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Single Field Inflation: Observables and Constraints

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Single Field Inflation: Observables and Constraints

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To my parents

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Single Field Inflation: Observables and Constraints

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One of the exciting aspects of cosmology is to understand the period of 'cosmic inflation' that powered the epoch of the Big Bang. Inflation has been very successful in explaining several puzzles of the standard big bang scenario. But the most important success of inflation is that it can explain the temperature fluctuations of cosmic microwave background and the large scale structures of the universe. Despite its great success, the details of the physics of inflation are still unknown. A large number of models of inflation successfully explain all the observations making it remarkably hard to distinguish between different models. We explore the possibility of differentiating between different inflationary models by studying two-point and three-point functions of primordial fluctuations produced during inflation.

First, we explore possible constraints on the inflationary equation state by considering current measurements of the power spectrum. Next, we explore the possibility of a single field slow-roll inflationary model with general initial state for primordial fluctuations. The two-point and three-point functions of primordial fluctuations are generally computed assuming that the fluctuations

are initially in the Bunch-Davies state. However, we show that the constraints on the initial state from observed power spectrum and local bispectrum are relatively weak and for slow-roll inflation a large number of initial states are consistent with the current observations. As the precision of the observations is increasing significantly, we may learn more about the initial state of the fluctuations in the near future.

Finally, we explore the consistency relations for the three-point functions, in the squeezed limit, of scalar and tensor perturbations in single-field inflation that in principle can be used to differentiate between single-field and multi-field inflation models. However, for slow-roll inflation, we find that it is possible to violate some of the consistency relations for initial states that are related to the Bunch-Davies state by Bogoliubov transformations and we identify the reason for the violation. Then we discuss the observational implications of this violation.

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Chapter 1

Introduction

It is believed that inflation [75, 91, 13], a period of exponential expansion of the universe, powered the epoch of the Big Bang. One of the exciting aspects of cosmology is to understand the period of inflation that has taken place 10^{-34} seconds after the Big Bang singularity. It is truly remarkable that we can address any meaningful question about an epoch shortly after the universe was born. A period of inflation naturally solves several puzzles of the standard big bang scenario. But the most important success of inflation is that it can explain the temperature fluctuations of cosmic microwave background (CMB) and the large scale structures (LSS) of the universe.

1.1 A brief history of the universe

Before we discuss the physics of inflation, let us first briefly summarize major events in the history of our universe. Based on our current knowledge, we can divide the history of our universe in two parts: A) First 10^{-14} seconds and B) From 10^{-14} seconds to today ($\sim$ 13.7 billion years). 10^{-14} seconds after the Big Bang singularity, the temperature of the universe is roughly 10 TeV. This is the energy scale that can be probed by accelerators and we are somewhat confident about the validity of the Standard Model of particle physics. Therefore, the history of the universe from 10^{-14} seconds to today is based on well understood physics. Whereas, before 10^{-14} seconds, the energy of the universe is well above 10 TeV and we do not have any direct experimental evidence for the physics of that era. However, we have some theoretical control over the physics above 10 TeV and the hope is that the very early universe can provide us important information about fundamental physics.[1]

1.1.1 From 10^{-14} seconds to today

1) 10^{-14} **s -** 10^{-10} **s (**$T \sim 10$ **TeV -** 100 **GeV)**

Before 10^{-10} seconds, the energy of the universe is above 100 GeV and the electroweak symmetry is restored and gauge bosons (Z, $W^{\pm}$) are massless. At $t \sim 10^{-10}$s electroweak symmetry is broken and the gauge bosons acquire mass. As the temperature of the universe drops below 100 GeV, the cross-section of weak interactions decreases.

2) 10^{-5} **s (**$T \sim 200$ **MeV)**

At this time a transition takes place in which free quarks and gluons combine to form baryons and mesons:

$$\text{quarks} + \text{gluons} \rightarrow \text{baryons} + \text{mesons}\ .$$

3) 0.2 **s (**$T \sim 1-2$ **MeV)**

An interaction process between particles falls out of equilibrium when the

[1]For a detailed discussions see [104, 62, 140] (technical) and [138] (popular).

interaction rate is smaller than the expansion rate of the universe. The cross-section of weak interactions decreases further and some weak interaction processes fall out of equilibrium. As a consequence, two important events take place at $t \sim 0.2$ s. (i) At $T \sim 1$ MeV primordial neutrinos decouple from the rest of the matter and travel freely without further scattering. (ii) Another set of weak interaction processes

$$n + e^{+} \Leftrightarrow p^{+} + \bar{\nu}_e \ ,$$

$$n + \nu_e \Leftrightarrow p^{+} + e^{-}$$

fall out of equilibrium. However, the neutron-to-proton ratio does not become constant after freeze out - the beta decay continues to reduce the ratio:

$$n \Rightarrow p^{+} + e^{-} + \bar{\nu}_e \ .$$

The abundances of the primordial elements are directly related to the number of surviving neutrons.

4) 1 s ($T \sim 0.5$ MeV)

At $t \sim 1$, the temperature of the universe is of order the electron mass. As the temperature drops below the electron rest mass, electrons and positrons efficiently annihilate

$$e^{-} + e^{+} \rightarrow \gamma + \gamma \ .$$

An initial electron-positron asymmetry of one part in a billion survives after annihilation. The photons produced after the annihilation are in thermal equilibrium. The temperature of the radiation is greater than the temperature of neutrinos which decoupled from the rest of the matter in the past.

5) $200 - 300$ **s** $(T \sim 0.05$ **MeV)**

At around $t \sim 200$ s strong interaction becomes efficient. As the temperature of the universe drops below 0.1 MeV, protons and surviving neutrons combine to form deuterium nuclei. Once deuterium nuclei are formed, the rest of BBN proceeds relatively quickly. The abundances of H, He and Li as predicted by Big Bang nucleosynthesis (BBN)[111, 135, 61, 130, 131, 132, 133, 106] are consistent with observations[20, 63, 85, 44].

6) 10^{11} **s** $(T \sim 1$ **eV)**

At this time energy densities of matter and radiation are equal and it corresponds to the beginning of the matter-dominated epoch. Charged particles and photons in the plasma are still in thermal equilibrium.

7) $10^{12} - 10^{13}$ **s** $(T \sim 0.1$ **eV)**

Around $t \approx 380,000$ years, free protons and electrons combine to form neutral hydrogen atoms. Photons decouple from the rest of the matter and form the cosmic microwave background that 13.7 billion years later gives us the earliest snapshot of our universe. Temperature fluctuations in the CMB provide direct information about the state of the primordial matter density at the last scattering surface. It is important to note that not all free protons and electrons form atoms. Due to the freeze out of recombination, one free electron (and proton) per 10,000 neutral atoms remain after the completion of recombination.

Neutral helium, which constitutes about 25% of the baryonic matter, are formed way before hydrogen. But the universe at that time was not transparent to the background radiation because of the presence of free electrons. The way helium recombination is seen in CMB is simply because helium consumes more electrons, leaving fewer electrons at the decoupling. Fewer electrons make photons propagate more freely, enhancing the Silk damping.

8) $10^{15} - 10^{17}$ **s** ($z \sim 25 - 6$)

Structure formation becomes more efficient after matter-radiation equality and small density perturbations $\rho(x,t) = \bar{\rho}(t)(1 + \delta(x,t))$ start to grow via gravitational instability and eventually form the large scale structure that we see today. Stars and galaxies form first and then they interact gravitationally to form larger structures like clusters and superclusters of galaxies. Both gravitational attraction and background pressure play significant roles during structure formation. Note that

$$\begin{aligned} &\delta \sim \ln a && \text{radiation} \\ &\delta \sim a && \text{matter} \\ &\delta \sim \text{const.} && \text{dark energy} \end{aligned}$$

and hence structure formation is more efficient after matter-radiation equality.

Around $z \sim 25$ high energy photons start coming out of the first starts. This high energy photons ionize the hydrogen in the inter-galactic medium. This process takes place from $z \sim 25$ to $z \sim 6$. At the same time most massive stars run out of nuclear fuel and explode as supernovae that create heavy elements (C,O,...) which are essential for the formation of life as we know it.

Around $z \sim 1$ dark energy starts dominating the energy density of the universe. At this time $\delta \sim$ constant and structure formation stops.

1.1.2 First 10^{-14} seconds

Before 10^{-14} seconds, the energy of the universe is above 10 TeV and the physics of that era is not accessible by direct experiments. However, we have some theoretical control over the physics above 10 TeV and below 10^{19} GeV. The very early universe can provide us important information about fundamental physics at very high energy scales. It is believed that the fluctuations of the CMB temperature are generated in the very early universe during a period of inflation. Recent observations indicate that inflation takes place at energies about 10^{16} GeV ($t \sim 10^{-34}$ s) [7].

Around the same energy scales, it is expected that Grand Unification of the electroweak and strong interactions takes place. Cosmic strings, monopoles that appear naturally in unified theories, might be important in the early universe. However, from the current observations it is somewhat clear that they do not play any significant role in the large scale structure formation.

Below the Planck energy 10^{19} GeV quantum gravity effects are not that important and because of that we can treat gravity classically. Around $t \sim 10^{-43}$ s, energy of the universe is comparable to the Planck energy and we no longer can trust classical general relativity. With our current understanding of physics this is where we lose all theoretical control.

1.2 Physics of inflation

The physics of inflation is rather simple. An exponential expansion of the universe during inflation is thought to be responsible for a flat and homogeneous universe on the large scale. On the other hand, quantum fluctuations produced during inflation grew via gravitational instability to form structures in the universe. In inflationary cosmological theories the temperature fluctuations of CMB and the large scale structures are directly related to the scalar curvature perturbations produced during inflation. Inflation naturally predicts an almost scale invariant power spectrum of primordial fluctuations[129, 78, 76, 29, 105] which accords with current observations from the size of the observable universe down to the scales of around a Mpc[81, 5]. Another exciting aspect of inflation is that it also predicts tensor fluctuations that can be directly observed in cosmological gravitational waves[128, 119, 7].

Our universe is homogeneous and isotropic on the large scale. Large scale evolution of the universe can be described by the Friedmann-Robertson-walker (FRW) metric [136, 118, 117, 66]

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \ ,$$

where, $a(t)$ is the scale factor. The curvature parameter $k = +1, 0, -1$ for positively curved, flat and negatively curved spacetimes respectively. Hubble parameter H is given by

$$H = \frac{\dot{a}(t)}{a(t)} \ .$$

Let us now assume that the universe is driven by a perfect fluid. In a frame which is comoving with the fluid, the energy momentum tensor of the fluid is given by

$$T_{\mu}^{\nu} = \begin{pmatrix} -\rho(t) & 0 & 0 & 0 \\ 0 & p(t) & 0 & 0 \\ 0 & 0 & p(t) & 0 \\ 0 & 0 & 0 & p(t) \end{pmatrix}$$

with the equation of state: $p(t) = w\rho(t)$. Where p is the pressure and ρ is the energy density and w is called the equation of state parameter. Evolution of the universe can be obtained from the Einstein's equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \ .$$

Now using the FRW metric, we obtained the Friedmann equations:

$$\begin{aligned} \frac{H^2}{8\pi G} &= \frac{1}{3}\rho(t) - \frac{k}{a^2(t)} \ , \\ \dot{H} + H^2 &= \frac{8\pi G}{6}\left[\rho(t) + 3p(t)\right] \ . \end{aligned}$$

Friedmann equations also lead to the continuity equation for the fluid driving the evolution

$$\dot{\rho}(t) + 3H[\rho(t) + p(t)] = 0 \ .$$

In table 1.1, we have shown the solutions of Friedmann equations for different epochs of interest.

	w	$\rho(a)$	$a(t)$	$H(t)$
Inflation	~ -1	const.	e^{Ht}	const.
Ideal fluid	$w > -1$	$a^{-3(1+w)}$	$t^{\frac{2}{3(1+w)}}$	$\frac{2}{3(1+w)t}$
Radiation	$\frac{1}{3}$	a^{-4}	$t^{1/2}$	$\frac{1}{2t}$
Matter	0	a^{-3}	$t^{2/3}$	$\frac{2}{3t}$
Dark energy	-1	const.	e^{Ht}	const.

Table 1.1: Cosmological solutions with flat FRW metric for different epochs.

The essential feature of inflation is that the expansion of universe undergoes a temporary accelerating period, where $\ddot{a} > 0$. The Friedman equations (along with the null energy condition) mean that a positive $\ddot{a}$ implies that $-1 < w < -1/3$. The scalar power spectrum is almost scale-invariant from the size of the observable universe down to scales of around a Mpc; as the authors show in [84], this places tight bounds on w: w must be close to -1 for those modes that contribute to the observed power spectrum. Therefore, during inflation

$$a \sim e^{Ht} , \qquad H = \text{const.}$$

However, in section 3 we will show that w is less tightly constrained for those periods of accelerated expansion corresponding to shorter length scales and we will explore the possibility of a significantly varying w before the start of decelerating expansion by dividing the inflationary epoch into two distinct periods.

1.2.1 Big Bang puzzles and inflation

Let us now discuss why we need an inflationary period right before the conventional Big Bang epoch. The theory of inflation has the important property of explaining away (at least to some degree) many of the puzzles of the very early universe in a broad class of general models [13, 75, 91].[2]

1) Flatness problem

Today our universe consists of mainly matter, radiation and dark energy. Let us define a critical energy density

$$\rho_c = \frac{3H_0^2}{8\pi G} \ .$$

H_0 is the present day value of the Hubble parameter. We can now define present ratio of the energy densities of different components with respect to the critical energy density:

$$\Omega_M = \frac{\rho_M^0}{\rho_c} \ , \qquad \Omega_R = \frac{\rho_R^0}{\rho_c} \ , \qquad \Omega_\Lambda = \frac{\rho_\Lambda^0}{\rho_c} \ .$$

Matter energy density consists of both baryonic matter and dark matter, so $\Omega_M = \Omega_B + \Omega_{DM}$. From Friedmann equations one can easily see

$$\left(\frac{H}{H_0}\right)^2 = \Omega_\Lambda + \Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_R \left(\frac{a_0}{a}\right)^4 + \Omega_k \left(\frac{a_0}{a}\right)^2 \ ,$$

where

$$\Omega_k = -\frac{k}{a_0^2 H_0^2} \ .$$

[2]For an alternative solution to some of these puzzles see [21]-[27].

Observations indicate that $\Omega_R \sim 10^{-4}$ and $\Omega_M = 0.315^{+0.016}_{-0.018}$, $\Omega_\Lambda = 0.685^{+0.018}_{-0.016}$[4]. Therefore, $\Omega_k \sim 0$ though it is still possible to have a non-zero Ω_k. Now if we define a quantity

$$\Omega(a) = \Omega_\Lambda + \Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_R \left(\frac{a_0}{a}\right)^4 \ ,$$

then today we have $\Omega(a_0) \approx 1$. But for the standard Big Bang cosmology without inflation $\Omega(a) = 1$ is an unstable fixed point and hence to achieve that today $\Omega(a)$ has to be extremely fine tuned in the early universe. It is easy to check that [33]

$$|\Omega(a_{BBN}) - 1| \sim \mathcal{O}\left(10^{-16}\right) \ ,$$
$$|\Omega(a_{GUT}) - 1| \sim \mathcal{O}\left(10^{-55}\right) \ .$$

This fine tuning problem can be avoided by having a period of inflation before the standard Big Bang epoch during which $a(t)$ increases by a very large factor e^N. Even if $|\Omega(a) - 1| \sim \mathcal{O}(1)$ in the beginning of inflation, it can be shown that

$$|\Omega(a_0) - 1| \approx e^{-2N} \left(\frac{a_I H_I}{a_0 H_0}\right)^2 \ ,$$

where a_I and H_I are the scale factor and Hubble parameter at the end of inflation. Therefore, the flatness problem can be avoided if

$$e^N >> \frac{a_I H_I}{a_0 H_0} \ .$$

Let us now make the assumption that both $a(t)$ and H do not change much from the end of inflation to the beginning of the radiation-dominated era. With that assumption one can show that minimum number of e-foldings needed to solve flatness problem is given by [140],

$$N > \ln\left[\frac{(\rho_I)^{1/4}}{0.037h\ \mathrm{eV}}\right] ,$$

where ρ_I is the energy density at the end of inflation and h is the Hubble constant in the units of 100km s^{-1} Mpc^{-1}. Therefore, with $h = 0.68$ and $(\rho_I)^{1/4} = 2 \times 10^{16}$ GeV [4, 7] flatness problem is solved with $N > 62$.

2) Horizon problem

Distance of the last scattering surface today is roughly $2/H_0$. This distance at the time of the last scattering is $d_L \approx 2H_0^{-1}(1 + z_L)^{-1}$, where $z_L \approx 1100$ is the redshift at the time of last scattering. For the standard Big Bang cosmology without inflation, the size of the horizon at the time of last scattering is $d_H \approx (4/3)H_0^{-1}(1 + z_L)^{-3/2}$. The horizon at the time of last scattering subtends an angle $\theta = 2d_H/d_L$ rad $\approx 2^o$. Therefore, at the time of last scattering, the regions that are separated by more than 2^o are causally independent. This is in contradiction with the observation of near-isotropy of the CMB [60, 113, 99, 98, 100] at large angular scales.

This apparent contradiction can be explained by inflation during which $a(t) \sim e^{H_I t}$ increases by a very large factor e^N. Now for the Big Bang cosmology with inflation, the size of the horizon at the time of last scattering is

$$d_H \approx \frac{a_L}{a_I H_I} e^N ,$$

where a_L is the scale factor at the time of last scattering. d_L is still given by $d_L \approx 2H_0^{-1}(a_L/a_0)$. The horizon at the time of last scattering subtends an angle

$$\theta \approx \left(\frac{a_0 H_0}{a_I H_I}\right) e^N \ .$$

Therefore inflation can be solved if

$$N > \ln\left[\frac{(\rho_I)^{1/4}}{0.037h \text{ eV}}\right] \ ,$$

where we have again made the assumption that both $a(t)$ and H do not change much from the end of inflation to the beginning of the radiation-dominated era. Therefore, with $h = 0.68$ and $(\rho_I)^{1/4} = 2 \times 10^{16}$ GeV [4, 7] the horizon problem is solved with $N > 62$.

2) Monopole problem

It is expected that Grand Unification of the electroweak and strong interactions takes place at energy 10^{16} GeV. All unified theories come with magnetic monopoles which creates a problem for the standard Big Bang cosmological models. It is expected that unified theories naturally produce one monopole per horizon at the time of symmetry breaking. These theories typically predict one monopole per 10^9 photons present in the universe today. Today there are roughly 10^9 microwave background photons present per nucleon [140]. So we should have at least one monopole per nucleon. However, observations indicate that there are fewer than 10^{-30} monopoles per nucleon.

Inflation provides a simple explanation. During inflation scale factor increases by a very large factor e^N. Because of that the density of monopoles decreases roughly by a factor of e^{-3N}. Therefore, monopole problem can be solved by having $N > \ln(10^{10}) \approx 23$ e-foldings of inflation.

1.2.2 Cosmological perturbations during inflation

In inflationary cosmological theories the temperature fluctuations of CMB and the large scale structures are directly related to the scalar curvature perturbations $\mathcal{R}$ produced during inflation. Inflation also predicts tensor fluctuations, denoted by h_{ij} that can be directly observed in cosmological gravitational waves. During inflation fluctuations "exit" the horizon (i.e. become larger than the physical size of the temporary event horizon). Outside the horizon, the fluctuations are "frozen-in" and both scalar and tensor perturbations remain at the constant value set by the inflationary epoch.Then, once the expansion begins to decelerate, these fluctuations "re-enter" the horizon in reverse order from their exit. This means that the smallest scale perturbations we observe were produced last, closest to the start of decelerated expansion.

The key success of the inflationary theory is its prediction of an almost scale invariant power spectrum of primordial fluctuations, which is in accord with the observations of the CMB [79, 142, 80, 74, 86, 35, 123, 36, 126, 87, 88, 37, 125, 4] and the LSS [143, 55, 12, 56, 54, 114, 82, 108]. However, the limited information we can extract about this power spectrum means that a wide class of inflationary models are viable. Moreover, the particular nature of that class of models and the constraints upon them are predicated on many assumptions about both the nature of inflation and the evolution history of the universe.

The two-point and three-point functions of primordial fluctuations are generally computed assuming that the fluctuations are initially in the Bunch-Davies state[42, 39]. The Bunch-Davies state is the minimum energy eigenstate of the Hamiltonian in the infinite past and it is a reasonable choice as an initial state but not unique. It is somewhat of a philosophical question whether initial conditions are integral part of a theory or should be analyzed separately. In spite of the great success of the inflationary theory, it is always important to verify the validity of different assumptions. There has been a great deal of work focused on departure from the Bunch-Davies state[89, 70, 50, 83, 102, 101, 59, 16, 11, 53, 67, 47, 72, 10, 73, 14, 65, 71, 15, 17, 41, 19]. In later chapters, we will discuss in detail the effect of relaxing the assumption of the Bunch-Davies state by choosing a general initial state built over the Bunch-Davies state. One aspect of prime interest is to understand how much information about the quantum state of primordial fluctuations in the beginning of inflation can be obtained.Recently, this area of research has instigated interest among physicists mainly because it has the exciting possibility of providing us a window for the physics before inflation.

1.3 Models of inflation

Despite its great success, the details of the physics of inflation are still unknown. Inflation is typically modeled by a (possibly multi-component) scalar field, the essential feature is that the expansion of universe undergoes a temporary accelerating period, where $\ddot{a} > 0$. A large number of phenomenological models of inflation successfully explain all the observations making it practically impossible to distinguish between different models.[3]

1.3.1 Single-field inflation

In these phenomenological models, only a single real scalar field contributes significantly to the energy density during inflation. A large number of phenomenological single field inflation models with different theoretical motivation have been proposed that are consistent with current observations. Interestingly the three-point functions of primordial fluctuations for single-field inflation, in squeezed limits, obey certain consistency relations that does not particularly depend on the details of the single-field model and hence can provide us with an important tool to falsify or establish single-field inflation [97, 58, 52, 57, 89, 120, 121, 18, 92, 115]. These three-point functions [97, 68, 127] are important observables that in principle can be used to differentiate between single-field and multi-field inflation models. Under very general assumptions: (i) it is effectively a single field inflation and (ii) there is no super-horizon evolution of the perturbations (i.e. both scalar and tensor perturbations are frozen outside the horizon), the three-point functions of comoving curvature perturbation ($\mathcal{R}_{\mathbf{k}}$) and tensor perturbation ($h^s_{\mathbf{k}}$), in the

[3]Recently, a lot of progress has been made in figuring out an effective field theory description of inflation [51, 141]. Also there are inflationary models that are derived from string theory [34].

squeezed limit $k_1, k_2 \gg k_3$ are known to obey some consistency relations which are of the form

$$\begin{aligned}
\langle \hat{\mathcal{R}}_{\mathbf{k}_1} \hat{\mathcal{R}}_{\mathbf{k}_2} \hat{\mathcal{R}}_{\mathbf{k}_3} \rangle &= (2\pi)^3 P_{\mathcal{R}}(k_1) P_{\mathcal{R}}(k_3)(n_s - 1)\delta^3(\sum \mathbf{k}) \ , \\
\langle \hat{\mathcal{R}}_{\mathbf{k}_1} \hat{\mathcal{R}}_{\mathbf{k}_2} \hat{h}^s_{\mathbf{k}_3} \rangle &= (2\pi)^3 P_{\mathcal{R}}(k_1) P_h(k_3) \left(2 - \frac{n_s}{2}\right) \frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}\epsilon^s_{ij}(\mathbf{k}_3)}{k_1^2} \delta^3(\sum \mathbf{k}) \ , \\
\langle \hat{h}^s_{\mathbf{k}_1} \hat{h}^{s'}_{\mathbf{k}_2} \hat{\mathcal{R}}_{\mathbf{k}_3} \rangle &= (2\pi)^3 P_h(k_1) P_{\mathcal{R}}(k_3) \ n_t \ \delta_{ss'}\delta^3(\sum \mathbf{k}) \ , \\
\langle \hat{h}^s_{\mathbf{k}_1} \hat{h}^{s'}_{\mathbf{k}_2} \hat{h}^{s''}_{\mathbf{k}_3} \rangle &= (2\pi)^3 P_h(k_1) P_h(k_3)\delta_{ss'} \left(\frac{3 - n_t}{2}\right) \frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}\epsilon^{s''}_{ij}(\mathbf{k}_3)}{k_1^2} \delta^3(\sum \mathbf{k}) \ .
\end{aligned}$$

Where n_s and n_t are the scalar and tensor spactral indices respectively and $\epsilon^s_{ij}(\mathbf{k})$ is the polarization tensor of the tensor modes. The other two squeezed limit three-point functions $\langle \hat{h}^s_{\mathbf{k}_1} \hat{\mathcal{R}}_{\mathbf{k}_2} \hat{h}^{s'}_{\mathbf{k}_3} \rangle$, $\langle \hat{h}^s_{\mathbf{k}_1} \hat{\mathcal{R}}_{\mathbf{k}_2} \hat{\mathcal{R}}_{\mathbf{k}_3} \rangle$ vanish in the limit $k_3/k_1 \to 0$. There has been a great deal of progress in measuring the three-point function (bispectrum) of scalar perturbation from the CMB and LSS indicating nearly gaussian primordial fluctuations. The current observational constraint on the non-Gaussianity parameter f_{NL}[30] is very weak and f^{loc}_{NL} remains the best constrained non-Gaussianity parameter: $f^{loc}_{NL} = 2.7 \pm 5.8$ (Planck)[6].

Single-field slow-roll inflation

Let us now discuss the simplest and most popular single-field inflation model which is so far consistent with all observations. We start with the Lagrangian of gravity and a minimally coupled real scalar field with a canonical kinetic term

$$S = \frac{1}{2} \int \sqrt{-g} d^4x \left[\frac{1}{8\pi G} R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi) \right] .$$

A homogeneous background solution (flat) has the form

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$$

with a background scalar field $\phi(\mathbf{x}, t) = \bar{\phi}(t)$. This background obeys the equations

$$3H^2 = 8\pi G\left[\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi})\right],$$
$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + V'(\bar{\phi}) = 0,$$

where H is the Hubble parameter $H = \dot{a}/a$. For slow-roll inflation $V(\bar{\phi})$ is approximately constant and slow roll parameters $|\epsilon|, |\eta| \ll 1$, where

$$\epsilon = -\frac{\dot{H}}{H^2} \approx \frac{\dot{\bar{\phi}}^2}{2H^2{M_{\rm pl}}^2} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2,$$
$$\eta = \frac{1}{8\pi G}\frac{V''}{V}.$$

One can also check that in the slow-roll limit

$$w = \frac{\frac{1}{2}\dot{\bar{\phi}}^2 - V(\bar{\phi})}{\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi})} \approx -1$$

and hence

$$a(t) \sim e^{Ht}\ .$$

1.3.2 Multi-field inflation

Theoretically there is no reason why inflation can not be driven by more than one scalar field. Physics of multi-field inflation is complicated and generally depends on the details of the model. Several multi-field inflation models have been proposed with different observational predictions. A great deal of work has been done to understand general as well as model specific features of multi-field inflation [137, 103, 116, 134, 69, 32, 38, 139, 46, 45, 31].

The most general multi-field inflation model with Einstein-Hilbert gravity and arbitrary numbers of real scalar fields is given by [140]:

$$S = \frac{1}{2}\int \sqrt{-g} d^4x \left[\frac{1}{8\pi G}R - g^{\mu\nu}\gamma_{nm}(\phi)\partial_\mu\phi^n\partial_\nu\phi^m - 2V(\phi)\right].$$

Where $V(\phi)$ is an arbitrary potential and $\gamma_{nm}(\phi)$ is an arbitrary real symmetric positive definite matrix. Again we will look for flat homogeneous background solution of the form

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$$

with a background scalar fields $\phi^n(\mathbf{x}, t) = \bar{\phi}^n(t)$. Equations of motion are

$$\begin{aligned} 3H^2 &= 8\pi G\left[\frac{1}{2}\gamma_{nm}(\bar{\phi})\dot{\bar{\phi}}^n\dot{\bar{\phi}}^m + V(\bar{\phi})\right], \\ \dot{H} &= -4\pi G\gamma_{nm}(\bar{\phi})\dot{\bar{\phi}}^n\dot{\bar{\phi}}^m \ . \end{aligned}$$

It can be checked easily that the equation of state parameter is given by

$$w = \frac{\frac{1}{2}\gamma_{nm}(\bar{\phi})\dot{\bar{\phi}}^n\dot{\bar{\phi}}^m - V(\bar{\phi})}{\frac{1}{2}\gamma_{nm}(\bar{\phi})\dot{\bar{\phi}}^n\dot{\bar{\phi}}^m + V(\bar{\phi})} \ .$$

Therefore the condition for having an exponential expansion is

$$V(\bar{\phi}) >> \frac{1}{2}\gamma_{nm}(\bar{\phi})\dot{\bar{\phi}}^n\dot{\bar{\phi}}^m \ .$$

We will not discuss multi-field inflation further and from now on we will mainly focus on single-field slow-roll inflation.

1.4 Outline

In chapter 2, we review quantization of the fluctuations for single-field slow-roll inflation. In chapter 3, we explore possible constraints on the inflationary equation state by considering current measurements of the power spectrum. Next, in chapter 4, we explore the possibility of a single field slow-roll inflationary model with general initial state for primordial fluctuations. The two-point and three-point functions of primordial fluctuations are generally computed assuming that the fluctuations are initially in the Bunch-Davies state. However, we show that the constraints on the initial state are relatively weak and for slow-roll inflation a large number of initial states are consistent with the current observations. In chapter 5, we introduce non-Gaussianity matrix $\mathcal{F}$ and then present some semiclassical arguments to reproduce the consistency relations for the three-point functions, in the squeezed limit, of scalar and tensor perturbations in single-field inflation that in principle can be used to differentiate between single-field and multi-field inflation models. In chapter 6, we calculate the three-point functions for slow-roll inflation with the Bunch-Davies state and coherent states to demonstrate that the consistency relations are obeyed. On the other hand, we also show that the consistency relations are violated for α-states which are states that are related to the Bunch-Davies state by Bogoliubov transformations. We end with a discussion on back-reaction of excited initial state and observational implications in chapter 7. Chapter 8 is devoted to concluding remarks. A detailed calculation of scalar three-point function with coherent states is shown in appendix

A. General results of the three-point functions with α-states are relegated to appendix B.

Chapter 2

Quantization of the fluctuations in inflationary universe

From a theoretical point of view, inflation is important because it gives us an opportunity to test predictions of quantum field theory in a curved space-time. Inflation has been very successful in explaining several puzzles of the standard big bang scenario. But the most important success of the inflationary theory, is its prediction of almost scale invariant power spectrum of primordial fluctuations[129, 78, 76, 29, 105]. In the inflationary scenario the temperature fluctuations of cosmic microwave background (CMB) and the large scale structure (LSS) are directly related to the curvature perturbations produced during inflation. Current observations of CMB strongly support the presence of an almost scale-invariant power spectrum.

