose that e is an edge of $A|G_t$ not contained in E_t^B and let p be a point in the interior of e. Note that x_0 is composed of a bounded number of legal segments of $\tilde{\Omega}_{t_0}$ and edges of $E^B_{t_0}$. To see this, recall that since x_0 is embedded it consists of a bounded number of natural edges of $A|G_{t_0}$, each of which is legal because $A|G_t$ has no interior illegal turns. Also, the number of edges of $E^B_{t_0}$ not appearing in $\tilde{\Omega}_{t_0}$ is bounded by $3 \cdot \text{rank}(B) - 3$ since there are no more of these edges than edges of $B|G_{t_0}$. Hence, each natural edge of $A|G_{t_0}$ crossed by x_0 is contained in a bounded number of legal segments of $\tilde{\Omega}_{t_0}$ plus edges of $E^B_{t_0}$ that are not contained in $\tilde{\Omega}_{t_0}$.

Let s be a legal segment of $\tilde{\Omega}_{t_0}$ that maps overs p more than twice. Then by forward invariance of $\tilde{\Omega}_t$ and legality of s, p must be contained in the core of $\tilde{\Omega}_t$. This contradicts the assumption that $\tilde{\Omega}_t$ is a forest. For an edge of $E^B_{t_0}$ we note that by forward invariance of $B|G_t$, no edge of $B|G_{t_0}$ can map over the edge $p(e)$. Hence, no edge of $E^B_{t_0}$ can map over e. We conclude that x_t crosses e no more that $2 \cdot |\text{legal segments of } \tilde{\Omega}_{t_0}|$ times. Since we have seen that this quantity is bounded by a constant depending only on the rank of G, we conclude that $\pi_A(\{G_t : t \geq t_0\})$ is uniformly bounded. □

Together, these lemmas complete the proof of our main theorem.

Theorem 5.3.3. *Let A and B be conjugacy classes of free factors of F_n with* $\text{rank}(A) \geq 2$. *Then either A and B are disjoint, $A \subset B$, or $\pi_A(B)$ is well-defined with uniformly bounded diameter.*

Proof. Suppose that A and B are free factors that are not disjoint and that A is not contained in B, up to conjugation. Let T be any free splitting of F_n with B as a vertex stabilizer and take $G \in \mathcal{X}_n$ to be a graph refining the splitting

T, so that $B|G$ is embedded in G. Let G_0 be the marked rose discussed above and construct the folding path $\{G_t : t \in [0,T]\}$ from G_0 to $G_T = G$ with subinterval $[a,b] \subset [0,T]$ satisfying conditions $(1),(2)$, and (3).

If $d_A(G_a, G_b)$ is larger than the bound determined in Lemma 5.3.2, then $A|G_a$ does not contain an embedded loop with edges in $\tilde{\Omega}_a \cup E_a^B$, so $\tilde{\Omega}_a \cup E_a^B$ is a forest. By Lemma 5.3.1, this implies that there is a marked graph where A and B are disjointly embedded, contradicting our assumption. Hence,

$$d_A(G_0, T) \leq d_A(G_0, G) + 4 \leq d_A(G_0, G_a) + d_A(G_a, G_b) + d_A(G_b, G) + 4$$

where the first and third terms are uniformly bounded by condition (3) in the properties of the folding path G_t and the second term is no larger than the bound determined in Lemma 5.3.2. Since T was an arbitrary splitting of F_n with vertex stabilizer B, this completes the proof. $\square$

Having shown that subfactor projections are well-defined, we collect some basic facts. First, for the free group F_n, let D denote the constant determined in Theorem 5.3.3 so that if B meets A then $\mathrm{diam}(\pi_A(B)) \leq D$. For free factors A, B each of which meet the rank ≥ 2 free factor C set

$$d_C(A,B) := d_C(\pi_C(A), \pi_C(B)) = \mathrm{diam}(\pi_C(A) \cup \pi_C(B)),$$

where d_C denotes distance in $\mathcal{F}(C)$. If, additionally, A and B are adjacent vertices of $\mathcal{F}_n$ then (up to switching A and B) $A \subset B$ and so $d_C(A,B) \leq 2D$, since each projection contains the projection of a graph where both A and B are embedded. This shows that the projection to $\mathcal{F}(C)$ is coarsely Lipschitz along paths in $\mathcal{F}_n$ all of whose vertices meet C. Finally, the following naturality property is a direct consequence of the definitions: if $f \in \mathrm{Out}(F_n)$ and A and B are free factors that meet the rank ≥ 2 free factor C then

$$d_{fC}(fA, fB) = d_C(A, B),$$

where we use the natural action of $\mathrm{Out}(F_n)$ on conjugacy classes of free factors.

