Vertex-Criticality and Bicriticality for Independent Domination and Total Domination In Graphs

by

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BSc, University of Victoria, 2004
MSc, University of Victoria, 2006

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Abstract

For any graph parameter, the removal of a vertex from a graph can increase the parameter, decrease the parameter, or leave the parameter unchanged. This document focuses on the case where the removal of a vertex decreases the parameter for the cases of independent domination and total domination. A graph is said to be *independent domination vertex-critical*, or i-critical, if $i(G-v) < i(G)$ for any vertex $v \in V(G)$, where $i(G)$ is the independent domination number of G. Likewise, a graph is said to be *total domination vertex-critical*, or γ_t-critical, if $\gamma_t(G-v) < \gamma_t(G)$ for any vertex $v \in V(G)$ such that $G-v$ has no isolated vertices, where $\gamma_t(G)$ is the total domination number of G. Following these notions, a graph is *independent domination bicritical*, or i-bicritical, if $i(G-\{u,v\}) < i(G)$ for any subset $\{u,v\} \subseteq V(G)$, and a graph is *total domination bicritical*, or γ_t-bicritical, if $\gamma_t(G-\{u,v\}) < \gamma_t(G)$ for any subset $\{u,v\} \subseteq V(G)$ such that $G-\{u,v\}$ has no isolated vertices. Additionally, a graph is called *strong independent domination bicritical*, or strong i-bicritical, if $i(G-\{u,v\}) = i(G)-2$ for any two independent vertices $\{u,v\} \subseteq V(G)$.

Construction results for i-critical graphs, i-bicritical graphs, strong i-bicritical graphs, γ_t-critical graphs, and γ_t-bicritical graphs are studied. Many known constructions are extended to provide necessary and sufficient conditions to build critical and bicritical graphs. New constructions are also presented, with a concentration on i-critical graphs. One particular construction shows that for any graph G, there exists an i-critical, i-bicritical, and strong i-bicritical graph H such that G is an induced subgraph of H. Structural properties of i-critical graphs, i-bicritical graphs, γ_t-critical graphs, and γ_t-bicritical graphs are investigated, particularly for the connectedness and edge-connectedness of critical and bicritical graphs. The coalescence construction which has appeared in earlier literature constructs a graph with a cut-vertex and this construction is studied in great detail for i-critical graphs, i-bicritical graphs, γ_t-critical graphs, and γ_t-bicritical graphs. It is also shown that strong i-bicritical graphs are 2-connected and thus the coalescence construction is not useful for such graphs.

Domination vertex-critical graphs (those graphs where $\gamma(G-v) < \gamma(G)$ for any vertex $v \in V(G)$) have been studied in previous publications. A well-known result states that $\text{diam}(G) \leq 2(\gamma(G)-1)$ for domination vertex-critical graphs. Here similar techniques are used to provide upper bounds on the diameter for i-critical graphs, strong i-bicritical graphs, and γ_t-critical graphs. The upper bound for the diameter of i-critical graphs trivially gives an upper bound for the diameter of i-bicritical graphs.

For a graph G, the *γ-graph of G*, denoted $G(\gamma)$, is the graph where the vertex set is the collection of minimum dominating sets of G. Adjacency between two minimum dominating sets in $G(\gamma)$ occurs if from one minimum dominating set a vertex can be removed and replaced with another vertex from $V(G)$ to arrive at the other minimum dominating sets. In the literature, two versions of adjacency have been defined:

- the single vertex replacement adjacency model: where the minimum dominating set D_1 is adjacent to the minimum dominating set D_2 if there exists a vertex $u \in D_1$ and a vertex $v \in D_2$ such that $D_2 = (D_1 - \{u\}) \cup \{v\}$, and

- the slide adjacency model: where the minimum dominating set D_1 is adjacent to the minimum dominating set D_2 if there exists a vertex $u \in D_1$ and a vertex $v \in D_2$ such that $D_2 = (D_1 - \{u\}) \cup \{v\}$ and $uv \in E(G)$.

In other words, one can think of adjacency between γ-sets D_1 and D_2 in $G(\gamma)$ as a swap of two vertices. In the slide adjacency model, these two vertices must be adjacent in G, hence the γ-graph obtained from the slide adjacency model is a subgraph of the γ-graph obtained in the single vertex replacement adjacency model. Results for both adjacency models are presented concerning the maximum degree, the diameter, and the order of the γ-graph when G is a tree.

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Introduction

1.1 Introduction, Definitions, and Examples

When studying any graph parameter, the concept of criticality is of interest. When changes are made to a graph, how does the graph parameter in question change (or not change)? There are of course many types of changes one could consider applying to a graph: deleting a vertex, deleting an edge, adding an edge, identifying two vertices, and contracting an edge, to name just a few. In this thesis vertex-criticality (the deletion of a vertex) with respect to various domination parameters is studied. All graphs considered will be finite, simple, and undirected. In general, graph theoretic definitions and notation as defined in [51] and domination definitions and notation as defined in [25] and [26] are followed.

A set of vertices $D \subseteq V(G)$ is called a *dominating set of* G if every vertex in $V(G) - D$ is adjacent to at least one vertex in D. The minimum cardinality of a dominating set of G is called the *domination number* and is denoted by $\gamma(G)$. If D is a dominating set of minimum cardinality, then D is called a γ-set. For a vertex

$v \in V(G)$ we say that D *dominates* v if either v is in D or v is adjacent to a vertex in D. Likewise, for a set of vertices $S \subseteq V(G)$ we say that D *dominates* S if every vertex in S is either in D or is adjacent to a vertex in D.

The domination number was first defined in 1958 by Berge [8], though he called it the "coefficient of external stability". Oystein Ore was the first to use the terminology of "dominating set" and "domination number" in his 1962 book *Theory of Graphs* [40], and is credited as publishing the first theorems on dominating sets. The notation of $\gamma(G)$ for the domination number was introduced by Cockayne and Hedetniemi in their 1977 survey paper on the state of domination problems at the time [15]. Since its introduction and popularization domination has been greatly studied with two published volumes devoted to the survey of various topics in domination ([25] and [26]). Additionally, many domination variants have been defined and investigated. Beyond the usual domination, two domination variants are studied in the results presented here. For a graph G without isolated vertices, a set of vertices $D \subseteq V(G)$ is called a *total dominating set of* G if every vertex in $V(G)$ is adjacent to at least one vertex in D. The minimum cardinality of a total dominating set of G is called the *total domination number* and is denoted by $\gamma_t(G)$. If D is a total dominating set of minimum cardinality, then D is called a γ_t-set. If G contains isolated vertices, then $\gamma_t(G)$ is undefined. Notice that every total dominating set is also a dominating set, and so $\gamma(G) \leq \gamma_t(G)$. A set of vertices $D \subseteq V(G)$ is called an *independent dominating set of* G if D is a dominating set that is also an independent set; that is, no two vertices of D are adjacent. The minimum cardinality of an independent dominating set is called the *independent domination number*, denoted by $i(G)$, and an independent dominating set of minimum cardinality is called an i-set. Notice that every maximal independent set is an independent dominating set. In addition, every independent dominating set is also a dominating set, and so $\gamma(G) \leq i(G) \leq \alpha(G)$ (where $\alpha(G)$ is the cardinality of a maximum independent set in G).

For a vertex $v \in V(G)$, the graph $G - v$ denotes the graph created from G by deleting the vertex v and all edges incident with v. Notice that for the domination number, the total domination number, and the independent domination number, the removal of a vertex from a graph has three possible outcomes: the domination parameter in question may decrease, increase, or not change. A graph G is *domination vertex-critical*, or *γ-vertex-critical*, if $\gamma(G - v) < \gamma(G)$ for every $v \in V(G)$. When considering the domination number, we can partition the vertex set of any graph G into the sets V_γ^0, V_γ^-, and V_γ^+, where

$$\begin{aligned} V_\gamma^0 &= \{v \in V(G) : \gamma(G - v) = \gamma(G)\} \\ V_\gamma^- &= \{v \in V(G) : \gamma(G - v) < \gamma(G)\} \\ V_\gamma^+ &= \{v \in V(G) : \gamma(G - v) > \gamma(G)\}. \end{aligned}$$

Thus if G is γ-vertex-critical, then $V(G) = V_\gamma^-$. If G is γ-vertex-critical and $\gamma(G) = k$ we say that G is *k-γ-vertex-critical.* If $\gamma(G - v) < \gamma(G)$ we say that the vertex v is a *γ-critical vertex.* Notice that the 4-cycle C_4 is γ-vertex-critical.

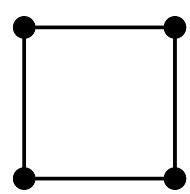

Figure 1.1: A γ-vertex-critical graph.

Likewise G is *total domination vertex-critical*, or *γ_t-vertex-critical*, if $\gamma_t(G - v) < \gamma_t(G)$ for every $v \in V(G)$ such that the graph $G - v$ contains no isolated vertices. Likewise, when considering the total domination number, we can partition the vertex set of G into the sets $V^0_{\gamma_t}$, $V^-_{\gamma_t}$, and $V^+_{\gamma_t}$, where

$$\begin{aligned} V^0_{\gamma_t} &= \{v \in V(G) : \gamma_t(G-v) = \gamma_t(G)\} \\ V^-_{\gamma_t} &= \{v \in V(G) : \gamma_t(G-v) < \gamma_t(G)\} \\ V^+_{\gamma_t} &= \{v \in V(G) : \gamma_t(G-v) > \gamma_t(G)\}. \end{aligned}$$

If G is γ_t-vertex-critical, then $V(G) = V^-_{\gamma_t}$. If G is γ_t-vertex-critical and $\gamma_t(G) = k$ we say that G is *k-γ_t-vertex-critical*. If $\gamma_t(G-v) < \gamma_t(G)$ we say that the vertex v is a *γ_t-critical vertex*. Notice that the 5-cycle C_5 is γ_t-vertex-critical.

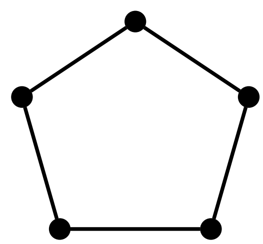

Figure 1.2: A γ_t-vertex-critical graph.

A graph G is *independent domination vertex-critical*, or *i-vertex-critical* if $i(G-v) < i(G)$ for every $v \in V(G)$. Again, for the independent domination number, we can partition the vertex set of G into the sets V^0_i, V^-_i, and V^+_i, where

$$\begin{aligned} V^0_i &= \{v \in V(G) : i(G-v) = i(G)\} \\ V^-_i &= \{v \in V(G) : i(G-v) < i(G)\} \\ V^+_i &= \{v \in V(G) : i(G-v) > i(G)\}. \end{aligned}$$

If G is i-vertex-critical, then $V(G) = V^-_i$. If G is i-vertex-critical and $i(G) = k$ we say that G is *k-i-vertex-critical*. Likewise, if $i(G-v) < i(G)$ (so $v \in V^-_i$) we say that the vertex v is an *i-critical vertex*. If $i(G-v) = i(G)$ (so $v \in V^0_i$) we say that the vertex v is an *i-stable vertex*. Notice that the cycle C_4 is i-vertex-critical (in addition to being γ-vertex-critical), and the cycle C_7 is also i-vertex-critical (and in fact is γ-vertex-critical too).

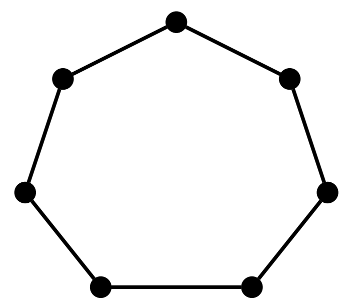

Figure 1.3: An i-vertex-critical graph.

For a set of vertices $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph of G induced by the vertices in S. For a set $S \subseteq V(G)$, $G - S$ is the graph $\langle V(G) - S \rangle$ and for a vertex $v \in V(G)$, $G - v$ is $\langle V(G) - \{v\} \rangle$. For graphs G and H with $V(G) \cap V(H) = \emptyset$ the *disjoint union of* G *and* H, written $G \cup H$, is the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. The graph $G_1 \cup G_2 \cup \cdots \cup G_k$ is defined recursively by $G_1 \cup G_2 \cup \cdots \cup G_k = (G_1 \cup \cdots \cup G_{k-1}) \cup G_k$.

The number of components in a graph G is denoted by $k(G)$. A *cut-vertex* is any vertex whose removal results in a graph with more components than G. That is, $x \in V(G)$ is a cut-vertex of G if $k(G - x) > k(G)$. A *vertex-cut* of a connected graph is a set of vertices $S \subseteq V(G)$ such that $G - S$ is disconnected. A k*-vertex-cut* is a vertex cut of cardinality k. The *connectivity* of a graph $G \neq K_n$ is the minimum cardinality of a vertex-cut of G (the connectivity of K_n is $n - 1$), and we say that G is k*-connected* if the connectivity of G is greater than or equal to k. That is, G is k-connected if we need to remove k or more vertices from G to create a disconnected graph. Likewise, G is k*-edge-connected* if we need to remove k or more edges from G in order to create a disconnected graph.

For a vertex $x \in V(G)$, the *open neighbourhood*, $N_G(x)$, is the set $\{y \mid xy \in E(G)\}$, and the *closed neighbourhood*, $N_G[x]$, is the set $N_G[x] = N_G(x) \cup \{x\}$.

Analogously, for a set $S \subseteq V(G)$, the *open neighbourhood of* S, $N_G(S)$, is the set $\{x \mid \exists y \in S$ such that $xy \in E(G)\}$, and the *closed neighbourhood of* S, $N_G[S]$, is the set $N_G[S] = N_G(S) \cup S$. When the graph G is obvious from context, we simply write $N(x)$, $N[x]$, $N(S)$, and $N[S]$. For a set of vertices $D \subseteq V(G)$ and a vertex $x \in D$, the *private neighbourhood of* x *with respect to* D, denoted $pn(x, D)$, is the set of vertices that are in the closed neighbourhood of x, but are not in the closed neighbourhood of any other vertex in D. That is, $pn(x, D) = N[x] - N[D - \{x\}]$. Likewise, for sets $S, D \subseteq V(G)$, the *private neighbourhood of* S *with respect to* D, denoted $pn(S, D)$, is the set $pn(S, D) = N[S] - N[D - S]$.

1.2 A Summary of Previous Results and Overview of New Results

Criticality for domination parameters was first studied by Sumner and Blitch [45]. They concentrated on γ-edge-critical graphs where $\gamma(G + uv) < \gamma(G)$ for every $uv \notin E(G)$. In their seminal paper Sumner and Blitch investigated γ-edge-critical graphs where $\gamma(G)$ is small; the 2-γ-edge-critical graphs were characterized (G is 2-γ-edge-critical if and only if $\overline{G}$ is the disjoint union of stars) and properties of 3-γ-edge-critical graphs were studied. In particular, they showed that every 3-γ-edge-critical graph contains a 3-cycle. They also showed that 3-γ-edge-critical graphs with an even order have a 1-factor. This result led to the further study of matchings in γ-edge-critical graphs and γ-vertex-critical graphs by Michael Plummer, Nawarat Ananchuen, and others ([2], [3], [4], [5], [6], [29], [48], [49] and elsewhere). For the 3-γ-edge critical graphs, Sumner and Blitch looked at the degree of vertices, and showed that the number of vertices of degree at most k is bounded above by a linear function of k. They investigated the diameter of 3-γ-edge-critical graphs and found that $\text{diam}(G) \leq 3$. Though they did not study γ-vertex-critical graphs, they did look at the effects of deleting a vertex in a γ-edge-critical graph. It was found that in a k-γ-edge-critical graph, $\gamma(G - v) \leq k$ for any $v \in V(G)$. Also, every k-γ-edge-critical graph contains a vertex $v \in V(G)$ such that $\gamma(G - v) = k - 1$.

Notice that the class of γ-vertex-critical graphs is distinct from the class of γ-edge-critical graphs. For example, the cycle C_7 is γ-vertex-critical but not γ-edge-critical. Any graph G where $\overline{G}$ is isomorphic to $K_{1,n_1} \cup K_{1,n_2}$, $n_1, n_2 \geq 1$, and n_1 and n_2 are not both equal to 1, is γ-edge-critical, but not γ-vertex-critical. (In fact, as mentioned above, G is 2-γ-edge-critical.) The cycle C_4 is a graph that is both γ-vertex-critical and γ-edge-critical. That being said, it is noted in [26] that every 2-γ-vertex-critical graph is also a 2-γ-edge-critical graph, but the converse does not hold. In their survey on domination, Ananchuen, Ananchuen, and Plummer [1] commented that for any graph property P, a P-vertex-critical graph can be changed into a P-edge-critical graph by adding all edges $uv \notin E(G)$ such that $P(G + uv) = P(G)$. In particular, a γ-vertex-critical graph can be changed into a γ-edge-critical graph by adding all edges $uv \notin E(G)$ such that $\gamma(G+uv) = \gamma(G)$. For example, the cycle C_7 is γ-vertex-critical but not γ-edge-critical, but by adding some edges we can arrive at a γ-edge-critical graph such that C_7 is a subgraph. Such a situation is pictured in Figure 1.4.

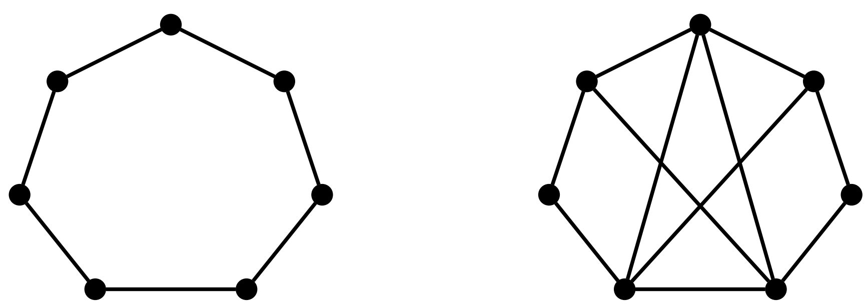

Figure 1.4: Graphs C_7 and a γ-edge-critical graph that contains C_7 as a subgraph.

Domination vertex-criticality can been generalized to (γ, k)-*criticality*. A graph G is said to be (γ, k)-critical if $\gamma(G - S) < \gamma(G)$ for any set of vertices $S \subseteq V(G)$ with $|S| = k$, and (l, k)-critical if G is (γ, k)-critical and $\gamma(G) = l$. Of course, the $(\gamma, 1)$-critical graphs are the γ-vertex-critical graphs. The $(\gamma, 2)$-critical graphs are commonly referred to as γ-*bicritical* graphs. These γ-bicritical graphs are further discussed in Chapter 3. This idea of generalizing γ-vertex-critical graphs by studying (γ, k)-critical graphs was introduced in 2010 by Mojdeh, Firoozi, and Hasni [37] and has been further studied ([36] and [22]).

Brigham, Chinn, and Dutton [9] were the first to focus on γ-vertex-critical graphs. They noted that the only 1-γ-vertex-critical graph is K_1 and the 2-γ-vertex-critical graphs are those that are isomorphic to K_{2n} with the edges of a 1-factor removed. They also gave a family of γ-vertex-critical graphs $G_{m,n}$, where $V(G_{m,n}) = \{v_1, v_1, \ldots, v_{(n-1)(m+1)}\}$ and $E(G_{m,n}) = \{v_i v_j \mid 1 \leq (i-j) \pmod{(n-1)(m+1)+1} \leq m/2\}$. The graph $G_{4,3}$ is isomorphic to the circulant $C_{11}\langle 1, 2\rangle$ and is 3-γ-vertex-critical.

Proposition 1.1. [9] *If G has a non-isolated vertex v such that $\langle N(v)\rangle$ is complete, then G is not γ-vertex-critical.*

Let n be the order of a graph, that is $n = |V(G)|$.

Proposition 1.2. [9] *If G has a γ-critical vertex, then $n \leq (\Delta+1)(\gamma-1)+1$.*

Brigham, Chinn, and Dutton also provided a characterization for γ-vertex-critical graphs having a minimum number of vertices, that is, when $n = \gamma + \Delta$. Investigating the order of critical graphs has proved popular. The order of γ-vertex-critical graphs was further studied by Fulman, Hanson, and MacGillivray [21], where they showed that the γ-vertex-critical graphs of maximum order are regular.

Theorem 1.3. [21] *If G is γ-vertex-critical with $|V(G)| = (\Delta+1)(\gamma-1)+1$, then G is regular.*

The order of γ-bicritical graphs, i-vertex-critical graphs, i-bicritical graphs, γ_t-vertex-critical graphs, and γ_t-bicritical graphs have all been investigated as well ([10], [30], [38], [47], and elsewhere).

Proposition 1.4. [9] *If G is γ-vertex-critical with e edges, then $n \leq (2e+3\gamma-\Delta)/3$.*

Brigham, Chinn, and Dutton investigated methods of constructing γ-vertex-critical graphs. In particular, they discussed the *coalescence* $G \cdot_{xy} H$. Let G and H be disjoint graphs with $x \in V(G)$ and $y \in V(H)$. The *coalescence of* G *and* H *with respect to* x *and* y is the graph $G \cdot_{xy} H$ with vertex set $V(G \cdot_{xy} H) = (V(G) - \{x\}) \cup (V(H) - \{y\}) \cup \{v\}$, where $v \notin V(G) \cup V(H)$, and edge set $E(G \cdot_{xy} H) = E(G - x) \cup E(H - y) \cup \{vw : xw \in E(G) \text{ or } yw \in E(H)\}$. We call v the *vertex of identification* of G and H, and we consider $V(G)$ and $V(H)$ as subsets of $V(G \cdot_{xy} H)$ and regard v as an element of both $V(G)$ and $V(H)$. Informally, $G \cdot_{xy} H$ is the graph obtained from $G \cup H$ by identifying x and y. If the context is clear, or if the vertices x and y are not important, we write $G \cdot H$ instead of $G \cdot_{xy} H$. The graph $G_1 \cdot G_2 \cdot \cdots \cdot G_k$ is defined recursively by $G_1 \cdot G_2 \cdot \cdots \cdot G_k = (G_1 \cdot G_2 \cdot \cdots \cdot G_{k-1}) \cdot G_k$. This construction is further discussed in Chapters 2, 3, 4, and 5.

Proposition 1.5. [9] *Let* G *and* H *be nontrivial graphs. Then* $\gamma(G) + \gamma(H) - 1 \leq \gamma(G \cdot_{xy} H) \leq \gamma(G) + \gamma(H)$. *Furthermore, if both* G *and* H *are* γ*-vertex-critical, or if* $G \cdot_{xy} H$ *is* γ*-vertex-critical, then* $\gamma(G \cdot_{xy} H) = \gamma(G) + \gamma(H) - 1$.

Proposition 1.6. [9] *The graph* $G \cdot_{xy} H$ *is* γ*-vertex-critical if and only if both* G *and* H *are* γ*-vertex-critical.*

The next result follows directly from the previous two propositions.

Theorem 1.7. [9] *A graph* G *is* γ*-vertex-critical if and only if each block of* G *is* γ*-vertex-critical. Further, if* G *is* γ*-vertex-critical with blocks* $G_1, G_2, \ldots, G_n$*, then* $\gamma(G) = \sum_{i=1}^{n} \gamma(G_i) - (n-1)$.

In addition to the coalescence construction, Brigham, Chinn, and Dutton found a method to take any graph G and create a γ-vertex-critical graph H such that G is an induced subgraph of H.

Theorem 1.8. [9] *For any graph* G *there is a* γ*-vertex-critical graph* H *such that* G *is an induced subgraph of* H.

In particular, this shows that the class of γ-vertex-critical graphs cannot be characterized by a finite list of forbidden subgraphs. This idea is revisited in Chapter 2.

Brigham, Chinn, and Dutton posed some questions about γ-vertex-critical graphs.

1. If G is a γ-vertex-critical graph, is $n \geq (\delta+1)(\gamma-1)+1$? This is trivially true when $n = (\Delta+1)(\gamma-1)+1$, the maximum possible value, and also holds when $n = \gamma + \Delta$, the minimum possible value.

2. If G is a γ-vertex-critical graph with $n = (\Delta+1)(\gamma-1)+1$, is G regular?

3. Sumner and Blitch [45] conjectured that $i(G) = \gamma(G)$ for γ-edge-critical graphs. Does $i(G) = \gamma(G)$ for γ-vertex-critical graphs? Again, the statement is true when the number of vertices is at the minimum or maximum value.

4. Let d be the diameter of a γ-vertex-critical graph. Is $d \leq 2(\gamma(G)-1)$? The relation holds when $n = \gamma + \Delta$ or $\gamma \leq 5$.

Fulman, Hanson, and MacGillivray [21] addressed all of these questions. As mentioned above, question 2 was answered affirmatively. They provided an example of the circulant $G = C_{17}\langle 1, 3, 5, 7, 10, 12, 14, 16\rangle$ as a 3-γ-vertex-critical, 8-regular graph where $n = 17 < 19 = (\delta+1)(\gamma-1)+1$, thus providing a negative answer for question 1. This example also has $i(G) = 5 \neq 3 = \gamma(G)$, and so also provides a negative answer to question 3. For question 4, a bound on the maximum diameter of a γ-vertex-critical graph was provided, which gave an affirmative answer. This bound and further results on the diameter of critical graphs are discussed in Chapter 5.

With respect to the domination number, many variations on criticality have been examined. The variation most commonly considered is that of γ-edge-critical graphs, the graphs studied by Sumner and Blitch in which $\gamma(G+uv) < \gamma(G)$ for every $uv \notin E(G)$. Other common variations include *γ-ER graphs*, where $\gamma(G-uv) >$

$\gamma(G)$ for every $uv \in E(G)$ ([24] and elsewhere), and *dot critical graphs*, in which $\gamma(G.uv) < \gamma(G)$ for every $\{u, v\} \subseteq V(G)$, where $G.uv$ denotes the graph that results from identifying vertices u and v and removing any resulting loops and multiple edges that are created ([11] and elsewhere). Here we concentrate only on vertex-criticality and hence for brevity we use the terms γ-critical, γ_t-critical, and i-critical to denote γ-vertex-critical, γ_t-vertex-critical, and i-vertex-critical graphs, respectively.

The remainder of this chapter contains basic results on i-critical and i-bicritical graphs, the main focus of this thesis. These results are those which will be used repeatedly in further chapters and require very little background to prove.

Chapter 2 discusses construction techniques for i-critical graphs. A survey of known construction techniques for γ-critical graphs and i-critical graphs is presented. Results which extend known constructions for i-critical graphs are provided, and new construction techniques are discussed.

A graph G is said to be *i-bicritical* if $i(G - \{u, v\}) < i(G)$ for any set of vertices $\{u, v\} \subseteq V(G)$, and *strong i-bicritical* if $i(G - \{u, v\}) = i(G) - 2$ for any set of vertices $\{u, v\} \subseteq V(G)$. Chapter 3 focuses on i-bicritical graphs and strong i-bicritical graphs. A survey of known results on γ-bicritical graphs is presented, and similar results are discussed for i-bicriticality and strong i-bicriticality. The connectivity of strong i-bicritical graphs is investigated, and a construction which produces a strong i-bicritical graph with a 2-vertex-cut is given. Other constructions for i-bicritical graphs and strong i-bicritical graphs are presented.

Recall that a graph G is said to be γ_t-critical if $\gamma_t(G - v) < \gamma_t(G)$ for every $v \in V(G)$ such that the graph $G - v$ contains no isolated vertices. Likewise, a graph G is said to be *γ_t-bicritical* if $\gamma_t(G - \{u, v\}) < \gamma_t(G)$ for every $\{u, v\} \subseteq V(G)$ such that the graph $G - \{u, v\}$ contains no isolated vertices. Chapter 4 investigates properties of γ_t-critical graphs and γ_t-bicritical graphs. A survey of known results is presented, and some results are extended. Constructions for γ_t-critical graphs and

γ_t-bicritical graphs are investigated. Some constructions are similar to those used for γ-critical graphs, γ-bicritical graphs, i-critical graphs, and i-bicritical graphs, and some constructions are unique to γ_t-critical graphs and γ_t-bicritical graphs.

As mentioned above, Brigham, Chinn, and Dutton [9] posed the question "is $\operatorname{diam}(G) \leq 2(\gamma(G) - 1)$ for a γ-critical graph G?". This lead Fulman, Hanson, and MacGillivray [21] to provide a tight upper bound for k-γ-critical graphs, giving an affirmative answer to the question. Chapter 5 concentrates on the maximum diameter of various critical graphs. In particular, a tight upper bound on the diameter of i-critical graphs is presented. From this bound, an easy upper bound on the diameter for i-bicritical graphs is obtained. In addition, an upper bound for γ_t-critical graphs is presented and examples which reach equality in this bound are given for the case of $\gamma_t(G) \equiv 2 \pmod 3$. An upper bound on the diameter of strong i-bicritical graphs is also presented.

Chapter 6 focuses on the *γ-graph*. The *γ-graph of a graph* G, $G(\gamma) = (V(\gamma), E(\gamma))$, is the graph where the vertex set $V(\gamma)$ is the collection of γ-sets of G. Adjacency between two γ-sets in $G(\gamma)$ can be defined in two different ways:

- Single vertex replacement adjacency model: where γ-set D_1 is adjacent to γ-set D_2 if there exists a vertex $u \in D_1$ and a vertex $v \in D_2$ such that $D_2 = (D_1 - \{u\}) \cup \{v\}$.