2.1 Scalar field in FRW universe

First, let us review the quantization of fluctuations in an inflationary universe with a single scalar field. We start with the Lagrangian of gravity and a minimally coupled real scalar field with a canonical kinetic term

$$S = \frac{1}{2}\int \sqrt{-g}d^4x \left[\frac{1}{8\pi G}R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi)\right]. \tag{2.1}$$

A homogeneous background solution has the form

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 \tag{2.2}$$

with a background scalar field $\phi(\mathbf{x},t) = \bar{\phi}(t)$. This background obeys the equations

$$3H^2 = 8\pi G\left[\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi})\right], \tag{2.3}$$

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + V'(\bar{\phi}) = 0, \tag{2.4}$$

where H is the Hubble parameter $H = \dot{a}/a$. For slow-roll inflation $V(\bar{\phi})$ is approximately constant and slow roll parameters $|\epsilon|, |\eta| \ll 1$, where

$$\epsilon = -\frac{\dot{H}}{H^2} \approx \frac{\dot{\bar{\phi}}^2}{2H^2{M_{\rm pl}}^2} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2, \tag{2.5}$$

$$\eta = \frac{1}{8\pi G}\frac{V''}{V}. \tag{2.6}$$

Next we consider perturbations around the homogeneous background solutions

$$\phi(\mathbf{x},t) = \bar{\phi}(t) + \delta\phi(\mathbf{x},t) \tag{2.7}$$

and the perturbed metric with scalar and tensor perturbations is given by

$$ds^2 = -(1+2\Phi)dt^2 + 2a(t)(\partial_i B)dx^i dt + a^2(t)[(1-2\Psi)\delta_{ij} + 2\partial_{ij}E + h_{ij}]dx^i dx^j. \tag{2.8}$$

Where h_{ij} is purely tensor perturbation and satisfies following conditions:

$$h_{ij} = h_{ji}, \qquad h_{ii} = 0, \qquad \partial_i h_{ij} = 0. \tag{2.9}$$

Tensor perturbations are gauge-invariant at linear order but scalar perturbations are not. We can avoid fictitious gauge modes of scalar perturbation by introducing gauge-invariant variables[28, 95]. One such variable is the comoving curvature perturbation

$$\mathcal{R} = \Psi + \frac{H}{\dot{\bar{\phi}}}\delta\phi. \tag{2.10}$$

Expanding the action (2.1), we get the gauge-invariant second order actions for scalar and tensor perturbations (with conformal time τ defined in the usual way)

$$S_2^{(s)} = \frac{1}{2}\int d\tau d^3x a^2 \frac{\dot{\bar{\phi}}^2}{H^2}\left[\mathcal{R}'^2 - (\partial_i \mathcal{R})^2\right] \ , \tag{2.11}$$

$$S_2^{(t)} = \frac{{M_{\rm pl}}^2}{8}\int d\tau d^3x a^2 \left[h'^2_{ij} - (\partial_l h_{ij})^2\right] \ . \tag{2.12}$$

Where, Planck mass $M_{\rm pl} = (8\pi G)^{-1/2}$ and $(...)' = \partial_\tau(...)$. We can define the Fourier transforms of the fields in the standard way,

$$\mathcal{R}(\mathbf{x},\tau) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\mathcal{R}_{\mathbf{k}}(\tau)e^{i\mathbf{k}.\mathbf{x}} \ , \tag{2.13}$$

$$h_{ij}(\mathbf{x},\tau) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{s=+,\times}\epsilon^s_{ij}(\mathbf{k})h^s_{\mathbf{k}}(\tau)e^{i\mathbf{k}.\mathbf{x}}, \tag{2.14}$$

where, $\epsilon^s_{ij}(\mathbf{k})$ is a real tensor (polarization tensor)[1] and it obeys $\epsilon^s_{ii}(\mathbf{k}) = \mathbf{k}^i\epsilon^s_{ij}(\mathbf{k}) = 0$ and $\epsilon^s_{ij}(\mathbf{k})\epsilon^{s'}_{ij}(\mathbf{k}) = 2\delta_{ss'}$. Because the fields $\mathcal{R}(\mathbf{x},\tau)$ and $h_{ij}(\mathbf{x},\tau)$ are real, we have the conditions:

$$\mathcal{R}^*_{\mathbf{k}}(\tau) = \mathcal{R}_{-\mathbf{k}}(\tau) \ , \qquad h^{*s}_{\mathbf{k}}(\tau) = h^s_{-\mathbf{k}}(\tau) \ , \qquad \epsilon^s_{ij}(\mathbf{k}) = \epsilon^s_{ij}(-\mathbf{k}) \ . \tag{2.15}$$

In terms of canonically normalized fields

$$v^0_{\mathbf{k}}(\tau) \equiv \frac{a(\tau)\dot{\bar{\phi}}}{H}\mathcal{R}_{\mathbf{k}}(\tau) \ , \qquad v^s_{\mathbf{k}}(\tau) \equiv \frac{a(\tau)}{\sqrt{2}}M_{\rm pl}h^s_{\mathbf{k}}(\tau) \tag{2.16}$$

[1]Note that $\epsilon^s_{ij}(\mathbf{k})$ depends only on the unit vector $\hat{\mathbf{k}}$.

the action $S_2 \equiv S_2^{(s)} + S_2^{(t)}$ is

$$S_2 = \frac{1}{2}\frac{1}{(2\pi)^3}\int d\tau d^3\mathbf{k}\left[v'^0_{\mathbf{k}}(\tau)v^{0*\prime}_{\mathbf{k}}(\tau) - k^2 v^0_{\mathbf{k}}(\tau)v^{0*}_{\mathbf{k}}(\tau) + \frac{z''}{z}v^0_{\mathbf{k}}(\tau)v^{0*}_{\mathbf{k}}(\tau)\right]$$
$$+ \sum_{s=+,\times}\frac{1}{2}\frac{1}{(2\pi)^3}\int d\tau d^3\mathbf{k}\left[v'^s_{\mathbf{k}}(\tau)v^{s*\prime}_{\mathbf{k}}(\tau) - k^2 v^s_{\mathbf{k}}(\tau)v^{s*}_{\mathbf{k}}(\tau) + \frac{a''}{a}v^s_{\mathbf{k}}(\tau)v^{s*}_{\mathbf{k}}(\tau)\right] \tag{2.17}$$

where $z = \frac{a(\tau)\dot{\phi}}{H}$. In the lowest order in the slow-roll expansion the last equation becomes

$$S_2 = \sum_{s=0,+,\times}\frac{1}{2}\frac{1}{(2\pi)^3}\int d\tau d^3\mathbf{k}\left[v'^s_{\mathbf{k}}(\tau)v^{s*\prime}_{\mathbf{k}}(\tau) - k^2 v^s_{\mathbf{k}}(\tau)v^{s*}_{\mathbf{k}}(\tau) + \frac{a''}{a}v^s_{\mathbf{k}}(\tau)v^{s*}_{\mathbf{k}}(\tau)\right]. \tag{2.18}$$

Note that the sum in the last equation is over $s = 0, +, \times$, where $s = 0$ corresponds to the scalar perturbations and $s = +, \times$ correspond to two polarization modes of the tensor perturbations. From this action we get the following equation for $v^s_{\mathbf{k}}$

$$v^{s\prime\prime}_{\mathbf{k}}(\tau) + \omega_k^2(\tau)v^s_{\mathbf{k}}(\tau) = 0 \qquad s = 0, +, \times \tag{2.19}$$

with $\omega_k^2(\tau) = k^2 - (a''/a)$. Let $u_k(\tau)$ and $u_k^*(\tau)$ be linearly independent complex solutions of equations of motion (3.1). Wronskian $W[u_k, u_k^*] = 2iIm[u_k'(\tau)u_k^*(\tau)] \neq 0$ and it is time-independent; so we can always normalize the mode function $u_k(\tau)$ by the condition

$$Im[u_k'(\tau)u_k^*(\tau)] = 1. \tag{2.20}$$

The general solution of equation (3.1) can be written as

$$v^s_{\mathbf{k}}(\tau) = \frac{1}{\sqrt{2}}\left[a^{s-}_{\mathbf{k}}u_k^*(\tau) + a^{s+}_{-\mathbf{k}}u_k(\tau)\right], \tag{2.21}$$

where $a^{s-}_{\mathbf{k}}$ and $a^{s+}_{-\mathbf{k}}$ are independent of τ and $a^{s+}_{\mathbf{k}} = (a^{s-}_{\mathbf{k}})^*$.

2.2 Quantization of fluctuations

We will work in the Heisenberg picture to quantize fields $v^s(\mathbf{x},\tau)$, (where $s = 0, +, \times$). We introduce the commutation relations

$$\left[\hat{v}^s(\mathbf{x},\tau), \hat{\pi}^{s'}(\mathbf{y},\tau)\right] = i\delta^3(\mathbf{x}-\mathbf{y})\delta_{ss'}, \tag{2.22}$$

where $\hat{\pi}^s = \hat{v}^{s\prime}$ is the canonical momentum. Now the equation (2.21) becomes

$$\hat{v}^s_{\mathbf{k}}(\tau) = \frac{1}{\sqrt{2}}\left[\hat{a}^s_{\mathbf{k}} u^*_k(\tau) + \hat{a}^{s\dagger}_{-\mathbf{k}} u_k(\tau)\right] \qquad \text{for} \qquad s = 0, +, \times. \tag{2.23}$$

The commutation relations (2.22) lead to commutation relations between $\hat{a}^{s\dagger}_{\mathbf{k}}$ and $\hat{a}^s_{\mathbf{k}}$

$$\left[\hat{a}^s_{\mathbf{k}_1}, \hat{a}^{s'\dagger}_{\mathbf{k}_2}\right] = (2\pi)^3\delta^3(\mathbf{k}_1 - \mathbf{k}_2)\delta_{ss'}, \qquad \left[\hat{a}^{s'\dagger}_{\mathbf{k}_1}, \hat{a}^{s\dagger}_{\mathbf{k}_2}\right] = \left[\hat{a}^{s'}_{\mathbf{k}_1}, \hat{a}^s_{\mathbf{k}_2}\right] = 0. \tag{2.24}$$

The Hamiltonian of the system of fluctuations is

$$\hat{H}(\tau) = \sum_{s=0,+,\times} \int \frac{d^3\mathbf{k}}{4(2\pi)^3} \left[\hat{a}^s_{\mathbf{k}} \hat{a}^s_{-\mathbf{k}} F^*_k(\tau) + \hat{a}^{s\dagger}_{\mathbf{k}} \hat{a}^{s\dagger}_{-\mathbf{k}} F_k(\tau) \right.$$
$$\left. + \left(\hat{a}^s_{\mathbf{k}} \hat{a}^{s\dagger}_{\mathbf{k}} + \hat{a}^{s\dagger}_{\mathbf{k}} \hat{a}^s_{\mathbf{k}}\right) E_k(\tau)\right] \tag{2.25}$$

where,

$$F_k(\tau) = (u'_k)^2 + \omega_k^2 u_k^2, \qquad E_k(\tau) = |u'_k|^2 + \omega_k^2 |u_k^2|. \tag{2.26}$$

Note that the sum in (2.25) is over $s = 0, +, \times$ and hence it contains both scalar and tensor fluctuations.

2.3 Bunch-Davies vacuum

Next we will define a "vacuum" state and find out the mode-function that describes the state. The Hamiltonian explicitly depends on the conformal time τ, making it impossible to define a vacuum in a time-independent way. We can define a vacuum by the standard condition: for all $\mathbf{k}$

$$\hat{a}^s_{\mathbf{k}}|0\rangle = 0 \qquad \text{for} \qquad s = 0, +, \times \ . \tag{2.27}$$

But this is not sufficient to specify the mode-function. There is no time-independent eigenstate of the Hamiltonian, so we take a particular moment $\tau = \tau_0$, and define vacuum as the lowest-energy eigenstate of the instantaneous Hamiltonian of the fluctuations at $\tau = \tau_0$ (we can always do that as long as $\omega_k^2(\tau_0) \geq 0$). That gives us the following initial conditions for the mode function

$$u'_k(\tau_0) = \pm i\sqrt{\omega_k(\tau_0)}e^{i\lambda(k)}, \qquad u_k(\tau_0) = \pm\frac{1}{\sqrt{\omega_k(\tau_0)}}e^{i\lambda(k)}, \tag{2.28}$$

where $\lambda(k)$ is some arbitrary time independent function of k. In the limit when τ_0 represents infinite past (i.e. $\tau_0 \to -\infty$), this vacuum is called the Bunch-Davies vacuum state. In this limit, $\omega_k^2 = k^2 \geq 0$ and we can define vacuum by equation(2.27) for all modes.

2.4 Power-spectrum for slow-roll inflation

For slow-roll inflation, $V(\bar{\phi})$ is approximately constant and the slow roll parameters $|\epsilon|, |\eta| \ll 1$. Therefore, the equation of state parameter $w \approx -1$

and $(a''/a) = (2/\tau^2)$. Solving equation (3.1) with normalization condition (2.20) and initial conditions (2.28) (with the + sign and $\tau_0 \to -\infty$ limit), we get

$$u_k(\tau) = \frac{e^{ik\tau}}{\sqrt{k}}\left(1 + \frac{i}{k\tau}\right). \tag{2.29}$$

With this mode function we can now compute power-spectrums of scalar and tensor perturbations for slow-roll inflation.

2.4.1 Scalar power-spectrum

Let us first compute the following quantity

$$\langle \hat{v}^s_{\mathbf{k}}(\tau)\hat{v}^{s'}_{\mathbf{k}'}(\tau)\rangle \equiv \langle 0|\hat{v}^s_{\mathbf{k}}(\tau)\hat{v}^{s'}_{\mathbf{k}'}(\tau)|0\rangle = \frac{1}{2}(2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')\delta_{ss'}|u_k(\tau)|^2, \tag{2.30}$$

where we got the last equation using (2.23). Before we proceed let us introduce some standard quantities

$$\langle \hat{\mathcal{R}}_{\mathbf{k}}(\tau)\hat{\mathcal{R}}_{\mathbf{k}'}(\tau)\rangle = (2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')P_{\mathcal{R}}, \quad \Delta^2_{\mathcal{R}} = \frac{k^3}{2\pi^2}P_{\mathcal{R}}, \quad n_s - 1 = \frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln k}, \tag{2.31}$$

where n_s is called the scalar spectral index or tilt. Using equations (2.29-2.31), in the superhorizon limit ($|k\tau| \ll 1$), we obtain [97]

$$\Delta^2_{\mathcal{R}} = \frac{H^4}{4\pi^2\ \dot{\bar{\phi}}^2}\ , \qquad n_s = 1 - 6\epsilon + 2\eta\ . \tag{2.32}$$

Where H is the Hubble parameter during inflation. Therefore slow-roll inflation predicts an almost scale-invariant scalar power spectrum (i.e. $n_s \approx 1$) which agrees with observation of the CMB and LSS.

2.4.2 Tensor power-spectrum

The power spectrum of two polarizations of h_{ij} is defined as

$$\langle \hat{h}^{s}_{\mathbf{k}}(\tau)\hat{h}^{s'}_{\mathbf{k}'}(\tau)\rangle = (2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')\delta_{ss'}P_h, \qquad \Delta_h^2 = \frac{k^3}{2\pi^2}P_h. \tag{2.33}$$

The power spectrum for tensor perturbations is defined as the sum of the power spectrum for the two polarizations

$$\Delta_t^2 \equiv 2\Delta_h^2. \tag{2.34}$$

Spectral index n_t for the tensor perturbations is defined in the following way

$$n_t = \frac{d\ln\Delta_t^2}{d\ln k}. \tag{2.35}$$

Finally, in the superhorizon limit ($|k\tau| \ll 1$), we get [97]

$$\Delta_t^2 = \frac{H^2}{\pi^2 \, {M_{\rm pl}}^2} \ , \qquad n_t = -2\epsilon \ . \tag{2.36}$$

Therefore, slow-roll inflation also predicts nearly scale-invariant tensor power-spectrum. Amplitude of tensor power-spectrum is rather small and the tensor-to-scalar ratio is given by

$$r \equiv \frac{\langle \hat{h}_{ij}(\mathbf{x})\hat{h}_{ij}(\mathbf{x})\rangle}{\langle \hat{\mathcal{R}}(\mathbf{x})\hat{\mathcal{R}}(\mathbf{x})\rangle} = \frac{4\Delta_h^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon \ . \tag{2.37}$$

Current bound on tensor-to-scalar ratio from Planck+WP is $r < 0.12$[5]. However, BICEP2 indicates that $r \approx 0.20$ [7]. A conclusive detection of primordial gravitational waves will provide an important test for single field slow-roll inflation (with canonical kinetic term) because there is a consistency relation between n_t and r

$$r = -8n_t \ . \tag{2.38}$$

Chapter 3

Constraining the inflationary equation of state

The key success of the inflationary theory is its prediction of an almost scale invariant power spectrum of primordial fluctuations, a prediction that is borne out by what we measure of the perturbations in the cosmological fluid today. However, the limited information we can extract about this power spectrum means that a wide class of inflationary models are viable. Moreover, the particular nature of that class of models and the constraints upon them are predicated on many assumptions about both the nature of inflation and the evolution history of the universe.

Whilst inflation is typically modeled by a scalar field, the essential feature is that the expansion of universe undergoes a temporary accelerating period, where $\ddot{a} > 0$. During accelerated expansion fluctuations "exit" the horizon (i.e. become larger than the physical size of the temporary event horizon). Then, once the expansion begins to decelerate, these fluctuations "re-enter" the horizon in reverse order from their exit. This means that the smallest scale perturbations we observe were produced last, closest to the start of decelerated expansion.

The top graphic in figure 3.1 schematically illustrates this, with a log-

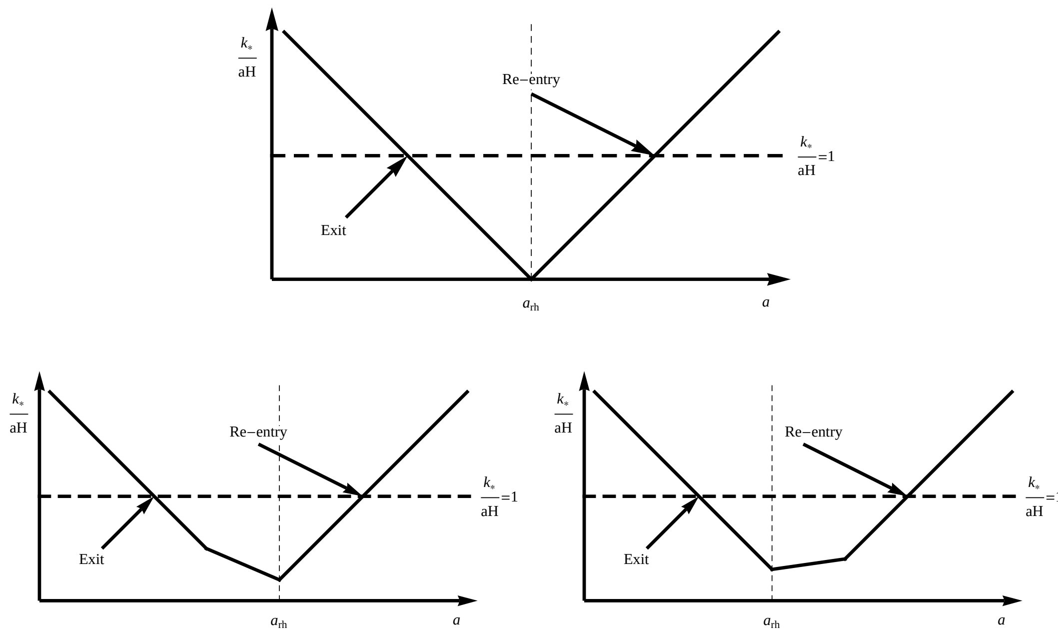

Figure 3.1: Schematic evolution of the quantity k/aH for a particular scale, k_*. Changing the equation of state before(after) the end of inflation changes the negative(positive) part of the slope

arithmic plot of the quantity k_*/aH. This quantity characterizes the size of the comoving horizon relative to a particular scale, k_*, with $k_*/aH = 1$ corresponding to the scale exiting (or re-entering) the horizon. The slope of the line is related to the dynamics (or, in the language we will be using presently, the equation of state) of the cosmological fluid – when the slope is negative(positive), the expansion is accelerating(decelerating).

In this chapter we eschew the aforementioned scalar field paradigm, and rather consider the equation of state, bounding, as best we can, the equation of state parameter w, defined by $p = w\rho$. The Friedman equations (along with the null energy condition) mean that a positive $\ddot{a}$ implies that $-1 < w < -1/3$. Changing the equation state changes the tilt of of the slope of the line in figure 3.1. We are primarily interested in changes to the negative part of the slope (i.e. the accelerated expansion) that are consistant the current observational constraints. One could also consider bounds on the equation of state during decelerated expansion, which has been done by, for example, [40] and [8]. The two possibilities are schematically shown respectively on the left and right of the lower half of figure 3.1. As we shall see, the broad cause of the uncertainty in the inflationary equation of state (and any uncertainity in the equation of state immediately after inflation) it that we only have a direct probe of the primordial spectrum for a subset of the e-foldings that correspond to our current observable universe.

We explore possible constraints on w outside of those directly imposed by the power spectrum. Said power spectrum is almost scale-invariant from the size of the observable universe down to scales of around a Mpc; as the authors show in [84], this places tight bounds on the deviation of w from -1. However, as we will describe below, w is less tightly constrained for those periods of accelerated expansion corresponding to shorter length scales.

We explore the possibility of a significantly varying w before the start of decelerating expansion by dividing the inflationary epoch into two distinct periods. The first, which for clarity we will refer to as "inflation", has $w \sim -1$ and is responsible for producing the nearly scale-invariant power spectrum we see on large scales. The second, from now on we call this "accelerated

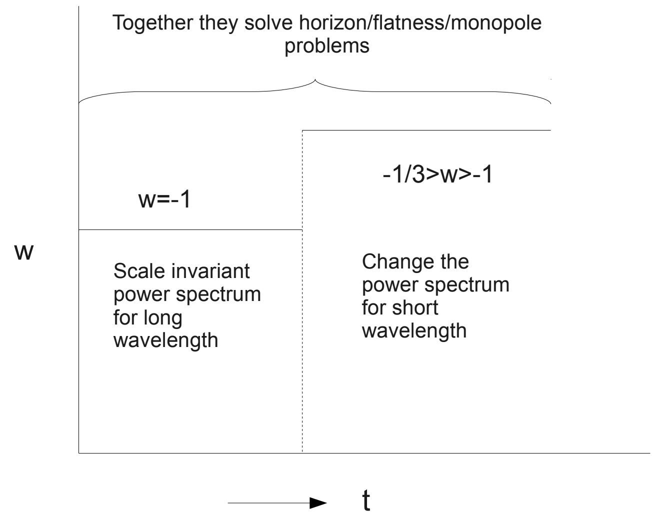

Figure 3.2: We divide the inflationary epoch into two distinct periods: (i) "inflation" with $w \sim -1$ and (ii) "accelerated expansion" with $-1 < w < -1/3$.

expansion", lasts as long as is necessary to be consistent with particular choices of the reheat scale (where deceleration begins at reheating) and the inflation-acceleration transition scale. (see figure 3.2.)

3.1 Fluctuations During Accelerated Expansion

3.1.1 Not Power-Law Inflation

Before marching onwards with our calculations, we should note that periods of expansion with $w \neq -1$ (but $< -1/3$) give rise to what is more commonly known as power-law inflation, where the scale factor grows like t^p ($p > 1$ is a function of w – the detailed solution can be found below). This is a well studied situation [1, 93, 94] in inflationary cosmology; it arises, for example, when we have a single scalar field with an exponential potential. We should note, however, that our setup is different in two important ways. Firstly, we are positing accelerating expansion in addition to inflation – i.e. it is inflation that provides the nearly scale-invariant power spectrum we need to explain CMB and matter power spectrum data. All we allow our period of accelerating expansion to do is to provide additional e-foldings as needed to solve the horizon problem and to change (possibly) the power spectrum at the highest wavenumbers.

Furthermore, our treatment of the quantum vacuum for the modes produced during accelerated expansion is different. Rather than assuming a Bunch-Davies vacuum state [42, 39], we instead take the modes during inflation as an initial condition on which to match those produced during accelerated expansion – this is equivalent to assuming a sudden change in the equation of state from $w \sim -1$ to $-1 < w < -1/3$ (see figure 3.2). For obvious reasons (i.e. we match only the shortest wavelength modes) this gives the same answer as assuming that the Bunch-Davies vacuum is the appropriate one to use for power law accelerating expansion. In the case of power-law inflation for sufficiently large powers (w sufficiently close to -1) one is close enough to de Sitter space such that the de Sitter invariant α-vacua give an appropriate family of privileged states from which the Bunch-Davies vacuum is chosen. For larger values of w (smaller powers), while one can construct the Bunch-Davies state, it no longer belongs to such a privileged invariant set, and thus is no longer an obvious choice for an appropriate vacuum.

Caveats and qualifications out the way, let us now move on to calculating the fluctuation spectrum.

3.1.2 Mode Functions

We are interested in the scalar fluctuations produced during an epoch of accelerated expansion (with the equation of state is given by $p = w\rho$ and $-1 < w < -1/3$). Following standard treatments we obtain the differential equation for the mode function

$$\frac{d^2 v_k}{d\tau^2} + \left(k^2 - \frac{1}{z}\frac{d^2 z}{d\tau^2}\right) v_k = 0 \ , \tag{3.1}$$

where τ is the conformal time and v is the Mukhanov variable which in terms of the gauge invariant comoving curvature perturbation $\mathcal{R}$ is given by

$$v \equiv z\mathcal{R} \ , \tag{3.2}$$

with z is defined in terms of the background scalar field as:

$$z = \frac{a\frac{d\phi}{dt}}{H} = \frac{a\dot{\phi}}{H} \ . \tag{3.3}$$

In terms of the equation of state parameter, $z^2 = \frac{3}{8\pi G}a^2(1+w)$.

Solving this equation allows us to find the mode functions during both inflation and our second period of accelerated expansion. This, in turn, will let us find an expression for the spectral index. We summarize the results below.

Solutions

During inflation, with $w = w_{\rm inf} \sim -1$ and $a = a_f e^{H_f(t-t_f)}$, the mode function is given by (in the Bunch-Davies vacuum):

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) \ . \tag{3.4}$$

From equation 3.2, this give $\mathcal{R}_k$ as:

$$\mathcal{R}_k = \sqrt{\frac{8\pi G}{3(1+w_{\rm inf})}}\frac{1}{a}\frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) \tag{3.5}$$

During an epoch of accelerated expansion (matching the scale factor and energy density to the prior inflationary period) the scale factor evolves as:

$$a = a_f\left[\frac{3}{2}H_f(1+w)(t-t_f) + 1\right]^{\frac{2}{3(1+w)}} \ , \tag{3.6}$$

with H given by:

$$H = H_f\left(\frac{a_f}{a}\right)^{\frac{3}{2}(1+w)} \ , \tag{3.7}$$

and τ defined in terms of a by:

$$\tau = \frac{1}{(3w+1)a_f H_f}\left[2\left(\frac{a}{a_f}\right)^{\frac{3w+1}{2}} - 3(w+1)\right] \tag{3.8}$$

This allows us to write equation 3.1 as:

$$\frac{d^2 v_k}{d\tau^2} + \left[k^2 - \frac{1-3w}{2\tau_f^2\left(1 - \frac{3w+1}{2}\frac{\tau-\tau_f}{\tau_f}\right)^2}\right]v_k = 0 \ , \tag{3.9}$$

where $\tau_f = -\frac{1}{a_f H_f}$. The general form of the solution is given by a linear combination of Hankel functions:

$$v_k = A(k)\sqrt{x}H_n^{(1)}\left[\frac{kx}{Q}\right] + B(k)\sqrt{x}H_n^{(2)}\left[\frac{kx}{Q}\right] \ . \tag{3.10}$$

With

$$Q = -\frac{3w+1}{2} > 0, \qquad n = \frac{3(1-w)}{4Q}, \qquad x = Q(\tau - \tau_f) + \tau_f \ . \tag{3.11}$$

Then $\mathcal{R}$ during this period is given by:

$$\mathcal{R}_k = \sqrt{\frac{8\pi G}{3(1+w)}}\frac{1}{a}\left[A(k)\sqrt{x}H_n^{(1)}\left[\frac{kx}{Q}\right] + B(k)\sqrt{x}H_n^{(2)}\left[\frac{kx}{Q}\right]\right] \tag{3.12}$$

The functions $A(k)$ and $B(k)$ can be found by matching this solution with that given in equation 3.5 – this, of course, assumes an instantaneous transition from inflation to a second period of acceleration. Doing this for all values of k is non-trivial, in particular for k corresponding to the horizon size, only numerical evaluation of $A(k)$ and $B(k)$ is possible. However, for modes asymptotic in $|k\tau_f|$ it is relatively straightforward to find the curvature perturbation. Far outside the horizon, $|k\tau_f| << 1$, the curvature perturbation is "frozen-in" and curvature perturbation remains at the constant value set by the inflationary epoch. Deep inside the horizon, $|k\tau_f| >> 1$ matching solutions with equation 3.5, we obtain (for $\tau > \tau_f$):

$$\begin{aligned}\mathcal{R}_k =& \frac{\pi}{2a}\sqrt{\frac{8G}{3Q(1+w_{\rm inf})}}\frac{\left(1-\frac{i}{k\tau_f}\right)}{\left(1-\frac{i}{2}\frac{1+\frac{1}{Q}}{k\tau_f}\right)} \\ & \exp\left[-i\left(n\pi/2+\pi/4-\frac{3k\tau_f}{2}\frac{1+w}{Q}\right)\right]\ \sqrt{x}\ H_n^{(2)}\left[\frac{kx}{Q}\right] \ . \end{aligned} \tag{3.13}$$

While it is relatively easy to extract observables (in particular the value of n_s) from the above expression (we do so below), we should note that if a secondary period on inflation produced visible consequences, the largest scales at which such effects could be observed do not correspond to either of the two limits discussed above. Rather, since inflationary perturbations would be responsible for large scale perturbations that we already measure, from the size of the observable universe down to scales less than $\sim$ 1Mpc, the largest scales from a secondary period of accelerated expansion would be those produced right after the end of inflation, when $k\tau_f \sim 1$. At these scales, we have to match our solutions numerically. We have carried out numerical investigation of the matching for $k\tau_f \sim 1$, and find that the form of v_k approaches the asymptotic results within a few percent for $0.5 \lesssim k\tau_f \lesssim 2$. Accordingly, we concentrate on the form of the mode function given in equation 3.13 in order to obtain an analytic expression for the primordial power spectrum of a secondary period of acceleration, though we note that a more precise numerical calculation would be needed in the event of an actual measurement.

