Embeddings of Configurations

by

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B.A., Oberlin College, 2009
M.Sc., University of Victoria, 2011

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ABSTRACT

In this dissertation, we examine the nature of embeddings with regard to both combinatorial and geometric configurations. A combinatorial $[r, k]$-configuration is a collection of abstract points and sets (referred to as blocks) such that each point is a member of r blocks, each block is of size k, and these objects satisfy a *linearity* criterion: no two blocks intersect in more than one point. A geometric configuration requires that the points and blocks be realized as points and lines within the Euclidean plane. We provide improvements on the current bounds for the asymptotic existence of both combinatorial and geometric configurations. In addition, we examine the largely new problem of embedding configurations within larger configurations possessing regularity properties. Additionally, previously undiscovered geometric $[r, k]$-configurations are found as near-coverings of combinatorial configurations.

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Chapter 1

Introduction

Geometric configurations are simple to define — a collection of points and lines on the plane, where each point meets a fixed number of lines, and each line meets a fixed number of points. This broad definition makes them appealing objects to study: they are simple to explain (even to one without a mathematical background), and possess very few restrictive properties. However, this lack of restriction is also a source of opacity in their analysis. The study of configurations is still an active area of research and there remain many unanswered questions regarding their structure.

The oldest known example of a nontrivial configuration is the Pappus configuration, discovered by Pappus of Alexandria, a Greek mathematician of aniquity [23]. This configuration is a collection of nine points and nine lines, all drawn on the plane. As seen in Figure 1.1, the points and lines are arranged so that each point meets three lines and every line meets three points. Such a configuration is called a '3-configuration', and the Pappus configuration is the smallest example of a geometric 3-configuration.

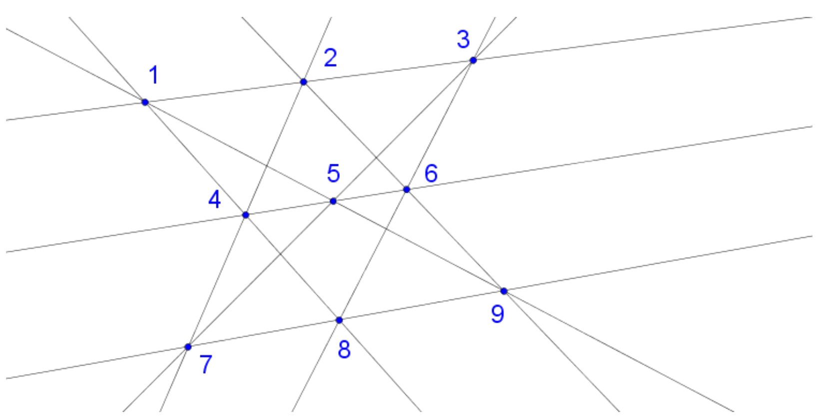


Figure 1.1: The Pappus Configuration

While other instances of geometric configurations made sporadic appearances afterwards, the formal notion and study of configurations did not appear until 1876, with the work of Theodor Reye [23]. Not long afterwards, abstract configurations were introduced. The fundamental property of a line in a geometry is that it is uniquely defined by two points. Similarly, in an abstract, or *combinatorial* configuration, the points are merely abstract elements without a geometry, and the 'lines' of the configuration are subsets of these points, where any pair of points lie in at most one line (subset). The classical example of such a configuration is the Fano plane. Shortly after the discovery of the Fano plane, the field of combinatorial geometry blossomed, along with the development of projective planes and design theory. Finite geometries and many block designs are more specific examples of combinatorial configurations, and have received a great deal of attention in the past century.

While projective planes and block designs have remained purely in the abstract realm, the difference between combinatorial configurations and their concrete *geometric* breathern was a source of some confusion in the early development of formal configuration theory. The distinction between the two was not made for quite some time [23]. In combinatorial configuration theory, research is generally concerned with determining the existence and enumeration of families of combinatorial configurations with a given set of desirable properties. The heart of geometric configuration theory lies within determining if a given combinatorial configuration admits an embedding as a geometric configuration.

In this dissertation, we seek to examine the nature of embeddings for both combinatorial and geometric configurations. If we have a partial configuration, does it embed in a larger configuration? The goal is to embed such a partial configuration as 'efficiently' as possible in a larger configuration. It is well-known that not every combinatorial configuration admits an embedding as a concrete geometric configuration. In fact, it is very unlikely that such an embedding exists for a given combinatorial configuration. This dissertation aims to find geometric configurations that carry some of the same structure as the given combinatorial configuration (while not being a pure embedding). We also introduce and explore coverings of combinatorial configurations by geometric configurations, with some positive results.

In Chapter 2, we begin by introducing a large amount of preliminary material. We will survey the current results in both combinatorial and geometric theory, as well as some useful results in design theory. In Chapter 3, we provide improved existence results for combinatorial configurations, and demonstrate constructive methods to embed partial configurations in larger regular configurations. In Chapter 4, we move on to geometric configurations, providing existence results for general regular configurations, and embedding results in the same style as combinatorial configurations. Chapter 4 also explores geometric configurations with rotational symmetry and configurations that exhibit similar structural properties to a given combinatorial configuration. Finally, we will provide concluding remarks and potential new directions for research in Chapter 5.

Before we begin, the author would like to thank Peter Dukes for his guidance throughout the doctoral program.

Chapter 2

Preliminaries

This chapter discusses the fundamental definitions and theorems in configuration theory. Section 2.1 is concerned with 'combinatorial' configurations, and also introduces some fundamental design theory terminology. This section also provides an asymptotic existence result for combinatorial configurations. This result will be improved upon in Chapter 3. In Section 2.2, we introduce 'geometric' configurations, along with some preliminary results that will prove useful in Chapter 4. The final section of this chapter establishes a connection between configuration theory and graph theory through the 'Levi graphs' of configurations. These graphs will prove especially useful in Section 4.4.

2.1 Combinatorial Configurations and Designs

2.1.1 Definitions of Configurations and Designs

Although the study of geometric configurations preceded that of their combinatorial counterparts by several hundred years, the latter subject is simpler in nature, and has also seen more significant progress. We begin then with the formal definition of a combinatorial configuration. Even here, there is a small amount of disagreement on the proper definition (compare [11] with [23, pg. 15]). We will use the more common definition, found in [23, pg. 15], among others. In [12], our definition is instead referred to as a *regular* configuration.

Definition 2.1.1. Given parameters $r, k \in \mathbb{N}$, an $[r, k]$-*combinatorial configuration* is an ordered triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ and $\mathcal{L}$ are disjoint, finite sets of elements, known respectively as *points* and *lines*, and $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$ is an incidence relation satisfying the following properties:

- Each point $p \in \mathcal{P}$ belongs to exactly r pairs $(p, \ell) \in \mathcal{I}$.
- Each line $\ell \in \mathcal{L}$ belongs to exactly k pairs $(p, \ell) \in \mathcal{I}$.
- For any $p, p' \in \mathcal{P}$ and $\ell, \ell' \in \mathcal{L}$, if $(p, \ell), (p', \ell), (p, \ell') and (p', \ell') \in \mathcal{I}$ then either $p = p'$ or $\ell = \ell'$. In other words, each pair of points are incident with at most one line.

If $|\mathcal{P}| = n$ and $|\mathcal{L}| = b$, then such a configuration is called an (n_r, b_k)-*combinatorial configuration*. Furthermore, if $r = k$ and $n = b$ we abbreviate these definitions: either as a k-*combinatorial configuration* or an (n_k)-*combinatorial configuration*.

Note that if every pair of points are incident with at most one line, then every pair of lines is incident with at most one point. In design theory, the term 'line' is replaced by the word 'block', and the values r and k are frequently referred to as the *replication number* and the *line* or *block size*, respectively. The third condition in the definition above is known as the *linearity* condition, as the condition that two points are incident with at most one line is a fundamental property of lines in geometry. For this reason, the elements of $\mathcal{L}$ are called lines. It is often convenient to abbreviate this definition by associating each line with its incidences. We may then reconsider lines as sets of points. The following definition is clearly equivalent to the original definition; however, it provides a slightly different perspective which will be useful in subsequent chapters.

Definition 2.1.2. Let $\mathcal{P}$ be a finite set of elements known as points and $\mathcal{L} \subset 2^{\mathcal{P}}$ be a set whose elements are known as lines. Then the pair $(\mathcal{P}, \mathcal{L})$ is an $[r, k]$-*combinatorial configuration* if $\mathcal{L}$ satisfies the following properties:

- Each point appears in exactly r lines.
- Each line contains exactly k points.
- (*Linearity*) Every pair of points appears in at most one line.

One other, less common definition provides a connection between combinatorial configurations and hypergraphs. Although the following definition will not be used in this thesis, it is given to provide some context and connections to graph theory.

Definition 2.1.3. Consider a hypergraph G with vertex set $\mathcal{P}$ and edge set $\mathcal{L}$ that satisfies the following criteria:

- Each point has degree r (the hypergraph is r-*regular*).
- Each edge contains k points (the hypergraph is k-*uniform*).
- (*Linearity*) Every pair of vertices is contained within at most one edge.

Such a graph G is an $[r, k]$-*combinatorial configuration*, or alternatively a r-*regular, k-uniform linear hypergraph.*

The equivalence of this definition to the previous two definitions is clear. We will use the second definition almost exclusively throughout this paper. At the heart of combinatorial configuration theory are existence and enumeration results: do there exist (n_r, b_k)-combinatorial configurations for given values of n, r, b and k, and if so, how many of them exist (up to isomorphism of the points, lines and incidence structure)?

The linearity condition imposed on combinatorial configurations is preserved by the removal of either points or lines from the configuration. Thus, the notion of a subconfiguration is a natural addition to the structure of a combinatorial configuration.

Definition 2.1.4. If $\mathcal{C}_0 = (\mathcal{P}_0, \mathcal{L}_0, \mathcal{I}_0)$ and $\mathcal{C} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ are two configurations, with $\mathcal{P}_0 \subset \mathcal{P}$ and $\mathcal{L}_0 \subset \mathcal{L}$ and $\mathcal{I}_0 \subset \mathcal{I}$, then $\mathcal{C}_0$ is a *subconfiguration* of $\mathcal{C}$. If $(p, \ell) \in \mathcal{I}$ implies $(p, \ell) \in \mathcal{I}_0$ for all $p \in \mathcal{P}_0$ and $\ell \in \mathcal{L}_0$, then $\mathcal{C}_0$ is an *induced subconfiguration* of $\mathcal{C}$.

Throughout our study of configurations, we will make frequent use of 'partial' configurations — configurations without a constant line size or replication number.

Definition 2.1.5. The pair $(\mathcal{P}, \mathcal{L})$ is a *partial* $[r, k]$-*configuration* if $\mathcal{L}$ only satisfies the linearity property of configurations, and the pair $(\mathcal{P}, \mathcal{L})$ does not necessarily have a constant replication number and/or block size. If $\mathcal{L}$ does not satisfy the linearity condition either, then $(\mathcal{P}, \mathcal{L})$ is an *incidence structure.*

The study of configurations has similar motivations to design theory. As a contrived example, suppose n individual people enter a tournament for Bridge (a game played with four players). The tournament consists of r rounds for each player, and no two players are to play together in the same game more than once. If an (n_r, b_4)-configuration exists, then such a tournament is possible, with each block representing the players in a single game. If the corresponding configuration is 'resolvable' (defined below), then the number r also correlates to the number of rounds necessary to hold the tournament. This scenario is more general than the usual 'round-robin' style tournaments, where *every* pair of players must play a game together (which is not always possible realistically).

Combinatorial configurations exhibit *duality*. If $\mathcal{C} = (\mathcal{P}, \mathcal{L})$ is a configuration with incidence relation $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$, then $\mathcal{C}^{\perp} = (\mathcal{L}, \mathcal{P})$ is a configuration as well, with incidence structure $\mathcal{I}^{\perp} \subset \mathcal{L} \times \mathcal{P}$ defined by $(\ell, p) \in \mathcal{I}^{\perp}$ if and only if $(p, \ell) \in \mathcal{I}$. If $\mathcal{C}$ is an (n_r, b_k)-configuration, then $\mathcal{C}^{\perp}$ is a (b_k, n_r)-configuration. This allows us to assume the inequality $r \geq k$ without loss to generality, if desired.

There are numerical constraints on the values of n, r, b and k in order for an (n_r, b_k)-configuration to exist.

Proposition 2.1.1. *For any (n_r, b_k)-configuration, $nr = bk$ and $n \geq r(k-1) + 1$.*

The first condition $nr = bk$ will be known as the *divisibility condition*. It follows from the fact that nr and bk are both the size of $\mathcal{I}$ (each of n points appears r times in the incidences of $\mathcal{I}$, and similarly each of the b blocks appears exactly k times among the incidences). This also implies that n must be a multiple of $k/\gcd(r, k)$. The second condition follows since a point p appears with $k-1$ other distinct points in r different blocks. Thus, we must have all these points, including p itself, in our configuration.

For a given r, k, what is the smallest number of points possible in an $[r, k]$-configuration? The above proposition gives a lower bound, but it is not always tight. In the case where $r = k$ and the bound is tight, the configuration is an example of a *projective plane* of order $k - 1$. It is well-known that projective planes do not exist for all orders (e.g. no order six projective plane exists).

In ordinary pairwise balanced combinatorial designs, any pair of points is usually required to lie within exactly one line. This stronger requirement has an intuitive motivation behind it: given a fixed number of n points, we wish to 'pack' blocks of size k into this set of points in an efficient manner — so that every pair of points appears in a line. This idea leads us to use various types of designs to construct small configurations.

Definition 2.1.6. A *balanced incomplete block design (BIBD)* with parameters (k, r, λ) is a finite set $\mathcal{P}$ of elements (again known as points) along with a family $\mathcal{B}$ of k-element subsets of $\mathcal{P}$ (called blocks) such that the number of blocks containing $p \in \mathcal{P}$ is r over

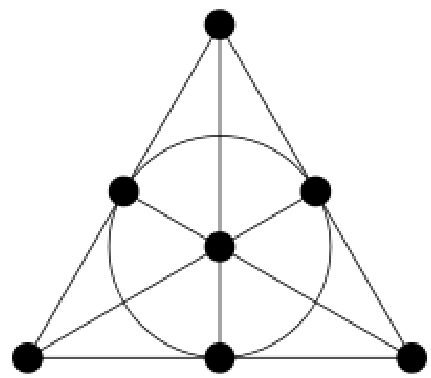


Figure 2.1: The Fano plane. There are seven points and seven lines (denoted by the six geometric lines and the circle). Each pair of points is contained within exactly one line.

all p, and each pair of points appears in exactly λ blocks. If $\lambda = 1$, then the parameter λ is omitted from the notation, and the design is also a configuration.

A classical example of a balanced incomplete block design is the *Fano plane,* a (7_3)-configuration or BIBD$(3,3)$ that is also a finite projective plane. This is represented graphically in Figure 2.1. The divisibility condition still applies to BIBDs; however, there is an additional restriction, known as the *local condition.* Given a fixed point p, note that *every* point is contained within a block containing p. There are r blocks containing p, and each block has $k-1$ other points. Thus, the total number of points in the BIBD is

$$n = r(k-1) + 1.$$

In the later sections, we will make use of resolvable transversal designs.

Definition 2.1.7. A *transversal design* TD(k,n) of order n and block size k is a triple $(\mathcal{P}, \mathcal{G}, \mathcal{B})$ such that:

- $\mathcal{P}$ is a set of kn points.
- $\mathcal{G}$ is a partition of $\mathcal{P}$ into k *groups* of size n.
- $\mathcal{B}$ is a family of k-subsets of X.
- Every pair of points in $\mathcal{P}$ belongs to either exactly one group or exactly one block, but not both.

0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
5	6	7	8	9	6	7	8	9	5	7	8	9	5	6	8	9	5	6	7	9	5	6	7	8
10	12	14	11	13	11	13	10	12	14	12	14	11	13	10	13	10	12	14	11	14	11	13	10	12
15	18	16	19	17	17	15	18	16	19	19	17	15	18	16	16	19	17	15	18	18	16	19	17	15
20	24	23	22	21	23	22	21	20	24	21	20	24	23	22	24	23	22	21	20	22	21	20	24	23

Table 2.1: An example of a $TD(5,5)$. The 25 blocks of size five are the columns of the table. The five groups of cardinality five are the elements contained in each row (ex: the first group has elements $\{0,1,2,3,4\}$). It is also an $RTD(5,5)$ — the parallel classes of blocks are separated by two lines in the table.

The term 'transversal' comes from the fact that every block is transverse to the groups: it contains exactly one point from each group. An example of a TD$(5,5)$ is given in Table 2.1.

A subclass of transversal designs are known as *resolvable transversal designs* or RTDs. A transversal design is *resolvable* if its set of blocks $\mathcal{B}$ can be partitioned into parallel classes, where each parallel class is itself a partition of the points $\mathcal{P}$. The transversal design in Table 2.1 is also an example of an RTD$(5,5)$. By restricting the blocks of an RTD(k,n) to any collection of r parallel classes, we obtain an example of an $[r,k]$-configuration on kn points for every r satisfying $1 \leq r \leq n$. Transversal designs in general are also useful for generating large BIBDs. Suppose a TD(k,n) is given, and a BIBD with block size k on n points is known. Then each group of the transversal design contains n points, and no two points appear in a block of the transversal design. By establishing an isomorphism between the points of a group and the points on the BIBD, we can add blocks of size k to each group of the transversal design. What results is a design in which every pair of points is contained within exactly one block — a BIBD with block size k and kn points. This is the core notion behind demonstrating the asymptotic existence of families of BIBDs. This constructive process is *Wilson's Fundamental Construction* [31]. Its importance in design theory cannot be overstated, however, we will not be utilizing this technique in subsequent chapters.

2.1.2 Asymptotic Existence Results for Designs and Configurations

As mentioned earlier, the trivial bound $n \geq r(k-1)+1$ on the number of points in a regular configuration is not always tight. For instance, there is no 7-configuration on $7(6)+1 = 43$ points (due to the nonexistence of a projective plane of order six). In fact, even in the case that a k-configuration on n points does exist, there is no guarantee that a k-configuration on $n+1$ points exists. However, if n is sufficiently large compared to k, then we can be assured that an (n_k)-configuration exists. Such an existence result is known as the asymptotic existence of k-configurations. Intuitively, with many points, there is a large degree of flexibility in how the lines may be arranged in the configuration. This increases the likelihood that a configuration is constructable.

Definition 2.1.8. Given a fixed $r, k \geq 2$, let $N(r,k)$ denote the smallest value for which an $[r,k]$-configuration exists for all $n \geq N(r,k)$ satisfying the divisibility condition. If $r = k$, this is shortened to $N(k)$.

Determining the value of $N(r,k)$ is still an outstanding problem in configuration theory, although reasonable bounds on $N(r,k)$ are known. To determine that $N(r,k)$ exists, we let $\mathcal{N}(r,k)$ denote the set

$$\mathcal{N}(r,k) := \{n \ : \ \text{an } (n_r, b_k)\text{-configuration exists}\}$$

Note that if $n, n' \in \mathcal{N}(r,k)$ then $n + n'$ is contained within $\mathcal{N}(r,k)$ as well. This is due to the property that the disjoint union of an (n_r, b_k)-configuration and an (n'_r, b'_k)-configuration is an $(n+n'_r, b+b'_k)$-configuration. The set of $[r,k]$-configurations are closed under the disjoint union operation; this means that $\mathcal{N}(r,k)$ forms a *numerical semigroup* — a subset of $\mathbb{N}$ that is closed under addition. We briefly introduce some asymptotic results regarding numerical semigroups.

Suppose S is a numerical semigroup with $\gcd(S) = 1$. Then the *Frobenius number* $g(S)$ of $S = \{s_1, s_2, ...\}$ is the largest value b for which the equation

$$a_1 s_1 + a_2 s_2 + \cdots + a_t s_t = b$$

has no solution for any finite subset $\{s_1, ..., s_t\} \subset S$. Such a value is known to exist precisely when $\gcd(S) = 1$ [19, pg. 400]. If s_1 and s_2 are relatively prime, then it is known that

$$g(\{s_1, s_2\}) = (s_1 - 1)(s_2 - 1) - 1,$$

so if $\{s_1, s_2\} \subset S$, then $g(S) \leq g(\{s_1, s_2\})$.

Returning to our examination of $\mathcal{N}(r, k)$, we find that every element of this set is a multiple of $k/\gcd(r, k)$. Denote this value by d. Then if the set $\mathcal{N}(r, k)/d$ has gcd equal to one, the Frobenius number $g(\mathcal{N}(r, k)/d)$ exists, and $N(r, k)/d$ is one larger than this Frobenius number. Thus, to demonstrate the asymptotic existence of combinatorial configurations, it sufficies to provide two examples of $[r, k]$-configurations on n_1, n_2 points, such that $\gcd(\frac{n_1}{d}, \frac{n_2}{d}) = 1$. From there, it follows that the Frobenius number of $\mathcal{N}(r, k)/d$ is no more than

$$(\frac{n_1}{d} - 1)(\frac{n_2}{d} - 1) - 1$$

so

$$N(r, k) \leq d\left(\frac{n_1}{d} - 1\right)\left(\frac{n_2}{d} - 1\right) = \frac{n_1 n_2}{d} - n_2 - n_1 + d.$$

Of course, this bound all depends upon the order of the two configurations found. Even if the two configurations have almost the smallest number of points possible (i.e. the number of points is of order $O(rk)$ in each), this provides a bound on $N(r, k)$ of the order $O(r^2k^2/d)$.

For example, if $r = k = 3$, then $d = 1$. The Fano plane, the Möbius-Kantor (8_3)-configuration

$$\begin{array}{ccc} \{1,2,3\} & \{2,5,7\} & \{3,4,6\} \\ \{1,4,5\} & \{2,6,8\} & \{3,5,8\} \\ \{1,6,7\} & & \{4,7,8\}, \end{array}$$

and the Pappus (9_3)-configuration provide examples on $7, 8$ and 9 points. It is known that $g(\{7, 8, 9\}) = 20$, so $N(3) \leq 21$. Examples of 3-configurations for all n between 7 and 21 have been provided, demonstrating that $N(3) = 7$. As we can see, applying the numerical semigroup argument above does not generally yield optimal results.

In order to demonstrate the finiteness of $N(r, k)$, we can apply a modification on an RTD to provide two examples of $[r, k]$-configurations with a difference of only d points between them. It is known that an RTD(k, ρ) exists for any prime $\rho > dr$. In such an RTD, there are ρk points and each group contains ρ points. Let S be any collection of dr parallel blocks, and g be any group of points. For each block in S, remove the point that is contained within group g from the block. This results in dr

points of g each with replication number $r-1$. Partition these points into sets of size k, and add these sets back into the configuration as blocks. What remains is a configuration with constant replication number and dr parallel blocks of size $k-1$. To complete the modification, add d isolated points to this configuration. Append each point to r distinct blocks of size $k-1$. The result of this modification is an $[r,k]$-configuration on $\rho k+d$ points. Thus, $N(r,k)$ is finite.

Other, more complex techniques exist to find better known bounds on general $N(r,k)$. Currently, one of the best general bounds for $N(r,k)$ is provided in the following theorem:

Theorem 2.1.2. *[14] For any fixed $r,k>3$, the value $N(r,k)$ is bounded above by*

$$drk\left((4t^2-16t)^2\gcd(r,k)-4t^2+16t\right)$$

where $d=\frac{k}{\gcd(r,k)}$ and $t=rk-r-k-1$.

This theorem provides a bound roughly on the order of $O(r^5k^6)$. In the next chapter, we provide a better bound on $N(r,k)$, improving this bound significantly.

2.2 Geometric Configurations

2.2.1 Definition of Geometric Configurations

As the nomenclature suggests, geometric configurations are dependent upon the axioms of a particular geometry. The term 'lines' in the combinatorial setting holds no real significance or value beyond the linearity property, as we were not working within a geometric setting. For the idea of a configuration to exist within the geometry, the notions of points and lines must be defined within the geometry. Such geometries are known as partial linear spaces. A *partial linear space* is an incidence structure $(\mathcal{P},\mathcal{L},\mathcal{I})$, for which the elements of $\mathcal{P}$ are called points, the elements of $\mathcal{L}$ are called lines, every line is incident with at least two points, and every pair of distinct points is incident with at most one line.

Definition 2.2.1. Let $\mathcal{P}$ and $\mathcal{L}$ be sets of points and lines (respectively) within a geometry $\mathcal{X}$. If each line $\ell\in\mathcal{L}$ is incident with exactly k points in $\mathcal{P}$, and each point p is incident with exactly r lines in $\mathcal{L}$, then the pair $\mathcal{G}=(\mathcal{P},\mathcal{L})$ is an $[r,k]$-*geometric configuration.*

Clearly the natural incidence relation established between points and lines of a geometric configuration demonstrates that a geometric configuration is also a combinatorial configuration, although the converse is not true. In fact, a large segment of geometric configuration theory is concerned with determining which combinatorial configurations have geometric counterparts. Usually, the geometries of interest are the Euclidean plane $\mathbb{R}^2$ and the real projective plane $\mathbb{P}^2$. We will assume our configurations lie in the Euclidean plane. However, many questions in the study of the existence of geometric configurations in $\mathbb{R}^2$ and $\mathbb{P}^2$ are the same.

We carry over the same definitions of the replication number and line size as before. Geometric configurations exhibit duality as well. A map that sends points to lines and vice versa is known as a *reciprocation,* and is illustrated in [16, pg. 132–136], among others.

In the geometric setting, the smallest 3-configuration is the Pappus configuration, containing 9 points and lines, and geometric 3-configurations exist for all larger n as well. The same existence and enumeration questions can be posed for geometric configuarations; however, one must take more care in determining whether two geometric configurations can be deemed isomorphic. Here, we state that two geometric configurations are isomorphic if there exists a *collineation* — a map preserving linearity — between them. Rotations, reflections and skew transformations are all examples of such collineations of the real plane.

2.2.2 Realizations of Combinatorial Configurations in the Plane

Each geometric configuration has an underlying combinatorial configuration; however, this notion has some subtlety that is worth mentioning in geometric configuration theory. For example, consider the following partial combinatorial configuration:

$$\begin{array}{cccc} \{1,2,3\} & \{1,4,8\} & \{2,4,7\} & \{3,6,8\} \\ \{4,5,6\} & \{1,5\} & \{2,6,9\} & \{3,5,7\} \\ \{7,8,9\} & & & \end{array}$$

To aid in our understanding of the structure of this partial configuration, note that if the point 9 were added to the line $\{1,5\}$, then the result is the Pappus configuration. Such a combinatorial partial configuration is *not* the underlying partial configuration of any geometric partial configuration. This is because, by the Pappus Hexagon Theorem (see [16, pg. 67]), any geometric partial configuration with lines dictated

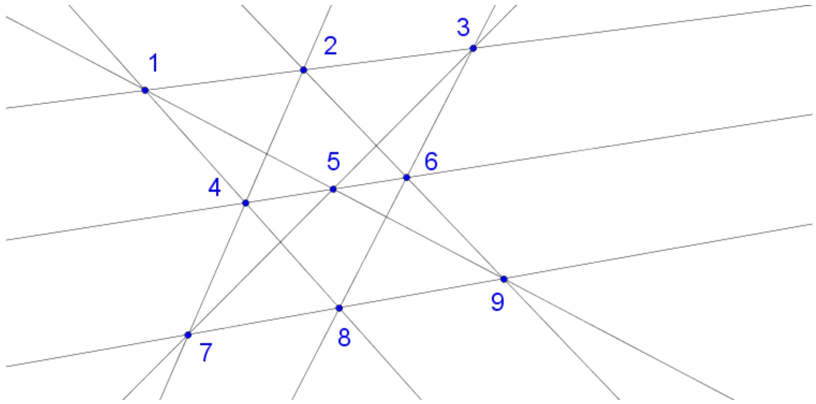


Figure 2.2: The Pappus configuration.

by the combinatorial partial configuration above must be such that the point 9 lies upon the line containing 1 and 5. Thus, 1 and 5 cannot be the only points on the line. What transpires is an undesirable incidence — the point 9 unintentionally belongs to the line determined by 1 and 5 (see Figure 2.2). In short, the geometric realization of a combinatorial partial configuration does not imply that a subconfiguration of the combinatorial partial configuration also admits a realization. These unwanted incidences become a large obstacle in answering the question of which combinatorial configurations appear as underlying configurations to some geometric configuration.

Definition 2.2.2. Consider a combinatorial configuration $\mathcal{C} = (\mathcal{P}, \mathcal{L})$ and a geometric configuration $\mathcal{G} = (\mathcal{P}', \mathcal{L}')$. We say that $\mathcal{G}$ is a *(strong) realization* of $\mathcal{C}$ if $\mathcal{C}$ is the underlying combinatorial configuration of $\mathcal{G}$.

Many constructions of large geometric configurations are axiomatic — they do not provide explicit equations for the lines in the configuration. As a result, it is challenging to determine if such a construction contains unwanted incidences. We can weaken the notion of a realization to allow for the possibility of unwanted incidences.