- Slide adjacency model: where γ-set D_1 is adjacent to γ-set D_2 if there exists a vertex $u \in D_1$ and a vertex $v \in D_2$ such that $D_2 = (D_1 - \{u\}) \cup \{v\}$ and $uv \in E(G)$.

Thus we can think of adjacency between γ-sets D_1 and D_2 in $G(\gamma)$ as a swap of two vertices. In the slide adjacency model, these two vertices must be adjacent in G, hence the γ-graph obtained from the slide adjacency model is a subgraph of the γ-graph obtained in the single vertex replacement adjacency model. Results for both adjacency models are presented concerning the maximum degree, the diameter, and the order of the γ-graphs of trees. The single vertex replacement adjacency model was first introduced by Subramanaian and Sridharan [43] in 2008, and the slide adjacency model was introduced independently by Fricke, Hedetniemi, Hedetniemi, and Hutson [20] in 2011. The single vertex replacement adjacency model was further studied in [33] and [42] and the slide adjacency model has been further studied in [16]. In this chapter upper bounds on $\Delta(G(\gamma))$, $\mathrm{diam}(G(\gamma))$, and the order of $G(\gamma)$ are given for the case that G is a tree, thus answering three questions posed by Fricke et al. [20].

1.3 Basic Results for i-Critical Graphs and i-Bicritical Graphs

To close the chapter, we present introductory results on i-critical graphs and i-bicritical graphs. Introductory results for γ_t-critical graphs and γ_t-bicritical graphs are contained in Chapter 4.

In her 1994 Master's Thesis [7], Suquin Ao was the first to define i-critical graphs. In this body of work she presented constructions for various families of i-critical graphs, many of these families also produced γ-critical graphs.

Observation 1.9. *If G is i-critical, then for any $v \in V(G)$, every minimum independent dominating set S of $G - v$ has $x \notin S$ for all $x \in N_G[v]$.*

Proposition 1.10. [7] *If G is i-critical, then for any vertex v there exists an i-set S such that $v \in S$.*

Note that the converse of Proposition 1.10 is not true. For example, every vertex of C_5 is contained in an i-set of C_5, but the graph is not i-critical.

Proposition 1.11. *For any graph G and vertex $v \in V(G)$, $i(G - v) \geq i(G) - 1$.*

Proof. Consider an i-set D of $G - v$. If D dominates v in G, then D is also an independent dominating set of G and so $i(G) \leq i(G - v)$. If D does not dominate v

in G, then $D \cup \{v\}$ is an independent dominating set of G and $i(G) \leq i(G-v)+1$. The result follows. □

The following result is a direct consequence of Proposition 1.11.

Proposition 1.12. [7] *If G is i-critical, then $i(G-v) = i(G) - 1$.*

Proposition 1.12 can be generalized for the deletion of any subset of vertices. Notice that this naturally leads to the consideration of (γ, k)-critical graphs.

Proposition 1.13. *For any graph G and vertices $S \subseteq V(G)$ with $|S| = k$, $i(G-S) \geq i(G) - k$.*

Proof. Consider an i-set D of $G - S$. If D is not an independent dominating set of G, then it is possible to add a vertex $x \in S$ that is not dominated by D to create a new independent set D'. If D' is not an independent dominating set of G, then it is possible to add a vertex $x' \in S$ that is not dominated by D' to create a new independent set D''. Continuing in this fashion, it is possible to arrive at an independent dominating set of G from D by adding at most the k vertices in S. Therefore $i(G) \leq i(G-S)+k$. □

In particular, Proposition 1.13 shows that if G is i-bicritical, then $i(G) - 2 \leq i(G - \{x,y\}) \leq i(G) - 1$ for any $\{x,y\} \subseteq V(G)$. The following result determines $i(G-\{x,y\})$ when G is i-bicritical and $xy \in E(G)$.

Proposition 1.14. *If $xy \in E(G)$, then $i(G - \{x,y\}) \geq i(G) - 1$.*

Proof. Consider $G - \{x,y\}$ where $xy \in E(G)$ and let D be an i-set of $G - \{x,y\}$. If D is an independent dominating set of G, then $i(G-\{x,y\}) \geq i(G)$. Otherwise, suppose D does not dominate at least one of x and y. Without loss of generality, suppose D does not dominate x. Then $D \cup \{x\}$ is an independent dominating set of G, and so $i(G) \leq |D \cup \{x\}| = i(G - \{x,y\}) + 1$. □

In fact, if $i(G-x) = i(G)-1$ we can say that $i(G-\{x,y\}) = i(G)-1$ when $xy \in E(G)$, even if G is not i-bicritical.

Proposition 1.15. *If $i(G-x) = i(G)-1$ for some $x \in V(G)$, then $i(G-\{x,y\}) = i(G)-1$ for all $y \in V(G)$ such that $xy \in E(G)$.*

Proof. Suppose that $i(G-x) = i(G)-1$ for some $x \in V(G)$. Then by the proof of Proposition 1.9, there exists an i-set S of $G-x$ such that $S \cap N_G(x) = \emptyset$. Hence if $xy \in E(G)$, then $y \notin S$ and S dominates $(G-x)-y \cong G-\{x,y\}$ and we have that $i(G-\{x,y\}) \leq |S| = i(G)-1$. Suppose that $i(G-\{x,y\}) < i(G)-1$, and consider an i-set of $G-\{x,y\}$. Then either $S \cap N_G(x) = \emptyset$ or $S \cap N_G(y) = \emptyset$ (for otherwise S dominates G). Without loss of generality, say that $S \cap N_G(x) = \emptyset$. But then $S \cup \{x\}$ independently dominates G and $|S \cup \{x\}| = i(G)-1$, a contradiction. Therefore $i(G-\{x,y\}) = i(G)-1$. □

Proposition 1.16. [7] *The only 2-i-critical graphs are K_{2n} less a perfect matching.*

Proposition 1.17. *The only 2-i-bicritical graphs are $\overline{K}_2$ and $K_1 \cup K_2$.*

Proof. Let G be a 2-i-bicritical graph. Since $i(G) = 2$, there exists an independent dominating set $\{x,y\} \subseteq V(G)$ such that $xy \notin E(G)$. Consider $G-\{x,y\}$. If $i(G-\{x,y\}) = 0$ then $G \cong K_1 \cup K_1$. Thus $i(G-\{x,y\}) = 1$ and there exists a vertex $w \in V(G-\{x,y\})$ that dominates $G-\{x,y\}$. In addition, w is not adjacent to at least one of x and y in G. Say $wy \notin E(G)$. Then $xw \in E(G)$ since $\{x,y\}$ is an independent dominating set. Consider $G-\{w,y\}$. Since $i(G-\{w,y\}) = 1$ there exists a vertex $z \in V(G-\{w,y\})$ that dominates $G-\{w,y\}$. Since w dominates $G-\{x,y\}$, $z \in N(w)$. Then $zy \notin E(G)$ for otherwise $i(G) = 1$.

Suppose $z \neq x$. Consider $G-\{w,z\}$. Since $i(G-\{w,z\}) = 1$, there exists a vertex $v \in V(G-\{w,z\})$ such that v dominates $G-\{w,z\}$. Notice that $v \neq y$ since $yw \notin E(G)$ and likewise $v \neq x$. Also, $vw \in E(G)$ since w dominates $G-\{x,y\}$ and

$vz \in E(G)$ since z dominates $G - \{w, y\}$. But then we have that v dominates G and $i(G) = 1$, a contradiction.

Suppose that $z = x$ and $N(w) - \{x\} \neq \emptyset$. Consider $G - \{w, x\}$. Since $i(G - \{w, x\}) = 1$ there exists a vertex $v \in V(G - \{w, x\})$ that dominates $G - \{w, x\}$. Then $vx \in E(G)$ since $x = z$ dominates $G - \{w, y\}$ and $vw \in E(G)$ since w dominates $G - \{x, y\}$. But then we have that v dominates G and $i(G) = 1$, a contradiction. Therefore $N(w) - \{x\} = \emptyset$ and $G \cong K_1 \cup K_2$. □

For any graph G, recall that the vertex set can be partitioned into sets V_i^0, V_i^-, and V_i^+, where

$$\begin{aligned} V_i^0 &= \{v \in V(G) : i(G - v) = i(G)\} \\ V_i^- &= \{v \in V(G) : i(G - v) < i(G)\} \\ V_i^+ &= \{v \in V(G) : i(G - v) > i(G)\}. \end{aligned}$$

Thus if G is i-critical, then $V(G) = V_i^-$. Using Proposition 1.11, it is easy to show that $V_i^+ = \emptyset$ if G is i-bicritical.

Proposition 1.18. *If G is i-bicritical, then either G is i-critical or $G - v$ is i-critical for any $v \in V_i^0$.*

Proof. Suppose that $V_i^+ \neq \emptyset$ and let $v \in V_i^+$. Then $i(G - v) > i(G)$. Let $u \in V(G) - \{v\}$. Then by Proposition 1.11 $i(G - \{u, v\}) = i((G - v) - u) \geq i(G - v) - 1 \geq i(G)$, a contradiction to the fact that G is i-bicritical. Thus $V_i^+ = \emptyset$.

If $V(G) = V_i^-$ then G is i-critical so suppose that $V_i^0 \neq \emptyset$ and let $v \in V_i^0$ such that $G - v$ is not i-critical. Then since $v \in V_i^0$, $i(G - v) = i(G)$. Let $u \in V(G - v)$ so that $i((G - v) - u) \geq i(G - v)$. Then $i(G - \{u, v\}) = i((G - v) - u) \geq i(G - v) = i(G)$, a contradiction to the fact that G is i-bicritical. The result follows. □

Proposition 1.19. [7] *If there exist distinct vertices $u, v \in V(G)$ such that $N[v] \subseteq N[u]$, then G is not i-critical.*

The above proposition immediately yields the following three results.

Proposition 1.20. [7] *If G has a vertex v with $\deg v \geq 1$ such that $\langle N[v] \rangle$ is complete, then G is not i-critical.*

Proposition 1.21. [7] *If G has a vertex v such that $\deg(v) = 1$, then G is not i-critical.*

Corollary 1.22. *No tree is i-critical.*

There are restrictions for the degrees of vertices in an i-bicritical graph as well.

Proposition 1.23. *If G has a vertex v such that $\deg(v) = 2$, then G is not i-bicritical.*

Proof. Suppose G is i-bicritical and let v be a vertex in G with $N(v) = \{x, y\}$. Consider an i-set S of $G - \{x, y\}$. Since v is an isolated vertex in $G - \{x, y\}$, $v \in S$. But then S is an independent dominating set of G with cardinality less than $i(G)$, a contradiction. □

Proposition 1.24. *If G is i-bicritical, then there does not exist $v \in V(G)$ such that $\langle N(v) \rangle$ has $K_{2,m}$ as a spanning subgraph.*

Proof. Suppose G is i-bicritical and let $v \in V(G)$ such that $\langle N(v) \rangle$ has $K_{2,m}$ as a spanning subgraph. Let $\{v_1, v_2\}$ be the vertices in the 2-partite set of $K_{2,m}$ and let D be an i-set of $G - \{v_1, v_2\}$. If any $x \in N[v] - \{v_1, v_2\}$ is also in D, then x would dominate both v_1 and v_2 and D would also be an independent dominating set of G. Thus $D \cap N[v] = \emptyset$, but this means that D does not dominate v, a contradiction. □

Recall that Sumner and Blitch [45] conjectured that $i(G) = \gamma(G)$ for γ-edge-critical graphs and Brigham, Chinn, and Dutton [9] posed the question "does $i(G) = \gamma(G)$ for γ-vertex-critical graphs?". Also recall that Fulman, Hanson, and MacGillivray [21] answered this question with a negative response. Ao [7] also answered the question (Question 3) about γ-edge-critical graphs with a negative response for $\gamma \geq 4$. For $\gamma(G) = 3$ van der Merwe, Mynhardt, and Haynes [46] showed that there exists a connected 3-γ-critical graph G with $i(G) = k$ for each $k \geq 3$. Despite this, there are considerations about criticality to be made when $\gamma(G) = i(G)$.

Proposition 1.25. *If $\gamma(G) = i(G)$, and G is i-critical, then G is γ-critical.*

Proof. For any graph G, $\gamma(G) \leq i(G)$. Let $v \in V(G)$. Thus $\gamma(G - v) \leq i(G - v) < i(G) = \gamma(G)$ and so G is γ-critical. □

Proposition 1.26. *If $\gamma(G) = i(G)$, and G is i-bicritical, then G is γ-bicritical.*

Proof. For any graph G, $\gamma(G) \leq i(G)$. Let $\{x, y\} \subseteq V(G)$. Then $\gamma(G - \{x, y\}) \leq i(G - \{x, y\}) < i(G) = \gamma(G)$. □

Proposition 1.27. *If $x, y \in V(G)$ have a common neighbour then $\gamma(G - \{x, y\}) \geq \gamma(G) - 1$.*

Proof. Let $x, y, z \in V(G)$ such that $xz, yz \in E(G)$. Let D be a γ-set of $G - \{x, y\}$. Then $\{z\} \cup D$ is a dominating set of G. Thus $\gamma(G) \leq \gamma(G - \{x, y\}) + 1$. □

The above proposition gives a result which shows how $\gamma(G)$ and $i(G)$ relate to each other when G is strong i-bicritical.

Proposition 1.28. *If G is strong i-bicritical, then $\gamma(G) < i(G)$.*

Proof. Suppose G is strong i-bicritical with $\gamma(G) = i(G)$ and let $x, y, z \in V(G)$ such that $xz, yz \in E(G)$ and $xy \notin E(G)$. Then $\gamma(G - \{x, y\}) \leq i(G - \{x, y\}) = i(G) - 2 = \gamma(G) - 2$, a contradiction to Proposition 1.27. Thus $\gamma(G) < i(G)$. $\square$

2

Construction Results

When studying any class of graphs, a major goal is to completely characterize the graphs in question. For γ-critical graphs Brigham, Chinn, and Dutton [9] presented a method so that given any graph G, one can construct a γ-critical graph H such that G is an induced subgraph of H. In other words, there is no finite list of forbidden induced subgraphs for γ-critical graphs, which indicates that the characterization problem is quite difficult. A consolation then would be the presentation of many examples of γ-critical graphs. For this, methods of constructing γ-critical graphs come in handy.

Brigham, Chinn, and Dutton [9] looked at constructions for γ-critical graphs. One construction they presented, called the *coalescence*, is very useful and can be adapted for constructing other types of critical graphs such as γ-bicritical, i-critical, i-bicritical, γ_t-critical, and γ_t-bicritical graphs. For γ-critical graphs, this construction combined with the upper bound on the diameter for k-γ-critical graphs [21] can be used to build k-γ-critical graphs of maximum diameter. (This result is presented in detail in Chapter 5.) The coalescence construction is introduced in Subsection 2.2.3.

In her Master's thesis, Ao [7] provided families of i-critical graphs. Many of these also produced γ-critical graphs. Brigham, Haynes, Henning, and Rall [10] also discussed families and constructions. These were used mainly to produce examples of γ-bicritical graphs, but many of the families presented were also γ-critical graphs, and many of the constructions relied on the use of γ-critical graphs.

Constructions for γ_t-critical graphs were investigated by Goddard, Haynes, Henning, and van der Merwe [23] and constructions for γ_t-bicritical graphs were investigated by Jafari Rad [30]. These constructions are studied in Chapter 4.

In this chapter, we extend the results for known constructions, showing both necessary and sufficient conditions for these constructions to produce i-critical graphs. We also present new constructions, showing sufficient and, in some cases, necessary conditions for these constructions to produce i-critical graphs. Constructions for i-bicritical graphs are investigated in Chapter 3, and constructions for γ_t-critical and γ_t-bicritical graphs are investigated in Chapter 4.

2.1 No Forbidden Subgraphs

As mentioned at the start of this chapter, Brigham, Chinn, and Dutton [9] showed that for any graph G there is a γ-critical graph H such that G is an induced subgraph of H. In this section we show similar results for i-critical graphs and i-bicritical graphs.

Let G be any graph with $i(G) \geq 3$. Construct H_1 as follows:

For each $v \in V(G)$, add independent vertices $\{v_1, v_2\}$ and add all edges between $V(G - v)$ and $\{v_1, v_2\}$. Additionally, for all pairs $x, y \in V(G)$ add all edges between $\{x_1, x_2\}$ and $\{y_1, y_2\}$. Notice that $i(H_1) = 3$. In her Master's thesis [7], Ao showed that H_1 is 3-i-critical.

Proposition 2.1. [7] *For any graph G with $i(G) \geq 3$, there exists a 3-i-critical graph*

H such that G is an induced subgraph of H.

Notice that if $i(G) \leq 2$ we can create the graph G' where $V(G') = V(G) \cup \{v_1\}$ if $i(G) = 2$ or $V(G') = V(G) \cup \{v_1, v_2\}$ if $i(G) = 1$ and $E(G') = E(G)$. Then $i(G') = 3$ and the graph H_1 can be constructed from G'. Since G is an induced subgraph of G', G is also an induced subgraph of H_1. Thus the condition $i(G) \geq 3$ can be dropped from the above result.

Proposition 2.2. *For any graph G, there exists a 3-i-critical graph H such that G is an induced subgraph of H.*

We now generalize the construction of H_1 to create k-i-critical graphs with $k \geq 3$.

Let G be any graph. Construct the graph H_j, $j \geq 1$, as follows:

If $i(G) \geq j + 2$, then for each vertex $v \in V(G)$ add independent vertices $\{v_1, v_2, \ldots, v_{j+1}\}$ and add all edges between $V(G-v)$ and $\{v_1, v_2, \ldots, v_{j+1}\}$. Additionally, for all pairs $x, y \in V(G)$ add all edges between $\{x_1, x_2, \ldots, x_{j+1}\}$ and $\{y_1, y_2, \ldots, y_{j+1}\}$. If $i(G) < j+2$, then let G' be the graph with $V(G') = V(G) \cup \{w_1, w_2, \ldots, w_{j+2-i(G)}\}$ and $E(G') = E(G)$. Then $i(G') = j + 2$ and we can construct H_j for G'. Notice that $i(H_j) = j + 2$.

Proposition 2.3. *The graph H_j is i-critical for $j \geq 1$.*

Proof. Consider $z \in V(H_j)$. If $z \in V(G)$, then $\{z_1, z_2, \ldots, z_{j+1}\}$ is an independent dominating set of $H_j - z$. If $z \in \{v_1, v_2, \ldots, v_j\}$ for some $v \in V(G)$, then $\{v\} \cup (\{v_1, v_2, \ldots, v_{j+1}\} - \{z\})$ is an independent dominating set of $H_j - z$. Thus $i(H_j - z) \leq j + 1 < i(H_j)$ and so H_j is i-critical. □

Notice that for $j = 1$ we have the graph H_1 and $i(H_1) = 3$. If for some $v \in V(G)$ we consider $G - \{v_1, v_2\}$ the possible i-sets of $G - \{v_1, v_2\}$ are of the form $\{u, u_1, u_2\}$ where $u \in V(G) - \{v\}$, or S where $S \subseteq V(G) - \{v\}$. If $i(G) \geq 4$ then $|S| \geq 3$ and $i(G - \{v_1, v_2\}) \geq 3$, so we are not guaranteed to have that H_1 is i-bicritical.

Proposition 2.4. *The graph H_j is i-bicritical for $j \geq 2$.*

Proof. Consider $\{x, y\} \subseteq V(H_j)$. If $\{x, y\} \subseteq V(G)$, then $\{x_1, x_2, \ldots, x_{j+1}\}$ is an independent dominating set of $H_j - \{x, y\}$. If $x \in V(G)$ and $y \in \{x_1, x_2, \ldots, x_{j+1}\}$, then $\{x_1, x_2, \ldots, x_{j+1}\} - \{y\}$ is an independent dominating set of $H_j - \{x, y\}$. If $x \in V(G)$ and $y \in \{z_1, z_2, \ldots, z_{j+1}\}$ for some $z \in V(G)$, then $\{x_1, x_2, \ldots, x_{j+1}\}$ is an independent dominating set of $H_j - \{x, y\}$. If $x \in \{u_1, u_2, \ldots, u_{j+1}\}$ for some $u \in V(G)$ and $y \in \{v_1, v_2, \ldots, v_{j+1}\}$ for some $v \in V(G)$, then $\{u\} \cup (\{u_1, u_2, \ldots, u_{j+1}\} - \{x\})$ is an independent dominating set of $H_j - \{x, y\}$. Finally, if $\{x, y\} \subseteq \{v_1, v_2, \ldots, v_{j+1}\}$ for some $v \in V(G)$, then $\{v\} \cup (\{v_1, v_2, \ldots, v_{j+1}\} - \{x, y\})$ is an independent dominating set of $H_j - \{x, y\}$. Hence in all cases, H_j is i-bicritical. □

Corollary 2.5. *For any graph G and for all $k \geq 3$, there exists a k-i-critical graph H such that G is an induced subgraph of H.*

Corollary 2.6. *For any graph G and for all $k \geq 4$, there exists a k-i-bicritical graph H such that G is an induced subgraph of H.*

The 2-i-critical graphs are characterized in Proposition 1.16, so for $k = 2$ we know exactly what the k-i-critical graphs look like. However, for $k \geq 3$ there is no characterization of k-i-critical graphs through a finite list of forbidden subgraphs. Likewise, the 2-i-bicritical graphs are characterized in Proposition 1.17 and so we exactly know the k-i-bicritical graphs for $k = 2$ but for $k \geq 4$ there is no characterization of k-i-bicritical graphs through a finite list of forbidden subgraphs. The structure of 3-i-bicritical graphs is unknown.

From the proof of Proposition 2.4 we can see that H_j may not be strong i-bicritical. Thus we provide the following construction.

Let G be any graph and let the maximal independent sets of G be I_1, I_2, …, I_k. Construct the graph H_j^I as follows:

For each I_l, $1 \leq l \leq k$, add the set of $\alpha(G) + 1 - |I_l|$ independent vertices $V_l = \{v_1, v_2, \ldots, v_{\alpha(G)+j-|I_l|}\}$ and add all edges between V_l and $V(G) - I_l$. For all pairs V_{l_1} and V_{l_2} add all edges between V_{l_1} and V_{l_2}.

Proposition 2.7. *For any G, $i(H_j^I) = \alpha(G) + j$, $j \geq 1$.*

Proof. If $x \in V(G)$, then x is only independent to vertices of V_l if $x \in I_l$. But $|V_l| + |I_l| = \alpha(G) + j$ for all $1 \leq l \leq k$. If $x \in V_l$, then x is only independent to vertices of V_l and vertices of I_l. But again $|V_l| + |I_l| = \alpha(G) + j$ for all $1 \leq l \leq k$. As these are the only maximal independent sets of H_j^I, $i(H_j^I) = \alpha(G) + j$. □

Proposition 2.8. *The graph H_j^I is i-critical for any G and all $j \geq 1$.*

Proof. Let $v \in H_j^I$. If $x \in V(G)$, suppose without loss of generality that $x \in I_1$. Then $(I_1 - \{x\}) \cup V_1$ dominates $H_j^I - x$ since $V_1 \neq \emptyset$ and V_1 dominates V_l for $2 \leq l \leq k$, V_1 dominates $N_G(x)$ and $I_1 - \{x\}$ dominates $G - N[x]$. But $|(I_1 - \{x\}) \cup V_1| = \alpha(G) + j - 1 = i(H_j^I) - 1$. Suppose without loss of generality that $x \in V_1$. Then I_1 dominates G. If $V_1 - \{x\} \neq \emptyset$, then $V_1 - \{x\}$ dominates V_l for $2 \leq l \leq k$. If $V_1 - \{x\} = \emptyset$, then for each V_l, $2 \leq l \leq k$, there exists a $z_l \in I_1$ such that z_l dominates V_l (since $I_1 \neq I_l$ for all $2 \leq l \leq k$). But $|I_1 \cup (V_1 - \{x\})| = \alpha(G) + j - 1 = i(H_j^I) - 1$. Thus H_j^I is i-critical. □

Proposition 2.9. *The graph H_j^I is i-bicritical for any G and all $j \geq 2$.*

Proof. Let $\{x, y\} \subseteq V(H_j^I)$.

Case 1: $\{x, y\} \subseteq V_1$

Then I_1 dominates G and for for each V_l, $2 \leq l \leq k$, there exists a $z_l \in I_1$ such that z_l dominates V_l (since $I_1 \neq I_l$ for all $2 \leq l \leq k$). Therefore $I_1 \cup (V_1 - \{x, y\})$ dominates $H_j^I - \{x, y\}$ and $|I_1 \cup (V_1 - \{x, y\})| = \alpha(G) + j - 2 = i(H_j^I) - 2$.

Case 2: $\{x, y\} \subseteq I_1$

Then $I_1 - \{x, y\}$ dominates $G - (N[x] \cup N[y])$, V_1 dominates $N(x) \cup N(y)$, and V_1 dominates V_l for all $2 \leq l \leq k$. Therefore $(I_1 - \{x, y\}) \cup V_1$ dominates $H_j^I - \{x, y\}$ and $|(I_1 - \{x, y\}) \cup V_1| = \alpha(G) + j - 2 = i(H_j^I) - 2$.

Case 3: $x \in I_1$ and $y \in V_1$

Then $I_1 - \{x\}$ dominates $G - N[x]$, $V_1 - \{y\} \neq \emptyset$ and so $V_1 - \{y\}$ dominates $N_G(x)$ and V_l for all $2 \leq l \leq k$. Therefore $(I_1 - \{x\}) \cup (V_1 - \{y\})$ dominates $H_j^I - \{x, y\}$ and $|(I_1 - \{x\}) \cup (V_1 - \{y\})| = \alpha G + j - 2 = i(H_j^I) - 2$.

Case 4: $x \in I_1$ and $y \in I_2$ but $xy \in E(G)$

Then $I_1 - \{x\}$ dominates $G - N[x]$ (and $y \notin I_1$), and V_1 dominates $N(x)$ and V_l for all $2 \leq l \leq k$. Therefore $(I_1 - \{x\}) \cup V_1$ dominates $H_j^I - \{x, y\}$ and $|(I_1 - \{x\}) \cup V_1| = \alpha(G) + j - 1 = i(H_j^I) - 1$.

Case 5: $x \in V_1$ and $y \in I_2$ but $y \notin I_1$

Then $I_2 - \{y\}$ dominates $G - N[y]$, and V_2 dominates $N_G(y)$, $V_1 - \{x\}$, and V_l for all $3 \leq l \leq k$. Therefore $(I_2 - \{y\}) \cup V_2$ dominates $H_j^I - \{x, y\}$ and $|(I_2 - \{y\}) \cup V_2| = \alpha(G) + j - 1 = i(H_j^I) - 1$.

Case 6: $x \in V_1$ and $y \in V_2$

Then I_1 dominates G and $V_1 - \{x\}$ dominates V_l for all $2 \leq l \leq k$ since $V_1 - \{x\} \neq \emptyset$. Therefore $I_1 \cup (V_1 - \{x\})$ dominates $H_j^I - \{x, y\}$ and $|I_1 \cup (V_1 - \{x\})| = \alpha(G) + j - 1 = i(H_j^I) - 1$.

Hence in all cases $i(H_j^I - \{x, y\}) < i(H_j^I)$, and thus H_j^I is i-bicritical. □

Corollary 2.10. *The graph H_j^I is strong i-bicritical for any G and all $j \geq 2$.*

Proof. The result follows from the proof of Proposition 2.9 since the only ways that $\{x, y\} \subseteq V(H_j^I)$ with $xy \notin E(H_j^I)$ are Cases 1, 2, and 3. □

Corollary 2.11. *For any graph G and for all $k \geq \alpha(G)+2$, there exists a strong k-i-bicritical graph H such that G is an induced subgraph of H.*

The condition $k \geq \alpha(G)+2$ in the above corollary can be removed through another construction.

Let G be any graph. Construct the graph H_j', $j \geq 5$, as follows:

If $i(G) \geq j$, then for each $x_1x_2 \notin E(G)$ add independent vertices $\{v_3, v_4, \ldots, v_j\}$ and add all edges between $V(G - \{x_1, x_2\})$ and $\{v_3, v_4, \ldots, v_j\}$. Additionally, for all $x_1x_2 \notin E(G)$ and $y_1y_2 \notin E(G)$ add all edges between $\{x_3, x_4, \ldots, x_j\}$ and $\{y_3, y_4, \ldots, y_j\}$. If $i(G) < j$, then let G' be the graph with vertex set $V(G') = V(G) \cup \{w_1, w_2, \ldots, w_{j-i(G)}\}$ and edge set $E(G') = E(G)$. Then $i(G') = j$ and we can construct H_j' for G'. Notice that $i(H_j') = j$.

Proposition 2.12. *The graph H_j' is strong i-bicritical for any G and all $j \geq 5$.*

Proof. Let $\{x, y\} \subseteq V(H_j')$ such that $xy \notin E(H_j')$.