The Spectral Index n_s

In order to construct a physical observable, we calculate the scalar spectral index. To do this we take the superhorizon limit, $|kx/Q| << 1$,[1] and use the asymptotic behavior of the Hankel function for $|y| << 1$ (this result, of course, only applies to modes which also have $|k\tau_f| >> 1$ – i.e. modes which

[1]One can check this is indeed the superhorizon limit by finding expressions for both the physical wavenumber, k/a, and the event horizon, $a\int_a^\infty \frac{1}{a'^2 H(a')}\mathrm{d}a'$, in terms of the quantities Q and x defined in equation 3.11.

were well inside the horizon at the end of inflation):

$$H_n^{(1,2)}[y] \to \mp\frac{i\Gamma[n]}{\pi}\left(\frac{y}{2}\right)^{-n} \quad . \tag{3.14}$$

From this we obtain:

$$\mathcal{R}_k \to i\Gamma[n]\sqrt{\frac{8G}{3(1+w_{\rm inf})}\frac{Q^{n+\frac{1}{2}}}{1+Q}}\left(k^{-n}a^{-1}x^{(\frac{1}{2}-n)}\right)$$
$$\exp\left[-i\left(n\pi/2+\pi/4-\frac{3k\tau}{2}\frac{1+w}{Q}\right)\right] \quad . \tag{3.15}$$

Therefore, $|\mathcal{R}_k|^2 \propto k^{-2n}$, and the scalar spectral index, n_s (defined through $|\mathcal{R}_k|^2 \propto k^{-4+n_s}$), is given by:

$$n_s - 1 = 3 - 2n = \frac{6(1+w)}{(1+3w)} \quad . \tag{3.16}$$

Clearly, for $-1 < w < -1/3$, we have $1 > n_s > -\infty$. Having found the relationship between w and n_s our next goal is to see how constraints can be brought to bear on these quantities. We do this by considering the thermal history of the universe.

3.2 Constraints From Thermal History

As discussed above, fluctuations produced during a period of accelerated expansion exit the horizon, and when they do so the associated curvature perturbation does not evolve (assuming that a single component fluid is driving the acceleration), until it re-enters the horizon at some later time, when the expansion is decelerating. Since our posited period of additional accelerated expansion takes place after inflation, it will affect the power spectrum at shorter scales. By considering the evolution history of the universe and the various bounds observations of the power spectrum place on said history, we can constrain some of the parameters of any additional period of accelerated expansion.

We assume a sequence of epochs after inflation, with a variety of different equations of state: during inflation we have $w = -1$, during our accelerated period of expansion we have $-1 < w_a < -1/3$, then from $t = t_{rh}$ (the time of reheating) the universe evolves in the conventional manner as an appropriate mixture of radiation, matter and Λ (this, of course, assumes an instantaneous reheat). Solving the Friedmann equations in the various different epochs and matching the scale factor and energy density across boundaries gives the solutions shown in table 3.1 (for clarity we've glossed over the details of scale factor evolution after reheating, but this will be taken into account below when finding constraints).

t	w	Solutions
$t < t_f$	~ -1	$a = a_f e^{H_f\left(t-t_f\right)}$ $\rho = \rho_f$ $H = \sqrt{\frac{8\pi G}{3}\rho_f}$
$t_f < t < t_{rh}$	$-1 < w_a < -1/3$	$a = a_f\left[\frac{3}{2}H_f\left(1+w_a\right)\left(t-t_f\right)+1\right]^{\frac{2}{3(1+w_a)}}$ $\rho = \rho_f\left(\frac{a_f}{a}\right)^{3(1+w_a)}$ $H = H_f\left(\frac{a_f}{a}\right)^{\frac{3}{2}(1+w_a)}$
$t > t_{rh}$	$1/3$	$a = a_{rh}\left[2H_{rh}\left(t-t_{rh}\right)+1\right]^{\frac{1}{2}}$ $\rho = \rho_{rh}\left(\frac{a_{rh}}{a}\right)^4$ $H = H_{rh}\left(\frac{a_{rh}}{a}\right)^2$

Table 3.1: Cosmological solutions for epochs of interest.

We can use the above history to give us a bound on the combined number of e-foldings from inflation and our accelerated period of expansion. Noting that during radiation-domination and other decelerating periods of expansion the particle horizon grows faster than the universe (and thus faster than the perturbations in the cosmological fluid), we see that in the far past modes that are currently inside the horizon were far outside it. This is the horizon problem, and it is solved by having a period of expansion where the horizon grows slower than the universe, which lasts long enough to ensure that scales within horizon today were also within the horizon in the early universe. The number of e-foldings that are required to solve the horizon problem is fixed by the amount of horizon growth relative to the size of the universe since deceleration began [140], or:

$$N_0 = \ln\left[\frac{a_{rh}H_{rh}}{a_0 H_0}\right] = \ln\left[\frac{\rho_{rh}^{1/4}}{0.037h\ \text{eV}}\right] \ . \tag{3.17}$$

h is the Hubble constant in units of 100km s^{-1} Mpc^{-1}. Furthermore, the number of e-foldings that took place between the time at which the mode corresponding to the size of the horizon today previously held that honor, and the time of reheating is given by:

$$N_{obs} = N_0 + \ln\left[\frac{H[t_{q0}]}{H[t_{rh}]}\right] \ . \tag{3.18}$$

t_{q0} is the time at which the mode corresponding to the size of the horizon today was produced. Since for our model this mode was produced during inflation, and since H is approximately constant during the inflationary epoch, we can replace t_{q_0} with t_f:

$$\ln\left[\frac{H[t_f]}{H[t_{rh}]}\right]_{max} = \frac{3}{2}\,(1+w_a)\ln\left[\frac{a_{rh}}{a_f}\right] = \frac{3}{2}\,(1+w_a)\,N_a\ . \tag{3.19}$$

N_a is the number of e-foldings from accelerated expansion. Thus we have:

$$N_{obs} = N_0 + \frac{3}{2}\,(1+w_a)\,N_a\ . \tag{3.20}$$

Incidentally, since $N_{obs} \geq N_0$ and since the observed scale-invariant spectrum of perturbations is at scales from the current size of the observable universe down, requiring that the primordial power spectrum (at large scales) is explained by inflation guarantees that the horizon problem will be solved. It's clear from equation 3.17 that the minimum required number of e-foldings is fewer for lower scale reheating. We cannot, however, push this scale arbitrarily low – we know, for example that the universe was radiation dominated at the time of big bang nucleosynthesis, which corresponds to a scale of around 1 MeV (in [77] Hannestad finds $T_{rh} > 4$ MeV or > 1 MeV if reheating direct to neutrinos). We thus work with a conservative lower bound on the reheat scale of $\rho_{rh}^{1/4} \sim 10$ MeV or – this would give $\gtrsim 19$ visible e-foldings (the precise bound depends also on the value of N_i). This analysis, and that which follows below, assumes instantaneous reheating – if this were not the case the true value of N_a would be somewhat smaller, with the rest of parameters adjusted accordingly. One might complain that baryogenesis places tighter bounds on T_{rh}; while this is true for most models of baryogenesis, it can be avoided in the Affleck-Dine scenario [9].

In our scenario the e-foldings within the observable universe are divided between inflation and accelerated expansion. We can place a lower limit on the portion of visible e-foldings coming from inflation by considering the form of the power spectrum, in particular the size of the deviation from scale invariance. To do this we use a recent reconstruction of the power spectrum from Peiris and Verde [112]. The authors reconstruct $n_s(q)$ for $0.0001 \leq q$ [h/Mpc] ≤ 3, finding that $0.7 \lesssim n_s(q) \lesssim 1.3$ – the values of this bound depend on the details of the reconstruction, and it is somewhat tighter for q away from the upper and lower limits. Let us define q_{max} as the physical wavenumber corresponding to the smallest scale perturbation produced by inflation and q_0 as the physical wavenumber corresponding to the horizon today, then the number of visible e-foldings produced by inflation is given by:

$$N_i = \ln\left[\frac{q_{max}}{q_0}\right] \ . \tag{3.21}$$

Then from equation 3.20 and noting $N_{obs} = N_a + N_i$, we have:

$$\begin{aligned} N_a &= -\frac{2}{3w_a+1}\left(N_0 - N_i\right) \\ &= -\frac{2}{3w_a+1}\left(\ln\left[\frac{a_{rh}H_{rh}}{a_0 H_0}\right] - \ln\left[\frac{q_{max}}{q_0}\right]\right) \\ &= -\frac{2}{3w_a+1}\ln\left[\frac{\rho_{rh}^{1/4}}{0.037h \text{ eV}}\frac{q_0}{q_{max}}\right] \ . \end{aligned} \tag{3.22}$$

N_a can also be expressed in terms of the ratio of the scale factors at the

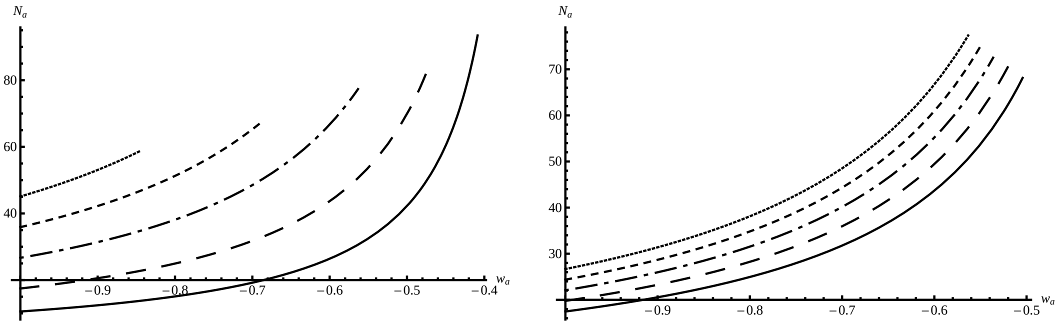

Figure 3.3: The left-hand plot shows the relationship between N_a and w for different choices of $\rho_{rh}^{1/4}$ (with $N_i \sim \ln 10^4$): 10^{16} MeV for the uppermost dotted line, 10^{12} MeV for the dashed line, 10^8 MeV for the dot-dash line, 10^4 MeV for the widely spaced dashed line and 10 MeV for the solid line. The right-hand plot shows the same for differing N_i (with $\rho_{rh}^{1/4} \sim 10^8$ MeV): $\ln 10^4$ for the uppermost dotted line, $\ln 10^5$ for the dashed line, $\ln 10^6$ for the dot-dash line, $\ln 10^7$ for the widely spaced dashed line and $\ln 10^8$ for the solid line.

beginning and end of accelerated expansion:

$$\begin{aligned} N_a =& \ln\left[\frac{a_{rh}}{a_f}\right] \\ =& \frac{1}{3(1+w_a)} \ln\left[\frac{\rho_f}{\rho_{rh}}\right] \ . \end{aligned} \tag{3.23}$$

Bounding $\rho_f^{1/4} \lesssim 10^{16}$ GeV (this bound comes from the amplitude of CMB fluctuations, 10^{-5} and the tilt-imposed upper limit of the slow roll parameter, $\epsilon \lesssim 0.05$, see e.g. [140] for details) we can use equations 3.22 and 3.23 to find allowed values of N_a and w_a for different choices of ρ_{rh}.

In figure 3.3 we illustrate the nature of the N_a-w_a relationship as we vary $\rho_{rh}^{1/4}$ (left-hand plot) and N_i (right-hand plot); note that ρ_f varies as one moves along the curves, each of which terminates when the upper limit for ρ_f is reached. It is clear that for a sufficiently low reheat temperature our current observations of the power spectrum could be consistent with having a period of quasi de Sitter inflation followed by a period of accelerated expansion with an equation of state significantly different from $w \sim -1$. Moreover, such a situation can be arranged with N_i sufficiently large such that the perturbations sourced by accelerated expansion are at length scales much smaller than those currently probed by experiment. On the other hand, if we wish to push up as close as possible to the current measurement of $N_i \sim \ln 10^4$ and thus be on the boundary of having an observable signature of accelerated expansion, we find the most dramatic result we can have (i.e. the largest w), comes from having a minimal reheat temperature, and would give us ~ 95 e-foldings of accelerated expansion with $w \sim -0.41$ and $n_s \sim -14.96$.

As our ability to measure the power spectrum on the shortest possible length scales improves, it is possible that we may be able to see the effects of a variation in the equation of state. However, given the difficulty of extracting information about the primordial power spectrum from scales much smaller than the scale at which local gravitational effects (i.e. not cosmology) dominates the physics it seems unlikely that any distinct signature of a second period of acceleration, unless its effects start at scales close to those at which we are currently confident of a scale-invariant spectrum. This restriction applies not only to the tilt, but also to other potential probes of the equation of state that rely on direct knowledge of the the spectrum (e.g. the running of n_s, higher order correlations etc.).

Although direct measurement of the spectrum is progressively harder at shorter and shorter distance scales, there are indirect probes that could be useful bounding the amplitude of primordial fluctuations. In particular, we know there must have been sufficient power at short distances to give rise to star production at an early enough times to produce the reionizened universe that we observe for $z \lesssim 6$ (some of the relevant issues are discussed in [64]). In principle this requirement may conflict with the most extreme scenarios we consider above, since a strongly red-tilted spectrum means that considerably less power is present at short scales when compared to the $\delta\rho/\rho \sim 10^{-5}$ that we have at large scales. However, although such constraints depend on the details of reionization, they seem unlikely to provide much stronger bounds on N_a and w_a for any given q_{max}/q_0 and $\rho_{rh}^{1/4}$.

Before moving on to our conclusions, let's consider what might happen if there was a secondary period of acceleration with a more complicated equation of state than that given by a constant w. To understand how such a period (distinguishable for inflation in principle) can be constrained in general it is helpful to consider again the relationships $N_a = N_{obs} - N_i$ and $N_{obs} \geq N_0$ (the latter is implied by equations 3.19). Together these give:

$$\begin{aligned} N_a \geq & N_0 - N_i \\ = & \ln\left[\frac{\rho_{rh}^{1/4}}{0.037h \text{ eV}}\right] - N_i \ . \end{aligned} \tag{3.24}$$

Given the lower bounds on the reheat temperature and the number of e-foldings of almost scale-invariance it's clearly possible to have at least a few e-foldings of arising from a second accelerating epoch (with $O(10)$ MeV reheating and scale invariance down to a Mpc, there are 8 unobserved e-foldings). Higher reheating scales would leave more possible e-foldings hidden. The precise nature of the accelerating expansion that is responsible for those e-foldings outside our current observations is almost free of constraints, subject only to the requirement that the energy density must change by a sufficient amount between the end of inflation and the start of decelerated expansion after reheating. This leaves a large set of possibilities available for an alternative acceleration mechanism between the end of inflation and the start radiation-domination, though one would expect that unless the equation of state parameter is changing rapidly compared to the Hubble constant, the space of possibilities would be constrained in a similar manner to the constant w case discussed here. In fact if one defines the parameter $w_{eff} = \frac{\int_1^{N_a} w}{N_a}$, one replace w with w_{eff} in the above expressions, to give the same restrictions, but now on N_a and w_{eff}.

Chapter 4

General initial states for perturbations

In spite of the great success of the inflationary theory, it is always important to verify the validity of different assumptions. The two-point and three-point functions of primordial fluctuations are generally computed assuming that the fluctuations are initially in the Bunch-Davies state[42, 39]. The Bunch-Davies vacuum state is the minimum energy eigenstate of the Hamiltonian in the infinite past and it is a reasonable choice as an initial state but not unique. It is somewhat of a philosophical question whether initial conditions are integral part of a theory or should be analyzed separately. The purpose of the next few chapters is to discuss the effect of relaxing the assumption of Bunch-Davies vacuum by choosing general initial states of both scalar and tensor perturbations built over the Bunch-Davies state. There has been a great deal of work focused on departure from the Bunch-Davies state[89, 70, 50, 83, 102, 101, 59, 16, 11, 53, 67, 47, 72, 10, 73, 14, 65, 71, 15, 17, 41, 19]. One aspect of prime interest is to understand how much information about the quantum state of primordial fluctuations in the beginning of inflation can be obtained. In order to achieve that it is essential to explore the dependence of different observables on initial states of the primordial fluctuations. Recently, this area of research has instigated interest among physicists mainly because it has the exciting possibility of providing us a window for the physics before inflation.

Constraints on the initial state from current measurements are relatively weak. For slow-roll inflation, a large number of states are consistent with the observations. However, renormalizability of the energy-momentum tensor of the primordial fluctuations imposes some restrictions on the initial state.

4.1 General initial states

Let us now define general initial states built over the Bunch-Davies vacuum state $|0\rangle$. It is important to note that we are in the Heisenberg picture where states are time-independent. We can use $\hat{a}_{\mathbf{k}}^{s\dagger}$ operators to build excited states over the Bunch-Davies vacuum state

$$|\psi_s\rangle = \frac{1}{\sqrt{n_1!n_2!...}} \left[\left(\hat{a}_{\mathbf{k}_1}^{s\dagger} \right)^{n_1} \left(\hat{a}_{\mathbf{k}_2}^{s\dagger} \right)^{n_2} ... \right] |0\rangle. \tag{4.1}$$

Again note that we have used index s to denote both scalar and tensor perturbations; $s = 0$ corresponds to scalar perturbations and $s = \times, +$ correspond to two polarization modes of tenser perturbations. We can write down a general excited state for a perturbation, using equation (4.1)

$$|G_s\rangle = \sum_{\psi_s} C_{\psi_s} |\psi_s\rangle. \tag{4.2}$$

Therefore, a general state can be written as a direct product

$$|G\rangle = |G_{s=0}\rangle \otimes |G_{s=+}\rangle \otimes |G_{s=\times}\rangle. \tag{4.3}$$

And with this initial state we can compute the power spectrum using

$$\langle \hat{v}^s_{\mathbf{k}}(\tau)\hat{v}^{s'}_{\mathbf{k}'}(\tau)\rangle = \frac{\langle G|\hat{v}^s_{\mathbf{k}}(\tau)\hat{v}^{s'}_{\mathbf{k}'}(\tau)|G\rangle}{\langle G|G\rangle}. \tag{4.4}$$

It is important to note that for a general state $|G\rangle$, one-point function $\langle \hat{v}^s_{\mathbf{k}}(\tau)\rangle$ may not be zero even at late time ($\tau \to 0$) and as a result both $\langle \hat{\mathcal{R}}_{\mathbf{k}}(\tau)\rangle$ and $\langle \hat{h}^s_{\mathbf{k}}(\tau)\rangle$ can be nonzero. In that case, the power spectrum should be defined in the following way,

$$\begin{aligned}\langle \hat{\mathcal{O}}^s_{\mathbf{k}}(\tau)\hat{\mathcal{O}}^{s'}_{\mathbf{k}'}(\tau)\rangle_{phy} \equiv& \langle(\hat{\mathcal{O}}^s_{\mathbf{k}}(\tau) - \langle\hat{\mathcal{O}}^s_{\mathbf{k}}(\tau)\rangle)(\hat{\mathcal{O}}^{s'}_{\mathbf{k}'}(\tau) - \langle\hat{\mathcal{O}}^{s'}_{\mathbf{k}'}(\tau)\rangle)\rangle \\ =& \langle\hat{\mathcal{O}}^s_{\mathbf{k}}(\tau)\hat{\mathcal{O}}^{s'}_{\mathbf{k}'}(\tau)\rangle - \langle\hat{\mathcal{O}}^s_{\mathbf{k}}(\tau)\rangle\langle\hat{\mathcal{O}}^{s'}_{\mathbf{k}'}(\tau)\rangle. \end{aligned} \tag{4.5}$$

With the initial state (4.3), we can calculate

$$\begin{aligned}\langle \hat{v}^s_{\mathbf{k}}(\tau)\hat{v}^{s'}_{\mathbf{k}'}(\tau)\rangle =& \frac{1}{2}(2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')|u_k(\tau)|^2\delta_{ss'} + A(s,s';\mathbf{k},\mathbf{k}')u^*_k u^*_{k'} \\ &+ A^*(s,s';-\mathbf{k},-\mathbf{k}')u_k u_{k'} + B(s,s';-\mathbf{k},\mathbf{k}')u_k u^*_{k'} \\ &+ B(s',s;-\mathbf{k}',\mathbf{k})u^*_k u_{k'}, \end{aligned} \tag{4.6}$$

where,

$$A(s,s';\mathbf{k},\mathbf{k}') = \frac{1}{2}\frac{\langle G|\hat{a}^s_{\mathbf{k}}\hat{a}^{s'}_{\mathbf{k}'}|G\rangle}{\langle G|G\rangle}, \qquad B(s,s';\mathbf{k},\mathbf{k}') = \frac{1}{2}\frac{\langle G|\hat{a}^{s\dagger}_{\mathbf{k}}\hat{a}^{s'}_{\mathbf{k}'}|G\rangle}{\langle G|G\rangle}. \tag{4.7}$$

And

$$\begin{aligned}\langle \hat{v}^s_{\mathbf{k}}(\tau)\rangle\langle\hat{v}^{s'}_{\mathbf{k}'}(\tau)\rangle =& b(s;\mathbf{k})b(s';\mathbf{k}')u^*_k u^*_{k'} + b^*(s;-\mathbf{k})b^*(s';-\mathbf{k}')u_k u_{k'} \\ &+ b^*(s;-\mathbf{k})b(s';\mathbf{k}')u_k u^*_{k'} + b(s;\mathbf{k})b^*(s';-\mathbf{k}')u^*_k u_{k'}, \end{aligned} \tag{4.8}$$

where,

$$b(s;\mathbf{k}) = \frac{1}{\sqrt{2}}\frac{\langle G|\hat{a}^s_{\mathbf{k}}|G\rangle}{\langle G|G\rangle} = \frac{1}{\sqrt{2}}\frac{\langle G_s|\hat{a}^s_{\mathbf{k}}|G_s\rangle}{\langle G_s|G_s\rangle}. \tag{4.9}$$

Now, if $s \neq s'$, it can be shown very easily that $A(s, s' \neq s; \mathbf{k}, \mathbf{k}') = b(s; \mathbf{k})b(s'; \mathbf{k}')$ and $B(s, s' \neq s; \mathbf{k}, \mathbf{k}') = b^*(s; \mathbf{k})b(s'; \mathbf{k}')$. And therefore,

$$\langle \hat{v}^s_{\mathbf{k}}(\tau)\hat{v}^{s' \neq s}_{\mathbf{k}'}(\tau)\rangle_{phy} = \langle \hat{v}^s_{\mathbf{k}}(\tau)\hat{v}^{s' \neq s}_{\mathbf{k}'}(\tau)\rangle - \langle \hat{v}^s_{\mathbf{k}}(\tau)\rangle\langle \hat{v}^{s' \neq s}_{\mathbf{k}'}(\tau)\rangle = 0 \tag{4.10}$$

That leads to

$$\langle \hat{v}^s_{\mathbf{k}}(\tau)\hat{v}^{s'}_{\mathbf{k}'}(\tau)\rangle_{phy} = \langle \hat{v}^s_{\mathbf{k}}(\tau)\hat{v}^{s}_{\mathbf{k}'}(\tau)\rangle_{phy}\delta_{ss'}. \tag{4.11}$$

Let us now calculate $\langle \hat{v}^s_{\mathbf{k}}(\tau)\hat{v}^{s'}_{\mathbf{k}'}(\tau)\rangle_{phy}$. Introducing $k_* = \sqrt{kk'}$, $\bar{k} = k + k'$ and $\Delta k = k - k'$ and using equation (2.29), we can write

$$\begin{aligned}
\langle \hat{v}^s_{\mathbf{k}}(\tau)\hat{v}^{s}_{\mathbf{k}'}(\tau)\rangle_{phy} =& \frac{1}{2}(2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')\frac{1}{k}\left(1 + \frac{1}{k^2\tau^2}\right) \\
&+ A_s(\mathbf{k}, \mathbf{k}')e^{-i\bar{k}\tau}\frac{1}{k_*}\left(1 - \frac{i\bar{k}}{\tau k_*^2} - \frac{1}{k_*^2\tau^2}\right) \\
&+ A^*_s(-\mathbf{k}, -\mathbf{k}')e^{i\bar{k}\tau}\frac{1}{k_*}\left(1 + \frac{i\bar{k}}{\tau k_*^2} - \frac{1}{k_*^2\tau^2}\right) \\
&+ B_s(-\mathbf{k}, \mathbf{k}')e^{i\Delta k\tau}\frac{1}{k_*}\left(1 - \frac{i\Delta k}{\tau k_*^2} + \frac{1}{k_*^2\tau^2}\right) \\
&+ B_s(-\mathbf{k}', \mathbf{k})e^{-i\Delta k\tau}\frac{1}{k_*}\left(1 + \frac{i\Delta k}{\tau k_*^2} + \frac{1}{k_*^2\tau^2}\right) \ ,
\end{aligned} \tag{4.12}$$

where,

$$\begin{aligned}
A_s(\mathbf{k}, \mathbf{k}') &= A(s, s; \mathbf{k}, \mathbf{k}') - b(s; \mathbf{k})b(s; \mathbf{k}'), \\
B_s(\mathbf{k}, \mathbf{k}') &= B(s, s; \mathbf{k}, \mathbf{k}') - b^*(s; \mathbf{k})b(s; \mathbf{k}').
\end{aligned} \tag{4.13}$$

In the superhorizon limit ($|k\tau|, |k'\tau| \ll 1$), we finally obtain

$$\begin{aligned}
\langle \hat{v}^s_{\mathbf{k}}(\tau)\hat{v}^{s}_{\mathbf{k}'}(\tau)\rangle_{phy} \approx& \frac{1}{2}(2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')\frac{1}{k}\left(1 + \frac{1}{k^2\tau^2}\right) + \left(\frac{1}{k_*^3\tau^2} + \frac{k^2 + k'^2}{2k_*^3}\right) \\
&\times\left[-A_s(\mathbf{k}, \mathbf{k}') - A^*_s(-\mathbf{k}, -\mathbf{k}') + B_s(-\mathbf{k}, \mathbf{k}') + B_s(-\mathbf{k}', \mathbf{k})\right] + \cdots
\end{aligned} \tag{4.14}$$

where the dots indicate terms of higher order.

4.1.1 Scalar power spectrum

In the superhorizon limit, from the last equation at the leading order we obtain

$$\langle\hat{\mathcal{R}}_{\mathbf{k}}(\tau)\hat{\mathcal{R}}_{\mathbf{k}'}(\tau)\rangle_{phy} \approx \frac{1}{2}(2\pi)^3\frac{H^4}{\dot{\bar{\phi}}^2k^3}\delta^3(\mathbf{k}+\mathbf{k}') + \frac{H^4}{\dot{\bar{\phi}}^2k_*^3}[-A_0(\mathbf{k},\mathbf{k}') - A_0^*(-\mathbf{k},-\mathbf{k}') + B_0(-\mathbf{k},\mathbf{k}') + B_0(-\mathbf{k}',\mathbf{k})]. \tag{4.15}$$

Let us now simplify the last equation by making some assumptions about the initial state. Our universe as we see it today, is homogeneous and isotropic on large scale. Demanding homogeneity in the superhorizon limit restricts the form of the power spectrum

$$\langle\hat{\mathcal{R}}_{\mathbf{k}}(\tau)\hat{\mathcal{R}}_{\mathbf{k}'}(\tau)\rangle_{phy} = P(\mathbf{k},\tau)\delta^3(\mathbf{k}+\mathbf{k}')\delta_{ss'}. \tag{4.16}$$

Where $P(\mathbf{k},\tau)$ is some arbitrary function of $\mathbf{k}$ and τ. If we also assume that the initial state is isotropic, then $P(\mathbf{k},\tau) = P(k,\tau)$. Comparing the last equation with the leading order term of equation (4.15), we also find that $P(k,\tau)$ does not depend on τ and hence $\langle\hat{\mathcal{R}}_{\mathbf{k}}(\tau)\hat{\mathcal{R}}_{\mathbf{k}'}(\tau)\rangle_{phy}$ is time-independent. All these assumptions about the initial state allow us to write

$$-A_0(\mathbf{k},\mathbf{k}') - A_0^*(-\mathbf{k},-\mathbf{k}') + B_0(-\mathbf{k},\mathbf{k}') + B_0(-\mathbf{k}',\mathbf{k}) = (2\pi)^3W_0(k)\delta^3(\mathbf{k}+\mathbf{k}'), \tag{4.17}$$

where, $W_0(k)$ is some arbitrary function of k. Therefore the scalar power spectrum is given by

$$\langle\hat{\mathcal{R}}_{\mathbf{k}}(\tau)\hat{\mathcal{R}}_{\mathbf{k}'}(\tau)\rangle = (2\pi)^3\frac{H^4}{\dot{\bar{\phi}}^2k^3}\left(\frac{1}{2}+W_0(k)\right)\delta^3(\mathbf{k}+\mathbf{k}'), \tag{4.18}$$

and

$$\Delta_{\mathcal{R}}^2 = \frac{H^4}{4\pi^2\ \dot{\phi}^2}\left(1+2W_0(k)\right)\ , \qquad n_s = 1-6\epsilon+2\eta+\frac{d\ln\left(1+2W_0(k)\right)}{d\ln k} \tag{4.19}$$

where, $W_0(k)$ is defined by equation (4.17). Let us note that here we have assumed that energies of these states are not large enough to affect the slow-roll parameters.