Definition 2.2.3. Consider a combinatorial configuration $\mathcal{C}$ and a geometric configuration $\mathcal{G}$ as before. We say that $\mathcal{G}$ is a *weak realization* or a *representation* of $\mathcal{C}$ if there exists a bijection $\mathcal{P} \to \mathcal{P}'$ and $\mathcal{L} \to \mathcal{L}'$, and every incidence in $\mathcal{C}$ is preserved by these maps in $\mathcal{G}$.

Table 2.2 gives a listing of the currently known enumeration results regarding combinatorial and geometric 3- and 4-configurations. This table incorporates nearly all known enumeration results for 3- and 4-configurations. As evidenced by the

n	(n_3)-Combinatorial	(n_3)-Geometric
7	1	0
8	1	0
9	3	3
10	10	9
11	31	31
12	229	229
13	2 036	?

n	(n_4)-Combinatorial	(n_4)-Geometric
13	1	0
14	1	0
15	4	0
16	19	0
17	1 972	0
18	971 191	2
19	269 224 652	0

Table 2.2: Enumeration results for known combinatorial and geometric 3-configurations and 4-configurations [12].

enumerations of 4-configurations, it is exceptionally rare that a combinatorial configuration admits a realization (although it is not proven that the proportion of realizable combinatorial configurations tends towards zero as $n \to \infty$). The enumeration data for configurations of $k \geq 5$ is virtually nonexistent.

One of the most celebrated results in the study of geometric realizations is stated below. It was first stated by Steinitz in his Ph.D. Thesis.

Theorem 2.2.1. *[30] Given any combinatorial* 3*-configuration* $\mathcal{C}$*, choose an arbitrary line* ℓ *and remove an arbitrary point* u *from this line. This new configuration* $\mathcal{C}^-$ *admits a weak realization.*

The version above is not exactly as stated in Steinitz's thesis. In fact, the original statement of Steinitz's configuration theorem above replaced the term 'weak realization' with 'realization', and is incorrect. This difference will be explained in the proof of the above theorem. The The proof of this theorem usually relies on the *Levi graph* of a configuration, so we outline the proof in the subsequent section.

In addition to the realization question, there are existence questions regarding geometric configurations. The asymptotic existence of geometric k-configurations is known, and proved in a similar fashion to that of combinatorial configurations. However, the bounds in this case are significantly worse, and usually not provided. We will provide a bound in subsequent chapters. Additionally, there is a great deal of study done on geometric configurations experiencing symmetry within the plane. Such configurations have a unique and visually appealing structure, as seen in Figure 2.3. A configuration with nontrivial rotational symmetry in the plane is known as *chiral*. If the configuration also admits reflective symmetry then it is *dihedral*. Very few familes of $[r, k]$-chiral configurations are known to exist for large values of r, k [7]. The rotational symmetry of a chiral configuration acts upon its points and lines,

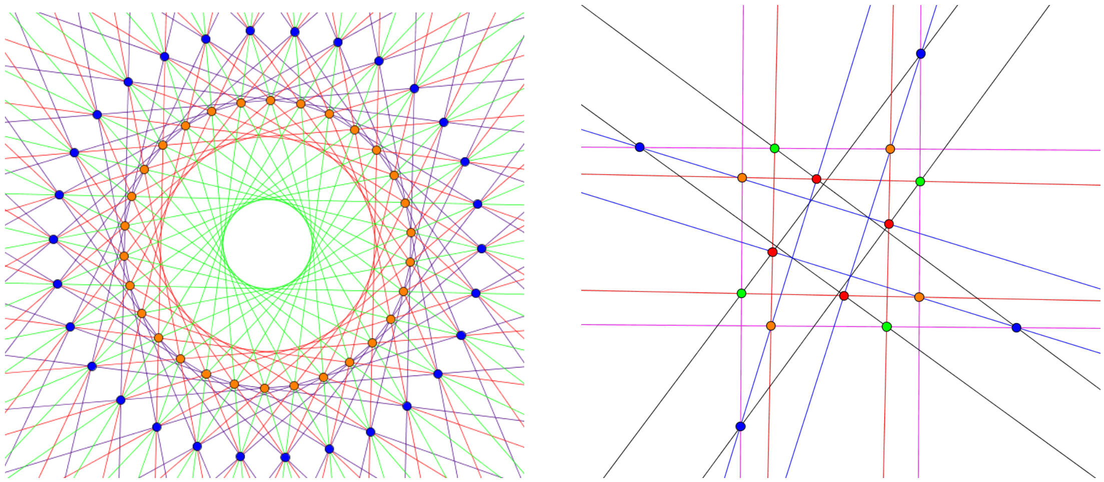


Figure 2.3: A $(60_6, 90_4)$-celestial (astral) configuration and a (12_3)-chiral (also astral) configuration. The first example is contained within [4, pg. 14]. The latter one appears in Grünbaum's book *Configurations on Points and Lines*, [23, pg. 25].

partitioning them into *symmetry classes* or *orbits*. Similar definitions may be applied to partial configurations as well. The orbits of points in a class trace out the vertices of a regular m-gon centred about the origin, for some m. The orbits of lines form the diagonals of some m-gon, centred about the origin. Note that each line in the configuration can meet at most two points in a particular orbit. This means the number of orbits of points must be at least $\lceil k/2 \rceil$, and likewise, the number of orbits of lines must exceed $\lceil r/2 \rceil$. An $[r, k]$-chiral configuration that meets these bounds is called *astral*. Astral configurations have received considerably more attention than chiral configurations, and their existence has been decided in the case when r, k are even.

Theorem 2.2.2 ([4])**.** *Astral configurations do not exist when r, k are even and $r, k \geq 6$. Astral configurations also do not exist when either r or k is 4 and the other parameter is at least 8. All other astral configurations for even r, k are known.*

A similar class of configurations that have received some attention are known as *celestial* configurations. The first instance of a celestial configuration appears in [24], and a more in-depth study can be found in [2].

Definition 2.2.4. A dihedral $[2r, 2k]$-configuration is *celestial* if, for every line, the $2k$ points incident with the line belong to k symmetry classes, and for every point, the $2r$ lines incidnet with the point belong to r symmetry classes. If there are h symmetry classes of points, then the configuration may be called an *h-celestial configuration*. A dihedral partial configuration is celestial if each line contains an even number of points which may be partitioned into pairs, and each pair of points belongs to a common symmetry class, and likewise each point contains an even number of lines which may be partitioned into pairs, and each pair of points belongs to a common symmetry class.

If the celestial configuration admits m-fold rotational symmetry, then the definition above implies that if a line is incident with a point p, then the line is a diagonal of the m-gon formed by the orbit of points containing p, and meets at a second point in this symmetry class. Figure 2.3 provides an example of a celestial configuration. Chapter 4 provides some new results on the existence of celestial configurations.

2.3 Levi Graphs

The incidence structures of combinatorial configurations can also be encoded as graphs.

Definition 2.3.1. Given a configuration $\mathcal{C} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ (either geometric or combinatorial), the *Levi graph* $L(\mathcal{C})$ is a bipartite graph with bipartition of vertices $(\mathcal{P}, \mathcal{L})$ and edge set $\mathcal{I}$.

The linearity condition implies that the graph has girth at least six (as a 4-cycle $p\ell qm$ directly implies p, q are both incident to ℓ and m). A bipartite graph with bipartition $(\mathcal{P}, \mathcal{L})$ is (r, k)-*biregular* if each vertex of $\mathcal{P}$ has degree r and each vertex of $\mathcal{L}$ has degree k. Thus, if $\mathcal{C}$ is an $[r, k]$-configuration, then the Levi graph $L(\mathcal{C})$ is (r, k)-biregular. This correspondence is bijective — every biregular bipartite graph with girth at least six can be interpreted as the Levi graph of a combinatorial configuration. A graph admits a geometric *strong* or *weak realization* if its corresponding combinatorial configuration admits a strong or weak realization, respectively.

When discussing properties of the Levi graph, the terms 'vertex' and 'edge' will be used to refer to the vertices and edges of the Levi graph. The terms 'point' and 'line' will be reserved for configurations (for example, the term 'vertex' will not be used to refer to a point of the configuration). Using this terminology, we restate and prove the theorem from the previous section.

Theorem 2.3.1 ([30, 27])**.** *Let $\mathcal{C}$ be a combinatorial 3-configuration. Let G be the Levi graph of $\mathcal{C}$, with one edge removed. The graph G admits a weak realization.*

Proof. Let u be a vertex of degree 2 in G, and let S be a spanning tree of G, with root vertex u. Choose an arbitrary leaf v_1 of S. Inductively define v_i to be a leaf of $S\backslash\{v_1, ..., v_{i-1}\}$ until all vertices of G have been listed. This ordering of the vertices ensures that each vertex v_i is adjacent to at most two vertices preceding it in the ordering (as v_i has degree at most three, and its parent in S must be listed after v_i). Let G_i be the induced subgraph of G on the vertex set $\{v_1, ..., v_i\}$. Clearly G_1 has a realization $\mathcal{G}_1$ on the plane. We demonstrate inductively that there exists a geometric configuration that is a weak realization of G_i. Suppose that the graph G_{i-1} has a weak realization $\mathcal{G}_{i-1}$ on the plane, and consider the vertex v_i:

- If v_i is not adjacent to any vertices preceding it in the ordering, then place the point or line arbitrarily down upon the plane. This is a weak realization of G_i, as G_i is equivalent to G_{i-1} with v_i as an isolated vertex.

- If v_i is adjacent to the vertex u preceding it in the ordering, then place the point (or line) denoting v_i on the line (or point) corresponding to u. This can always be done, and the result is a geometric configuration $\mathcal{G}_i$ that is a weak realization of G_i.

- If v_i is adjacent to the vertices u, w preceding it in the ordering, then place the point (or line) denoting v_i at the intersection of the lines u, w (or as the line joining point u, w). This can always be done, and the result is a geometric configuration $\mathcal{G}_i$ that is a weak realization of G_i.

□

Note that while we may place v_i at the intersection of any two lines, there is no guarantee that other lines will not also intersect this point. Thus, the geometric configuration is not necessarily strong.

By and large, the construction method given above seems to yield a truly strong realization; however, counterexamples do exist. An example of this is the (16_3)-combinatorial configuration

$$\begin{array}{ccccccc} \{1,2,3\} & \{1,4,8\} & \{2,4,7\} & \{3,6,8\} & \{9,B,C\} & \{B,D,F\} & \{C,D,G\} \\ \{4,5,6\} & \{1,5,9\} & \{2,6,9\} & \{3,5,7\} & \{A,D,E\} & \{B,E,G\} & \{C,E,F\} \\ \{7,8,A\} & & & & \{A,F,G\}. & & \end{array}$$

To aid in our understanding of this configuration, note that if the A and 9 from the lines $\{7, 8, A\}$ and $\{9, B, C\}$ were swapped, the result would be the disjoint union of the Pappus configuration and the Fano plane. If we let $\mathcal{C}^-$ denote the above configuration with the point A removed from the line $\{7, 8, A\}$ then any ordering of vertices from the proof above yields a configuration isomorphic to that shown in Figure 2.4. This realization is not strong, as it contains a line meeting points $7, 8, 9, A$.

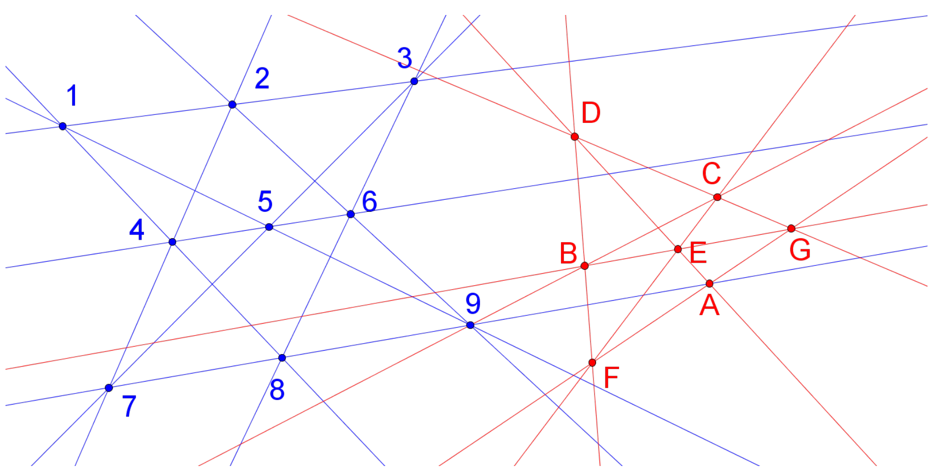


Figure 2.4: A (16_3)-configuration such that the configuration minus any intersection admits a weak, but not strong, realization due to Steinitz' Configuration Theorem.

Chapter 3

Combinatorial Configurations

In this chapter, the term 'configurations' will exclusively refer to combinatorial configurations. We begin by providing two asymptotic existence results for $[r, k]$-configurations. Recall from the previous chapter that $N(r, k)$ is the least value for which an (n_r, b_k)-configuration exists for all $n \geq N(r, k)$ that satisfies the divisibility condition $nr = bk$. The first asymptotic result (Theorem 3.1.1) provides an upper bound on $N(r, k)$ that is an improvement on previously known bounds. The second asymptotic result (Theorem 3.1.2) further improves this upper bound under the condition that r is substantially larger than k (this will be made more precise later). Both of these proofs are constructive in nature — we provide explicit examples of (n_r, b_k)-configurations. In Section 3.2, we explore the idea of embedding a partial configuration within an $[r, k]$-configuration, providing bounds on the minimum number of points needed in the $[r, k]$-configuration. This result is then used to answer an open question regarding designs.

3.1 Existence Results

Recall from the previous chapter that the existence of an (n_r, b_k)-configuration depends upon the divisibility condition $nr = bk$, and the inequality $n \geq r(k-1)+1$. The latter condition imposes a lower bound for $N(r, k)$. We now provide an upper bound for $N(r, k)$, for any $r, k \geq 2$.

Theorem 3.1.1. *$N(r, k) < k^2 \cdot \max\{r+1, \frac{r}{2}+k\}$ for all $r \geq k$.*

This theorem improves previously existing results and is constructive in nature. Note that this provides a bound roughly on the order of $O(k^2 r + k^3)$, whereas the lower bound $r(k-1)+1$ is of order $O(rk)$. Therefore $N(r,k)$ cannot have an upper bound of a smaller order than $O(rk)$ (as this is the order of the lower bound on $N(r,k)$). If the imbalance between r and k is large, then an upper bound of order $O(rk)$ is obtained from the following result.

Theorem 3.1.2. *Given a fixed k, there exists a value $R(k)$ such that $N(r,k) \leq 2rk+r$ for all $r \geq R(k)$.*

In both of these theorems, we will also provide connected combinatorial $[r,k]$-configurations on n points for all $n > N(r,k)$ (a configuration is *connected* if it is not the disjoint union of two subconfigurations). Both theorems are strengthened by this connectivity property. The general concept behind both theorems is to create a configuration resembling a resolvable transversal design with block size k. Several constructions will be utilized to prove these theorems. To assist in the notation, we define the following:

$$[n] := \{1, 2, ..., n\},$$

$$d := k/\gcd(r,k).$$

The divisibility condition on $[r,k]$-configurations implies that the number of points must be a multiple of d.

3.1.1 Construction of the $[r,k]$-Configurations $A(w,\lambda,r,k)$

As mentioned, the proof of our first asymptotic existence result Theorem 3.1.1 is constructive: we provide explicit examples of (n_r, b_k)-configurations for each n larger than $k^2 \cdot \max r+1, \frac{r}{2}+k$. These constructions belong to a family of configurations which we will denote by $A(w,\lambda,r,k)$.

Due to the dual nature of configurations, we will assume that $r \geq k$. The configuration $A(w,\lambda,r,k)$ takes parameters $w \in \mathbb{N}$ satisfying

$$w \geq k \cdot \max\left\{r+1,\, \frac{r}{2}+k\right\}$$

and $\lambda \in \{0, ..., \gcd(r,k)-1\}$. Given these parameters, the configuration $A(w,\lambda,r,k)$ will be an $[r,k]$-configuration with $wk+\lambda d$ points.

We begin our construction with the point set $\mathbb{Z}_k \times \mathbb{Z}_w$. Define

$$b_{mc} := \{(x,\, mx+c) \;:\; x \in \mathbb{Z}_k\}, \qquad \text{for all } m \in [w],\, c \in \mathbb{Z}_w$$

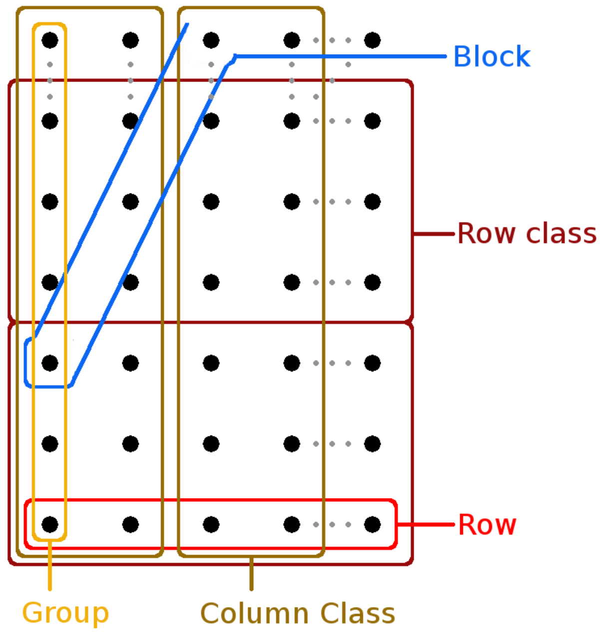


Figure 3.1: A diagram illustrating the definitions proposed in the construction $A(w, \lambda, r, k)$. Here the lower left dot is considered $(0, 0)$. The highlighted line represents $b_{2,2}$, having slope 2 and intercept 2. The value of $\frac{r}{\gcd(r,k)}$ is equal to 3, since the row classes contain three rows, and $\gcd(r, k) = 2$, since each column class contains two columns. The column classes continue, and partition all the columns (and all the points within those columns as well). If $\lambda = 2$, then the only two row classes are those illustrated: H_0 and H_1. Each $G_\alpha \cap H_\beta$ contains $2 \cdot 3 = 6$ points.

to be the set of lines. We will refer to m as the *slope* of the line, and c as the *intercept*. These lines as a collection do not preserve the linearity condition unless w is a prime (which we do not require), so the collection of points and lines do not technically form a configuration, and is merely an incidence structure. However, a suitable subset of these lines is indeed linear, as we will now show. Partition the lines into *parallel classes with slope* m

$$B_m := \{b_{mc} \ : \ c \in \mathbb{Z}_w\}, \quad m \in [w].$$

The *rows* of the incidence structure may be thought of as lines of slope zero. Due to their special role, we denote these blocks by

$$h_c := \{(x, c) \ : \ x \in \mathbb{Z}_k\}, \qquad \text{for each } c \in \mathbb{Z}_w.$$

The columns of the array correspond with groups in the sense of transversal designs.

These will be denoted

$$g_x := \{(x, c) \ : \ c \in \mathbb{Z}_w\}, \qquad \text{for each } x \in \mathbb{Z}_k.$$

Finally, we will partition the k groups into d *column classes* of size $\gcd(r, k)$:

$$G_\alpha := \left\{g_{\alpha \gcd(r,k)},\, g_{\alpha \gcd(r,k)+1},\, \ldots,\, g_{(\alpha+1)\gcd(r,k)-1}\right\}, \qquad \text{for all } \alpha \in \{0, \ldots, d-1\},$$

and partition the first $\lambda \frac{r}{\gcd(r,k)}$ rows into λ *row classes* of size $\frac{r}{\gcd(r,k)}$:

$$H_\beta := \left\{h_{\beta(\frac{r}{\gcd(r,k)})},\, \ldots,\, h_{(\beta+1)(\frac{r}{\gcd(r,k)})-1}\right\}, \qquad \text{for all } \beta \in \{0, \ldots, \lambda-1\}.$$

Figure 3.1 illustrates all of these definitions.

Restrict the set of lines to those only contained within a parallel class of slope $0 < m < w/k$. We claim that the set of lines contained within these parallel classes is linear. Suppose p, q are two points contained within two lines $b_{m_1c_1}$ and $b_{m_2c_2}$. Then we may rewrite p, q as

$$p := (x,\, m_1x + c_1) = (x,\, m_2x + c_2) \qquad (\text{mod } w),$$

$$q := (y,\, m_1y + c_1) = (y,\, m_2y + c_2) \qquad (\text{mod } w),$$

for some $x, y \in \mathbb{Z}_k$. These two equalities directly imply that

$$m_1(x-y) = m_2(x-y) \qquad (\text{mod } w),$$

or

$$(m_1 - m_2)(x-y) = 0 \qquad (\text{mod } w).$$

If $x = y$, then p and q belong to the same column (as they have the same first coordinate), which implies $p = q$, as no line contains two different points in the same group. If $m_1 = m_2$ then clearly $b_{m_1c_1} = b_{m_2c_2}$. Otherwise, $|x - y| < k$ and $0 < |m_1 - m_2| < \frac{w}{k}$. Therefore, the product $|x-y| \cdot |m_1 - m_2|$ does not exceed $w - 1$, and thus cannot be congruent to 0 modulo w. This implies that either $p = q$ or $b_{m_1c_1} = b_{m_2c_2}$, which in turn implies that the set of lines contained in a parallel class with slope $0 < m < w/k$ is indeed linear. If $\gcd(r, k) = 1$, then $\lambda = 0$ and restricting the set of lines in the incidence structure to any collection of r parallel classes each with slope $< w/k$ will result in an $[r,k]$-configuration with wk points, which we will denote $A(w,0,r,k)$. We now assume that $\gcd(r,k) > 1$.

For any $\alpha \in \{0,...,d-1\}$ and $\beta \in \{0,...,\lambda-1\}$, let $G_\alpha \cap H_\beta$ denote the set of r points contained within a group in G_α and a row in H_β. There are $d\lambda \leq k$ such intersections of row and column classes. For every pair α, β, associate a unique parallel class $B_{m(\alpha,\beta)}$ with slope $m(\alpha,\beta)$ satisfying

$$\frac{r}{\gcd(r,k)} \leq m(\alpha,\beta) < \frac{r}{\gcd(r,k)} + k.$$

Note that $\frac{r}{\gcd(r,k)} + k \leq \frac{w}{k}$, since $w \geq k(\frac{r}{2}+k)$. Thus the parallel classes $B_{m(\alpha,\beta)}$ all have slope between 0 and w/k (these classes are part of the collection of w/k parallel classes we have restricted to). We show that any line in $B_{m(\alpha,\beta)}$ meets at most one point in $G_\alpha \cap H_\beta$. Suppose a line b_{mc} in $B_{m(\alpha,\beta)}$ meets group $g_x \in G_\alpha$ at a point p. Then we may write p as

$$p = (x,\, mx + c).$$

Given $y \in [\gcd(r,k)]$, consider the point q where b_{mc} meets g_{x+y}:

$$q = (x+y,\, mx + c + my).$$

These two points have a difference in the second coordinate equal to my. Since $y \in [\gcd(r,k)]$ and $\frac{r}{\gcd(r,k)} \leq m < \frac{r}{\gcd(r,k)} + k$, we have

$$\frac{r}{\gcd(r,k)} \leq my,$$

and

$$\begin{aligned} my &< \left(\frac{r}{\gcd(r,k)} + k\right)(\gcd(r,k)) \\ &\leq \frac{3r}{2} + k\gcd(r,k) - \frac{r}{2} \\ &\leq k\left(\frac{3r}{2k} + \gcd(r,k)\right) - \frac{r}{\gcd(r,k)} \\ &\leq k\left(\frac{r}{2} + k\right) - \frac{r}{\gcd(r,k)} \\ &\leq w - \frac{r}{\gcd(r,k)}. \end{aligned}$$

The third and fourth inequalities follow from the inequality $2 \leq \gcd(r,k) \leq k$. The row class H_β contains $\frac{r}{\gcd(r,k)}$ *consecutive* rows, so the difference in second-coordinate between any two points in H_β lies within the interval $[-\frac{r}{\gcd(r,k)}+1, \frac{r}{\gcd(r,k)}-1]$. Since my does not lie within this interval (modulo w), we conclude that p and q cannot both lie within the same row class. Thus $q \notin H_\beta$. A similar conclusion can be drawn for any point where b_{mc} meets g_{x-y}. This result holds for all $y \in [\gcd(r,k)]$. The $\gcd(r,k)$ groups contained within a column class are consecutive, and G_α consists of some subset of $\{g_{x-\gcd(r,k)}, ..., g_{x+\gcd(r,k)}\}$. Therefore, b_{mc} meets every other point in a column within G_α in a row that is not contained within H_β. That is, b_{mc} only meets $G_\alpha \cap H_\beta$ at p. It follows that every line in $B_{m(\alpha,\beta)}$ meets at most one point in $G_\alpha \cap H_\beta$.

Now restrict our collection of w/k parallel classes further to any r-subset of parallel classes that contains the classes $B_{m(\alpha,\beta)}$. This results in an $[r,k]$-configuration on wk points. For every pair α, β, remove the points in $G_\alpha \cap H_\beta$ from the lines within parallel class $B_{m(\alpha,\beta)}$. Only one point has been removed from exactly r different parallel lines. Add an isolated point to the configuration, and append it to each of these r lines of size $k-1$ within $B_{m(\alpha,\beta)}$. This does not destroy the linearity of the configuration, as these r lines are all parallel. However, it does destroy regularity (each point no longer has replication number r). After all λd points have been added to the now partial configuration (one for each pair α, β), the points within any row class H_β now have replication number $r-1$ (each one has been removed from exactly one line). All other points have replication number r, and all lines have size k. To complete the construction, add the rows $h_0, ..., h_{\lambda-1}$ to the partial configuration as lines. The result is a configuration with constant replication number r and line size k. This $[r,k]$-configuration contains $wk + \lambda d$ points, and we denote such a configuration by $A(w, \lambda, r, k)$.

The first asymptotic existence result Theorem 3.1.1 follows quickly from this construction. Consider any n larger than $k^2 \cdot \max\{r+1, \frac{r}{2}+k\}$ that is also a multiple of d. Write n in the form $wk + \lambda d$, where $\lambda \leq \gcd(r,k)$. Then w is large enough to guarantee the existence of the configuration $A(w, \lambda, r, k)$. Therefore, there exists an $[r,k]$-configuration on n points.

As an explicit example of this construction, consider $A(21, 2, 6, 3)$. This construction yields an $(65_6, 130_3)$-configuration. Initially, the incidence structure contains 63 points arranged as $\mathbb{Z}_3 \times \mathbb{Z}_{21}$. The value of λ is 2, so we have two row classes, H_0 and H_1, each containing 2 rows. Note that $G_0 = \{g_0, g_1, g_2\}$ is the only column class. Thus, $\alpha = 0$ and $0 \leq \beta < 2$, so the number of pairs α, β is 2. Define $B_{m(0,0)}$ and

$x0, y1, z2$	$x0, y4, z8$	$x0, y5, z10$	$x0, y6, z12$	$\infty_{00}, x2, x4$	$x0, \infty_{01}, z6$	$x0, y0, z0$
$x1, y2, z3$	$x1, y5, z9$	$x1, y6, z11$	$x1, y7, z13$	$\infty_{00}, y3, z5$	$x1, y4, z7$	$x1, y1, z1$
$x2, y3, z4$	$x2, y6, z10$	$x2, y7, z12$	$x2, y8, z14$	$x2, y4, z6$	$\infty_{01}, y5, z8$	$x2, y2, z2$
$x3, y4, z5$	$x3, y7, z11$	$x3, y8, z13$	$x3, y9, z15$	$x3, y5, z7$	$\infty_{01}, y6, z9$	$x3, y3, z3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$x18, y19, z20$	$x18, y1, z5$	$x18, y2, z7$	$x18, y3, z9$	$x18, y20, \infty_{00}$	$x18, y0, \infty_{01}$	
$x19, y20, z0$	$x19, y2, z6$	$x19, y3, z8$	$x19, y4, z10$	$x19, \infty_{00}, z2$	$x19, y1, z4$	
$x20, y0, z1$	$x20, y3, z7$	$x20, y4, z9$	$x20, y5, z11$	$x20, \infty_{00}, z3$	$x20, \infty_{01}, z5$	

Table 3.1: The 130 lines of $A(21, 2, 6, 3)$. The columns of lines correspond to the lines of B_1, B_4, B_5 and B_6, followed by the class B_2 after the points $x0, y0, z0, x1, y1, z1$ are replaced with ∞_{00} (since these six points lie in the set $G_0 \cap H_0$). The sixth column of lines is the class B_3 after the points $x2, y2, z2, x3, y3, z3$ are replaced with ∞_{01} (as these six points lie within $G_0 \cap H_1$). Finally, the last column contains the four rows that are added to the configuration.

$B_{m(0,1)}$ to be equal to B_2 and B_3 respectively. Finally, include the additional parallel classes B_1, B_4, B_5 and B_6 (all of these have slope less than w/k and thus are linear). We list the lines of this configuration in Table 3.1. To ease in notation, we write the coordinates $(0, i)$, $(1, i)$ and $(2, i)$ as xi, yi and zi respectively. So the coordinate $(1, 17)$ is denoted $y17$. The isolated point created for $G_0 \cap H_0$ is denoted ∞_{00} and the other isolated point is ∞_{01}.