Example 6 (Distance 2 factors with well-defined projection)**.** Let A, B be conjugacy classes of free factors in F_n with representatives (still denoted A, B) such that $A \cap B = x \neq 1$ is proper in each factor. (In Chapter 3, we said that A and B meet, but we now have a much more general notion of meeting.) Then A and B are distinct and $d_{\mathcal{F}}(A, B) = 2$. In this case, however, there is an obvious way to associate to B a free factor of A; use the intersection x of A and B. In fact, basic covering space theory (see Lemma 3.1.1) shows that for any $G \in \mathcal{X}_n$ where B is embedded, $x \in \pi_A(G)$. Hence, $x \in \pi_A(B)$ and $\mathrm{diam}(\pi_A(B)) \leq 8$.

5.4 Properties

The following properties of subfactor projection are obtained just as in [9]. The point here is that our conclusions hold for more general pairs of free factors, so long as we project into the free factor complex rather than free splitting complex of a free factor. Some proofs are provided for completeness and as a verification that they apply in our more general setting. We first have the following version of Lemma 4.12 of [9].

Lemma 5.4.1. *Suppose that A is nearly embedded in $G \in \mathcal{X}_n$. Then there is a $G' \in \mathcal{X}_n$ where A is embedded and a path in $\mathcal{X}_n$ from G to G' with the property that for any free factor B which A meets, the projection of this path to $\mathcal{F}(B)$ has uniformly bounded diameter.*

Proof. We refer to the proof of Lemma 5.3.1. Since A is nearly embedded in G, $\tilde{\Omega} \subset A|G$ is a forest. Let T be a maximal tree containing $\tilde{\Omega}$ and set E to be the set of edge of $A|G$ not contained in T. Recall that $p : A|G \to G$ maps edges of E bijectively to edges of $p(E)$. If the image of $B|G$ in G crosses no edge of $p(E)$ then B is carried by the subgraph $G \backslash p(E)$ of $G' = A|G \vee_{x=p(x)} (G \backslash p(E))$. This contradicts our assumption that A meets B. Hence, the image of $B|G$ crosses the image of some edge e of E in $A|G$. In the language of [9], B is *good* for A. The required path G_t from G' to G is then the path determined by folding the morphism $G' \to G$ given in Lemma 5.3.1. This path makes only bounded progress in $\mathcal{F}(B)$, indeed in $\mathcal{S}(B)$, as shown in [9]. The point is that the splitting of B determined by the preimage of the midpoint of $p(e)$ through the map $B|G_t \to G_t$ is unaltered along the path. □

Theorem 5.4.2. *Given F_n, there is an $M \geq 0$ so that if A and B are overlapping free factors of rank ≥ 2 then for any splitting T that meets both factors*

$$\min\{d_A(B,T), d_B(A,T)\} \leq M.$$

Proof. We follow the proof of Proposition 4.14 of [9] and use Lemma 5.4.1 above. Assume that both $d_A(B,T)$ and $d_B(A,T)$ are very large (relative to

D) and let $G \in \mathcal{X}_n$ be a refinement of T (as a splitting of F_n). Define a folding path G_t, $t \in [0, S]$ from G_0 to $G_S = G$, where G_0 is any graph with A embedded. Since $d_B(A, T)$ and, hence, $d_B(G_0, G_S)$ is large, by Lemma 5.2.2 there is a subinterval $[t_1, t_2]$ where B is nearly embedded and where G_t makes large progress in $\mathcal{F}(B)$, i.e. $d_B(G_{t_1}, G_{t_2})$ is large.

Since B is nearly embedded in G_{t_2} and $d_A(B, G)$ is big by assumption, Lemma 5.4.1 and Lemma 5.2.2 imply that there is an subinterval $[t_3, t_4] \subset [t_2, S]$, where A is nearly embedded. Hence, G_{t_3} is a graph where A is nearly embedded and has very large distance in $\mathcal{F}(B)$ from G_0, where A is embedded. This contradicts Lemma 5.4.1 and the fact that $\operatorname{diam}(\pi_B(A)) \leq D$.

□

Finally, we note the following version of the Bounded Geodesic Image Theorem. The proof in [9] follows through without change after using the more general conditions for projection that are explained in this note.

Theorem 5.4.3 (Bounded Geodesic Image Theorem). *For $n \geq 3$, there is $M \geq 0$ so that if A is a free factor of F_n of rank ≥ 2 and γ is a geodesic of $\mathcal{F}_n$ with each vertex of γ meeting A (i.e having nontrivial projection to $\mathcal{F}(A)$) then* $\operatorname{diam}(\pi_A(\gamma)) \leq M$.