If $\{x, y\} = \{x_1, x_2\} \subseteq V(G)$ then $\{x_3, x_4, \ldots, x_j\}$ is an independent dominating set of $H_j' - \{x, y\}$. If, without loss of generality, $x \in V(G)$ and $y \notin V(G)$ then there exists $x_2 \in V(G)$ such that $xx_2 \notin E(G)$ and $y \in \{x_3, x_4, \ldots, x_j\}$. Then $\{x_2, x_3, x_4, \ldots, x_j\} - \{y\}$ is an independent dominating set of $H_j' - \{x, y\}$. If $\{x, y\} \subseteq V(H - G)$, then there exists $x_1, x_2 \in V(G)$ with $x_1x_2 \notin E(G)$ such that $\{x, y\} \subseteq \{x_3, x_4, \ldots, x_j\}$. Then $\{x_1, x_2, x_3, x_4, \ldots, x_j\} - \{x, y\}$ is an independent dominating set of $H_j' - \{x, y\}$. Thus H_j' is strong i-bicritical. □

Corollary 2.13. *For any graph G and for all $k \geq 5$ there exists a strong k-i-bicritical graph H such that G is an induced subgraph of H.*

2.2 Constructions

We now investigate various construction techniques that yield i-critical graphs. Constructions for i-bicritical graphs and strong i-bicritical graphs can be found in Chapter 3, and constructions for γ_t-critical graphs and γ_t-bicritical graphs can be found in Chapter 4.

2.2.1 Disjoint Union

Let G and H be graphs with $V(G) \cap V(H) = \emptyset$. The *disjoint union of G and H*, written $G \cup H$, is the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. The graph $G_1 \cup G_2 \cup \cdots \cup G_k$ is defined recursively by $G_1 \cup G_2 \cup \cdots \cup G_k = (G_1 \cup \cdots \cup G_{k-1}) \cup G_k$. Note that $i(G_1 \cup G_2 \cup \cdots \cup G_k) = \sum_{j=1}^{k} i(G_j)$.

Proposition 2.14. *The graph $G_1 \cup G_2 \cup \cdots \cup G_k$ is i-critical if and only if all of G_1, G_2, ..., G_k are i-critical.*

Proof. If G_1, G_2, ..., G_k are all i-critical, then for some $v \in V(G_j)$, $1 \leq j \leq k$, $(G_1 \cup \cdots \cup G_k) - v \cong G_1 \cup \cdots \cup G_j - v \cup \cdots \cup G_k$. Thus $i((G_1 \cup \cdots \cup G_k) - v) = i(G_1) + \cdots + i(G_j) - 1 + \cdots + i(G_k) < i(G_1 \cup \cdots \cup G_k)$ and so $G_1 \cup \cdots \cup G_k$ is i-critical.

Suppose for the converse that some G_j, $1 \leq j \leq k$, is not i-critical. Let $v \in V(G_j)$ be a vertex such that $i(G_j - v) = i(G_j)$ and consider the graph $(G_1 \cup \cdots \cup G_k) - v$. Since $(G_1 \cup \cdots \cup G_k) - v \cong G_1 \cup \cdots \cup G_j - v \cup \cdots \cup G_k$, we have that $i((G_1 \cup \cdots \cup G_k) - v) = i(G_1) + \cdots + i(G_j - v) + \cdots + i(G_k) = i(G_1) + \cdots + i(G_j) + \cdots + i(G_k) = i(G_1 \cup \cdots \cup G_k)$ and thus $G_1 \cup \cdots \cup G_k$ is not i-critical, a contradiction. □

2.2.2 Join

The *join of G and H*, written $G + H$, is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

The graph $G_1 + G_2 + \cdots + G_k$ is defined recursively by $(G_1 + \cdots + G_{k-1}) + G_k$. Note that $i(G_1 + G_2 + \cdots + G_k) = \min\{i(G_1), i(G_2), \ldots, i(G_k)\}$. The graph $K_{3,3} + K_{3,3}$ is pictured in Figure 2.1.

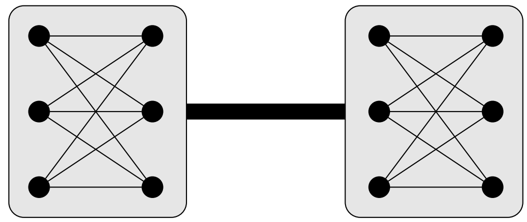

Figure 2.1: The graph $K_{3,3} + K_{3,3}$.

Ao introduced the join construction for i-critical graphs [7]. Here she showed that if $G_1, G_2, \ldots, G_k$ are i-critical and $i(G_1) = i(G_2) = \cdots = i(G_k)$, then $G_1 + G_2 + \cdots G_k$ is i-critical. The converse is shown below.

Proposition 2.15. *The graph $G = G_1 + G_2 + \cdots + G_k$ is i-critical if and only if all of G_1, G_2, …, G_k are i-critical and $i(G_1) = i(G_2) = \cdots = i(G_k)$.*

Proof. Let $G = G_1 + G_2 + \cdots + G_k$.

Suppose without loss of generality that G is i-critical but G_1 is not i-critical and let $v \in V(G_1)$ such that $i(G_1 - v) \geq i(G_1)$. Let D be an i-set of $G - v$. Then $D \cap V(G_j) \neq \emptyset$ for only one j, $1 \leq j \leq k$. If $D \cap V(G_1) \neq \emptyset$, then $i(G - v) = i(G_1 - v) \geq i(G_1) \geq i(G)$. If $D \cap V(G_j) \neq \emptyset$ for $j \neq 1$, then $i(G - v) = i(G_j) \geq i(G)$. In either case we have a contradiction. Hence we can conclude that all of G_1, G_2, …, G_k are i-critical.

For the second part of the statement, suppose without loss of generality that $i(G_1) > i(G_j)$ for all j, $2 \leq j \leq k$, and let $v \in V(G_1)$. Let D be an i-set of $G - v$. Again, $D \cap V(G_j) \neq \emptyset$ for only one j, $1 \leq j \leq k$. If $D \cap V(G_1) \neq \emptyset$, then $i(G - v) = i(G_1 - v) = i(G_1) - 1 \geq i(G)$. If $D \cap V(G_j) \neq \emptyset$ for $j \neq 1$, then $i(G - v) = i(G_j) \geq i(G)$. In either case we have a contradiction to G being i-critical, and we can conclude that $i(G_1) = i(G_2) = \cdots = i(G_k)$.

For the converse, suppose that $i(G_1) = i(G_2) = \cdots = i(G_k)$ and all of G_1, G_2, …, G_k are i-critical. Consider $v \in V(G)$ and without loss of generality suppose that $v \in V(G_1)$. Let D be an i-set of $G_1 - v$. Then D dominates $G_1 - v$ and by construction D dominates G_2, G_3, …, G_k. Thus D is an independent dominating set of $G - v$ and so $i(G - v) \leq |D| = i(G_1 - v) = i(G_1) - 1 < i(G_1) = i(G)$. Therefore G is i-critical. □

Notice that $\gamma(G_1 + G_2 + \cdots + G_k) = \gamma_t(G_1 + G_2 + \cdots + G_k) = 2$ and so the join construction is of no use to construct γ-critical, γ-bicritical, γ_t-critical, and γ_t-bicritical graphs.

2.2.3 Coalescence

Let G and H be disjoint graphs with $x \in V(G)$ and $y \in V(H)$. The *coalescence of G and H with respect to x and y* is the graph $G \cdot_{xy} H$ with vertex set $V(G \cdot_{xy} H) = (V(G) - \{x\}) \cup (V(H) - \{y\}) \cup \{v\}$, where $v \notin V(G) \cup V(H)$, and edge set $E(G \cdot_{xy} H) = E(G - x) \cup E(H - y) \cup \{vw : xw \in E(G) \text{ or } yw \in E(H)\}$. We call v the *vertex of identification* of G and H, and we consider $V(G)$ and $V(H)$ as subsets of $V(G \cdot_{xy} H)$ and regard v as an element of both $V(G)$ and $V(H)$. Informally, $G \cdot_{xy} H$ is the graph obtained from $G \cup H$ by identifying x and y. If the context is clear, or if the vertices x and y are not important, we write $G \cdot H$ instead of $G \cdot_{xy} H$. The graph $G_1 \cdot G_2 \cdot \cdots \cdot G_k$ is defined recursively by $G_1 \cdot G_2 \cdot \cdots \cdot G_k = (G_1 \cdot G_2 \cdot \cdots \cdot G_{k-1}) \cdot G_k$. This construction which first appeared in [9] and [21] was found to be useful in building γ-critical graphs with maximum diameter [21]. The graphs $C_4 \cdot C_4$ and an example of $C_4 \cdot C_4 \cdot C_4 \cdot C_4 \cdot C_4$ are pictured in Figure 2.2 and Figure 2.3, respectively. (Note that there are other possible configurations of $C_4 \cdot C_4 \cdot C_4 \cdot C_4 \cdot C_4$.)

Ao looked at the coalescence construction for i-critical graphs in her Master's thesis [7].

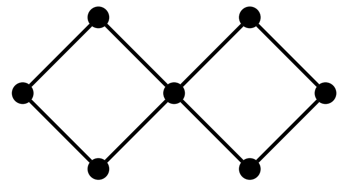

Figure 2.2: The graph $C_4 \cdot C_4$.

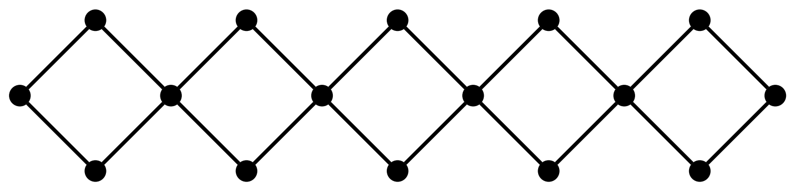

Figure 2.3: The graph $C_4 \cdot C_4 \cdot C_4 \cdot C_4 \cdot C_4$.

Proposition 2.16. [7] *If G and H are disjoint nontrivial graphs, then for any coalescence $G \cdot H$, $i(G) + i(H) - 1 \leq i(G \cdot H) \leq \min\{i(G) + \alpha(H), i(H) + \alpha(G)\}$, where $\alpha(G)$ is the independence number of G.*

Theorem 2.17. [7] *If G and H are i-critical, then $i(G \cdot H) = i(G) + i(H) - 1$.*

Theorem 2.18. [7] *If $G \cdot H$ is i-critical, then $i(G \cdot H) = i(G) + i(H) - 1$.*

Theorem 2.19. [7], [18] *The graph $G \cdot H$ is i-critical if and only if both G and H are i-critical. Furthermore, $i(G \cdot H) = i(G) + i(H) - 1$ if $G \cdot H$ is i-critical.*

A straightforward proof by induction yields the following result using Theorem 2.19 as the base case.

Proposition 2.20. [7] *The graph G is i-critical if and only if every block of G is i-critical. Furthermore, if the blocks of G are labelled $G_1, G_2, \ldots, G_k$, then $i(G) = \left(\sum_{j=1}^{k} i(G_j)\right) - (k-1)$.*

This construction is revisited in Chapter 3 where it is used to construct i-bicritical graphs. It is also shown in Proposition 3.49 that this construction cannot be used to construct strong i-bicritical graphs. Chapter 4 uses this construction to create

γ_t-critical graphs and γ_t-bicritical graphs. In Chapter 5 the coalescence is used to provide examples of k-i-critical graphs that obtain the maximum diameter.

2.2.4 Generalized Coalescence

Let G_1, G_2, and H be disjoint graphs such that for $j = 1, 2$, G_j has a subgraph $H_j \cong H$. The *generalized coalescence of G_1 and G_2 with respect to H_1 and H_2* is the graph $G_1(H_1) \odot G_2(H_2)$ created by identifying the vertices of H_1 with the corresponding vertices in H_2. Notice that $G_1(H_1) \odot G_2(H_2)$ is a generalization of the coalescence as $G_1(\{x\}) \odot G_2(\{y\}) \cong G_1 \cdot_{xy} G_2$.

Little is known about this generalized coalescence for the purpose of constructing critical and bicritical graphs. Note that G is chordal if and only if G is complete or $G = G_1(H_1) \odot G_2(H_2)$ where $H_1 \cong H_2 \cong K_n$ for some n, and G_i, $i = 1, 2$, is chordal. Proposition 3.49 shows that $G_1(\{x\}) \odot G_2(\{y\})$ is not a valid construction to produce strong i-bicritical graphs. Proposition 3.52 shows that $G_1(H_1) \odot G_2(H_2)$ is not a valid construction to produce strong i-bicritical graphs when $H_1 \cong H_2 \cong \overline{K}_2$. Proposition 3.57 provides an example when $G_1(H_1) \odot G_2(H_2)$ is strong i-bicritical (and also i-bicritical and i-critical) where $H_1 \cong H_2 \cong K_2$.

2.2.5 Joined Coalescence

Let G_1, G_2, and H be disjoint graphs such that for $j = 1, 2$, G_j has a subgraph $H_j \cong H$. We define the *joined coalescence of G_1 and G_2 with respect to H_1 and H_2* to be the graph $G_1(H_1)\widehat{\odot}G_2(H_2)$ obtained from G_1 and G_2 by identifying vertices of H_1 with the corresponding vertices of H_2 and adding the set of edges $\{x_1x_2 : x_1 \in V(G_1) - V(H_1) \text{ and } x_2 \in V(G_2) - V(H_2)\}$. Notice that for any i-set S of $G_1(H_1)\widehat{\odot}G_2(H_2)$, $S \subseteq V(G_i)$ for exactly one i. Thus $i(G_1(H_1)\widehat{\odot}G_2(H_2)) = \min\{i(G_1), i(G_2)\}$. The joined coalescence was introduced by Ao [7]. She presented sufficient conditions for $G_1(H_1)\widehat{\odot}G_2(H_2)$ to be i-critical. The graph $K_{3,3}(K_{2,2})\widehat{\odot}K_{3,3}(K_{2,2})$ is pictured in

Figure 2.4.

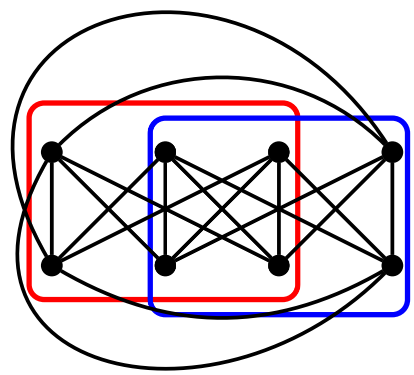

Figure 2.4: The graph $K_{3,3}(K_{2,2})\widehat{\odot}K_{3,3}(K_{2,2})$.

Let $\alpha(G)$ denote the independence number of G.

Proposition 2.21. [7] *Let G_1, G_2, and H be disjoint graphs such that for $j = 1, 2$, G_j has a subgraph $H_j \cong H$. If G_1 and G_2 are k-i-critical and $\alpha(H) \leq k - 2$, then $G_1(H_1)\widehat{\odot}G_2(H_2)$ is also k-i-critical.*

This construction can be generalized to combine more than two graphs. Let $H_{1,2}$ be a subgraph of $G_1(H_1)\widehat{\odot}G_2(H_2)$ and let H_3 be a subgraph of G_3 where $H_{1,2} \cong H_3$. The graph $(G_1(H_1)\widehat{\odot}G_2(H_2))(H_{1,2})\widehat{\odot}G_3(H_3)$ is obtained by identifying vertices of $H_{1,2}$ with corresponding vertices of H_3 and adding edges $\{x_{1,2}x_3 : x_{1,2} \in V(G_1(H_1)\widehat{\odot}G_2(H_2) - H_{1,2})$ and $x_3 \in V(G_3 - H_3)\}$. This can be generalized similarly for more than three graphs as the graph

$$G_{\widehat{\odot}} = ((((G_1(H_1)\widehat{\odot}G_2(H_2))(H_{1,2}))\widehat{\odot}G_4(H_4))\cdots\widehat{\odot}G_{m-1}(H_{m-1}))(H_{1,2,\ldots,m-1})\widehat{\odot}G_m(H_m).$$

Note that this construction is associative.

Proposition 2.22. *For each $H \in \{H_1, H_2, \ldots, H_m, H_{1,2}, H_{1,2,3}, \ldots, H_{1,2,\ldots,m-1}\}$, suppose $\alpha(H) \leq k-2$. Then*

$$G_{\widehat{\odot}} = (((G_1(H_1)\widehat{\odot}G_2(H_2))(H_{1,2})\widehat{\odot}G_4(H_4)) \cdots \widehat{\odot}G_{m-1}(H_{m-1}))(H_{1,2,\ldots,m-1})\widehat{\odot}G_m(H_m)$$

is k-i-critical if and only if $k = \min\{i(G_1), i(G_2), \ldots, i(G_m)\}$ and every vertex x in $G_{\widehat{\odot}}$ is in some $V(G_j)$ where $i(G_j - x) = k-1$.

Proof. Suppose $k = \min\{i(G_1), i(G_2), \ldots, i(G_m)\}$ and every vertex x in $G_{\widehat{\odot}}$ is in some $V(G_j)$ where $i(G_j - x) = k-1$. Since $k = \min\{i(G_1), i(G_2), \ldots, i(G_m)\}$, $i(G_{\widehat{\odot}}) = k$. Let D be an i-set of $G_j - x$, thus $|D| = k-1$. Since $\alpha(H) \leq k-2$, there is a vertex $y \in V(G_j - H_j)$ such that $y \in D$. Therefore D independently dominates $G - x$ and so $i(G_{\widehat{\odot}} - x) = k - 1 < i(G_{\widehat{\odot}})$ and $G_{\widehat{\odot}}$ is k-i-critical.

Suppose $G_{\widehat{\odot}}$ is k-i-critical. Then by construction, $k = \min\{i(G_1), i(G_2), \ldots, i(G_m)\}$. Consider $G_{\widehat{\odot}} - x$. Say $x \in V(G_j)$, $1 \leq j \leq m$. Let D be an i-set of $G_{\widehat{\odot}} - x$, and so $|D| = k-1$. By construction of $G_{\widehat{\odot}}$, $D \subseteq V(G_l)$ for some $1 \leq l \leq m$. If $x \notin V(G_l)$ then D dominates G, a contradiction. Thus $x \in V(G_l)$ and D dominates $G_l - x$, and so $i(G_l - x) \leq k-1$. Therefore $i(G_l) = k$, and $i(G_l - x) = k-1$. □

There is a simpler version of this construction where each contributing graph $G_1, G_2, \ldots, G_m$ has a subgraph H_j, $1 \leq j \leq m$ where $H_j \cong H$ and the constructed graph is obtained by identifying corresponding vertices of $H_1, H_2, \ldots, H_m$ with each other and adding the edges $\{x_j x_l : x_j \in V(G_j - H_j) \text{ and } x_l \in V(G_l - H_l), j \neq l, 1 \leq j, l, \leq m\}$. In this case, we denote the constructed graph more simply by $G_1(H_1)\widehat{\odot}G_2(H_2)\widehat{\odot} \cdots \widehat{\odot}G_m(H_m)$.

Corollary 2.23. *Let $G_1, G_2, \ldots, G_m$ be graphs such that each G_j has a subgraph $H_j \cong H$, $1 \leq j \leq m$, and $\alpha(H) \leq k-2$. Then $G = G_1(H_1)\widehat{\odot}G_2(H_2)\widehat{\odot} \cdots \widehat{\odot}G_m(H_m)$ is k-i-critical if and only if $k = i(G_1) = i(G_2) = \cdots = i(G_m)$ and every vertex*

$x \in V(G_j - H_j)$ is i-critical in G_j and every vertex $x \in H$ is i-critical in some G_j, $1 \leq j \leq m$.

2.2.6 Wreath Product

The *wreath product of G with H*, written $G[H]$, is the graph with vertex set $V(G[H]) = \{(g,h) : g \in V(G), h \in V(H)\}$ and edge set $E(G[H]) = \{(g_1,h_1)(g_2,h_2) : g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)\}$. Notice that, in general, $G[H] \not\cong H[G]$. For example, $\overline{K}_2[C_3] \cong C_3 \cup C_3$, a disconnected graph, but $C_3[\overline{K}_2] \cong K_{2,2,2}$, a connected graph. The graph $G[H]$ can be thought of as the graph obtained by replacing each vertex of G by a copy of H and adding all edges between two copies of H if and only if the corresponding vertices in G are adjacent. The graph $C_5[C_4]$ is pictured in Figure 2.5.

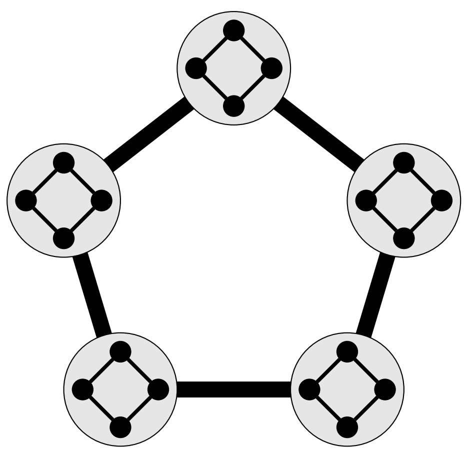

Figure 2.5: The graph $C_5[C_4]$.

Proposition 2.24. *For any graphs G and H, $i(G[H]) = i(G)i(H)$.*

Proof. Let D be an i-set of $G[H]$ and let $S_1 = \{g : (g,h) \in D$ for some $h \in V(H)\}$. Since D is an independent dominating set of $G[H]$, S_1 is an independent dominating set of G. Thus $|S_1| \geq i(G)$. For a fixed $g \in V(G)$, let $S_g = \{h : (g,h) \in D\}$. Since D is an independent dominating set of $G[H]$, either $S_g = \emptyset$ or S_g is an independent dominating set of H. Thus $|S_g| = 0$ or $|S_g| \geq i(H)$. But this implies that $|D| \geq i(G)i(H)$.

Let S_1 be an i-set of G and let S_2 be an i-set of H. Let $D = \{(g, h) : g \in S_1 \text{ and } h \in S_2\}$. Then D is an independent dominating set of $G[H]$ and $|D| = i(G)i(H)$. Therefore $i(G[H]) = i(G)i(H)$. □

Proposition 2.25. *The graph $G[H]$ is i-critical if and only if every vertex of G is in an i-set of G and H is i-critical.*

Proof. Let $v = (g, h) \in V(G[H])$. Let S_1 be an i-set of G containing g, let S_g be an i-set of $H - h$, and let S_2 be an i-set of H. Then $D = \{(g, u) : u \in S_g\} \cup \{(x, y) : x \neq g, x \in S_1, y \in S_2\}$ is an independent dominating set of $G[H] - v$, and $i(G[H]) \leq |D| = i(H) - 1 + (i(G) - 1)(i(H)) = i(G)i(H) - 1 = i(G[H]) - 1$. Thus $G[H]$ is i-critical.

Suppose $G[H]$ is i-critical and let $(g, h) \in V(G[H])$. Let S be an i-set of $G[H]$ such that $(g, h) \in S$. Let $S' = \{u \in V(G) : \exists v \in V(H) \text{ with (u, v)} \in \text{S}\}$. Then S' is an i-set of G with $g \in S'$. Let $S_g = \{v \in V(H) : (g, v) \in S\}$. Then S_g is an i-set of H and $S_g - \{h\}$ is an independent dominating set of $H - h$ with cardinality $i(H) - 1$. Therefore H is i-critical. □

The wreath product can be generalized to say that instead of replacing each vertex of G by a copy of H, we replace the vertices of G by copies of different graphs. Formally, let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let H_1, H_2, $\ldots$, H_n be graphs which are pairwise vertex disjoint. We define the graph $G[H_1, H_2, \ldots, H_n]$ as the graph with vertex set $V(H_1) \cup V(H_2) \cup \cdots \cup V(H_n)$ and edge set $E(H_1) \cup E(H_2) \cup \cdots \cup E(H_n) \cup \{h_i h_j : h_i \in V(H_i), h_j \in V(H_j), \text{ and } v_i v_j \in E(G)\}$. The following result can be obtained through a proof similar to that of Proposition 2.24 and Proposition 2.25.

Proposition 2.26. *Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The graph $G[H_1, H_2, \ldots, H_n]$ is i-critical if and only if every vertex of G is in an i-set of G, each of $H_1, H_2, \ldots, H_n$ is i-critical, and for any two index sets $J, K \subseteq \{1, 2, \ldots, n\}$ such that the sets $S_1 = \{v_j : j \in J\}$ and $S_2 = \{v_k : k \in K\}$ are i-sets of G we have that $\sum_{j\in J} i(H_j) = \sum_{k\in K} i(H_k)$. Note that $i(G[H_1, H_2, \ldots, H_n]) = \sum_{j\in J} i(H_j)$ where J is an index set $J \subseteq \{1, 2, \ldots, n\}$ such that $S = \{v_j : j \in J\}$ is an i-set of G.*

The next result investigates the validity of using the wreath product to construct γ-critical graphs.

Proposition 2.27. *For any graphs G and H, $\gamma(G[H]) = \gamma(G)$ if $\gamma(H) = 1$ and $\gamma(G[H]) = \gamma_t(G)$ if $\gamma(H) \geq 2$.*

Proof. Suppose $\gamma(H) = 1$ and let $v \in V(H)$ be a dominating vertex in H. Let S be a γ-set of G. Then $D = \{(x, v) : x \in S\}$ is a dominating set of $G[H]$. Therefore $\gamma(G[H]) \leq |D| = \gamma(G)$. Now let D be a γ-set of $G[H]$ and let $S = \{g \in V(G) : \exists h \in V(H) \text{ with } (g, h) \in D\}$. Since D is a dominating set of $G[H]$, S is a dominating set of G and so $\gamma(G) \leq \gamma(G[H])$. Thus $\gamma(G[H]) = \gamma(G)$ if $\gamma(H) = 1$.

Suppose $\gamma(H) \geq 2$. Let S be a γ_t-set of G and let $v \in H$. Then $D = \{(x, v) : x \in S\}$ is a dominating set of $G[H]$ and so $\gamma(G[H]) \leq \gamma_t(G)$. Now let D be a γ-set of $G[H]$ and let $(g, h) \in D$. If $(x, y) \notin D$ for every $x \in N_G(g)$ and every $y \in V(H)$, then $|D \cap \{(g, v) : v \in V(H)\}| \geq \gamma(H) \geq 2$. Thus for every vertex $g \in V(G)$, either $|D \cap \{(g, v) : v \in V(H)\}| \geq 2$ or there exists an $x \in N_G(g)$ and $y \in V(H)$ such that $(x, y) \in D$. Let I be the set of vertices $I = \{g \in V(G) : (g, h) \in D$ and there does not exist $x \in N_G(g)$ with $(x, y) \in D$ for some $y \in V(H)$. Let $h \in V(H)$ and for each $x \in I$ let g_x be any vertex in $N_G(x)$. Then let $D_1 = D - \{(x, y) : x \in I\} \cup \{(x, h) : x \in I\} \cup \{(g_x, h) : x \in I\}$. Then D_1 is a total dominating set of $G[H]$. Let $S_1 = \{u \in V(G) : \exists\, v \in V(H) \text{ with } (u, v) \in D_1\}$. Then D_1 is a total dominating set of G and so $\gamma_t(G) \leq |S_1| \leq |D_1| \leq \gamma(G[H])$. Thus $\gamma(G[H]) = \gamma_t(G)$.

□

Proposition 2.28. *For any graphs G and H, $\gamma_t(G[H]) = \gamma_t(G)$.*

Proof. If $|V(H)| = 1$, then $G[H] \cong G$ and so clearly $\gamma_t(G[H]) = \gamma_t(G)$. Thus suppose that $|V(H)| \geq 2$.

Consider a γ_t-set S of G. For a fixed $h \in V(H)$, let $D = \{(g, h) : g \in S\}$. Then D is a total dominating set of $G[H]$ and so $\gamma_t(G[H]) \leq \gamma_t(G)$.

Now consider a γ_t-set S of $G[H]$ and let $(g, h) \in S$. Either $(x, y) \notin S$ for every $x \in N_G(g)$ and every $y \in V(H)$ and so $|S \cap \{(g, v) : v \in V(H)\}| \geq 2$, or there exists an $x \in N_G(g)$ and $y \in V(H)$ such that $(x, y) \in S$. Let I be the set of vertices $g \in V(G)$ such that $(g, h) \in S$ for some $h \in V(H)$ and there is no $x \in N_G(g)$ with $(x, y) \in S$ for some $y \in V(H)$. Let $h \in V(H)$ and for each $x \in I$ let g_x be any vertex in $N_G(x)$. Then let $S_1 = S - \{(x, y) : x \in I\} \cup \{(x, h) : x \in I\} \cup \{(g_x, h) : x \in I\}$. Then S_1 is a total dominating set of $G[H]$ and $|S_1| \leq |S|$. Let $D = \{u \in V(G) : \exists v \in V(H) \text{ with } (u, v) \in S_1\}$. Then D is a total dominating set of G and so $\gamma_t(G) \leq |D| = |S_1| \leq |S| = \gamma_t(G[H])$. Therefore $\gamma_t(G) = \gamma_t(G[H])$. □

Proposition 2.29. *If $\gamma(H) \geq 3$, then $G[H]$ is not γ-critical.*

Proof. Let $h \in V(H)$ and $g \in V(G)$. Since $\gamma(H) \geq 3$, we have that $\gamma(H - h) \geq \gamma(H) - 1 \geq 2$. But then by the proof of Proposition 2.27, we have that $\gamma(G[H] - (g, h)) = \gamma_t(G) = \gamma(G[H])$ and so $G[H]$ is not γ-critical. □

Corollary 2.30. *If $G[H]$ is γ-critical, then $\gamma(H) = 1$ or $\gamma(H) = 2$.*

Proposition 2.31. *If $G[H]$ is γ-critical, then H is γ-critical.*

Proof. Suppose H is not γ-critical and let $h \in V(H)$ such that $\gamma(H - h) \geq \gamma(H)$. Consider any $g \in V(G)$. If $\gamma(H) \geq 2$, then by the proof of Proposition 2.27, $\gamma(G[H] - (g, h)) = \gamma_t(G[H]) = \gamma(G[H])$. If $\gamma(H) = 1$, then $\gamma(G[H] - (g, h)) = \gamma(G) = \gamma(G[H])$. In either case, $G[H]$ is not γ-critical. □

Corollary 2.32. *If $G[H]$ is γ-critical, then $\gamma(H) = 2$ or $H \cong K_1$.*

Proposition 2.33. *If $\gamma(H) = 2$ and $G[H]$ is γ-critical, then $\gamma_t(G - N[x]) = \gamma_t(G) - 2$ for every $x \in V(G)$.*

Proof. Suppose $G[H]$ is γ-critical and $\gamma(H) = 2$, and let $V(G) = \{g_1, g_2, \ldots, g_r\}$ and $V(H) = \{h_1, h_2, \ldots, h_s\}$. Consider $G[H] - (g_k, h_k)$ for some $g_k \in V(G)$ and some $h_k \in V(H)$. Let S be a γ-set of $G[H] - (g_k, h_k)$. Then $|S| = \gamma(G[H] - (g_k, h_k)) = \gamma(G[H]) - 1 = \gamma_t(G) - 1$. If there is a vertex $(x, y) \in S$ with $xg_k \in E(G)$ then S is also a dominating set of $G[H]$, a contradiction to the criticality of $G[H]$. Therefore, there is a vertex $(g_k, h_l) \in S$ for some $l \neq k$ since $\gamma(H - h_k) = 1$. Furthermore, h_l is a dominating vertex of $H - h_k$ and (g_k, h_l) is the only vertex of the form (g_k, v) in S, otherwise we can create a dominating set of $G[H] - (g_k, h_k)$ with fewer vertices. Now $S_1 = S - (g_k, h_l)$ dominates the subgraph $G[H] - \{(w, z) : w \in N_G[g_k]\}$ and $|S_1| = \gamma_t(G) - 2$.