4.1.2 Tensor power spectrum

For the tensor modes, in the superhorizon limit, at the leading order we have

$$\begin{aligned}\langle \hat{h}^s_{\mathbf{k}}(\tau)\hat{h}^{s'}_{\mathbf{k}'}(\tau)\rangle_{phy} \approx & (2\pi)^3\frac{H^2}{{M_{\rm pl}}^2k^3}\delta^3(\mathbf{k}+\mathbf{k}')\delta_{ss'} + \frac{2H^2}{{M_{\rm pl}}^2k_*^3}[-A_s(\mathbf{k},\mathbf{k}') \\ & - A_s^*(-\mathbf{k},-\mathbf{k}') + B_s(-\mathbf{k},\mathbf{k}') + B_s(-\mathbf{k}',\mathbf{k})]\delta_{ss'}. \end{aligned} \tag{4.20}$$

We can make similar assumptions about the initial state of tensor modes to obtain

$$-A_s(\mathbf{k},\mathbf{k}') - A_s^*(-\mathbf{k},-\mathbf{k}') + B_s(-\mathbf{k},\mathbf{k}') + B_s(-\mathbf{k}',\mathbf{k}) = (2\pi)^3 W_s(k)\delta^3(\mathbf{k}+\mathbf{k}')\ . \tag{4.21}$$

Therefore, the tensor power spectrum is given by

$$\langle \hat{h}^s_{\mathbf{k}}(\tau)\hat{h}^{s'}_{\mathbf{k}'}(\tau)\rangle_{phy} = (2\pi)^3\frac{H^2}{{M_{\rm pl}}^2k^3}\left(1+2W_s(k)\right)\delta^3(\mathbf{k}+\mathbf{k}')\delta_{ss'}, \tag{4.22}$$

and

$$\Delta_t^2 = \frac{H^2}{\pi^2\ {M_{\rm pl}}^2}\left(1+\sum_{s=\times,+}W_s(k)\right)\ , \qquad n_t = -2\epsilon + \frac{d\ln\left(1+\sum_{s=\times,+}W_s(k)\right)}{d\ln k} \tag{4.23}$$

where, $W_s(k)$ is defined by equation (4.21).

In the special case: $W_0(k) = W_+(k) = W_\times(k)$ which is the case when a pre-inflationary dynamics excites both the scalar modes and the tensor modes in the same way. In that case, the tenser-to-scalar ratio remains unchanged

$$r = 16\epsilon \ . \tag{4.24}$$

However, both n_s and n_t get corrected and hence the consistency relation $r = -8n_t$ is no longer true. Detection of primordial gravitational waves will provide us an important tool for probing the initial state.

4.2 Constraints from observations

Constraints on the initial state from current measurements are relatively weak. Our universe as we see it today, is homogeneous and isotropic on large scale. Temperature fluctuations of CMB and LSS are directly related to the curvature perturbations produced during inflation. In general, the one-point functions $\langle\hat{\mathcal{R}}_{\mathbf{k}}(\tau)\rangle, \langle\hat{h}^s_{\mathbf{k}}(\tau)\rangle \neq 0$, even though $\hat{\mathcal{R}}_{\mathbf{k}}(\tau)$ and $\hat{h}^s_{\mathbf{k}}(\tau)$ are perturbations. Without loss of generality we can impose the restriction that $\langle\hat{\mathcal{R}}_{\mathbf{k}}(\tau)\rangle = \langle\hat{h}^s_{\mathbf{k}}(\tau)\rangle = 0$ for the superhorizon modes. From that we can write down

$$\langle G|\hat{a}^{s\dagger}_{-\mathbf{k}}|G\rangle = \langle G|\hat{a}^s_{\mathbf{k}}|G\rangle. \tag{4.25}$$

Also the present observations of the CMB temperature inhomogeneities indicates the presence of almost scale-invariant spectrum of curvature perturbations. That leads to the constraint

$$\frac{d\ln\left(1+2W_0(k)\right)}{d\ln k} << 1 \tag{4.26}$$

for all the observable modes. Currently only bound on the initial state of tensor perturbations comes from the tensor-to-scalar ratio r. Current bound on tensor-to-scalar ratio from Planck+WP is $r < 0.12$[5]. However, more recent observations (BICEP2) indicate that $r \approx 0.20$ [7]. Therefore,

$$\frac{16\epsilon\left(1+\sum_{s=\times,+} W_s(k)\right)}{\left(1+2W_0(k)\right)} \approx 0.20\ . \tag{4.27}$$

4.2.1 Some examples

Now we will give some examples to show that it is possible to construct states that are identical to the Bunch-Davies state. These states must obey

$$W_0(k) = W_+(k) = W_\times(k) = 0\ . \tag{4.28}$$

It is also easy to check that coherent states

$$\hat{a}^s_{\mathbf{k}}|C\rangle = C(\mathbf{k};s)|C\rangle \tag{4.29}$$

where $\langle C|C\rangle = 1$. Constraints (4.25) and (4.28) lead to the condition

$$C^*(-\mathbf{k};s) = C(\mathbf{k};s). \tag{4.30}$$

To give another nontrivial example we look for states of the form $|G_s\rangle = |0\rangle + |\psi\rangle$, where $|\psi\rangle$ is an excited state (or a combination of excited states).

The simplest example of such a state is $|G_s\rangle = |0\rangle + \int d^3\mathbf{k}\ \alpha(k)\ \hat{a}^{s\dagger}_{\mathbf{k}}|0\rangle$. But it is easy to check that this state does not work. $\alpha(k)$ has to be real to satisfy equation (4.25). Then one can show that $\alpha(k)$ has to vanish to satisfy equation (4.28). But

$$|G_s\rangle = |0\rangle + \int d^3\mathbf{k}\ \alpha(k)\ \hat{a}^{s\dagger}_{\mathbf{k}}|0\rangle + \int d^3\mathbf{k}_1 d^3\mathbf{k}_2\ \beta(k_1)\beta(k_2)\ \hat{a}^{s\dagger}_{\mathbf{k}_1}\hat{a}^{s\dagger}_{\mathbf{k}_2}|0\rangle, \quad (4.31)$$

where $\alpha(k)$ and $\beta(k)$ are real functions, makes it possible to construct states that satisfy both constraints (4.25) and (4.28) and hence produce a scale-invariant power spectrum. Equation (4.25) is already satisfied and equation (4.28) leads to

$$\alpha(k)\alpha(k') + (4N-1)\beta(k)\beta(k') = 0 \quad (4.32)$$

with $N = (2\pi)^3 \int d^3\mathbf{k}\,(\beta(k))^2$. So all the states given by equation (4.31) with $\alpha(k)$ and $\beta(k)$ obeying equation (4.32) produce scale invariant power spectrum. Example of one such state is

$$\begin{aligned}|G_s\rangle = |0\rangle + A\sqrt{1 - \frac{32\pi^4 A^2}{\gamma^3}} \int d^3\mathbf{k}\ e^{-\gamma k}\ \hat{a}^{s\dagger}_{\mathbf{k}}|0\rangle \\ +A^2 \int d^3\mathbf{k}_1 d^3\mathbf{k}_2\ e^{-\gamma(k_1+k_2)}\ \hat{a}^{s\dagger}_{\mathbf{k}_1}\hat{a}^{s\dagger}_{\mathbf{k}_2}|0\rangle, \quad (4.33)\end{aligned}$$

where, A and γ are real constants and $\frac{32\pi^4 A^2}{\gamma^3} \leq 1$. Renormalizability of the energy-momentum tensor of the fluctuations can impose more constraints on these states. Here we should note that it is possible to construct infinite number of such states (with even more complicated combination of excited states) that obey constraints (4.25) and (4.28) and hence can produce a scale-invariant power-spectrum.

4.3 Constraints from renormalizability of the energy-momentum tensor

Let us now define the energy-momentum tensor of the perturbations. The Einstein's equations for the full system is $G_{\mu\nu} - 8\pi G T_{\mu\nu} \equiv \Pi_{\mu\nu} = 0$. Following [2], we can perform a perturbative expansion of the Einstein's equations:

$$\Pi_{\mu\nu} = \Pi^{(0)}_{\mu\nu} + \Pi^{(1)}_{\mu\nu} + \Pi^{(2)}_{\mu\nu} + ... \tag{4.34}$$

Evolution of the background is given by the lowest order equation $\Pi^{(0)}_{\mu\nu} = 0$. The first order Einstein's equations $\Pi^{(1)}_{\mu\nu} = 0$ give the equations of motion for the perturbations. Therefore, we can write

$$G^{(0)}_{\mu\nu} = 8\pi G_N T^{(0)}_{\mu\nu} - \Pi^{(2)}_{\mu\nu} + ... \ , \tag{4.35}$$

where $\Pi^{(2)}_{\mu\nu}$ has to be computed with the perturbations that solve the equations of motion $\Pi^{(1)}_{\mu\nu} = 0$. From the last equation, it is clear that the energy-momentum tensor of the perturbations is given by $8\pi G_N \mathcal{T}_{\mu\nu} = -\Pi^{(2)}_{\mu\nu}$. Obviously both scalar and tensor perturbations will contribute to the energy-momentum tensor:

$$\mathcal{T}_{\mu\nu} = \mathcal{T}^s_{\mu\nu} + \mathcal{T}^t_{\mu\nu} \ . \tag{4.36}$$

We will promote $\mathcal{T}_{\mu\nu}$ to an operator and estimate $\langle \hat{\mathcal{T}}_{\mu\nu} \rangle$ for single-field slow-roll inflation with a general initial state for the perturbations.

4.3.1 Energy momentum tensor for scalar perturbations

In the longitudinal gauge, the metric with only scalar perturbation has the form

$$ds^2 = -(1+2\Phi)dt^2 + a^2(t)\left[(1-2\Psi)\delta_{ij}\right]dx^i dx^j. \tag{4.37}$$

At first order we get $\Phi = \Psi$. Other equations of motion for the perturbations are

$$\dot{\Psi} + H\Psi = 4\pi G\dot{\bar{\phi}}\delta\phi, \tag{4.38}$$

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + V''(\bar{\phi})\delta\phi - \left(\frac{\nabla^2}{a^2}\right)\delta\phi = -2\Psi V'(\bar{\phi}) + 4\dot{\Psi}\dot{\bar{\phi}}, \tag{4.39}$$

$$-\left(4\pi G\dot{\bar{\phi}}^2 + \frac{\nabla^2}{a^2}\right)\Psi = 4\pi G\left(-\dot{\bar{\phi}}\delta\dot{\phi} + \ddot{\bar{\phi}}\delta\phi\right). \tag{4.40}$$

We have the good old gauge invariant variable $\mathcal{R}$ defined as

$$\mathcal{R} = \Psi + \frac{H}{\dot{\bar{\phi}}}\delta\phi. \tag{4.41}$$

Following [2], we can write down the energy-momentum tensor for the perturbations $\mathcal{T}^s_{\mu\nu} = -\Pi^{(2)}_{\mu\nu}$ in terms of Ψ and $\delta\phi$

$$\mathcal{T}^s_{00} = \frac{1}{8\pi G}\left[12H(\Psi\dot{\Psi}) - 3(\dot{\Psi})^2 + 9a^{-2}(\nabla\Psi)^2\right]$$
$$+\left[\frac{1}{2}(\delta\dot{\phi})^2 + \frac{1}{2}a^{-2}(\nabla\delta\phi)^2 + \frac{1}{2}V''(\bar{\phi})(\delta\phi)^2 + 2V'(\bar{\phi})(\Psi\delta\phi)\right], \tag{4.42}$$

$$\mathcal{T}^s_{ij} = a^2\delta_{ij}\left(\frac{1}{8\pi G}\left[(24H^2 + 16\dot{H})\Psi^2 + 24H(\Psi\dot{\Psi}) + (\dot{\Psi})^2 + 4\Psi\ddot{\Psi} - \frac{4}{3a^2}(\nabla\Psi)^2\right]\right.$$
$$\left.+\left[4\dot{\bar{\phi}}^2\Psi^2 + \frac{1}{2}(\delta\dot{\phi})^2 - \frac{1}{6a^2}(\nabla\delta\phi)^2 - 4\dot{\bar{\phi}}\delta\dot{\phi}\Psi - \frac{1}{2}V''(\bar{\phi})\delta\phi^2 + 2V'(\bar{\phi})\Psi\delta\phi\right]\right). \tag{4.43}$$

Our goal is to find out constraints on the initial state from renormalizability of the energy-momentum tensor. So we want to consider the contributions of large-k fluctuations of $\mathcal{T}^s_{\mu\nu}$. At large-k ($k >> aH$),

$$\mathcal{R}_k \sim \frac{H}{a\dot{\bar{\phi}}} \frac{e^{ik\tau}}{\sqrt{k}}. \tag{4.44}$$

From equation (4.38) we have for large-k (and for quasi-de Sitter inflation)

$$\Psi_k \sim -\frac{iH\epsilon a}{k} \mathcal{R}_k, \tag{4.45}$$

where, ϵ is the slow roll parameter. In this approximation, for $\delta\phi$ we obtain,

$$\delta\phi \sim \mathcal{R} \frac{\dot{\bar{\phi}}}{H} \left(1 + \frac{iH\epsilon a}{k}\right) \sim \mathcal{R} \frac{\dot{\bar{\phi}}}{H}. \tag{4.46}$$

By inspection, it is very clear that all terms in equation (4.60) containing Ψ are suppressed by powers of Ha/k compared to terms without Ψ. Therefore in large-k limit we have,

$$\mathcal{T}^s_{00} \approx \left[\frac{1}{2}(\delta\dot{\phi})^2 + \frac{1}{2}a^{-2}(\nabla\delta\phi)^2 + \frac{1}{2}V''(\bar{\phi})(\delta\phi)^2\right], \tag{4.47}$$

$$\mathcal{T}^s_{ij} \approx a^2\delta_{ij} \left[\frac{1}{2}(\delta\dot{\phi})^2 - \frac{1}{6}a^{-2}(\nabla\delta\phi)^2 \frac{1}{2}V''(\bar{\phi})(\delta\phi)^2\right]. \tag{4.48}$$

4.3.2 Renormalization of energy-momentum tensor

Using equation (4.46), we can write down $\mathcal{T}^s_{\mu\nu}$ (for large-k) in terms of gauge invariant variable $\mathcal{R}$

$$\langle \hat{\mathcal{T}^s}_{00} \rangle \approx \left(\frac{\dot{\bar{\phi}}}{H}\right)^2 \left[\frac{1}{2}a^{-2}\langle(\hat{\mathcal{R}}')^2\rangle + \frac{1}{2}a^{-2}\langle(\nabla\hat{\mathcal{R}})^2\rangle + \frac{1}{2}V''(\bar{\phi})\langle\hat{\mathcal{R}}^2\rangle\right], \tag{4.49}$$

$$\langle \hat{\mathcal{T}^s}_{ij} \rangle \approx a^2\delta_{ij} \left(\frac{\dot{\bar{\phi}}}{H}\right)^2 \left[\frac{1}{2}a^{-2}\langle(\hat{\mathcal{R}}')^2\rangle - \frac{1}{6}a^{-2}\langle(\nabla\hat{\mathcal{R}})^2\rangle - \frac{1}{2}V''(\bar{\phi})\langle\hat{\mathcal{R}}^2\rangle\right]. \tag{4.50}$$

For the vacuum state, we have the following expressions for the unregularized energy-momentum tensor

$$\langle 0|\hat{\mathcal{T}^s}_{00}|0\rangle \approx \frac{1}{4\pi^2}\int^{\infty} dk H^4 \left[\frac{3\eta}{2k} + \frac{1}{2}\tau^2 k(1+3\eta) + \tau^4 k^3\right], \tag{4.51}$$

$$\langle 0|\hat{\mathcal{T}^s}_{ij}|0\rangle \approx - a^2\delta_{ij}\frac{1}{4\pi^2}\int^{\infty} dk H^4 \left[\frac{3\eta}{2k} + \frac{1}{6}\tau^2 k(1+9\eta) - \frac{1}{3}\tau^4 k^3\right], \tag{4.52}$$

where η is the second slow-roll parameter. To obtain the renormalized value of $\langle\hat{\mathcal{T}^s}_{\mu\nu}\rangle$, one can use any regularization method (for example adiabatic regularization). Detailed discussions of adiabatic regularization method can be found in [110, 43]. Adiabatic regularization of $\langle\hat{\mathcal{T}^s}_{\mu\nu}\rangle$ can be done by subtracting adiabatic modes up to order four.

$$\langle 0|\hat{\mathcal{T}^s}_{\mu\nu}|0\rangle_{ren} = \langle 0|\hat{\mathcal{T}^s}_{\mu\nu}|0\rangle - \langle 0|\hat{\mathcal{T}^s}_{\mu\nu}|0\rangle_{adi}, \tag{4.53}$$

where, $\langle 0|\hat{\mathcal{T}^s}_{\mu\nu}|0\rangle_{adi}$ is calculated using the adiabatic mode functions of order four

$$u_k^{adi;4}(\tau) = \frac{1}{\sqrt{W^{(4)}}} e^{i\int^{\tau} W^{(4)} d\tau}, \qquad \mathcal{R}_k^{adi}(\tau) = \left(\frac{H}{a\dot{\bar{\phi}}}\right) u_k^{adi;4}(\tau), \tag{4.54}$$

where,

$$W^{(4)} = k - \frac{1}{k\tau^2} + \frac{1}{k^3\tau^4}. \tag{4.55}$$

Therefore we have the following expression for $\langle 0|\hat{t}_{\mu\nu}|0\rangle_{adi}$

$$\langle 0|\hat{\mathcal{T}^s}_{00}|0\rangle_{adi} \approx \frac{1}{4\pi^2}\int^{\infty} dk H^4 \left[\frac{3\eta}{2k} + \frac{1}{2}\tau^2 k(1+3\eta) + \tau^4 k^3\right] + \text{UV finite}, \tag{4.56}$$

$$\langle 0|\hat{\mathcal{T}^s}_{ij}|0\rangle_{adi} \approx - a^2\delta_{ij}\frac{1}{4\pi^2}\int^{\infty} dk H^4 \left[\frac{3\eta}{2k} + \frac{1}{6}\tau^2 k(1+9\eta) - \frac{1}{3}\tau^4 k^3\right] + \text{UV finite}. \tag{4.57}$$

Hence, $\langle 0|\hat{\mathfrak{T}^s}_{\mu\nu}|0\rangle_{ren}$ does not have any ultraviolet divergences. For a general initial state $|G\rangle$ we can renormalize energy-momentum tensor by

$$\langle G|\hat{\mathfrak{T}^s}_{\mu\nu}|G\rangle_{ren} \equiv \langle G|\hat{\mathfrak{T}^s}_{\mu\nu}|G\rangle - \langle 0|\hat{\mathfrak{T}^s}_{\mu\nu}|0\rangle_{adi}. \tag{4.58}$$

Note that in the right hand side, we do not have $\langle G|\hat{\mathfrak{T}^s}_{\mu\nu}|G\rangle_{adi}$. But we have $\langle 0|\hat{\mathfrak{T}^s}_{\mu\nu}|0\rangle_{adi}$. It can be understood easily in Minkowski space limit. In Minkowski space, $\langle G|\hat{\mathfrak{T}^s}_{\mu\nu}|G\rangle_{adi} = \langle G|\hat{\mathfrak{T}^s}_{\mu\nu}|G\rangle$. So, if we had $\langle G|\hat{\mathfrak{T}^s}_{\mu\nu}|G\rangle_{adi}$ in equation (4.58), then $\langle G|\hat{\mathfrak{T}^s}_{\mu\nu}|G\rangle_{ren} = 0$ for any state $|G\rangle$. Clearly that can not be right.

4.3.3 Energy momentum tensor for tensor perturbations

Similarly we can define the energy momentum tensor for the scalar perturbations. The metric with only scalar perturbation is given by

$$ds^2 = -dt^2 + a^2(t)\left[\delta_{ij} + h_{ij}\right] dx^i dx^j. \tag{4.59}$$

The energy-momentum tensor of tensor perturbations $\mathfrak{T}^t_{\mu\nu}$ is given by

$$8\pi G \mathfrak{T}^t_{00} = \frac{\dot{a}}{a}\dot{h}_{kl}h_{kl} + \frac{1}{8}\left(\dot{h}_{kl}\dot{h}_{kl} + \frac{1}{a^2}\partial_m h_{kl}\partial_m h_{kl}\right), \tag{4.60}$$

$$8\pi G \mathfrak{T}^t_{ij} = a^2\delta_{ij}\left[\frac{3}{8a^2}\partial_m h_{kl}\partial_m h_{kl} - \frac{3}{8}\dot{h}_{kl}\dot{h}_{kl}\right] + \frac{1}{2}a^2\dot{h}_{ik}\dot{h}_{kj} + \frac{1}{4}\partial_i h_{kl}\partial_j h_{kl} - \frac{1}{2}\partial_l h_{ki}\partial_l h_{jk}. \tag{4.61}$$

For a general initial state $|G\rangle$ we can renormalize this energy-momentum tensor by

$$\langle G|\hat{\mathfrak{T}^t}_{\mu\nu}|G\rangle_{ren} \equiv \langle G|\hat{\mathfrak{T}^t}_{\mu\nu}|G\rangle - \langle 0|\hat{\mathfrak{T}^t}_{\mu\nu}|0\rangle_{adi}. \tag{4.62}$$

4.3.4 Constraint on the initial state

We will impose the constraint on the initial state $|G\rangle$ that $\langle G|\hat{\mathfrak{T}}_{\mu\nu}|G\rangle_{ren}$ does not have any ultraviolet divergences. That means the state $|G\rangle$ does not introduce any new ultra-violet divergences to the energy-momentum tensor. Therefore, to make sure that $\langle G|\hat{\mathfrak{T}}_{\mu\nu}|G\rangle_{ren}$ has desired UV behavior, we will impose the following constraint on the initial state

$$\langle G|\hat{\mathfrak{T}}_{\mu\nu}|G\rangle = \langle 0|\hat{\mathfrak{T}}_{\mu\nu}|0\rangle + \text{UV finite} \,, \tag{4.63}$$

If we impose constraint (4.63) on the coherent state $\hat{a}_{\mathbf{k}}|C\rangle = C(\mathbf{k};s)|C\rangle$ (with $C^*(-\mathbf{k};s) = C(\mathbf{k};s)$), we find that $C(\mathbf{k};s)$ goes to zero faster than $\frac{1}{k^{5/2}}$ for large $\mathbf{k}$. One can check that state (4.33) also satisfies constraint (4.63).

Chapter 5

Consistency relations for the three-point functions

5.1 Introduction

Despite its great success, the details of the physics of inflation are still unknown. A large number of models of inflation successfully explain all the observations making it practically impossible to distinguish between different models. The three-point functions of primordial fluctuations[97, 68, 127] are important observables that in principle can be used to differentiate between single-field and multi-field inflation models. In particular, the three-point functions for single-field inflation, in squeezed limits, obey certain consistency relations that can provide us with an important tool to falsify or establish single-field inflation [97, 58, 52, 57, 89, 120, 121, 18, 92, 115].

In this chapter, we will assume that there is effectively one single scalar degree of freedom during inflation; discussion of this section does not particularly depend on the details of the single-field model. Inflation generates both scalar ($\mathcal{R}$) and tensor (h_{ij}) perturbations and the power spectrums of the perturbations $P_{\mathcal{R}}$ and P_h, defined as

$$\langle \hat{\mathcal{R}}_{\mathbf{k}} \hat{\mathcal{R}}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^3(\mathbf{k}+\mathbf{k}') P_{\mathcal{R}} \ , \qquad \langle \hat{h}^s_{\mathbf{k}} \hat{h}^{s'}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^3(\mathbf{k}+\mathbf{k}') \delta_{ss'} P_h \ , \quad (5.1)$$

depend on the details of the model. In the last equation, $\mathcal{R}_{\mathbf{k}}$ and $h^s_{\mathbf{k}}$ are the Fourier transforms of $\mathcal{R}$ and h_{ij}, respectively. Whereas, for single-field inflation the three-point functions of the perturbations, in the squeezed limit (i.e. $k_1, k_2 \gg k_3$), are known to obey some consistency relations which are of the form

$$\langle \hat{A}_{\mathbf{k}_1} \hat{B}_{\mathbf{k}_2} \hat{C}_{\mathbf{k}_3} \rangle = (2\pi)^3 \mathcal{F}_{AC} P_A(k_1) P_C(k_3) \delta_{s(A),s(B)} \frac{\epsilon^{s(C)}_{ij}(\mathbf{k}_3) k_{1;i} k_{1;j}}{k_1^2} \delta^3 \left(\sum \mathbf{k} \right) , \tag{5.2}$$

where $\hat{A}_{\mathbf{k}}, \hat{B}_{\mathbf{k}}, \hat{C}_{\mathbf{k}}$ are either scalar perturbation $\hat{\mathcal{R}}_{\mathbf{k}}$ or tensor perturbation $\hat{h}^s_{\mathbf{k}}$. We have used the notations that for scalar perturbations $s(\mathcal{R}) = 0$ and $\epsilon^0_{ij}(\mathbf{k}) \equiv \delta_{ij}$. For tensor perturbations, $s(h)$ is the polarization of the mode and $\epsilon^s_{ij}(\mathbf{k})$ is the polarization tensor that obeys $\epsilon^s_{ii}(\mathbf{k}) = \mathbf{k}^i \epsilon^s_{ij}(\mathbf{k}) = 0$ and $\epsilon^s_{ij}(\mathbf{k}) \epsilon^{s'}_{ij}(\mathbf{k}) = 2\delta_{ss'}$. $\mathcal{F}_{AC}$ is a measure of non-Gaussianity which can be calculated in terms of other observables using some very general arguments. Note that

$$f^{loc}_{NL} \equiv -\frac{5}{12} \mathcal{F}_{\mathcal{R}\mathcal{R}} \ . \tag{5.3}$$

5.2 Consistency relations

In this section, we will make some semiclassical arguments to reproduce the consistency relations (5.2) along with the non-Gaussianity parameters $\mathcal{F}_{AC}$. Our goal is to compute $\langle \hat{A}_{\mathbf{k}_1}(\tau) \hat{B}_{\mathbf{k}_2}(\tau) \hat{C}_{\mathbf{k}_3}(\tau) \rangle$ in the squeezed limit (i.e. $k_1, k_2 \gg k_3$), after k_1, k_2 modes have crossed the horizon; thus k_3 mode crossed the horizon in the distant past. Let us now clearly state all the assumptions that we are going to make: (1) it is effectively a single field inflation[1] and (2) there is no super-horizon evolution of the perturbations and hence both scalar and tensor perturbations are frozen outside the horizon. We will also use the exact squeezed limit $k_3 \to 0$.

Mode k_3 crosses the horizon long before modes k_1, k_2 and hence we can treat mode k_3 classically. It will contribute to the background metric and modes k_1 and k_2 evolve in this perturbed background which is given by

$$ds^2 = -dt^2 + a^2(t) g^B_{ij} d\mathbf{x}_i d\mathbf{x}_j \ , \tag{5.4}$$

where,

$$g^B_{ij} = e^{-2\mathcal{R}_B(\mathbf{x})\delta_{ij} + h_{B;ij}(\mathbf{x})} \tag{5.5}$$

is the contributions from modes far outside the horizon with $\mathcal{R}_B$ and $h_{B;ij}$ are given by

$$\mathcal{R}_B(\mathbf{x},\tau) = \int_{k \ll k_1,k_2} \frac{d^3\mathbf{k}}{(2\pi)^3} \mathcal{R}_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}} \ , \tag{5.6}$$

$$h_{B;ij}(\mathbf{x},\tau) = \int_{k \ll k_1,k_2} \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{s=+,\times} \epsilon^s_{ij}(\mathbf{k}) h^s_{\mathbf{k}}(\tau) e^{i\mathbf{k}.\mathbf{x}} \ . \tag{5.7}$$

Therefore, in the squeezed limit ($k_3 \ll k_1 \approx k_2$), we can make the approximation

$$\langle \hat{A}_{\mathbf{k}_1}(\tau) \hat{B}_{\mathbf{k}_2}(\tau) \hat{C}_{\mathbf{k}_3}(\tau) \rangle \approx \langle \langle \hat{A}_{\mathbf{k}_1}(\tau) \hat{B}_{\mathbf{k}_2}(\tau) \rangle_{\mathbf{k}_3} \hat{C}_{\mathbf{k}_3}(\tau) \rangle \ , \tag{5.8}$$

where, $\langle \hat{A}_{\mathbf{k}_1}(\tau) \hat{B}_{\mathbf{k}_2}(\tau) \rangle_{\mathbf{k}_3}$ is the two-point function in the perturbed background (5.4). It is clear from the metric (5.4) that long wavelength mode (neglecting

[1] This can be easily generalized for single-clock inflations.

the gradients) is equivalent to a change of coordinates

$$\mathbf{x}_i \to \mathbf{x}'_i = \Lambda_{ij}\mathbf{x}_j \ , \qquad \text{with} \qquad \Lambda_{ij} = e^{-\mathcal{R}_B\delta_{ij}+h_{B;ij}/2} \ . \tag{5.9}$$

5.2.1 Three scalars correlator

First, we need to find out how $\mathcal{R}_{\mathbf{k}}$ transforms under the change of coordinate (5.9). It is easy to check that Fourier transform of $\mathcal{R}(\mathbf{x}'(\mathbf{x}), \tau)$ is given by

$$\int d^3\mathbf{x}\ \mathcal{R}(\mathbf{x}'(\mathbf{x}), \tau)e^{-i\mathbf{k}.\mathbf{x}} = \det(\Lambda^{-1})\mathcal{R}_{\Lambda^{-1}\mathbf{k}} \ . \tag{5.10}$$

Therefore, under this change of the coordinates (5.9), $\mathcal{R}_{\mathbf{k}}$ transforms as

$$\mathcal{R}_{\mathbf{k}} \to \det(\Lambda^{-1})\mathcal{R}_{\Lambda^{-1}\mathbf{k}} \ , \qquad \text{with} \qquad (\Lambda^{-1})_{ij} = e^{\mathcal{R}_B\delta_{ij}-h_{B;ij}/2} \ . \tag{5.11}$$

Using the identity $\det(e^A) = e^{tr(A)}$ and $tr(h_{ij}) = 0$, we obtain

$$\det(\Lambda) = \exp\left[tr(-\mathcal{R}_B\delta_{ij} + h_{B;ij}/2)\right] = \exp(-3\mathcal{R}_B), \tag{5.12}$$

$$|\Lambda^{-1}\mathbf{k}| = \left[(\Lambda^{-1})_{ij}(\Lambda^{-1})_{il}\mathbf{k}_j\mathbf{k}_l\right]^{1/2} \approx k\left(1 + \mathcal{R}_B - h_{B;ij}\frac{\mathbf{k}_i\mathbf{k}_j}{2k^2}\right). \tag{5.13}$$

Also recall that

$$\delta^3(\Lambda^{-1}\mathbf{k}_1 + \Lambda^{-1}\mathbf{k}_2) = \det(\Lambda)\delta^3(\mathbf{k}_1 + \mathbf{k}_2). \tag{5.14}$$

Next we will compute $\langle\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\rangle$ in the perturbed background (5.4) after modes k_1, k_2 cross the horizon. In the unperturbed background, the two point function in the super-horizon limit is given by $\langle\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\rangle \propto \frac{1}{k^{(4-n_s)}}\delta^3(\mathbf{k}_1+$

$\mathbf{k}_2$), where n_s is the scalar spectral index at $k = k_1$. Now using equations (5.12-5.14), we finally obtain

$$
\begin{aligned}
\langle \hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\rangle_{\mathbf{k}_3} &= \det(\Lambda^{-1})^2 \langle \hat{\mathcal{R}}_{\Lambda^{-1}\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\Lambda^{-1}\mathbf{k}_2}(\tau)\rangle \\
&= \det(\Lambda^{-1})^2 (2\pi)^3 P_{\mathcal{R}}(|\Lambda^{-1}\mathbf{k}_1|)\delta^3(\Lambda^{-1}\mathbf{k}_1 + \Lambda^{-1}\mathbf{k}_2) \\
&= \det(\Lambda^{-1})^2 (2\pi)^3 P_{\mathcal{R}}(k_1) \frac{\det(\Lambda)}{\left(1 + \mathcal{R}_B - h_{B;ij}\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}}{2k_1^2}\right)^{4-n_s}} \delta^3(\mathbf{k}_1 + \mathbf{k}_2) \\
&= (2\pi)^3 P_{\mathcal{R}}(k_1) \frac{\det(\Lambda^{-1})}{\left(1 + \mathcal{R}_B - h_{B;ij}\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}}{2k_1^2}\right)^{4-n_s}} \delta^3(\mathbf{k}_1 + \mathbf{k}_2) \\
&= (2\pi)^3 P_{\mathcal{R}}(k_1) \left[1 + (n_s - 1)\mathcal{R}_B + \left(2 - \frac{n_s}{2}\right) h_{B;ij}\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}}{k_1^2} + ...\right] \\
&\qquad \times \delta^3(\mathbf{k}_1 + \mathbf{k}_2). \quad (5.15)
\end{aligned}
$$

So far we have treated $\mathcal{R}_{\mathbf{k}_3}$ and $h^s_{\mathbf{k}_3}$ as classical fields but now we will promote both $\mathcal{R}_{\mathbf{k}_3}$ and $h^s_{\mathbf{k}_3}$ to quantum operators. Replacing $\mathcal{R}_B$ in the last equation by (5.6), we obtain

$$
\begin{aligned}
\langle \hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)\rangle &\approx \langle\langle \hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\rangle_{\mathbf{k}_3}\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)\rangle \\
&= -(2\pi)^3 P_{\mathcal{R}}(k_1)\delta^3(\mathbf{k}_1 + \mathbf{k}_2)(1 - n_s)\int_{k\ll k_1,k_2} \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}.\mathbf{x}} \langle \hat{\mathcal{R}}_{\mathbf{k}}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)\rangle \\
&\approx -(2\pi)^3 P_{\mathcal{R}}(k_1)P_{\mathcal{R}}(k_3)\delta^3(\mathbf{k}_1 + \mathbf{k}_2)(1 - n_s)\int_{k\ll k_1,k_2} d^3\mathbf{k} e^{i\mathbf{k}.\mathbf{x}}\delta^3(\mathbf{k} + \mathbf{k}_3) \\
&\approx (2\pi)^3 P_{\mathcal{R}}(k_1)P_{\mathcal{R}}(k_3)(n_s - 1)\delta^3(\sum \mathbf{k}) \ , \quad (5.16)
\end{aligned}
$$

where, in the squeezed limit $\sum \mathbf{k} \equiv \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 \approx \mathbf{k}_1 + \mathbf{k}_2$. Comparing the last equation with equation (5.2), we find $\mathcal{F}_{\mathcal{R}\mathcal{R}} = (n_s - 1)$. Therefore, in the squeezed limit we have,

$$f_{NL}^{loc} \approx \frac{5}{12}(1 - n_s). \quad (5.17)$$

Note that for running spectral index, n_s in the last equation stands for spectral index at $k = k_1$.