We conclude this section with a remark: Using Theorem 5.1.1 one can give a coarse lower bound on distance in $\operatorname{Out}(F_n)$ exactly as in [47]. Since these lower bounds do not cover all distance in $\operatorname{Out}(F_n)$, i.e. they do not give upper bounds, we do not provide the details here. However, we do note that similarly to [9] one needs to bound the size of a collection of rank ≥ 2 free factors where pairwise projections are not defined. This is done in [9] by finding a finite coloring of the free factor complex so that between similarly colors factors one may project one of the factors to the *splitting complex* of the other. As is a theme of this chapter, if we consider projections to factor complexes things become simpler. In particular, if distinct factors $A, B \in \mathcal{F}_n^0$ represent the same subgroup of $H_1(F_n; \mathbb{Z}/2)$, then A and B must overlap. Hence, we can provide the following coloring of $\mathcal{F}_n^0$: define $\mathcal{H}$ to be the set of proper subgroups of $H_1(F_n; \mathbb{Z}/2)$ and let $c : F_n^0 \to \mathcal{H}$ be defined by

$$c(A) = H_1(A; \mathbb{Z}/2) \leq H_1(F_n; \mathbb{Z}/2).$$

Then, as explained above, if A and B are distinct free factors with rank ≥ 2 and $c(A) = c(B)$, then A and B overlap.

5.5 Filling free factors

In our construction of fully irreducible outer automorphism of $\mathrm{Out}(F_n)$, it will be important to know when a collection of free factors has the property that every free factor of F_n has nonempty projection to some factor in the collection. Say that a collection of free factors $\{A_1, \ldots, A_n\}$ of F_n *fills* if for any free factor $C < F_n$, there is an $1 \leq i \leq n$ so that C meets A_i. When $\mathrm{rank}(A_i) \geq 2$ for each i, this is equivalent to saying that C has nonempty projection to $\mathcal{F}(A_i)$ for some $1 \leq i \leq n$. In this section, we present a few examples of filling and non-filling pairs of free factors.

Example 7 (Non-filling free factors)**.** Let $F_3 = F(x, y, z)$ and consider the following rank 2 free factors: $A = \langle x, y\rangle$, $B = \langle y, z\rangle$, and $C = \langle x, z\rangle$. Then no pair of these factors fills F_n. In fact, the rank one factor $\langle xyz\rangle$ is disjoint from each of A, B, and C.

This shows that our definition of filling is distinct from some other notions that appear in the literature. For example, it is clear that for any nontrivial F_n-tree T, either T^A or T^B is nontrivial.

Example 8 (Filling free factors via homology)**.** Using the notation in the previous example, set $w = zyzyz^2yz$. It is easy to see that w is primitive and that $W = \langle x, w\rangle$ is a free factor of F_3. We claim that A and W fill F_3. Since $A \cap W = \langle x\rangle$, we have $d_{\mathcal{F}}(A, W) = 2$. Hence, A and W are distance 2 factors that fill.

To see that A and W filling, suppose that there is a conjugacy class $[\gamma]$, where γ is primitive in F_3 and disjoint from both A and W. Let $\mathcal{H} : F(x, y, z) \to \mathbb{Z}^3$ be the abelianization map and identify $\mathbb{Z}^3$ with $H_1(F_3)$ using the generators x, y, z. Then, disjointness of γ from A and W implies that we have the following direct sum decompositions:

$$H_1(F_3) = \mathcal{H}(A) \oplus \mathcal{H}(\gamma) = \mathcal{H}(W) \oplus \mathcal{H}(\gamma).$$

To see this, note that since A and γ is disjoint there is a choice of conjugates of these factors so that $F_3 = A * \gamma$, now apply $\mathcal{H}$. We now check these decompositions in the coordinates

$$\mathcal{H}(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{H}(y) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \quad \mathcal{H}(z) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let $\mathcal{H}(\gamma) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and note that $\mathcal{H}(w) = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$. Since $H_1(F_3) = \mathcal{H}(A) \oplus \mathcal{H}(\gamma)$, we must have

$$\det \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{pmatrix} = \pm 1.$$

So $c = \pm 1$. By replacing γ by its inverse, we may suppose that $c = 1$. However, $H_1(F_3) = \mathcal{H}(W) \oplus \mathcal{H}(\gamma)$ implies that

$$\pm 1 = \det \begin{pmatrix} 1 & 0 & a \\ 0 & 3 & b \\ 0 & 5 & 1 \end{pmatrix} = 3 - 5b.$$

Since, $3 - 5b = \pm 1$ has no solution with $b \in \mathbb{Z}$, we have a contradiction. Hence, A and W fill F_3 as required.

Example 9 (Cyclic factors that fill via distance)**.** Let $\alpha, \beta \in C_n^0$ with $d_{\mathcal{C}}(\alpha, \beta) \geq 3$ (where $\mathcal{C}_n$ is defined in Section 5.6). Then by definition of distance in $\mathcal{C}_n$, there is no rank one fact that is disjoint from both α and β. This immediately implies that α and β fill in the sense that there is no factor of F_n disjoint from both α and β.