3.1.2 Construction of the $[r, k]$-Configurations $A'(\rho, \mu, \lambda, r, k)$

The second asymptotic existence result Theorem 3.1.2 demonstrates that, if r is sufficiently large relative to k then an (n_r, b_k)-configuration exists, provided $n \geq 2rk+r$. The proof of this result follows somewhat similarly to Theorem 3.1.1. We provide a family of constructions $A'(\rho, \mu, \lambda, r, k)$. This family requires a certain level of imbalance between r, k: in particular, $2r > k^2$. Furthermore, our parameters have changed. The value ρ is a prime larger than $2r$ that takes the role of w in the previous example. A new parameter μ is also added to the construction, where μ is any integral value between 0 and $(\frac{1}{2k} - \frac{2}{r})\rho$. As before, λ is a whole number less than $\gcd(r, k)$.

The initialization of the construction $A'(\rho, \mu, \lambda, r, k)$ is similar to the construction $A(\rho, \lambda, r, k)$, and we utilize the same definitions as before. Here the collection of all lines does indeed form a linear set, since ρ is prime. To see this, consider the lines b_{mc}

over the point set $\mathbb{Z}_k \times \mathbb{Z}_\rho$. Let p, q be two points in lines $b_{m_1c_1}$ and $b_{m_2c_2}$. Then

$$p = (x,\, m_1x + c_1) = (x,\, m_2x + c_2),$$

$$q = (y,\, m_1y + c_1) = (y,\, m_2y + c_2).$$

As in the case of $A(\rho, \lambda, r, k)$, this implies that

$$(m_1 - m_2)(x - y) = 0 \qquad (\text{mod } \rho).$$

However, $\mathbb{Z}_\rho$ is a field, so this directly implies either $x = y$ or $m_1 = m_2$. In the former case, we come to the conclusion that $p = q$, and in the latter case we have $b_{m_1c_1} = b_{m_2c_2}$. Thus, the set of all lines is linear. However, we will not yet consider these lines as belonging to the configuration $A'(\rho, \mu, \lambda, r, k)$.

For each $\beta \in [\lambda]$, define I_β to be the set of $\frac{r}{\gcd(r,k)}$ lines in a parallel class B_β with intercept c satisfying $0 \le c < \frac{r}{\gcd(r,k)}$.

Lemma 3.1.3. *Assume $\lambda > 0$. Then the number of parallel classes B_m containing a line meeting $G_\alpha \cap I_\beta$ in more than one point is bounded above by $2(r - \frac{r}{\gcd(r,k)})$, for any choice of $\alpha \in [d]$ and $\beta \in [\lambda]$.*

Proof. Note that $\gcd(r, k) \neq 1$ (since $0 < \lambda < \gcd(r, k)$). We first demonstrate that if a parallel class B_m contains a line meeting $G_\alpha \cap I_\beta$ in more than one point, then it also contains a line meeting $G_0 \cap I_\beta$ in more than one point. This will allow us to only consider the case where $\alpha = 0$.

Consider the transformation:

$$f : G_\alpha \cap I_\beta \to G_0 \cap I_\beta, \qquad f(x, y) = (x - \alpha \cdot \gcd(r, k), y - \beta\alpha \cdot \gcd(r, k))$$

This bijection is merely a translation that sends the points of $G_\alpha \cap I_\beta$ to $G_0 \cap I_\beta$. For any block b containing points $p_1, ..., p_{\gcd(r,k)}$ in $G_\alpha \cap I_\beta$, let $f(b)$ denote the block containing points $f(p_1), ..., f(p_{\gcd(r,k)})$. Since f is a translation, it follows that b and $f(b)$ are parallel — the map f merely translates points, so b and $f(b)$ share the same slope. Thus, if a block b_{mc} meets two points p, q in $G_\alpha \cap I_\beta$, then $f(b_{mc})$ meets the two points $f(p), f(q)$, which are each contained within $G_0 \cap I_\beta$. Since $f(b_{mc})$ is parallel to b_{mc}, it follows that if the parallel class B_m contains a block meeting two points in $G_\alpha \cap I_\beta$, then it also contains a block meeting two points in $G_0 \cap I_\beta$. Therefore, we may assume without loss of generality that $\alpha = 0$. Define the points

$$\mathbf{0} := (0,0), \qquad \mathbf{Z} := \left(0, \frac{r}{\gcd(r,k)} - 1\right)$$

in $G_0 \cap I_\beta$. We claim that if a line b_{mc} contains two points $p, q \in G_0 \cap I_\beta$, then there exists a parallel line containing either $\mathbf{0}$ or $\mathbf{Z}$ and another point in $G_0 \cap I_\beta$. Consider such a line b_{mc} in parallel class B_m that meets two points $p, q \in G_0 \cap I_\beta$, with $p \in b_{\beta c_1}$ and $q \in b_{\beta c_2}$, for some $c_1, c_2 \in [0, \frac{r}{\gcd(r,k)} - 1]$. Then we may write

$$p := (x,\ \beta x + c_1), \qquad q := (y,\ \beta y + c_2),$$

$$q - p = (y - x,\ \beta(y - x) + c_2 - c_1),$$

where $x, y \in [0, \gcd(r,k) - 1]$. Assume without loss to generality that $y \geq x$. It then follows that $q - p \in G_0$, since $0 \leq y - x < \gcd(r,k)$. The second coordinate of $q - p$ lies within the interval

$$\left[\beta(y-x) - \left(\frac{r}{\gcd(r,k)} - 1\right),\ \beta(y-x) + \left(\frac{r}{\gcd(r,k)} - 1\right)\right].$$

Note that the second-coordinates of points in $g_{y-x} \cap I_\beta$ lie within the interval

$$\left[\beta(y-x),\ \beta(y-x) + \left(\frac{r}{\gcd(r,k)} - 1\right)\right].$$

Then the line b_{m0} meets $p - p = \mathbf{0}$ and $q - p$. If $q - p$ has a second coordinate in the interval

$$\left[\beta(y-x),\ \beta(y-x) + \left(\frac{r}{\gcd(r,k)} - 1\right)\right],$$

then it lies within $g_{y-x} \cap I_\beta$, and thus it lies within $G_0 \cap I_\beta$. Now suppose the second coordinate of $q - p$ is instead contained within the interval

$$\left[\beta(y-x) - \left(\frac{r}{\gcd(r,k)} - 1\right),\ \beta(y-x)\right].$$

In this case, $b_{m,(\frac{r}{\gcd(r,k)}-1)}$ contains $\mathbf{Z}$ and $q - p + \mathbf{Z}$. The latter point lies within $g_{y-x} \cap I_\beta$. Therefore, if the parallel class B_m contains a line b_{mc} meeting $G_0 \cap I_\beta$ in two distinct points, then either B_m contains a line meeting $\mathbf{0}$ and another point of $G_0 \cap I_\beta$ or B_m contains a line meeting $\mathbf{Z}$ and another point of $G_0 \cap I_\beta$. This completes the claim.

This means that in order to count the number of parallel classes B_m containing a line meeting $G_0 \cap I_\beta$ in two points, it suffices to count the number of lines meeting $\mathbf{0}$ and another point of $G_0 \cap I_\beta$ and the number of lines meeting $\mathbf{Z}$ and another point of $G_0 \cap I_\beta$. The number of points in $G_0 \cap I_\beta$ that are not in the same column as $\mathbf{0}$ or $\mathbf{Z}$ is

$$(\gcd(r,k)-1)\cdot\frac{r}{\gcd(r,k)} = r - \frac{r}{\gcd(r,k)}.$$

Thus, the maximum number of lines meeting $\mathbf{0}$ or $\mathbf{Z}$ as well as another point of $G_0 \cap I_\beta$ is bounded above by

$$2\left(r - \frac{r}{\gcd(r,k)}\right).$$

□

If $2r > k^2$ and $\lambda > 0$, then the above lemma implies that there are at most

$$2r - \frac{2r}{\gcd(r,k)} \leq 2r - k$$

parallel classes containing a line that meets a point of any $G_\alpha \cap I_\beta$ in more than one point. Since $\rho > 2r$, it follows that for each of the $d\lambda \leq k$ distinct pairs of α, β, we may choose a unique parallel class $B_{m(\alpha,\beta)}$ that does not contain a line meeting $G_\alpha \cap I_\beta$ in two or more points. In the case that $\lambda = 0$, there are no pairs α, β to consider.

Add the λd parallel classes $B_{m(\alpha,\beta)}$ to the incidence structure. For each pair α, β, remove the points in $G_\alpha \cap I_\beta$ from the lines in $B_{m(\alpha,\beta)}$. Note that this creates exactly r 'shortened' lines of size $k-1$. Add an isolated point in the incidence structure and append it to each of these r lines in $B_{m(\alpha,\beta)}$ of size $k-1$. Since all of the lines in $B_{m(\alpha,\beta)}$ are parallel, the linearity of the incidence structure is preserved. After this procedure is completed for all pairs, add the lines in I_β to the structure. These lines belong to the parallel class B_β, and thus adding them to the structure does not destroy the linearity property. The outcome is a partial configuration with $\rho k + \lambda d$ points. The λd added points have a replication number of r, while the points in the original point set $\mathbb{Z}_k \times \mathbb{Z}_\rho$ each posses a replication number of λd. The number of (potentially modified) parallel classes that we utilized in this construction so far is $\lambda d + \lambda$ (λ of the form B_β and λd of the form $B_{m(\alpha,\beta)}$). This value is less than or equal to $2k$.

Next, define

$$Y := \left\lfloor \frac{r}{k} - 1 \right\rfloor.$$

For each $y \in [Y]$, choose a previously unmodified (i.e. not utilized so far in the construction) parallel class B^y, and let μ_y be any integer in the interval $[0, \frac{\rho}{r}]$. Choose $\mu_y r$ lines from B^y and label this set as I^y. For each $x \in \mathbb{Z}_k$ and $y \in [Y]$, choose k other unmodified parallel classes B^{xy} (all distinct). This selection requires a total of

$$Yk + Y \leq \left(\frac{r}{k} - 1\right) k + \frac{r}{k} \leq 2r - 2k$$

unmodified parallel classes. The last inequality is a consequence of the fact that $2r > k^2$. Since $\rho > 2r$, it follows that there are a sufficient number of unmodified parallel classes to make such a selection feasible. Add the parallel classes B^{xy} to the partial configuration. This keeps the replication number of the points in $\mathbb{Z}_k \times \mathbb{Z}_\rho$ constant, and less than r (since the total number of pairs x, y is less than $r - k$, and the points already possess a replication number of $\lambda d \leq k$). Remove the $\mu_y r$ points in $g_x \cap I^y$ from the parallel class B^{xy}. This creates $\mu_y r$ shortened lines of size $k - 1$. For each x, add μ_y isolated points to the partial configuration, and append each one to r distinct short lines in B^{xy} (linearity is preserved since the lines within B^{xy} are parallel). Finally, add the lines of I^y to the partial configuration. The consequence of this is an $[r, k]$-configuration with $\sum \mu_y k$ additional points. The values of μ_y can be chosen so that $\sum \mu_y = \mu$, since each μ_y is an integer between 0 and ρ/r, and

$$Y \cdot \lfloor \rho/r \rfloor \geq \left(\frac{r}{k} - 2\right)\left(\frac{\rho}{r} - 1\right) \geq \left(\frac{1}{k} - \frac{2}{r}\right)\rho - \frac{r}{k} + 2 \geq \left(\frac{1}{k} - \frac{2}{r}\right)\rho.$$

The third inequality is just an expansion of the second expression, and the final inequality is due to the fact that $r/k > 2$. Furthermore, the replication number of each of the initial ρk points of $\mathbb{Z}_k \times \mathbb{Z}_\rho$ is constant. Finally, add a sufficient number of remaining parallel classes to the partial configuration until the replication number of all points in $\mathbb{Z}_k \times \mathbb{Z}_\rho$ is r. The result is an $[r, k]$-configuration containing $\rho k + \mu k + \lambda d$ points. We will denote this configuration by $A'(\rho, \mu, \lambda, r, k)$. This configuration will be instrumental in proving the second asymptotic existence result, Theorem 3.1.2. We now turn to the proof of this theorem.

3.1.3 Proof of a Second Asymptotic Existence Result

The construction $A'(\rho, \mu, \lambda, r, k)$, along with a variation of a number theoretic result known as Bertrand's Postulate will yield Theorem 3.1.2.

Theorem 3.1.4. *[25, pg. 494] Bertrand's Postulate: Given any $\varepsilon > 0$, there exists a value $R'(\varepsilon)$ such that a prime exists in the interval $[x,\ (1+\varepsilon)x]$ for all $x \geq R'(\varepsilon)$.*

We now turn to the proof of Theorem 3.1.2.

Proof. Given a fixed k, let $R := R(k)$ be any integer sufficiently large to guarantee the existence of a prime in the interval

$$\left[x,\ \left(1+\frac{1}{2k}-\frac{2}{R}\right)x\right]$$

for every $x \geq R(k)$. Such an $R(k)$ exists by Bertrand's Postulate. Let r be any integer larger than $R(k)$, and let $\rho_1, \rho_2, ...$ be the sequence of consecutive primes larger than $2r$. Then $A'(\rho_i, \mu, \lambda, r, k)$ can be utilized to generate a configuration containing n points, for any n satisfying the divisibility condition and contained within the interval

$$[\rho_i k,\ \rho_i k + \mu k + \lambda d].$$

The constraints on μ and λ imply that this interval contains the subintervals

$$\left[\rho_i k,\ \left(1+\frac{1}{2k}-\frac{2}{r}\right)\rho_i k\right] \supset \left[\rho_i k,\ \left(1+\frac{1}{2k}-\frac{2}{R}\right)\rho_i k\right] \supset [\rho_i k,\ \rho_{i+1} k]$$

where the last inclusion is due to Bertrand's Postulate, since ρ_{i+1} must lie in the interval between ρ_i and $(1+\frac{1}{2k}-\frac{2}{R})\rho_i$. This sequence of intervals $[\rho_i k,\ \rho_{i+1} k]$ covers all multiples of d larger than $\rho_1 k$. Since ρ_1 lies somewhere in the interval

$$\left[2r,\ \left(1+\frac{1}{2k}-\frac{2}{2r}\right)2r\right] = \left[2r,\ 2r+\frac{r}{k}-2\right] \subset \left[2r,\ 2r+\frac{r}{k}\right],$$

we have that

$$N(r,k) \leq 2rk + r.$$

□

The value of $r := R(k)$ is known to exist, but may be quite large compared to k.

Example 3.1.1. As an example of how the construction $A'(\rho, \mu, \lambda, r, k)$ may be used in practice, we examine the case when $r = 240\,000$ and $k = 30$. Here, Theorem 3.1.1 provides an upper bound of roughly 1.08×10^8, while the trivial lower bound is set at 6.96×10^6. In [29, pg. 354], it is shown that for all $x \geq 2\,010\,760$ there exists a prime between x and $(1+\frac{1}{16597})x$. Given any ρ, the value of μ must be chosen to lie between 0 and $(\frac{1}{2k}-\frac{2}{r})\rho > \frac{1}{16597}\rho$. Thus, the sequence $\rho_1, \rho_2, \rho_3, \ldots$ of consecutive primes larger than 2 010 760 are such that $\rho_{i+1} < \rho_i + \frac{1}{16597}\rho_i$, and therefore every n larger than $\rho_1 k$ can be written in the form

$$\rho_i k + \mu k + \lambda d$$

for some suitable ρ_i, μ and λ in the construction $A'(\rho_i, \mu, \lambda, r, k)$. Since $\rho_1 = 2\,010\,881$, it follows that

$$N(240000, 30) \leq 2010881 \cdot 30 \leq 6.04 \times 10^7.$$

This gives a better bound than Theorem 3.1.1 provides, even though $R(30)$ may be significantly larger than 240 000. In fact, the theorem in [29] can be used as above for any r, k combination such that

$$k \in [8, 1417]$$

$$r \in \left[\frac{k^2}{2}, 1\,005\,440\right]$$

to show that $N(r, k) \leq 2010881k$.

The proof of Theorem 3.1.2 relies heavily on two critical ideas:

- The size of a prime gap is small relative to the size of the corresponding primes.
- Adding up to $\mu k + \lambda d$ points to an initial configuration on point set $\mathbb{Z}_k \times \mathbb{Z}_\rho$. The addition of these isolated points requires the modification of pre-existing parallel classes. A large number of parallel classes (dependent on r) relative to k allows for the addition of more isolated points to the configuration.

Such a proof technique does not fare well for general r, k. First, we cannot utilize stronger versions of Bertrand's Postulate. This means that, for a given prime ρ, slightly larger than r, we must be able to generate configurations with any number of points within the interval $[\rho k, 2\rho k]$ (satisfying the divisibility conditions). Thus, we must be able to add up to ρk points to the $[r, k]$-configuration $A'(\rho, 0, 0, r, k)$. However, μk is bounded above by $\rho/2$ (and this assumes that $2r > k^2$).

Although this proof technique cannot generalize to any r, k, it can be utilized to a moderate degree in the case that $\gcd(r, k) = 1$. In this case, $\lambda = 0$, and the value of Y may be increased to $\lfloor r/k \rfloor$. The restriction that $2r > k^2$ may also be removed in this scenario. Thus, μ may potentially be as large as

$$Y \cdot \lfloor \rho/r \rfloor \geq \left(\frac{r}{k} - 1\right)\left(\frac{\rho}{r} - 1\right) \geq \left(\frac{\rho}{k} - \frac{r}{k} - \frac{\rho}{r}\right).$$

As a final note on this optimization, we may assume that ρ is less than $k(r+k)$. From Theorem 3.1.1, $[r,k]$-configurations on more than $k^2(r+k)$ points are already known to exist, and therefore, we need not consider cases where ρ is larger than $k(r+k)$. Thus, when $\gcd(r,k)=1$, the value of μk may be as large as

$$\rho - r - \frac{k^2(r+k)}{r}.$$

If $r=4$ and $k=3$, then $Y=1$. If $\rho=11$, then μ may be as large 2, and this generates configurations on $33, 36$, and 39 points. If $\rho=13$, then μ may be as large as 3, and this generates configurations on $39, 42, 45$, and 48 points. We may then invoke Theorem 3.1.1 to demonstrate the existence of configurations on more than 48 points. Thus, $N(4,3) \leq 33$. A similar argument can be utilized to show that $N(5,3) \leq 33$, provided a configuration on 48 points exists.

3.2 Embedding Configurations

Suppose we are given a partial configuration $\mathcal{C}_0 = (\mathcal{P}, \mathcal{L}_0)$, with constant line size k, but not necessarily constant replication number. It is natural to ask if this partial configuration is in fact a subconfiguration of an $[r,k]$-configuration on the same number of points. An affirmative answer yields an $[r,k]$-configuration $\mathcal{C} = (\mathcal{P}, \mathcal{L}_0)$ with $\mathcal{L} \supset \mathcal{L}_0$. It is quickly evident that not every configuration yields a completion, even in the cases where the number of points satisfies the divisibility conditions for an $[r,k]$-configuration to exist. If $\mathcal{C}_0$ contains n points, and $\mathcal{L}_0$ is dense (that is, $|\mathcal{L}_0|$ is close to nr/k), then it is certainly possible that the additional lines required cannot be positioned in such a way as to create an $[r,k]$-configuration. As a trivial example, the following collection of seven lines given below cannot be completed as an (8_3)-configuration:

$$\begin{array}{lll} \{1,2,3\} & \{2,4,6\} & \{3,4,7\} \\ \{1,4,5\} & \{2,5,7\} & \{3,5,8\} \\ \{1,6,7\}. & & \end{array}$$

A certain level of sparsity is desired in $\mathcal{L}_0$ if we hope to complete a configuration.

An alternative to completing a partial configuration is the idea of *embedding* $\mathcal{C}_0 = (\mathcal{P}_0, \mathcal{L}_0)$ in an $[r,k]$-configuration with only a marginal increase in the number of points. What occurs is a tradeoff: we no longer require sparsity in the size of $\mathcal{L}_0$ if we are allowed to add points to the partial configuration. In order for an embedding to exist, there must be certain criteria that both $\mathcal{C}_0$, and any configuration $\mathcal{C}$ containing $\mathcal{C}_0$ as a subconfiguration must satisfy:

- The necessary existence conditions on $\mathcal{C}$ must be met: $nr = bk$ and $n \geq r(k-1)+1$.
- The replication number of every point in $\mathcal{C}_0$ must be less than or equal to r.

If the maximum replication number over all points in $\mathcal{C}_0$ is no more than r, while the line size is constant, then we may refer to $\mathcal{C}_0$ as a *partial* $[r,k]$*-configuration.* If a point p_i in a partial $[r,k]$-configuration has replication number r_i, then its *deficiency* is $r - r_i$. The *total deficiency* of a partial $[r,k]$-configuration is the sum of the deficiencies over all points.

The following theorem shows that every partial $[r,k]$-configuration $\mathcal{C}_0$ can be embedded in a larger $[r,k]$-configuration $\mathcal{C}$, and gives bounds on the maximum number of points required to add to $\mathcal{C}_0$ in order to obtain such an embedding. This type of embedding is an *induced* embedding — it contains $\mathcal{C}_0$ as an induced subconfiguration. The theorem below does not completely answer the embedding question (as it requires some assumptions on the total deficiency that are not necessary for an embedding to exist). However, it does most of the work in solving some general embedding questions. Several more general embedding results are proved as corollaries to this theorem.

Theorem 3.2.1. *Let $\mathcal{C}_0$ be a partial $[r,k]$-configuration on n points with total deficiency F. Let $d = k/\gcd(r,k)$. If F is at least $d(r^2+kr)$ and a multiple of k, then $\mathcal{C}_0$ is an induced subconfiguration on an $[r,k]$-configuration $\mathcal{C}$ containing fewer than $n + \frac{(2k+1)F}{r} + 3rk^2$ points.*

This can also be considered as a generalization of a similar result in graph theory proved by Erdős and Kelly in [20]. The graph theoretic result demonstrates that any graph on n vertices is an induced subgraph of an r-regular graph ($r < n$) with no more than $2n$ points. Our next result will extend this to embeddings as linear hypergraphs. The *degree sequence* of a hypergraph on vertices $v_1, ..., v_n$ is the sequence $(d_1, ..., d_n) \in \mathbb{N}^n$ such that d_i is the degree of v_i. The vertices are labeled so that the degree sequence is monotonic and increasing. If $\mathcal{C}$ is a partial configuration on n points with constant block size k, then it may be considered as a k-uniform linear hypergraph (see Section 2.1). Let $(r_1, ..., r_n)$ be the degree sequence of $\mathcal{C}$ as a linear hypergraph, so that r_i is the replication number of point v_i in $\mathcal{C}$. Suppose there exists a k-uniform linear hypergraph $\mathcal{C}'$ on $n+m$ vertices (for some $m \in \mathbb{N}$) with degree sequence

$$(r - r_1, r - r_2, ..., r - r_n, r, r, r, ..., r).$$

Let v_i' be the vertex with degree $r - r_i$, for each $i \in [n]$. Then the linear hypergraph formed by taking the disjoint union of $\mathcal{C}$ and $\mathcal{C}'$ and identifying vertices v_i and v_i', for each $i \in [n]$ is a k-uniform, r-regular hypergraph containing $\mathcal{C}$ as an induced hypergraph. In other words, the configuration $\mathcal{C}$ may be embedded in an $[r, k]$-configuration on $n+m$ points. The problem of determining whether a given sequence can be realized as the degree sequence of a linear hypergraph has been examined by Bhave, Bam and Deshpande in [9], but has not been fully answered.

As in the previous section, we will utilize a construction resembling resolvable transversal designs to aid in our embeddings.

3.2.1 The Configurations $E(\rho, r, k)$ and $E'(\rho, r, k)$

Let ρ be any prime larger than r. Then as before, create a partial configuration with point set $\mathbb{Z}_k \times \mathbb{Z}_\rho$. Define the lines, parallel classes, rows and groups of the configuration as in $A(\rho, 0, r, k)$. Since ρ is prime, the set of all lines forms a $[\rho, k]$-configuration, and restricting to any set of r parallel classes yields an $[r, k]$-configuration. We will restrict to the set of parallel classes with slope in the interval $[0, r-1]$. Denote this configuration as $E(\rho, r, k)$. Note that this means $E(\rho, r, k)$ has point set $\mathbb{Z}_k \times \mathbb{Z}_\rho$, and lines

$$b_{mc} = \{(x, y) \in \mathbb{Z}_k \times \mathbb{Z}_\rho \ : \ y = mx + c\} \text{ for all } m \in 0, 1, ..., r-1 \text{ and } c \in \mathbb{Z}_\rho.$$

Let H denote the set of points within the first r rows $h_0, ..., h_{r-1}$. For every $i \in [k]$, remove the points of $H \cap g_{i-1}$ from the lines within the parallel class B_i. Add k isolated points $p_1, ..., p_k$ to the configuration, and append p_i to the r lines in B_i of size $k-1$. Since the lines within B_i are all parallel, linearity of the configuration is preserved. Adding the rows $h_0, ..., h_{r-1}$ to the configuration as lines will again return an $[r, k]$-configuration, this time on $\rho k + k$ points. This configuration will be denoted $E'(\rho, r, k)$.

3.2.2 Proof of the Embedding Theorem

We begin the proof with the classical version of Bertrand's Postulate, as given in [25, pg. 494].

Theorem 3.2.2. *Bertrand's Postulate: For any* $x \in \mathbb{Z}^+$ *there exists a prime within the interval* $[x, 2x]$.

We will also require the following lemma, also found in [17].

Lemma 3.2.3. *Provided* $\rho \geq r + k$, *and* $r, k \geq 2$, *there exists an ordering of the lines within* $E(\rho, r, k)$ *so that any* r *consecutive lines are pairwise disjoint.*

Proof. Simply list the lines b_{mc} of $E(\rho, r, k)$ in the 'natural' order:

$$b_{00}, b_{01}, ..., b_{0,\rho-1}, b_{10}, b_{11}, ..., b_{1,\rho-1}, ..., b_{r-1,0}, b_{r-1,1}, ..., b_{r-1,\rho-1}$$

Blocks with the same first index are clearly disjoint (they belong to a common parallel class). Suppose that there exists a point $(x, y) \in b_{mc_1} \cap b_{m+1,c_2}$ for some $m \in \{0, 1, ..., r-2\}$ and $c_1, c_2 \in \mathbb{Z}_\rho$. Then

$$y \equiv mx + c_1 \equiv (m+1)x + c_2 \pmod{\rho}$$

The latter equivalence implies that

$$c_1 \equiv x + c_2 \pmod{\rho}.$$

Since $0 \leq x \leq k-1$, it follows that c_2 lies within the interval $[c_1 - (k-1), c_1]$ (with values taken modulo ρ). However, the r parallel classes following b_{mc_1} in the ordering have second index in the interval $[c_1 + 1, c_1 + r]$ (taken modulo ρ). The interval $[c_1 - (k-1), c_1]$ contains the first $k-1$ integers preceding c_1, and the interval $[c_1 + 1, c_1 + r]$ contains the first r integers following c_1 (again, values are taken modulo ρ). Since $\rho \geq r + k$, it follows that these two intervals are disjoint. Therefore, none of the r parallel classes following b_{mc_1} in the ordering can contain a second index equal to c_2. This implies that any r consecutive lines in the ordering are pairwise disjoint. □

We now prove the main theorem regarding embeddings. We restate it first.

Theorem 3.2.1. *Let* $\mathcal{C}_0$ *be a partial* $[r, k]$*-configuration on* n *points with total deficiency* F. *Let* $d = k/\gcd(r, k)$. *If* F *is at least* $d(r^2 + kr)$ *and a multiple of* k, *then* $\mathcal{C}_0$

is an induced subconfiguration on an $[r,k]$*-configuration* $\mathcal{C}$ *containing fewer than* $n + \frac{(2k+1)F}{r} + 3rk^2$ *points.*

Proof. Let ρ be any prime such that $\rho > F/(dr)$ and $\rho < 2F/(dr)$ (which is guaranteed to exist by Bertrand's Postulate). Create d configurations $E(\rho, r, k-1)$, and label them $E_1, ..., E_d$. Recall that each one contains r parallel classes. Let B_{ij} denote the parallel class of slope j within E_i. Order the parallel classes

$$B_{11}, B_{21}, B_{31}, ..., B_{d1}, B_{12}, B_{22}, ..., B_{d2}, ..., B_{1r}, ..., B_{dr}.$$

Apply the 'natural' ordering to the lines within each parallel class above (i.e. blocks with a lower intercept are placed earlier in the ordering). If $d > 1$, then clearly any r consecutive lines in this ordering are pairwise disjoint (as any two such lines are either parallel or belong to different copies of $E(\rho, r, k-1)$). If, on the other hand, $d = 1$, then the ordering of the lines becomes equivalent to the natural ordering presented in Lemma 3.2.3, and since $\rho > F/(dr) \geq r + k$, this lemma ensures that any r consecutive blocks are pairwise disjoint. This creates a total of $d(k-1)\rho$ points and $dr\rho$ lines of size $k-1$.