Now the set S_1 can be used to create a total dominating set of $G - N_G[g_k]$. If $(g_j, a), (g_j, b) \in S_1$ for any vertex $v_j \in V(G)$ then the set $S_2 = (S_1 - \{(g_j, b)\}) \cup \{(c, b)\}$ where $c \in N_G(g_j)$ is also a dominating set of $G[H] - \{(w, z) : w \in N_G[g_k]\}$. Thus we can create a total dominating set D of $G[H] - \{(w, z) : w \in N_G[g_k]\}$ with cardinality at most $\gamma_t(G) - 2$. Finally, the set $D_1 = \{u : \exists (u, u_1) \in D\}$ is a total dominating set of $G - N_G[g_k]$ with cardinality at most $\gamma_t(G) - 2$. Since $\gamma_t(G - N_G[g_k]) \geq \gamma_t(G) - 2$, we have that $\gamma_t(G - N_G[g_k]) = \gamma_t(G) - 2$. □

The cycle C_6 is an example of a graph such that $\gamma_t(G - N[x]) = \gamma_t(G) - 2$ for every vertex $x \in V(G)$. Notice that the cycle C_4 is 2-γ-critical and the graph $C_6[C_4]$ is γ-critical.

Proposition 2.34. *If G is a graph such that $\gamma_t(G - N[x]) = \gamma_t(G) - 2$ for every $x \in V(G)$ and H is a 2-γ-critical graph, then $G[H]$ is γ-critical.*

Proof. Let $g \in V(G)$ and $h \in V(H)$ and consider $G[H]-(g,h)$. Let S be a γ_t-set of $G-N[g]$ and let h_1 be a dominating vertex of $H-h$. Then $S_1 = \{(x,h_1) : x \in S\} \cup \{(g,h_1)\}$ is a dominating set of $G[H]-(g,h)$ and $|S_1| = \gamma_t(G)-2+1 = \gamma_t(G)-1 = \gamma(G[H])-1$. Thus $G[H]$ is γ-critical. □

2.2.7 Weighting Construction

The construction presented in this section is a combination of the coalescence and a generalized version of the wreath product. Given a graph L and a graph R we construct the weighted graph G_{wt} as follows:

Let $v \in V(L)$ and $v \in V(R)$. Replace each vertex $u \in V(L-v)$ with $\{(u,u_1), (u,u_2), \ldots, (u,u_x)\}$, $x \geq 2$. (That is, replace u with a copy of $\overline{K}_x$.) For $u, w \in V(L-v)$ add all edges between $\{(u,u_1),(u,u_2),\ldots,(u,u_x)\}$ and $\{(w,w_1),(w,w_2),\ldots,(w,w_x)\}$ exactly when $uw \in E(L)$. Replace each vertex $u \in V(R-v)$ with $\{(u,u_1), (u,u_2), \ldots, (u,u_z)\}$, $z \geq 2$. (That is, replace u with a copy of $\overline{K}_z$.) For $u, w \in V(R-v)$ add all edges between $\{(u,u_1),(u,u_2),\ldots,(u,u_z)\}$ and $\{(w,w_1),(w,w_2),\ldots,(w,w_z)\}$ exactly when $uw \in E(R)$. Replace v with $\{(v,v_1),(v,v_2),\ldots,(v,v_y)\}$, $y \geq 2$. (That is, replace v with a copy of $\overline{K}_y$.) Add all edges between $\{(v,v_1),(v,v_2),\ldots,(v,v_y)\}$ and $\{(u,u_1),(u,u_2),\ldots,(u,u_x)\}$ exactly when $vu \in E(L)$ and all edges between $\{(v,v_1),(v,v_2),\ldots,(v,v_y)\}$ and $\{(u,u_1),(u,u_2),\ldots,(u,u_z)\}$ exactly when $vu \in E(R)$. The weighting construction with $L = K_{3,3}$, $R = P_4$ and $x = y = z = 2$ is pictured in Figure 2.6 and the weighting construction with $L = R = K_{3,3_{[v]}}$ and $x = y = 2$ and $z = 4$ is pictured in Figure 2.7. (The explanation on how to obtain the graph $K_{3,3_{[v]}}$ is provided in Chapter 3 through the expansion of G via v construction.)

We first provide restrictions on x, y, and z in order for G_{wt} to be i-critical.

Proposition 2.35. *If G_{wt} is i-critical, $i(L-v) < i(L)$, and $i(R-v) < i(R)$, then $x = y = z$.*

Proof. If G_{wt} is i-critical, then for any vertex $w \in V(G_{wt})$ there exists an independent

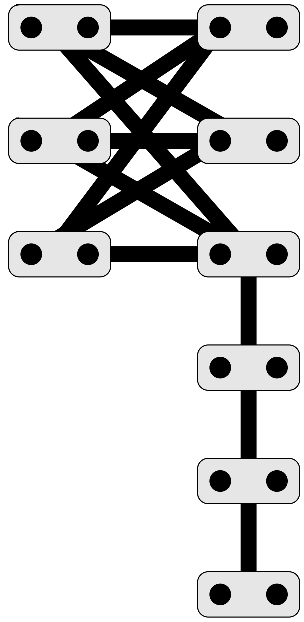

Figure 2.6: The weighting construction with $L = K_{3,3}$, $R = P_4$, and $x = y = z = 2$.

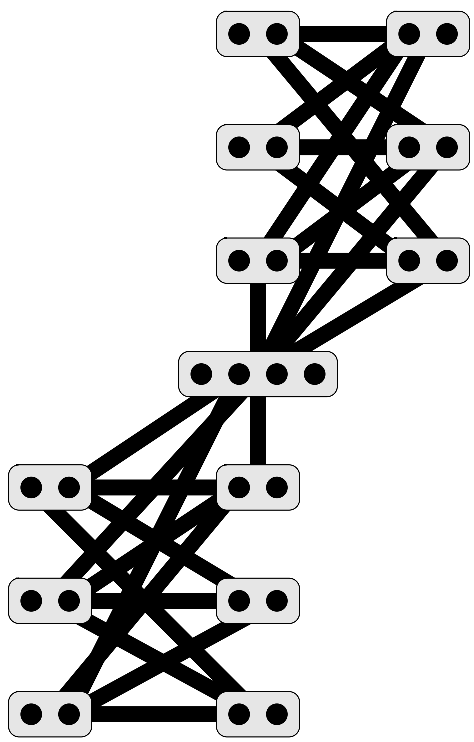

Figure 2.7: The weighting construction with $L = R = K_{3,3_{[v]}}$ and $x = y = 2$ and $z = 4$.

dominating set D of G_{wt} with $w \in D$. Recall that this set D can be created by finding an i-set D' of $G_{wt} - w$ and then $D = D' \cup \{w\}$.

Thus if $w = (v, v_j)$ for some j, $1 \leq j \leq y$, we have that $|D| = (i(L) - 1)x + (i(R)-1)z+y$. If $w = (u, u_j)$ for some j, $1 \leq j \leq x$, where $u \in V(L)$ and $uv \in E(L)$, we have that $|D| = i(L)x + (i(R) - 1)z$. If $w = (u, u_j)$ for some j, $1 \leq j \leq z$, where $u \in V(R)$ and $uv \in E(R)$, we have that $|D| = (i(L) - 1)x + i(R)z$.

Therefore $(i(L)-1)x+(i(R)-1)z+y = i(L)x+(i(R)-1) = z(i(L)-1)x+i(R)z$, and so $x + z - y = x = z$ and $y = x = z$. □

Proposition 2.36. *The graph G_{wt} is not i-critical if $i(L-v) < i(L)$ and $i(R-v) = i(R)$. Likewise, G_{wt} is not i-critical if $i(L-v) = i(L)$ and $i(R-v) < i(R)$.*

Proof. Without loss of generality, suppose that $i(L-v) < i(L)$ and $i(R-v) = i(R)$. If G_{wt} is i-critical, then for any vertex $w \in V(G_{wt})$ there exists an independent dominating set D of G_{wt} with $w \in D$. Recall that this set D can be created by finding an i-set D' of $G_{wt} - w$ and then $D = D' \cup \{w\}$.

Thus if $w = (v, v_j)$ for some j, $1 \leq j \leq y$, we have that $|D| = (i(L)-1)x+(i(R)-1)z + y$. If $w = (u, u_j)$ for some j, $1 \leq j \leq x$, where $u \in V(L)$ and $uv \in E(L)$, we have that $|D| = i(L)x+i(R)z$. If $w = (u, u_j)$ for some j, $1 \leq j \leq z$, where $u \in V(R)$ and $uv \in E(R)$, we have that $|D| = (i(L) - 1)x + i(R)z$.

Therefore $(i(L) - 1)x + (i(R) - 1)z + y = i(L)x + i(R)z = (i(L) - 1)x + i(R)z$ and so $x = y = 0$, a contradiction. □

Proposition 2.37. *If G_{wt} is i-critical, $i(L-v) = i(L)$, and $i(R-v) = i(R)$, then $y = x + z$.*

Proof. If G_{wt} is i-critical, then for any vertex $w \in V(G_{wt})$ there exists an independent dominating set D of G_{wt} with $w \in D$. Recall that this set D can be created by finding an i-set D' of $G_{wt} - w$ and then $D = D' \cup \{w\}$.

Thus if $w = (v, v_j)$ for some j, $1 \leq j \leq y$, we have that $|D| = (i(L)-1)x+(i(R)-1)z+y$. If $w = (u, u_j)$ for some j, $1 \leq j \leq x$, where $u \in V(L)$ and $uv \in E(L)$, we have that $|D| = i(L)x + i(R)z$. If $w = (u, u_j)$ for some j, $1 \leq j \leq z$, where $u \in V(R)$ and $uv \in E(R)$, we have that $|D| = i(L)x + i(R)z$.

Therefore $(i(L) - 1)x + (i(R) - 1)z + y = i(L)x + i(R)z$ and so $y = x + z$. □

Proposition 2.38. *If every vertex of L is in an i-set of L, every vertex of R is in an i-set of R, and either*

- *$i(L - v) < i(L)$, $i(R - v) < i(R)$, and $y = x = z$ or*
- *$i(L - v) = i(L)$, $i(R - v) = i(R)$, and $y = x + z$,*

then G_{wt} is i-critical.

Proof. Suppose that $i(L - v) < i(L)$ and $i(R - v) < i(R)$. Consider $w \in V(G_{wt})$.

Suppose $w = (v, v_j)$ for some j, $1 \leq j \leq y$. Let D_L be an i-set of L such that $v \in D_L$ and let D_R be an i-set of R such that $v \in D_R$. Let $D_1 = (\{(v, v_j) : 1 \leq j \leq y\} - \{w\}) \cup \{(u, u_j) : 1 \leq j \leq x, u \in D_L - \{v\}\} \cup \{(u, u_j) : 1 \leq j \leq z, u \in D_R - \{v\}\}$. Then D_1 is an independent dominating set of $G_{wt} - w$ and $|D_1| = (i(L) - 1)x + (i(R) - 1)z + y - 1$.

Suppose $w = (u, u_j)$ for some j, $1 \leq j \leq x$ where $u \in V(L - v)$. Let D_L be an i-set of L such that $u \in L$ and D_R be an i-set of $R - v$. If $v \in D_L$ let $D_2 = \{(v, v_j) : 1 \leq j \leq y\} \cup (\{(u, u_j) : 1 \leq j \leq x\} - \{w\}) \cup \{(t, t_j) : 1 \leq j \leq x, t \in D_L - \{u, v\}\} \cup \{(t, t_j) : 1 \leq j \leq z, t \in D_R\}$. Then D_2 is an independent dominating set of $G_{wt} - w$ and $|D_2| = (i(L) - 1)x + (i(R) - 1)z + y - 1$. If $v \notin D_L$ let $D_3 = (\{(u, u_j) : 1 \leq j \leq x\} - \{w\}) \cup \{(t, t_j) : 1 \leq j \leq x, t \in D_L - \{u\}\} \cup \{(t, t_j) : 1 \leq j \leq z, t \in D_R\}$. Then D_3 is an independent dominating set of $G_{wt} - w$ and $|D_3| = (i(L) - 1)x + (i(R) - 1)z + x - 1$.

Suppose $w = (u, u_j)$ for some j, $1 \leq j \leq z$ where $u \in V(R - v)$. Let D_R be an i-set of R such that $u \in D_R$ and let D_L be an i-set of $L - v$. If $v \in D_R$ let $D_4 = \{(v, v_j) : 1 \leq j \leq y\} \cup (\{(u, u_j) : 1 \leq j \leq z\} - \{w\}) \cup \{(t, t_j) : 1 \leq j \leq z, t \in D_R - \{u, v\}\} \cup \{(t, t_j) : 1 \leq j \leq x, t \in D_L\}$. Then D_4 is an independent dominating set of $G_{wt} - w$ and $|D_4| = (i(L) - 1)x + (i(R) - 1)z + y - 1$. If $v \notin D_R$ let $D_5 = (\{(u, u_j) : 1 \leq j \leq z\} - \{w\}) \cup \{(t, t_j) : 1 \leq j \leq z, t \in D_R - \{u\}\} \cup \{(t, t_j) : 1 \leq j \leq x, t \in D_L\}$. Then D_5 is an independent dominating set of $G_{wt} - w$ and $|D_5| = (i(L) - 1)x + (i(R) - 1)z + z - 1$.

But in this case $x = y = z$ and so $|D_1| = |D_2| = |D_3| = |D_4| = |D_5| = (i(L) - 1)x + (i(R) - 1)z + y - 1 = i(G_{wt}) - 1$.

Now suppose that $i(L - v) = i(L)$ and $i(R - v) = i(R)$. Consider $w \in V(G_{wt})$.

Suppose $w = (v, v_j)$ for some j, $1 \leq j \leq y$. Let D_L be an i-set of L such that $v \in D_L$ and let D_R be an i-set of R such that $v \in D_R$. Let $D_1 = (\{(v, v_j) : 1 \leq j \leq y\} - \{w\}) \cup \{(u, u_j) : 1 \leq j \leq x, u \in D_L - \{v\}\} \cup \{(u, u_j) : 1 \leq j \leq z, u \in D_R - \{v\}\}$. Then D_1 is an independent dominating set of $G_{wt} - w$ and $|D_1| = (i(L) - 1)x + (i(R) - 1)z + y - 1$.

Suppose $w = (u, u_j)$ for some j, $1 \leq j \leq x$ where $u \in V(L - v)$. Let D_L be an i-set of L such that $u \in L$. If $v \in D_L$ let D_R be an i-set of R such that $v \in D_R$ and let $D_2 = \{(v, v_j) : 1 \leq j \leq y\} \cup (\{(u, u_j) : 1 \leq j \leq x\} - \{w\}) \cup \{(t, t_j) : 1 \leq j \leq x, t \in D_L - \{u, v\}\} \cup \{(t, t_j) : 1 \leq j \leq z, t \in D_R - \{v\}\}$. Then D_2 is an independent dominating set of $G_{wt} - w$ and $|D_2| = (i(L) - 1)x + (i(R) - 1)z + y - 1$. If $v \notin D_L$ let D_R be an i-set of $R - v$ and let $D_3 = (\{(u, u_j) : 1 \leq j \leq x\} - \{w\}) \cup \{(t, t_j) : 1 \leq j \leq x, t \in D_L - \{u\}\} \cup \{(t, t_j) : 1 \leq j \leq z, t \in D_R\}$. Then D_3 is an independent dominating set of $G_{wt} - w$ and $|D_3| = (i(L) - 1)x + (i(R) - 1)z + x + z - 1$.

Suppose $w = (u, u_j)$ for some j, $1 \leq j \leq z$ where $u \in V(R - v)$. Let D_R be an i-set of R such that $u \in D_R$. If $v \in D_R$ let D_L be an i-set of L such that $v \in L$ and let $D_4 = \{(v, v_j) : 1 \leq j \leq y\} \cup (\{(u, u_j) : 1 \leq j \leq z\} - \{w\}) \cup \{(t, t_j) : 1 \leq j \leq$

$z, t \in D_R - \{u, v\}\} \cup \{(t, t_j) : 1 \leq j \leq x, t \in D_L - \{v\}\}$. Then D_4 is an independent dominating set of $G_{wt} - w$ and $|D_4| = (i(L) - 1)x + (i(R) - 1)z + y - 1$. If $v \notin D_R$ let D_L be an i-set of $L - v$ and let $D_5 = (\{(u, u_j) : 1 \leq j \leq z\} - \{w\}) \cup \{(t, t_j) : 1 \leq j \leq z, t \in D_R - \{u\}\} \cup \{(t, t_j) : 1 \leq j \leq x, t \in D_L\}$. Then D_5 is an independent dominating set of $G_{wt} - w$ and $|D_5| = (i(L) - 1)x + (i(R) - 1)z + x + z - 1$.

But in this case $y = x + z$ and so $|D_1| = |D_2| = |D_3| = |D_4| = |D_5| = (i(L) - 1)x + (i(R) - 1)z + y - 1 = i(G_{wt}) - 1$. □

2.2.8 An Expansion-Join Construction

Let G_1 and G_2 be any graphs with $x \in V(G_1)$ and $y \in V(G_2)$. Create the graph $G_{1(x)} +_{xx'yy'} G_{2(y)}$ as follows:

Add the vertex x' to G_1 and add all edges $\{x'v : v \in N_{G_1}(x)\}$, add the vertex y' to G_2 and add all edges $\{y'v : v \in N_{G_2}(y)\}$. Add all edges between $\{x, x'\}$ and $\{y, y'\}$. The graph $C_{4x} +_{xx'yy'} C_{4y}$ is pictured in Figure 2.8.

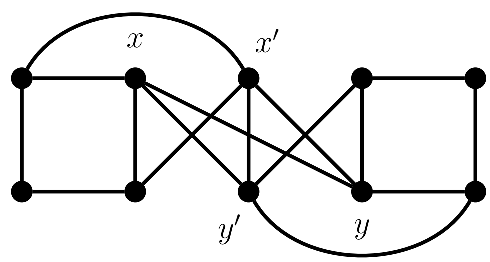

Figure 2.8: The graph $C_{4x} +_{xx'yy'} C_{4y}$.

Proposition 2.39. *The graph $G = G_{1(x)} +_{xx'yy'} G_{2(y)}$ has $i(G) \geq i(G_1) + i(G_2)$.*

Proof. Let D be an i-set of G and let $D_1 = D \cap (V(G_1) \cup \{x'\})$ and $D_2 = D \cap (V(G_2) \cup \{y'\})$. If $x \in D$, then $D \cap N_{G_1}(x) = \emptyset$ and $D \cap \{y, y'\} = \emptyset$. Thus $x' \in D$ and $D_1 - \{x\}$ is an independent dominating set of G_1 and D_2 is an independent dominating set of $G_2 - y$. Therefore $|D| \geq i(G_1) + 1 + i(G_2) - 1 = i(G_1) + i(G_2)$. Likewise, $|D| \geq i(G_1) + i(G_2)$ if $x' \in D$, or $y \in D$, or $y' \in D$.

If $x \notin D$ and $D \cap N_{G_1}(x) = \emptyset$, then using the same argument above we have that $x' \notin D$. Thus $|D \cap \{y, y'\}| \geq 1$ and so $D \cap \{y, y'\} = \{y, y'\}$. Then $D_2 - \{y'\}$ is an independent dominating set of G_2 and D_1 is an independent dominating set of $G_1 - x$. Therefore $|D| \geq i(G_1) - 1 + i(G_2) + 1 = i(G_1) + i(G_2)$.

If $x \notin D$ and $D \cap N_{G_1}(x) \neq \emptyset$, then D_1 is an independent dominating set of G_1 and $x' \notin D$. If $y \in D$ or $y' \in D$, then $D \cap \{y, y'\} = \{y, y'\}$. Thus $D_2 - \{y'\}$ is an independent dominating set of G_2 and so $|D| \geq i(G_1) + i(G_2) + 1$. If $y \notin D$, then $y' \notin D$ and so D_2 is an independent dominating set of G_2. Therefore $|D| \geq i(G_1) + i(G_2)$. Likewise $|D| \geq i(G_1) + i(G_2)$ if $x' \notin D$ or $y \notin D$ or $y' \notin D$. $\square$

Proposition 2.40. *The graph $G = G_{1(x)} +_{xx'yy'} G_{2(y)}$ is i-critical if G_1 and G_2 are i-critical.*

Proof. We first show that $i(G) = i(G_1) + i(G_2)$. Let D_1 be an i-set of G_1 such that $x \in D_1$ and let D_2 be an i-set of $G_2 - y$. Then $D = D_1 \cup D_1 \cup \{x'\}$ is an independent dominating set of G and so $|D| = i(G_1) + i(G_2)$.

Let $v \in V(G)$ and consider $G - v$.

If $v = x$ or $v = x'$, let D_1 be an i-set of G_1 such that $x \in D_1$ and let D_2 be an i-set of $G_2 - y$. Then $D = (D_1 - \{x\}) \cup (\{x, x'\} - \{v\}) \cup D_2$ is an independent dominating set of $G - v$ and $|D| = i(G_1) + i(G_2) - 1$. Likewise we can create an i-set D of $G - v$ with $|D| = i(G_1) + i(G_2) - 1$ if $v = y$ or $v = y'$.

If $v \in N_{G_1}(x)$, let D_1 be an i-set of $G_1 - v$ and let D_2 be an i-set of G_2 with $y \notin D_2$. Then $D = D_1 \cup D_2$ is an independent dominating set of $G - v$ and $|D| = i(G_1) - 1 + i(G_2)$. Likewise we can create an i-set D of $G - v$ with $|D| = i(G_1) - 1 + i(G_2)$ if $v \in N_{G_2}(y)$.

If $v \in V(G_1 - x)$ and $v \notin N_{G_1}(x)$, let D_1 be an i-set of $G_1 - v$. If $x \in D_1$, let D_2 be an i-set of $G_2 - y$. Then $D = D_1 \cup \{x'\} \cup D_2$ is an independent dominating set of $G - v$ and $|D| = i(G_1) + i(G_2) - 1$. If $x \notin D_1$, then $D \cap N_{G_1}(x) \neq \emptyset$. Let D_2 be an i-set of G_2 with $y \notin D_2$. Then $D = D_1 \cup D_2$ is an independent dominating set of $G - v$ and $|D| = i(G_1) + i(G_2) - 1$. Likewise we can create an i-set D of $G - v$ with $|D| = i(G_1) + i(G_2) - 1$ if $v \in V(G_2 - y)$ and $v \notin N_{G_2}(y)$.

Therefore G is i-critical. □

2.2.9 The Expansion Construction

Let G_1, G_2, …, G_k be any graphs with $x_j \in V(G_j)$, $1 \leq j \leq k$. Construct the graph G_{exp} as follows:

For each x_j, $1 \leq j \leq k$, add the set of independent vertices $\{y_{j2}, y_{j3}, \ldots, y_{jk}\}$ and add all edges between $N_{G_j}(x_j)$ and $\{y_{j2}, y_{j3}, \ldots, y_{jk}\}$. For all $1 \leq i \leq j \leq k$ add all edges between $\{x_i, y_{i2}, y_{i3}, \ldots, y_{ik}\}$ and $\{x_j, y_{j2}, y_{j3}, \ldots, y_{jk}\}$.

Proposition 2.41. *For any graphs G_1, G_2, …, G_k, $i(G_{exp}) \geq i(G_1) + i(G_2) + \ldots + i(G_k)$.*

Proof. Let D be an i-set of G_{exp}, and let $D_j = D \cap (V(G_j) \cup \{y_{12}, y_{13}, \ldots, y_{1k}\})$ for $1 \leq j \leq k$.

Without loss of generality, suppose $x_1 \in D$. Then $D \cap N_{G_1}(x_1) = \emptyset$ and so $D \cap \{y_{jl} : 2 \leq j \leq k, 2 \leq l \leq k\} = \emptyset$. Thus $\{y_{12}, y_{13}, \ldots, y_{1k}\} \subseteq D$ and $D_1 - \{y_{12}, y_{13}, \ldots, y_{1k}\}$ is an independent dominating set of G_1. Also, for all j, $2 \leq j \leq k$, D_j is an independent dominating set of $G_j - x_j$. Therefore $|D| \geq i(G_1) + k - 1 + \sum_{j=2}^{k}(i(G_j) - 1) = i(G_1) + i(G_2) + \ldots + i(G_k)$.

Without loss of generality, suppose that $x_1 \notin D$ and $D \cap N_{G_1}(x_1) = \emptyset$. Then $D \cap \{x_j, y_{j2}, y_{j3}, \ldots, y_{jk}\} = \{x_j, y_{j2}, y_{j3}, \ldots, y_{jk}\}$ for some j, $2 \leq j \leq k$, and $D \cap \{x_l, y_{l2}, y_{l3}, \ldots, y_{lk}\} = \emptyset$ for all l, $1 \leq l \leq k$ with $l \neq j$. Then $D_j - \{y_{j2}, y_{j3}, \ldots, y_{jk}\}$

is an independent dominating set of G_j and D_l is an independent dominating set of $G_l - x_l$ for all $1 \leq l \leq k$ with $l \neq j$. Therefore $|D| \geq \sum_{l=1}^{k}(i(G_l) - 1) + 1 + k - 1 = i(G_1) + i(G_2) + \ldots + i(G_k)$.

Without loss of generality, suppose that $x_1 \notin D$ and $D \cap N_{G_1}(x_1) \neq \emptyset$. Then D_1 is an independent dominating set of G_1. If $x_j \in D$ for some j, $2 \leq j \leq k$, or $y_{jl} \in D$ for some j and l, $2 \leq j \leq k$ and $2 \leq l \leq k$, then $D \cap \{x_j, y_{j2}, y_{j3}, \ldots, y_{jk}\} = \{x_j, y_{j2}, y_{j3}, \ldots, y_{jk}\}$ and $D \cap \{y_{lm} : 1 \leq l \leq k, l \neq j, 2 \leq m \leq k\} = \emptyset$. Then $D_j - \{y_{j2}, y_{j3}, \ldots, y_{jk}\}$ is an independent dominating set of G_j and D_l, $2 \leq l \leq k$ with $l \neq j$, is an independent dominating set of $G_l - x_l$. Therefore $|D| \geq i(G_1) + \sum_{l=2}^{k}(i(G_l) - 1) + 1 + k - 1 = i(G_1) + i(G_2) + \ldots + i(G_k)$. If $x_j \notin D$ for all $1 \leq j \leq k$ and $y_{jl} \notin D$ for all $1 \leq j \leq k$ and $2 \leq l \leq k$, then D_j is an independent dominating set of G_j for all $1 \leq j \leq k$. Therefore $|D| \geq i(G_1) + i(G_2) + \ldots + i(G_k)$. □

Proposition 2.42. *The graph G_{exp} is i-critical if all of G_1, G_2, …, G_k are i-critical.*

Proof. We first show that $i(G_{exp}) = i(G_1) + i(G_2) + \ldots + i(G_k)$. Let D_1 be an i-set of G_1 such that $x_1 \in D_1$. For $2 \leq j \leq k$, let D_j be an i-set of $G_j - x_j$. Then $D = D_1 \cup \{y_{12}, y_{13}, \ldots, y_{1k}\} \cup D_2 \cup \ldots \cup D_k$ and $|D| = i(G_1) + k - 1 + \sum_{j=2}^{k}(i(G_j) - 1) = i(G_1) + i(G_2) + \ldots + i(G_k)$.

Let $v \in V(G)$ and consider $G_{exp} - v$.