It is also important to note that if two rank 1 free factors do not fill, this does not imply that they are contained, up to conjugacy, in a free factor of F_n. (See Example 10.) This is completely different from the situation in the surface topology where if curves α and β are disjoint from a curve δ, then there is some subsurface $Y \subset S$ that contains both α and β and misses δ.

Example 10 (Distance two factors in $\mathcal{C}_n$ that are not contained in a proper free factor)**.** Consider $F_3 = F(x, y, z)$. To construct a cyclic factor $w \in \mathcal{C}_n$ with distance 2 from $\langle y \rangle$, we take a cyclic factor in $\langle x, z \rangle$ that is sufficiently complicated and then twist it around y. Specifically, take w to be the conjugacy class of the cyclic factor $\langle y^2 x z^2 x z \rangle$. Since both x and w are disjoint from y, $d_{\mathcal{C}}(x, w) = 2$.

To show that no conjugates of x and w are contained in any proper free factor of F_n we use Whitehead's algorithm. See [45] for the required definitions and theorems. First, let $W(x)$ and $W(w)$ be the Whitehead graphs for the cyclically reduced elements x and w of F_3 with respect to the basis x, y, z. (See Figure 5.1.) The Whitehead graph of the pair $W(\{x, w\})$ is the superposition of $W(x)$ and $W(w)$ and inspection shows that this graph does not have a cut vertex. By [45], the absence of a cut vertex implies that the conjugacy classes of x and w are not *separable*, i.e. there is no free splitting of F_n such that both x and w are each conjugate into vertex stabilizers. Hence, x and w cannot be conjugated into any proper free factor of F_3.

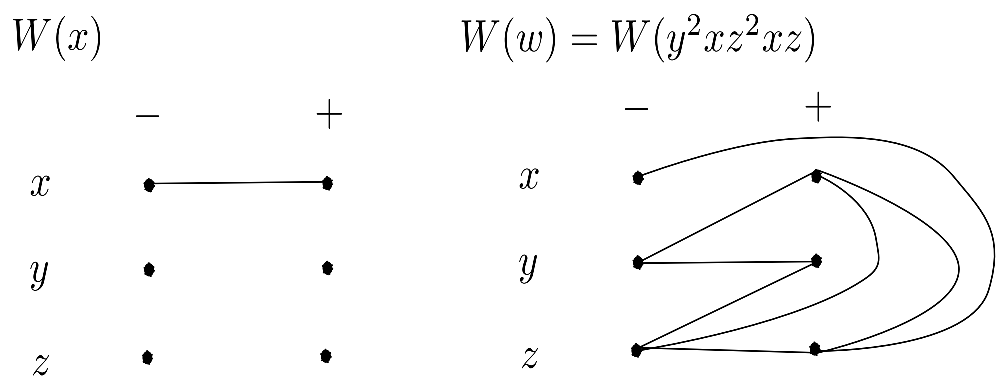

Figure 5.1: Whitehead graphs of x and w.

5.6 Constructing Fully Irreducible Automorphisms

We consider the following modification of the free factor complex. Let $\mathcal{C}_n$ for $n \geq 2$ be the graph defined as follows: the vertices of $\mathcal{C}_n$ are conjugacy classes of rank 1 free factors of F_n and two vertices $v, w \in \mathcal{C}_n^0$ are joined by an edge if they can be represented by elements x and y in F_n, respectively, such that $\langle x, y \rangle$ is a rank 2 free factor of F_n; that is edges are determined by disjointness of vertices. This graph is obviously quasi-isometric to the free factor complex of F_n. For a free factor $A < F_n$, let X_A denote the set of vertices of $\mathcal{C}_n$ that fail to project to $\mathcal{F}(A)$, i.e. that are disjoint from A. The complex $\mathcal{C}_n$ has the following advantage over $\mathcal{F}_n$: for any free factor A the diameter of X_A in $\mathcal{C}_n$ is ≤ 2. In fact, X_A is contained in a 1-neighborhood of any rank 1 free factor of A. We remark that for γ_1, γ_2 adjacent vertices of $\mathcal{C}_n$ that are not contained in X_A, $d_A(\gamma_1, \gamma_2) \leq 2D$. We also have the corresponding version of the Bounded Geodesic Image Theorem for $\mathcal{C}_n$. We state it here for later reference.

Proposition 5.6.1. *For $n \geq 3$, there is an $M \geq 0$ so that if A is a free factor of F_n of rank ≥ 2 and γ is a geodesic in $\mathcal{C}_n$ with each vertex of γ meeting A, i.e. γ is disjoint from X_A, then* $\operatorname{diam}(\pi_A(\gamma)) \leq M$.

To make effective use of the graph $\mathcal{C}_n$, we need the following lemma.