Append the point p_1 to the first f_1 lines in this ordering. Since $f_1 < r$, the first f_1 lines are pairwise disjoint, so appending p_1 to each of these lines does not destroy linearity. Then append the point p_2 to the next f_2 lines in this ordering. Repeat this procedure until all n points have replication number r. Since there are $dr\rho > F$ lines of size $k-1$ in $E_1 \cup \cdots \cup E_d$, it is possible to complete this procedure without exhausting all the lines of $E_1 \cup \cdots \cup E_d$. The result is a partial configuration where all points have replication number r, but some lines are of size $k-1$. The total number of lines of size $k-1$ is $dr\rho - F$. Since $dr\rho$ is a multiple of k and F is a multiple of k (by assumption), then the total number of lines of size $k-1$ is also a multiple of k. Define y to be equal to $dr\rho - F$. Next, add $k\lfloor y/(kr) \rfloor$ isolated points $q_1, ..., q_{k\lfloor y/(kr)\rfloor}$ to the partial configuration. Proceeding where we left off in the ordering of the lines above, append q_1 to the next r lines of size $k-1$ in the ordering, and repeat this for all q_i until the $k\lfloor y/(kr) \rfloor$ points each have replication number r. This operation requires a total of

$$rk \left\lfloor \frac{y}{kr} \right\rfloor \leq y$$

lines of size $k-1$. The total number of remaining lines of size $k-1$ is

$$y - rk\left\lfloor \frac{y}{rk} \right\rfloor,$$

which is a multiple of k (since y is a multiple of k). Furthermore, the total number of remaining lines of size $k-1$ is no more than rk, since

$$y - rk\left\lfloor \frac{y}{rk} \right\rfloor \leq y - rk\left(\frac{y}{rk} - 1\right) \leq rk.$$

For each line of size $k-1$, add a distinct isolated point to the configuration, and append it to the line. Let L denote the set of points added to the partial configuration. The result is a partial configuration with constant line size k, and all but $|L|$ points have replication number r (and each point in L has replication number 1). To finish the construction, partition the points of L into k groups $L_1, ..., L_k$ so that each $|L_i|$ is also multiple of k, and $|L_i| - |L_j|$ is either 0 or k, for any $i, j \in [k]$. In other words,

$$|L_i| = k\left\lfloor \frac{L}{k^2} \right\rfloor \quad \text{or} \quad k\left(\left\lfloor \frac{L}{k^2} \right\rfloor + 1\right).$$

Such a partition can be found by simply grouping the points of L into sets of size k (which is possible as $|L|$ is a multiple of k), and distributing these sets as evenly as possible into the L_is. Assume without loss of generality that $|L_1| \leq |L_2| \leq \cdots \leq |L_k|$. Add an array of points to the partial configuration with k rows and $k\sigma - |L_1|$ columns, for some prime σ satisfying $r < \sigma \leq 2r$ (which exists, again by Bertrand's Postulate). Place each column in a unique line (of size k), and group the rows of the array into sets $G_1, ..., G_k$.

For every $i \in [k]$, the set $G_i \cup L_i$ contains only points of replication number 1, and $|G_i \cup L_i| = k\sigma$ or $k\sigma + k$. Additionally, no two points within $G_i \cup L_i$ are contained within a common line. For each $i \in [k]$, if $|G_i \cup L_i| = k\sigma$, then the configuration $E(\sigma, r-1, k)$, also on $k\sigma$ points, can be used to arrange the points of $G_i \cup L_i$ into an $[r-1, k]$-configuration. If $|G_i \cup L_i| = k\sigma + k$, then the configuration $E'(\sigma, r-1, k)$ can be used instead. The result is that all points originally of replication number 1 now have replication number $1 + (r-1) = r$, thus yielding an $[r, k]$-configuration containing $\mathcal{C}_0$ as an induced subconfiguration.

We now take a moment to examine the total number of points in this larger configuration. The configuration $\mathcal{C}_0$ contains n points. Each of the d copies of

$E(\rho, r, k-1)$ added to the configuration contains $(k-1)\rho$ points. There are $k\lfloor \frac{y}{kr} \rfloor$ points of the form q_i in the configuration as well. The set L contains no more than rk points, and finally k configurations either of the form $E(\sigma, r-1, k)$ or $E'(\sigma, r-1, k)$ appear in the configuration. Thus, the total number of points in this configuration $\mathcal{C}$ does not exceed

$$n + d(k-1)\rho + k\left\lfloor \frac{y}{kr} \right\rfloor + rk + k(\sigma k + k).$$

Since $\sigma \leq 2r$, $y = dr\rho - F$ and $\rho < 2F/(dr)$, the total number of points in $\mathcal{C}$ does not exceed

$$n + d(k-1)\frac{2F}{dr} + k\left\lfloor \frac{dr\rho - F}{kr} \right\rfloor + rk + k(2rk + k),$$

which is bounded above by the simpler expression,

$$n + \frac{2kF}{r} + k\left\lfloor \frac{F}{kr} \right\rfloor + 2rk^2 + rk + k^2.$$

One final simplification of this expression (using the fact that $rk^2 > rk + k^2$, since $rk > r + k$), yields

$$n + \frac{(2k+1)F}{r} + 3rk^2. \qquad \square$$

A diagram of this construction is given in Figure 3.2.

Theorem 3.2.1 proves that an embedding exists if the total deficiency is larger than $d(r^2 + rk)$ and a multiple of k. The following corollary demonstrates that these conditions are not necessary for an embedding to exist (although a slightly worse bound is obtained).

Corollary 3.2.4. *Let $\mathcal{C}_0$ be a partial $[r,k]$-configuration on n points with total deficiency F. Then $\mathcal{C}_0$ is an induced subconfiguration of an $[r,k]$-configuration $\mathcal{C}$ with fewer than $n + \frac{(2k+1)F}{r} + 6rk^2$ points.*

Proof. Given $\mathcal{C}_0$, add some isolated points to the configuration until the total deficiency of this new partial $[r,k]$-configuration is at least $d(r^2+rk)$ and a multiple of k. The total number of points needed to satisfy such criteria is no more than $d(r+k)+k < d(r+2k)$. Call this new configuration with the additional isolated points $\mathcal{C}_1$. It contains fewer than $n + d(r+2k)$ points with total deficiency less than $F + rd(r+2k)$. It also has $\mathcal{C}_0$ as an induced subconfiguration. From Theorem 3.2.1, it follows that $\mathcal{C}_1$ is an induced subconfiguration of an $[r,k]$-configuration on

$$(n + r + 2k) + \frac{(2k+1)(F + dr(r+2k))}{r} + 3rk^2$$

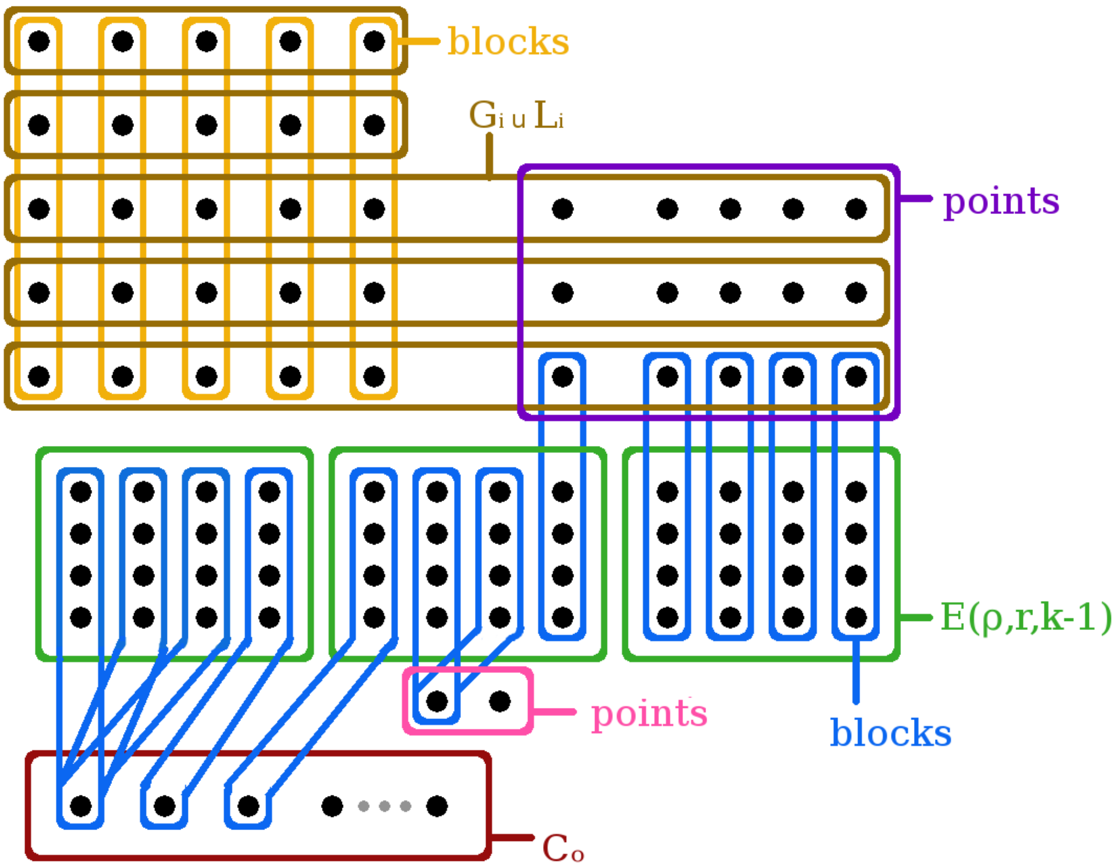


Figure 3.2: A diagram of the embedding construction. Initial configuration $\mathcal{C}_0$ is given in the lower-left corner. Several copies of $E(\rho, r, k-1)$ are created, and many of their lines are joined with the deficient points in $\mathcal{C}_0$. Some of the other lines are added to isolated points (indicated in pink). Each of the remaining lines are then appended to a distinct isolated point (although not every one of these is shown in the diagram). These points indicated in purple in the upper-right and are partitioned into sets L_i (denoted by the rows of the points in the diagram). A new array $k \times \sigma$ array of points is added to the configuration (indicated by the grid in the upper left corner). The columns of this grid are enclosed in lines, as shown. The rows are denoted by the sets G_i. Together $G_i \cup L_i$ is a set of points, no two contained within a line. This set of points is filled with an $[r-1, k]$-configuration.

points. By expanding and simplifying, the value above is equal to

$$n + \frac{(2k+1)F}{r} + dr + 2dk + r + 2k + 3rk^2.$$

Using the inequality $d < k$ yields a further simplification of

$$n + \frac{(2k+1)F}{r} + kr + 2k^2 + r + 2k + 3rk^2.$$

Since $2k \leq k^2$, and $r \geq 3$ it follows that $2k + 2k^2 \leq rk^2$. Likewise, $kr + r < rk^2$. Therefore, a final simplification yields at most

$$n + \frac{(2k+1)F}{r} + 5rk^2$$

points. □

Since the total deficiency of a partial $[r,k]$-configuration on n points cannot exceed nr, we obtain the following additional corollary by replacing F in the previous theorem by nr.

Corollary 3.2.5. *Let $\mathcal{C}_0$ be a partial $[r,k]$-configuration on n points. Then $\mathcal{C}_0$ is an induced subconfiguration on an $[r,k]$-configuration $\mathcal{C}$ with fewer than $(2k+2)n + 5rk^2$ points.*

The embedding technique in Theorem 3.2.1 gives a configuration $\mathcal{C}$ containing $\mathcal{C}_0$ as an *induced* subconfiguration. If an induced subconfiguration is not necessary (and we merely require that $\mathcal{C}_0$ be a subconfiguration of an $[r,k]$-configuration), then this bound may be improved. Intuitively, if $\mathcal{C}_0$ has a large number of deficient points, then we may find a set of k points, no two of which belong to a common block. These k points may be placed in a block together. This reduces the total deficiency of the configuration, which allows for a smaller embedding.

Corollary 3.2.6. *Any partial $[r,k]$-configuration $\mathcal{C}_0$ on n points can be embedded in an $[r,k]$-configuration $\mathcal{C}$ with fewer than $n + 2rk^3 + 6rk^2$ points.*

Proof. Let X denote the points of $\mathcal{C}_0$ that have replication number strictly less than r. We claim that if $|X| \geq rk^2$, then there exists k points in X such that no two appear in a common block. To see this, suppose that $|X| \geq rk^2$. Then choose a point $v_1 \in X$, and let V_1 denote the set of all points of X that appear in a block with v_1. Since v_1 is contained within no more than $(r-1)$ blocks, and each block contains $(k-1)$ other points, it follows that $|V_1| \leq (r-1)(k-1)$. For each $i \in \{2, ..., k\}$, inductively define v_i to be a point in

$$X \setminus \bigcup_{j=1}^{i-1} V_j,$$

and V_i to be the set of all points of X that appear in a block with v_i. Since $|X| \geq rk^2$, and $|V_1 \cup \cdots \cup V_i| < i(r-1)(k-1)$, it follows that such a selection of the v_is is possible. No two of the k points $v_1, ..., v_k$ appear in a common block. Therefore, we may add the block $\{v_1, ..., v_k\}$ to the configuration without destroying linearity. Since each point belongs in X, this does not increase the replication number of any point beyond r. We may repeat this process of adding blocks to $\mathcal{C}_0$ until the total number of points with replication number strictly less than r is less than rk^2. Call this new partial $[r, k]$-configuration with additional blocks $\mathcal{C}_1$. It also has n points, but has total deficiency less than r^2k^2 (since the number of points with nonzero deficiency is no more than rk^2). Thus, by Corollary 3.2.4, we may embed $\mathcal{C}_1$ in an $[r, k]$-configuration on at most

$$n + \frac{(2k+1)r^2k^2}{r} + 5rk^2$$

points. This simplifies to at most

$$n + 2rk^3 + 6rk^2$$

points. □

This final corollary extends a previous graph-theoretic result by Akiyama, Era and Harary [1] on embeddings. The authors of [1] find that every graph n with maximum degree r can be embedded in an r-regular graph on fewer than $n + r + 2$ points. By considering a graph of maximum degree as a partial $[r, 2]$-configuration, Corollary 3.2.6 provides an embedding in an r-regular graph on $n + 44r$ points. As one might expect, we have lost some efficiency in our embedding in order to provide a more general result.

3.2.3 Applications to Embeddings into Designs

One can extend the embedding question into the realm of design theory as well. Given a configuration $\mathcal{C}_0$, does it exist as a subconfiguration of some BIBD? This question is of a different nature: we may add points to the configuration, but as we add more points, the required replication number of these points increases as well (because each point added must be in a block with each point already in the configuration). Relying on the result Theorem 3.2.1, [18] demonstrates that, when $k = 4$, every configuration can be embedded in some BIBD$(v, 4)$. In fact, this embedding is quadratic in the number of points in the original configuration, positively answering a question appearing in [28].

Theorem 3.2.7. *Let $n \equiv 1 \pmod 6$, with $n > 44$. Suppose there exists an r-regular configuration $\mathcal{C}$ of line size 4 on $2n$ points, with $r < n/9$. Then there exists a $BIBD(v, 4)$ containing $\mathcal{C}$, where $v < 32n^2$.*

Combining the above theorem with Theorem 3.2.1, we can obtain a more general result.

Corollary 3.2.8. *Let $\mathcal{C}_0$ be a configuration with line size 4. Then $\mathcal{C}_0$ is a subconfiguration of some $BIBD(v, 4)$, where $v \leq 32(5n + 36r)^2 \sim O(n^2)$.*

Proof. Given $\mathcal{C}_0$, let r be any multiple of 4 that is larger than the maximum replication number in $\mathcal{C}_0$. Since the maximum replication number in $\mathcal{C}_0$ is bounded above by $\frac{n-1}{3}$, we have

$$r \leq \frac{n-1}{3} + 3.$$

Let $\mathcal{C}'$ be an $[r, k]$-configuration containing $\mathcal{C}_0$ as a subconfiguration, as determined in Theorem 3.2.1. The number of points N' on $\mathcal{C}'$ can be bounded above by

$$9n + 56r.$$

If N' is congruent to 2 modulo 12, then N' is even and $\frac{1}{2}N'$ is congruent to 1 modulo 6. Thus, we may apply Theorem 3.2.7 on this configuration to yield a BIBD$(v, 4)$ containing $\mathcal{C}'$ as a subconfiguration. Suppose N' is not congruent to 2 modulo 12. Then let $\mathcal{C}''$ be any $[r, k]$-configuration on N'' points such that $N' + N''$ is congruent to 2 modulo 12. Since $k/\gcd(r, k) = 1$, there exists such a $\mathcal{C}''$ on no more than

$$N(r, k) + 11 \leq 16 \max\left\{r + 1, \frac{r}{2} + 4\right\} + 11 \leq 16r + 27$$

points. Therefore, the disjoint union configuration $\mathcal{C} := \mathcal{C}' \sqcup \mathcal{C}''$ is an $[r, k]$-configuration on no more than

$$9n + 72r + 27$$

points, and it contains $\mathcal{C}_0$ as a subconfiguration. Since $N' + N''$ is congruent to 2 modulo 12, it follows that $N' + N''$ is even and $\frac{1}{2}(N' + N'')$ is congruent to 1 modulo 6. The application of Theorem 3.2.7 on $\mathcal{C}$ yields a BIBD$(v, 4)$ containing $\mathcal{C}$ (and $\mathcal{C}_0$) as a subconfiguration, where

$$v \leq 32\left(\frac{9n + 72r + 27}{2}\right)^2 \leq 32\left(5n + 36r\right)^2.$$

□

3.2.4 Subconfigurations of $[r, k]$-Configurations

Theorem 3.2.1 answers in the affirmative the question of whether a given partial configuration can be embedded within some $[r, k]$-configuration. A question in the spirit of the converse to this is whether a fixed (n_r, b_k)-configuration contains some $[r_0, k_0]$-configuration as a subconfiguration. This question has been largely answered, although not utilizing the terminology used in configuration theory. Given a graph, G, and a function $f : V(G) \to \mathbb{Z}^+$, then a spanning subgraph F of G is an *f-factor* if $\deg_F(v) = f(v)$ for each $v \in V(G)$. The following two theorems are proven by Folkman and Fulkerson.

Theorem 3.2.9. *([21]) Let G be a bipartite graph with bipartition (P, B) and let $f : V(G) \to \mathbb{Z}^+$ be a function. Then G has an f-factor if and only if $\sum_{v\in P} f(v) = \sum_{v\in B} f(v)$ and for all $S \subset P$ and $T \subset B$, we have*

$$\sum_{v\in T} f(v) + \sum_{v\in S} \left(\deg_{G-T}(v) - f(v)\right) \geq 0.$$

Corollary 3.2.10. *If the degree of every vertex in G is divisible by g, then for the function*

$$f(v) = \frac{\deg(v)}{g},$$

G admits an f-factor.

As there is a bijective correspondence between combinatorial configurations and bipartite, girth at least six graphs, these statements can be reformulated in terms of configuration theory. If $G = (\mathcal{P}, \mathcal{L})$ denotes the Levi graph of a configuration $(\mathcal{P}, \mathcal{L})$, and $f(p) = r_0$ for all $p \in \mathcal{P}$, while $f(\ell) = k_0$ for all $\ell \in \mathcal{L}$, then an f-factor of G corresponds to an $[r_0, k_0]$-subconfiguration on n points (where n is the order of G).

Since the degrees of the vertices in $G := L(\mathcal{C})$ for some (n_r, b_k)-configuration $\mathcal{C}$ are all a multiple of $\gcd(r,k)$, Corollary 3.2.10 admits an interpretation into configuration theory.

Corollary 3.2.11. *Let g be a factor of $\gcd(r,k)$. Then any (n_r, b_k)-configuration contains an $[\frac{r}{g}, \frac{k}{g}]$-subconfiguration on n points. Otherwise, an $[r_0, k_0]$-subconfiguration on n points exists if and only if, for every subset $T \subset \mathcal{L}$ and $S \subset \mathcal{P}$, we have*

$$k_0|T| + (r - r_0)|S| \geq e(T,S),$$

where $e(T,S)$ denotes the number of edges in $L(\mathcal{C})$ between T and S.

The first part of this corollary follows directly from Corollary 3.2.10. The inequality provided in the latter half stems from the following equalities:

$$\sum_{v \in T} f(v) = k_0|T|, \qquad \sum_{v \in S} f(v) = r_0|S|, \qquad \sum_{v \in S} \deg_{G-T}(v) = r|S| - e(T,S).$$

Chapter 4

Geometric Configurations

We now turn our attention to geometric configurations and their realizations. Because of the common nomenclature between geometric and combinatorial configurations, it is vital to distinguish the context when referring to configurations. Such confusion between combinatorial and geometric configurations exists even in the present day, with some authors referring extensively to configurations without distinguishing whether combinatorial or geometric variants are being referenced. Grünbaum resolves some of these discrepancies in [23, pg. 8–14]. In this section, a configuration is assumed to be geometric unless otherwise specified. We begin in the same spirit as combinatorial configurations — providing existence and embedding results. After demonstrating that any partial configuration can be embedded within an $[r, k]$-configuration, we go further and demonstrate that any partial configuration can be embedded within a *chiral* $[r, k]$-configuration, with m-fold symmetry, for any $m \geq 3$. In section 4.4, we discuss which combinatorial configurations (or partial configurations) can be covered by a geometric configuration (the definition of a covering in terms of configuration theory is also introduced in this section). Furthermore, this section introduces some new families of geometric configurations.

4.1 Existence Results

The existence question on geometric configurations does not fair nearly as well as their combinatorial cousins. In fact, an asymptotic existence result for general $[r, k]$-configurations is startlingly absent from the literature on the subject. Grünbaum demonstrated the asymptotic existence of (n_k)-configurations, using a series of tools

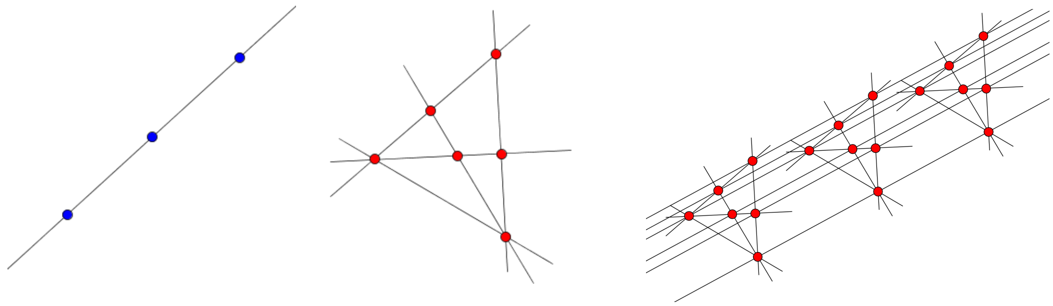


Figure 4.1: The configuration on the right is the Cartesian product of the two configurations on the left.

collectively known as 'The Grünbaum Calculus' [27, pg. 243–251]. These operations on configurations yield larger configurations. Once a sufficient number of k-configurations are determined, we can use the fact that the set of geometric k-configurations forms a semigroup under the disjoint union operation to demonstrate the asymptotic existence of k-configurations. Our proof of the asymptotic existence of $[r, k]$-configurations will differ slightly from Grümbaum's; however, we will still rely on three fundamental operations on geometric configurations, each given in Pisanski and Servatius' book *Configurations from a Graphical Viewpoint* [27, pg. 247–248].

- **Duality:** The existence of an (n_r, b_k)-configuration implies the existence of an (b_k, n_r)-configuration.

- **Disjoint Union:** Let $\mathcal{G}$ and $\mathcal{G}'$ be $[r, k]$-configurations on n and n' points respectively. Then their disjoint union $\mathcal{G} \sqcup \mathcal{G}'$ is an $[r, k]$-configuration on $n + n'$ points.

- **Cartesian Product:** Let $\mathcal{G}$ be an (n_r, b_k)-configuration and $\mathcal{G}'$ be an $(n'_{r'}, b'_k)$-configuration. For each point $p_i \in \mathcal{G}$, let $\mathbf{v}_i$ be the vector with tail at the origin and head at the point p_i. Create n identically sized copies of $\mathcal{G}'$, labelled $\mathcal{G}'_i$, and translate it by the vector $\mathbf{v}_i$. Collinearity is preserved under translations, so any set of collinear points in $\mathcal{G}'$ remains collinear in $\mathcal{G}'_i$. If p_{i_1}, p_{i_2} and p_{i_3} are collinear in $\mathcal{G}$, then the copies $q_{i_1} \in \mathcal{G}'_{i_1}$, $q_{i_2} \in \mathcal{G}'_{i_2}$ and $q_{i_3} \in \mathcal{G}'_{i_3}$ of any point $q \in \mathcal{G}'$ are collinear. For each line $\{p_{i_1}, ..., p_{i_k}\}$ in $\mathcal{G}$, create n' lines, joining the points $q_{i_1}, ..., q_{i_k}$ for every $q \in \mathcal{G}'$ and copy $q_{i_j} \in \mathcal{G}'_{i_j}$. The result is an $[r + r', k]$-configuration on $nb' + n'b$ points. This configuration is denoted $\mathcal{G} \times \mathcal{G}'$. Note that this operation is not commutative, and bears strong similarities to the lexicographic product of graphs (in fact, the Levi graph of the product is the lexicographic product of the Levi graphs of the two original configurations).

In addition, we will rely on some examples of *highly incident configurations* — configurations in which $r, k \geq 5$. Very little is known about such configurations. Berman's two papers [3, 7] provide some families of highly incident configurations that are not obtained via the Grünbaum Calculus applied to smaller configurations.

Theorem 4.1.1. *([7]) Given values of r, k, there exists a $[2r, 2k]$-configuration on*

$$m \cdot \binom{r-1+k}{k-1}$$

points, where m is any value larger than $r + k$.

The family of constructions Berman provides are one of the smallest known for general r, k. There are many values for r and k for which the combination above is never congruent to $k/\gcd(r, k)$ modulo k. As such, disjoint unions of these types of configurations cannot be used to demonstrate the asymptotic existence of $[r, k]$-configurations. This can be seen by using Lucas' Theorem regarding combinations [26].

Theorem 4.1.2 (Lucas' Theorem)**.** *The value $\binom{M}{N}$ is a multiple of a prime p if and only if the base p representation of N contains a digit that is greater than the corresponding digit of the base p representation of M.*

This theorem implies, for instance, that the combination in Theorem 4.1.1 is a multiple of 2 when $k = 6$ and r is congruent to 6 modulo 8. However, in such cases, $k/gcd(r, k)$ is either 1 or 3, so the divisibility condition imposed upon configurations does not preclude configurations on an odd number of points (but Berman's family of configurations for such values of k and r will only contain an even number of points). Therefore, Theorem 4.1.1 does not guarantee the asymptotic existence of $[r, k]$-configurations.

Another family of $[r, k]$-configurations is the family of generalized Gray configurations [27, pg. 254]. This configuration, along with some variants given below, will be instrumental in proving our existence and embedding theorems.

Definition 4.1.1. Let $\ell_1, ..., \ell_r$ be arbitrary $[1, k]$-configurations (i.e. a single line with k points), and no pair of lines ℓ_i, ℓ_j are parallel (where $i \neq j$). Define L_k^r

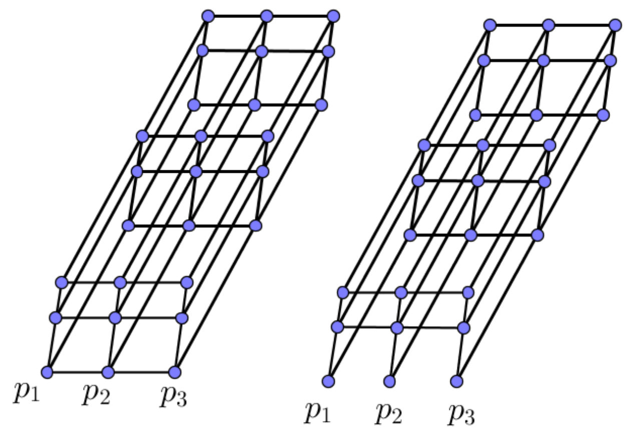


Figure 4.2: The left configuration is an example of an $L_3^3[p_1, p_2, p_3]$ configuration, while the right partial configuration is an example of an $L_3^3(p_1, p_2, p_3)$ partial configuration.

to be the Cartesian product $\prod \ell_i$. Such a configuration is known as a generalized *Gray configuration*. It is an $[r, k]$-configuration on k^r points. Given k collinear points $p_1, ..., p_k$, we may choose ℓ_1 to be the $[1, k]$-configuration consisting of the line containing points $p_1, ..., p_k$. In this case, define $L_k^r[p_1, ..., p_k]$ to be the Cartesian product $\prod \ell_i$, and define $L_k^r(p_1, ..., p_k)$ to be equal to $L_k^r[p_1, ..., p_k]$, with the line joining $p_1, ..., p_k$ removed (but the points $p_1, ..., p_k$ remain in the configuration). The partial configuration $L_k^r(p_1, ..., p_k)$ is not a configuration, as the k points $p_1, ..., p_k$ have replication number $r - 1$. The lines $\ell_1, ..., \ell_r$ in any configuration from the families above will be referred to as the *fundamental* lines of the configuration.