5.2.2 Two scalars and a graviton correlator

We can perform a similar calculation for $\langle \hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{h}^s_{\mathbf{k}_3}(\tau)\rangle$. Replacing $h_{B;ij}$ in equation (5.15) by (5.7), we can write

$$\begin{aligned}
\langle \hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{h}^s_{\mathbf{k}_3}(\tau)\rangle &\approx \langle\langle \hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\rangle_{\mathbf{k}_3}\hat{h}^s_{\mathbf{k}_3}(\tau)\rangle \\
&= P_{\mathcal{R}}(k_1)\delta^3(\sum \mathbf{k})\left(2-\frac{n_s}{2}\right)\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}}{k_1^2} \\
&\qquad \times \int_{k\ll k_1,k_2} d^3\mathbf{k}e^{i\mathbf{k}.\mathbf{x}}\sum_{s'=+,\times}\epsilon^{s'}_{ij}(\mathbf{k})\langle \hat{h}^s_{\mathbf{k}}(\tau)\hat{h}^{s'}_{\mathbf{k}_3}(\tau)\rangle \\
&\approx (2\pi)^3 P_{\mathcal{R}}(k_1)P_h(k_3)\delta^3(\sum \mathbf{k})\left(2-\frac{n_s}{2}\right)\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}}{k_1^2} \\
&\qquad \times \int_{k\ll k_1,k_2} d^3\mathbf{k}\ \epsilon^s_{ij}(\mathbf{k})e^{i\mathbf{k}.\mathbf{x}}\delta^3(\mathbf{k}+\mathbf{k}_3) \\
&\approx (2\pi)^3 P_{\mathcal{R}}(k_1)P_h(k_3)\left(2-\frac{n_s}{2}\right)\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}\epsilon^s_{ij}(\mathbf{k}_3)}{k_1^2}\delta^3(\sum \mathbf{k})\ . \qquad (5.18)
\end{aligned}$$

Comparing the last equation with equation (5.2), we find

$$\mathcal{F}_{\mathcal{R}h} = \left(2-\frac{n_s}{2}\right)\ . \qquad (5.19)$$

5.2.3 Two gravitons and a scalar correlator

It is straight forward to use a similar argument to compute the other two three-point functions. First let us note that the Fourier transform of $h_{ij}(\mathbf{x},\tau)$ is given by,

$$\sum_{s=+,\times}\epsilon^s_{ij}(\mathbf{k})h^s_{\mathbf{k}}(\tau) = \int d^3\mathbf{x}h_{ij}(\mathbf{x},\tau)\ e^{-i\mathbf{k}.\mathbf{x}}. \qquad (5.20)$$

Under the coordinate transformation (5.9), $h_{ij}(\mathbf{x},\tau) \to h_{ij}(\mathbf{x}'(\mathbf{x}),\tau)$ and the fourier transform of $h_{ij}(\mathbf{x}'(\mathbf{x}),\tau)$ is given by,

$$\int d^3\mathbf{x} h_{ij}(\mathbf{x}'(\mathbf{x}),\tau)\ e^{-i\mathbf{k}.\mathbf{x}} = \det(\Lambda^{-1}) \sum_{s'=+,\times} \epsilon^{s'}_{ij}(\Lambda^{-1}\mathbf{k}) h^{s'}_{\Lambda^{-1}\mathbf{k}}(\tau). \tag{5.21}$$

Therefore, under this coordiante transformation, $h^s_{\mathbf{k}}(\tau)$ transforms as

$$h^s_{\mathbf{k}}(\tau) \to \frac{1}{2}\det(\Lambda^{-1}) \sum_{s'=+,\times} \epsilon^{s'}_{ij}(\Lambda^{-1}\mathbf{k})\epsilon^s_{ij}(\mathbf{k}) h^{s'}_{\Lambda^{-1}\mathbf{k}}(\tau). \tag{5.22}$$

We need to compute $\langle \hat{h}^s_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\rangle$ in the perturbed background (5.4) after modes k_1, k_2 cross the horizon. Now, using the last equation, we obtain

$$\begin{aligned}
&\langle \hat{h}^s_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\rangle_{\mathbf{k}_3} \\
=&\frac{1}{4}\det(\Lambda^{-1})^2 \sum_{s_1,s_2=+,\times} \epsilon^{s_1}_{ij}(\Lambda^{-1}\mathbf{k}_1)\epsilon^s_{ij}(\mathbf{k}_1)\epsilon^{s_2}_{kl}(\Lambda^{-1}\mathbf{k}_2)\epsilon^{s'}_{kl}(\mathbf{k}_2)\langle \hat{h}^{s_1}_{\Lambda^{-1}\mathbf{k}_1}(\tau)\hat{h}^{s_2}_{\Lambda^{-1}\mathbf{k}_2}(\tau)\rangle \\
=&\frac{1}{4}\det(\Lambda^{-1})^2(2\pi)^3 P_h(|\Lambda^{-1}\mathbf{k}_1|) \\
&\qquad \times \sum_{s_1,s_2=+,\times} \epsilon^{s_1}_{ij}(\Lambda^{-1}\mathbf{k}_1)\epsilon^s_{ij}(\mathbf{k}_1)\epsilon^{s_2}_{kl}(\Lambda^{-1}\mathbf{k}_2)\epsilon^{s'}_{kl}(\mathbf{k}_2)\delta_{s_1 s_2}\delta^3(\Lambda^{-1}\mathbf{k}_1+\Lambda^{-1}\mathbf{k}_2) \\
=&\frac{(2\pi)^3}{4}\det(\Lambda^{-1})P_h(|\Lambda^{-1}\mathbf{k}_1|)\epsilon^s_{ij}(\mathbf{k}_1)\epsilon^{s'}_{kl}(\mathbf{k}_1) \\
&\qquad \times \sum_{s_1=+,\times} \epsilon^{s_1}_{ij}(\Lambda^{-1}\mathbf{k}_1)\epsilon^{s_1}_{kl}(\Lambda^{-1}\mathbf{k}_1)\delta^3(\mathbf{k}_1+\mathbf{k}_2)
\end{aligned} \tag{5.23}$$

In the unperturbed background, the two point function of gravitons in the super-horizon limit can be written as $\langle \hat{h}^s_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\rangle \propto \frac{1}{k^{(3-n_t)}}\delta^3(\mathbf{k}_1+\mathbf{k}_2)$, where

n_t is the tensor spectral index. Therefore,

$$\begin{aligned}\langle \hat{h}^s_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\rangle_{\mathbf{k}_3} = &\frac{(2\pi)^3}{4}P_h(k_1)\epsilon^s_{ij}(\mathbf{k}_1)\epsilon^{s'}_{kl}(\mathbf{k}_1)\Pi_{ij,kl}(\Lambda^{-1}\mathbf{k}_1) \\ &\times \frac{\det(\Lambda^{-1})}{\left(1+\mathcal{R}_B - h_{B;ij}\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}}{2k_1^2}\right)^{3-n_t}}\delta^3(\mathbf{k}_1+\mathbf{k}_2) \\ =&\frac{(2\pi)^3}{4}P_h(k_1)\epsilon^s_{ij}(\mathbf{k}_1)\epsilon^{s'}_{kl}(\mathbf{k}_1)\Pi_{ij,kl}(\Lambda^{-1}\mathbf{k}_1) \\ &\times\left[1+n_t\mathcal{R}_B+\left(\frac{3}{2}-\frac{n_t}{2}\right)h_{B;ij}\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}}{k_1^2}+...\right]\delta^3(\mathbf{k}_1+\mathbf{k}_2)\ ,\end{aligned} \tag{5.24}$$

where, $\Pi_{ij,lm}(\mathbf{k})$ is defined as

$$\Pi_{ij,lm}(\mathbf{k}) = \sum_{s=+,\times}\epsilon^s_{ij}(\mathbf{k})\epsilon^s_{lm}(\mathbf{k}). \tag{5.25}$$

A formula can be obtained for $\Pi_{ij,lm}(\mathbf{k})$ by using the conditions that $\Pi_{ij,lm}(\mathbf{k})$ is a tensor function of $\hat{\mathbf{k}}$ (because polarization tensor $\epsilon^s_{ij}(\mathbf{k})$ depends only on the direction of vector $\mathbf{k}$), symmetric in i and j and in l and m and $\Pi_{ij,lm}(\mathbf{k}) = \Pi_{lm,ij}(\mathbf{k})$. $\Pi_{ij,lm}(\mathbf{k})$ also obeys the conditions $\mathbf{k}_i\Pi_{ij,lm}(\mathbf{k}) = 0$ and $\Pi_{ij,ij}(\mathbf{k}) = 4$. The last condition comes from the normalization of the polarization tensor $\epsilon^s_{ij}(\mathbf{k})$. Finally we have,

$$\begin{aligned}\Pi_{ij,lm}(\mathbf{k}) =&\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}-\delta_{ij}\delta_{lm}+\delta_{ij}\hat{\mathbf{k}}_l\hat{\mathbf{k}}_m+\delta_{lm}\hat{\mathbf{k}}_i\hat{\mathbf{k}}_j \\ &-\delta_{il}\hat{\mathbf{k}}_j\hat{\mathbf{k}}_m-\delta_{im}\hat{\mathbf{k}}_j\hat{\mathbf{k}}_l-\delta_{jl}\hat{\mathbf{k}}_i\hat{\mathbf{k}}_m-\delta_{jm}\hat{\mathbf{k}}_i\hat{\mathbf{k}}_l+\hat{\mathbf{k}}_i\hat{\mathbf{k}}_j\hat{\mathbf{k}}_l\hat{\mathbf{k}}_m.\end{aligned} \tag{5.26}$$

That leads to

$$\epsilon^s_{ij}(\mathbf{k}_1)\epsilon^{s'}_{kl}(\mathbf{k}_1)\Pi_{ij,kl}(\Lambda^{-1}\mathbf{k}_1) = 4\delta_{ss'}+O(h_{ij}^2) \tag{5.27}$$

yielding

$$\langle \hat{h}^s_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\rangle_{\mathbf{k}_3} =(2\pi)^3 P_h(k_1)\delta_{ss'}\delta^3(\mathbf{k}_1+\mathbf{k}_2) \times \left[1+n_t\mathcal{R}_B+\left(\frac{3}{2}-\frac{n_t}{2}\right)h_{B;ij}\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}}{k_1^2}+...\right] . \quad (5.28)$$

Again, we will promote both $\mathcal{R}_{\mathbf{k}_3}$ and $h^s_{\mathbf{k}_3}$ to quantum operators. Using equations (5.7) and (5.28), we obtain

$$\begin{aligned}\langle \hat{h}^s_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)\rangle &\approx \langle\langle \hat{h}^s_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\rangle_{\mathbf{k}_3}\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)\rangle \\ &= (2\pi)^3 P_h(k_1)\delta_{ss'}\delta^3(\sum\mathbf{k})n_t\int_{k\ll k_1,k_2}\frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}.\mathbf{x}}\langle\hat{\mathcal{R}}_{\mathbf{k}}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)\rangle \\ &\approx (2\pi)^3 P_h(k_1)P_{\mathcal{R}}(k_3)\delta_{ss'}\delta^3(\sum\mathbf{k})n_t\int_{k\ll k_1,k_2}d^3\mathbf{k}e^{i\mathbf{k}.\mathbf{x}}\delta^3(\mathbf{k}+\mathbf{k}_3) \\ &\approx (2\pi)^3 P_h(k_1)P_{\mathcal{R}}(k_3)\ n_t\ \delta_{ss'}\delta^3(\sum\mathbf{k})\ , \quad (5.29)\end{aligned}$$

where, in the squeezed limit $\sum\mathbf{k}\equiv\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3\approx\mathbf{k}_1+\mathbf{k}_2$. Comparing the last equation with equation (5.2), we find $\mathcal{F}_{h\mathcal{R}}=n_t$. Note that for running spectral index, n_t in the last equation stands for the tensor spectral index at $k=k_1$.

5.2.4 Three gravitons correlator

Using equations (5.7) and (5.28), we obtain

$$\begin{aligned}
\langle \hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\hat{h}^{s''}_{\mathbf{k}_3}(\tau)\rangle &\approx \langle\langle \hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\rangle_{\mathbf{k}_3}\hat{h}^{s''}_{\mathbf{k}_3}(\tau)\rangle \\
&= (2\pi)^3 P_h(k_1)\delta_{ss'}\delta^3(\sum \mathbf{k})\left(\frac{3-n_t}{2}\right)\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}}{k_1^2} \\
&\quad \times \int_{k\ll k_1,k_2}\frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}.\mathbf{x}}\sum_{s_1=+,\times}\epsilon^{s_1}_{ij}(\mathbf{k})\langle \hat{h}^{s''}_{\mathbf{k}}(\tau)\hat{h}^{s_1}_{\mathbf{k}_3}(\tau)\rangle \\
&\approx (2\pi)^3 P_h(k_1)P_h(k_3)\delta_{ss'}\delta^3(\sum \mathbf{k})\left(\frac{3-n_t}{2}\right)\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}}{k_1^2} \\
&\quad \times \int_{k\ll k_1,k_2} d^3\mathbf{k}\ \epsilon^{s''}_{ij}(\mathbf{k})e^{i\mathbf{k}.\mathbf{x}}\delta^3(\mathbf{k}+\mathbf{k}_3) \\
&\approx (2\pi)^3 P_h(k_1)P_h(k_3)\delta_{ss'}\left(\frac{3-n_t}{2}\right)\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}\epsilon^{s''}_{ij}(\mathbf{k}_3)}{k_1^2}\delta^3(\sum \mathbf{k}).
\end{aligned} \tag{5.30}$$

Comparing the last equation with equation (5.2), we find

$$\mathcal{F}_{hh} = \left(\frac{3-n_t}{2}\right)\ , \tag{5.31}$$

with n_t being the tensor spectral index at $k = k_1$.

From the discussion of this section it is also obvious that both $\langle \hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{h}^{s'}_{\mathbf{k}_3}(\tau)\rangle$ and $\langle \hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)\rangle$ are zero in the squeezed limit $k_1, k_2 \gg k_3$ because there is no cross-correlation between scalar and tensor perturbations, i.e., $\langle \hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\rangle = 0$. More precisely,

$$\frac{\langle \hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{h}^{s'}_{\mathbf{k}_3}(\tau)\rangle}{P_{\mathcal{R}}(k_2)P_h(k_3)}\ \&\ \frac{\langle \hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)\rangle}{P_{\mathcal{R}}(k_3)P_h(k_1)} \to 0 \tag{5.32}$$

in the limit $k_3/k_1 \to 0$. Therefore, three-point functions of scalar and tensor perturbations in the squeezed limit, indeed obey consistency relations of the form (5.2) where $\mathcal{F}_{AC}$ is given

$$\mathcal{F} \equiv \begin{pmatrix} \mathcal{F}_{\mathcal{R}\mathcal{R}} & \mathcal{F}_{\mathcal{R}h} \\ \mathcal{F}_{h\mathcal{R}} & \mathcal{F}_{hh} \end{pmatrix} = \begin{pmatrix} n_s - 1 & 2 - \frac{n_s}{2} \\ n_t & \frac{3-n_t}{2} \end{pmatrix} , \tag{5.33}$$

where all the quantities are evaluated at $k = k_1$.

It is important to note that there is an implicit assumption in the derivation of the consistency relations which plays a crucial role. In our derivation, we have taken the squeezed limit $k_3 \to 0$ first and that allows us to approximate the effect of k_3-mode as a perturbation to the background metric (5.4). But in an honest calculation of squeezed limit three-point function, one should compute the three-point function first and then take the squeezed limit $k_3 \to 0$. So, we have made the assumption that the terms that we ignored by taking the squeezed limit first are small. However, we will show that this assumption is not always valid when the perturbations are in excited initial states.

Chapter 6

Non-Gaussianities and general initial states

In the last chapter, we showed that under very general assumptions: (i) it is effectively a single field inflation and (ii) there is no super-horizon evolution of the perturbations (i.e. both scalar and tensor perturbations are frozen outside the horizon), the three-point functions of comoving curvature perturbation ($\mathcal{R}_{\mathbf{k}}$) and tensor perturbation ($h^s_{\mathbf{k}}$), in the squeezed limit $k_1, k_2 \gg k_3$ obey certain consistency relations which are of the form

$$
\begin{aligned}
\langle \hat{\mathcal{R}}_{\mathbf{k}_1}\hat{\mathcal{R}}_{\mathbf{k}_2}\hat{\mathcal{R}}_{\mathbf{k}_3}\rangle &= (2\pi)^3 P_{\mathcal{R}}(k_1)P_{\mathcal{R}}(k_3)(n_s-1)\delta^3(\sum \mathbf{k}) \ , \\
\langle \hat{\mathcal{R}}_{\mathbf{k}_1}\hat{\mathcal{R}}_{\mathbf{k}_2}\hat{h}^s_{\mathbf{k}_3}\rangle &= (2\pi)^3 P_{\mathcal{R}}(k_1)P_h(k_3)\left(2-\frac{n_s}{2}\right)\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}\epsilon^s_{ij}(\mathbf{k}_3)}{k_1^2}\delta^3(\sum \mathbf{k}) \ , \\
\langle \hat{h}^s_{\mathbf{k}_1}\hat{h}^{s'}_{\mathbf{k}_2}\hat{\mathcal{R}}_{\mathbf{k}_3}\rangle &= (2\pi)^3 P_h(k_1)P_{\mathcal{R}}(k_3)\ n_t\ \delta_{ss'}\delta^3(\sum \mathbf{k}) \ , \\
\langle \hat{h}^s_{\mathbf{k}_1}\hat{h}^{s'}_{\mathbf{k}_2}\hat{h}^{s''}_{\mathbf{k}_3}\rangle &= (2\pi)^3 P_h(k_1)P_h(k_3)\delta_{ss'}\left(\frac{3-n_t}{2}\right)\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}\epsilon^{s''}_{ij}(\mathbf{k}_3)}{k_1^2}\delta^3(\sum \mathbf{k}) \ .
\end{aligned}
$$

The other two squeezed limit three-point functions $\langle \hat{h}^s_{\mathbf{k}_1}\hat{\mathcal{R}}_{\mathbf{k}_2}\hat{h}^{s'}_{\mathbf{k}_3}\rangle$, $\langle \hat{h}^s_{\mathbf{k}_1}\hat{\mathcal{R}}_{\mathbf{k}_2}\hat{\mathcal{R}}_{\mathbf{k}_3}\rangle$ vanish in the limit $k_3/k_1 \to 0$. There has been a great deal of progress in measuring the three-point function (bispectrum) of scalar perturbation from the CMB and LSS indicating nearly gaussian primordial fluctuations. The current observational constraint on the non-Gaussianity parameter f_{NL}[30] is very weak and f^{loc}_{NL} remains the best constrained non-Gaussianity parameter:

$f_{NL}^{loc} = 2.7 \pm 5.8$ (Planck)[6].

One aspect of prime interest is to understand how much information about the quantum state of primordial fluctuations in the beginning of inflation can be obtained. In order to achieve that it is essential to explore the dependence of different observables on initial states of the primordial fluctuations. Recently, this area of research has instigated interest among physicists mainly because it has the exciting possibility of providing us a window for the physics before inflation. In this paper, we explore how the single-field consistency relations depend on initial states of the perturbations. The primary motivation is two-fold: first as the precision of the observations is increasing significantly, we may learn more about the initial state of the fluctuations in the near future. Second, in order for the consistency relations to be applied as a tool to falsify or establish single-field inflation, it is important to know if they are valid for general initial states.

The three-point functions of primordial fluctuations are generally computed assuming that the fluctuations are initially in the Bunch-Davies state[42, 39]. For slow-roll inflation it is well known that all the consistency relations are obeyed for the Bunch-Davies initial state [97].In this chapter, we will compute the three-point functions of scalar and tensor perturbations for slow-roll inflation. We will show that the consistency relations are obeyed for coherent initial states; in fact all the three-point functions of scalar and tensor perturbations with a coherent state as the initial state are identical to the three-point functions with the Bunch-Davies initial state. It is perhaps not very surprising since one can think of a coherent state as zero-point quantum fluctuations around some classical state. Interactions will generate non-trivial one-point functions, however, that will contribute only to the classical part. The quantum fluctuations contribute to the physically relevant part of the three-point correlations and hence they remain unchanged.

On the other hand, the consistency relation of scalar three-point function is known to be violated for initial states that are related to the Bunch-Davies state by Bogoliubov transformations (we will call them α-states)[67, 14, 65, 19].[1,2] We will show that when both scalar and tensor perturbations are initially in α-states, all four consistency relations can be violated. For the derivation of the consistency relations, it is necessary to take the squeezed limit first and then calculate the three-point functions. However, in an honest calculation of the squeezed limit three-point function for a particular model, one should compute the three-point function first and then take the squeezed limit. So, there is an implicit assumption that the terms that are ignored by taking the squeezed limit first are small. However, for α-states the correction terms are large in the squeezed limit and hence this assumption is not valid. Let us also note that the other two squeezed limit three point functions $\langle \hat{h}^s_{\mathbf{k}_1} \hat{\mathcal{R}}_{\mathbf{k}_2} \hat{h}^{s'}_{\mathbf{k}_3} \rangle$ and $\langle \hat{h}^s_{\mathbf{k}_1} \hat{\mathcal{R}}_{\mathbf{k}_2} \hat{\mathcal{R}}_{\mathbf{k}_3} \rangle$ remain vanishingly small even for α-states because there is still no cross-correlation between scalar and

[1]These states are also called Bogoliubov states in the literature because of obvious reason but we think the name α-states is infinitesimally better because of their similarity with the α-states of de Sitter space.

[2]The consistency relation is also violated for models that involve non-attractor dynamics[109, 49, 48].

tensor perturbations.

6.1 Three-point functions and general initial states

In this section, we will set up the calculation for $\langle \hat{O}_{\mathbf{k}_1}(\tau)\hat{O}_{\mathbf{k}_2}(\tau)\hat{O}_{\mathbf{k}_3}(\tau)\rangle$ (where $\hat{O}_{\mathbf{k}}$ is either scalar perturbation $\hat{\mathcal{R}}_{\mathbf{k}}$ or tensor perturbation $\hat{h}^s_{\mathbf{k}}$) with a general initial state, then we will review the results for the Bunch-Davies state. For a general state $|G\rangle$, the operator $\hat{O}_{\mathbf{k}}$ can have a non vanishing expectation value and hence physically relevant part of the three-point function is given by,

$$\langle \hat{O}_{\mathbf{k}_1}(\tau)\hat{O}_{\mathbf{k}_2}(\tau)\hat{O}_{\mathbf{k}_3}(\tau)\rangle_{phy} \equiv \langle \{\hat{O}_{\mathbf{k}_1}(\tau) - \langle \hat{O}_{\mathbf{k}_1}(\tau)\rangle\}\{\hat{O}_{\mathbf{k}_2}(\tau) - \langle \hat{O}_{\mathbf{k}_2}(\tau)\rangle\} \\ \{\hat{O}_{\mathbf{k}_3}(\tau) - \langle \hat{O}_{\mathbf{k}_3}(\tau)\rangle\}\rangle \ , \quad (6.1)$$

where $\langle \hat{A}\rangle \equiv \langle G|\hat{A}|G\rangle$. Using time-dependent perturbation theory, for any operator (e.g. $\hat{O}_{\mathbf{k}_1}(\tau)\hat{O}_{\mathbf{k}_2}(\tau)$...) we have,

$$\langle G|\hat{O}_{\mathbf{k}_1}(\tau)\hat{O}_{\mathbf{k}_2}(\tau)...|G\rangle = \langle G|\left(\bar{T}e^{i\int_{\tau_0}^{\tau} H^I_{int}(\tau')d\tau'}\right)\hat{O}^I_{\mathbf{k}_1}(\tau)\hat{O}^I_{\mathbf{k}_2}(\tau)... \\ \times \left(Te^{-i\int_{\tau_0}^{\tau} H^I_{int}(\tau')d\tau'}\right)|G\rangle, \quad (6.2)$$

where all fields are in the interaction picture and $H^I_{int}(\tau)$ is the interacting part of the Hamiltonian in the interaction picture. T and $\bar{T}$ are the time and anti-time ordered product respectively. τ_0 is the conformal time at the beginning of inflation. At first order in perturbation theory, we obtain,

$$\begin{aligned}&\langle G|\hat{O}_{\mathbf{k}_1}(\tau)\hat{O}_{\mathbf{k}_2}(\tau)...|G\rangle\\&\quad=\langle G|\hat{O}^I_{\mathbf{k}_1}(\tau)\hat{O}^I_{\mathbf{k}_2}(\tau)...|G\rangle-i\int_{\tau_0}^{\tau}d\tau'\langle G|\left[\hat{O}^I_{\mathbf{k}_1}(\tau)\hat{O}^I_{\mathbf{k}_2}(\tau)...,H^I_{int}(\tau')\right]|G\rangle.\end{aligned} \tag{6.3}$$

So far our discussion is very general and does not depend on the details of the inflationary model.

For slow-roll inflation, we can use equations (6.1,6.3) to calculate different three-point functions. We are interested in the late time behavior of the three-point functions i.e we will take the usual limit $\tau\to 0$. For simplicity we will assume that for the free theory, the operator expectation value $\langle\hat{O}_{\mathbf{k}}(\tau)\rangle$ in state $|G\rangle$ vanishes at late time.[3] Therefore, at first order in slow-roll parameters, three-point functions are given by,

$$\begin{aligned}\langle G|\hat{O}_{\mathbf{k}_1}(\tau)\hat{O}_{\mathbf{k}_2}(\tau)\hat{O}_{\mathbf{k}_3}(\tau)|G\rangle_{phy}&=\langle G|\hat{O}_{\mathbf{k}_1}(\tau)\hat{O}_{\mathbf{k}_2}(\tau)\hat{O}_{\mathbf{k}_3}(\tau)|G\rangle\\&-\left(\langle G|\hat{O}_{\mathbf{k}_1}(\tau)|G\rangle\langle G|\hat{O}_{\mathbf{k}_2}(\tau)\hat{O}_{\mathbf{k}_3}(\tau)|G\rangle+\text{cyclic perm}\right),\end{aligned} \tag{6.4}$$

where all the quantities in the right hand side should be computed using equation (6.3).

6.2 Non-Gaussianities: Bunch-Davies state

Next, as a warm up exercise we will calculate the three-point functions for slow-roll inflation with the Bunch-Davies initial state (defined in section

[3]We should note that interactions can generate non vanishing one-point functions even for these states.