Lemma 5.6.2. *Let A and B be free factors of F_n with* $\operatorname{rank}(A) \geq 2$ *and $\pi_A(B) \neq \emptyset$. Then there is a cyclic (i.e. rank 1) factor $\gamma \subset B$ with $\pi_A(\gamma) \neq \emptyset$.*

Proof. If B is rank 1 there is nothing to show, and if $B \subset A$ then any rank 1 subfactor will do. Hence, we may assume that $\operatorname{rank}(B) \geq 2$ and that $\pi_B(A) \neq \emptyset$. Choose a cyclic factor γ of B that is at distance $> D + 4$ from $\pi_B(A)$ in $\mathcal{F}(B)$. If $\pi_A(\gamma) = \emptyset$ then there is a marked graph G containing subgraphs representing A and γ, respectively. Then by definition

$$\pi_B(G) \subset \pi_B(A) \quad \text{and} \quad \pi_B(G) \subset \pi_B(\gamma),$$

implying that $d_B(A, \gamma) \leq \operatorname{diam}(\pi_B(A)) + \operatorname{diam}(\pi_B(\gamma)) \leq D+4$, a contradition. □

The following proposition shows how subfactor projections can be used to build up distance in the graph $\mathcal{C}_n$. In the mapping class group situation, this is proven for the curve complex in [37]. The idea originates in [33].

Proposition 5.6.3. *Let $\{A_i\}$ be a collection of free factors and let X_i be the set of vertices of $\mathcal{C}_n$ that do not project to A_i, i.e. $X_i = X_{A_i}$. Let M be the constant determined in Proposition 5.6.1. Assume that*

1. *X_i and X_{i+1} are disjoint in $\mathcal{C}_n$ and*

2. *$d_{A_i}(x_{i-1}, x_{i+1}) > 2M$ for any $x_{i-1} \in X_{i-1}$ and $x_{i+1} \in X_{i+1}$.*

Then the X_i are pairwise disjoint and for any $x_j \in X_j$ and $x_{j+k} \in X_{j+k}$, any geodesic $[x_j, x_{j+k}]$ contains a vertex from X_i for $j \leq i \leq j+k$.

Proof. The proof is adapted from [37]. There are two reasons for providing the details here. First, the argument is an illustration of how the general subfactor projections discussed in this note and the complex $\mathcal{C}_n$ are many ways analogous to subsurface projections and the curve complex. Second, there are several subtleties that make subfactor projections different; for example, there is no canonical "boundary curve" of A contained in X_A.

The proposition is proven by induction on k; for $k = 1$ there is nothing to prove. Let $x_j \in X_j$ and $x_{j+k} \in X_{j+k}$ be given and consider a geodesic $[x_j, x_{j+k}]$. Select any $x_{j+k-2} \in X_{j+k-2}$. We first show that there exists a geodesic $[x_j, x_{j+k-2}]$ that avoids vertices of X_{j+k-1}. To see this, start with a geodesic $[x_j, x_{j+k-2}]$ that contains a vertex x_{j+k-1} of X_{j+k-1} and decompose it as

$$[x_j, x_{j+k-2}] = [x_j, x_{j+k-1}] \cup [x_{j+k-1}, x_{j+k-2}].$$

Suppose that we have chosen x_{j+k-1} to be the first vertex of X_{j+k-1} that appears along $[x_j, x_{j+k-2}]$ so that $[x_j, x_{j+k-1}]$ is disjoint from X_{j+k-1} except at its last vertex. The induction hypotheses now implies that $[x_j, x_{j+k-1}]$ meets X_{k+j-2} at a vertex x'_{k+j-2} and we can write

$$[x_j, x_{j+k-2}] = [x_j, x'_{j+k-2}] \cup [x'_{j+k-2}, x_{j+k-1}] \cup [x_{j+k-1}, x_{j+k-2}].$$

By assumption, these last two geodesics have length at least 1 and since the diameter of each X_i is less than or equal to 2 we may replace the union of the last two geodesics with a geodesic $\{x'_{j+k-2}, a_{j+k-2}, x_{j+k-2}\}$, where a_{j+k-2} is a cyclic factor of A_{j+k-2} whose projection to A_{j+k-1} is nonempty. This is possible by Lemma 5.6.2. Hence, we have produced a geodesic from x_j to x_{j+k-2} that avoids X_{j+k-1}.