Figure 4.2 depicts two examples of such configurations. Note that L_k^r, $L_k^r[p_1, ..., p_k]$, and $L_k^r(p_1, ..., p_k)$ all refer to *families* of configurations. Each configuration in these families depends upon the choice of the fundamental lines. The term 'fundamental' in the above definition is not used elsewhere in the literature. We create this definition only to aid in the proofs of the next three theorems.

Theorem 4.1.3. *Given any value $r \geq k$, there exists an integer $N'(r, k)$ such that an $[r, k]$-configuration on n points exists for all $n \geq N'(r, k)$ satisfying the divisibility condition $nr = bk$.*

Proof. Note that any configuration of the form L_k^r contains k^r points, and thus it suffices to demonstrate the existence of an $[r, k]$-configuration on m points, with m congruent to $d := k/\gcd(r, k)$ modulo k. To do this, we will first place d isolated points down on the plane in general position. Draw r lines arbitrarily through each point. This creates a total of rd lines, which is a multiple of k. Partition these lines into groups $G_1, ..., G_{r/\gcd(r,k)}$ of size k. For each group G_i, draw $k-1$ lines $\ell_{i1}, ..., \ell_{i,k-1}$ on the plane in general position (so that each line does not meet any points in the partial configuration), but do not include these lines in the partial configuration. Each line ℓ_{ij} meets the k lines of G_i in distinct coordinates: $p_{ij1}, ..., p_{ijk}$. For each pair i, j, choose a configuration of the form $L_k^r(p_{ij1}, ..., p_{ijk})$ such that the only points of the partial configuration that are incident with a line of the chosen configuration from $L_k^r(p_{ij1}, ..., p_{ijk})$ are the points $p_{ij1}, ..., p_{ijk}$. Such a selection is possible since we may choose the position of the fundamental lines of the configuration, with the exception of the fundamental line containing $p_{ij1}, ..., p_{ijk}$. The points $p_{ij1}, ..., p_{ijk}$ now have replication number r — one incidence is formed by one of the lines in G_i, while the other $r-1$ incidences are generated by the configuration of the form $L_k^r(p_{ij1}, ..., p_{ijk})$.

The resulting configuration is indeed an $[r, k]$-configuration. The configuration depends on the location of our initial d isolated points, and the choice of configurations $L_k^r(p_{ij1}, ..., p_{ijk})$. The family of configurations constructed via the manner described above will be denoted M_k^r. The number of points in any configuration belonging to M_k^r will be denoted by m.

There are d isolated points in the initial stage of the construction, and $\frac{r}{\gcd(r,k)}(k-1)$ lines of the form ℓ_{ij}. Since any configuration in the family $L_k^r(p_{ij1}, ..., p_{ijk})$ contains k^r points, it follows that

$$m = d + \frac{r}{\gcd(r,k)}(k-1) \cdot k^r.$$

The greatest common divisor of m and k^r is d. Therefore, a configuration L_k^r and M_k^r can be used as generators (under the disjoint union operation) to find examples of $[r, k]$-configurations for all sufficiently large multiples of d. $\square$

The bounds on $N'(r, k)$ are exceptionally large: roughly on the order of $(k^r)^2$ (due to the Frobenius number of m and k^r in the above proof, as illustrated in Chapter 2). This bound can be improved by using M_k^r and Berman's configurations as generators. Further improvements can be made by arranging the d initial points, dr lines, and the lines ℓ_{ij} in specific arrangements to allow for the L_k^r configurations to have more than just one line coincident with the ℓ_{ij}. However, these improvements are modest relative to the exponential bound. Significant new developments in the method of constructing families of highly incident configurations are required before this bound can be substantially improved upon.

It is known that $N'(3,3) = 9$ and Bokowski proves in [13] that $20 \leq N'(4,4) \leq 27$. There are no better bounds known for any larger r, k values.

4.2 Embedding Configurations

It is trivial to see that most partial configurations do not admit embeddings into geometric configurations on the same number of points — any generic set of isolated points on the plane will fail to have a set of k collinear points, for all $k \geq 3$. However, as in the combinatorial case, every geometric partial configuration may be embedded in a larger $[r, k]$-configuration.

Theorem 4.2.1. *Every geometric partial configuration $\mathcal{G}_0$ with n points and constant line size k appears as the subconfiguration of some $[r, k]$-configuration, provided no replication number of a point in $\mathcal{G}_0$ exceeds r.*

Proof. Let $\mathcal{G}_0$ have point set $\mathcal{P}_0 := \{p_1, ..., p_n\}$, where r_i is the replication number of p_i, and $f_i := r - r_i$ is its deficiency. For example, Figure 4.3(a) illustrates a partial configuration on 7 points. If we wish to embed this configuration into a 3-configuration, then four points have deficiency 1 (and thus have replication number 2), and one point has deficiency 2 (and thus has a replication number of one).

Suppose first that n is a multiple of $d = k/\gcd(r, k)$. For each p_i, draw f_i lies through each point (in general position). Since n is a multiple of d, the total number of incidences, nr, is a multiple of k. Each line of size k in $\mathcal{G}_0$ generates k incidences, so the total number of deficiencies F is also a multiple of k. Partition the F lines of size one into groups of size k: $G_1, ..., G_{F/k}$. Figure 4.3(b) contains an example of this step. Here there are six lines added to the configuration, and they are placed in groups G_1, G_2 of size $k = 3$.

With this partitioning, we proceed as we did in Theorem 4.1.3. Create $k-1$ lines $\ell_{i,1}, ..., \ell_{i,k-1}$ for each group G_i, ensuring that these lines do not meet any point already existing in the partial configuration. Do not include these lines within our partial configuration. Each line ℓ_{ij} intersects the k lines of G_i in k places: $p_{ij1}, ..., p_{ijk}$. Figure 4.3(c) illustrates this, where $k - 1 = 2$ lines are created for each group. The lines for group G_i are labelled ℓ_{i1}, ℓ_{i2}. They are dashed in the diagram to emphasize that these lines are not to be added to the configuration. Each of the lines ℓ_{i1} and ℓ_{i2} meets the lines of the group G_i in $k = 3$ distint places, indicated by the points $p_{i11}, p_{i12}, p_{i13}$ and $p_{i21}, p_{i22}, p_{i23}$.

For each pair i, j, append a partial configuration from $L_k^r(p_{ij1}, ..., p_{ijk})$ to the existing partial configuration (see Figure 4.3(d)). As in the previous theorem, an appropriate choice of the fundamental lines of this choice from $L_k^r(p_{ij1}, ..., p_{ijk})$ ensures that no unwanted incidences occur. The result is an $[r, k]$-configuration with

$$n + (k-1)\frac{F}{k}k^r$$

points.

This concludes the case where n is a multiple of d. If n is not a multiple of d, then we initially add some isolated points to the partial configuration so that the new partial configuration contains a number of points equal to a multiple of d. Once this is done, then we perform the embedding procedure outlined above. In this case, F is bounded above by $(n + d)r$. This provides a upper bound of

$$n + d + (k-1)(n + d)rk^{r-1}$$

points for a geometric $[r, k]$-configuration containing $\mathcal{G}_0$. □

As a corollary, we can deduce that every partial configuration (even without constant line size) can be embedded in an $[r, k]$-configuration. This is done by simply adding isolated points onto each line of deficient line size $< k$, and then beginning the embedding construction above. There are at most $b \leq nr/k$ lines in the configuration $\mathcal{G}_0$.

Corollary 4.2.2. *Every partial configuration $\mathcal{G}_0$ on n points and b lines (not necessarily of constant line size) can be embedded in an $[r, k]$-configuration with no more than*

$$[(k-1)b + n + d] + (k-1)[(k-1)b + n + d]rk^{r-1}$$

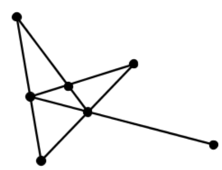
points.

Again, marginal improvements can be made to this result by positioning the lines added to the configuration in a particular fashion; however, these improvements are insignificant relative to the k^{r-1} component of the bound.

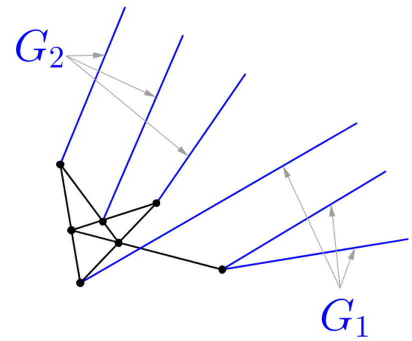
The converse question investigated in the previous chapter has a more optimistic result, as it does not rely on the geometry of the configuration:

Proposition 4.2.3. *An (n_r, b_k)-configuration $\mathcal{G}$ has an $[r_0, k_0]$-subconfiguration if and only if its underlying combinatorial configuration has an $[r_0, k_0]$-subconfiguration.*

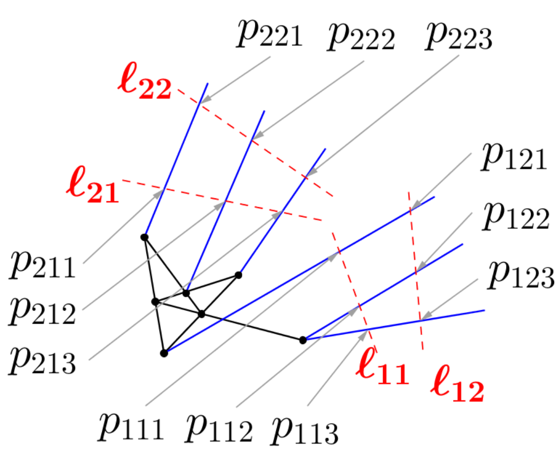
Figure 4.3



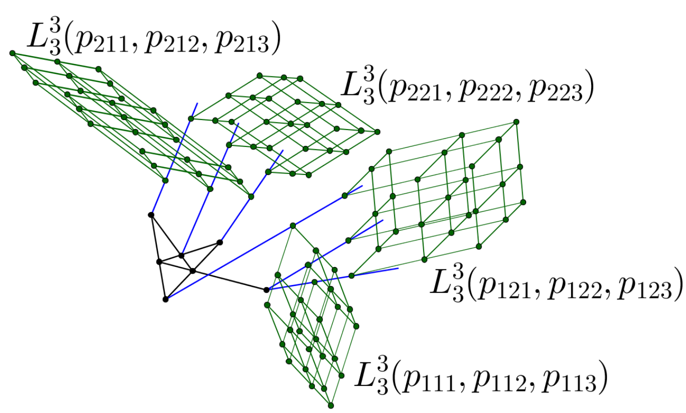
(a) The partial configuration $\mathcal{C}_0$. Several points have deficiencies as a 3-configuration.



(b) The blue lines created are of size one, but now each point in $\mathcal{C}_0$ has replication number 3. These six blue lines are placed into two groups G_1 and G_2, each of size $k = 3$.



(c) The lines $\ell_{1,1}, \ell_{1,2}, \ell_{2,1}$ and $\ell_{2,2}$ are created as dashed red lines (they are not a part of the final configuration).



(d) The configurations of the form $L_3^3(p_{ij1}, p_{ij2}, p_{ij3})$ are added to the configuration, for each $i, j \in \{1, 2\}$. The result is a 3-configuration containing $\mathcal{C}_0$ as an induced subconfiguration.

4.3 Chiral Embeddings

A geometric configuration is chiral if it exhibits nontrivial rotational symmetry in the plane, and does not exhibit mirror symmetry. Usually we add the condition that the configuration is also *connected* in this case (it is not a disjoint union of smaller configurations). This additional structure on configurations allows for a greater depth of study, and as such there is significantly more known about chiral configurations than configurations in general. In what follows we demonstrate that every partial configuration can be embedded in a chiral $[r, k]$-configuration. This provides numerous more examples of highly incident, chiral configurations (although admittedly, as before, these examples are somewhat contrived and have an excessive number of points).

Theorem 4.3.1. *Every partial configuration $\mathcal{G}_0$ on n points is a subconfiguration of some $[r, k]$-configuration with m-fold rotational symmetry, for any $r, k \geq 3$ (provided they both exceed the maximum replication number and line size of $\mathcal{G}_0$ respectively) and any $m > 1$.*

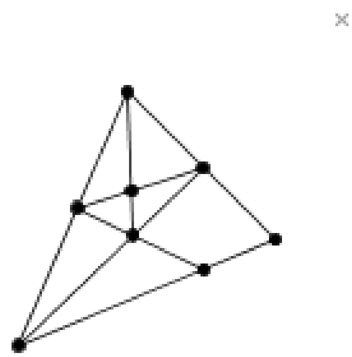
Proof. We assume the case where every line is of constant size k (we may add some initial points to the partial configuration to ensure this is the case, as was done in Corollary 4.2.2). Figure 4.4(a) gives an example of such a partial configuration. Next, for each deficiency at point p_i, create f_i isolated lines in general position, for a total of $F := \sum f_i$ lines of size one. Give each line of size one an arbitrary orientation. Create km copies of this partial configuration, $\mathcal{G}_1, ..., \mathcal{G}_{km}$ where $\mathcal{G}_i$ is an isometric copy, rotated by $2\pi i/km$ about the origin. This disjoint union, denoted by $\mathcal{G}$, is a partial configuration with km-fold symmetry. Figure 4.4(b) provides an example of this operation. Note that a partial configuration with km-fold symmetry also has m-fold symmery. Intuitively, we will take the km partial configurations 'k at a time' (i.e. considering k adjacent partial configurations) and fill the deficiencies on these k partial configurations. To fill these deficiencies we will append copies of the configuration $L^r_k(p_1, ..., p_k)$ to the configuration, in a manner somewhat similar to Theorem 4.2.1 (although more care will be needed to ensure that rotational symmetry is preserved). Then, by rotating the resulting structure m times (by an angle of $2\pi/m$ each time) about the origin, an $[r, k]$-configuration will emerge, with m-fold symmetry. Next, place an arbitrary m-gon M_1 on the plane, centred at the origin (the vertices and edges of this figure are *not* part of the configuration). Ensure that M_1 is sufficiently large, so that every line of size one meets an edge of M_1. If the m-gon lies in general position, no two lines of $\mathcal{G}$ meet on an edge of M_1, and no line of $\mathcal{G}$ meets a vertex of M_1. Choose an arbitrary edge e of M_1, and an arbitrary line ℓ^α of size one. Let $\ell_1^\alpha, ..., \ell_{km}^\alpha$ be the km copies of this line, rotated around the plane. Exactly k of these lines intersect e oriented from the interior of M_1 to the exterior. Suppose without loss to generality that these k lines are $\ell_1^\alpha, ..., \ell_k^\alpha$. Let $p_i^{e\alpha}$ denote the location where ℓ_i^α and e meet. Append a structure of the form $L_k^r(p_1^{e\alpha}, ..., p_k^{e\alpha})$ to the partial configuration, again ensuring that no lines belonging to this structure meet any other points of the configuration beyond $p_1^{e\alpha}, ..., p_k^{e\alpha}$. Each point $p_i^{e\alpha}$ has replication number r, since it is incident with $r-1$ lines belonging to the structure of the form $L_k^r(p_1^{e\alpha}, ..., p_k^{e\alpha})$, and 1 line from ℓ_i^α. Then copy and rotate the structure $L_k^r(p_1^{e\alpha}, ..., p_k^{e\alpha})$ a total of m times about the origin by $2\pi/m$. The lines $\ell_1^\alpha, ..., \ell km^\alpha$ now have size two, and all points have replication number r. Repeat this procedure for any other lines of size one, until all lines are now of size two. This procedure is performed in Figure 4.4(c), causing nine lines originally of size one to become lines of size two.

Before M_1 was chosen, the partial configuration consisted of km disjoint subconfigurations. However, now the partial configuration contains no more than m disjoint subconfigurations. This is because the lines $\ell_1^\alpha, ..., \ell_k^\alpha$ belonging to $\mathcal{G}_1, ..., \mathcal{G}_k$ are incident with $p_1^{e\alpha}, ..., p_k^{e\alpha}$. These points all belong to the (connected) partial configuration of the form $L_k^r(p_1^{e\alpha}, ..., p_k^{e\alpha})$. Thus, $\mathcal{G}_1, ..., \mathcal{G}_k$ are all connected to each other, as are the rotations of these configurations by $2\pi/m$.

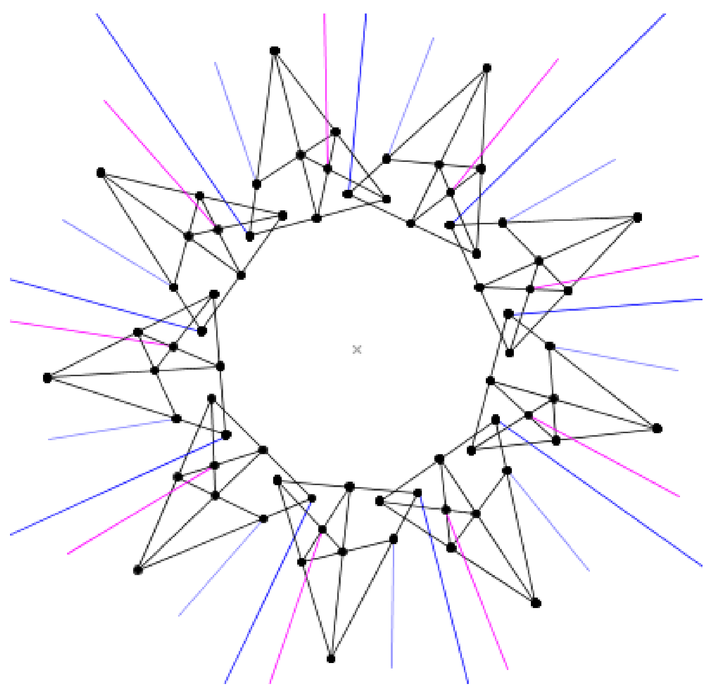
Next, we consider another m-gon M_2 placed on the plane; however, we must position M_2 in a slightly more specific manner. Choose an arbitrary line ℓ of size two. The km copies of this line can be partitioned into m groups, where each group consists of k lines that meet a common edge of M_1. Let ℓ' and ℓ'' be two lines belonging to different groups, where ℓ' is a rotation by $2\pi/m$ of ℓ''. Then place M_2 on the plane so that a single edge of M_2 meets both ℓ' and ℓ''. Two partial configurations $\mathcal{G}_i$ and $\mathcal{G}_j$ are connected if their corresponding copies of the line ℓ belong to the same group. In what follows, we will repeat the procedure outlined with M_1. Thus, either $\mathcal{G}_i$ and $\mathcal{G}_{i+1}$ are connected under the operation performed with M_1, or they will be connected under the operation performed with M_2. This implies that the entire partial configuration will be connected. Figure 4.4(d) illustrates this step. The resulting configuration is indeed connected. The nine lines of size two in the previous diagram are now of size three.

Choose an arbitrary edge e of M_2, and an arbitrary line ℓ^β of size two. Let $\ell_1^\beta, ..., \ell_{km}^\beta$ be the km copies of this line, rotated around the plane. Exactly k of these lines intersect e oriented from the interior of M_2 to the exterior. As before, suppose

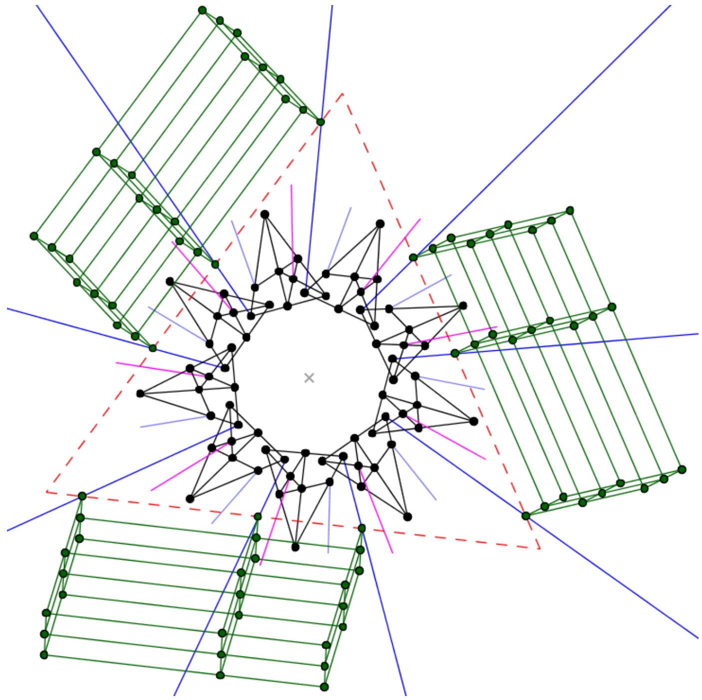
Figure 4.4



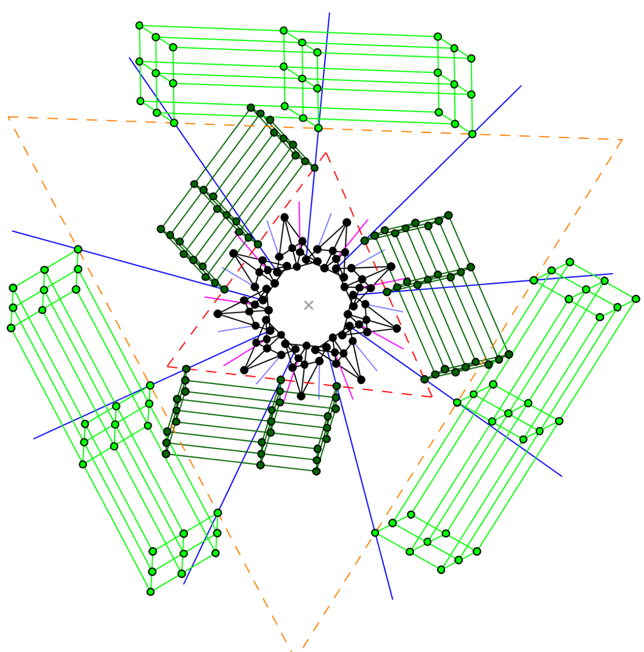
(a) The partial configuration $\mathcal{C}_0$, along with the origin (marked as an ‘x’). Several points have deficiencies as a 3-configuration. This partial configuration will be embedded in a 3-configuration with 3-fold symmetry.



(b) New lines are created of size one and joined to each deficient point. Then the resulting partial configuration is rotated $km = 9$ times about the origin.



(c) The triangle M_1 is shown with red, dashed lines. Three blue lines (each of size one) meet each side of the triangle. The points where these three lines meet M_1 are used to append a partial configuration of the form $L_3^3(p_1, p_2, p_3)$ to the configuration. Then this appended partial configuration is rotated about the origin in order to maintain 3-fold symmetry. The blue lines now each have size two.



(d) We repeat the previous step with a new triangle M_2. The blue lines in this partial configuration now have size three. The resulting partial configuration is also connected. This procedure is then repeated for each of the remaining lines of size one (not shown).

without loss of generality that these k lines are $\ell_1^\beta, ..., \ell_k^\beta$. Let $p_i^{e\beta}$ denote the location where ℓ_i^β and e meet. Append a choice of $L_k^r(p_1^{e\beta}, ..., p_k^{e\beta})$ to the partial configuration. Then copy and rotate the structure $L_k^r(p_1^{e\beta}, ..., p_k^{e\beta})$ a total of m times about the origin by $2\pi/m$. The lines $\ell_1^\beta, ..., \ell_{km}^\beta$ now have size three, and all points have replication number r. Repeat this procedure for any other lines of size two, until all lines are now of size three. The resulting partial configuration is connected.

All that remains is to fill the deficient lines of size 3 until they obtain size k. This can be done by creating m-gons $M_3, ..., M_{k-1}$. Since the partial configuration is already connected, we need not be as particular about the location of these m-gons. As we have done for M_1 and M_2, attach copies of a member of $L_k^r(p_1, ..., p_k)$ to sets of k deficient lines, and rotate these copies m times by $2\pi/m$. The result is a full $[r, k]$-configuration with m-fold rotational symmetry. □

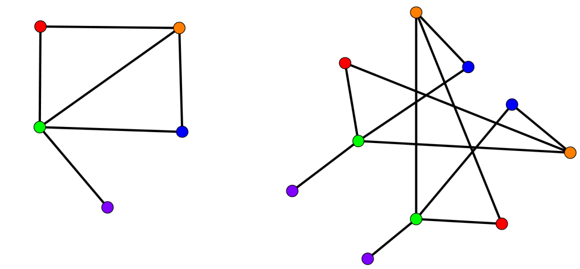
This embedding technique provides a new method for constructing configurations which exhibit rotational symmetry. There are many other types of such configurations, such as astral and celestial configurations (defined in Chapter 2). Another family of configurations with rotational symmetry are *floral* configurations, first discovered by Jürgen Bokowski and then elaborated upon in [6]. However, beyond the property of rotational symmetry, there is little in common between such configurations and our chiral embedding process.

4.4 Covering Configurations

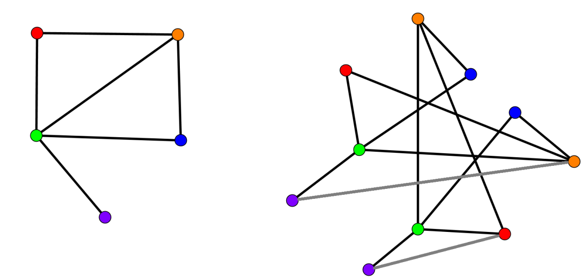
As seen from Table 2.2, very few combinatorial configurations admit geometric realizations. Even Steinitz' Theorem [30] for 3-configurations, introduced in Chapter 2, does not guarantee a strong realization. What follows is an effort to find a geometric configuration that at least shares some of the properties of a given combinatorial configuration. While a combinatorial configuration may not have a geometric embedding, it can nearly be covered by a geometric configuration, as we will now show.

First we define the notion of a *covering* in configuration theory — it is the same as the notion of a covering on their corresponding Levi graphs.

Definition 4.4.1. Given two graphs G and H with vertex sets $V(G)$ and $V(H)$, we say that G is a *(strong) covering* of H if there exists a surjective map $\phi : V(G) \to V(H)$ such that, for any vertex $v \in V(G)$, the restriction of ϕ to the neighbourhood $N(v)$ of vertices adjacent to v in G is a bijection onto the vertices of the neighbourhood



(a) The left graph is covered by the graph on the right. Each coloured vertex on the right maps onto the same coloured vertex on the left. The set of vertices in the neighbourhood of each vertex on the right is in bijective correspondence to the set of vertices in the neighbourhood of the vertex it maps to on the left.



(b) The left graph is weakly covered by the graph on the right (note the addition of two additional edges from the previous figure). The set of vertices in the neighbourhood of each vertex on the right contains the set of vertices in the neighbourhood of the vertex it maps to on the left as a subgraph. Note the additional two edges (in grey) in the graph on the right (compared to the previous subfigure).

Figure 4.5: Coverings and weak coverings illustrated.

of $\phi(v)$ in H. We say G is a *weak covering* (of H) if the restriction of ϕ to a subset of the vertices of the neighbourhood $N(v)$ is a bijection onto the vertices of the neighbourhood of $\phi(v)$.

Figure 4.5 illustrates the differences between coverings and weak coverings. A weak covering has additional 'unwanted' edges. This will correspond to the notion of unwanted incidences in geometric realizations of combinatorial configurations. As mentioned, this terminology extends to configurations through their Levi graphs.

Configurations with rotational symmetry are potentially coverings of smaller combinatorial configurations. If the group $\mathbb{Z}_m$ acts on the geometric configuration $\mathcal{G}$ via rotations of $2\pi/m$, then the graph $L(\mathcal{G})$ is acted upon by $\mathbb{Z}_m$ in the same manner. Since the geometric rotation clearly maps points to points (and lines to lines), the graph $L(\mathcal{G})/\mathbb{Z}_m$ is still bipartite. This quotient graph is a well-known object, and is found in Grünbaum's book [23, pg.38], among others. If the quotient graph is also simple and has girth at least six, then $\mathcal{G}$ is a covering of the combinatorial configuration with Levi graph isomorphic to $L(\mathcal{G})/\mathbb{Z}_m$. With this idea in mind, configurations with rotational symmetry will prove to be useful when constructing coverings of combinatorial configurations.

Since chiral configurations experience rotational symmetry, the points can be partitioned into orbits under this rotation, and each orbit of points forms the vertices of a regular m-gon (where m is the order of the rotation group). Similarly, the lines of a chiral configuration can be partitioned into classes as well, and each class forms the diagonals of regular m-gons, centred around the origin. In the graph $L(\mathcal{G})/\mathbb{Z}_m$, each vertex corresponds to a full orbit of points or lines. When the notion of point and line are not relevant (for instance, when referring to properties of graphs in general), we will refer to both points and lines as *objects.* Two vertices in this quotient graph are joined by a single edge if a line in one orbit meets a point in the other orbit. It is also possible for two vertices in this quotient graph to be joined by two edges, if a line from the corresponding orbit meets two distinct points in the corresponding orbit of points. See Figure 4.6 for an example.