(2.3))

$$|0\rangle \equiv |0\rangle_{\rm scalar} \otimes |0_{s=+}\rangle \otimes |0_{s=\times}\rangle \ , \tag{6.5}$$

to demonstrate that all the consistency relations (5.2,5.33) are satisfied and in the leading order we obtain

$$\mathcal{F} \equiv \begin{pmatrix} \mathcal{F}_{\mathcal{R}\mathcal{R}} & \mathcal{F}_{\mathcal{R}h} \\ \mathcal{F}_{h\mathcal{R}} & \mathcal{F}_{hh} \end{pmatrix} = \begin{pmatrix} -6\epsilon + 2\eta & 3/2 \\ -2\epsilon & 3/2 \end{pmatrix} \ . \tag{6.6}$$

6.2.1 Three scalars correlator

At leading order in the slow-roll parameters, the third order action for scalar fluctuations is given by [97]

$$S_3 = -8\pi G \int d^3x d\tau \ a^3(\tau) \left(\frac{\dot{\phi}}{H}\right)^4 H \mathcal{R}_c'^2 \partial^{-2} \mathcal{R}_c', \tag{6.7}$$

where, $\mathcal{R}_c$ is the redefined field

$$\mathcal{R} = \mathcal{R}_c - \frac{1}{4}\left(3\epsilon - 2\eta\right) \mathcal{R}_c^2 - \frac{1}{2}\epsilon \ \partial^{-2}\left(\mathcal{R}_c \partial^2 \mathcal{R}_c\right). \tag{6.8}$$

In momentum space the last equation becomes

$$\mathcal{R}_{\mathbf{k}} = \mathcal{R}_{c,\mathbf{k}} - \frac{1}{4}\left(3\epsilon - 2\eta\right) \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathcal{R}_{c,\mathbf{p}} \mathcal{R}_{c,\mathbf{k}-\mathbf{p}} - \frac{1}{2}\epsilon \ \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{(\mathbf{k}-\mathbf{p})^2}{k^2} \mathcal{R}_{c,\mathbf{p}} \mathcal{R}_{c,\mathbf{k}-\mathbf{p}}. \tag{6.9}$$

The interaction Hamiltonian can be found from $S_3 = -\int d\tau H_{int}$. In momentum space H_{int} is given by

$$\begin{aligned} H_{int}(\tau) = &- \frac{8\pi G}{(2\pi)^6} a^3(\tau) \left(\frac{\dot{\phi}}{H}\right)^4 H \\ &\times \int d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 \left(\frac{1}{p_3^2}\right) \mathcal{R}'_{\mathbf{p}_1}(\tau) \mathcal{R}'_{\mathbf{p}_2}(\tau) \mathcal{R}'_{\mathbf{p}_3}(\tau) \delta^3(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3). \end{aligned} \tag{6.10}$$

For the Bunch-Davies state, only the first term of equation (6.4) contributes and we obtain,

$$\langle 0|\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)|0\rangle_{phy} = \langle 0|\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_3}(\tau)|0\rangle - i\int_{\tau_0}^{\tau} d\tau' \langle 0|\left[\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_3}(\tau), H^I_{int}(\tau')\right]|0\rangle. \tag{6.11}$$

The first term in the last equation can be written using the redefined field (6.9)

$$\begin{aligned}
&\langle 0|\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_3}(\tau)|0\rangle = \\
&-\frac{1}{4}(3\epsilon - 2\eta)\left(\int \frac{d^3\mathbf{p}}{(2\pi)^3}\langle 0|\hat{\mathcal{R}}^I_{c,\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{p}}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_3-\mathbf{p}}(\tau)|0\rangle + \text{cyc perm}\right) \\
&-\frac{1}{2}\epsilon\left(\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{(\mathbf{k}_3-\mathbf{p})^2}{k_3^2}\langle 0|\hat{\mathcal{R}}^I_{c,\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{p}}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_3-\mathbf{p}}(\tau)|0\rangle + \text{cyc perm}\right).
\end{aligned} \tag{6.12}$$

$\hat{\mathcal{R}}^I_{c,\mathbf{k}}(\tau)$ behaves like the free field, and can be written as

$$\hat{\mathcal{R}}^I_{c,\mathbf{k}}(\tau) = \frac{1}{\sqrt{2}}\left[\hat{a}^0_{\mathbf{k}}\mathcal{R}^*_k(\tau) + \hat{a}^{0\dagger}_{-\mathbf{k}}\mathcal{R}_k(\tau)\right], \tag{6.13}$$

where $\mathcal{R}_k(\tau) = \left(\frac{H}{a\dot{\phi}}\right)\frac{e^{ik\tau}}{\sqrt{k}}\left(1 + \frac{i}{k\tau}\right)$ and operator $\hat{a}^0_{\mathbf{k}}$ annihilates $|0\rangle_{\text{scalar}}$. At the leading order in the slow-roll parameters equation (A.2) becomes

$$\langle 0|\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_3}(\tau)|0\rangle = -(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)P_R(k_2)P_R(k_1) \times \left[\frac{1}{2}\left(3\epsilon - 2\eta + \epsilon\frac{k_1^2+k_2^2}{k_3^2}\right)\right] + \text{cyc perm}. \tag{6.14}$$

Next term in the equation (6.11) can be easily computed yielding[4]

$$-i\int_{\tau_0}^{\tau} d\tau' \langle 0| \left[\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_3}(\tau), H^I_{int}(\tau')\right] |0\rangle = -(2\pi)^3 P_R(k_2)P_R(k_1)$$
$$\times\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\left(\frac{4\epsilon}{(k_1+k_2+k_3)}\frac{k_1^2k_2^2}{k_3^3}\right) + \text{cyc perm.} \tag{6.15}$$

Therefore, finally we have,

$$\langle 0|\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)|0\rangle_{phy} = -(2\pi)^3\delta^3(\sum\mathbf{k})P_R(k_2)P_R(k_1)$$
$$\times\left[\frac{1}{2}\left(3\epsilon-2\eta+\epsilon\frac{k_1^2+k_2^2}{k_3^2}\right)+\frac{4\epsilon}{(k_1+k_2+k_3)}\frac{k_1^2k_2^2}{k_3^3}\right] + \text{cyc perm,} \tag{6.16}$$

where $\sum\mathbf{k} = \mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3$. Therefore, in the squeezed limit, f_{NL}^{loc} is given by,

$$f_{NL}^{loc} \approx \frac{5}{12}(1-n_s). \tag{6.17}$$

6.2.2 Two scalars and a graviton correlator

At leading order in the slow-roll parameters, the relevant part of the action is given by [97]

$$S_3 = \frac{1}{2}\int d^3x d\tau\ a(\tau)^2\left(\frac{\dot{\bar{\phi}}^2}{H^2}\right)h_{ij}\partial_i\mathcal{R}_c\partial_j\mathcal{R}_c\ , \tag{6.18}$$

where, $\mathcal{R}_c$ is again a redefined field which has a form similar to (6.8), however, for this computation only the leading part is important and hence $\mathcal{R}_c = \mathcal{R}$.

---

[4]For large τ_0, all exponentials with τ_0 will oscillate. When performing the calculations, we can either use the average value (i.e. zero) for them or we can choose an integration contour such that the oscillating pieces decrease exponentially for large τ_0 [97].

In momentum space H_{int} is given by

$$H_{int}(\tau) = \frac{a^2(\tau)}{2(2\pi)^6}\left(\frac{\dot{\phi}^2}{H^2}\right)\sum_{s'=+,\times}\int d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 \epsilon^{s'}_{ij}(\mathbf{p}_3)\mathbf{p}_{1,i}\mathbf{p}_{2,j} \times \mathcal{R}_{\mathbf{p_1}}(\tau)\mathcal{R}_{\mathbf{p_2}}(\tau)h^{s'}_{\mathbf{p_3}}(\tau)\delta^3(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3). \qquad (6.19)$$

And hence at first order in perturbation theory, the three-point function is given by

$$\langle 0|\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{h}^s_{\mathbf{k}_3}(\tau)|0\rangle_{phy} = \langle 0|\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{h}^{s,I}_{\mathbf{k}_3}(\tau)|0\rangle - i\int_{\tau_0}^{\tau} d\tau' \langle 0|\left[\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{h}^{s,I}_{\mathbf{k}_3}(\tau), H^I_{int}(\tau')\right]|0\rangle. \qquad (6.20)$$

The first term in the last equation vanishes. The second term can be computed easily, yielding

$$\langle 0|\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{h}^s_{\mathbf{k}_3}(\tau)|0\rangle_{phy} = (2\pi)^3\delta^3(\sum \mathbf{k})\frac{H^6}{2{M_{\rm pl}}^2\dot{\phi}^2}\epsilon^s_{ij}(\mathbf{k}_3)\mathbf{k}_{1,i}\mathbf{k}_{2,j}\mathcal{I}(k_1,k_2,k_3) \qquad (6.21)$$

where,

$$\mathcal{I}(k_1,k_2,k_3) = \frac{1}{(k_1k_2k_3)^3}\left(-k_t + \frac{k_1k_2k_3}{k_t^2} + \frac{k_1k_2+k_2k_3+k_1k_3}{k_t}\right) \qquad (6.22)$$

with $k_t = k_1+k_2+k_3$.

Let us consider two limiting cases. In the limit $k_3 << k_1, k_2$, we recover

$$\langle 0|\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{h}^s_{\mathbf{k}_3}(\tau)|0\rangle_{phy} = (2\pi)^3\delta^3(\sum \mathbf{k})P_{\mathcal{R}}(k_1)P_h(k_3)\frac{\epsilon^s_{ij}(\mathbf{k}_3)\mathbf{k}_{1,i}\mathbf{k}_{1,j}}{k_1^2}\left(\frac{3}{2}\right). \qquad (6.23)$$

Where we have used the fact that $\epsilon^s_{ij}(\mathbf{k}_3)\mathbf{k}_{1,i}\mathbf{k}_{2,j} = -\epsilon^s_{ij}(\mathbf{k}_3)\mathbf{k}_{1,i}\mathbf{k}_{1,j}$. Recall that $n_s \sim 1$ and hence this result is consistent with (5.2) and (5.33).

Also note that in the limit $k_3 << k_1, k_2$, we get

$$\frac{\langle 0|\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{h}^s_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)|0\rangle_{phy}}{P_h(k_1)P_{\mathcal{R}}(k_3)} \approx \mathcal{O}\left(\frac{k_3^2}{k_1^2}\right) \tag{6.24}$$

and hence consistent with (5.2).

6.2.3 Two gravitons and a scalar correlator

At leading order in the slow-roll parameters, the relevant part of the action is given by

$$S_3 = -\frac{1}{4}\int d^3x d\tau\ a(\tau)^3 H \left(\frac{\dot{\bar{\phi}}^2}{H^2}\right) h'_{ij}h'_{ij}\partial^{-2}\mathcal{R}'_c\ , \tag{6.25}$$

where, following [97] we have done further field redefinition

$$\mathcal{R} = \mathcal{R}_c + \frac{1}{32}h_{ij}h_{ij} - \frac{1}{16}\partial^{-2}\left(h_{ij}\partial^2 h_{ij}\right) + ... \tag{6.26}$$

where dots represent terms that are negligible outside the horizon. In momentum space the last equation becomes,

$$\mathcal{R}_{\mathbf{k}} = \mathcal{R}_{c,\mathbf{k}} + \frac{1}{32}\sum_{s,s'=+,\times}\int \frac{d^3\mathbf{p}}{(2\pi)^3} h^s_{\mathbf{p}} h^{s'}_{\mathbf{k}-\mathbf{p}}\epsilon^s_{ij}(\mathbf{p})\epsilon^{s'}_{ij}(\mathbf{k}-\mathbf{p})\left(1 - \frac{2(\mathbf{k}-\mathbf{p})^2}{k^2}\right)\ . \tag{6.27}$$

In momentum space, the interaction Hamiltonian is given by

$$\begin{aligned} H_{int}(\tau) = &- \frac{1}{4(2\pi)^6}a^3(\tau)H\left(\frac{\dot{\bar{\phi}}^2}{H^2}\right)\sum_{s_1,s_2=+,\times}\int d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3\left(\frac{1}{p_3^2}\right) \\ &\times h^{s_1}_{\mathbf{p}_1}{}'(\tau) h^{s_2}_{\mathbf{p}_2}{}'(\tau)\mathcal{R}'_{c,\mathbf{p}_3}(\tau)\epsilon^{s_1}_{ij}(\mathbf{p}_1)\epsilon^{s_2}_{ij}(\mathbf{p}_2)\delta^3(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\ . \end{aligned} \tag{6.28}$$

Hence at first order in perturbation theory, the three-point function is given by

$$\begin{aligned}\langle 0|\hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)|0\rangle_{phy} &= \langle 0|\hat{h}^{s,I}_{\mathbf{k}_1}(\tau)\hat{h}^{s',I}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^{I}_{\mathbf{k}_3}(\tau)|0\rangle \\ &- i\int_{\tau_0}^{\tau} d\tau'\langle 0|\left[\hat{h}^{s,I}_{\mathbf{k}_1}(\tau)\hat{h}^{s',I}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^{I}_{\mathbf{k}_3}(\tau), H^{I}_{int}(\tau')\right]|0\rangle.\end{aligned} \tag{6.29}$$

The first term of the last equation is nonzero

$$\begin{aligned}\langle 0|\hat{h}^{s,I}_{\mathbf{k}_1}(\tau)\hat{h}^{s',I}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^{I}_{\mathbf{k}_3}(\tau)|0\rangle =& \frac{(2\pi)^3}{16}P_h(k_1)P_h(k_2)\epsilon^{s}_{ij}(\mathbf{k}_1)\epsilon^{s'}_{ij}(\mathbf{k}_2) \\ &\times\left(\frac{k_3^2-k_1^2-k_2^2}{k_3^2}\right)\delta^3(\sum\mathbf{k})\ .\end{aligned} \tag{6.30}$$

The second term can also be computed using the interaction Hamiltonian (6.28), yielding

$$\begin{aligned}-i\int_{\tau_0}^{\tau} d\tau'\langle 0|\left[\hat{h}^{s,I}_{\mathbf{k}_1}(\tau)\hat{h}^{s',I}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^{I}_{\mathbf{k}_3}(\tau), H^{I}_{int}(\tau')\right]|0\rangle &= -\frac{(2\pi)^3}{2}P_h(k_1)P_h(k_2) \\ &\times\epsilon^{s}_{ij}(\mathbf{k}_1)\epsilon^{s'}_{ij}(\mathbf{k}_2)\left(\frac{k_1^2k_2^2}{k_3^3k_t}\right)\delta^3(\sum\mathbf{k})\ .\end{aligned} \tag{6.31}$$

Therefore,

$$\begin{aligned}\langle 0|\hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)|0\rangle_{phy} &= \frac{(2\pi)^3}{2}P_h(k_1)P_h(k_2)\epsilon^{s}_{ij}(\mathbf{k}_1)\epsilon^{s'}_{ij}(\mathbf{k}_2) \\ &\times\left[\frac{1}{8}\left(\frac{k_3^2-k_1^2-k_2^2}{k_3^2}\right)-\left(\frac{k_1^2k_2^2}{k_3^3k_t}\right)\right]\delta^3(\sum\mathbf{k})\ .\end{aligned} \tag{6.32}$$

In the limit $k_3 << k_1 = k_2$, we obtain

$$\begin{aligned}\langle 0|\hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)|0\rangle_{phy} &\approx -\frac{(2\pi)^3}{2}P_h(k_1)P_h(k_1)\delta_{ss'}\left(\frac{k_1^3}{k_3^3}\right)\delta^3(\sum\mathbf{k}) \\ &= (2\pi)^3\ P_h(k_1)P_{\mathcal{R}}(k_3)n_t\delta_{ss'}\delta^3(\sum\mathbf{k})\ ,\end{aligned} \tag{6.33}$$

where we have used the fact that $n_t = -2\epsilon$.

Note that in the other squeezed limit $k_3 << k_1 = k_2$, we obtain

$$\frac{\langle 0|\hat{h}^s_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{h}^{s'}_{\mathbf{k}_3}(\tau)|0\rangle_{phy}}{P_h(k_3)P_{\mathcal{R}}(k_2)} \approx \mathcal{O}\left(\frac{k_3^2}{k_1^2}\epsilon\right) \ . \tag{6.34}$$

Therefore, two gravitons and a scalar three-point functions in the squeezed limit agree with the consistency conditions (5.2) and (5.33).

6.2.4 Three gravitons correlator

The third order action for the 3-gravitons interaction is given by[5]

$$S_3 = \frac{{M_{\rm pl}}^2}{4}\int d^3x d\tau \ a(\tau)^2 \left(h_{ik}h_{jl} - \frac{1}{2}h_{ij}h_{kl}\right)\partial_k\partial_l h_{ij} \ , \tag{6.35}$$

The interaction Hamiltonian can be found from $S_3 = -\int d\tau H_{int}$. In momentum space H_{int} is given by

$$\begin{aligned} H_{int}(\tau) =& \frac{{M_{\rm pl}}^2 a^2(\tau)}{4(2\pi)^6}\int d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 \delta^3\left(\sum \mathbf{p}\right) \\ & \times \sum_{s_1,s_2,s_3} h^{s_1}_{\mathbf{p}_1}(\tau)h^{s_2}_{\mathbf{p}_2}(\tau)h^{s_3}_{\mathbf{p}_3}(\tau)T(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;s_1,s_2,s_3) \ , \end{aligned} \tag{6.36}$$

where,

$$T(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;s_1,s_2,s_3) = \left(\epsilon^{s_1}_{ik}(\mathbf{p}_1)\epsilon^{s_2}_{jl}(\mathbf{p}_2) - \frac{1}{2}\epsilon^{s_1}_{ij}(\mathbf{p}_1)\epsilon^{s_2}_{kl}(\mathbf{p}_2)\right)\epsilon^{s_3}_{ij}(\mathbf{p}_3)\mathbf{p}_{3,k}\mathbf{p}_{3,l} \ . \tag{6.37}$$

At first order in the perturbation theory, we obtain

$$\begin{aligned} \langle 0|\hat{h}^s_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\hat{h}^{s''}_{\mathbf{k}_3}(\tau)|0\rangle_{phy} =& \langle 0|\hat{h}^{s,I}_{\mathbf{k}_1}(\tau)\hat{h}^{s',I}_{\mathbf{k}_2}(\tau)\hat{h}^{s'',I}_{\mathbf{k}_3}(\tau)|0\rangle \\ & - i\int_{\tau_0}^{\tau} d\tau'\langle 0|\left[\hat{h}^{s,I}_{\mathbf{k}_1}(\tau)\hat{h}^{s',I}_{\mathbf{k}_2}(\tau)\hat{h}^{s'',I}_{\mathbf{k}_3}(\tau), H^I_{int}(\tau')\right]|0\rangle. \end{aligned} \tag{6.38}$$

[5]For detailed discussions see [97, 122, 124, 96].

The first term in the last equation vanishes. The second term is nonzero and the final result is

$$\langle 0|\hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\hat{h}^{s''}_{\mathbf{k}_3}(\tau)|0\rangle_{phy} = (2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{H^4}{2{M_{\rm pl}}^4}\mathcal{I}(k_1,k_2,k_3) \times \left[T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;s,s',s'') + \text{all permutations}\right] \ , \tag{6.39}$$

where,

$$\mathcal{I}(k_1,k_2,k_3) = \frac{1}{(k_1k_2k_3)^3}\left(-k_t + \frac{k_1k_2k_3}{k_t^2} + \frac{k_1k_2+k_2k_3+k_1k_3}{k_t}\right) \tag{6.40}$$

with $k_t = k_1+k_2+k_3$. The three-point function (6.39) can be simplified further (see [97])

$$\langle 0|\hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\hat{h}^{s''}_{\mathbf{k}_3}(\tau)|0\rangle_{phy} = (2\pi)^3\delta^3(\sum\mathbf{k})\frac{H^4}{2{M_{\rm pl}}^4}\mathcal{I}(k_1,k_2,k_3) \times\left(-\epsilon^{s}_{ii'}(\mathbf{k}_1)\epsilon^{s'}_{jj'}(\mathbf{k}_2)\epsilon^{s''}_{ll'}(\mathbf{k}_3)t_{ijl}t_{i'j'l'}\right) \ , \tag{6.41}$$

where,

$$t_{ijl} = \mathbf{k}_{1,l}\delta_{ij} + \mathbf{k}_{2,i}\delta_{jl} + \mathbf{k}_{3,j}\delta_{il} \ . \tag{6.42}$$

In the squeezed limit $k_3 << k_1, k_2$, we obtain,

$$\langle 0|\hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\hat{h}^{s''}_{\mathbf{k}_3}(\tau)|0\rangle_{phy} = (2\pi)^3\delta^3(\sum\mathbf{k})P_h(k_1)P_h(k_3) \times \delta_{ss'}\left(\frac{3}{2}\right)\frac{\mathbf{k}_{1;i}\mathbf{k}_{1;j}\epsilon^{s''}_{ij}(\mathbf{k}_3)}{k_1^2} \ . \tag{6.43}$$

Therefore, all the three-point functions of scalar and tensor perturbations for slow-roll inflation with the Bunch-Davies initial state are consistent with the consistency relations (5.2,5.33) and the non-Gaussianity matrix $\mathcal{F}$ is given by

$$\mathcal{F} \equiv \begin{pmatrix}\mathcal{F}_{\mathcal{R}\mathcal{R}} & \mathcal{F}_{\mathcal{R}h}\\ \mathcal{F}_{h\mathcal{R}} & \mathcal{F}_{hh}\end{pmatrix} = \begin{pmatrix}-6\epsilon+2\eta & 3/2\\ -2\epsilon & 3/2\end{pmatrix} \ . \tag{6.44}$$

6.3 Non-Gaussianities from coherent states

Now we will calculate the three-point functions for slow-roll inflation with a non-Bunch-Davies initial state of fluctuations. First we will consider coherent states. Coherent states are special class of states because one can think of a coherent state as zero-point quantum fluctuations around some classical oscillation. Coherent states are defined as

$$\hat{a}^s_{\mathbf{k}}|C\rangle = C(\mathbf{k};s)|C\rangle \ , \qquad s = 0 \ , + \ , \times \ , \tag{6.45}$$

where again $s = 0$ corresponds to scalar perturbations and two polarizations of tensor perterbations are $s = + \ , \times$. The functions $C(\mathbf{k};s)$ are not entirely free of constraints; these states will not introduce any new UV-divergences to the energy-momentum tensor only if $C(\mathbf{k};s)$ goes to zero faster than $\frac{1}{k^{5/2}}$ for large $\mathbf{k}$.

A coherent state $|C\rangle$ can be represented as an excited state built over the Bunch-Davies state:

$$|C\rangle = \left[\prod_{s=0,\times,+}\prod_{\mathbf{k}} e^{-\frac{|C(\mathbf{k};s)|^2}{2}} \exp\left(C(\mathbf{k};s)\hat{a}^{s\dagger}_{\mathbf{k}}\right)\right]|0\rangle \ . \tag{6.46}$$

From the last equation it is clear that

$$\langle 0|C\rangle = \prod_{s=0,\times,+}\prod_{\mathbf{k}}\left(e^{-\frac{|C(\mathbf{k};s)|^2}{2}}\right) \ . \tag{6.47}$$

Note that the Bunch-Davies state is a special coherent state with $C(\mathbf{k};s) = 0$.

Without loss of generality, we can impose the restriction that $\langle \hat{\mathcal{R}}_{\mathbf{k}}(\tau)\rangle = \langle \hat{h}^s_{\mathbf{k}}(\tau)\rangle = 0$, in the superhorizon limit (before we introduce three point interactions). That leads to the condition

$$C^*(-\mathbf{k}; s) = C(\mathbf{k}; s). \tag{6.48}$$

However, we will show that interactions will generator non vanishing one-point functions even for these states.

Coherent state is a special state because it closely resembles classical harmonic oscillation. We do not know anything about the physics before inflation, a priori any excited state is as good an initial state as the Bunch-Davies state. In particular, it can be shown explicitly that at late time, all the three-point functions for coherent initial state are identical to that with the Bunch-Davies initial state (6.16).

$$\langle C|\hat{O}_{1;\mathbf{k}_1}(\tau)\hat{O}_{2;\mathbf{k}_2}(\tau)\hat{O}_{3;\mathbf{k}_3}(\tau)|C\rangle_{phy} = \langle 0|\hat{O}_{1;\mathbf{k}_1}(\tau)\hat{O}_{2;\mathbf{k}_2}(\tau)\hat{O}_{3;\mathbf{k}_3}(\tau)|0\rangle_{phy} \tag{6.49}$$

where $\hat{O}_{n;\mathbf{k}_n}(\tau)$, (with $n = 1, 2, 3...$) are either scalar perturbation $\hat{\mathcal{R}}_{\mathbf{k}}$ or tensor perturbation $\hat{h}^s_{\mathbf{k}}$. It is not very difficult to understand why that is the case. One can think of coherent state as zero-point quantum fluctuations around some classical state (see figure 6.1). So, the field $\hat{O}^{coh}_{\mathbf{k}}(\tau)$ in the coherent state can be written as $\hat{O}^{coh}_{\mathbf{k}}(\tau) = O^{cl}(\tau) + \hat{O}^{vac}_{\mathbf{k}}(\tau)$, where classical part $O^{cl}(\tau)$ is obviously the expectation value $\langle \hat{O}^{coh}_{\mathbf{k}}(\tau)\rangle$; and $\hat{O}^{vac}_{\mathbf{k}}(\tau)$ is the original quantum field but now in the vacuum state. Interactions will generate non-zero one-point function even at late time, however, that will contribute only to the classical part. Only the quantum fluctuations contribute to the physically relevant part of three-point correlations and hence they remain unchanged (6.49). Let us

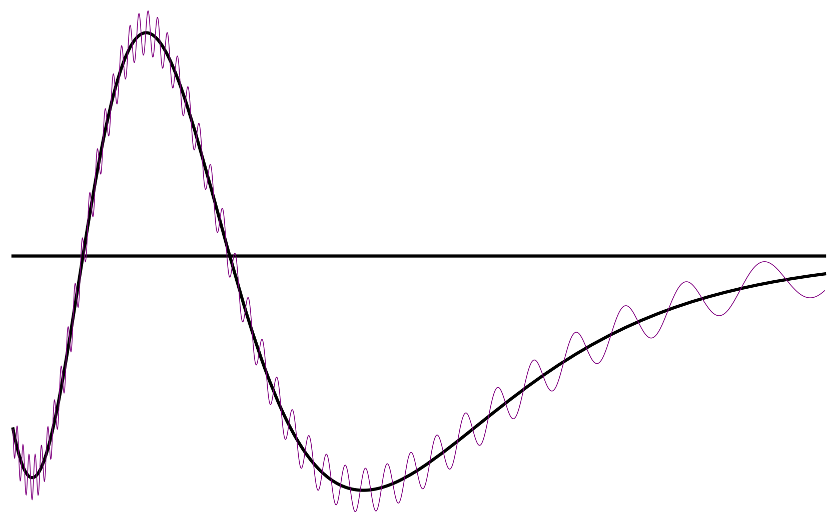

Figure 6.1: Coherent state as zero-point quantum fluctuations around some classical state

now make this discussion precise by performing a tree-level computation.[6]

Before we proceed, let us note few things. First of all, a coherent state is annihilated by an operator $\hat{c}^s_{\mathbf{k}}$,

$$\hat{c}^s_{\mathbf{k}}|C\rangle = 0 \ , \qquad \text{where,} \qquad \hat{c}^s_{\mathbf{k}} = \hat{a}^s_{\mathbf{k}} - C(\mathbf{k};s) \tag{6.50}$$

and

$$\left[\hat{c}^s_{\mathbf{k}_1}, \hat{c}^{s'\dagger}_{\mathbf{k}_2}\right] = (2\pi)^3\delta^3(\mathbf{k}_1 - \mathbf{k}_2)\delta_{ss'} \ , \qquad s, s' = 0 \ , + \ , \times \ . \tag{6.51}$$

Any operator $\hat{O}_{n;\mathbf{k}_n}(\tau)$ of the free theory can be written in terms of operators $\hat{c}^s_{\mathbf{k}_n}$ and $\hat{c}^{s\dagger}_{-\mathbf{k}_n}$

$$\hat{O}_{n;\mathbf{k}_n}(\tau) = \frac{1}{\sqrt{2}}\left[\hat{c}^s_{\mathbf{k}_n} u^*_{n;k_n}(\tau) + \hat{c}^{s\dagger}_{-\mathbf{k}_n} u_{n;k_n}(\tau)\right] + \bar{O}_{n;\mathbf{k}_n}(\tau) \ , \tag{6.52}$$

[6]See appendix A for a detailed calculation of scalar three-point function with coherent states.

where $u_{n;k_n}(\tau)$ is the mode function associated with the operator $\hat{O}_{n;\mathbf{k}_n}(\tau)$ and for slow-roll inflation $u_{n;k_n}(\tau) = \frac{e^{ik\tau}}{\sqrt{k}}\left(1+\frac{i}{k\tau}\right)$ (up to a factor which is not important for our purpose). $\bar{O}_{n;\mathbf{k}_n}(\tau)$ is the classical part

$$\bar{O}_{n;\mathbf{k}_n}(\tau) = \langle C|\hat{O}_{n;\mathbf{k}_n}(\tau)|C\rangle = \frac{1}{\sqrt{2}}\left[C(\mathbf{k}_n)u^*_{n;k_n}(\tau) + C^*(-\mathbf{k}_n)u_{n;k_n}(\tau)\right] \ . \tag{6.53}$$

Now one can easily show that for the free theory

$$\begin{aligned}&\langle C|\hat{O}_{1;\mathbf{k}_1}(\tau_1)\hat{O}_{2;\mathbf{k}_2}(\tau_2)...|C\rangle \\ &\qquad = \langle 0|\left(\hat{O}_{1;\mathbf{k}_1}(\tau_1)+\bar{O}_{1;\mathbf{k}_1}(\tau_1)\right)\left(\hat{O}_{2;\mathbf{k}_2}(\tau_2)+\bar{O}_{2;\mathbf{k}_2}(\tau_2)\right)...|0\rangle \ .\end{aligned} \tag{6.54}$$

Note that at late time ($\tau \to 0$), $\bar{O}_{n;\mathbf{k}_n}(\tau) = 0$ because of the condition (6.48). Now let us turn on interactions and compute the three-point function $\langle C|\hat{O}_{1;\mathbf{k}_1}(\tau)\hat{O}_{2;\mathbf{k}_2}(\tau)\hat{O}_{3;\mathbf{k}_3}(\tau)|C\rangle$ in the limit $\tau \to 0$. In first order in perturbation theory, we obtain

$$\begin{aligned}\langle C|\hat{O}_{1;\mathbf{k}_1}(\tau)\hat{O}_{2;\mathbf{k}_2}(\tau)\hat{O}_{3;\mathbf{k}_3}(\tau)|C\rangle &= \langle C|\hat{O}^I_{1;\mathbf{k}_1}(\tau)\hat{O}^I_{2;\mathbf{k}_2}(\tau)\hat{O}^I_{3;\mathbf{k}_3}(\tau)|C\rangle \\ &- i\int_{\tau_0}^{\tau} d\tau'\langle C|\left[\hat{O}^I_{1;\mathbf{k}_1}(\tau)\hat{O}^I_{2;\mathbf{k}_2}(\tau)\hat{O}^I_{3;\mathbf{k}_3}(\tau), H^I_{int}(\tau')\right]|C\rangle.\end{aligned} \tag{6.55}$$

All the fields are now in the interaction picture. In the interaction picture fields behave like free fields and hence can be written in the form (6.52). The first term in the last equation is evaluated at time $\tau \to 0$ and hence from equation (6.54) we get

$$\langle C|\hat{O}^I_{1;\mathbf{k}_1}(\tau)\hat{O}^I_{2;\mathbf{k}_2}(\tau)\hat{O}^I_{3;\mathbf{k}_3}(\tau)|C\rangle = \langle 0|\hat{O}^I_{1;\mathbf{k}_1}(\tau)\hat{O}^I_{2;\mathbf{k}_2}(\tau)\hat{O}^I_{3;\mathbf{k}_3}(\tau)|0\rangle \ . \tag{6.56}$$

Before we proceed further, a few comments are in order: one can naively assume that the quantity $\langle 0|\hat{O}^I_{1;\mathbf{k}_1}(\tau)\hat{O}^I_{2;\mathbf{k}_2}(\tau)\hat{O}^I_{3;\mathbf{k}_3}(\tau)|0\rangle$ vanishes. However, it is important to note that this quantity can be non-zero because some of the relevant three-point interactions are written in terms of redefined fields which generally have a quadratic piece (see section 6.2.1 for an example).