Since $[x_j, x_{j+k-2}]$ avoids X_{j+k-1}, Proposition 5.6.1 implies that $d_{A_{j+k-1}}(x_j, x_{j+k-2}) \leq M$. Hence,

$$\begin{aligned} d_{A_{j+k-1}}(x_j, x_{j+k}) &\geq d_{A_{j+k-1}}(x_{j+k-2}, x_{j+k}) - d_{A_{j+k-1}}(x_j, x_{j+k-2}) \\ &> 2M - M \geq M. \end{aligned}$$

Another application of Proposition 5.6.1 gives that any geodesic $[x_j, x_{j+k}]$ must contain a vertex that misses A_{j+k-1}, hence there is a vertex $x_{j+k-1} \in X_{j+k-1}$ with $x_{j+k-1} \in [x_j, x_{j+k}]$. This implies that we may write $[x_j, x_{j+k}] = [x_j, x_{j+k-1}] \cup [x_{j+k-1}, x_{j+k}]$ and applying the induction hypothesis to $[x_j, x_{j+k-1}]$ we conclude that the geodesic $[x_j, x_{j+k}]$ contains a vertex from each X_i for $j \leq i \leq j+k$. Also, if $X_j \cap X_{j+k}$ contained a vertex x then the geodesic $[x, x]$

would have to intersect X_{j+1}, contradicting our hypothesis. This concludes the proof. $\square$

The next theorem is similar to Proposition 3.3 in [37], where pseudo-Anosov mapping classes are constructed using the curve complex.

Theorem 5.6.4. *Let A and B be rank ≥ 2 free factors of F_n that fill and let $f, g \in \mathrm{Out}(F_n)$ satisfy the following:*

1. *$f(A) = A$ and $f|_A \in \mathrm{Out}(A)$ has translation length $> 2M + 4D$, and*
2. *$g(B) = B$ and $g|_B \in \mathrm{Out}(B)$ has translation length $> 2M + 4D$.*

Then the subgroup $\langle f, g\rangle \leq \mathrm{Out}(F_n)$ is free of rank 2 and any nontrivial automorphism in $\langle f, g\rangle$ that is not conjugate to a power of f or g is fully irreducible. Moreover, any finitely generated subgroup of $\langle f, g\rangle$ consisting entirely of such automorphisms has the property that any orbit map into $\mathcal{F}_n$ is a quasi-isometric embedding.

Before beginning the proof we make the following remark: By [8], an outer automorphism has positive translation length in $\mathcal{F}_n$ (or $\mathcal{C}_n$) if and only if it is fully irreducible. Hence, if f and g fix the free factors A and B, respectively, and their restrictions are fully irreducible, then conditions (1) and (2) are satisfied after passing to a sufficiently high power. If there were to exists a uniform lower bound on the translation length of a fully irreducible automorphism in $\mathcal{F}_n$, depending only on n, then such a power would be independent of f and g.

Proof. We sketch the proof as the details are similar to [37]. First, note that we have chosen translation lengths sufficiently large so that any geodesic of $\mathcal{C}_n$ joining vertices of X_B and fX_B must contain a vertex of X_A, and similarly any geodesic joining vertices of X_A and gX_A must contain a vertex of X_B. To see this, note that since A and B fill, $X_A \cap X_B = \emptyset$. Also, if b is a rank 1 free factors of B that meets A, which exists by Lemma 5.6.2,

$$\operatorname{diam}(\pi_A(X_B)) \leq 2 \cdot \max\{d_A(b,\beta) : \beta \in X_B\} \leq 2D.$$

Hence, for any $\beta \in X_B$ and $\beta' \in fX_B$, let $a_\beta \in \pi_A(\beta)$ so that

$$\begin{aligned} d_A(\beta,\beta') &\geq d_A(\beta, f\beta) - d_A(f\beta,\beta') \\ &\geq d_A(a_\beta, fa_\beta) - 2\cdot \operatorname{diam}(\pi_A(\beta)) - d_A(f\beta,\beta') \\ &> (2M+4D) - 2D - 2D \geq 2M > M. \end{aligned}$$

Then by the Proposition 5.6.1, any geodesic from β to β' must contain a vertex that misses A, i.e. that is contained in X_A. The proof now proceeds by using Theorem 5.6.3 to show that elements not conjugate to powers of f or g act with positive translation length on $\mathcal{C}_n$ and are therefore fully irreducible.

For any $w \in \langle f, g\rangle$ in reduced form, write $w = s_1 \dots s_n$ where each s_i is a *syllable* of w, i.e. a maximal power of either f or g. Suppose for simplicity that s_1 is a power of f and s_n is a power of g (so in particular n is even) and set $X_i = s_1 \dots s_{i-1}X_A$ for i odd and $X_i = s_1 \dots s_{i-1}X_B$ for i even. By naturality of the $\operatorname{Out}(F_n)$-action, these are precisely the sets of vertices of $\mathcal{C}_n$ that fail to project to the free factors $A_i = s_1 \dots s_{i-1}A$ and $B_i = s_1 \dots s_{i-1}B$, respectively. Using that fact that X_A is fixed by f and X_B is fixed by g, it is quickly verified that the sets X_i satisfy the conditions of Theorem 5.6.3 for $1 \leq i \leq n+1$. We conclude that for $\alpha \in X_A$ and $w\alpha \in wX_A = X_{n+1}$, the geodesic $[\alpha, w\alpha]$ contains at least $n+1$ vertices and so

$$d_{\mathcal{C}}(\alpha, w\alpha) \geq n = |w|_s,$$

where $|\cdot|_s$ denotes the number of syllables. In general, one shows that either $d_{\mathcal{C}}(\alpha, w\alpha) \geq |w|_s$ or $d_{\mathcal{C}}(\beta, w\beta) \geq |w|_s$, where $\beta \in X_B$, depending on the first and last syllable of w.