Without loss of generality, suppose $v \in \{x_1, y_{12}, y_{13}, \ldots, y_{1k}\}$ and let D_1 be an i-set of G_1 such that $x_1 \in D_1$. For $2 \leq j \leq k$, let D_j be an i-set of $G_j - x_j$. Then $D = (D_1 - v) \cup (\{x_1, y_{12}, y_{13}, \ldots, y_{1k}\} - \{v\}) \cup D_2 \cup \ldots \cup D_k$ is an independent dominating set of $G_{exp} - v$ and $|D| = i(G_1) - 1 + k - 1 + \sum_{j=2}^{k}(i(G_j) - 1) = i(G_1) + i(G_2) + \ldots + i(G_k) - 1 = i(G_{exp}) - 1$.

Without loss of generality, suppose $v \in N_{G_1}(x_1)$. Let D_1 be an i-set of $G_1 - v$. Thus $x_1 \notin D_1$ and $D_1 \cap N_{G_1}(x_1) \neq \emptyset$. For $2 \leq j \leq k$, let D_j be an i-set of G_j such that $x_j \notin D_j$. Then $D = D_1 \cup D_2 \cup \ldots \cup D_k$ is an independent dominating set of $G_{exp} - v$ and $|D| = i(G_1) + i(G_2) + \ldots + i(G_k) - 1$.

Without loss of generality, suppose $v \in V(G_{exp} - x_1)$ and $v \notin N_{G_1}(x_1)$. Let D_1 be an i-set of $G_1 - v$. If $x_1 \in D_1$, let D_j be an i-set of $G_j - x_j$ for $2 \leq j \leq k$. Then $D = D_1 \cup \{y_{12}, y_{13}, \ldots, y_{1k}\} \cup D_2 \cup \ldots \cup D_k$ is an independent dominating set of $G_{exp} - v$ and $|D| = i(G_1) - 1 + k - 1 + \sum_{j=2}^{k}(i(G_j) - 1) = i(G_1) + i(G_2) + \ldots + i(G_k) - 1 = i(G) - 1$. If $x_1 \notin D_1$, let D_j be an i-set of G_j such that $x_j \notin D_j$ for $2 \leq j \leq k$. Then $D = D_1 \cup D_2 \cup \ldots \cup D_k$ is an i-set of $G_{exp} - v$ and $|D| = i(G_1) + i(G_2) + \ldots + i(G_k) - 1$.

Hence in all cases G_{exp} is i-critical. □

Necessary conditions for the graph G_{exp} to be i-critical are unknown.

2.3 Summary and Directions for Future Work

Section 2.1 gave a construction which shows that for any graph G and all $k \geq 3$ there exists a k-i-critical graph H such that G is an induced subgraph of H, which extended a known result by Ao [7]. This same construction showed that for any graph G and all $k \geq 4$ there exists a k-i-bicritical graph H such that G is an induced subgraph of H. A variation on this construction showed that for any graph G and all $k \geq 5$ there exists a strong k-i-bicritical graph H such that G is an induced subgraph of G. Section 2.2.1 provided necessary and sufficient conditions for the disjoint union $G_1 \cup G_2 \cup \cdots \cup G_k$ to be i-critical. Sufficient conditions for the join $G_1 + G_2 + \cdots + G_k$ were known by Ao [7], and the main result of Section 2.2.2 extended this to necessary conditions. The coalescence construction $G \cdot H$ was extended to the generalized coalescence construction $G_1(H_1) \odot G_2(H_2)$ and while not much is known about this construction, it is further studied in Chapter 3 for strong i-bicritical graphs in the cases of $H_1 \cong H_2 \cong K_2$ and $H_1 \cong H_2 \cong \overline{K_2}$. The joined coalescence construction $G_1(H_1) \widehat{\odot} G_2(H_2)$ was investigated in Section 2.2.5.

Again, Ao [7] was aware of sufficient conditions for the construction to produce an i-critical graph and here necessary conditions were provided. The wreath product $G[H]$ was introduced in Section 2.2.6 and necessary and sufficient conditions were provided for the construction to produce an i-critical graph. The wreath product was also investigated as a possible construction for γ-critical graphs and the results showed that only trivial cases yield γ-critical graphs. The weighting construction G_{wt} was introduced as a new construction to create i-critical graphs and necessary and sufficient conditions to do so were provided. Finally, two new constructions, the expansion-join and the expansion, were presented and sufficient conditions to produce i-critical graphs were provided.

We close this chapter with a collection of open questions:

1. Proposition 1.25 of Chapter 1 says that if $\gamma(G) = i(G)$ and G is i-critical, then G is γ-critical.

    - Characterize the graphs for which $i(G) = \gamma(G)$ and G is i-critical.

    Fulman, Hanson, and MacGillivray [21] found that not all γ-critical graphs have $\gamma(G) = i(G)$.

    - Are all γ-critical graphs with $\gamma(G) = i(G)$ also i-critical?

    If so, then γ-criticality is closely related to i-criticality when $\gamma(G) = i(G)$. Likewise, Proposition 1.26 says that if $\gamma(G) = i(G)$ and G is i-bicritical, then G is γ-bicritical.

    - Characterize the graphs for which $i(G) = \gamma(G)$ and G is i-bicritical.

    Proposition 1.28 says that an i-bicritical graph with $i(G) = \gamma(G)$ will not be strong i-bicritical.

2. Corollary 2.5 of Chapter 2 shows that for any graph G and for all $k \geq 3$, there exists a k-i-critical graph H such that G is an induced subgraph of H. Notice that the 2-i-critical graphs are completely characterized. There are gaps in the knowledge for i-bicritical graphs though. Corollary 2.6 shows that for any graph G and for all $k \geq 4$, there exists a k-i-bicritical graph H such that G is an induced subgraph of H. On the other hand, the 2-i-bicritical graphs are completely characterized.

   - Characterize the 3-i-bicritical graphs, or show that for any graph G there exists a 3-i-bicritical graph H such that G is an induced subgraph of H.

   The situation is the same for strong i-bicritical graphs. Corollary 2.13 shows that for any graph G and for all $k \geq 5$ there exists a strong k-i-bicritical graph H such that G is an induced subgraph of H, while the strong 3-i-bicritical graphs have been completely characterized.

   - Characterize the strong 4-i-bicritical graphs, or show that for any graph G there exists a strong 4-i-bicritical graph H such that G is an induced subgraph of H.

3. As mentioned earlier in this chapter, little is known about the generalized coalescence construction $G_1(H_1) \odot G_2(H_2)$. The case $G_1(\{x\}) \odot G_2(\{y\})$ is equivalent to the basic coalescence $G_1 \cdot_{xy} G_2$ and this has been well-studied for γ-critical, i-critical, i-bicritical, γ_t-critical, and γ_t-bicritical graphs.

   - Investigate the generalized coalescence construction for critical graphs when H_1 and H_2 are more than single vertices.

3

Bicritical Graphs

3.1 A History of Domination Bicritical Graphs and Independent Domination Bicritical Graphs

Recall that a graph G is *domination bicritical*, or *γ-bicritical*, if the removal of any two vertices decreases the domination number, that is, if $\gamma(G - \{u, v\}) < \gamma(G)$ for any $\{u, v\} \subseteq V(G)$. Domination bicriticality was introduced by Brigham, Haynes, Henning, and Rall in 2005 [10]. Here they provided many examples to show the existence of γ-bicritical graphs.

Proposition 3.1. [10] *The circulant $C_8\langle 1, 4\rangle$ is 3-γ-critical and 3-γ-bicritical.*

The *Cartesian product* of G and H, denoted by $G\Box H$, is the graph with vertex set $\{(g, h) : g \in V(G), h \in V(H)\}$ and edge set $\{(g_1, h_1)(g_2, h_2) : g_1 = g_2$ and $\mathrm{h_1h_2} \in$ E(H) or $\mathrm{h_1} = \mathrm{h_2}$ and $\mathrm{g_1g_2} \in$ E(G)$\}$. The graph $K_3\Box K_3$ is pictured in Figure 3.2.

Proposition 3.2. [10] *The Cartesian product $K_t\Box K_t$ for $t \geq 3$ is t-γ-critical and t-γ-bicritical.*

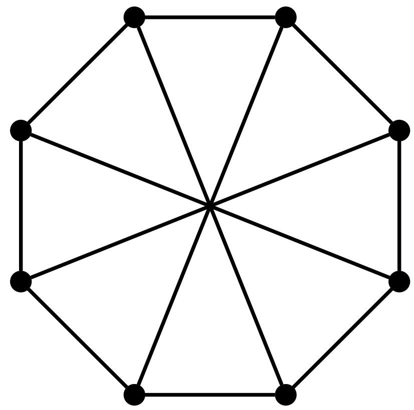

Figure 3.1: The circulant $C_8\langle 1, 4\rangle$.

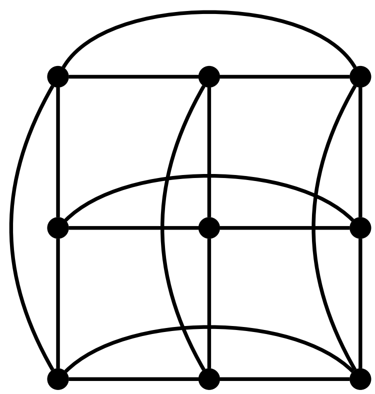

Figure 3.2: The graph $K_3 \square K_3$.

Proposition 3.3. [10] *Any graph formed from the complete bipartite graph $K_{2t,2t}$ where $t \geq 3$ by removing the edges of t disjoint 4-cycles is 4-γ-critical and 4-γ-bicritical.*

The graph obtained by removing the edges of three disjoint 4-cycles from $K_{6,6}$ is pictured in Figure 3.3.

Despite the pattern hinted at in the previous three examples, there exist graphs that are γ-bicritical and not γ-critical. A construction that creates such graphs is discussed in Section 3.1.1.

As we have seen before, for a set $S \subseteq V(G)$, there are lower bounds on $\gamma(G - S)$. In the case of γ-bicritical graphs, we have $|S| = 2$.

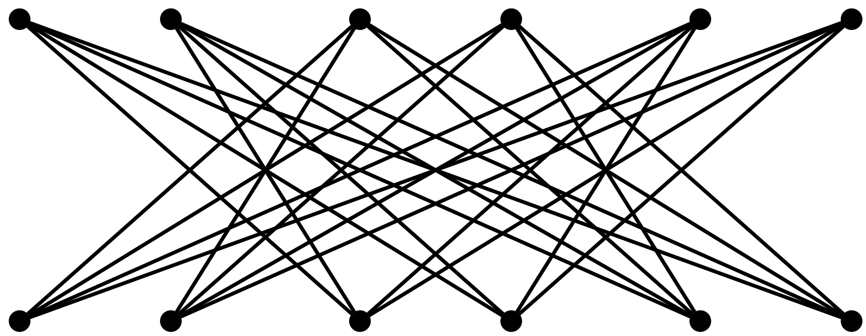

Figure 3.3: The graph $K_{6,6}$ minus the edges of 3 disjoint 4-cycles.

Observation 3.4. [10] *For a γ-bicritical graph G and $\{x, y\} \subseteq V(G)$, $\gamma(G) - 2 \leq \gamma(G - \{x, y\}) \leq \gamma(G) - 1$.*

Observation 3.5. [10] *If G is any graph and $\{x, y\} \subseteq V(G)$ such that $\gamma(G - \{x, y\}) = \gamma(G) - 2$, then $d_G(x, y) \geq 3$.*

In particular, Observation 3.5 says that if vertices u and v are adjacent, then $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$.

Although a γ-bicritical graph G need not be γ-critical, it is not far from being a γ-critical graph.

Observation 3.6. [10] *If G is a γ-bicritical graph, then $V(G) = V^- \cup V^0$, that is, $V^+ = \emptyset$. Furthermore, either G is γ-critical, or $G - v$ is γ-critical for all $v \in V^0$.*

Section 1.2 discussed results concerning bounds on the order of γ-critical graphs. Brigham et al. found similar results for γ-bicritical graphs.

Proposition 3.7. [10] *If G is a γ-bicritical graph of order n, then $n \leq (\Delta(G) + 1)(\gamma(G) - 1) + 2$.*

Proposition 3.8. [10] *If G is a regular γ-bicritical graph of order n, then $n \leq (\Delta(G) + 1)(\gamma(G) - 1) + 1$.*

Proposition 3.9. [10] *If G is a connected γ-bicritical graph, then $\gamma(G) \geq 3$.*

Proposition 3.10. [10] *If G is a connected γ-bicritical graph, then $\delta(G) \geq 3$.*

The bounds in the previous two propositions are sharp.

A large portion of this chapter focuses on constructions for i-bicritical graphs. For the γ-bicritical graphs, Brigham et al. [10] gave some constructions. In particular, they studied the coalescence construction, $G \cdot H$, that was seen in Chapter 2. Below is a modified version of the result stated in their paper. (For an explanation on why a modified version is presented here, see Proposition 3.32.)

Proposition 3.11. [10] *The graph $G \cdot H$ is γ-bicritical if G and H are both γ-critical and γ-bicritical.*

Using this result, Brigham et al. created k-γ-bicritical graphs of diameter $k-1$. Let G be the graph obtained from $K_{6,6}$ by removing the edges of three disjoint 4-cycles (i.e. the graph in Figure 3.3). Let H be the circulant $C_8\langle 1, 4\rangle$ (i.e., the graph in Figure 3.1). From Proposition 3.3 and Proposition 3.1 earlier in this section, both G and H are γ-critical and γ-bicritical. Thus by Proposition 2.19 and Proposition 3.11, $G \cdot H$ is γ-critical and γ-bicritical. Notice that $\gamma(G \cdot H) = 3 + 4 - 1 = 6$ and $\operatorname{diam}(G \cdot H) = 5$. In addition, the graph $H \cdot H$ is γ-critical and γ-bicritical with $\gamma(H \cdot H) = 3 + 3 - 1 = 5$ and $\operatorname{diam}(H \cdot H) = 4$. Using these ideas, by using k copies of H and considering the graph $H \cdot H \cdot \cdots \cdot H$, we have $\gamma(H \cdot H \cdot \cdots \cdot H) = 3k - (k-1) = 2k + 1$ and $\operatorname{diam}(H \cdot H \cdot \cdots \cdot H) = 2k$. Additionally, by using one copy of G and k copies of H and considering the graph $G \cdot H \cdot H \cdot \cdots \cdot H$, we have $\gamma(G \cdot H \cdot H \cdot \cdots \cdot H) = 4 + 3k - k = 4 + 2k$ and $\operatorname{diam}(G \cdot H \cdot H \cdot \cdots \cdot H) = 3 + 2k$. Thus for $k \geq 3$ there exists an example of a k-γ-bicritical graph G of diameter $k-1$. This leads to the following conjecture.

Conjecture 3.12. [10] *If G is a connected k-γ-bicritical graph, then* $\operatorname{diam}(G) \leq k-1$.

Brigham et al. also investigated the connectivity and edge-connectivity of γ-bicritical graphs.

Proposition 3.13. [10] *If G is a connected γ-bicritical graph, then G is 2-edge-connected.*

This is also true of γ-critical graphs. That is, if G is γ-critical, then G is 2-edge-connected.

Proposition 3.14. [10] *If G is a connected graph that is 3-γ-bicritical or 4-γ-bicritical, then G is 3-edge-connected.*

Theorem 3.15. [10] *Let G be a connected γ-bicritical graph. If G is cubic or claw-free, then G is 3-edge-connected.*

They then focused on 3-γ-bicritical graphs.

Proposition 3.16. [10] *If G is a connected 3-γ-bicritical graph, then G is 3-connected.*

Observation 3.17. [10] *A cubic graph G is 3-γ-bicritical if and only if G is isomorphic to the circulant $C_8\langle 1, 4\rangle$.*

Brigham et al. left several open problems:

- Is it true that every γ-bicritical graph has a minimum dominating set containing any two specified vertices of the graph?
- If G is a connected γ-bicritical graph, is it true that G is 3-edge-connected? In particular, if G is a connected 5-γ-bicritical graph, it it true that G is 3-edge-connected?
- Characterize the 3-γ-bicritical graphs.
- Characterize the connected cubic γ-bicritical graphs.
- Is it true that if G is a connected k-γ-bicritical graph, then $\text{diam}(G) \leq k - 1$?
- Is it true that if G is a connected γ-bicritical graph, then $\gamma(G) = i(G)$?

For the first question, it is not true that every γ-bicritical graph has a minimum dominating set containing any two specified vertices of the graph. For example, consider the circulant $C_8\langle 1, 4\rangle$. For any two adjacent vertices on the 8-cycle, there is no γ-set that contains both these vertices.

Chen, Fujita, Furuya, and Sohn further studied γ-bicritical graphs [13]. Here they answered the second question posed by Brigham et al [10], namely: If G is a connected γ-bicritical graph, is it true that G is 3-edge-connected? In particular, if G is a connected 5-γ-bicritical graph, it it true that G is 3-edge-connected? Chen et al. give a construction of a 5-γ-bicritical graph with edge-connectivity 2. Through other constructions they were able to show the following.

Theorem 3.18. [13] *Let t be an integer with $t \geq 5$. There exist infinitely many connected t-γ-bicritical graphs G that have edge-connectivity equal to 2.*

The remainder of this chapter discusses i-bicritical graphs. A graph G is *independent domination bicritical*, or *i-bicritical*, if $i(G - \{u, v\}) < i(G)$ for any $\{u, v\} \subseteq V(G)$. By Proposition 1.13, we have that for any graph G, $i(G - \{u, v\}) \geq i(G) - 2$, and so if G is i-bicritical we have that $i(G) - 2 \leq i(G - \{u, v\}) \leq i(G) - 1$. Proposition 1.14 shows that if $uv \in E(G)$, then $i(G - \{u, v\}) \geq i(G) - 1$, and so in an i-bicritical graph we have that $i(G - \{u, v\}) = i(G) - 1$ for adjacent vertices u and v. Thus we define a *strong independent domination bicritical graph*, or *strong i-bicritical graph*, to be a graph G for which $i(G - \{u, v\}) = i(G) - 2$ for all $\{u, v\} \subseteq V(G)$ where u and v are independent.

Recall from Proposition 1.17 that the only 2-i-bicritical graphs are $\overline{K_2}$ and $K_1 \cup K_2$, thus for the remainder of this chapter we discuss i-bicritical graphs with $i(G) \geq 3$. Other facts that are of use in this chapter are Proposition 1.18, which states that if G is i-bicritical then G is i-critical or $G - v$ is i-critical for all $v \in V^0$, and Proposition 1.23, which states that if G is i-bicritical then there is no vertex $v \in V(G)$ with $\deg(v) = 2$.

In this chapter, we will discuss i-bicritical graphs in Section 3.1.1 and strong i-bicritical graphs in Section 3.2. Structural properties of these graphs will be investigated and constructions for i-bicritical graphs and strong i-bicritical graphs will be seen. Many constructions will be the same as what was seen in Chapter 2, and a construction will be introduced that creates bicritical graphs but not critical graphs.

Independent domination bicritical graphs were introduced by Xu, Xu, and Zhang [52]. Here they presented introductory results and examples, some of which we have already seen, and a construction.

Observation 3.19. [52] *The circulant $C_8\langle 1,4\rangle$ is not i-bicritical.*

Observation 3.20. *The complete bipartite graph $K_{n,n}$, $n \geq 3$, is i-bicritical.*

Observation 3.21. *The complete bipartite graph $K_{n,n+1}$, $n \geq 3$, is i-bicritical.*

Notice that $K_{n,n}$ and $K_{n,n+1}$ are not γ-bicritical as $\gamma(K_{n,n}) = 2 = \gamma(K_{n,n+1})$. Thus by the above three observations we know that the class of i-bicritical graphs differs from the class of γ-bicritical graphs.

Proposition 3.22. [52] *The graph $K_n \square K_n$ is i-critical and i-bicritical.*

The *generalized Petersen graph* $G(n,k)$ is the graph with vertex set $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and edge set $\{v_iv_{i+1}\} \cup \{u_iu_{i+k}\} \cup \{v_iu_i\}$, where addition is performed modulo n. The graph $G(5,1)$ is the usual Petersen graph. The graph $G(7,2)$ is pictured in Figure 3.4.

Proposition 3.23. *The graph $G(7,2)$ is i-bicritical.*

Proof. We first determine $i(G(7,2))$. Suppose $i(G(7,2)) = 4$ and let D be an i-set of $G(7,2)$. Then $|D \cap \{v_1,\ldots,v_7\}| \geq 2$ or $|D \cap \{u_1,\ldots,u_7\}| \geq 2$. Without loss of generality, suppose that $|D \cap \{v_1,\ldots,v_7\}| \geq 2$. If $|D \cap \{v_1,\ldots,v_7\}| \geq 3$, then $D \cap \{v_1,\ldots,v_7\}$ dominates at most three of $\{u_1,\ldots,u_7\}$ and it is not possible

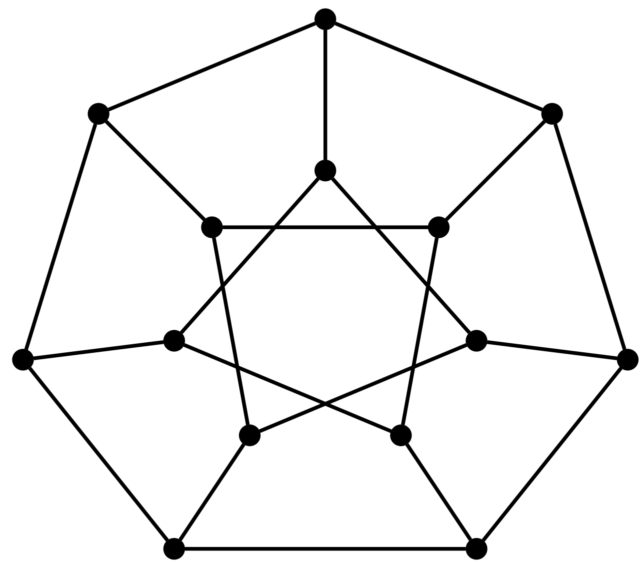

Figure 3.4: The generalized Petersen graph $G(7, 2)$.

to dominate the remaining four vertices of $\{u_1, \ldots, u_7\}$ with a single vertex. Thus $|D \cap \{v_1, \ldots, v_7\}| = 2$. But then there is no independent selection of two vertices from $\{v_1, \ldots, v_7\}$ and two vertices from $\{u_1, \ldots, u_7\}$ that dominate $G(7, 2)$. However $\{v_1, v_3, u_2, u_5, u_6\}$ is an independent dominating set of $G(7, 2)$ and thus $i(G(7, 2)) = 5$.

Let $\{x, y\} \subseteq V(G(7, 2))$ and consider $G(7, 2) - \{x, y\}$. Up to symmetry, there are seven cases. If $\{x, y\} = \{v_1, v_2\}$, then $\{v_4, v_6, u_2, u_3\}$ is an independent dominating set of $G(7, 2) - \{x, y\}$. If $\{x, y\} = \{v_1, v_3\}$, then $\{v_5, v_7, u_1, u_2\}$ is an independent dominating set of $G(7, 2) - \{x, y\}$. If $\{x, y\} = \{v_1, v_4\}$, then $\{v_6, u_2, u_3\}$ is an independent dominating set of $G(7, 2) - \{x, y\}$. If $\{x, y\} = \{v_1, u_1\}$, then $\{v_3, v_6, u_4, u_7\}$ is an independent dominating set of $G(7, 2) - \{x, y\}$. If $\{x, y\} = \{v_1, u_2\}$, then $\{v_3, v_5, v_7, u_6\}$ is an independent dominating set of $G(7, 2) - \{x, y\}$. If $\{x, y\} = \{v_1, u_3\}$, then $\{v_2, v_4, u_6, u_7\}$ is an independent dominating set of $G(7, 2) - \{x, y\}$. If $\{x, y\} = \{v_1, u_4\}$, then $\{v_4, v_6, u_2, u_3\}$ is an independent dominating set of $G(7, 2) - \{x, y\}$. Thus in all cases $G(7, 2)$ is i-bicritical. □

3.1.1 Constructions for i-Bicritical Graphs

Corollary 2.6 shows that for any graph G and any $k \geq 4$ there is a k-i-bicritical graph H such that G is an induced subgraph of H. Even for a fixed value $i(G) = k$ it is challenging to characterize the k-i-bicritical graphs as there is no finite list of forbidden induced subgraphs. Since the characterization problem is difficult it is useful to know ways to create i-bicritical graphs. A collection of methods to construct i-bicritical graphs is presented in this section. Notice that many of the constructions presented rely on the use of already known i-bicritical graphs to create new i-bicritical graphs.

Recall that the *disjoint union of* G *and* H, denoted $G \cup H$, is the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. The graph $G_1 \cup G_2 \cup \ldots \cup G_k$ is defined recursively by $G_1 \cup G_2 \cup \ldots \cup G_k = (G_1 \cup \ldots \cup G_{k-1}) \cup G_k$. Note that $i(G_1 \cup G_2 \cup \ldots \cup G_k) = \sum_{j=1}^{k} i(G_j)$.

Proposition 3.24. *The graph $G_1 \cup G_2 \cup \cdots \cup G_k$ is i-bicritical if and only if all of G_1, G_2, ..., G_k are i-bicritical or isomorphic to K_1 and at most one G_j, $1 \leq j \leq k$, is not i-critical.*

Proof. Suppose all of G_1, G_2, ..., G_k are i-bicritical or isomorphic to K_1 and at most one G_j, $1 \leq j \leq k$, is not i-critical. Consider $(G_1 \cup \cdots \cup G_k) - \{u, v\}$ for some $\{u, v\} \subseteq V(G_1 \cup \cdots \cup G_k)$. If $\{u, v\} \subseteq V(G_j)$ for some $1 \leq j \leq k$, then $i((G_1 \cup \cdots \cup G_k) - \{u, v\}) = i(G_1 \cup \cdots \cup G_j - \{u, v\} \cup \cdots \cup G_k) \leq i(G_1) + \cdots + i(G_j) - 1 + \cdots + i(G_k) = i(G_1 \cup \cdots \cup G_k) - 1$.

If $u \in V(G_j)$ and $v \in V(G_l)$ for some $1 \leq j < l \leq k$, then $i((G_1 \cup \cdots \cup G_k) - \{u, v\}) = i(G_1 \cup \cdots \cup G_j - u \cup \cdots \cup G_l - v \cup \cdots \cup G_k) \leq i((G_1 \cup \cdots \cup G_k) - \{u, v\}) - (i(G_j) + i(G_l)) + (i(G_j) + i(G_l) - 1) = i(G_1 \cup \cdots \cup G_k) - 1$ since at most one of G_j and G_l is not i-critical. Hence in either case, $G_1 \cup \cdots \cup G_k$ is i-bicritical.

For the converse, suppose $G_1 \cup G_2 \cup \cdots \cup G_k$ is i-bicritical and consider $(G_1 \cup G_2 \cup$

$\cdots \cup G_k) - \{u, v\}$ for some $\{u, v\} \subseteq V(G_1 \cup G_2 \cup \cdots \cup G_k)$.

If $\{u, v\} \subseteq V(G_j)$ for some $1 \leq j \leq k$, then $i((G_1 \cup G_2 \cup \cdots \cup G_k) - \{u, v\}) = i(G_1 \cup \cdots \cup G_j - \{u, v\} \cup \cdots \cup G_k) = i(G_1) + \cdots + i(G_j - \{u, v\}) + \cdots + i(G_j) \leq i(G_1 \cup G_2 \cup \cdots \cup G_k) - 1$ since $G_1 \cup G_2 \cup \cdots \cup G_k$ is i-bicritical. But then we have that $i(G_j - \{u, v\}) \leq i(G_j) - 1$ and so G_j is i-bicritical. Therefore every G_t, $1 \leq t \leq k$, with at least two vertices is i-bicritical.

Consider the case where $u \in V(G_j)$ and $v \in V(G_l)$ for some $1 \leq j < l \leq k$, and suppose that u is a vertex such that $i(G_j - u) = i(G_j)$ and v is a vertex such that $i(G_l - v) = i(G_l)$. Then $i((G_1 \cup G_2 \cup \cdots \cup G_k) - \{u, v\}) = i(G_1 \cup \cdots \cup G_j - u \cup \cdots \cup G_l - v \cup \cdots \cup G_k) = i(G_1) + \cdots + i(G_j - u) + \cdots + i(G_l - v) + \cdots + i(G_k) = i(G_1) + \cdots + i(G_j) + \cdots + i(G_l) + \cdots + i(G_k) = i(G_1 \cup G_2 \cup \cdots \cup G_k)$, a contradiction to $G_1 \cup G_2 \cup \cdots \cup G_k$ being i-bicritical. Therefore at most one of G_j and G_l has an i-stable vertex, and we can conclude that at most one G_t, $1 \leq t \leq k$, is not i-critical. □

Recall that the *join of G and H*, denoted $G + H$, is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$. The graph $G_1 + G_2 + \cdots + G_k$ is defined recursively by $(G_1 + \cdots + G_{k-1}) + G_k$. Note that $i(G_1 + G_2 + \cdots + G_k) = \min\{i(G_1), i(G_2), \ldots, i(G_k)\}$.

Proposition 3.25. *The graph $G_1 + G_2 + \cdots + G_k$ is i-bicritical if and only if all of G_1, G_2, …, G_k are i-bicritical and either*

(a) $i(G_1) = i(G_2) = \cdots = i(G_k)$ and all but one of G_1, G_2, …, G_k are i-critical or

(b) $i(G_1) - 1 = i(G_2) = \cdots = i(G_k)$ and G_1 has no edges and all of G_2, G_3, …, G_k are i-critical.

Proof. Let $G = G_1 + G_2 + \cdots + G_k$.