Definition 4.4.2. The *reduced Levi graph* $R(\mathcal{G})$ of a geometric configuration $\mathcal{G}$ with m-fold rotational symmetry is the graph $L(\mathcal{G})/\mathbb{Z}_m$. It is bipartite.

This definition does not exactly match the definition that appears in Grünbaum's book [23, pg. 38]. Our definition is slightly simpler and carries less information about the configuration; however, it is similar enough that we will use the same terminology. For a more detailed discussion of reduced Levi graphs, and the related concept of *voltage graphs*, see [10, 22], among others.

In some cases, an orbit of lines will intersect an orbit of points in two places. In such a situation, we may define the *span* of the line with respect to the orbit of points.

Definition 4.4.3. Given a regular m-gon with points $u_0, ..., u_{m-1}$, a line ℓ meeting u_i and $u_{i+\alpha}$ has *span* α with respect to the orbit u of these points. Given a set of m lines $\ell_1, ..., \ell_{m-1}$ that are each rotations of $2\pi/m$ about the origin, a point u meeting ℓ_i and $\ell_{i+\alpha}$ has *span* α with respect to the orbit ℓ of lines.

For now, we only consider celestial configurations — if the orbit u is adjacent to ℓ (that is, every line of orbit ℓ meets some point on orbit u), then it meets the orbit u twice (see Definition 2.2.4). This is also equivalent to stating that if the

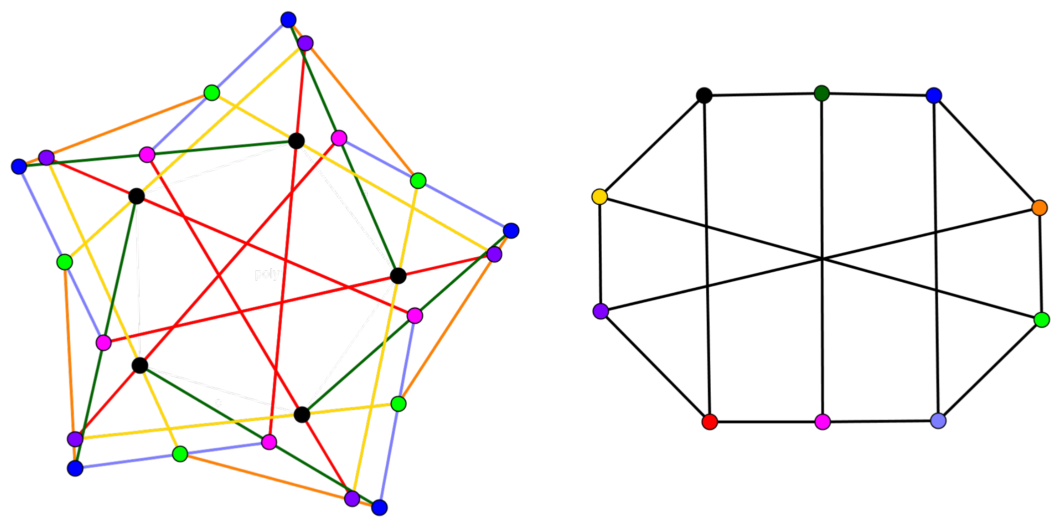


Figure 4.6: An example of a reduced Levi graph. The configuration on the left is a (25_3)-chiral configuration, and its reduced Levi graph appears on the right. The colours of the vertices in the Levi graph correspond to the orbits of objects in the configuration.

corresponding vertices u, ℓ are adjacent in $R(\mathcal{G})$, then there is a double-edge joining them. For celestial configurations, we may encode additional information into the reduced Levi graph. Given such a celestial configuration $\mathcal{G}$ with m-fold rotational symmetry, consider any edge uv where u corresponds to an orbit of points, and v corresponds to an orbit of lines. Then we may apply a label to the edge uv that is equal to the span of v with respect to the orbit u. This is a well-defined labelling, since if the orbit of lines v is of span α with respect to the orbit of points u, then the orbit of points u is of span α with respect to the lines v as well.

Definition 4.4.4. Given an m-fold celestial configuration $\mathcal{G}$. Suppose that, for any two vertices $u, \ell \in R(\mathcal{G})$ either u is not adjacent to ℓ (written $u \nsim \ell$) or there exists a double edge joining u and ℓ. Then define $R^*(\mathcal{G})$ to be the reduced Levi graph where each double edge of $R(\mathcal{G})$ is replaced by a single edge. Label each edge of this graph with the span between the two incident orbits.

With these tools and definitions in mind, we can begin our analysis of such configurations.

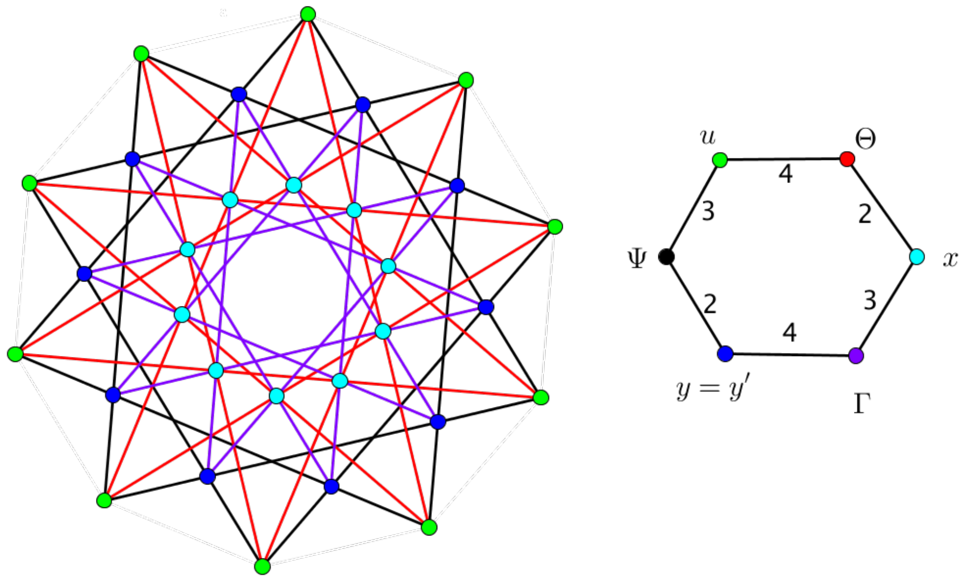


Figure 4.7: The graph of $R^*(\mathcal{G})$ for the geometric configuration $\mathcal{G}$ determined by the orbits $u, x, y, \Theta, \Psi, \Gamma$ with spans $\alpha = 4$, $\beta = 3$, $\delta = 2$ and $m = 10$.

4.4.1 The Point Completion Lemma

The foundation of our work on these types of celestial configurations rests upon the Point Completion Lemma, formulated by Leah Berman in [3]. We state the lemma as it appears.

Lemma 4.4.1. *Point Completion Lemma (PCL): Given a regular m-gon $\mathcal{M}$ with vertices $u_0, u_1, ..., u_{m-1}$ and diagonals $\Theta_i = u_i \vee u_{i+\alpha}$ of span α and $\Psi_i = u_i \vee u_{i+\beta}$ of span β. Define $x_i = \Theta_i \wedge \Theta_\delta$ to lie at the intersection of two lines. The points $x_0, ..., x_{m-1}$ form an m-gon $\mathcal{N}$. Let Γ_i be diagonals of $\mathcal{N}$ of span β, that is $\Gamma_i = x_i \vee x_{i+\beta}$, and let $y_i = \Gamma_i \wedge \Gamma_{i-\alpha}$ and $y'_i = \Psi_i \wedge \Psi_{i+\delta}$. Then $y_i = y'_i$.*

The six orbits $u, x, y, \Theta, \Psi, \Gamma$ given in the lemma together form a 4-configuration. An illustration of a configuration formed this way, along with its corresponding reduced Levi graph appears in Figure 4.7. Notice that while the graph is 2-regular, the corresponding geometric configuration is a 4-configuration. This is due to the fact that each edge in $R^*(\mathcal{G})$ actually corresponds to a double-edge in $R(\mathcal{G})$.

Thus, the underlying six-cycle C_6 can be given a labelling to make it the reduced Levi graph for some 3-celestial configuration. In particular, this labelling is such that the opposite edges of the six-cycle share a label, and the m-gon must be such that m is larger than twice the maximum label on the cycle. This restriction on m is necessary due to the fact that orbits of objects that are span α and span $m - \alpha$ are coincident. Thus, we want to be sure that two edges with different labels imply a truly different construction. Some 3-celestial 4-configurations with a reduced Levi graph equal to C_6 have been studied in [10, 23]. In [23], Grünbaum refers to the 3-celestial 4-configurations obtained by the PCL as *trivial* celestial 4-configurations. Celestial configurations exist with reduced Levi graph isomorphic to C_6, where the opposing edges do not share a common label; however, the existence of these configurations does not immediately follow from the PCL. In other words, the PCL gives a labelling on C_6 that is sufficient to guarantee the existence of the corresponding celestial configuration, but the labelling determined by the PCL is not necessary. This idea will be discussed further at the end of the chapter.

Lemma 4.4.2. *PCL reformulated: Given any distinct positive integers $\alpha_1, \alpha_2, \alpha_3$, and m such that $m \geq 2\max\{\alpha_i\}$. Consider the graph G that is a six-cycle with vertices $u_1...u_6$. Give each edge u_iu_{i+1} and $u_{i+3}u_{i+4}$ a common label α_i. The resulting graph is the reduced Levi graph $R^*(\mathcal{G})$ of a 3-celestial 4-configuration.*

Thus we have shown in the above lemma that the underlying graph C_6 can be given a labelling that admits a strong covering by the Levi graph of a 3-celestial 4-configuration. We next make repeated use of the PCL to extend this result to larger cycles.

Theorem 4.4.3. *PCL Extension: Given $n \geq 3$ and a $2n$-cycle G with vertices labelled*

$$vu_1u_2...u_{n-1}wu_{-n+1}u_{-n+2}...u_{-1}v$$

Let $e_{\pm i}$ denote the edge between $u_{\pm i}$ and $u_{\pm(i+1)}$, for each $i \in \mathbb{N}$ less than $n-1$. Then if G has an edge-labelling such that

- *e_i and e_{-i} share a common label α_i*
- *If n is even, then vu_1 and $u_{n-1}w$ have a common label β. The edges vu_{-1} and wu_{-n+1} have a common label δ as well.*
- *If n is odd, then vu_1 and wu_{-n+1} have a common label β. The edges vu_{-1} and $u_{n-1}w$ have a common label δ as well.*
- *No two adjacent edges share a common label.*

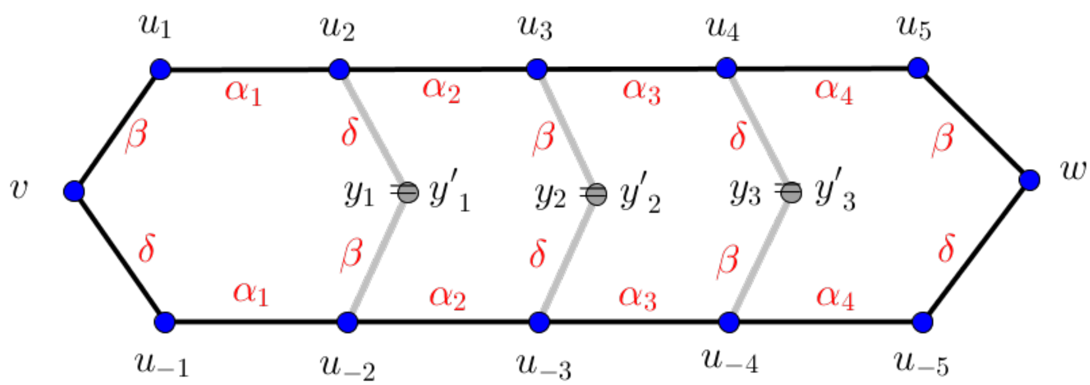


Figure 4.8: The PCL Extension, outlined in the case where $n = 5$.

Then this labelled cycle G is the reduced Levi graph $R^*(\mathcal{G})$ for some n-celestial configuration with m-fold symmetry, for all sufficiently large m.

Proof. This proof will be constructive in nature. Assume by duality that v corresponds to an orbit of points. Begin with an arbitrarily placed m-gon v of points, for any $m > 2\max\{\alpha_i, \beta, \delta\}$. Place the orbits u_1 and u_{-1} of lines down on the plane as span β and δ with respect to v. Continue by placing the orbits u_2 and u_{-2} on the plane, each of span α_1 with respect to u_1 and u_{-1} respectively. Next, add an orbit y_2 of points on the plane of span δ with respect to u_2 and an orbit y_2' of points on the plane of span β with respect to u_{-2}. By the PCL, we have that $y_2' = y_2$ (the orbits $vu_1u_2y_2u_{-2}u_{-1}v$ forms a 6-cycle satisfying the conditions of the PCL). Next place the orbit of objects u_3 down on the plane with span α_2 with respect to u_2, and likewise place the orbit of objects u_{-3} down with respect to u_{-2}. Similarly to before, let y_3 be an orbit of objects on the plane with span β with respect to u_3 and an orbit of objects y_3' of lines on the plane with span δ with respect to u_{-3}. As before, $y_3 = y_3'$ by the PCL, applied to the 6-cycle $y_2u_2u_3y_3u_{-3}u_{-2}y_2$. Note that the labels u_3y_3 and $u_{-3}y_3$ are reversed from their counterparts u_2y_2 and $u_{-2}y_2$, satisfying the conclusions of the hypothesis when $n = 4$. Continue this process for all $i \leq n-1$. By the conditions placed upon the hypothesis, $y_{n-1} = y_{n-1}'$. By renaming the orbit y_{n-1} as the orbit w, and removing all of the orbits y_i for each i from 2 to $n-2$ inclusive, we obtain an n-celestial 4-configuration with a reduced Levi graph $R^*(\mathcal{G})$ that possesses the labelling of G. □

Any labelled cycle that satisfies the premises of the PCL Extension will be referred to as a *PCL cycle.*

For every even cycle of length 6 or more, the graph possesses a strong covering which is geometrically realizable. If we are simply given the underlying cycle, we can apply a labelling using the PCL Extension so that the cycle possesses a strong covering graph which is realizable on the plane as a geometric configuration. In [2] and [23], the 4-configurations formed by the PCL Extension are also referred to as *trivial* celestial 4-configurations.

Definition 4.4.5. If a given graph H (possibly with labelled edges) admits a covering by a celestial partial configuration $\mathcal{G}$ such that the graphs $R^*(\mathcal{G})$ and H without any labels are equal, and any labels on $R^*(\mathcal{G})$ and H agree, then $\mathcal{G}$ is a *celestial covering* of H. The graph $R^*(\mathcal{G})$ is also referred to as a celestial covering. The same notions of strong and weak coverings apply in this scenario.

The corollary below is a reformulation of a well-known result, found in Grünbaum's book [23, pgs. 203–210].

Corollary 4.4.4. *Any $2n$-cycle can be labelled in such a way that the cycle admits a celestial covering.*

The statement of the above corollary varies from that found in [23]. Grünbaum refers to such $2n$-cycles as the class of trivial celestial 4-configurations. Here, we focus more on the notion that the $2n$-cycle (which is the Levi graph of a connected $[2,2]$-configuration) can be *covered* by a celestial configuration.

We can also apply this modified PCL repeatedly to achieve other labellings on cycles that still admit geometric coverings. Given two edges e, e' in a graph G, their *distance* $\delta(e, e')$ is the minimum number of internal vertices over all paths P containing e and e' as the first and last edges respectively. If $e = e'$, then $\delta(e, e') = 0$.

Theorem 4.4.5. *PCL Swapping: Let C be a cycle of length $2n$ with a labelling that admits a weak celestial covering. Choose two edges e, e' that are an even distance apart with labels β, δ. Then the cycle with labels β and δ swapped admits a weak celestial covering as well.*

Proof. Given such a C, let e_1 and e_k be two edges an even distance apart with labels β and δ respectively, and let $P = e_1, e_2, ..., e_k$ be a path of length k (which is odd) with ends v and w and internal vertices $u_1, ..., u_{k-1}$. Create a new path $P' = e_{-1}e_{-2}...e_{-k}$ joining v and w with internal vertices $u_{-1}, ..., u_{-k+1}$. Then $P \cup P'$ forms a cycle of length $2k$. Apply the label δ to e_{-1} and β to e_{-k}, and apply a label to all other e_{-i} that is the same as e_i, and denote this label as α_i. With this labelling, $P \cup P'$ is a

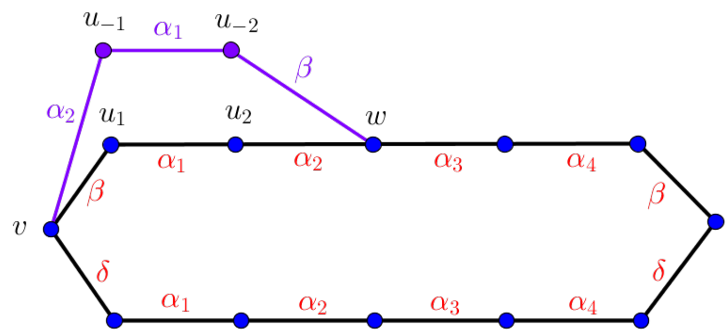


Figure 4.9: PCL Swapping illustrated. The path $P = vu_1u_2w$ can be deleted and replaced with the path $P' = vu_{-1}u_{-2}w$.

PCL cycle. Let $\mathcal{G}$ be a weak celestial covering of C. Consider the orbit v, and place objects u_{-1} at span δ with respect to this orbit. Then construct the orbit of objects u_{-2} at span α_2 with respect to u_{-1}. Continue in this fashion until orbit u_{-k+1} is placed. Create a new orbit w' of objects at span δ with respect to u_{-k+1}. By the PCL Extension applied to $P \cup P'$, it follows that $w' = w$. Removing the orbits of objects corresponding to internal vertices along path P results in a weak celestial covering $\mathcal{G}$ with reduced Levi graph equal to $(C \backslash P) \cup P'$ — a cycle with the same labelling as C, with the exception that β and δ are swapped. □

A swap given by the above theorem will be called a *PCL swap*. Any cycle with a labelling that can be obtained from a sequence of PCL swaps applied to a cycle satisfying the hypotheses of the PCL Extension will also be referred to as a *PCL cycle*. By applying repeated PCL swaps to a cycle, we obtain the following two corollaries.

Corollary 4.4.6. *If n is odd, then a labelled cycle C of length $2n$ with antipodal edges sharing a common label is a PCL cycle. It admits a weak celestial covering.*

Corollary 4.4.7. *Consider any labelled cycle C of length $2n$. If the edges can be partitioned into pairs (e, e') such that e, e' share a common label and are of odd distance apart, then C is a PCL cycle.*

This second corollary will prove to be especially useful as we move beyond the case where the underlying graph is a cycle. In any given cycle, a partition of the edges in the manner outlined in Corollary 4.4.7 will be a partition into *PCL pairs*. There may be multiple ways to partition the edges into PCL pairs, but an arbitrary partition will suffice when needed.

We now move from cycles to consider arbitrary simple bipartite graphs.

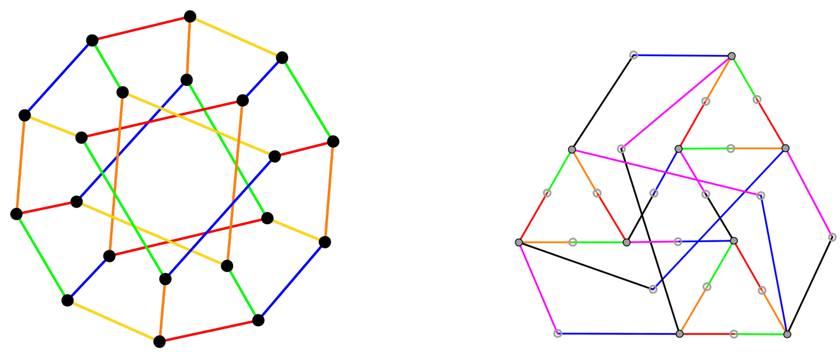


Figure 4.10: A 3-regular feasible graph, and a 4,2-biregular feasible graph. Each colour represents a unique label. The 3-regular feasible graph also appears in [5, pg. 75].

Definition 4.4.6. A labelled bipartite graph G is *feasible* if every cycle is a PCL cycle. It is *proper* if no two adjacent edges share a common label.

We will assume our labellings are proper. To demonstrate that feasible graphs that are not cycles exist, see Figure 4.10 for two examples. Example 4.4.1. and Example 4.4.2. will provide families of feasible graphs as well. Many of the following results can be applied even if the labelling is not proper; however, they tend to yield degenerate cases.

Theorem 4.4.8. *Every 2-connected feasible graph G admits a weak celestial m-fold covering, for all sufficiently large m.*

Proof. Let m be any number larger than twice the maximum value of the labels of the edges of G. We proceed by induction on the number of vertices n of G. If $n = 1$, then any orbit of m points placed as vertices of a regular m-gon is a strong celestial covering. Suppose now that G has n vertices, and let v be any vertex of G, and suppose by duality that it denotes an orbit of points. Remove v from G, and let $\mathcal{G}'$ be the weak celestial m-fold covering of $G \backslash v$ (which exists since $G \backslash v$ is also feasible, as it is simply a subgraph of a feasible graph). Let $u_1, ..., u_d$ denote the neighbours of v in G, and e_i denote the edge $u_i v$ with label α_i. Create an orbit of points v_i of span α_i with respect to the orbit u_i. It remains to show that $v_1 = ... = v_d$. Since the graph G is feasible and 2-connected, the path $u_1 v u_2$ lies on some PCL-cycle. All of the orbits of this cycle except v already exist within $\mathcal{G}'$ by the inductive hypothesis. By Corollary 4.4.7, it follows that $v_1 = v_2$. From this, we have $v_1 = ... = v_d$, and these orbits all coincide as the orbit corresponding to v. The resulting configuration $\mathcal{G}$ is a weak celestial covering of G (weak because we are uncertain if the orbit v admits any accidental incidences with orbits of lines). $\square$

Corollary 4.4.9. *Every feasible graph G admits a weak celestial m-fold covering, where m is any integer larger than twice the maximum label of G.*

Proof. Let $G_1, ..., G_c$ be the 2-connected components of G, and $\mathcal{G}_1, ..., \mathcal{G}_c$ their corresponding weak celestial coverings. Let T denote the block tree of G. Starting with any block of the tree, place the corresponding $\mathcal{G}_i$ down on the plane. For any component G_j adjacent to G_i in T with an edge corresponding to the cut vertex v, place $\mathcal{G}_j$ down on the plane, and scale and/or rotate the orbit associated to v so that it coincides with the orbit associated to v in $\mathcal{G}_i$. Repeating this procedure until all blocks have their corresponding geometric (possibly partial) configurations on the plane, the resulting configuration, denoted $\mathcal{G}$, is a weak celestial covering of G. □

Every underlying simple bipartite graph possesses a feasible labelling — simply apply the same label to every edge. Not every underlying simple graph possesses a proper feasible labelling. For instance, a six-cycle with a path of length three joining a pair of antipodal vertices is a simple example of a graph that does not possess a labelling where all cycles are PCL cycles.

If an $[r, k]$-combinatorial configuration $\mathcal{C}$ with Levi graph $L(\mathcal{C})$ has a feasible labelling, then $L(\mathcal{C}) = R^*(\mathcal{G})$ for some geometric $[2r, 2k]$-configuration (again, this doubling is due to the fact that each edge in $R^*(\mathcal{G})$ represents a double-edge in $R(\mathcal{G})$).

Developing individual examples of feasible graphs may be a challenging endeavour. We provide two families of biregular feasible graphs. The first example is derived from [3], while the second family leads to an entirely new, larger class of configurations.

Example 4.4.1. For a given r, k, consider two subsets $S, T \subset \mathbb{N}$, with $S = \{x_1, ..., x_{r-1}\}$ and $T = \{y_1, ..., y_k\}$, where no two x_i, y_j are equal. Let P be the set of all ordered pairs (σ', τ) where $\sigma, \sigma' \subset S$, $\tau \subset T$, and $|\sigma'| = |\tau| - 1$. Let B be the set of all ordered pairs (σ, τ) with $\sigma \subset S$ and $|\sigma| = |\tau|$. Let G be the bipartite graph with bipartition (P, B). An edge joins $(\sigma', \tau') \in P$ and $(\sigma, \tau) \in B$ if and only if either

- $\tau = \tau'$ and $\sigma = \sigma' \cup \{x\}$, for some $x \in S \backslash \sigma$ or
- $\sigma = \sigma'$ and $\tau = \tau' \backslash \{y\}$, for some $y \in \tau'$.

In the former case, we give the edge a label of x, while in the latter, we give it a label of y. Then the graph $G = (P, B)$ is feasible and r, k-biregular (it is the reduced Levi graph of a celestial configuration). To demonstrate biregularity, consider any $(\sigma', \tau') \in P$. Then there are $|S| - |\sigma'|$ elements x to add to σ' to become $(\sigma' \cup x, \tau') \in B$.

Alternatively, there are $|\tau'|$ elements y in τ' to remove to find an element $(\sigma', \tau' \backslash y) \in B$. In total, $(\sigma', \tau') \in P$ is adjacent to $|S| - |\sigma'| + |\tau'|$ elements in B. Since σ' and τ' differ in cardinality by one, and $|S|$ has $r-1$ elements, it follows that this point is adjacent to r elements of B. Now, suppose $(\sigma, \tau) \in B$. Then there are $|\sigma|$ elements x in σ that may be removed to become an element $(\sigma \backslash x, \tau) \in P$. Likewise, there are $T - |\tau|$ elements y to add to τ in order to become an element $(\sigma, \tau \cup y) \in P$. This implies that (σ, τ) is adjacent to $T - |\tau| + |\sigma| = k$ elements of P. Thus, G is biregular.

Next we show it is feasible. Let C be a cycle in G. Then choose a point $(\sigma_1, \tau_1) \in P$. We may write C in the form

$$(\sigma_1, \tau_1)\,,\ (\sigma_2, \tau_2)\,,\ (\sigma_3, \tau_3) \ldots,\ (\sigma_n, \tau_n),\ (\sigma_1, \tau_1).$$

For a given edge (σ_i, τ_i) and $(\sigma_{i+1}, \tau_{i+1})$ with label z, there are four possibilities:

- $\sigma_{i+1} = \sigma_i \cup z$, and $\tau_i = \tau_{i+1}$. In this case, $z \in S$ and $(\sigma_i, \tau_i) \in P$.
- $\sigma_{i+1} = \sigma_i \backslash z$, and $\tau_i = \tau_{i+1}$. In this case, $z \in S$ and $(\sigma_i, \tau_i) \in B$.
- $\tau_{i+1} = \tau_i \cup z$, and $\sigma_i = \sigma_{i+1}$. In this case $z \in T$ and $(\sigma_i, \tau_i) \in B$.
- $\tau_{i+1} = \tau_i \backslash z$ and $\sigma_i = \sigma_{i+1}$. In this case $z \in T$ and $(\sigma_i, \tau_i) \in P$.

For any given edge of C with the label $z \in S \cup T$ between (σ_i, τ_i) and $(\sigma_{i+1}, \tau_{i+1})$, there must exist another edge with label z. This is because z appears either in $\sigma_i \cup \tau_i$ or $\sigma_{i+1} \cup \tau_{i+1}$, but not both. Assume without loss to generality that $i = 0$ (modulo n) and $z \in S$. Thus $z \in \sigma_n$, but $z \notin \sigma_1$. Then since C is a cycle, there must be some j such that σ_j does not contain z, while σ_{j+1} does. Let j be the smallest value for which $z \notin \sigma_j$, but $z \in \sigma_{j+1}$. Then it remains to show that these two labels form a PCL pair — that they are an odd distance apart. Since $z \notin \sigma_1$, but it is in σ_n, it follows that $(\sigma_n, \tau_n) \in P$ and $(\sigma_1, \tau_1) \in B$. Likewise, $z \notin \sigma_j$, but $z \in \sigma_{j+1}$. Therefore, $(\sigma_j, \tau_j) \in B$, so j is odd. From this it follows that the edge $(\sigma_n, \tau_n), (\sigma_1, \tau_1)$ and the edge $(\sigma_j, \tau_j), (\sigma_{j+1}, \tau_{j+1})$ are of odd distance apart. Therefore, C is a PCL cycle, and G is feasible.

In [3], Berman provides an analysis on the number of vertices in each of the partitions of G. There are $\binom{r+k-1}{k-1}$ vertices of G corresponding to orbits of points, and $\binom{r+k-1}{r-1}$ vertices of G corresponding to orbits of lines. Thus, the celestial $[2r, 2k]$-configuration contains $m \cdot \binom{r+k-1}{k-1}$ points.

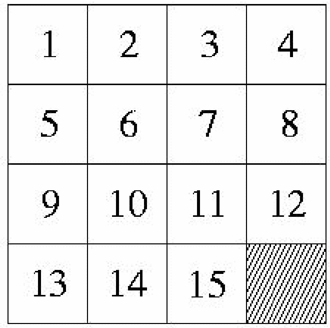


Figure 4.11: A picture of the original 15-puzzle. One square of the grid is empty, allowing any tiles adjacent to the empty space to move to this position (transposing the empty space with an adjacent tile).