To finish the proof, observe that any $w \in \langle f, g \rangle$ that is not a conjugate to a power of f or g has a conjugate w' with an even number of syllables and w' has the property that $|w'^n|_s = n|w'|_s$. Hence, w' has positive translation length in $\mathcal{C}_n$, as does its conjugate w. This shows that w is fully irreducible.

The statement about quasi-isometric orbit maps follows as in [37]. □

Finally, we present an example to show that the filling hypothesis in Theorem 5.6.4 cannot be removed.

Example 11 (Non-IWIP example)**.** Let $F_3 = F(x, y, z)$ and consider the rank 2 free factors $A = \langle x, y \rangle$ and $B = \langle y, z \rangle$. We saw in Example 7, A and B do not fill, hence Theorem 5.6.4 does not apply. Here we give an example to show that the filing hypothesis in necessary.

Define $\phi_A \in \mathrm{Out}(F_n)$ by the assignments $x \mapsto xy$, $y \mapsto yxy$, and $z \mapsto y^{-1}z$. Then ϕ_A is an outer automorphism of F_3 that fixes A and whose restriction to A is fully irreducible. Similarly define $\phi_B \in \mathrm{Out}(F_3)$ by $y \mapsto yz$, $z \mapsto zyz$, and $x \mapsto xy^{-1}z^{-1}$. Then ϕ_B fixes B and the restriction of ϕ_B to B is fully irreducible. (In both cases, the restriction is the well known map

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

on homology.) However, $\langle \phi_A, \phi_B \rangle \leq \mathrm{Out}(F_3)$ contains no fully irreducible automorphisms. In fact, every automorphism of $\langle \phi_A, \phi_B \rangle \leq \mathrm{Out}(F_3)$ fixes the conjugacy class of the primitive element $\gamma = xz$:

$$\phi_A(\gamma) = \phi_A(x)\pi_A(z) = xyy^{-1}z = xz = \gamma$$

and

$$\phi_B(\gamma) = \phi_B(x)\phi_B(z) = xy^{-1}z^{-1}zyz = ax = \gamma.$$

Bibliography

[1] Yael Algom-Kfir. Strongly contracting geodesics in Outer space. *Geometry & Topology*, 15:2181–2233, 2011.

[2] Javier Aramayona and Juan Suoto. Automorphisms of the graph of free splittings. *The Michigan Mathematical Journal*, 60(3):483–493, 2011.

[3] Jason Behrstock. Asymptotic geometry of the mapping class group and Teichmüller space. *Geom. Topol.*, 10:1523–1578, 2006.

[4] Jason Behrstock, Bruce Kleiner, Yair Minsky, and Lee Mosher. Geometry and rigidity of mapping class groups. *Geometry & Topology*, 16:781–888, 2012.

[5] Jason A Behrstock and Yair N Minsky. Dimension and rank for mapping class groups. *Annals of Mathematics*, 167:1055–1077, 2008.

[6] Mladen Bestvina. A Bers-like proof of the existence of train tracks for free group automorphisms. *arXiv preprint arXiv:1001.0325*, 2010.

[7] Mladen Bestvina, Kenneth Bromberg, and Koji Fujiwara. Constructing group actions on quasi-trees and applications to mapping class groups. *arXiv preprint arXiv:1006.1939*, 2010.

[8] Mladen Bestvina and Mark Feighn. Hyperbolicity of the complex of free factors. *arXiv preprint arXiv:1107.3308*, 2011.

[9] Mladen Bestvina and Mark Feighn. Subfactor projections. *Journal of Topology*, 2014.

[10] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Annals of Mathematics*, pages 1–51, 1992.

[11] M.R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319. Springer, 2009.

[12] Jeffrey Brock and Benson Farb. Curvature and rank of Teichmüller space. *American Journal of Mathematics*, pages 1–22, 2006.

[13] Matt Clay and Alexandra Pettet. Twisting out fully irreducible automorphisms. *Geometric and Functional Analysis*, 20(3):657–689, 2010.

[14] Matt T. Clay, Christopher J. Leininger, and Johanna Mangahas. The geometry of right-angled Artin subgroups of mapping class groups. *Groups Geom. Dyn.*, 6(2):249–278, 2012.

[15] John Crisp and Bert Wiest. Embeddings of graph braid and surface groups in right-angled artin groups and braid groups. *Algebr. Geom. Topol*, 4:439–472, 2004.

[16] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986.

[17] Benson Farb and Lee Mosher. Convex cocompact subgroups of mapping class groups. *Geom. Topol.*, 6:91–152 (electronic), 2002.