Suppose without loss of generality that G_1 is not i-bicritical and let $\{x, y\} \subseteq V(G_1)$ such that $i(G_1 - \{x, y\}) \geq i(G_1)$. Let D be an i-set of $G - \{x, y\}$. Now $D \cap V(G_j) \neq \emptyset$ for only one j, $1 \leq j \leq k$. If $D \cap V(G_1) \neq \emptyset$, then $i(G - \{x, y\}) = i(G_1 - \{x, y\}) \geq i(G_1) \geq i(G)$. If $D \cap V(G_j) \neq \emptyset$ for $j \neq 1$, then $i(G - \{x, y\}) = i(G_j) \geq i(G)$. In either case, we conclude that all of G_1, G_2, ..., G_k are i-bicritical.

Suppose without loss of generality that G_1 and G_2 are not i-critical and let $x \in V(G_1)$ and $y \in V(G_2)$ such that $i(G_1 - x) \geq i(G_1)$ and $i(G_2 - y) \geq i(G_2)$. Let D be an i-set of $G - \{x, y\}$. Again, $D \cap V(G_j) \neq \emptyset$ for only one j, $1 \leq j \leq k$. If $D \cap V(G_1) \neq \emptyset$, then $i(G - \{x, y\}) = i(G_1 - x) \geq i(G_1) \geq i(G)$. Likewise if $D \cap V(G_2) \neq \emptyset$. If $D \cap V(G_j) \neq \emptyset$ for $j \neq 1, 2$, then $i(G - \{x, y\}) = i(G_j) \geq i(G)$. In any case, we can conclude that at most one of G_1, G_2, ..., G_k is not i-critical.

Suppose without loss of generality that $i(G_1) \geq i(G_2) + 2$. Let $\{x, y\} \subseteq V(G_1)$ and let D be an i-set of $G - \{x, y\}$. If $D \cap V(G_1) \neq \emptyset$, then $i(G - \{x, y\}) = i(G_1 - \{x, y\}) \geq i(G_1) - 2 \geq i(G_2) \geq i(G)$. If $D \cap V(G_j) \neq \emptyset$ for $j \neq 1$, then $i(G - \{x, y\}) = i(G_j) \geq i(G)$. In either case, we conclude that the independent domination numbers of G_1, G_2, ..., G_k differ by at most one.

Suppose without loss of generality that $i(G_1) + 1 = i(G_2) = i(G_3)$. Let $x \in V(G_2)$ and $y \in V(G_3)$ and let D be an i-set of $G - \{x, y\}$. If $D \cap V(G_2) \neq \emptyset$, then $i(G - \{x, y\}) = i(G_2 - x) \geq i(G_2) - 1 \geq i(G_1) \geq i(G)$. Likewise if $D \cap V(G_3) \neq \emptyset$. If $D \cap V(G_j) \neq \emptyset$ for $j \neq 2, 3$, then $i(G - \{x, y\}) = i(G_j) \geq i(G)$. In any case, G is not i-bicritical and so either $i(G_1) = i(G_2) = \cdots = i(G_k)$ or $i(G_1) - 1 = i(G_2) = \cdots = i(G_k)$.

Suppose that $i(G_1) - 1 = i(G_2) = \cdots = i(G_k)$ and G_1 has at least one edge. Let $xy \in E(G_1)$ and let D be an i-set of $G - \{x, y\}$. If $D \cap V(G_1) \neq \emptyset$, then $i(G - \{x, y\}) = i(G_1 - \{x, y\}) \geq i(G_1) - 1 = i(G_2) = i(G)$. If $D \cap V(G_j) \neq \emptyset$ for $j \neq 1$, then $i(G - \{x, y\}) = i(G_j) = i(G)$. In either case, G is not i-bicritical and so G_1 has no edges.

Suppose that $i(G_1) - 1 = i(G_2) = \cdots = i(G_k)$ and without loss of generality suppose that G_2 is not i-critical. Let $x \in V(G_1)$ and $y \in V(G_2)$ such that $i(G_2 - y) \geq i(G_2)$. Let D be an i-set of $G - \{x, y\}$. If $D \cap V(G_1) \neq \emptyset$, then $i(G - \{x, y\}) = i(G_1 - x) = i(G_1) - 1 = i(G_2) = i(G)$. If $D \cap V(G_2) \neq \emptyset$, then $i(G - \{x, y\}) = i(G_2 - y) \geq i(G_2) = i(G)$. If $D \cap V(G_j) \neq \emptyset$ for $j \neq 1, 2$, then $i(G - \{x, y\}) = i(G_j) = i(G)$. In any case, G is not i-bicritical and so all of G_2, G_3, ..., G_k are i-critical if $i(G_1) - 1 = i(G_2) = \cdots = i(G_k)$.

Now suppose that all of G_1, G_2, ..., G_k are i-bicritical and either (a) or (b) holds. Let $\{x, y\} \subseteq V(G)$.

If, without loss of generality, $\{x, y\} \subseteq V(G_1)$, let D be an i-set of $G_1 - \{x, y\}$. If $i(G_1) - 1 = i(G_2) = \cdots = i(G_k)$, then $i(G - \{x, y\}) \leq |D| = i(G_1 - \{x, y\}) = i(G_1) - 2 < i(G)$. If $i(G_1) = i(G_2) = \cdots = i(G_k)$, then $i(G - \{x, y\}) \leq |D| = i(G_1 - \{x, y\}) \leq i(G_1) - 1 < i(G)$.

Suppose, without loss of generality, that $x \in V(G_1)$ and $y \in V(G_2)$. If $i(G_1) - 1 = i(G_2) = \cdots = i(G_k)$, then all of G_2, G_3, ..., G_k are i-critical. Let D be an i-set of $G_2 - y$. Then $i(G - \{x, y\}) \leq |D| = i(G_2 - y) = i(G_2) - 1 < i(G)$. If $i(G_1) = i(G_2) = \cdots = i(G_k)$, then at most one of G_1, G_2, ..., G_k is not i-critical. Suppose G_1 is i-critical and let D be an i-set of $G_1 - x$. Then $i(G - \{x, y\}) \leq |D| = i(G_1 - x) = i(G_1) - 1 < i(G)$.

Therefore in any case $G = G_1 + G_2 + \cdots + G_k$ is i-bicritical. □

Recall that the *coalescence of* G *and* H *respect to* x *and* y is the graph $G \cdot_{xy} H$ with vertex set $V(G \cdot_{xy} H) = (V(G) - \{x\}) \cup (V(H) - \{y\}) \cup \{v\}$, where $v \notin V(G) \cup V(H)$, and edge set $E(G \cdot_{xy} H) = E(G - x) \cup E(H - y) \cup \{vw : xw \in E(G) \text{ or } yw \in E(H)\}$. The graph $G_1 \cdot G_2 \cdot \cdots \cdot G_k$ is defined recursively by $G_1 \cdot G_2 \cdot \cdots \cdot G_k = (G_1 \cdot G_2 \cdot \cdots \cdot G_{k-1}) \cdot G_k$. If the context is clear, or if the vertices x and y are not important, $G \cdot H$ is used instead of $G \cdot_{xy} H$. In the following results we show that there are two cases for $i(G \cdot_{xy} H)$ when $G \cdot_{xy} H$ is i-bicritical: $i(G \cdot_{xy} H) = i(G) + i(H) - 1$ or $i(G \cdot_{xy} H) = i(G) + i(H)$. Note that this differs from the i-critical case where the only option is $i(G \cdot_{xy} H) = i(G) + i(H) - 1$ if $G \cdot_{xy} H$ is i-critical.

Proposition 3.26. *The graph $G \cdot_{xy} H$ is i-bicritical if G and H are both i-bicritical and at most one of G and H is not i-critical. In this case, $i(G \cdot_{xy} H) = i(G) + i(H) - 1$.*

Proof. Suppose that $i(G \cdot_{xy} H) \leq i(G) + i(H) - 2$. Consider S an i-set of $G \cdot_{xy} H$, and let $S_1 = S \cap V(G)$ and $S_2 = S \cap V(H)$. Recall that v is the vertex of identification of G and H. If $v \in S$ then S_1 is an independent dominating set of G and S_2 is an independent dominating set of H. But then $|S_1| \geq i(G)$ and $|S_2| \geq i(H)$ and so $|S| \geq i(G) + i(H) - 1$, a contradiction. If $v \notin S$, then S_1 or S_2 dominates v. Without loss of generality, say S_1 dominates v. Then S_1 is an independent dominating set of G and so $|S_1| \geq i(G)$. If S_2 also dominates v, then S_2 is an independent dominating set of H and so $|S_2| \geq i(H)$. But then $|S_1 \cup S_2| \geq i(G) + i(H)$, a contradiction. If S_2 does not dominate v, then S_2 is an independent dominating set of $H - y$ and so $|S_2| \geq i(H) - 1$. But then $|S_1 \cup S_2| \geq i(G) + i(H) - 1$, a contradiction. Thus $i(G \cdot_{xy} H) \geq i(G) + i(H) - 1$.

Suppose that G and H are i-bicritical, G is also i-critical, and there exists an i-set S of H such that $y \in S$. We first show that $i(G \cdot_{xy} H) = i(G) + i(H) - 1$. Let S_1 be an i-set of $G - x$ and let S_2 be an i-set of H such that $y \in S_2$. Since G is i-critical, we have that $S_1 \cup S_2$ is an independent dominating set of $G \cdot_{xy} H$ and $i(G \cdot_{xy} H) \leq |S_1 \cup S_2| = i(G) + i(H) - 1$.

We now show that $G \cdot_{xy} H$ is i-bicritical. Consider $G \cdot_{xy} H - \{u_1, u_2\}$.

Case 1: $\{u_1, u_2\} \subseteq V(G)$.

Let S_1 be an i-set of $G - \{u_1, u_2\}$ and let S_2 be an i-set of $H - y$. Since y is critical in H, we have that $N_H(y) \cap S_2 = \emptyset$. Thus $S = S_1 \cup S_2$ is an independent dominating set of $G \cdot_{xy} H$ and $i(G \cdot_{xy} H) \leq |S| \leq i(G) - 1 + i(H) - 1 = i(G) + i(H) - 2 = i(G \cdot_{xy} H) - 1$.

Case 2: $\{u_1, u_2\} \subseteq V(H)$.

Let S_1 be an i-set of $G-x$ and let S_2 be an i-set of $H-\{u_1,u_2\}$. Since G is i-critical, we have that $N_G(x)\cap S_1=\emptyset$. Thus $S=S_1\cup S_2$ is an independent dominating set of $G\cdot_{xy}H$ and $i(G\cdot_{xy}H)\leq |S|\leq i(G)-1+i(H)-1=i(G)+i(H)-2=i(G\cdot_{xy}H)-1$.

Case 3: $u_1\in V(G)-\{x\}$ and $u_2\in V(H)-\{y\}$.

Let S_2 be an i-set of $H-\{u_2,y\}$. If S_2 dominates y, let S_1 be an i-set of $G_1-\{u_1,x\}$. Then $S=S_1\cup S_2$ is an independent dominating set of $G\cdot_{xy}H-\{u_1,u_2\}$ and so $i(G\cdot_{xy}H)\leq |S|\leq i(G)-1+i(H)-1=i(G)+i(H)-2=i(G\cdot_{xy}H)-1$. If S_2 does not dominate y, let S_1 be an i-set of $G-u_1$. Then $S=S_1\cup S_2$ is an independent set of $G\cdot_{xy}H-\{u_1,u_2\}$ and so $i(G\cdot_{xy}H)\leq |S|\leq i(G)-1+i(H)-1=i(G)+i(H)-2=i(G\cdot_{xy}H)-1$.

Hence in any case, $G\cdot_{xy}H$ is i-bicritical. □

Proposition 3.27. *If x is in an i-set of G and y is in an i-set of H, then $i(G\cdot_{xy}H)=i(G)+i(H)-1$.*

Proof. Let D_1 be an i-set of G such that $x\in D_1$ and let D_2 be an i-set of H such that $y\in D_2$. Then $D=D_1\cup D_2$ is an independent dominating set of $G\cdot_{xy}H$ and so $i(G\cdot_{xy}H)=i(G)+i(H)-1$. □

Proposition 3.28. *If x is not in any i-set of G and y is not in any i-set of H, then $i(G\cdot_{xy}H)=i(G)+i(H)$.*

Proof. If x is not in any i-set of G, this implies that x is not i-critical in G. Likewise y is not i-critical in H. Let D be an i-set of $i(G\cdot_{xy}H)$. Let $D_1=D\cap V(G)$ and $D_2=D\cap V(H)$. If $v\in D$, then D_1 is an independent dominating set of G and D_2 is an independent dominating set of H. But since $v\in D_1$ and $v\in D_2$ we have that $|D_1|>i(G)$ and $|D_2|>i(H)$. But then $|D|>i(G)+i(H)-1$. Suppose then that $v\notin D$. If D_1 dominates v then D_1 is an independent dominating set of G and so $|D_1| \geq i(G)$. But then D_2 is an independent dominating set of $H - y$ and so $|D_2| \geq i(H)$ since y is not i-critical. But then $|D| \geq i(G) + i(H)$. If D_1 does not dominate v, then D_2 is an independent dominating set of H and so $|D_2| \geq i(H)$. But then D_1 is an independent dominating set of $G - x$ and so $|D_1| \geq i(G)$ since x is not i-critical. But then $|D| \geq i(G) + i(H)$. In all cases we have that $i(G \cdot_{xy} H) \geq i(G) + i(H)$.

Finally, let D_1 be an i-set of G and D_2 be an i-set of H. Thus $x \notin D_1$ and $y \notin D_2$ and so $D = D_1 \cup D_2$ is an independent dominating set of $G \cdot_{xy} H$. Therefore $i(G \cdot_{xy} H) = i(G) + i(H)$. □

Notice that, without loss of generality, if x is in an i-set of G and y is not in any i-set of H, it possible to have either $i(G \cdot_{xy} H) = i(G) + i(H) - 1$ or $i(G \cdot_{xy} H) = i(G) + i(H)$. For example, if we let $G = K_{3,3}$ and $H = K_{3,4}$ where x is any vertex of $K_{3,3}$ and y is a vertex of degree 3 in $K_{3,4}$, then $G \cdot_{xy} H$ has $i(G \cdot_{xy} H) = 5 = i(G) + i(H) - 1$. Let $G_{[v]}$ be the graph with vertex set $V(G_{[v]}) = V(G) \cup \{v'\}$ and edge set $E(G_{[v]}) = E(G) \cup \{uv' : u \in N_G[v]\}$. This construction is called the *expansion of G via v* and is studied later in this section. If we let $G = K_{3,3}[v]$ (the expansion via v of $K_{3,3}$, where v is any vertex in $K_{3,3}$) and $H = K_{3,4}$ where x is v', the vertex added to $K_{3,3}$ in the expansion, and y is a vertex of degree 3 in $K_{3,4}$, then $G \cdot_{xy} H$ has $i(G \cdot_{xy} H) = 6 = i(G) + i(H)$. These two cases are pictured below in Figure 3.5.

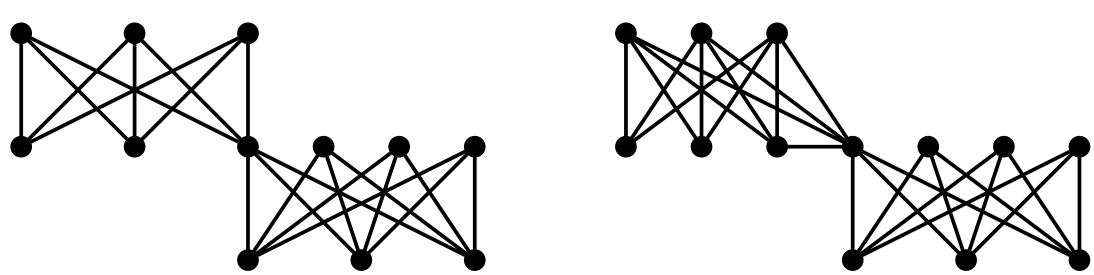

Figure 3.5: The graphs $K_{3,3} \cdot K_{3,4}$ and $K_{3,3_{[v]}} \cdot K_{3,4}$.

Proposition 3.29. *If x is in an i-set of G, y is not in any i-set of H, and*

- *$i(G \cdot_{xy} H) = i(G) + i(H)$, then x is not i-critical in G*
- *$i(G \cdot_{xy} H) = i(G) + i(H) - 1$, then x is i-critical in G.*

Proof. Suppose $i(G \cdot_{xy} H) = i(G) + i(H)$ and suppose x is i-critical in G. Let D_1 be an i-set of $G - x$ and let D_2 be an i-set of H. Then $D = D_1 \cup D_2$ is an i-set of $G \cdot_{xy} H$. But $|D| = i(G) + i(H) - 1$, a contradiction. Therefore x is not critical in G.

Suppose $i(G \cdot_{xy} H) = i(G) + i(H) - 1$ and let D be an i-set of $G \cdot_{xy} H$. Let $D_1 = D \cap V(G)$ and $D_2 = D \cap V(H)$. If $v \in D$ then D_1 is an independent dominating set of G and D_2 is an independent dominating set of H. But then $|D_1| \geq i(G)$ and $|D_2| > i(H)$ and so $|D| \geq i(G) + i(H) + 1 - 1 = i(G) + i(H)$, a contradiction. Thus suppose that $v \notin D$. If D_1 dominates v, then D_1 is an independent dominating set of G and D_2 is an independent dominating set of $H - y$ and so $|D_1| \geq i(G)$ and $|D_2| \geq i(H)$. But then $|D| \geq i(G) + i(H)$, a contradiction. If D_1 does not dominate x, then D_1 is an independent dominating set of $G - x$ and D_2 is an independent dominating set of H. Thus $|D_1| \geq i(G) - 1$ and $|D_2| \geq i(H)$. But since $i(G \cdot_{xy} H) = i(G) + i(H) - 1$ we have that $|D_1| = i(G) - 1$ and $|D_2| = i(H)$, and so x is i-critical in G. □

Proposition 3.30. *If $G \cdot_{xy} H$ is i-bicritical with $i(G \cdot_{xy} H) = i(G) + i(H)$, then x is not critical in G and y is not critical in H and either:*

- *G is i-bicritical or*
- *there exists $\{w, z\} \subseteq V(G)$ such that $i(G - \{w, z\}) = i(G)$ and there exists an i-set D_1 of $G - \{w, z\}$ such that $x \in D_1$ and an i-set D_2 of H with $y \in D_2$ or*
- *there exists $\{w, z\} \subseteq V(G)$ such that $i(G - \{w, z\}) = i(G)$ and $i(G - \{w, z, x\}) = i(G) - 1$ and there exists an i-set D of H such that $y \notin D$*

and likewise for H.

Proof. The fact that x is not i-critical in G and y is not i-critical in H follows from Proposition 3.28 and Proposition 3.29.

Without loss of generality, let $\{w, z\} \subseteq V(G)$ such that $i(G - \{w, z\}) \geq i(G)$. Let D be an i-set of $G \cdot_{xy} H - \{w, z\}$. Thus $i(G) + i(H) - 1 \geq |D| \geq i(G) + i(H) - 2$. Let $D_1 = D \cap V(G)$ and $D_2 = D \cap V(H)$.

Suppose $v \in \{w, z\}$. Then D_1 is an independent dominating set of $G - \{w, z\}$ and D_2 is an independent dominating set of $H - y$. Therefore $|D| \geq i(G) + i(H)$, a contradiction.

Suppose $v \notin \{w, z\}$ and $v \in D$. Then D_1 is an independent dominating set of $G - \{w, z\}$ and D_2 is an independent dominating set of H. Therefore $|D| \geq i(G) + i(H) - 1$ and if equality is attained we have $|D_1| = i(G)$ and $|D_2| = i(H)$.

Suppose $v \notin \{w, z\}$ and $v \notin D$. If D_1 dominates v, then D_1 is an independent dominating set of $G - \{w, z\}$ and D_2 is an independent dominating set of $H - y$. But then $|D| \geq i(G) + i(H)$, a contradiction. If D_1 does not dominate v, then D_2 is an independent dominating set of H and D_1 is an independent dominating set of $G - \{w, z, x\}$. Thus $|D_2| \geq i(H)$ and $|D_1| \geq i(G - \{w, z, x\}) \geq i((G - \{w, z\}) - x) \geq i(G - \{w, z\}) - 1 \geq i(G) - 1$. But then $|D| \geq i(G) + i(H) - 1$ and if equality is attained we have $|D_1| = i(G) - 1$ and $|D_2| = i(H)$. □

Proposition 3.31. *If $G \cdot_{xy} H$ is i-bicritical with $i(G \cdot_{xy} H) = i(G) + i(H) - 1$ then x is i-critical in G, y is i-critical in H, both G and H are i-bicritical, and at most one of G and H is not i-critical.*

Proof. Let $u \in N_G(x)$. Then $i(G) + i(H) - 2 = i(G \cdot_{xy} H - \{u, v\}) = i(G - \{u, x\}) + i(H - y) \geq i(G) - 1 + i(H) - 1 = i(G) + i(H) - 2$. Therefore $i(G - \{u, x\}) = i(G) - 1$ and $i(H - y) = i(H) - 1$ and so y is i-critical in H. Likewise for $u \in N_H(y)$ we can conclude that x is i-critical in G.

Without loss of generality, suppose $\{w, z\} \subseteq V(G)$ such that $i(G - \{w, z\}) \geq i(G)$. Let D be an i-set of $G \cdot_{xy} H - \{w, z\}$ and let $D_1 = D \cap V(G)$ and $D_2 = D \cap V(H)$.

If $v \in \{w, z\}$ then $i(G \cdot_{xy} H - \{w, z\}) = i(G - \{w, z\}) + i(H - y) = i(G) + i(H) - 1$, a contradiction.

If $v \notin \{w, z\}$ and $v \in D$ then D_1 is an independent dominating set of $G - \{w, z\}$ and D_2 is an independent dominating set of H. Therefore $|D| \geq i(G) + i(H) - 1$, a contradiction.

Thus suppose that $v \notin \{w, z\}$ and $v \notin D$. If D_1 dominates v, then D_1 is an independent dominating set of $G - \{w, z\}$ and D_2 is an independent dominating set of $H - y$. Therefore $|D| \geq i(G) + i(H) - 1$, a contradiction. If D_1 does not dominate v, then D_2 is an independent dominating set of H and D_1 is an independent dominating set of $G - \{w, z, x\}$. Thus $|D_2| \geq i(H)$ and $|D_1| \geq i(G - \{w, z, x\}) = i((G - \{w, z\}) - x) \geq i(G - \{w, z\}) - 1 \geq i(G) - 1$. Therefore $|D| \geq i(G) + i(H) - 1$, a contradiction. Therefore both G and H are i-bicritical.

Suppose that G and H are both not i-critical and let $w \in V(G - x)$ such that $i(G - w) \geq i(G)$ and let $z \in V(H - y)$ such that $i(H - z) \geq i(H)$. Let D be an i-set of $G \cdot_{xy} H - \{w, z\}$ and let $D_1 = D \cap V(G)$ and $D_2 = D \cap V(H)$.

If $v \in D$, then D_1 is an independent dominating set of $G - w$ and D_2 is an independent dominating set of $H - z$. Therefore $|D| \geq i(G) + i(H) - 1$, a contradiction.

Suppose $v \notin D$. Let D_1 dominates v, then D_1 is an independent dominating set of $G - w$ and D_2 is an independent dominating set of $H - \{y, z\}$. But then $|D_1| \geq i(G)$ and $|D_2| \geq i(H - \{y, z\}) = i((H - z) - y) \geq i(H - z) - 1 \geq i(H) - 1$. Therefore $|D| \geq i(G) + i(H) - 1$, a contradiction. If D_1 does not dominate v, then D_2 is an independent dominating set of $H - z$ and D_1 is an independent dominating set of $G - \{w, x\}$. But then $|D_2| \geq i(H)$ and $|D_1| \geq i(G - \{w, x\}) = i((G - w) - x) \geq i(G - w) - 1 \geq i(G) - 1$. Therefore $|D| \geq i(G) + i(H) - 1$, a contradiction. Therefore at most one of G and H is not i-critical. □

Using a proof similar to Proposition 3.31, we can show the following.

Proposition 3.32. *If $G \cdot_{xy} H$ is γ-bicritical with $\gamma(G \cdot_{xy} H) = \gamma(G) + \gamma(H) - 1$ then x is γ-critical in G, y is γ-critical in H, both G and H are γ-bicritical, and at most one of G and H is not γ-critical.*

This differs from the statement given by Brigham et al. which says that a graph is γ-bicritical if and only if each block is γ-bicritical and γ-critical. A γ-bicritical graph with a block that is not γ-critical is shown in Figure 3.6. This graph is also i-bicritical and the same block that is not γ-critical is also not i-critical. Note that this graph has $\gamma(G) = i(G) = 5$ and the block that is neither γ-critical nor i-critical is the block that is isomorphic to $K_3 \square K_{3_{[v]}}$ (the expansion of $K_3 \square K_3$ via v).

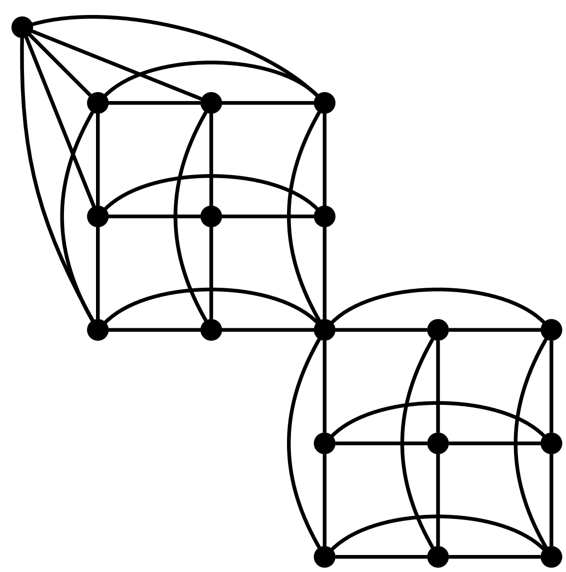

Figure 3.6: A γ-bicritical and i-bicritical graph with a block that is neither γ-critical nor i-critical.

Recall that if G is i-bicritical, then there is no vertex $v \in V(G)$ such that $\deg(v) = 2$. It is possible though for G to be i-bicritical and have a vertex of degree one. An example of such a graph is $K_2 \cdot_{xy} K_{n,n+1}$, $n \geq 3$, where x is any vertex of K_2 and y is any vertex of $K_{n,n+1}$ of degree n. The graph $K_2 \cdot_{xy} K_{3,4}$ is shown in Figure 3.7.

Additionally, this example shows that an i-bicritical graph can have a cut-edge and so the edge connectivity need not be very large.

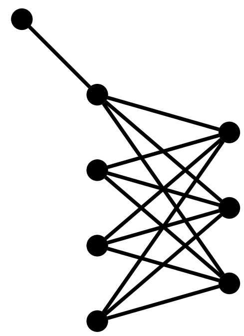

Figure 3.7: An i-bicritical graph with a vertex of degree 1.

Let G_1 be a graph with subgraph $H_1 \cong H$ and G_2 be a graph with subgraph $H_2 \cong H$. Recall that the graph $G_1(H_1)\widehat{\odot}G_2(H_2)$ is the graph obtained from G_1 and G_2 by identifying vertices of H_1 with the corresponding vertices of H_2 and adding the set of edges $\{x_1x_2 : x_1 \in V(G_1) - V(H_1)$ and $x_2 \in V(G_2) - V(H_2)\}$. Let $H_{1,2}$ be a subgraph of $G_1(H_1)\widehat{\odot}G_2(H_2)$ and let H_3 be a subgraph of G_3 where $H_{1,2} \cong H_3$. Recall that the graph $(G_1(H_1)\widehat{\odot}G_2(H_2))(H_{1,2})\widehat{\odot}G_3(H_3)$ is obtained by identifying vertices of $H_{1,2}$ with corresponding vertices of H_3 and adding edges $\{x_{1,2}x_3 : x_{1,2} \in V(G_1(H_1)\widehat{\odot}G_2(H_2) - H_{1,2})$ and $x_3 \in V(G_3 - H_3)\}$. This can be generalized similarly for more than three graphs as the graph

$$G_{\widehat{\odot}} = (((G_1(H_1)\widehat{\odot}G_2(H_2))(H_{1,2})\widehat{\odot}G_4(H_4)) \cdots \widehat{\odot}G_{m-1}(H_{m-1}))(H_{1,2,\ldots,m-1})\widehat{\odot}G_m(H_m).$$

Also recall that $i(G_{\widehat{\odot}}) = \min\{i(G_1), i(G_2), \ldots, i(G_m)\}$.