The second term in equation (6.55) is more complicated because it depends on the full history. The interaction Hamiltonian in momentum space, for the cases we are interested in, can be written in the following form

$$\begin{aligned} H_{int}(\tau') =& \lambda(\tau') \int d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 f(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \\ & \times \hat{M}_{1';\mathbf{p}_1}(\tau')\hat{M}_{2';\mathbf{p}_2}(\tau')\hat{M}_{3';\mathbf{p}_3}(\tau')\delta^3(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \ , \end{aligned} \tag{6.57}$$

where, $\lambda(\tau')$ and $f(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ are functions that we will keep unspecified. $\hat{M}_{n;\mathbf{p}_n}(\tau')$'s are either scalar and tensor perturbations $\hat{\mathcal{R}}_{\mathbf{k}}(\tau')$ and $\hat{h}^s_{\mathbf{k}}(\tau')$ or their derivatives $\partial_{\tau'}\hat{\mathcal{R}}_{\mathbf{k}}(\tau')$ and $\partial_{\tau'}\hat{h}^s_{\mathbf{k}}(\tau')$ (in the interaction picture). Similar to (6.52), they can be expressed in the following way

$$\hat{M}_{n;\mathbf{k}_n}(\tau') = \frac{1}{\sqrt{2}} \left[\hat{c}^s_{\mathbf{k}_n} v^*_{n;k_n}(\tau') + \hat{c}^{s\dagger}_{-\mathbf{k}_n} v_{n;k_n}(\tau') \right] + \bar{M}_{n;\mathbf{k}_n}(\tau') \ , \tag{6.58}$$

where $v_{n;k_n}(\tau')$ is the mode function associated with the operator $\hat{M}_{n;\mathbf{k}_n}(\tau')$ and $\bar{M}_{n;\mathbf{k}_n}(\tau') = \langle C|\hat{M}_{n;\mathbf{k}_n}(\tau')|C\rangle$.

Now, let us evaluate the quantity (in the leading order)[7]

$$
\begin{aligned}
\langle C| & \left[\hat{O}^I_{1;\mathbf{k}_1}(\tau)\hat{O}^I_{2;\mathbf{k}_2}(\tau)\hat{O}^I_{3;\mathbf{k}_3}(\tau), \hat{M}_{1';\mathbf{p}_1}(\tau')\hat{M}_{2';\mathbf{p}_2}(\tau')\hat{M}_{3';\mathbf{p}_3}(\tau')\right]|C\rangle \\
&= \langle 0| \left[\hat{O}^I_{1;\mathbf{k}_1}(\tau)\hat{O}^I_{2;\mathbf{k}_2}(\tau)\hat{O}^I_{3;\mathbf{k}_3}(\tau), \hat{M}_{1';\mathbf{p}_1}(\tau')\hat{M}_{2';\mathbf{p}_2}(\tau')\hat{M}_{3';\mathbf{p}_3}(\tau')\right]|0\rangle \\
&+ \Big(\langle 0|\hat{O}^I_{1;\mathbf{k}_1}(\tau)\hat{O}^I_{2;\mathbf{k}_2}|0\rangle\langle 0|\left[\hat{O}^I_{3;\mathbf{k}_3}(\tau), \hat{M}_{1';\mathbf{p}_1}(\tau')\right]|0\rangle\bar{M}_{2';\mathbf{p}_2}(\tau')\bar{M}_{3';\mathbf{p}_3}(\tau') \\
&\qquad + \text{cyclic perm}(1',2',3') + \text{cyclic perm}(1,2,3)\Big) \,. \qquad (6.59)
\end{aligned}
$$

One can also check that in the first order in perturbation theory

$$
\begin{aligned}
\langle C|\hat{O}_{1;\mathbf{k}_1}(\tau)|C\rangle &= -i\int_{\tau_0}^{\tau} d\tau'\lambda(\tau')\int d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\delta^3(\textstyle\sum \mathbf{p}) \\
&\times \Big(\langle 0|\left[\hat{O}^I_{1;\mathbf{k}_1}(\tau), \hat{M}_{1';\mathbf{p}_1}(\tau')\right]|0\rangle\bar{M}_{2';\mathbf{p}_2}(\tau')\bar{M}_{3';\mathbf{p}_3}(\tau') + \text{cyc perm}(1',2',3')\Big) \,. \qquad (6.60)
\end{aligned}
$$

Finally one can easily show that

$$
\begin{aligned}
\langle C|\hat{O}_{1;\mathbf{k}_1}(\tau)\hat{O}_{2;\mathbf{k}_2}(\tau)\hat{O}_{3;\mathbf{k}_3}(\tau)|C\rangle &= \langle 0|\hat{O}_{1;\mathbf{k}_1}(\tau)\hat{O}_{2;\mathbf{k}_2}(\tau)\hat{O}_{3;\mathbf{k}_3}(\tau)|0\rangle_{phy} \\
&+ \Big(\langle C|\hat{O}_{1;\mathbf{k}_1}(\tau)|C\rangle\langle C|\hat{O}_{2;\mathbf{k}_2}(\tau)\hat{O}_{3;\mathbf{k}_3}(\tau)|C\rangle + \text{cyc perm}(1,2,3)\Big) \qquad (6.61)
\end{aligned}
$$

Therefore, in the tree-level, using equation (6.4) we obtain

$$
\langle C|\hat{O}_{1;\mathbf{k}_1}(\tau)\hat{O}_{2;\mathbf{k}_2}(\tau)\hat{O}_{3;\mathbf{k}_3}(\tau)|C\rangle_{phy} = \langle 0|\hat{O}_{1;\mathbf{k}_1}(\tau)\hat{O}_{2;\mathbf{k}_2}(\tau)\hat{O}_{3;\mathbf{k}_3}(\tau)|0\rangle_{phy} \,. \qquad (6.62)
$$

Therefore, the non-Gaussianity matrix $\mathcal{F}$ remains the same[8]

$$
\mathcal{F} \equiv \begin{pmatrix} \mathcal{F}_{\mathcal{R}\mathcal{R}} & \mathcal{F}_{\mathcal{R}h} \\ \mathcal{F}_{h\mathcal{R}} & \mathcal{F}_{hh} \end{pmatrix} = \begin{pmatrix} -6\epsilon + 2\eta & 3/2 \\ -2\epsilon & 3/2 \end{pmatrix} \,. \qquad (6.63)
$$

[7]Note that $\lambda(\tau')$ in $H_{int}(\tau')$ is already slow-roll suppressed and hence we only need the leading contribution.

[8]One can check that the power-spectrums with coherent states are identical to that with the Bunch-Davies state and hence this is consistent with equation (5.33).

6.4 Non-Gaussianities from α-states: violation of consistency relations

Another special class of non-Bunch-Davies initial states are α-states; these states are related to the Bunch-Davies state by Bogoliubov transformations.[9] These states are annihilated by operator $\hat{b}^s_{\mathbf{k}}$:

$$\hat{b}^s_{\mathbf{k}}|\alpha\rangle = 0 \ , \qquad \text{where} \qquad \hat{b}^s_{\mathbf{k}} = \alpha^*_s(k)\hat{a}^s_{\mathbf{k}} + \beta_s(k)\hat{a}^{s\dagger}_{-\mathbf{k}} \tag{6.64}$$

for $s = 0, \times, +$. $\alpha_s(k)$ and $\beta_s(k)$ are arbitrary complex functions of k that satisfy

$$|\alpha_s(k)|^2 - |\beta_s(k)|^2 = 1 \qquad \text{for} \qquad s = 0, \times, + \ . \tag{6.65}$$

A state $|\alpha\rangle$ can be written explicitly as an excited state built over the Bunch-Davies state in the following way

$$|\alpha\rangle = \left[\prod_{s=0,\times,+}\prod_{\mathbf{k}} \frac{1}{|\alpha_s(k)|^{1/2}} \exp\left(-\frac{\beta_s(k)}{2\alpha^*_s(k)}\hat{a}^{s\dagger}_{\mathbf{k}}\hat{a}^{s\dagger}_{-\mathbf{k}}\right)\right]|0\rangle \ . \tag{6.66}$$

Note that

$$\langle 0|\alpha\rangle = \prod_{s=0,\times,+}\prod_{\mathbf{k}}\left(\frac{1}{|\alpha_s(k)|^{1/2}}\right) \ . \tag{6.67}$$

Few comments are in order: it can be shown that α-states are normalizable only if $|\beta_s(k)|^2 \to 0$ faster than k^{-3} at $k \to \infty$. However, the condition that these states do not introduce any new divergences to the energy-momentum tensor requires $|\beta_s(k)|^2 \to 0$ faster than k^{-4} for large k (see chapter 4.1).

[9] These states are also called Bogoliubov states.

The Bunch-Davies state is a special case of α-states with: $\alpha_s(k) = 1$ and $\beta_s(k) = 0$. And one can also show that for any coherent state $|C\rangle$ and any α-state $|\alpha\rangle$

$$\langle\alpha|C\rangle = \prod_{s=0,\times,+}\prod_{\mathbf{k}}\left(\frac{e^{-\frac{|C(\mathbf{k};s)|^2}{2}}}{|\alpha_s(k)|^{1/2}}\right)\exp\left(-\frac{\beta_s^*(k)}{2\alpha_s(k)}C(\mathbf{k};s)C(-\mathbf{k};s)\right)\ . \quad (6.68)$$

In these states, it is more convenient to express scalar and tensor perturbations in terms of operators $\hat{b}^s_{\mathbf{k}}$ and $\hat{b}^{s\dagger}_{\mathbf{k}}$:

$$\hat{\mathcal{R}}_{\mathbf{k}}(\tau) = \frac{1}{\sqrt{2}}\left[\hat{b}^0_{\mathbf{k}}\tilde{\mathcal{R}}^*_k(\tau) + \hat{b}^{0\dagger}_{-\mathbf{k}}\tilde{\mathcal{R}}_k(\tau)\right], \qquad \hat{h}^s_{\mathbf{k}}(\tau) = \frac{1}{\sqrt{2}}\left[\hat{b}^s_{\mathbf{k}}\tilde{h}^{s*}_k(\tau) + \hat{b}^{s\dagger}_{-\mathbf{k}}\tilde{h}^s_k(\tau)\right] \quad (6.69)$$

where,

$$\tilde{\mathcal{R}}_k(\tau) = \left(\frac{H}{a\dot{\phi}}\right)\left(\alpha_0(k)u_k(\tau) + \beta_0(k)u^*_k(\tau)\right), \quad (6.70)$$

$$\tilde{h}^s_k(\tau) = \left(\frac{\sqrt{2}}{aM_{\rm pl}}\right)\left(\alpha_s(k)u_k(\tau) + \beta_s(k)u^*_k(\tau)\right), \quad (6.71)$$

with $u_k(\tau) = \frac{e^{ik\tau}}{\sqrt{k}}\left(1 + \frac{i}{k\tau}\right)$. Practically, computations with α-states are similar to that with the Bunch-Davies state but we have to replace the mode function $u_k(\tau)$ by $\alpha_s(k)u_k(\tau) + \beta_s(k)u^*_k(\tau)$ with appropriate s.

In this section, we will keep the discussion general and not specify the functional forms of $\beta_s(k)$. It is important to note that in this section we will assume that the energies of these states are not large enough to affect the slow-roll parameters. The power spectrum and the spectral index of scalar perturbations with α-states are obtained to be

$$P_{\mathcal{R}}(k) = \frac{H^4}{2\dot{\phi}^2 k^3} \left|\alpha_0(k) - \beta_0(k)\right|^2 \ , \tag{6.72}$$

$$n_s - 1 = 2\eta - 6\epsilon + \frac{d}{d\ln k} \ln \left|\alpha_0(k) - \beta_0(k)\right|^2 . \tag{6.73}$$

Similarly, the power spectrum and the spectral index of tensor perturbations with α-states are obtained to be

$$P_h(k) = \frac{1}{k^3}\frac{H^2}{M_{pl}^2} \left|\alpha_s(k) - \beta_s(k)\right|^2 , \tag{6.74}$$

$$n_t = -2\epsilon + \frac{d}{d\ln k} \ln \sum_{s=+,\times} \left|\alpha_s(k) - \beta_s(k)\right|^2 . \tag{6.75}$$

Next we will calculate the three-point functions in the squeezed limit with α states to show that they still can be written as (5.2), however, consistency relation (5.33) is violated. In this section we will only present the squeezed limit results; general results are relegated to appendix B.

6.4.1 Three scalars correlator

The calculation for the scalar three-point function with α-state as the initial state is identical to the computation of section (6.2.1) and hence we only present the result. The interaction Hamiltonian has already been computed (6.10); the redefined field $\mathcal{R}_c$ is given by equation (6.9). In the squeezed limit ($k_3 << k_1 = k_2$), we obtain

$$\begin{aligned}\langle\alpha|\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)|\alpha\rangle_{phy} \approx{}& (2\pi)^3 P_R(k_3) P_R(k_1) \\ &\times \left[4\epsilon\left(\frac{k_1}{k_3}\right)\Phi(k_1,k_3) - 6\epsilon + 2\eta\right]\delta^3(\textstyle\sum \mathbf{k}).\end{aligned} \tag{6.76}$$

Therefore, in the squeezed limit, f_{NL}^{loc} is given by,

$$f_{NL}^{loc} \approx \frac{5}{12}\left[-4\epsilon\left(\frac{k_1}{k_3}\right)\Phi(k_1,k_3)+6\epsilon-2\eta\right] \ , \tag{6.77}$$

where $\Phi(k_1,k_3)$ is given by,

$$\Phi(k_1,k_3)=\alpha_0(k_1)\beta_0(k_1)\left(\frac{\alpha_0^*(k_1)-\beta_0^*(k_1)}{\alpha_0(k_1)-\beta_0(k_1)}\right)\left(\frac{\alpha_0(k_3)+\beta_0(k_3)}{\alpha_0(k_3)-\beta_0(k_3)}\right)+c.c. \tag{6.78}$$

In general the first term in equation (6.77) is large in the limit $k_3 << k_1$ and hence the consistency condition is violated. In section 7, we will estimate how large this violation can be. But before that let us comment on why the consistency relation is violated. For the derivation of the consistency relations, it is necessary to take the squeezed limit first and then calculate the three-point functions. However, in an honest calculation of the squeezed limit three-point function for a particular model, one should compute the three-point function first and then take the squeezed limit. So, there is an implicit assumption that the terms that are ignored by taking the squeezed limit first are small. The three-point function (this is true for all the three point functions) with α-states contains terms like (where τ_0 is the conformal time in the beginning of inflation)

$$i\int_{\tau_0}^{0} d\tau e^{i\tau(-k_1+k_2-k_3)}+c.c=2\left(\frac{1-\cos(-k_1+k_2-k_3)\tau_0}{-k_1+k_2-k_3}\right) \tag{6.79}$$

that are absent for the Bunch-Davies state. Now if we take the limit $\tau_0\to-\infty$ first and then $k_3\to 0$, we obtain

$$i\int_{\tau_0}^{0} d\tau e^{i\tau(-k_1+k_2-k_3)}+c.c\sim-\frac{2}{k_3} \tag{6.80}$$

which is large in the squeezed limit. However, if incorrectly we take the limit $k_3 \to 0$ first, then we obtain

$$i \int_{\tau_0}^{0} d\tau e^{i\tau(-k_1+k_2-k_3)} + c.c \approx 0. \tag{6.81}$$

Therefore, the terms that we missed by taking the squeezed limit $k_3 \to 0$ first are rather large and hence the consistency relations are violated.

6.4.2 Two scalars and a graviton correlator

The interaction Hamiltonian in the momentum space is given by equation (6.19). The two scalars and a graviton three-point function in the squeezed limit ($k_3 << k_1 = k_2$) can be calculated easily, yielding

$$\langle\alpha|\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{h}^s_{\mathbf{k}_3}(\tau)|\alpha\rangle_{phy} = (2\pi)^3\delta^3(\sum \mathbf{k})P_{\mathcal{R}}(k_1)P_h(k_3)\frac{\epsilon^s_{ij}(\mathbf{k}_3)\mathbf{k}_{1,i}\mathbf{k}_{1,j}}{k_1^2} \times \left[-2\left(\frac{k_1}{k_3}\right)\Theta(k_1,k_3) + \frac{3}{2} + ...\right], \tag{6.82}$$

where, function $\Theta(k_1, k_3)$ depends on the initial state and it is given by

$$\Theta(k_1,k_3) = \alpha_0(k_1)\beta_0(k_1)\left(\frac{\alpha_0^*(k_1)-\beta_0^*(k_1)}{\alpha_0(k_1)-\beta_0(k_1)}\right)\left(\frac{\alpha_s(k_3)+\beta_s(k_3)}{\alpha_s(k_3)-\beta_s(k_3)}\right) + c.c. \tag{6.83}$$

Few comments are in order. In general the first term in equation (6.82) is large in the limit $k_3 \to 0$ and hence the consistency condition is violated. However, when the scalar perturbations are initially in the Bunch-Davies state (but tensor perturbations are in an α-state), $\Theta(k_1, k_3) = 0$ and hence the consistency condition is respected.

Note that in the squeezed limit $k_3 << k_2 = k_1$ the other three-point function

$$\frac{\langle\alpha|\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{h}^s_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)|\alpha\rangle_{phy}}{P_h(k_1)P_{\mathcal{R}}(k_3)} \approx \mathcal{O}\left(|\beta_s(k_1)|\frac{k_3}{k_1}\right) . \tag{6.84}$$

In section 7, we will show that $|\beta_s(k_1)| << 1$ and hence this three-point function remains vanishingly small.

6.4.3 Two gravitons and a scalar correlator

The interaction Hamiltonian in momentum space is given by equation (6.28). The two scalars and a graviton three-point function in the squeezed limit ($k_3 << k_1 = k_2$) can be calculated easily, yielding

$$\begin{aligned}\langle\alpha|\hat{h}^s_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)|\alpha\rangle_{phy} \approx (2\pi)^3 P_h(k_1)P_{\mathcal{R}}(k_3)\delta_{ss'}\delta^3(\sum \mathbf{k})\\ \times\left[4\epsilon\left(\frac{k_1}{k_3}\right)\Psi(k_1,k_3) - 2\epsilon + ...\right] ,\end{aligned} \tag{6.85}$$

where, $\Psi(k_1,k_3)$ depends on the initial state

$$\Psi(k_1,k_3) = \alpha_s(k_1)\beta_s(k_1)\left(\frac{\alpha^*_s(k_1)-\beta^*_s(k_1)}{\alpha_s(k_1)-\beta_s(k_1)}\right)\left(\frac{\alpha_0(k_3)+\beta_0(k_3)}{\alpha_0(k_3)-\beta_0(k_3)}\right) + c.c. \tag{6.86}$$

Note that the consistency condition (5.33) is again violated unless tensor perturbations are in the Bunch-Davies state. Whereas it is easy to check that in the squeezed limit $k_3 << k_2 = k_1$, the other three-point function remains vanishingly small

$$\frac{\langle\alpha|\hat{h}^s_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{h}^{s'}_{\mathbf{k}_3}(\tau)|\alpha\rangle_{phy}}{P_h(k_3)P_{\mathcal{R}}(k_2)} \approx \mathcal{O}\left(\frac{k_3}{k_1}\epsilon|\beta_0(k_1)|\right) \tag{6.87}$$

and hence it still obeys the consistency condition (5.2).

6.4.4 Three gravitons correlator

The interaction Hamiltonian in momentum space is given by equation (6.36). The three gravitons three-point function in the squeezed limit ($k_3 << k_1 = k_2$) can be calculated easily, yielding

$$\langle\alpha|\hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\hat{h}^{s''}_{\mathbf{k}_3}(\tau)|\alpha\rangle_{phy} = (2\pi)^3\delta^3(\sum\mathbf{k})P_h(k_1)P_h(k_3)\delta_{ss'}\frac{\epsilon^{s''}_{ij}(\mathbf{k}_3)\mathbf{k}_{1,i}\mathbf{k}_{1,j}}{k_1^2}$$
$$\times\left[-2\left(\frac{k_1}{k_3}\right)\Theta(k_1,k_3)+\frac{3}{2}+...\right]. \qquad (6.88)$$

Where, function $\Theta(k_1, k_3)$ depends on the initial state and it is given by

$$\Theta(k_1,k_3)=\alpha_s(k_1)\beta_s(k_1)\left(\frac{\alpha^*_s(k_1)-\beta^*_s(k_1)}{\alpha_s(k_1)-\beta_s(k_1)}\right)\left(\frac{\alpha_{s''}(k_3)+\beta_{s''}(k_3)}{\alpha_{s''}(k_3)-\beta_{s''}(k_3)}\right)+c.c. \qquad (6.89)$$

In general the first term in equation (6.82) is large in the limit $k_3 \to 0$ and hence the consistency condition is violated.

Let us now consider a special case: $\beta_0(k) = \beta_+(k) = \beta_\times(k) = \beta(k)$; which is the case when a pre-inflationary dynamics excites both the scalar modes and the tensor modes in the same way. Therefore, the non-Gaussianity $\mathcal{F}$ matrix, defined in (5.33), is given by,

$$\mathcal{F}=2f(k_1,k_3)\begin{pmatrix}2\epsilon & -1\\ 2\epsilon & -1\end{pmatrix}+\begin{pmatrix}-6\epsilon+2\eta & 3/2\\ -2\epsilon & 3/2\end{pmatrix} \qquad (6.90)$$

where,

$$f(k_1,k_3)=\left(\frac{k_1}{k_3}\right)\left[\alpha(k_1)\beta(k_1)\left(\frac{\alpha^*(k_1)-\beta^*(k_1)}{\alpha(k_1)-\beta(k_1)}\right)\left(\frac{\alpha(k_3)+\beta(k_3)}{\alpha(k_3)-\beta(k_3)}\right)+c.c\right]. \qquad (6.91)$$

In particular f_{NL}^{loc} is given by,

$$f_{NL}^{loc} \approx \frac{5}{12} \left[-4\epsilon f(k_1, k_3) + 6\epsilon - 2\eta\right] \ . \tag{6.92}$$

Now if we want to preserve scale-invariance, the function $\beta(k)$ has to be approximately constant for all the observable modes. In that case, it is obvious that the $\mathcal{F}$-matrix for α-states is not consistent with (5.33) because the first term in equation (6.90) dominates in the squeezed limit $k_3 << k_1 = k_2$. In the next chapter, we will estimate how large $f(k_1, k_3)$ can be for states with energies not too large to affect the slow-roll parameters.

Chapter 7

Constraints from back-reaction for slow-roll inflation

We will now consider the back-reaction of excited initial states for slow-roll inflation. Before we proceed, let us once more define the energy-momentum tensor of the perturbations. The Einstein's equations for the full system is $G_{\mu\nu} - 8\pi G T_{\mu\nu} \equiv \Pi_{\mu\nu} = 0$. Following [2], we can perform a perturbative expansion of the Einstein's equations:

$$\Pi_{\mu\nu} = \Pi^{(0)}_{\mu\nu} + \Pi^{(1)}_{\mu\nu} + \Pi^{(2)}_{\mu\nu} + ... \tag{7.1}$$

Evolution of the background is given by the lowest order equation $\Pi^{(0)}_{\mu\nu} = 0$. The first order Einstein's equations $\Pi^{(1)}_{\mu\nu} = 0$ give the equations of motion for the perturbations. Therefore, we can write

$$G^{(0)}_{\mu\nu} = 8\pi G_N T^{(0)}_{\mu\nu} - \Pi^{(2)}_{\mu\nu} + ... \ , \tag{7.2}$$

where $\Pi^{(2)}_{\mu\nu}$ has to be computed with the perturbations that solve the equations of motion $\Pi^{(1)}_{\mu\nu} = 0$. From the last equation, it is clear that the energy-momentum tensor of the perturbations is given by $8\pi G_N \mathcal{T}_{\mu\nu} = -\Pi^{(2)}_{\mu\nu}$. Obviously both scalar and tensor perturbations will contribute to the energy-momentum tensor:

$$\mathcal{T}_{\mu\nu} = \mathcal{T}^s_{\mu\nu} + \mathcal{T}^t_{\mu\nu} \ . \tag{7.3}$$

Explicit forms of $\mathcal{T}^s_{\mu\nu}$ and $\mathcal{T}^t_{\mu\nu}$ for single-field inflation can be found in [2].

We will promote $\mathcal{T}_{\mu\nu}$ to an operator and estimate $\langle \hat{\mathcal{T}}_{\mu\nu} \rangle$ for single-field slow-roll inflation with α-states. $\langle \hat{\mathcal{T}}_{\mu\nu} \rangle$ contains UV-divergences and hence should be properly renormalized using any regularization method (for example adiabatic regularization).[1] For our purpose, for a general initial state $|G\rangle$ it is sufficient to define the renormalized energy momentum tensor of the fluctuations in the following way:

$$\langle G|\hat{\mathcal{T}}_{\mu\nu}|G\rangle_{ren} = \langle G|\hat{\mathcal{T}}_{\mu\nu}|G\rangle - \langle 0|\hat{\mathcal{T}}_{\mu\nu}|0\rangle \tag{7.4}$$

since a well-behaved initial state should not introduce any new ultra-violet divergences to the energy-momentum tensor.

Our goal is not to perform an exact computation but to estimate how large $\beta_0(k)$ and $\beta_s(k)$ can be without causing large back-reaction. From that we will estimate how large the deviations from non-Gaussianity consistency relations can be for α-states. For a particular state, undoubtably an exact computation will be more useful.

Before we proceed let us explicitly write down $T^{(0)}_{\mu\nu}$ for single field inflation:

$$T^{(0)}_{00} = \rho^{(0)} = \frac{1}{2}\dot{\bar{\phi}}^2 + V\left(\bar{\phi}\right) \ , \tag{7.5}$$

$$T^{(0)}_{ij} = \delta_{ij} a^2 p^{(0)} = \delta_{ij} a^2 \left(\frac{1}{2}\dot{\bar{\phi}}^2 - V\left(\bar{\phi}\right)\right) \ . \tag{7.6}$$

[1]Detailed discussions of adiabatic regularization method can be found in [110, 43].

In the beginning of inflation i.e. at $\tau = \tau_0$, following [89] the leading contribution to the energy-momentum tensor of scalar fluctuations is given by

$$\langle \hat{\mathfrak{T}}^s_{00} \rangle \approx \frac{1}{2}\left(\frac{\dot{\bar{\phi}}}{Ha}\right)^2 \left[\langle (\hat{\mathcal{R}}')^2 \rangle + \langle (\nabla \hat{\mathcal{R}})^2 \rangle\right] \ , \tag{7.7}$$

$$\langle \hat{\mathfrak{T}}^s_{ij} \rangle \approx \delta_{ij}\left(\frac{\dot{\bar{\phi}}}{H}\right)^2 \left[\frac{1}{2}\langle (\hat{\mathcal{R}}')^2 \rangle - \frac{1}{6}\langle (\nabla \hat{\mathcal{R}})^2 \rangle\right] \ . \tag{7.8}$$

Similarly, the leading contribution to the energy-momentum tensor of tensor fluctuations is given by (see chapter 4.1)

$$\langle \hat{\mathfrak{T}}^t_{00} \rangle \approx \frac{{M_{\rm pl}}^2}{8a^2}\left[\langle (\hat{h}'_{kl})^2 \rangle + \langle (\partial_m \hat{h}_{kl})^2 \rangle\right] \ , \tag{7.9}$$

$$\begin{aligned}\langle \hat{\mathfrak{T}}^t_{ij} \rangle =& \frac{3{M_{\rm pl}}^2}{8}\delta_{ij}\left[-\langle (\hat{h}'_{kl})^2 \rangle + \langle (\partial_m \hat{h}_{kl})^2 \rangle\right] \\ &+ {M_{\rm pl}}^2\left[\frac{1}{2}\langle \hat{h}'_{ik}\hat{h}'_{kj} \rangle + \frac{1}{4}\langle (\partial_i \hat{h}_{kl})(\partial_j \hat{h}_{kl}) \rangle - \frac{1}{2}\langle (\partial_l \hat{h}_{ki})(\partial_l \hat{h}_{jk}) \rangle\right] .\end{aligned} \tag{7.10}$$

We will now compute these quantities for α-states. We will assume that both $\beta_0(k)$ and $\beta_s(k)$ are nonzero and approximately constant for $k_0 < k < k_*$, where, $k_0 = a_0 H$, a_0 being the scale factor at the initial time $\tau = \tau_0$. For $k > k_*$, $\beta_0(k)$ and $\beta_s(k)$ drop to zero very fast.[2] We have assumed that modes inside the horizon ($k > k_0$) at $\tau = \tau_0$ are uncorrelated with modes outside the horizon ($k < k_0$) and only modes inside the horizon are excited at $\tau = \tau_0$ by some pre-inflationary causal dynamics. For $k_0 < k < k_*$, spectral indices remain unchanged

$$n_s \approx 1 - 6\epsilon + 2\eta \ , \qquad n_t \approx -2\epsilon \ . \tag{7.11}$$

[2]States like these are relevant if we want to preserve the scale invariance of scalar and tensor power spectrums.