[18] Stefano Francaviglia and Armando Martino. Metric properties of outer space. *Publicacions Matemàtiques*, 55(2):433–473, 2011.

[19] U. Hamenstädt and S. Hensel. Spheres and Projections for $\mathrm{Out}(F_n)$. *ArXiv e-print 1109.2687*, September 2011.

[20] Ursula Hamenstädt. Word hyperbolic extensions of surface groups. *preprint*, page arXiv:0807.4891v2, 2005.

[21] Ursula Hamenstaedt. Geometry of the mapping class groups iii: Quasi-isometric rigidity. *arXiv preprint math/0512429*, 2005.

[22] Michael Handel and Lee Mosher. Lipschitz retraction and distortion for subgroups of Out (f_n). *arXiv preprint arXiv:1009.5018*, 2010.

[23] Michael Handel and Lee Mosher. The free splitting complex of a free group I: Hyperbolicity. *arXiv preprint arXiv:1111.1994*, 2011.

[24] Allen Hatcher. Homological stability for automorphism groups of free groups. *Commentarii Mathematici Helvetici*, 70(1):39–62, 1995.

[25] Allen Hatcher and Karen Vogtmann. The complex of free factors of a free group. *Quarterly Journal of Mathematics*, 49(196):459–468, 1998.

[26] Allen Hatcher, Karen Vogtmann, et al. Isoperimetric inequalities for automorphism groups of free groups. *Pacific J. Math*, 173(2):425–441, 1996.

[27] Sebastian Hensel, Damian Osajda, and Piotr Przytycki. Realisation and dismantlability. *arXiv preprint arXiv:1205.0513*, 2012.

[28] S. Hermiller and J. Meier. Algorithms and geometry for graph products of groups. *Journal of Algebra*, 171(1):230–257, 1995.

[29] Arnaud Hilion and Camille Horbez. The hyperbolicity of the sphere complex via surgery paths. *arXiv preprint arXiv:1210.6183*, 2012.

[30] Ilya Kapovich and Martin Lustig. Ping-pong and Outer space. *Journal of Topology and Analysis*, 2(02):173–201, 2010.

[31] Ilya Kapovich and Kasra Rafi. On hyperbolicity of free splitting and free factor complexes. *arXiv preprint arXiv:1206.3626*, 2012.

[32] Michael Kapovich. RAAGs in Ham. *Geometric And Functional Analysis*, pages 1–23, 2011.

[33] Richard P. Kent, IV and Christopher J. Leininger. Uniform convergence in the mapping class group. *Ergodic Theory and Dynamical Systems*, 28(4):1177–1196, 2008.

[34] Thomas Koberda. Right-angled artin groups and a generalized isomorphism problem for finitely generated subgroups of mapping class groups. *Geometric And Functional Analysis*, pages 1–50, 2010.

[35] François Laudenbach. Sur les 2-spheres d'une variete de dimension 3. *The Annals of Mathematics*, 97(1):57–81, 1973.

[36] Johanna Mangahas. Uniform uniform exponentional growth of subgroups of the mapping class group. *Geom. Funct. Anal.*, 19(5):1468–1480, 2010.

[37] Johanna Mangahas. A recipe for short-word pseudo-Anosovs. *American Journal of Mathematics*, 135(4):1087 – 1116, 2013.

[38] Johanna Mangahas and Samuel J. Taylor. Convex cocompactness in mapping class groups via quasiconvexity in right-angled artin groups. *arXiv preprint arXiv:1306.5278*, 2013.

[39] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.

[40] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000.

[41] Yair N. Minsky. The classification of Kleinian surface groups, I: Models and bounds. *Annals of Mathematics*, 171:1–107, 2010.

[42] Kasra Rafi. A combinatorial model for the Teichmüller metric. *GAFA Geometric And Functional Analysis*, 17(3):936–959, 2007.

[43] Lucas Sabalka and Dmytro Savchuk. Submanifold projection. *arXiv preprint arXiv:1211.3111*, 2012.

[44] Peter Shalen. Representations of 3-manifold groups. In RB Sher and RJ Daverman, editors, *Handbook of Geometric Topology*. North Holland,, 2001.

[45] J.R. Stallings. Topology of finite graphs. *Invent. Math.*, 71(3):551–565, 1983.

[46] Samuel J. Taylor. A note on subfactor projections. *To appear in Algebraic and Geometric Topology; arXiv preprint arXiv:1307.1495*, 2013.

[47] Samuel J. Taylor. Right-angled Artin groups and Out (F_n) I: quasi-isometric embeddings. *To appear in Groups, Geometry, and Dynamics; arXiv preprint arXiv:1303.6889*, 2013.

[48] Stephen Wang. Representations of surface groups and right-angled artin groups in higher rank. *Algebr. Geom. Topol*, 7:1099–1117, 2007.