Proposition 3.33. *For each $H \in \{H_1, H_2, \ldots, H_m, H_{1,2}, H_{1,2,3}, \ldots, H_{1,2,\ldots,m-1}\}$, suppose $\alpha(H) \leq k - 3$. Then*

$$G_{\widehat{\odot}} = (((G_1(H_1)\widehat{\odot}G_2(H_2))(H_{1,2})\widehat{\odot}G_4(H_4)) \cdots \widehat{\odot}G_{m-1}(H_{m-1}))(H_{1,2,\ldots,m-1})\widehat{\odot}G_m(H_m)$$

is k-i-bicritical if and only if $k = \min\{i(G_1), i(G_2), \ldots, i(G_m)\}$ and every pair of vertices $\{x, y\}$ in $G_{\widehat{\odot}}$ is either

- *contained in some $V(G_j)$, $1 \leq j \leq m$, where $i(G_j - \{x, y\}) \leq k - 1 < i(G_j)$ or,*
- *without loss of generality, x is in some G_j, $1 \leq j \leq m$, such that $i(G_j - x) = k - 1$ and $y \notin V(G_j)$.*

Proof. Suppose $G_{\widehat{\odot}}$ is k-i-bicritical. Then by construction, $k = \min\{i(G_1), i(G_2), \ldots, i(G_m)\}$. Consider $G_{\widehat{\odot}} - \{x, y\}$. Let D be an i-set of $G_{\widehat{\odot}} - \{x, y\}$, and so $|D| \leq k-1$. By construction of $G_{\widehat{\odot}}$, $D \subseteq V(G_j)$ for some $1 \leq j \leq m$. If $\{x, y\} \cap V(G_j) = \emptyset$, then D is an independent dominating set of $G_{\widehat{\odot}}$, a contradiction. Say $\{x, y\} \subseteq V(G_j)$. Then D is an independent dominating set of $G_j - \{x, y\}$ and so $i(G_j - \{x, y\}) \leq |D| \leq k-1 < i(G_j)$. Suppose, without loss of generality, that $v \in V(G_j)$ and $y \notin V(G_j)$. Then D is an independent dominating set of $G_j - x$ and so $i(G_j - x) \leq |D| \leq k - 1 < i(G_j)$.

Suppose that $k = \min\{i(G_1), i(G_2), \ldots, i(G_m)\}$ and for each $\{x, y\} \subseteq V(G_{\widehat{\odot}})$ either $\{x, y\} \subseteq V(G_j)$ for some $1 \leq j \leq m$ where $i(G_j - \{x, y\}) \leq k - 1$, or, without loss of generality, $x \in V(G_j)$ for some $1 \leq j \leq m$ and $y \notin V(G_j)$ and $i(G_j - x) = k - 1$. By construction $i(G_{\widehat{\odot}}) = k$. Consider $G_{\widehat{\odot}} - \{x, y\}$. If $\{x, y\} \subseteq V(G_j)$, then $i(G_j - \{x, y\}) \leq k - 1$. Let D be an i-set of $G_j - \{x, y\}$. Otherwise $x \in V(G_j)$ and $y \notin V(G_j)$, and $i(G_j - x) = k - 1$. In this case, let D be an i-set of $G_j - x$. In both cases, D is an independent dominating set of $G_{\widehat{\odot}} - \{x, y\}$ and so $i(G_{\widehat{\odot}} - \{x, y\}) \leq |D| \leq k - 1 < i(G_{\widehat{\odot}})$. □

For the simpler construction, $G_1(H_1) \widehat{\odot} G_2(H_2) \widehat{\odot} \cdots \widehat{\odot} G_m(H_m)$, we have the following result.

Corollary 3.34. *Let $G_1, G_2, \ldots, G_m$ and H be disjoint graphs such that for $j = 1, 2, \ldots, m$, G_j has a subgraph $H_j \cong H$. Suppose $\alpha(H) \leq k - 3$. Then the graph*

$G = G_1(H_1)\widehat{\odot}G_2(H_2)\widehat{\odot}\cdots\widehat{\odot}G_m(H_m)$ is k-i-bicritical if and only if $k = i(G_1) = i(G_2) = \cdots = i(G_m)$ and for every $\{x, y\} \subseteq V(G)$ either

- *$\{x, y\} \subseteq V(G_j)$ for some $1 \leq j \leq m$ where $i(G_j - \{x, y\}) \leq k - 1$ or,*
- *without loss of generality, x is in some G_j, $1 \leq j \leq m$, such that $i(G_j - x) = k - 1$ and $y \notin V(G_j)$.*

Recall that the *wreath product of G with H*, written $G[H]$, is the graph with vertex set $V(G[H]) = \{(g, h) : g \in V(G), h \in V(H)\}$ and edge set $E(G[H]) = \{(g_1, h_1)(g_2, h_2) : g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)\}$. Proposition 2.24 shows that $i(G[H]) = i(G)i(H)$.

Proposition 3.35. *The graph $G[H]$ is i-bicritical if and only if every vertex of G is in an i-set of G and H is both i-critical and i-bicritical with $i(H) \geq 3$.*

Proof. Consider $v_1 = (g_1, h_1) \in V(G[H])$ and $v_2 = (g_2, h_2) \in V(G[H])$.

Case 1: $g_1 = g_2 = g$.

That is, h_1 and h_2 are in the same copy of H in $G[H]$. Let S_1 be an i-set of G containing g, let S_g be an i-set of $H - \{h_1, h_2\}$, and let S_2 be an i-set of H. Then $D = \{(g, h) : h \in S_g\} \cup \{(x, y) : x \neq g, x \in S_1, y \in S_2\}$ is an independent dominating set of $G[H] - \{v_1, v_2\}$. But $i(G[H] - \{v_1, v_2\}) \leq |D| \leq i(H) - 1 + (i(G) - 1)(i(H)) = i(G)i(H) - 1 = i(G[H]) - 1$.

Case 2: $g_1 \neq g_2$ and $g_1g_2 \in E(G)$.

Without loss of generality, say that G has an i-set that contains g_1. Let S_1 be an i-set of G such that $g_1 \in S_1$, let S_{g_1} be an i-set of $H - h_1$, and let S_2 be an i-set of H. Then $D = \{(g_1, h) : h \in S_{g_1}\} \cup \{(x, y) : x \neq g_1, x \in S_1, y \in S_2\}$ is an independent dominating set of $G[H] - \{v_1, v_2\}$. But $i(G[H]) \leq |D| = i(H) - 1 + (i(G) - 1)(i(H)) = i(G)i(H) - 1 = i(G[H]) - 1$.

Case 3: $g_1 \neq g_2$ and $g_1g_2 \notin E(G)$.

Without loss of generality, say that G has an i-set that contains g_1. Let S_1 be an i-set of G such that $g_1 \in S_1$.

If $g_2 \in S_1$, let S_{g_1} be an i-set of $H - h_1$, let S_{g_2} be an i-set of $H - h_2$, and let S_2 be an i-set of H. Then $D = \{(g_1, h) : h \in S_{g_1}\} \cup \{(g_2, h) : h \in S_{g_2}\} \cup \{(x, y) : x \neq g_1, x \neq g_2, x \in S_1, y \in S_2\}$ is an independent dominating set of $G[H] - \{v_1, v_2\}$. But $i(G[H] - \{v_1, v_2\}) \leq |D| = i(H) - 1 + i(H) - 1 + (i(G) - 1)(i(H)) = i(G)i(H) - 2 = i(G[H]) - 2$.

If $g_2 \notin S_1$, let S_{g_1} be an i-set of $H - h_1$ and let S_2 be an i-set of H. Then $D = \{(g_1, h) : h \in S_g\} \cup \{(x, y) : x \neq g_1, x \in S_1, y \in S_2\}$ is an independent dominating set of $G[H] - \{v_1, v_2\}$. But $i(G[H] - \{v_1, v_2\}) \leq |D| = i(H) - 1 + (i(G) - 1)(i(H)) = i(G)i(H) - 1 = i(G[H]) - 1$.

Thus in all cases $G[H]$ is i-bicritical.

Suppose $G[H]$ is i-bicritical with $x, y \in V(G[H])$ where $x = (g_1, h_1)$ and $y = (g_2, h_2)$. Let S be an i-set of $G[H]$ such that $|S \cap \{x, y\}| \geq 1$. Suppose without loss of generality that $x \in S$. Let $S' = \{u \in V(G) : \exists v \in V(H) \text{ with } (\mathrm{u}, \mathrm{v}) \in \mathrm{S}\}$. Then S' is an i-set of G with $g_1 \in S'$. Therefore for any $g_1, g_2 \in V(G[H])$ there is an i-set S' of G such that $|S' \cap \{g_1, g_2\}| \geq 1$.

Suppose that H is not i-bicritical and let $h_1, h_2 \in V(H)$ such that $i(H - \{h_1, h_2\}) \geq i(H)$. Let $x = (g, h_1)$ and $y = (g, h_2)$ for some $g \in V(G)$. Let S be an i-set of $G[H] - \{x, y\}$ and let $S' = \{u \in V(G) : \exists v \in V(H) \text{ with } (\mathrm{u}, \mathrm{v}) \in \mathrm{S}\}$. Then S' is an independent dominating set of G if $i(H) \geq 3$. For each $v \in V(G)$ let $S_v = \{h : (v, h) \in S\}$. If $S_g = \emptyset$, then S is an independent dominating set of G with cardinality less than $i(G)$, a contradiction. If $S_g \neq \emptyset$ then S_g is an independent dominating set of $H - \{h_1, h_2\}$ and so $|S_g| \geq i(H)$. But then for all other $S_v \neq \emptyset$, S_v is an independent dominating set of H and so $|S_v| \geq i(H)$. Then $i(G[H] - \{x, y\}) \geq i(G)i(H)$, a contradiction. Therefore H is i-bicritical.

Suppose that H is not i-critical and let $h \in V(H)$ such that $i(H - h) \geq i(H)$.

Let $g_1, g_2 \in V(G)$ and let $x = (g_1, h)$ and $y = (g_2, h)$. Consider S an i-set of $G[H] - \{x, y\}$. Let $S' = \{u \in V(G) : \exists v \in V(H) \text{ with } (\mathrm{u}, \mathrm{v}) \in \mathrm{S}\}$. Then S is an independent dominating set of G if $i(H) \geq 2$. For each $u \in V(G)$ let $S_u = \{v : (u, v) \in S\}$. If $S_{g_1} = \emptyset$ and $S_{g_2} = \emptyset$, then S is an independent dominating set of G, a contradiction. If, without loss of generality, $S_{g_1} \neq \emptyset$ and $S_{g_2} = \emptyset$, then S_{g_1} is an independent dominating set of $H - h$ and so $|S_{g_1}| \geq i(H - h) \geq i(H)$. For all other $S_u \neq \emptyset$, $|S_u| \geq i(H)$ and so $i(G[H] - \{x, y\}) \geq i(G)i(H)$, a contradiction. If $S_{g_1} \neq \emptyset$ and $S_{g_2} \neq \emptyset$ then S_{g_1} and S_{g_2} are independent dominating sets of $H - h$ and so $|S_{g_1}| \geq i(H - h) \geq i(H)$ and $|S_{g_2}| \geq i(H - h) \geq i(H)$. For all other $S_u \neq \emptyset$, $|S_u| \geq i(H)$ and so $i(G[H] - \{x, y\}) \geq i(G)i(H)$, a contradiction. Therefore H is i-critical. □

We now revisit a graph that is useful for constructing i-bicritical graphs only. This construction was introduced by Brigham et al. [10] as a way of producing γ-bicritical graphs that are not γ-critical. For a graph G and a vertex $v \in V(G)$, the *expansion of G via v*, denoted $G_{[v]}$, is the graph with vertex set $V(G_{[v]}) = V(G) \cup \{v'\}$ (where $v' \notin V(G)$) and edge set $E(G_{[v]}) = E(G) \cup \{uv' : u \in N_G[v]\}$. Note that $i(G_{[v]}) = i(G)$. Also, $G_{[v]}$ is not i-critical since $G_{[v]} - v' \cong G$ (and likewise $G_{[v]} - v \cong G$). The expansion via v construction was presented in [10], where the authors use the construction to build γ-bicritical graphs.

Proposition 3.36. [10] *If G is γ-bicritical and γ-critical, then $G_{[v]}$ is γ-bicritical.*

Proposition 3.37. [52] *If G is i-bicritical and i-critical, then $G_{[v]}$ is i-bicritical.*

In fact, the conditions in these two propositions can be relaxed. The proofs for γ-bicritical and i-bicritical are virtually identical, so we only present the proof for the modified statement about i-bicritical graphs.

Proposition 3.38. *If G is γ-bicritical and $v \in V(G)$ such that $\gamma(G - v) = \gamma(G) - 1$, then $G_{[v]}$ is γ-bicritical.*

Proposition 3.39. *If G is i-bicritical and $v \in V(G)$ such that $i(G-v) = i(G) - 1$, then $G_{[v]}$ is i-bicritical.*

Proof. Let $\{x, y\} \subseteq V(G_{[v]})$.

Case 1: $\{x, y\} \cap \{v, v'\} = \emptyset$

Let D be an i-set of $G - \{x, y\}$. Since D dominates v in G, D dominates v' in $G_{[v]}$. Thus D is an independent dominating set of $G_{[v]} - \{x, y\}$ and $i(G_{[v]} - \{x, y\}) \leq |D| \leq i(G) - 1 = i(G_{[v]}) - 1$.

Case 2: $|\{x, y\} \cap \{v, v'\}| = 1$

Without loss of generality, say that $x = v'$. Then $G_{[v]} - \{x, y\} = G_{[v]} - \{v', y\} = G - y$. Since G is i-critical we have that $i(G_{[v]} - \{x, y\}) = i(G - y) < i(G) = i(G_{[v]})$.

Case 3: $\{x, y\} = \{v, v'\}$

Then $G_{[v]} - \{x, y\} = G - v$. Since G is i-critical we have that $i(G_{[v]} - \{x, y\}) = i(G - v) < i(G) = i(G_{[v]})$

Thus in all cases $G_{[v]}$ is i-bicritical. □

If, in addition to being i-bicritical, G is an i-critical graph, then the only stable vertices of $G_{[v]}$ are v and v'. To see this, let $x \in V(G)$ such that $x \neq v$ and let D be an i-set of $G - x$. Since D dominates v in $G - x$, D dominates v' in $G_{[v]} - x$. Thus D is an independent dominating set of $G_{[v]} - x$ and $i(G_{[v]} - x) \leq |D| < i(G) = i(G_{[v]})$.

Notice that the expansion via v construction is useful in creating i-bicritical graphs with exactly two i-stable vertices (the vertices v and v' have $i(G - v) = i(G - v') = i(G)$), that is, graphs with $|V_i^0| = 2$. Graphs that are both i-critical and i-bicritical (such as $K_{n,n}$ or $K_n \square K_n$) are examples that have no i-stable vertices, that is, $|V_i^0| = 0$. For $n \geq 3$, the complete bipartite graph $K_{n,n+1}$ is an example of an i-bicritical graph with $n+1$ i-stable vertices, that is, a graph with $|V_i^0| = n + 1 \geq 4$. It is left as an open problem to investigate i-bicritical graphs with $|V_i^0| = k$ for a fixed k. In particular, does there exist an i-bicritical graph G with $|V_i^0| = 2$?

The expansion via v construction is also useful to rule out the existence of many end-vertices in an i-bicritical graph.

Proposition 3.40. *If G is an i-bicritical graph then each $v \in V(G)$ is adjacent to at most one end-vertex.*

Proof. Recall that for any i-bicritical graph, $V_i^+ = \emptyset$. Suppose G is an i-bicritical graph and suppose $v \in V(G)$ is adjacent to at least two end-vertices. Suppose $i(G-v) = i(G) - 1$, and let S be an i-set of $G - v$. Then the isolated vertices adjacent to v in G are all contained in S and so S is also an independent dominating set of G, a contradiction. Therefore v is not an i-critical vertex. Let $x \in V(G)$, $x \neq v$. Suppose $i(G-x) \geq i(G)$. By Proposition 1.18, $G - x$ is an i-critical graph, but then $G - x$ has an end-vertex, a contradiction to Proposition 1.21. Hence the only stable vertex in G is v. Therefore if G is i-bicritical with a vertex adjacent to at least two end-vertices, then G has at most one stable vertex.

Now consider the graph $G_{[x]}$. By Proposition 3.39, $G_{[x]}$ is i-bicritical since x is an i-critical vertex in G. But then $G_{[x]}$ is a graph where v is adjacent to at least two end-vertices. Also $V^0 = \{v, x, x'\}$, a contradiction to the previous paragraph. Thus if G is i-bicritical then each vertex $v \in V(G)$ is adjacent to at most one end-vertex. □

Corollary 3.41. *No tree is i-bicritical.*

Proof. Suppose T is an i-bicritical tree. Then by Proposition 3.40, each vertex in T is adjacent to at most one leaf. Let $P = v_1 v_2 \cdots v_t$ be a longest path in T. Then $\deg(v_2) = \deg(v_{t-1}) = 2$, a contradiction to Proposition 1.23. □

The above proof corrects an error made by Xu et al. [52], which claimed the result.

3.2 Strong i-Bicritical Graphs

A *strong i-bicritical graph* is a graph G such that $i(G - \{u, v\}) = i(G) - 2$ for any $\{u, v\} \subseteq V(G)$ such that $uv \notin E(G)$. Note that by Proposition 1.14, if $i(G - \{u, v\}) = i(G) - 2$ for any two vertices, then G is isomorphic to $\overline{K_n}$. In this section we discuss properties of strong i-bicritical graphs and present methods to construct strong i-bicritical graphs.

Observation 3.42. *The complete bipartite graph $K_{n,n}$, $n \geq 3$, is strong i-bicritical.*

Notice that the complete bipartite graph $K_{n,n+1}$ is not strong i-bicritical. Thus there are i-bicritical graphs that are not strong i-bicritical.

Observation 3.43. *If G is strong i-bicritical, then $S \cap (N(x) \cup N(y)) = \emptyset$ for every minimum independent dominating set S of $G - \{x, y\}$ where $xy \notin E(G)$.*

Proposition 3.44. [52] *If G is strong i-bicritical, then for any $\{x, y\} \subseteq V(G)$ with $xy \notin E(G)$ there exists an i-set S such that $\{u, v\} \subseteq S$.*

Proof. Consider an i-set S of $G - \{x, y\}$. Then $|S| = i(G) - 2$ and by Observation 3.43 $S \cap (N_G(x) \cup N_G(y)) = \emptyset$. Therefore $S \cup \{x, y\}$ is an independent dominating set of G of cardinality $i(G)$. □

To see that the converse of Proposition 3.44 is not true, consider the graph C_{3n+1}. For any two independent vertices x and y in C_{3n+1} there exists an i-set S such that $\{x, y\} \subseteq S$. However, C_{3n+1} is not strong i-bicritical since for $\{x, y\} \subset V(C_{3n+1})$ where $d(x, y) = 2$, we have that $i(C_{3n+1} - \{x, y\}) = i(C_{3n+1})$.

Corollary 2.13 shows that for any graph G and any $k \geq 5$ there is a strong k-i-bicritical graph H such that G is an induced subgraph of H. This makes characterizing the strong k-i-bicritical graphs, $k \geq 5$, difficult as there is no finite list of forbidden subgraphs. The strong 3-i-bicritical graphs are easily characterized though in the next result.

Proposition 3.45. *The only strong* 3-*i-bicritical graphs are* $\overline{K_3}$ *and* $K_{3,3,\ldots,3}$.

Proof. Let G be a strong 3-i-bicritical graph. If G has no edges, then $G \cong \overline{K_3}$. Hence assume otherwise. Since $i(G) = 3$, there exists an independent set $\{x_1, y_1, z_1\}$ that dominates G. Suppose there exists a vertex $z_2 \in V(G)$ $(z_2 \neq x_1, y_1)$ such that $z_1z_2 \notin E(G)$. Consider $G-\{z_1, z_2\}$. Then $i(G-\{z_1, z_2\}) = 1$ and there exists a vertex $v \in V(G)$ such that v dominates $G-\{z_1, z_2\}$. Note that $\{v, z_1, z_2\}$ is an independent dominating set of G. In particular, we have that $vz_2 \notin E(G)$. Consider $G - \{v, z_2\}$. Then $i(G-\{v, z_2\}) = 1$ and there exists a vertex $w \in V(G)$ $(w \neq x_1, y_1, z_1)$ such that w dominates $G-\{v, z_2\}$. Note that $wz_2 \notin E(G)$ and $wv \notin E(G)$ since $\{v, w, z_2\}$ is an independent dominating set of G. But v dominates $G - \{z_1, z_2\}$ and so $wv \in E(G)$, a contradiction. Therefore there does not exist a vertex z_2 $(\neq x_1, y_1)$ such that $z_1z_2 \notin E(G)$.

Likewise there is no vertex x_2 such that $x_1x_2 \notin E(G)$ nor a vertex y_2 such that $y_1y_2 \notin E(G)$. Since $\{x_1, y_1, z_1\}$ was an arbitrary i-set of G, and every vertex of G is contained in some i-set by Proposition 1.10, we conclude that $G \cong K_{3,3,\ldots,3}$. □

Thus for the remainder of this section, we discuss strong i-bicritical graphs with $i(G) \geq 4$.

Proposition 3.46. *If G is strong i-bicritical, then G is i-critical.*

Proof. Note first that if G is strong i-bicritical, then $i(G) \neq 1$ so every vertex has at least one vertex that is nonadjacent. We want to show that $i(G - x) = i(G) - 1$ for all $x \in V(G)$. Consider $G - \{x, y\}$ for $\{x, y\} \subseteq V(G)$ with $xy \notin E(G)$. Since G is strong i-bicritical we have that $i(G - \{x, y\}) = i(G) - 2$. Consider an i-set S of $G - \{x, y\}$. By Observation 3.43, $S \cap (N_G(x) \cup N_G(y)) = \emptyset$. Therefore $S \cup \{y\}$ dominates $G - x$ and $|S \cup \{y\}| = i(G) - 1$. Therefore G is i-critical. □

Proposition 3.47. *If G is strong i-bicritical, then G is i-bicritical.*

Proof. Note first that if G is i-bicritical, then $i(G) \neq 1$ so every vertex has at least one vertex that is nonadjacent. We want to show that $i(G - \{x, y\}) \leq i(G) - 1$ for all $\{x, y\} \subseteq V(G)$. If $xy \notin E(G)$, then $i(G - \{x, y\}) = i(G) - 2$ since G is strong i-bicritical. If $xy \in E(G)$, then $i(G - \{x, y\}) = i(G) - 1$ by Proposition 3.46 and Proposition 1.14. □

Notice that the converse of Propositions 3.46 and 3.47 does not hold. For example, $K_{3,3} \cdot K_{3,3}$ is a graph that is i-critical and i-bicritical, but is not strong i-bicritical. In fact, Proposition 3.49 will show that since $K_{3,3} \cdot K_{3,3}$ has a cut-vertex, it is not strong i-bicritical.

Proposition 3.48. *If G is strong i-bicritical, then $\delta(G) \geq 3$.*

Proof. If there is a vertex of degree one in G, then G is not i-critical by Proposition 1.21. But then G is not strong i-bicritical by Proposition 3.46. If there is a vertex of degree two in G, then G is not i-bicritical by Proposition 1.23. But then by Proposition 3.47, G is not strong i-bicritical. Therefore $\delta(G) \geq 3$. □

The next collection of results investigate the connectivity of strong i-bicritical graphs.

Proposition 3.49. *If G is strong i-bicritical, then G has no cut-vertex.*

Proof. Let v be a cut-vertex of G, let $G_1 - v$ be one component of $G - v$, and let $G_2 - v = (G - v) - G_1$. Then G_1 is the graph induced by $V(G_1 - v) \cup \{v\}$ and G_2 is the graph induced by $V(G_2 - v) \cup \{v\}$. Since G is strong i-bicritical, we have that G is i-critical and $i(G) = i(G_1) + i(G_2) - 1$. Furthermore, G_1 is i-critical and G_2 is i-critical. Let $x \in V(G_1)$ such that $xv \in E(G)$ and let $y \in V(G_2)$ such that $yv \in E(G)$. Then $xy \notin E(G)$ and $i(G - \{x, y\}) = i(G_1) + i(G_2) - 3$. Let D be an

i-set of $G - \{x, y\}$. Thus $v \notin D$. Let $D_1 = D \cap V(G_1)$ and $D_2 = D \cap V(G_2)$. Then $|D| = |D_1| + |D_2|$ and so either $|D_1| \leq i(G_1) - 2$ or $|D_2| \leq i(G_2) - 2$. Without loss of generality, suppose $|D_1| \leq i(G_1) - 2$. Since D dominates $G - \{x, y\}$, either D_1 or D_2 (or possibly both D_1 and D_2) dominate v.

Suppose D_1 dominates v. Then D_1 dominates $G_1 - x$. But then $|D_1| \geq i(G) - 1$, a contradiction.

Suppose D_2 dominates v. Then D_1 dominates $G_1 - \{x, v\}$. But since $xv \in E(G_1)$, we have that $|D_1| \geq i(G_1 - \{x, v\}) \geq i(G_1) - 1$, a contradiction.

Therefore G is not strong i-bicritical. □

Corollary 3.50. *Let G be a graph with cut-vertex v. Let $x \in V(G)$ such that $xv \in E(G)$, let $y \in V(G)$ such that $yv \in E(G)$, and suppose x and y are in different components of $G - v$. If G is i-critical, then $i(G - \{x, y\}) \geq i(G) - 1$.*

Corollary 3.51. *If G is strong i-bicritical, then G is 2-connected.*

Notice that Proposition 3.49 shows that the coalescence construction does not work for strong i-bicritical graphs, and so there is no result analogous to Propositions 2.20 and 3.11 for strong i-bicritical graphs.

Proposition 3.52. *If G is strong i-bicritical, then G has no 2-vertex-cut $\{u, v\}$ such that $uv \notin E(G)$.*

Proof. Suppose that G is strong i-bicritical with a 2-vertex-cut $\{u, v\}$ such that $uv \notin E(G)$. Let G_1 be the graph induced by the vertices of one component of $V(G - \{u, v\})$ along with $\{u, v\}$. Let G_2 be the graph induced by $V(G - G_1) \cup \{u, v\}$.

Notice that $i(G) - 2 = i(G - \{u, v\}) = i(G_1 - \{u, v\}) + i(G_2 - \{u, v\}) \geq i(G_1) - 2 + i(G_2) - 2 = i(G_1) + i(G_2) - 4$ and so $i(G) \geq i(G_1) + i(G_2) - 2$.

Let D be an i-set of G such that $\{u, v\} \subseteq D$. Let $D_1 = D \cap V(G_1)$ and $D_2 = D \cap V(G_2)$ and let $d_1 = |D_1|$ and $d_2 = |D_2|$. Thus $i(G) = d_1 + d_2 - 2$. Let S be an i-set of $G - \{u, v\}$ and let $S_1 = S \cap V(G_1)$ and $S_2 = S \cap V(G_2)$. Let $s_1 = |S_1|$ and $s_2 = |S_2|$. Thus $i(G - \{u, v\}) = |S| = s_1 + s_2 = i(G) - 2 = d_1 + d_2 - 4$. Notice that $S \cap N_{G_1}(\{u, v\}) = \emptyset$ and $S \cap N_{G_2}(\{u, v\}) = \emptyset$. Suppose without loss of generality that $s_1 < d_1 - 2$. Then $D_2 \cup S_1$ is an independent dominating set of G and $|D_2 \cup S_1| < d_1 + d_2 - 2$, a contradiction. Therefore $s_1 \geq d_1 - 2$ and $s_2 \geq d_2 - 2$ and since $d_1 + d_2 - 4 = s_1 + s_2$, we have that $s_1 = d_1 - 2$ and $s_2 = d_2 - 2$. Now let S' be an i-set of G_1. Then $S' \cup S_2$ is an independent dominating set of G (since $S \cap N_{G_2}(\{u, v\}) = \emptyset$). But $|S' \cup S_2| = i(G_1) + d_2 - 2 = i(G) = d_1 + d_2 - 2$ and so $d_1 = i(G_1)$. Likewise $d_2 = i(G_2)$ and so $i(G) = i(G_1) + i(G_2) - 2$.

Let $x \in V(G_1 - \{u, v\})$ such that $xu \in E(G_1)$ and let $y \in V(G_2 - \{u, v\})$ such that $yu \in E(G_2)$. Now let D be an i-set of $G - \{x, y\}$, and so $|D| = i(G_1) + i(G_2) - 4$. Let $D_1 = D \cap V(G_1)$ and $D_2 = D \cap V(G_2)$. Notice that $v \notin D$.

Suppose that $v \notin D$. If $|D_1| \geq i(G_1) - 1$ we have that $|D_2| \leq i(G_2) - 3$, a contradiction since D_2 is an independent dominating set of $G_2 - y$ or $G_2 - \{v, y\}$. Therefore $|D_1| \leq i(G_1) - 2$. If D_1 does not dominate v, then D_1 is an independent dominating set of $G_1 - \{v, x\}$ or $G_1 - \{x, u, v\}$ but $i(G_1 - \{v, x\}) \geq i(G_1) - 2$ and $i(G_1 - \{x, u, v\}) = i((G_1 - \{x, u\}) - v) \geq i(G_1 - \{x, u\}) - 1 \geq i(G_1) - 2$. Therefore $|D_1| = i(G_1) - 2$ and D_2 dominates v. But then D_2 is an independent dominating set of $G_2 - y$ or $G_2 - \{u, y\}$ and $i(G_2 - y) \geq i(G_2) - 1$ and $i(G_2 - \{u, y\}) \geq i(G_2) - 1$. But then $|D| \geq i(G_1) + i(G_2) - 3$, a contradiction.

Suppose that $v \in D$. Notice that $D \cap N_{G_1}(x) = \emptyset$. Since $|D| = i(G_1) + i(G_2) - 4$, we have that $|D_1| \leq i(G_1) - 2$ or $|D_2| \leq i(G_2) - 2$. Without loss of generality, suppose that $|D_1| \leq i(G_1) - 2$. Then $D = D_1 \cup \{x\}$ is an independent dominating set of G_1. But $|D| \leq i(G_1) - 1$, a contradiction.