The next example explores a different class of feasible graphs, and therefore yields previously undiscovered highly incident geometric configurations. This illustrates that feasible graphs are neither trivial to construct, nor are they completely determined by the previous example. It is interesting to note that this example relies on *puzzle graphs*, a type of graph that was previously considered a purely recreational mathematical object.

Example 4.4.2. The 15-puzzle, or sliding puzzle, is a common puzzle game patented by Noyes Chapman in 1880. In this game, we have 15 tiles, arranged in a 4×4 grid, with one square of the grid empty. We may transpose a tile with the empty space to create a new arrangement of tiles. The goal of the puzzle game is to place the tiles in order from a particular rearrangement. This game can be generalized to *Puz-graphs*, first explored by Wilson (who also developed Wilson's Fundamental Construction in design theory) [32]. Given a graph G on n vertices, consider the set of vertex-labellings on G with $n-1$ distinct labels from the set $[n-1]$, and one vertex given the 'empty' label (denoted by 0). All possible labellings form the set of vertices in the Puz-graph $\text{Puz}(G)$. Two different labellings u, v are adjacent in $\text{Puz}(G)$ if and only if v can be obtained from u by a transposition of the empty label with a neighbouring label. In the 15-puzzle scenario, the graph G is the 4×4 grid, and the Puz-graph vertices are different arrangements of the tiles within the 15-puzzle. Two vertices are adjacent if there is a single move in the 15-puzzle that joins the two tilings. There exists a natural projection map from $\text{Puz}(G)$ to G that sends an arrangement of tiles to the vertex of G that contains the empty tile in the arrangement.

Let G be a bipartite graph. It is known, from [32], that $\text{Puz}(G)$ contains two isomorphic components and is bipartite. To avoid confusion between the vertices of G and Puz(G), we will refer to the vertices of Puz(G) as *arrangements of tiles* or *tilings* and an edge in Puz(G) as a *move* or *transposition*. The *tiles* of Puz(G) are the labelled vertices of G. Given a move uv in Puz(G), let α be the label of the tile transposed with the empty tile. Then grant the move uv the label α. We redefine Puz(G) to include this labelling. For example, consider the 4-cycle C_4, and the Puz-graph Puz(C_4). One component of the Puz-graph is shown in Figure 4.12, along with some of the tilings correponding to the vertices of Puz(C_4). There are $4! = 24$ total vertices in the Puz-graph, each corresponding to a different arrangement of the tiles $0, 1, 2, 3$ on the vertices of C_4. Two tilings are adjacent in the Puz-graph if and only if the arrangements of the tiles differ by a transposition of the empty tile 0 in C_4 with a tile it is adjacent to in C_4. For instance, if the tiles $0, 1, 2, 3$ appear cyclically along the vertices of C_4, then this tiling is adjacent to the tilings $1, 0, 2, 3$ and $3, 1, 2, 0$ (the empty tile 0 is swapped with one of the cyclically adjacent tiles 1 or 3). Since the empty tile in C_4 is only ever adjacent to two other tiles, it follows that the corresponding Puz-graph is 2-regular. It is in fact, two disjoint 12-cycles, as indicated by Figure 4.12.

Proposition 4.4.10. *If G is bipartite then the graph Puz(G) is feasible.*

Proof. Let be a cycle C of Puz(G), and let W be the projection of C onto G via the natural projection map described above. The empty label moves between bipartitions of G as it traverses W. Let (U, V) be the bipartition of G, and then define $C = u_1, v_1, u_2, v_2, ..., u_c, v_c$ where u_i (resp. v_i) is an arrangement of tiles where the empty tile is contained within U (resp. V). Given an edge label α in C, note that the tile α must begin and end in the same bipartition as the empty tile traverses W. Therefore, the number of moves of C labelled by α is even, and alternates between an edge of the form u_iv_i and an edge of the form v_ju_{j+1} (as the tile moves from V to U and then back from U to V). Thus, any move u_iv_i labelled α may be paired with the next move v_ju_{j+1} labelled by α. Such a pairing may be considered a PCL pair, as these two moves are an odd distance apart. Each cycle C can then be partitioned into PCL pairs and is therefore a PCL cycle. This makes Puz(G) feasible. □

However, Puz(G) does not represent a strong geometric realization. To see this, let G equal the four-cycle C_4. Then Puz(C_4) consists of two disjoint 12-cycles. If we restrict to either component, the resulting 12-cycle contains edges labelled $1, 2, 3, 1, 2, 3, ...$ Traversing this 12-cycle represents moving the empty tile three times around the cycle C_4. If x_1, x_2, x_3, x_4 are the vertices of C_4, then this means we may identify the labellings of Puz(G) where the labels are permuted under $(x_1x_3)(x_2x_4)$. After this identification is made, the resulting puzzle graph is C_6, with edges labelled $1, 2, 3, 1, 2, 3$. See Figure 4.12.

A detailed analysis of more complex realizations of puzzle graphs can be found in the appendix.

It is helpful to establish some operations that may be performed on feasible graphs to yield larger graphs that are still feasible. There are some operations that may be performed on feasible graphs to yield new varieties of feasible graphs. Clearly the disjoint union operation remains an option. We provide two less trivial examples of operations below.

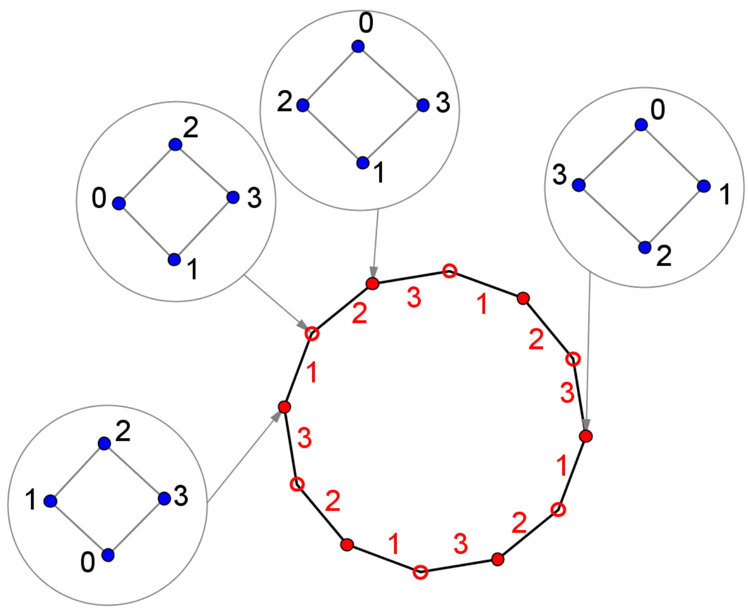


Figure 4.12: One component of the puzzle graph Puz(C_4), with its corresponding edge-labelling. Each vertex is labelled with its corresponding arrangement of tiles on C_4 (some of these labellings are shown in the grey circles). Notice that adjacent labellings in the puzzle graph correspond to labellings where the empty label '0' has been swapped with an adjacent label α. The edge on the puzzle graph is given the same label α as the label on the vertex in C_4 that is swapped with '0'. Antipodal points on this cycle correspond to coincident orbits of objects, and may be identified.

4.4.2 k-fold Replication

Given a feasible bipartite graph $G = (A, B)$, let $A := \{v_1, ..., v_n\}$. Create k copies $G_1, ..., G_k$ of G and let v_{ij} denote the copy of v_i contained within G_j. Create new isolated vertices $u_1, ..., u_n$ and add edges $v_{ij}u_i$ to the graph with label α_j, for any $\alpha_j \in \mathbb{N}$. See Figure 4.13 for an example of this operation. The resulting graph G' is also feasible. To show this, let C be any cycle in G'. If C is entirely contained within some G_i, then C is clearly a PCL cycle. Suppose C spans many copies of G. Then we may write C as

$$v_{i_t j_1} P_1 v_{i_1 j_1} u_{i_1} v_{i_1 j_2} P_2 v_{i_2 j_2} u_{i_2} v_{i_2 j_3} P_3 \cdots P_t v_{i_t j_t} u_{i_t} v_{i_t j_1}$$

with $t \in \mathbb{N}$, each $j_1, ..., j_t \in [k]$, and $i_1, ..., i_t \in [n]$. Each P_i is a path within some copy $G_1, ..., G_k$ of G. Intuitively, some portion of the cycle C 'enters' the copy G_{j_x} by way of some vertex $v_{i_x j_x}$, passes through the point u_{i_x}, and then enters another copy of G through the vertex $v_{i_x j_{x+1}}$. Note that each path P_x is of even length, since the vertices v_{ij} all belong to copies of the same partition A. Thus the edges $u_{i_{x-1}} v_{i_{x-1} j_x}$ and $v_{i_x j_x} u_{i_x}$ are of a common label and an odd distance apart, and can be grouped as PCL pairs. It remains to show that the edges of the paths in each copy of G can be partitioned into PCL pairs.

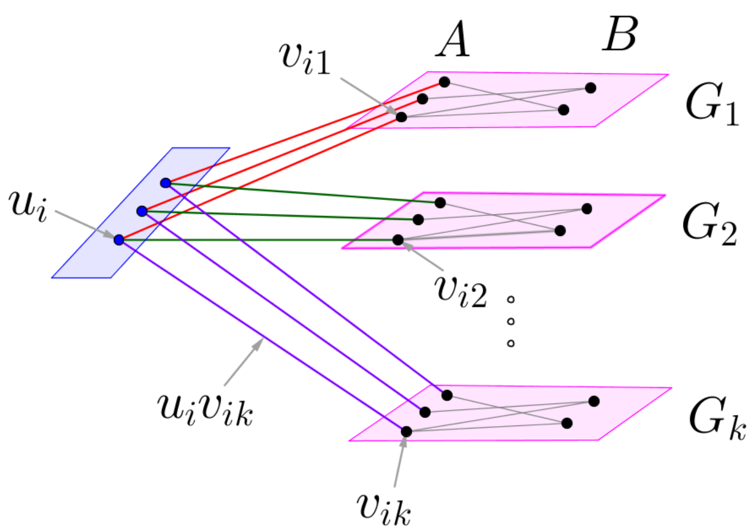


Figure 4.13: The k-fold replication operation. Each G_j is a copy of the graph G. The vertices v_{ij} belong to copies of the partition A of G (as indicated at the start of Section 4.4.2.). Each coloured edge joining a u_i with some v_{ij} in G_j represents a separate label α_j.

Consider the paths P_x and their natural projections P'_x down onto G. Since the last vertex of P'_x is equal to the first vertex of P'_{x+1}, the sequence $P' := P'_1 P'_2 ... P'_t$ is a closed walk, possibly with repeated edges (even consecutively repeated edges). Let a portion Q of P' be known as a *backtracking* walk if it has edges of the form $e_1 e_2 ... e_k e_k e_{k-1} ... e_1$. It is maximal if there is no backtracking walk containing Q as a subwalk. For each maximal backtracking walk $e_1 e_2 ... e_k e_k e_{k-1} .... e_1$, pair identical edges (i.e. (e_i, e_i)) of this walk as PCL pairs in C (note that although the pair (e_k, e_k) are adjacent in P', they will not be in C, as each edge will lift to a different copy of G). With the edges in all backtracking walks considered, let P'' be the closed walk P' with all backtracking walks removed. This walk can now be decomposed into cycles $C'_1, ..., C'_y$ for some $y \in \mathbb{N}$. Each of these cycles in G are PCL cycles. Thus, each edge e in P_x is projected onto P'_x as e' and belongs to either a backtracking walk or a unique cycle C'_i. In the latter case, this cycle is PCL, so the projected edge e' is part of a PCL pair with another edge f' in C'_i. This edge f' belongs to some other path P'_w (where w may equal x), and lifts up to an edge f in P_w. Since (e', f') is a PCL pair, they are of odd distance and share a common label. The edges e, f also share a common label. The distance between e and f is also odd, since we are only inserting paths of the form $v_{i_x j_x} u_{i_x} v_{i_x j_{x+1}}$, which are of length two, in between the paths P_x. Thus, the parities of $\delta(e', f')$ and $\delta(e, f)$ are the same. Therefore, all the edges of C can be placed in PCL pairs, so the cycle C is a PCL cycle. Thus, G' is a feasible graph, known as the *k-fold replication* of G. If G is (r, k)-biregular on $n + b$ vertices, then G' is $(r + 1, k)$-biregular with $k(n + b) + n$ vertices.

4.4.3 Bonding Operation

Another slightly more complex operation on feasible graphs will be known as *bonding* two feasible graphs. Given feasible bipartite graphs G and H, let U be a subset of one bipartition of G and V be a subset of a bipartiton of H, with the property that $|U| = |V| = m$ for some $m \in \mathbb{N}$. Let $U = \{u_1, ..., u_m\}$ and $V = \{v_1, ..., v_m\}$.

Create m copies $G_1, ..., G_m$ of G and m copies $H_1, ..., H_m$ of H. Let U_i and V_i be the copies of U and V in G_i and H_i. Denote the copy of u_j in G_i by u_{ij}, and likewise for v_{ij}. Add the edges $u_{ij} v_{ji}$ to the configuration. Let E be the set of all edges formed this way, so $|E| = m^2$. Give all the edges of E the same label α. Each u in some U_i is joined with exactly one v in some V_j, and this association is invertible. See Figure 4.14 for an example of such a construction. It remains to show that the resulting graph is feasible.

Let C be any cycle in the construction. If C is entirely contained within some copy of G or H, then clearly C is a PCL cycle. Otherwise, C can be written as

$$uP_1e_1P_2...P_ce_cu,$$

where u is a vertex in some U_j, the edges e_i are contained within E, and the paths P_{2i+1} belong to some copy of G, while the paths P_{2i} belong to some copy of H. The paths P_i are of even length, since they begining and end with a vertex in the same bipartition (either in a copy of G or H). The value of c must also be even, since the path must begin and end in a copy of G (and each path transitions from a copy of G to H or vice versa). Thus, the edges $e_1, e_2, ..., e_c$ may be partitioned into pairs (e_{2i-1}, e_{2i}) that share a common label α and are odd distance apart (PCL pairs). It remains to partition the edges within the paths. For any even x, consider the segment $e_{x-1}P_xe_x$ of the cycle C. Then $e_{x-1} = u_{ij}v_{ji}$, and P_x is contained entirely within H_j. Then the last vertex of P_x is a vertex v_{jk}, for some k, and thus $e_x = v_{jk}u_{kj}$. The initial vertex u_{ij} and final vertex u_{kj} of this segment are copies of the same vertex u_j within G_i and G_k respectively. Consider the sequence of paths $P_1P_3...P_{c-1}$, and their natural projections $P'_1P'_3...P'_{c-1}$ down onto G. Then since the final vertex of P_x is a copy of the initial vertex of P_{x+2}, it follows that $P'_1P'_3...P'_{c-1}$ is a closed walk (potentially with backtracking segments) within G. As in the case with the k-fold replication operation, we may pair the edges in C that project down onto a maximal backtracking walk. The remainder of the closed walk can then be decomposed into PCL cycles $C'_1..., C'_y$, for some $y \in \mathbb{N}$. If e is some edge of P_x and its projection e' belongs to a unique PCL cycle C'_t, then it is paired with some edge f' in some C'_w. The lift of f' back to some P_w is unique. Since e' and f' share a common label, so do e and f. Since e' and f' are of odd distance, and the segments of the form $e_{x-1}P_xe_x$ are of even length, it follows that e and f are also of odd distance. Therefore, (e, f) is a PCL pair in the cycle C. Likewise, the edges of $P_2P_4....P_c$ may be partitioned into PCL pairs. This partition of C into PCL pairs demonstrates that C is a PCL cycle, and the construction yields a feasible graph.

We denote this graph $\bigwedge\!\!\!\bigwedge_{G,H}(U, V)$, as the *bonding* of G and H at U and V. In the case that $G = H$, then we simply write $\bigwedge\!\!\!\bigwedge_G(U, V)$ or $\bigwedge\!\!\!\bigwedge(U, V)$ when G is implied. We may also extend this operation to include the case where $|U| \neq |V|$. In this scenario, we add a prestep to the operation: let $|U| = m$ and $|V| = m'$, and $g := \gcd(m, m')$.

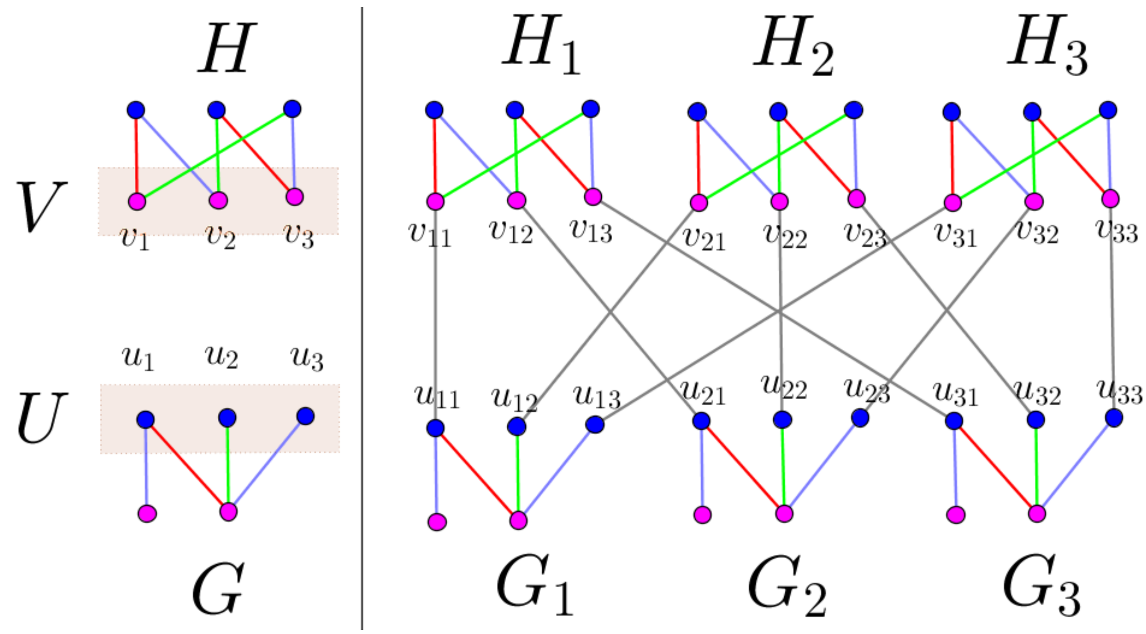


Figure 4.14: The bonding operation on the feasible graphs G, H shown at left. The sets U and V of vertices in G and H respectively are also highlighted on the left. Each colour on the edges denotes a different label. On the right is an illustration of the graph $\bowtie_{G,H}(U,V)$. Note that each vertex u_{ij} is joined with the vertex v_{ji} by a new edge. All of these edges share a common label, and belong to the set E, defined at the start of Section 4.4.3.

Then let G' be the disjoint union of m'/g copies of G, and U' be the disjoint union of the copies of U. Likewise, let H' be the disjoint union of m/g copies of H. Then define $\bowtie_{G,H}(U,V)$ as $\bowtie_{G',H'}(U',V')$. Furthermore, if $\mathcal{G}, \mathcal{H}$ are configurations (either combinatorial or geometric), then $\bowtie_{\mathcal{G},\mathcal{H}}(U,V)$ will refer to $\bowtie_{L(\mathcal{G}),L(\mathcal{H})}(U,V)$.

This operation can be useful in balancing partial configurations that have some deficient points. Suppose $\mathcal{G}$ is an $[r-1,k]$-combinatorial configuration with point set P and $\mathcal{H}$ is an $[r,k-1]$-combinatorial configuration with L denoting the set of lines. Then bonding the points of $\mathcal{G}$ with the lines of $\mathcal{H}$ causes each point in a copy of $\mathcal{G}$ to be joined to exactly one line in a copy of $\mathcal{H}$. Thus, the replication number of each point in a copy of $\mathcal{G}$ is r, and the line size of each line in a copy of $\mathcal{H}$ is k. Thus, $\bowtie_{\mathcal{G},\mathcal{H}}(P,L)$ is an $[r,k]$-configuration.

If U, V are subsets of different bipartitions of the graph G, then the bonding operation $G_1 := \bowtie_G(U,V)$ comes equipped with a natural projection map $\phi : V(G_1) \to V(G)$. Note that each vertex $u \in U$ lifts to $2|U|$ vertices in G_1 under this projection. Exactly $|U|$ of these vertices are incident with some edge in E, and are therefore incident with some vertex in the lift $\phi_1^{-1}(V)$. This is due to the fact that half of the newly created copies of G have edges in E meeting a copy of G at every vertex in U, while the other half of the copies of G meet E at every point in V (and not in U). We define U_1 to be the set of $|U|^2$ vertices corresponding to a copy of U that are 'unmatched' with a vertex in some copy of V. Dually, we define V_1 to be the set of $|V|^2$ vertices corresponding to a copy of V that are unmatched with a vertex in some copy of U.

We may think of $\bowtie_G(U, V)$ as an attempt to match the vertices in U to the vertices in V. As stated in the preceding paragraph, this attempt is unsuccessful: only half of the points in some copy of U are joined to a vertex in some copy of V. We can recursively apply the bonding operation on G_1:

$$G_2 := \bowtie_{G_1}(U_1, V_1) := \bowtie_G^2(U, V).$$

This again comes equipped with a natural projection $\phi_1 : V(G_2) \to V(G_1)$, and through this map a projection $\Phi_1 : V(G_2) \to V(G)$ where $\Phi_1 := \phi \circ \phi_1$. This recursive bonding application again attempts to match the unmatched vertices of some copy of U to some unmatched vertex of a copy of V. Each vertex $u \in U$ lifts under ϕ^{-1} to $2|U|$ vertices, where $|U|$ are unmatched. Each of these $|U|$ unmatched vertices lifts again under ϕ_1^{-1} to $2|U|^2$ vertices, where $|U|^2$ are unmatched in G_2. However, $u \in U$ lifts under Φ_1 to a total of $4|U|^2$ matched or unmatched vertices in G_2. Thus, $\frac{3}{4}$ of the vertices in G_2 that project down to u are adjacent to some vertex that projects down into V. This extends to all of U: exactly $\frac{3}{4}$ of the vertices in G_2 that project down into U are matched with some vertex in G_2 that projects down into V. If we recursively let U_i denote the set of vertices belonging to a copy of U that are unmatched with any vertex in a copy of V, and likewise for V_i, then we define

$$G_i := \bowtie_{G_{i-1}}(U_{i-1}, V_{i-1}) := \bowtie_G^{i-1}(U, V).$$

Repeated applications of the bonding operation yield the following proposition:

Proposition 4.4.11. *Given a feasible graph G where U, V are equal sized subsets of different bipartitions. Let G_s be defined as the s-fold recursive graph $\bowtie_G^s(U, V)$. If N denotes the number of vertices in G_s that project down to U, then the number of vertices in G_s that project down to U and are adjacent to some vertex that projects* *down to V is*

$$\left(1 - \frac{1}{2^s}\right) N.$$

With enough applications of the bonding operation, we can conclude that almost all vertices in some copy of U are adjacent to some vertex in a copy of V (and vice versa). This will form the foundation of our theorem regarding covering configurations.

4.4.4 Covering Configurations

Since an arbitrary combinatorial configuration does not often lend itself to a geometric realization, we next look for geometric configurations with Levi graphs that share some similar properties to a given combinatorial configuration. In Theorem 4.4.12, we seek to answer whether every combinatorial configuration can be covered by a geometric configuration. Although an affirmative result is still unknown, we provide a 'nearly' affirmative result, making extensive use of the bonding operation.

Theorem 4.4.12. *Given an $\varepsilon > 0$ and a combinatorial (possibly partial) configuration with Levi graph H. There exists a celestial geometric (possibly partial) configuration $\mathcal{G}$ with an underlying Levi graph G and a map $\psi : V(G) \to V(H)$ such that, if uv are adjacent in H, then at least $(1-\varepsilon)|\phi^{-1}(u)|$ of the vertices in $\phi^{-1}(u)$ are adjacent to exactly one vertex in $\phi^{-1}(v)$.*

Proof. Let $q \in \mathbb{N}$ be such that $2^{-q} < \varepsilon$. Suppose H has a bipartition (X, Y). We proceed by induction on the number of edges m in H. If $m = 0$, then the theorem is trivial, with $|X|$ arbitrary orbits of points and $|Y|$ arbitrary orbits of lines, with no intersection between them. Assume the theorem is true for some graph H with $m-1$ edges, and consider some combinatorial configuration on m edges. Let uv be an arbitrary edge in H. By the inductive hypothesis, the theorem holds for $H' := H \backslash uv$. Let G' be the underlying feasible reduced Levi graph of the celestial configuration $\mathcal{G}'$ that satisfies the conclusions of the theorem with H', and let Φ' be the projection map from G' to H'.

Let U', V' be the lifts of $u, v \in H$ respectively in G'. Consider the graph $G := /\!\!\!\backslash\!\!\!\backslash^q_{G'}(U', V')$, along with the projection $\phi : V(G) \to V(G')$, and the projection $\Phi = \Phi' \circ \phi$. Then the only edges formed by the bonding operation are between copies of vertices in U' with vertices in a copy of V'. Since the vertices in U' and V' all project down onto u and v (respectively) under Φ', it follows that the only edges created are the desired ones — edges joining some lift of u with some lift of v. Additionally, from

Proposition 4.4.11, the number of vertices $\phi^{-1}(U')$ that are incident with exactly one vertex in $\phi^{-1}(V')$ is $(1-2^{-q})|\phi^{-1}(U')|$, which is greater than $(1-\varepsilon)|\phi^{-1}(U')|$. Therefore, the proportion of vertices in $\Phi^{-1}(u)$ that are incident with some vertex in $\Phi^{-1}(v)$ is at least $(1-\frac{1}{2^q}) > (1-\varepsilon)$. This completes the proof. □

Of course, the graph G is likely to be exceptionally large under this constructive process, and in fact the number of orbits of G grows exponentially with both the number of edges in H and the value of q in the proof. As $\varepsilon \to 0$, the number of vertices in the construction of G tends towards infinity. In fact, it is a simple matter to find an infinite feasible graph G that is an infinite cover of any H. The *universal covering graph of* H is an infinite tree. Start with a single vertex u_v, corresponding to $v \in H$, and join a leaf to u_v corresponding to each neighbour of v in H. For each leaf in the resulting tree, add a new adjacent vertex for each neighbour of the corresponding vertex in H. We proceed in this manner ad infinitum. This graph is acyclic, and thus any arbitrary labelling of the edges will yield a feasible infinite graph. This feasible graph can produce an infinite celestial configuration that has H as a quotient.

4.4.5 Generating New Families of $[2r, 2k]$-configurations

Another application of the bonding operation and k-fold replication on feasible graphs is its ability to construct new highly incident configurations. After applying the bonding operation $\mathbb{A}_{\mathcal{G}}(P, B)$ on an $[2r, 2k]$-configuration $\mathcal{G}$ with Levi graph $G := (P, B)$, the resulting structure is no longer a configuration — some points (resp. lines) in have replication number $2r$ (resp. line size $2k$) while others have replication number $2r+2$ (resp. line size $2k+2$). However, we may apply a variation of the k-fold replication operation to 'fill' the deficient points and lines. Simply apply a $(2k+2)$-fold replication followed by a $(2r+2)$-fold replication: first only to the vertices corresponding to points of replication number r in $\mathbb{A}_G(U, V)$, and then to all the lines of size k in the resulting configuration. In the end, a balanced $[2r+2, 2k+2]$-configuraiton is obtained. The number of orbits of points after applying the bonding operation on $\mathcal{G}$ with n orbits of points and b orbits of lines is

$$\frac{2nb}{\gcd(n, b)}$$

and two applications of the replication operation yield a $[2r+2, 2k+2]$-configuration on

$$\left[r + \frac{r}{k} + 1\right](k+1)\left(\frac{2nb}{\gcd(n, b)}\right)$$

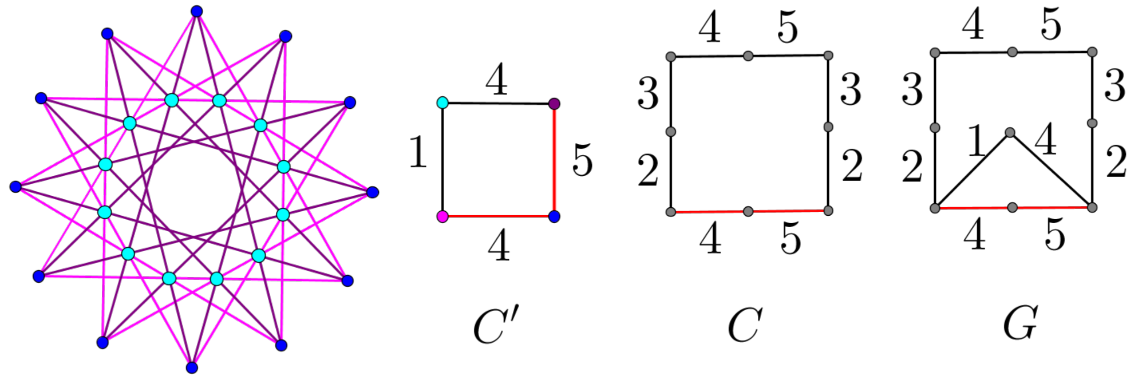


Figure 4.15: An example of a 4-configuration with $R^*(\mathcal{G})$ that is not a PCL-cycle. The left graph is the reduced Levi graph of the 4-configuration shown. It is a 4-cycle C'. The 8-cycle C shown in the diagram is feasible. The path P' of C' indicated with red edges is isomorphic to the path P of C, also in red. By identifying C and C' along these paths, we obtain a larger graph G that still admits a realization, although it is not feasible. The deletion of the red path in G will yield a 4-configuration with a non-feasible reduced Levi graph. This technique can be used to generate more examples of non-feasible graphs that are still the reduced Levi graph of a celestial partial configuration. The (24_4)-configuration shown on the left is given in [23, pg. 3].

orbits of points. Since the bonding operation and k-fold replication can result in significantly larger configurations, these operations do not yield chiral configurations of minimal size; however, they do provide a new, large family of highly incident configurations that were previously unknown.