Note that the squeezed limit three-point functions will have the nontrivial k_1/k_3 term only when $k_1 < k_*$. Let us now compute the renormalized energy-momentum tensor of scalar fluctuations at $\tau = \tau_0$.

$$\langle\alpha|\hat{\mathfrak{T}}^s_{00}|\alpha\rangle \approx \frac{1}{4\pi^2 a_0^4}\int^{k_*} k^3 dk\left(1+2|\beta_0(k)|^2\right) , \tag{7.12}$$

where again a_0 is the scale factor at the initial time $\tau = \tau_0$. Note that we have ignored the terms with exponential factors $e^{2ik\tau_0}$ or $e^{-2ik\tau_0}$ because they oscillate rapidly. Now using (7.4), we obtain,

$$\langle\alpha|\hat{\mathfrak{T}}^s_{00}|\alpha\rangle_{ren} \approx \frac{1}{2\pi^2 a_0^4}\int^{k_*} k^3 dk|\beta_0(k)|^2 \approx \frac{H^4}{8\pi^2}\left(\frac{k_*}{k_0}\right)^4 |\beta_0(k_*)|^2 , \tag{7.13}$$

where $k_0 = a_0 H$. Similarly for other components of the energy-momentum tensor, we obtain

$$\langle\alpha|\hat{\mathfrak{T}}^s_{ij}|\alpha\rangle_{ren} \approx \delta_{ij}\frac{a_0^2 H^4}{24\pi^2}\left(\frac{k_*}{k_0}\right)^4 |\beta_0(k_*)|^2 . \tag{7.14}$$

We can perform a similar computation for tensor perturbations and at $\tau = \tau_0$ we obtain

$$\langle\alpha|\hat{\mathfrak{T}}^t_{00}|\alpha\rangle_{ren} \approx \frac{H^4}{8\pi^2}\left(\frac{k_*}{k_0}\right)^4 \sum_{s=+,\times} |\beta_s(k_*)|^2 , \tag{7.15}$$

$$\langle\alpha|\hat{\mathfrak{T}}^t_{ij}|\alpha\rangle_{ren} \approx \delta_{ij}\frac{a_0^2 H^4}{24\pi^2}\left(\frac{k_*}{k_0}\right)^4 \sum_{s=+,\times} |\beta_s(k_*)|^2 . \tag{7.16}$$

Before we proceed few comments are in order. Note that both scalar and tensor perturbations behave like radiation and their energy densities decay as $1/a^4$. And also one can check that

$$p^s = \frac{1}{3}\rho^s , \qquad p^t = \frac{1}{3}\rho^t , \tag{7.17}$$

as expected for radiations. When initial states of scalar and tensor perturbations are the same i.e. $\beta_s(k) = \beta_0(k)$, it is easy to show that

$$\rho^t = 2\rho^s \ , \qquad p^t = 2p^s \ , \tag{7.18}$$

hence tensor perturbations contribute more to the energy-momentum tensor.

The back-reaction will not alter the background evolution if $T^{(0)}_{\mu\nu} >> \langle \hat{\mathfrak{T}}_{\mu\nu} \rangle$. For slow-roll inflation $\frac{1}{2}\dot{\bar{\phi}}^2 << V\left(\bar{\phi}\right)$ and hence the energy densities ρ^s and ρ^t must be small compare to the kinetic energy of inflation for the background evolution to remain unaltered [65]. That leads to

$$\sum_{s=0,+,\times} |\beta_s(k_*)|^2 << \frac{4\pi^2 \dot{\bar{\phi}}^2}{H^4} \left(\frac{k_0}{k_*}\right)^4 \ . \tag{7.19}$$

It is impotent to note that as long as $\rho^s + \rho^t << V\left(\bar{\phi}\right)$, we will have slow-roll inflation. However, the slow-roll parameter

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{\dot{\bar{\phi}}^2}{2H^2 {M_{\rm pl}}^2} + \frac{H^2}{6\pi^2 {M_{\rm pl}}^2} \left(\frac{k_*}{k_0}\right)^4 \left(\frac{a_0}{a}\right)^4 \sum_{s=0,+,\times} |\beta_s(k_*)|^2 \tag{7.20}$$

is now affected by the excited state when the second term is comparable to the first term. It will not affect the background evolution but it will influence the evolution of perturbations and hence it should be treated more carefully.

For simplicity, we will assume that $\beta_s(k) = \beta_0(k) = \beta(k)$, which is the case when a pre-inflationary dynamics excites both the scalar modes and the tensor modes in the same way. Therefore from the last equation we obtain,

$$|\beta(k_*)|^2 << \frac{4\pi^2 \dot{\bar{\phi}}^2}{3H^4} \left(\frac{k_0}{k_*}\right)^4 \ . \tag{7.21}$$

Using the fact[3] that $k_1 < k_*$ and $k_3 > k_0$ and

$$\Delta_{\mathcal{R}}^2 = \frac{H^4|\alpha(k_*) - \beta(k_*)|^2}{4\pi^2\dot{\bar{\phi}}^2} \tag{7.22}$$

where, we also have the usual condition $|\alpha(k_*)|^2 - |\beta(k_*)|^2 = 1$, we obtain

$$|\beta(k_*)|^2 << \frac{|\alpha(k_*) - \beta(k_*)|^2}{3\Delta_{\mathcal{R}}^2}\left(\frac{k_3}{k_1}\right)^4 \ . \tag{7.23}$$

As explained in [65], $|\beta(k_*)| >> 1$ has already been ruled out if we want to avoid a step in the scalar power spectrum. For $|\beta(k_*)| << 1$ or $|\beta(k_*)| \sim \mathcal{O}(1)$ it is easy to check that $|\alpha(k_*) - \beta(k_*)|^2 \sim \mathcal{O}(1)$ and we obtain

$$|\beta(k_*)| << \frac{1}{\sqrt{3}\Delta_{\mathcal{R}}}\left(\frac{k_3}{k_1}\right)^2 \ . \tag{7.24}$$

Note that from observation $\Delta_{\mathcal{R}}^2 = 2.2 \times 10^{-9}$ [5] and even with $k_3/k_1 \sim 10^{-2}$, from the last equation we get $|\beta(k_*)| << 1$. Therefore with this constraints, we obtain

$$f(k_1, k_3) = \left(\frac{k_1}{k_3}\right)\left[\alpha(k_1)\beta(k_1)\left(\frac{\alpha^*(k_1) - \beta^*(k_1)}{\alpha(k_1) - \beta(k_1)}\right)\left(\frac{\alpha(k_3) + \beta(k_3)}{\alpha(k_3) - \beta(k_3)}\right) + c.c\right]$$
$$\Rightarrow |f(k_1, k_3)| << 2\left(\frac{k_3}{k_1}\right)\frac{1}{\sqrt{3}\Delta_{\mathcal{R}}} \ . \tag{7.25}$$

Note that the last equation is linear in k_3/k_1 and hence $f(k_1, k_3) \to 0$ in the limit $k_3/k_1 \to 0$. However, the consistency relations are violated for the physically relevant case, i.e. when k_3/k_1 is small but finite; using the observed value of $\Delta_{\mathcal{R}}^2$ and $k_3/k_1 \sim 10^{-2}$, we finally get,

$$|f(k_1, k_3)| << 200 \ . \tag{7.26}$$

[3] Recall that our squeezed limit corresponds to $k_3 << k_1 = k_2$.

For three-scalars correlator, this corresponds to $|f_{NL}^{loc}| \lesssim 1$ and hence it is unobservable in the near future [65].[4] However, $f(k_1, k_3)$ is large enough to violate all the consistency relations. It could be interesting to consider more general single field inflation models where sound speed can be small.

[4]Even for $|f(k_1, k_3)| \sim 200$, the signal to noise ratio for Planck is $S/N < 0.9$ and hence can not be detected.

Chapter 8

Conclusions

In spite of the great success of the inflationary theory, it is always important to verify the validity of different assumptions. First, we have explored the possibilities of a secondary period of accelerated expansion in the universe's history. With such a period immediately following inflation we show that the bounds imposed by considering current measurements of the power spectrum are relatively weak – with reasonable reheat temperatures and inflationary scales it is relatively easy to fit in a secondary period of accelerated expansion with a vastly different equation of state to inflation and still have a nearly scale-invariant spectrum at scales from the size of the universe down to a Mpc and below.

Then we have explored the possibility of a general initial state for primordial fluctuations. Constraints on initial state from current measurements of power spectrum and bispectrum are relatively weak and for slow roll inflation, a large number of states are consistent with the observations. The Bunch-Davies state is just one such example. Coherent states are also interesting examples of states that are consistent with current observations. It is impossible to differentiate between these coherent states and the Bunch-Davies vacuum state just from the two-point and three-point functions.

Then, we have studied the consistency relations of the two-point and the three-point functions of scalar and tensor perturbations in single-field inflation with general initial conditions for the perturbations. The three-point functions of the perturbations, in the squeezed limit (i.e. $k_1, k_2 \gg k_3$), are known to obey certain consistency relations which are of the form

$$\langle \hat{A}_{\mathbf{k}_1} \hat{B}_{\mathbf{k}_2} \hat{C}_{\mathbf{k}_3} \rangle = (2\pi)^3 \mathcal{F}_{AC} P_A(k_1) P_C(k_3) \delta_{s(A),s(B)} \frac{\epsilon_{ij}^{s(C)}(\mathbf{k}_3) k_{1;i} k_{1;j}}{k_1^2} \delta^3 \left(\sum \mathbf{k} \right) ,$$

where $\hat{A}_{\mathbf{k}}, \hat{B}_{\mathbf{k}}, \hat{C}_{\mathbf{k}}$ are either scalar perturbation $\hat{\mathcal{R}}_{\mathbf{k}}$ or tensor perturbation $\hat{h}^s_{\mathbf{k}}$. We have used the notation that for the scalar perturbations $s(\mathcal{R}) = 0$ and $\epsilon^0_{ij}(\mathbf{k}) \equiv \delta_{ij}$. For the tensor perturbations, $s(h)$ is the polarization of the mode and $\epsilon^s_{ij}(\mathbf{k})$ is the polarization tensor. $\mathcal{F}_{AC}$ is a measure of non-Gaussianity and it is given by

$$\mathcal{F} \equiv \begin{pmatrix} \mathcal{F}_{\mathcal{R}\mathcal{R}} & \mathcal{F}_{\mathcal{R}h} \\ \mathcal{F}_{h\mathcal{R}} & \mathcal{F}_{hh} \end{pmatrix} = \begin{pmatrix} n_s - 1 & 2 - \frac{n_s}{2} \\ n_t & \frac{3-n_t}{2} \end{pmatrix} .$$

For slow-roll inflation, we find that all the three-point functions of scalar and tensor perturbations with a coherent state as the initial state are identical to three-point functions with the Bunch-Davies initial state. On the other hand, there is a violation of the consistency relations for α-states, which are states that are related to the Bunch-Davies state by Bogoliubov transformations. For slow-roll inflation, the back-reaction of the initial state of the primordial fluctuations imposes some restrictions on how large the violations can be. The energy density stored in the excited initial state has to be small compared to the kinetic energy of inflation for the slow-roll parameters to be unaffected. In particular, for the scalar three-point function, this imposes a constraint: $|f_{NL}^{loc}| \lesssim 1$ and hence it is unobservable in the near future; however, it is large enough to violate the consistency relation.

For single field slow-roll inflation with canonical kinetic term, tensor tilt and tensor-to-scalar ratio obey certain consistency relation: $r + 8n_t = 0$. When the perturbations are initially in excited states generated by some pre-inflationary dynamics, the tenser-to-scalar ratio r, n_s and n_t can get corrected and hence the consistency relation no longer holds. It is important to note that even with excited initial states slow-roll inflation predicts $r + 8n_t \sim \mathcal{O}(\epsilon, \eta)$. Recent detection of primordial gravitational waves can provide a crucial test for slow-roll inflation with the Bunch-Davies initial condition for the perturbations.

Let us conclude by saying that although a complete characterization of the three-point functions with different initial states is a challenging task, the three-point functions for different shapes of momentum-space triangles can be a useful tool for probing the initial state. Observations made by the current generation of cosmological experiments may contain valuable information about the initial state of primordial fluctuations and that would provide a window for the physics before inflation.

Appendices

Appendix A

Computation of scalar three-point function with coherent state

Using time-dependent perturbation theory we have,

$$\begin{aligned}
&\langle C|\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)|C\rangle \\
&= \langle C|\left(\bar{T}e^{i\int_{\tau_0}^{\tau}H_{int}^I(\tau')d\tau'}\right)\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_3}(\tau)\left(Te^{-i\int_{\tau_0}^{\tau}H_{int}^I(\tau')d\tau'}\right)|C\rangle \\
&= \langle C|\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_3}(\tau)|C\rangle \\
&\qquad - i\int_{\tau_0}^{\tau}d\tau'\langle C|\left[\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_3}(\tau), H^I_{int}(\tau')\right]|C\rangle. \qquad \text{(A.1)}
\end{aligned}$$

Now all the fields are in the interaction picture and H_{int} is given by equation (6.10). T and $\bar{T}$ are the time and anti-time ordered product respectively. τ_0 is the conformal time at the beginning of inflation and we will take the limit $\tau_0 \to -\infty$. Throughout the calculation we will assume that $\mathbf{k}_i \neq 0$. We will also take the usual limit $\tau \to 0$. The first term in (A.1) can be written using the redefined field (6.9)

$$\begin{aligned}
&\langle C|\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_3}(\tau)|C\rangle = \langle C|\hat{\mathcal{R}}^I_{c,\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_3}(\tau)|C\rangle \qquad \text{(A.2)} \\
&-\frac{1}{4}(3\epsilon-2\eta)\left(\int\frac{d^3\mathbf{p}}{(2\pi)^3}\langle C|\hat{\mathcal{R}}^I_{c,\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{p}}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_3-\mathbf{p}}(\tau)|C\rangle + \text{c.p.}\right) \\
&-\frac{1}{2}\epsilon\left(\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{(\mathbf{k}_3-\mathbf{p})^2}{k_3^2}\langle C|\hat{\mathcal{R}}^I_{c,\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{p}}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_3-\mathbf{p}}(\tau)|C\rangle + \text{c.p.}\right),
\end{aligned}$$

where c.p. stands for cyclic permutation. $\hat{\mathcal{R}}^I_{c,\mathbf{k}}(\tau)$ behaves like the free field, and can be written as

$$\hat{\mathcal{R}}^I_{c,\mathbf{k}}(\tau) = \frac{1}{\sqrt{2}}\left[\hat{a}^0_{\mathbf{k}}\mathcal{R}^*_k(\tau) + \hat{a}^{0\dagger}_{-\mathbf{k}}\mathcal{R}_k(\tau)\right], \tag{A.3}$$

where $\mathcal{R}_k(\tau) = \left(\frac{H}{a\dot{\phi}}\right)\frac{e^{ik\tau}}{\sqrt{k}}\left(1+\frac{i}{k\tau}\right)$. Using the commutation relation

$$\left[\hat{\mathcal{R}}^I_{c,\mathbf{k}}(\tau), \hat{a}^\dagger_{\mathbf{k}'}\right] = \frac{1}{\sqrt{2}}(2\pi)^3\mathcal{R}^*_k(\tau)\delta^3(\mathbf{k}-\mathbf{k}'), \tag{A.4}$$

the first term in the right hand side of equation (A.2) can be calculated

$$\begin{aligned}\langle C|\hat{\mathcal{R}}^I_{c,\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_3}(\tau)|C\rangle &= \sqrt{2}(2\pi)^3C(\mathbf{k}_3)Re[\mathcal{R}_{k_3}(\tau)]P_R(k_1)\delta^3(\mathbf{k}_1+\mathbf{k}_2)\\ &+\left[\frac{1}{2}(2\pi)^3\mathcal{R}_{k_2}(\tau)\mathcal{R}^*_{k_2}(\tau)\langle C|\hat{\mathcal{R}}^I_{c,\mathbf{k}_1}(\tau)|C\rangle\delta^3(\mathbf{k}_3+\mathbf{k}_2)+\mathbf{k}_1\leftrightarrow\mathbf{k}_2\right].\end{aligned} \tag{A.5}$$

In the limit $\tau \to 0$, $\langle C|\hat{\mathcal{R}}^I_{c,\mathbf{k}}(\tau)|C\rangle = 0$ because of the constraint (4.30) and $\mathcal{R}_k(\tau)$ is purely imaginary. Therefore,

$$\langle C|\hat{\mathcal{R}}^I_{c,\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{c,\mathbf{k}_3}(\tau)|C\rangle = 0. \tag{A.6}$$

Last two terms can also be computed and in the limit $\tau \to 0$ we get,

$$\begin{aligned}\langle C|\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_3}(\tau)|C\rangle = -(2\pi)^3&\left[\frac{1}{2}(3\epsilon-2\eta)\left(P_R(k_2)P_R(k_1)+\text{c.p.}\right)\right.\\ &\left.+\frac{1}{2}\epsilon\left(P_R(k_2)P_R(k_1)\frac{k_1^2+k_2^2}{k_3^2}+\text{c.p.}\right)\right]\delta^3(\textstyle\sum\mathbf{k}).\end{aligned} \tag{A.7}$$

Next we will compute,

$$\int_{\tau_0}^{\tau} d\tau' \langle C|\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_3}(\tau)H^I_{int}(\tau')|C\rangle = -2\epsilon\int_{\tau_0}^{\tau} d\tau' a^3(\tau')\left(\frac{\dot{\bar{\phi}}^2}{H}\right)$$
$$\times\int d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 \delta^3(\mathbf{p})\left(\frac{1}{p_3^2}\right)[(Re[\mathcal{R}'_{p_3}(\tau')]Re[\mathcal{R}'_{p_2}(\tau')]C(\mathbf{p}_3)C(\mathbf{p}_2)$$
$$\mathcal{R}^*_{k_3}(\tau)\mathcal{R}'_{k_3}(\tau')P_R(k_1)\delta^3(\mathbf{k}_1+\mathbf{k}_2)\delta^3(\mathbf{k}_3+\mathbf{p}_1)+\text{p-cyclic})+\text{k-cyclic}]$$
$$+2\epsilon\frac{(2\pi)^3}{i(k_1+k_2+k_3)}\delta^3(\mathbf{k})\left[P_R(k_1)P_R(k_2)\frac{k_1^2k_2^2}{k_3^3}+\text{c.p.}\right]. \tag{A.8}$$

Where, $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$, $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3$ and we have used the following equation

$$\mathcal{R}^*_k(\tau)\mathcal{R}'_k(\tau') = -\left(\frac{H^3}{\dot{\bar{\phi}}^2}\right)\frac{1}{a(\tau')k}e^{ik\tau'}. \tag{A.9}$$

Similarly,

$$\int_{\tau_0}^{\tau} d\tau' \langle C|H^I_{int}(\tau')\hat{\mathcal{R}}^I_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^I_{\mathbf{k}_3}(\tau)|C\rangle = -2\epsilon\int_{\tau_0}^{\tau} d\tau' a^3(\tau')\left(\frac{\dot{\bar{\phi}}^2}{H}\right)$$
$$\times\int d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 \delta^3(\mathbf{p})\left(\frac{1}{p_3^2}\right)[(Re[\mathcal{R}'_{p_3}(\tau')]Re[\mathcal{R}'_{p_2}(\tau')]C(\mathbf{p}_3)C(\mathbf{p}_2)\mathcal{R}_{k_3}(\tau)$$
$$\mathcal{R}^{*\prime}{}_{k_3}(\tau')P_R(k_1)\delta^3(\mathbf{k}_1+\mathbf{k}_2)\delta^3(\mathbf{k}_3+\mathbf{p}_1)+\text{p-cyclic})+\text{k-cyclic}]$$
$$-2\epsilon\frac{(2\pi)^3}{i(k_1+k_2+k_3)}\delta^3(\mathbf{k})\left[P_R(k_1)P_R(k_2)\frac{k_1^2k_2^2}{k_3^3}+\text{c.p.}\right]. \tag{A.10}$$

Now we have to compute the last term in equation (6.4)

$$\langle C|\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)|C\rangle\langle C|\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)|C\rangle = -2\epsilon\int_{\tau_0}^{\tau} d\tau' a^3(\tau')\left(\frac{\dot{\bar{\phi}}^2}{H}\right)$$
$$\times\int d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 \delta^3(\mathbf{p})\left(\frac{1}{p_3^2}\right)(2Re[\mathcal{R}'_{p_3}(\tau')]Re[\mathcal{R}'_{p_2}(\tau')]C(\mathbf{p}_3)C(\mathbf{p}_2)$$
$$Im[\mathcal{R}^*_{k_1}(\tau)\mathcal{R}'_{k_1}(\tau')]P_R(k_2)\delta^3(\mathbf{k}_2+\mathbf{k}_3)\delta^3(\mathbf{k}_1+\mathbf{p}_1)+\text{p-cyclic}). \tag{A.11}$$

Putting all the terms together in equation (6.4), we have

$$\langle C|\hat{\mathcal{R}}^{phy}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}^{phy}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}^{phy}_{\mathbf{k}_3}(\tau)|C\rangle = -(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)P_R(k_2)P_R(k_1) \times \left[\frac{1}{2}\left(3\epsilon - 2\eta + \epsilon\frac{k_1^2+k_2^2}{k_3^2}\right) + \frac{4\epsilon}{(k_1+k_2+k_3)}\frac{k_1^2k_2^2}{k_3^3}\right] + \text{c.p.} \quad \text{(A.12)}$$

Appendix B

Three-point functions with α-states

Here we will present full expressions of all the three-point functions with α-states.

B.1 Three scalars correlator

The three scalar three-point function at late time ($\tau \to 0$) is given by

$$\begin{aligned}
\langle\alpha|\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)|\alpha\rangle_{phy} = -\delta^3(\sum\mathbf{k})\frac{(2\pi)^3 H^6\delta^3(\sum\mathbf{k})}{4k_1k_2k_3{M_{\rm pl}}^2\dot{\phi}^2}\left(\sum_i\frac{1}{k_i^2}\right) \\
\left[\,(\alpha_0^*(k_1)-\beta_0^*(k_1))\,(\alpha_0^*(k_2)-\beta_0^*(k_2))\,(\alpha_0^*(k_3)-\beta_0^*(k_3))\right. \\
\times\left\{(\alpha_0(k_1)\alpha_0(k_2)\alpha_0(k_3)+\beta_0(k_1)\beta_0(k_2)\beta_0(k_3))\,\frac{1}{k_1+k_2+k_3}\right. \\
+(\alpha_0(k_1)\alpha_0(k_2)\beta_0(k_3)+\beta_0(k_1)\beta_0(k_2)\alpha_0(k_3))\,\frac{1}{-k_1-k_2+k_3} \\
+(\alpha_0(k_1)\beta_0(k_2)\alpha_0(k_3)+\beta_0(k_1)\alpha_0(k_2)\beta_0(k_3))\,\frac{1}{-k_1+k_2-k_3} \\
\left.\left.+(\alpha_0(k_1)\beta_0(k_2)\beta_0(k_3)+\beta_0(k_1)\alpha_0(k_2)\alpha_0(k_3))\,\frac{1}{k_1-k_2-k_3}\right\}+c.c.\right] \\
+\left\{\frac{(2\pi)^3}{2}P_{\mathcal{R}}(k_1)P_{\mathcal{R}}(k_2)\left(2\eta-3\epsilon-\epsilon\frac{k_1^2+k_2^2}{k_3^2}\right)+\text{c.p.}\right\} \qquad \text{(B.1)}
\end{aligned}$$

where c.p. stands for cyclic permutations and c.c. stands for complex conjugation.

B.2 Two scalars and a graviton correlator

The two scalars and a graviton three-point function at late time ($\tau \to 0$) is given by

$$\langle\alpha|\hat{\mathcal{R}}_{\mathbf{k}_1}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_2}(\tau)\hat{h}^s_{\mathbf{k}_3}(\tau)|\alpha\rangle_{phy} = (2\pi)^3\delta^3(\sum\mathbf{k})\frac{H^6}{{M_{\rm pl}}^2\dot{\phi}^2}\frac{\epsilon^s_{ij}(\mathbf{k}_3)\mathbf{k}_{1,i}\mathbf{k}_{2,j}}{4(k_1k_2k_3)^3}$$
$$\times\left(\alpha^*_0(k_1)-\beta^*_0(k_1)\right)\left(\alpha^*_0(k_2)-\beta^*_0(k_2)\right)\left(\alpha^*_s(k_3)-\beta^*_s(k_3)\right)$$
$$\times\{\left(\alpha_0(k_1)\alpha_0(k_2)\alpha_s(k_3)+\beta_0(k_1)\beta_0(k_2)\beta_s(k_3)\right)\mathcal{I}_0(k_1,k_2,k_3)$$
$$+\left(\alpha_0(k_1)\alpha_0(k_2)\beta_s(k_3)+\beta_0(k_1)\beta_0(k_2)\alpha_s(k_3)\right)\mathcal{I}_1(k_1,k_2,k_3)$$
$$+\left(\alpha_0(k_1)\beta_0(k_2)\alpha_s(k_3)+\beta_0(k_1)\alpha_0(k_2)\beta_s(k_3)\right)\mathcal{I}_1(k_1,k_3,k_2)$$
$$+\left(\alpha_0(k_1)\beta_0(k_2)\beta_s(k_3)+\beta_0(k_1)\alpha_0(k_2)\alpha_s(k_3)\right)\mathcal{I}_1(k_3,k_2,k_1)\}+c.c.\ , \tag{B.2}$$

where, *c.c.* stands for complex conjugate and

$$\mathcal{I}_0(k_1,k_2,k_3) = \left(-k_t+\frac{k_1k_2k_3}{k_t^2}+\frac{k_1k_2+k_2k_3+k_1k_3}{k_t}\right)\ , \tag{B.3}$$

$$\mathcal{I}_1(k_1,k_2,k_3) = \left(k_1+k_2-k_3+\frac{k_1k_2k_3}{(k_1+k_2-k_3)^2}+\frac{-k_1k_2+k_2k_3+k_1k_3}{k_1+k_2-k_3}\right)\ . \tag{B.4}$$

B.3 Two gravitons and a scalar correlator

Two graviton and a scalar three-point function at late time ($\tau \to 0$) is given by

$$\begin{aligned}
\langle\alpha|\hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\hat{\mathcal{R}}_{\mathbf{k}_3}(\tau)|\alpha\rangle_{phy} &= -\frac{(2\pi)^3H^4}{4k_1k_2k_3^3{M_{\rm pl}}^4}\epsilon^{s}_{ij}(\mathbf{k}_1)\epsilon^{s'}_{ij}(\mathbf{k}_2)\delta^3(\sum\mathbf{k}) \\
&\times\Big[\,(\alpha^*_s(k_1)-\beta^*_s(k_1))\,(\alpha^*_{s'}(k_2)-\beta^*_{s'}(k_2))\,(\alpha^*_0(k_3)-\beta^*_0(k_3)) \\
&\times\Big\{(\alpha_s(k_1)\alpha_{s'}(k_2)\alpha_0(k_3)+\beta_s(k_1)\beta_{s'}(k_2)\beta_0(k_3))\,\frac{1}{k_1+k_2+k_3} \\
&+(\alpha_s(k_1)\alpha_{s'}(k_2)\beta_0(k_3)+\beta_s(k_1)\beta_{s'}(k_2)\alpha_0(k_3))\,\frac{1}{-k_1-k_2+k_3} \\
&+(\alpha_s(k_1)\beta_{s'}(k_2)\alpha_0(k_3)+\beta_s(k_1)\alpha_{s'}(k_2)\beta_0(k_3))\,\frac{1}{-k_1+k_2-k_3} \\
&+(\alpha_s(k_1)\beta_{s'}(k_2)\beta_0(k_3)+\beta_s(k_1)\alpha_{s'}(k_2)\alpha_0(k_3))\,\frac{1}{k_1-k_2-k_3}\Big\}+c.c.\Big] \\
&+\frac{(2\pi)^3}{16}P_h(k_1)P_h(k_2)\epsilon^{s}_{ij}(\mathbf{k}_1)\epsilon^{s'}_{ij}(\mathbf{k}_2)\left(\frac{k_3^2-k_1^2-k_2^2}{k_3^2}\right)\delta^3(\sum\mathbf{k}) \qquad \text{(B.5)}
\end{aligned}$$

B.4 Three gravitons correlator

Similarly, three gravitons correlation function can be calculated in the α-states and the final result is

$$\begin{aligned}\langle\alpha|\hat{h}^{s}_{\mathbf{k}_1}(\tau)\hat{h}^{s'}_{\mathbf{k}_2}(\tau)\hat{h}^{s''}_{\mathbf{k}_3}(\tau)|\alpha\rangle_{phy} = &(2\pi)^3\delta^3(\sum\mathbf{k})\frac{H^4}{2{M_{\rm pl}}^4}\\ &\times\frac{1}{2(k_1k_2k_3)^3}\left(-\epsilon^{s}_{ii'}(\mathbf{k}_1)\epsilon^{s'}_{jj'}(\mathbf{k}_2)\epsilon^{s''}_{ll'}(\mathbf{k}_3)t_{ijl}t_{i'j'l'}\right)\\ &\left(\alpha^*_s(k_1)-\beta^*_s(k_1)\right)\left(\alpha^*_{s'}(k_2)-\beta^*_{s'}(k_2)\right)\left(\alpha^*_{s''}(k_3)-\beta^*_{s''}(k_3)\right)\\ &\times\{\left(\alpha_s(k_1)\alpha_{s'}(k_2)\alpha_{s''}(k_3)+\beta_s(k_1)\beta_{s'}(k_2)\beta_{s''}(k_3)\right)\mathcal{I}_0(k_1,k_2,k_3)\\ &+\left(\alpha_s(k_1)\alpha_{s'}(k_2)\beta_{s''}(k_3)+\beta_s(k_1)\beta_{s'}(k_2)\alpha_{s''}(k_3)\right)\mathcal{I}_1(k_1,k_2,k_3)\\ &+\left(\alpha_s(k_1)\beta_{s'}(k_2)\alpha_{s''}(k_3)+\beta_s(k_1)\alpha_{s'}(k_2)\beta_{s''}(k_3)\right)\mathcal{I}_1(k_1,k_3,k_2)\\ &+\left(\alpha_s(k_1)\beta_{s'}(k_2)\beta_{s''}(k_3)+\beta_s(k_1)\alpha_{s'}(k_2)\alpha_{s''}(k_3)\right)\mathcal{I}_1(k_3,k_2,k_1)\}\\ &+c.c.\,, \end{aligned} \tag{B.6}$$

where, *c.c.* again stands for complex conjugate and

$$\mathcal{I}_0(k_1,k_2,k_3) = \left(-k_t+\frac{k_1k_2k_3}{k_t^2}+\frac{k_1k_2+k_2k_3+k_1k_3}{k_t}\right)\,, \tag{B.7}$$

$$\mathcal{I}_1(k_1,k_2,k_3) = \left(k_1+k_2-k_3+\frac{k_1k_2k_3}{(k_1+k_2-k_3)^2}+\frac{-k_1k_2+k_2k_3+k_1k_3}{k_1+k_2-k_3}\right)\,. \tag{B.8}$$

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