If G is an r, k-biregular bipartite feasible graph, then it is the reduced Levi graph $R^*(\mathcal{G})$ of some $[2r, 2k]$-configuration. If the labels on the edges of G are all odd, then by choosing the rotation group $\mathbb{Z}_m$ to be such that m is even, we can construct a $[r, k]$-configuration on half of the number of points and lines. For each orbit $u_0, ..., u_{m-1}$ of objects, remove the objects with odd subscripts. Since m is even, this deletion is well-defined modulo m. The result is an $[r, k]$-configuration. This is because, if the edge labels of G are odd, every object u that is span α with respect to v originally met objects v_i and $v_{i+\alpha}$ within that orbit. However, exactly one of these subscripts is odd, and therefore every object still in u meets exactly one of the objects in orbit v after the deletion. Thus, we obtain an $[r, k]$-configuration with half as many points and lines. This deletion procedure is an extension of the *odd deletion* procedure for 4-configurations, discussed by Berman and Grünbaum in [8].

It is also worth noting that a labelling need not correspond to a PCL cycle in order to be realizable. In Figure 4.15, a 4-configuration $\mathcal{G}$ is shown that possesses a graph $R^*(\mathcal{G})$ that is not a PCL-cycle. However, this set of labels does not work for a general m, only in this case where $m = 12$. Known examples of reduced Levi graphs of celestial configurations that are not feasible can be utilized to create further examples of celestial configurations with non-feasible reduced Levi graphs. For instance, suppose C is a feasible cycle on $2n$ vertices, and C' is a cycle that is the reduced Levi graph of a celestial 4-configuration. If P and P' are paths of equal length on C and C', and both P and P' share a common labelling (that is, they are isomorphic as labelled paths), then let G be the graph of $C \cup C'$, with the paths P and P' identified. G is also the reduced Levi graph of a celestial configuration, even if C' is not feasible (see Figure 4.15 for an example). The proof of this is nearly identical to that of the Swapping PCL (see Theorem 4.4.5), so we omit it here. In addition, the graph $G - P$ is then a cycle. If C' is not feasible, then $G - P$ may not be feasible; however, it is still the reduced Levi graph of a celestial configuration. As shown in Figure 4.15, we can use some sporadic examples of 2-celestial 4-configurations to modify other types of celestial configurations. This type of modification allows us to expand the class of celestial configurations beyond those with a feasible reduced Levi graph. Many examples of 3-celestial and 4-celestial 4-configurations are provided by Berardinelli and Berman in [2], and these examples may be used to modify feasible reduced Levi graphs to provide new examples of celestial configurations. Such sporadic examples could be useful in building even more unique variations of celestial configurations that are highly incident; however, some care must be taken. Replacing a path P_1 with a path P_2 in a graph G that is not a cycle only yields a feasible graph if every cycle containing an edge of P_1 remains a PCL cycle after the replacement with P_2. Alternatively, it is possible that a 'nearly' feasible graph — one in which all but a few cycles are PCL cycles could still be realizable, if the few remaining non-PCL cycles are labeled to be sporadic cases of realizable non-PCL cycles.

Chapter 5

Conclusions and Further Research

The theorems presented in this thesis provide an answer to some of the fundamental questions regarding both the existence of configurations and the properties of embeddings in both the combinatorial and geometric realms. However, there remain numerous unanswered questions and conjectures.

5.1 Further Questions on Combinatorial Configurations

Theorem 3.1.1 and Theorem 3.1.2 both concern themselves with providing optimal bounds on the asymptotic existence of combinatorial configurations. While the bound in Theorem 3.1.1 is not far from optimal, some improvements on this bound can still theoretically be made, reducing the bound on $N(r,k)$ from order $O(rk^2)$ to $O(rk)$ for all possible r,k. In Theorem 3.1.2, an optimal bound is obtained; however, the value of $R(k)$ outlined in this theorem relies on Bertrand's Postulate, and may be excessive.

Regarding embeddings, there are other types of embedding questions that may be asked. For instance, can an initial partial configuration $\mathcal{C}_0$ be embedded in a larger, regular $[r,k]-$configuration $\mathcal{C}$ in such a way so that the lines of $\mathcal{C}$ may be partitioned into copies of $\mathcal{C}_0$? This question likely requires some further restrictions on the permissible line sizes and replication numbers of each line and point in $\mathcal{C}_0$. Alternatively, we could apply colours to the lines and/or points of the initial partial configuration $\mathcal{C}_0$, and ask if such a configuration may be embedded in a larger coloured configuration, where the restriction of points and/or lines of each colour determine a configuration.

Due to the extremely flexible nature of configurations, the author conjectures that there is an affirmative answer to these questions, and many other similar questions regarding configurations. The aim would then be to find efficient bounds on the size of these larger configurations.

Another area of potential research for configuration theory is to examine partial configurations. For instance, if $R, K \subseteq 2^{\mathbb{N}}$, then we define an $[R, K]$-configuration to be a configuration where every line size is a member of K, and every replication number is a member of R. If $R = \{r\}$ and $K = \{k\}$ for some $r, k \geq 2$, then an $[R, K]$-configuration is equivalent to an $[r, k]$-configuration. A $[\{3, 4\}, \{3, 4\}]$-combinatorial configuration on 10 points is given below:

$$\begin{array}{ccc} \{1,2,3,4\} & \{1,5,6\} & \{1,7,8\} \\ \{2,5,7,9\} & \{2,6,8\} & \{3,6,10\} \\ \{3,8,9\} & \{4,6,7\} & \{4,9,10\} \\ \{5,8,10\} & & \end{array}$$

Let $N(R, K)$ be the value for which an $[R, K]$-configuration exists for all $n \geq N(R, K)$ satisfying certain divisibility conditions, similar to those for $[r, k]$-configurations. Can we find a bound for $N(R, K)$ that is a nontrivial improvement upon $N(r, k)$? Clearly such existence bounds are less than $N(r, k)$ for any $r \in R$ and $k \in K$. Perhaps a more interesting question would be to impose stricter criteria on such $[R, K]$-configurations, by requiring lines of size $k \in K$ to appear with a specified frequency (or requiring a replication number r to appear with a given frequency). Formally, for each $r \in R$, let p_r be a real number between 0 and 1, such that $\sum p_r = 1$, and for each $k \in K$, let q_k be a real number between 0 and 1 such that $\sum q_k = 1$. Then the author conjectures that, for any $\varepsilon > 0$ and sufficiently large values of n, b, there exists a connected $[R, K]$-configuration on n points and b lines satisfying the following properties:

- for each $r \in R$, the number of points with replication number r is contained within the interval $p_r n \pm \varepsilon n$,
- for each $k \in K$, the number of lines with line size k is contained within the interval $q_k b \pm \varepsilon b$,

provided n, b satisfy some trivial conditions. A similar result is known to exist for some types of pairwise balanced designs [15], but the size of n required to guarantee such a design is not provided. It would be interesting to see how such a bound would fare in configuration theory, relative to $N(R, K)$.

As we see, most problems regarding pairwise balanced designs can be applied to configurations as well, since configurations contain PBDs (with $\lambda = 1$) as a subclass. Studying the differences between configurations and designs could prove useful in illuminating the obstructions to current open problems in design theory.

Other directions for research that are not borrowed from design theory include an analysis of configurations with Levi graphs posessing certain properties. For instance, providing examples of Levi graphs that are t-connected, or possess girth t, or with bounded diameter, et cetera.

5.2 Further Questions on Geometric Configurations

A satisfactory existence result on regular geometric configurations remains elusive. As we have seen, even finding non-trivial examples of highly incident configurations is challenging. Our study in embeddings has provided many more examples of undiscovered geometric configurations — even those exhibiting chiral symmetry. However, these configurations all include a variation of the Gray configuration L_k^r as a subconfiguration and may be viewed as contrived by some. These embeddings are also quite large, as they rely on previously existing $[r, k]$-configurations (which are again not optimal). With the lack of a suitable result regarding the embedding of combinatorial configurations as geometric configurations, we instead turned to examining geometric configurations whose Levi graph is a covering of a given combinatorial configuration.

Even here, a truly positive result was not found. Instead, an asymptotically optimal answer was found utilizing the graph bonding operation. This operation relied on celestial configurations. Other similar types of operations may also exist on these types of geometric configurations, similar in nature to the bonding operation and the k-fold replication operation.

One significant improvement that may be made is to find other theorems like the Point Completion Lemma to expand the class of labelled Levi graphs that are able to be feasibly constructed as geometric configurations. Such a discovery would potentially allow for significant improvements in the number of known highly incident configurations.

Conversely, one could instead focus on obstructions to geometric realizations. Clearly, the Levi graph of any geometric configuration cannot contain the Levi graph of the Fano plane as a subgraph. What other subgraphs provide obstructions? While many combinatorial configurations may possess a few common traits that prohibit geometric realization, there are likely numerous or even infinite sporadic cases of unrealizable configurations as well. It may be simpler to provide obstructions to the existence of a chiral realization. In Chapter 4, we provided a translation of the Point Completion Lemma from its geometric setting to a graph theoretic formulation. Such a translation for obstructions as 'forbidden subgraphs' would be beneficial to remove the complications that arise from working within a geometry.

Of course, any question posed for combinatorial configurations above can also be posed for geometric configurations. It is likely that with a sufficient number of points, such existence questions also yield affirmative answers. Other directions involve extending the currently known results into other geometries or dimensions, such as configurations of points and lines in $\mathbb{R}^3$, or points and circles on the sphere S^2. Such configurations have scarcely been examined. However, so little is known about geometric configurations in the simplest case (as points and lines in $\mathbb{R}^2$), and it is doubtful that more exotic types of configurations would yield new and surprising results.

While the history of both combinatorial and geometric configurations is long, there are still many gaps that remain in their study. This dissertation resolves some of the holes in configuration theory, and aims to provide some understanding on the structure and substructures of both combinatorial and geometric configurations.

Appendix A

Puzzle Graph Analysis

We continue our analysis from Example 4.4.2 of geometric realizations of Puz(G) for a bipartite graph G.

A.1 The Complete Bipartite Graph $K_{r,k}$

Let (P, B) denote the bipartition of points, where P contains all degree r vertices of $K_{r,k}$. In the geometric realization of Puz($K_{r,k}$), any tiling A corresponds to an orbit of objects. However, there are several tilings in Puz($K_{r,k}$) that correspond to an orbit of objects coincident with A. We can partition the tilings of Puz($K_{r,k}$) into equivalence classes, where the tilings of a class all correspond to the same orbit of objects. We begin by analyzing the tilings equivalent to a given tiling.

Proposition A.1.1. *Let A be any tiling, with v_0 corresponding to the vertex with the empty tile. Let C be any 4-cycle $v_0v_1v_2v_3$ in $K_{r,k}$. If $A' := (v_1v_3)(v_0v_2)A$ denotes the tiling with the tiles v_0, v_2 permuted and the tiles v_1, v_3 permuted, then A' is equivalent to A.*

Proof. Create the following definitions:

$$\begin{array}{cccc} A_0 := A & A_1 := (v_0v_1)A_0 & A_2 := (v_1v_2)A_1 & A_3 := (v_2v_3)A_2 \\ A_4 := (v_3v_0)A_3 & A_5 := (v_0v_1)A_4 & A_6 := (v_1v_2)A_5 & \end{array}$$

Then one can show that $A_6 = A'$. Furthermore, the moves $A_0A_1, A_1A_2, ..., A_5A_6$ within Puz($K_{r,k}$) contain labels $v_1, v_2, v_3, v_1, v_2, v_3$ respectively. Therefore, by the PCL, it follows that the orbit corresponding to A_0 is coincident with the orbit corresponding to A_6. Thus, A is equivalent to A'. An illustration of this proof for $K_{2,2} = C_4$ is given in Figure 4.12. □

Given an arrangement of tiles A with the empty tile contained within P, let Σ be a permutation of the tiles of P. Then Σ can be written as a sequence of transpositions $\sigma_s \sigma_{s-1} \cdots \sigma_1$ such that each transposition exchanges the empty tile with another tile. For example, if $\Sigma = (v_1 v_2)$, then we may write

$$\Sigma = (v_0 v_2)(v_1 v_2)(v_0 v_1).$$

The first transposition swaps the empty tile v_0 with v_1, while the second transposition moves it from v_1 to v_2, and the third transposition moves it from v_2 back to v_0, leaving the tiles on v_1 and v_2 permuted.

Let Π be a permutation of the same parity as Σ, written as a sequence of transpositions $\Pi = \pi_p \pi_{p-1} \cdots \pi_1$. If $p > s$, then choose an arbitrary permutation π of the tiles of B, and append π^{p-s} to Π (note that this does not change the permutation Π, since $p - s$ is even). If, on the other hand, $s > p$, then let σ be any transposition containing the empty tile and append σ^{s-p} to Σ. Again, this does not change the permutation Σ. We may then assume that $p = s$. By the proposition above, A is equivalent to $\sigma_1 \pi_1 A$, which is equivalent to $\sigma_2 \pi_2 \sigma_1 \pi_1 A$, and so forth. Since the tiles transposed by Σ and Π are disjoint, it follows that A is equivalent to $\Sigma \Pi A$. A similar argument can be utilized in the case that A is an tiling where the empty tile is contained within B instead.

Consider any arrangement of tiles A within a component of $\text{Puz}(K_{r,k})$. Then there are $r!$ possible choices of Σ and $k!/2$ allowable permutations Π for each choice of Σ. Thus, the size of the equivalence class containing A is at least $r!k!/2$ (and is some multiple of this value). Suppose A is an orbit of points, so that the empty tile is contained within P. There are $r \cdot (r + k - 1)!$ total tilings with the empty tile contained within P, and each of them correspond to an arragement of $\text{Puz}(K_{r,k})$. Therefore, if the size of the equivalence class is $r!k!/2$, then the total number of such equivalence classes is

$$\frac{r \cdot (r + k - 1)!}{r!k!/2}.$$

Half of these equivalence classes belong to one of the two components of $\text{Puz}(K_{r,k})$.

Restricting to a single component of $\mathrm{Puz}(K_{r,k})$ yields a geometric configuration with

$$\frac{r(r+k-1)!}{r!k!} = \binom{r+k-1}{r-1}$$

orbits of points. By a similar argument, there are

$$\binom{r+k-1}{k-1}$$

orbits of lines. The result is a $[2r, 2k]$-geometric configuration, provided each equivalence class does indeed have size $r!k!/2$.

In fact, these configurations are identical to the ones found by L. Berman, illustrated in Example 4.4.1. To see this, we define $S = [r-1]$ and $T = [k] + r - 1$ from Example 4.4.1. Identify each equivalence class of $\mathrm{Puz}(K_{r,k})$ with the pair (σ, τ), where $\sigma \subset S$ is the subset of the tiles $[r-1]$ within any tiling of the equivalence class that are also contained within P. Then $\tau \subset T$ is defined as the subset of tiles $[k] + r - 1$ within any tiling of the equivalence class that are also contained within B.

A.2 The Dutch Windmill Graph D^t_{2m}

For bipartite graphs that are not complete, a more nuanced approach is needed, as any set of four vertices does not necessarily belong to a cycle. The Dutch Windmill Graph D^t_{2m} consists of t copies of the cycle C_{2m}, with one vertex in common amongst all such cycles. This graph is clearly bipartite. Let $C^1, ..., C^t$ denote the t cycles of D^t_{2m}, and orient each cycle arbitrarily, so that the 0th vertex of each cycle is the central vertex of the windmill. Define P^t_{2m} to be the component of the Puz-graph of D^t_{2m} containing the following tiling: the empty tile v_0 is positioned at the centre of the windmill and the ith vertex of C^j is given the tile v^j_i. As in the case of the complete bipartite graph, we deem to tilings to be equivalent if they correspond to the same orbit of objects in the geometric realization. Our goal is to analyze the equivalence classes of P^t_{2m}.

Note that in this Puz-graph component, any move of the empty tile only permutes the tiles within a particular cycle, and these tiles can never be permuted outside of the cycle they initially belong. In other words, in every tiling of P^t_{2m}, the tile v^j_i is contained within the cycle C^j. Most tilings in P^t_{2m} have degree two. In fact, the only tilings that have a larger degree are those where the empty tile is contained at the centre of the windmill. We will refer to these tilings as *central* tilings. Each central tiling has degree $2t$. Starting at any central tiling and moving the empty tile along a particular cycle C^j in the positive direction will result in another central tiling after exactly $2m$ moves. The difference between these two central tilings is a permutation of tiles given as

$$\sigma_j = (v^j_{2m-1} v^j_{2m-2} \cdots v^j_2 v^j_1).$$

It follows that the central tilings are in correspondence with the group of permutations:

$$\Sigma := \langle \sigma_j \;:\; \sigma_j^{2m-1} = \mathbf{1} \text{ and } \sigma_i\sigma_j = \sigma_j\sigma_i \rangle.$$

This group Σ is isomorphic to $\mathbb{Z}^t_{2m-1}$, and as such has $(2m-1)^t$ elements. In P^t_{2m}, two central tilings corresponding to elements σ and σ' are joined by a path of length $2m$ if and only if $\sigma = \sigma'\sigma_j^{\pm 1}$, for some j. This path corresponds to the empty tile moving in either direction along the cycle C^j.

For a given element $\sigma \in \Sigma$, and $j \in [t]$, let C^j_σ denote the cycle in P^t_{2m} formed by starting at the central tiling corresponding to σ, and repeatedly moving the empty tile around C^j until the we return to the tiling corresponding to σ. This requires moving around C^j a total of $2m-1$ times, implying that C^j_σ is a $(2m)(2m-1)$-cycle, containing $2m-1$ central tilings. Note that the moves along C^j_σ are labelled in order $v^j_1, v^j_2, ..., v^j_{2m-1}$ a total of $2m$ times. By the PCL Extension Theorem, it follows that for every path along this cycle of length $2(2m-1)$, the initial and terminating vertices correspond to coincident orbits of points or lines. Suppose without loss to generality that the central tiling σ corresponds to an orbit of points. Two arragements of distance $2(2m-1)$ along C^j_σ are equivalent. This partitions the tilings corresponding to orbits of points into $2m-1$ classes. Each class contains a single central tiling. The tilings corresponding to lines are partitioned into $2m-1$ classes as well.

Identify all these tilings in each cycle C^j_σ. Every tiling in P^t_{2m} corresponding to a point is identified with a unique central tiling. The resulting identification yields a feasible, $(2t, 2)$-biregular, bipartite graph P'. The number of tilings of P' corresponding to orbits of points is equal to the number of central tilings of P^t_{2m}. Thus, the graph P' is the reduced Levi graph of a $[4t, 4]$-configuration, with $(2m-1)^t$ orbits of points, and $t(2m-1)^t$ orbits of lines.

An example of this construction is illustrated in Figure A.1. Here, we consider the Puz-graph of D^2_4. Even one component of this Puz-graph is rather large, so the figure only depicts one cycle of the Puz-graph, along with its neighbours. Each arrangement of tiles in this Puz-graph corresponds to a unique labelling of the vertices of D^2_4, and some of these labellings are provided in the diagram. Notice that the antipodal points on the 12-cycle shown in the figure correspond to coincident orbits, by the PCL. After applying the necessary quotient operation on these vertices, the resulting reduced Levi graph contains 9 vertices corresponding to orbits of points and 18 vertices corresponding to orbits of lines.

Note that it is not necessary to have all the cycles in the Dutch windmill graph be the same length, provided they are all even cycles. If we have cycles of length $2m_1, ..., 2m_t$, then the resulting geometric configuration will contain

$$\prod(2m_i - 1)$$

orbits of points. In fact, it is not even necessary that the spokes of the windmills are cycles, provided they are bipartite. For instance, a similar analysis could be conducted on the case where t different complete bipartite graphs all share a single vertex in common. Figure A.2 shows the celestial configuration with 13-fold symmetry that possesses this reduced Levi graph. Even for the simplest Dutch Windmill graph (beyond a single cycle when $t = 1$), the resulting geometric configuration is difficult to depict.

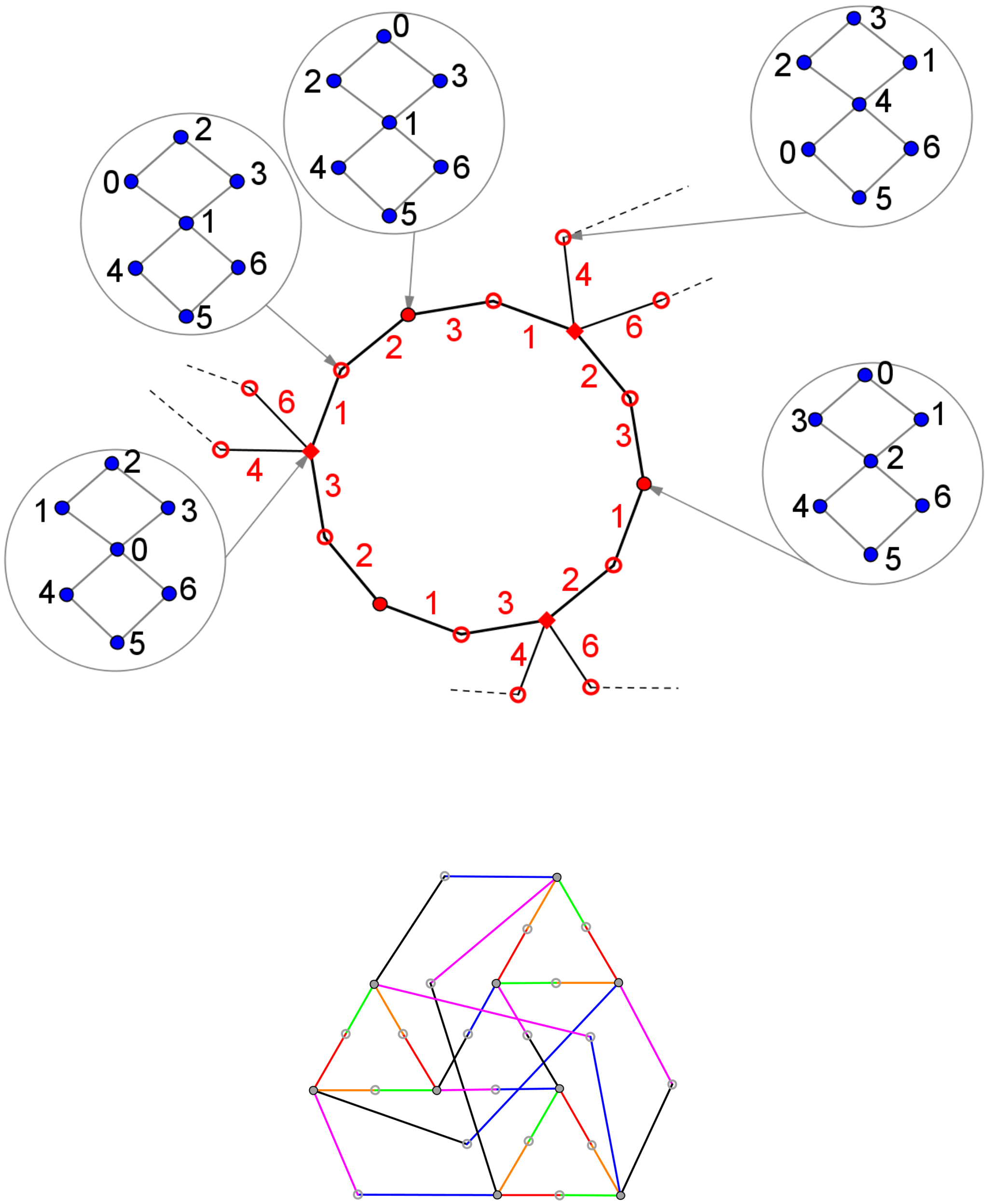


Figure A.1: The first diagram demonstrates a portion of the Puz-graph Puz(D_4^2). Five examples of labellings on D_4^2 are shown in the gray circles. The label 0 corresponds to the empty label. The red vertices belong to the Puz-graph (each one corresponds to a unique tiling), with hollow vertices corresponding to orbits of lines, while the filled vertices correspond to orbits of points. Diamond-shaped vertices correspond to central tilings. The five examples of labellings on D_4^2 correspond to five different tilings of the Puz-graph, as shown. The second diagram illustrates the resulting feasible reduced Levi graph that is obtained once the vertices of the Puz-graph are identified. Each colour represents a unique label.

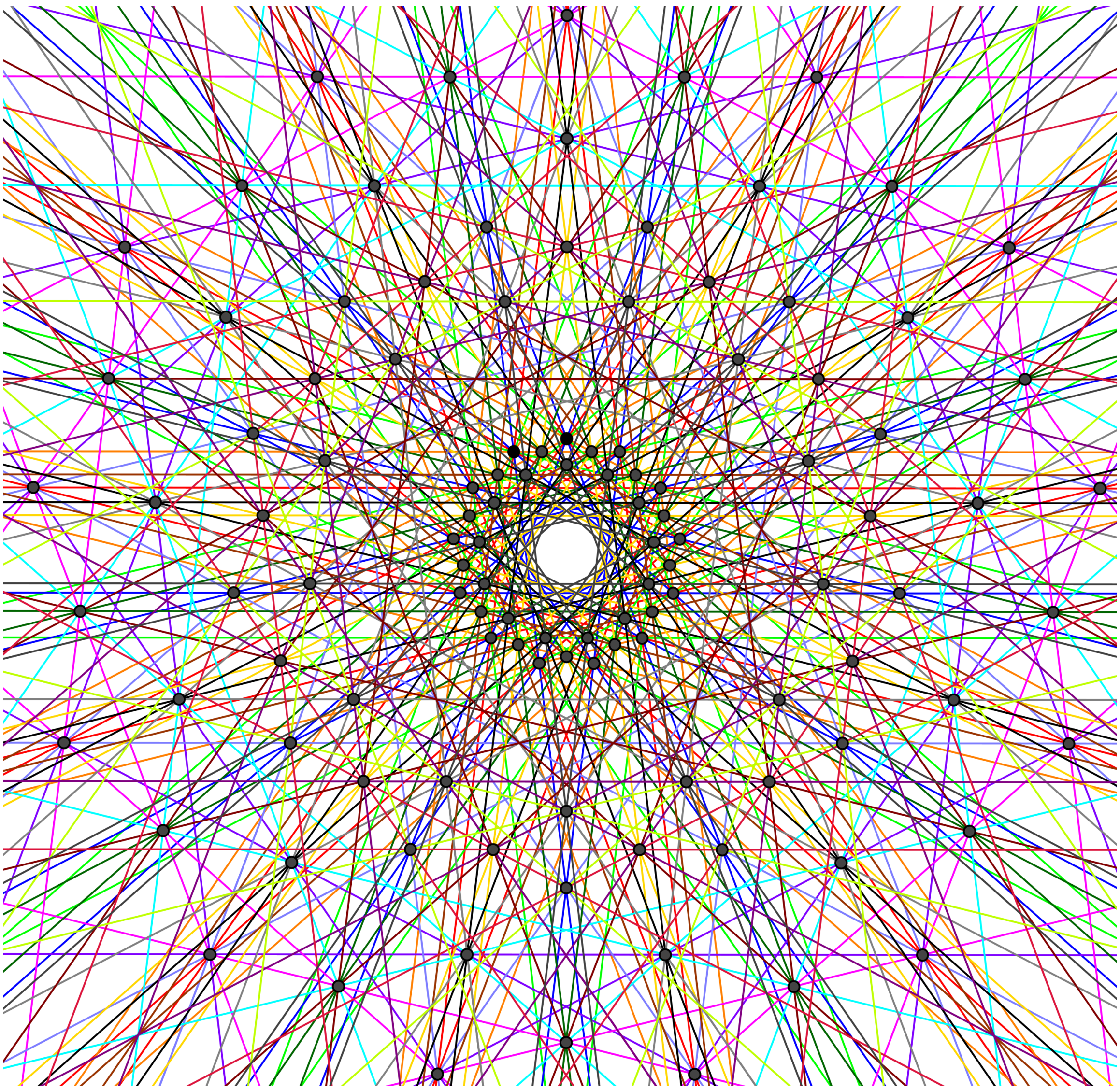


Figure A.2: The realization of the Puz-graph of D_4^2 as a celestial $(117_8, 234_4)$-configuration with 13-fold dihedral symmetry.

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