Therefore if G is strong i-bicritical there does not exist a 2-vertex-cut $\{u, v\}$ such that $uv \notin E(G)$. □

Let G be a strong i-bicritical graph with a 2-vertex-cut $\{u, v\}$ such that $uv \in E(G)$. Let G_1 be the graph induced by the vertices of one component of $V(G - \{u, v\})$ along with $\{u, v\}$. Let G_2 be the graph induced by $V(G - G_1) \cup \{u, v\}$.

Proposition 3.53. *If G is strong i-bicritical with a 2-vertex-cut $\{u, v\}$ such that $uv \in E(G)$, then $i(G) \geq i(G_1) + i(G_2) - 1$.*

Proof. Let D be an i-set of $G - \{u, v\}$ and let $D_1 = D \cap V(G_1)$ and $D_2 = D \cap V(G_2)$. Then $i(G - \{u, v\}) \geq i(G) - 1$. Also D_1 is an independent dominating set of $G_1 - \{u, v\}$ and D_2 is an independent dominating set of $G_2 - \{u, v\}$. Thus $|D_1| \geq i(G_1) - 1$ and $|D_2| \geq i(G_2) - 1$. Therefore $i(G) \geq i(G_1) + i(G_2) - 1$. □

Proposition 3.54. *If u is i-critical in G_1 and v is i-critical in G_2, then G is not strong i-bicritical.*

Proof. Suppose that u is i-critical in G_1 and v is i-critical in G_2. Let D_1 be an i-set of $G_1 - u$ and D_2 be an i-set of $G_2 - v$. Then $v \notin D_1$ and $u \notin D_2$ and $D = D_1 \cup D_2$ is an independent dominating set of G with $|D| = i(G_1) + i(G_2) - 2$, and so G is not strong i-bicritical. □

Proposition 3.55. *If u is i-critical in G_1 and G_2 and v is i-critical in neither G_1 nor G_2, then G is not strong i-bicritical.*

Proof. Suppose that G is strong i-bicritical and u is i-critical in G_1 and G_2 and v is i-critical in neither G_1 nor G_2. Let D_1 be an i-set of G_1 such that $u \in D_1$ and let D_2 be an i-set of G_2 such that $u \in D_2$. Then $D_1 \cup D_2$ is an independent dominating set of G of size $i(G_1) + i(G_2) - 1$. Therefore $i(G) = i(G_1) + i(G_2) - 1$. Consider $z \in V(G_2)$ such that $vz \notin E(G)$. Then there exists an i-set D of $G - \{v, z\}$ with $|D| = i(G) - 2$. Notice that $u \notin D$. Then $S = D \cup \{v\}$ is an independent dominating set of $G - z$ with $|S| = i(G) - 1$ and so $i(G - z) = i(G) - 1$. Therefore $i(G-z) = i(G_1)+i(G_2)-2$. Notice that $S_1 = S \cap V(G_1)$ is an independent dominating set of G_1 and $S_2 = S \cap V(G_2)$ is an independent dominating set of $G_2 - z$. Therefore $|S_1| \geq i(G_1)$ and $|S_2| \geq i(G_2) - 1$. Suppose that $|S_1| = i(G_1)$ and $|S_2| = i(G_2) - 1$. Then $D \cap V(G_1)$ is an independent dominating set of $G_1 - y$ of size $i(G_1) - 1$ and so y is i-critical in G_1, a contradiction. Thus suppose $|S_1| > i(G_1)$ and $|S_2| < i(G_2) - 1$. But then $i(G_2 - z) < i(G_2) - 1$ a contradiction. □

There are three cases left to consider for the existence of a 2-vertex-cut $\{u, v\}$ with $uv \in E(G)$ in strong i-bicritical graphs: without loss of generality that u and v are both i-critical in G_1 and neither u nor v is i-critical in G_2, without loss of generality that u is i-critical in G_1 but not G_2 and v is i-critical in neither G_1 nor G_2, and that neither u nor v are i-critical in G_1 and G_2.

Proposition 3.56. *Let G be a strong i-bicritical graph with a 2-vertex-cut $\{u, v\}$ where $uv \in E(G)$. If u and v are i-critical in G_1 and neither u nor v are i-critical in G_2, then*

- $i(G) = i(G_1) + i(G_2) - 1$
- $i(G_1 - \{u, v\}) = i(G_1) - 1$
- $i(G_2 - \{u, v\}) = i(G_2) - 1$
- $i(G_1 - \{x, y\}) = i(G_1) - 2$ *for every* $xy \notin E(G_1)$ *where* $x \notin \{u, v\}$ *and* $y \notin \{u, v\}$
- *for every* $xy \notin E(G_2)$ *where* $x \notin \{u, v\}$ *and* $y \notin \{u, v\}$, $i(G_2 - \{x, y\}) = i(G_2) - 2$ *or* $i(G_2 - \{x, y, u\}) = i(G_2) - 2$ *or* $i(G_2 - \{x, y, v\}) = i(G_2) - 2$
- $i(G_1 - x) = i(G_1) - 1$ *for every* $x \in V(G_1 - \{u, v\})$ *where either* $xu \notin E(G_1)$ *or* $xv \notin E(G_1)$

- $i(G_2 - x) = i(G_2) - 1$ *for every* $x \in V(G_2 - \{u, v\})$ *where either* $xu \notin E(G_2)$ *or* $xv \notin E(G_1)$.

Proof. Let S_2 be any i-set of G_2. If $u \in S_2$ then let S_1 be an i-set of G_1 that contains u. (Note that such an i-set exists since u is i-critical in G_1.) Likewise, if $v \in S_2$ then let S_1 be an i-set of G_1 that contains v. If $\{u, v\} \cap S_2 = \emptyset$, then let S_1 be an i-set of $G_1 - u$. Since u is i-critical in G_1, $|S_1| = i(G_1) - 1$ and $v \notin S_1$. In any of these cases, $S_1 \cup S_2$ is an independent dominating set of G of cardinality $i(G_1) + i(G_2) - 1$. Thus by Proposition 3.53, $i(G) = i(G_1) + i(G_2) - 1$.

Now consider $G - \{u, v\}$ and let S be an i-set of $G - \{u, v\}$. Since G is strong i-bicritical, it is also i-bicritical and so $|S| = i(G_1) + i(G_2) - 2$. Now $|S \cap V(G_1)| \geq i(G_1) - 1$ and $|S \cap V(G_2)| \geq i(G_2) - 1$ and so $|S \cap V(G_1)| = i(G_1) - 1$ and $|S \cap V(G_2)| = i(G_2) - 1$. Thus we have that $i(G_1 - \{u, v\}) = i(G_1) - 1$ and $i(G_2 - \{u, v\}) = i(G_2) - 1$.

Let $\{x, y\} \subseteq V(G_1)$ where $xy \notin E(G_1)$ and $u, v \notin \{x, y\}$ and let S be an i-set of $G - \{x, y\}$. Then $|S| = i(G_1) + i(G_2) - 3$ since G is strong i-bicritical. Suppose $u \in S$. Then $S \cap V(G_1)$ independently dominates $G_1 - \{x, y\}$ and $S \cap V(G_2)$ independently dominates G_2 and so $|S \cap V(G_1)| \geq i(G_1) - 2$ and $|S \cap V(G_2)| \geq i(G_2)$. Therefore $|S \cap V(G_1)| = i(G_1) - 2$ and $|S \cap V(G_2)| = i(G_2)$. Likewise, if $v \in S$ then $|S \cap V(G_1)| = i(G_1) - 2$. Now suppose $S \cap \{u, v\} = \emptyset$. Suppose both u and v are dominated by vertices in $S \cap V(G_1)$. Then $S \cap V(G_1)$ is an independent dominating set of $G_1 - \{x, y\}$ and $S \cap V(G_2)$ is an independent dominating set of $G_2 - \{u, v\}$, and so $|S \cap V(G_1)| \geq i(G_1) - 2$ and $|S \cap V(G_2)| \geq i(G_2) - 1$. Therefore $|S \cap V(G_1)| = i(G_1) - 2$ and $|S \cap V(G_2)| = i(G_2) - 1$. Suppose both u and v are dominated by vertices in $S \cap V(G_2)$. Then $S \cap V(G_2)$ is an independent dominating set of G_2 and $S \cap V(G_1)$ is an independent dominating set of $G_1 - \{u, v, x, y\}$. Thus $|S \cap V(G_2)| \geq i(G_2)$ and since $uv \in E(G_1)$ $|S \cap V(G_1)| \geq i(G_1) - 3$ and so $|S \cap V(G_2)| = i(G_2)$ and $|S \cap V(G_1)| = i(G_1) - 3$. Without loss of generality, say that u is not dominated by $S \cap V(G_1)$. Then $(S \cap V(G_1)) \cup \{u\}$ is an independent dominating set of $G_1 - \{x, y\}$

and $(S \cap V(G_1)) \cup \{u\}$ has cardinality $i(G_1) - 2$. Finally, without loss of generality, suppose u is dominated by $S \cap V(G_1)$ but not $S \cap V(G_2)$ and v is dominated by $S \cap V(G_2)$ but not $S \cap V(G_1)$. Then $S \cap V(G_1)$ is an independent dominating set of $G_1 - \{x, y, v\}$ and $S \cap V(G_2)$ is an independent dominating set of $G_2 - u$. Since u is not i-critical in G_2 we have that $|S \cap V(G_2)| \geq i(G_2)$ and $|S \cap V(G_1)| \geq i(G_1) - 3$. Then $|S \cap V(G_2)| = i(G_2)$ and $|S \cap V(G_1)| = i(G_1) - 3$. Notice that the only way this is possible is if x, y, and v are pairwise independent in G_1. In this case, $(S \cap V(G_1)) \cup \{v\}$ is an independent dominating set of $G_1 - \{x, y\}$ of cardinality $i(G_1) - 2$. In all cases then, $i(G_1 - \{x, y\}) = i(G_1) - 2$ when $xy \notin E(G_1)$ and $u, v \notin \{x, y\}$.

Let $\{x, y\} \subseteq V(G_2)$ where $xy \notin E(G_2)$ and $u, v \notin \{x, y\}$ and let S be an i-set of $G - \{x, y\}$. Then $|S| = i(G_1) + i(G_2) - 3$ since G is strong i-bicritical. Suppose $u \in S$. Then $S \cap V(G_1)$ is an independent dominating set of G_1 and $S \cap V(G_2)$ is an independent dominating set of $G_2 - \{x, y\}$. Thus $|S \cap V(G_1)| \geq i(G_1)$ and $|S \cap V(G_2)| \geq i(G_2) - 2$ and therefore $|S \cap V(G_1)| = i(G_1)$ and $|S \cap V(G_2)| = i(G_2) - 2$. Likewise, if $v \in S$, then we have that $|S \cap V(G_1)| = i(G_1)$ and $|S \cap V(G_2)| = i(G_2) - 2$ and $S \cap V(G_2)$ is an independent dominating set of $G_2 - \{x, y\}$. Now suppose $S \cap \{u, v\} = \emptyset$. Suppose both u and v are dominated by $S \cap V(G_2)$. Then $S \cap V(G_2)$ is an independent dominating set of $G_2 - \{x, y\}$ and $S \cap V(G_1)$ is an independent dominating set of $G_1 - \{u, v\}$ and so $|S \cap V(G_2)| \geq i(G_2) - 2$ and $|S \cap V(G_1)| \geq i(G_1) - 1$. Therefore $|S \cap V(G_2)| = i(G_2) - 2$ and $|S \cap V(G_1)| = i(G_1) - 1$. Suppose both u and v are dominated by vertices in $S \cap V(G_1)$. Then $S \cap V(G_1)$ is an independent dominating set of G_1 and $S \cap V(G_2)$ is an independent dominating set of $G_2 - \{u, v, x, y\}$. Thus $|S \cap V(G_1)| \geq i(G_1)$. Since $uv \in E(G_2)$, $|S \cap V(G_2)| \geq i(G_2) - 3$ and so $|S \cap V(G_1)| = i(G_1)$ and $|S \cap V(G_2)| = i(G_2) - 3$. Without loss of generality, say that u is not dominated by $S \cap V(G_2)$. Then $(S \cap V(G_2)) \cup \{u\}$ is an independent dominating set of $G_2 - \{x, y\}$ of cardinality $i(G_2) - 2$. Finally, without loss of generality, suppose u is dominated by $S \cap V(G_2)$ but not by $S \cap V(G_1)$

and v is dominated by $S \cap V(G_1)$ but not by $S \cap V(G_2)$. Then $S \cap V(G_1)$ is an independent dominating set of $G_1 - u$ and $S \cap V(G_2)$ is an independent dominating set of $G_2 - \{v, x, y\}$. Then $|S_1| = i(G_1) - 1$ and $|S_2| = i(G_2) - 2$.

Let $x \in V(G_1 - \{u, v\}$ where $xu \notin E(G)$, and consider $G - \{x, u\}$. Let S be an i-set of $G - \{x, u\}$ and so $|S| = i(G_1) + i(G_2) - 3$. Therefore $N(u) \cap S = \emptyset$, and in particular $v \notin S$. Let $S_1 = S \cap V(G_1)$ and let $S_2 = S \cap V(G_2)$. Suppose v is dominated by a vertex in S_2. Then S_2 is an independent dominating set of $G_2 - \{u\}$ and S_1 is an independent dominating set of $G_1 - \{x, u, v\}$. Thus $|S_1| \geq i(G_1) - 2$ and $|S_2| \geq i(G_2)$, a contradiction. Suppose v is dominated by a vertex in S_1. Then S_1 is an independent dominating set of $G_1 - \{x, u\}$ and S_2 is an independent dominating set of $G_2 - \{u, v\}$. Thus $|S_1| = i(G_1) - 2$ and $|S_2| = i(G_2) - 1$. But then $S_1 \cup \{u\}$ is an independent dominating set of $G_1 - x$ of cardinality $i(G_1) - 1$. A similar argument is used if $xv \notin E(G)$.

Let $x \in V(G_2 - \{u, v\}$ where $xu \notin E(G)$, and consider $G - \{x, u\}$. Let S be an i-set of $G - \{x, u\}$ and so $|S| = i(G_1) + i(G_2) - 3$. Therefore $N(u) \cap S = \emptyset$, and in particular $v \notin S$. Let $S_1 = S \cap V(G_1)$ and let $S_2 = S \cap V(G_2)$. Suppose v is dominated by a vertex in S_2. Then S_2 is an independent dominating set of $G_2 - \{x, u\}$ and S_1 is an independent dominating set of $G_1 - \{u, v\}$. Thus $|S_2| = i(G_2) - 2$ and $|S_1| = i(G_1) - 1$. But then u is i-critical in G_2, a contradiction. Suppose v is dominated by a vertex in S_1 and $N(v) \cap S_2 = \emptyset$. Then S_1 is an independent dominating set of $G_1 - u$ and S_2 is an independent dominating set of $G_2 - \{x, u, v\}$. Thus $S_2 \cup \{u\}$ is an independent dominating set of $G_2 - x$ of cardinality $i(G_2) - 1$. A similar argument is used if $xv \notin E(G)$. □

The following construction creates a strong i-bicritical graph with a 2-vertex-cut.

Proposition 3.57. *Let G_1 be a strong i-bicritical graph and let $G_2 = H_{[v]}$ where H is a strong i-bicritical graph. Then the graph $G = G_1(\langle\{u, u'\}\rangle) \odot G_2(\langle\{v, v'\}\rangle)$ is a* *strong i-bicritical graph, where uu' is any edge in G_1 and vv' is the edge in G_2 created by v and its copy v'.*

Proof. Without loss of generality, assume u is identified with v and u' is identified with v'. Notice that by construction both u and u' are i-critical in G_1 and neither v nor v' is i-critical in G_2.

Let S be an i-set of G and let $S_1 = S \cap V(G_1)$ and $S_2 = S \cap V(G_2)$. Suppose u and u' are dominated by S_1. Then $|S_1| \geq i(G_1)$ and $|S_2 - \{v, v'| \geq i(G_2) - 1$. Suppose that u is dominated by S_1 and u' is not dominated by S_1. Then S_2 dominates v'. Thus by construction, $|S_1| \geq i(G_1) - 1$ and $|S_2| \geq i(G_2)$, as if S_2 dominates v' then S_2 also dominates v. Likewise if u is not dominated by S_1 and u' is, then $|S_1| \geq i(G_1) - 1$ and $|S_2| \geq i(G_2)$. If neither u nor u' are dominated by S_1, then v and v' are dominated by S_2. Then $|S_2| \geq i(G_2)$ and $|S_1| \geq i(G_1) - 1$. Hence $i(G) \geq i(G_1) + i(G_2) - 1$. Let S_1 be an i-set of G_1 and let S_2 be an i-set of $G_2 - \{v, v'\}$. Then $|S_1| = i(G_1)$ and $|S_2| = i(G_2) - 1$ as H is strong i-bicritical and thus H is i-critical. Also $S_2 \cap (N(v) \cup N(v')) = \emptyset$ and so $S_1 \cup S_2$ is an independent dominating set of G of cardinality $i(G_1) + i(G_2) - 1$. Therefore $i(G) = i(G_1) + i(G_2) - 1$.

Consider $G - \{x, y\}$. If $\{x, y\} \subseteq V(G_1)$ where $u \notin \{x, y\}$ and $u' \notin \{x, y\}$ then let S_1 be an i-set of $G_1 - \{x, y\}$ and let S_2 be an i-set of $G_2 - \{v, v'\}$. Since G_1 is strong i-bicritical $|S_1| = i(G_1) - 2$ and since H is strong i-bicritical $|S_2| = i(G_2) - 1$ and $(N(v) \cup N(v')) \cap S_2 = \emptyset$. Then $S = S_1 \cup S_2$ is an independent dominating set of $G - \{x, y\}$ and $|S| = i(G_1) + i(G_2) - 3$.

If $\{x, y\} \subseteq V(G_2)$ where $v \notin \{x, y\}$ and $v' \notin \{x, y\}$ then let S_2 be an i-set of $G_2 - \{x, y\}$. If $v \in S_2$ then let S_1 be an i-set of $G_1 - u$. If $v' \in S_2$ then let S_1 be any i-set of $G_1 - u'$. If $S_2 \cap \{v, v'\} = \emptyset$, then let S_1 be any i-set of $G_1 - \{u, u'\}$. In all cases, $|S_1| = i(G_1) - 1$, $|S_2| = i(G_2) - 2$, and $S_1 \cup S_2$ is an independent dominating set of $G - \{x, y\}$ of cardinality $i(G_1) + i(G_2) - 3$.

Now suppose that $x \in V(G_1 - \{u, u'\})$ and without loss of generality that $y = u$.

Let S_1 be an i-set of $G_1 - \{x, u\}$ and let S_2 be an i-set of $G_2 - \{v, v'\}$. Then $|S_1| = i(G_1) - 2$ and $|S_2| = i(G_2) - 1$ and $(N(v) \cup N(v')) \cap S_2 = \emptyset$. Then $S_1 \cup S_2$ is an independent dominating set of $G - \{x, y\}$ of cardinality $i(G_1) + i(G_2) - 3$.

Suppose that $x \in V(G_2 - \{v, v'\})$ and without loss of generality that $y = v$. Let S_2 be an i-set of $G_2 - \{x, v, v'\}$. Since H is a strong i-bicritical graph $|S_2| = i(G_2) - 2$ and $(N(v) \cup N(v')) \cap S_2 = \emptyset$. Let S_1 be an i-set of $G_1 - u$, and so $|S_1| = i(G_1) - 1$. Then $S_1 \cup S_2$ is an independent dominating set of G with cardinality $i(G_1) + i(G_2) - 3$. Therefore G is strong i-bicritical. □

3.2.1 Constructions for Strong i-Bicritical Graphs

In this section we provide methods to construct strong i-bicritical graphs analogous to the constructions presented in Chapter 2 and Section 3.1.1. Namely, we consider the disjoint union, the join, the circle arc construction, and the wreath product.

Proposition 3.58. *The graph $G_1 \cup G_2 \cup \cdots \cup G_k$ is strong i-bicritical if and only if all of G_1, G_2, …, G_k are strong i-bicritical or isomorphic to K_1.*

Proof. If all of G_1, G_2, …, G_k are strong i-bicritical, then from Proposition 3.46 all G_1, G_2, …, G_k are i-critical. Consider $\{u, v\} \subseteq V(G_1 \cup G_2 \cup \cdots \cup G_k)$ with $uv \notin E(G_1 \cup G_2 \cup \cdots \cup G_k)$. If $\{u, v\} \subseteq V(G_j)$ for some $1 \le j \le k$, then $uv \notin E(G_j)$ and $(G_1 \cup G_2 \cup \cdots \cup G_k) - \{u, v\} \cong G_1 \cup \cdots \cup G_j - \{u, v\} \cup \cdots \cup G_k$ and $i((G_1 \cup G_2 \cup \cdots \cup G_k) - \{u, v\}) = i(G_1 \cup \cdots \cup G_j - \{u, v\} \cup \cdots \cup G_k) = i(G_1) + \cdots + i(G_j) - 2 + \cdots + i(G_k) = i(G_1 \cup G_2 \cup \cdots \cup G_k) - 2$ since G_j is strong i-bicritical.

If $u \in V(G_j)$ and $v \in V(G_l)$ for some $1 \le j < l \le k$, then $(G_1 \cup G_2 \cup \cdots \cup G_k) - \{u, v\} \cong G_1 \cup \cdots \cup G_j - u \cup \cdots \cup G_l - v \cup \cdots \cup G_k$ and $i((G_1 \cup G_2 \cup \cdots \cup G_k) - \{u, v\}) = i(G_1 \cup \cdots \cup G_j - u \cup \cdots \cup G_l - v \cup \cdots \cup G_k) = i(G_1) + \cdots + i(G_j) - 1 + \cdots + i(G_j) - 1 + \cdots + i(G_k) = i(G_1 \cup G_2 \cup \cdots \cup G_k) - 2$, since G_j and G_l are both strong i-bicritical and thus i-critical. Thus in either case $G_1 \cup G_2 \cup \cdots \cup G_k$ is strong i-bicritical.

For the converse, suppose that $G_1 \cup G_2 \cup \cdots \cup G_k$ is strong i-bicritical and consider $(G_1 \cup G_2 \cup \cdots \cup G_k) - \{u, v\}$ where $\{u, v\} \subseteq V(G_1 \cup G_2 \cup \cdots \cup G_l)$ and $uv \notin E(G_1 \cup G_2 \cup \cdots \cup G_k)$.

Consider the case where $u \in V(G_j)$ and $v \in V(G_l)$ for some $1 \leq j < l \leq k$. If u is a vertex such that $i(G_j - u) = i(G_j)$, then $i((G_1 \cup \cdots \cup G_k) - \{u, v\}) = i(G_1) + \cdots + i(G_j - u) + \cdots + i(G_j - v) + \cdots + i(G_k) \geq i(G_1) + \cdots + i(G_j) + \cdots + i(G_l) - 1 + \cdots + i(G_k) > i(G_1 \cup \cdots \cup G_k) - 2$ and so $G_1 \cup \cdots \cup G_k$ is not strong i-bicritical. Hence both G_j and G_l are critical and we can conclude that every G_t, $1 \leq t \leq k$, is i-critical.

Now consider the case where $\{u, v\} \subseteq V(G_j)$ for some $1 \leq j \leq k$. Then $i((G_1 \cup \cdots \cup G_k) - \{u, v\}) = i(G_1) + \cdots + i(G_j - \{u, v\}) + \cdots + i(G_k)$. If $i(G_j - \{u, v\}) > i(G_j) - 2$, then $G_1 \cup \cdots \cup G_k$ is not strong i-bicritical. Therefore $i(G_j - \{u, v\}) = i(G_j) - 2$ and G_j is thus strong i-bicritical. Therefore all of G_1, G_2, $\ldots$, G_k are strong i-bicritical. □

Proposition 3.59. *The graph $G_1 + G_2 + \cdots + G_k$ is strong i-bicritical if and only if all of G_1, G_2, $\ldots$, G_k are strong i-bicritical and $i(G_1) = i(G_2) = \cdots = i(G_k)$.*

Proof. Suppose without loss of generality that G_1 is not strong i-bicritical and let $\{x, y\} \subseteq V(G_1)$ such that $xy \notin E(G_1)$ and $i(G_1 - \{x, y\}) \geq i(G_1) - 1$. Let D be an i-set of $G - \{x, y\}$. Notice $D \cap V(G_j) \neq \emptyset$ for only one j, $1 \leq j \leq k$. If $D \cap V(G_1) \neq \emptyset$, then $i(G - \{x, y\}) = i(G_1 - \{x, y\}) \geq i(G_1) - 1 \geq i(G) - 1$. If $D \cap V(G_j) \neq \emptyset$ for $j \neq 1$, then $i(G - \{x, y\}) = i(G_j) \geq i(G)$. Therefore all of G_1, G_2, $\ldots$, G_k are strong i-bicritical.

Suppose without loss of generality that $i(G_1) - 1 \geq i(G_2)$. Let $\{x, y\} \subseteq V(G_1)$ such that $xy \notin E(G_1)$ and let D be an i-set of $G - \{x, y\}$. If $D \cap V(G_1) \neq \emptyset$, then $i(G - \{x, y\}) = i(G_1 - \{x, y\}) = i(G_1) - 2 \geq i(G_2) - 1 \geq i(G) - 1$. If $D \cap V(G_j) \neq \emptyset$

for $j \neq 1$, then $i(G-\{x,y\}) = i(G_j) \geq i(G)$. In either case G is not strong i-bicritical and so $i(G_1) = i(G_2) = \ldots = i(G_k)$.

Suppose that G_1, G_2, …, G_k are all strong i-bicritical and that $i(G_1) = i(G_2) = \ldots i(G_k)$. Notice that the only $\{x,y\} \subseteq V(G)$ such that $xy \notin E(G)$ is when $\{x,y\} \subseteq V(G_j)$ for some j, $1 \leq j \leq k$. Without loss of generality, suppose that $\{x,y\} \subseteq V(G_1)$ such that $xy \notin E(G_1)$. Let D be an i-set of $G_1 - \{x,y\}$. Then $i(G-\{x,y\}) \leq |D| = i(G_1 - \{x,y\}) = i(G_1) - 2 = i(G) - 2$ and so G is strong i-bicritical. □

Proposition 3.60. *For each $H \in \{H_1, H_2, \ldots, H_m, H_{1,2}, H_{1,2,3}, \ldots, H_{1,2,\ldots,m-1}\}$, suppose $\alpha(H) \leq k-3$. Then*

$$G_{\widehat{\odot}} = (((G_1(H_1)\widehat{\odot}G_2(H_2))(H_{1,2})\widehat{\odot}G_4(H_4)) \cdots \widehat{\odot}G_{m-1}(H_{m-1}))(H_{1,2,\ldots,m-1})\widehat{\odot}G_m(H_m)$$

is strong k-i-bicritical if and only if $k = \min\{i(G_1), i(G_2), \ldots, i(G_m)\}$ and every pair of independent vertices $\{x,y\}$ in $G_{\widehat{\odot}}$ is contained in some $V(G_j)$, $1 \leq j \leq m$, where $i(G_j - \{x,y\}) = k-2$.

Proof. Suppose $G_{\widehat{\odot}}$ is strong k-i-bicritical. Then by construction, $k = \min\{i(G_1), i(G_2), \ldots, i(G_m)\}$. Consider $G_{\widehat{\odot}} - \{x,y\}$ where $xy \notin E(G_{\widehat{\odot}})$. Let D be an i-set of $G_{\widehat{\odot}} - \{x,y\}$, and so $|D| = k-2$. By construction of $G_{\widehat{\odot}}$, $D \subseteq V(G_j)$ for some $1 \leq j \leq m$ and $\{x,y\} \subseteq V(G_l)$ for some $1 \leq l \leq m$. If $\{x,y\} \cap V(G_j) = \emptyset$, then D is an independent dominating set of $G_{\widehat{\odot}}$, a contradiction. Say $\{x,y\} \subseteq V(G_j)$. Then D is an independent dominating set of $G_j - \{x,y\}$ and so $i(G_j - \{x,y\}) \leq |D| = k-2 = i(G_j) - 2$.

Suppose that $k = \min\{i(G_1), i(G_2), \ldots, i(G_m)\}$ and for each $\{x,y\} \subseteq V(G_{\widehat{\odot}})$ where $xy \notin E(G_{\widehat{\odot}})$ then $\{x,y\} \subseteq V(G_j)$ for some $1 \leq j \leq m$, where $i(G_j - \{x,y\}) = k-2$. By construction $i(G_{\widehat{\odot}}) = k$. Consider $G_{\widehat{\odot}} - \{x,y\}$ where $xy \notin E(G_{\widehat{\odot}})$. By construction, $\{x,y\} \subseteq V(G_j)$ for some $1 \leq j \leq m$. Let D be an i-set of G_j, and so $|D| = k - 2$. Then D is an independent dominating set of $G_{\widehat{\odot}} - \{x, y\}$ and so $i(G_{\widehat{\odot}} - \{x, y\}) = k - 2 = i(G_{\widehat{\odot}}) - 2$. □

For the simpler construction $G_1(H_1)\widehat{\odot}G_2(H_2)\widehat{\odot}\cdots G_m(H_m)$ we have a very similar result.

Corollary 3.61. *Let $G_1, G_2, \ldots, G_m$ and H be disjoint graphs such that for $j = 1, 2, \ldots, m$, G_j has a subgraph $H_j \cong H$. Suppose $\alpha(H) \leq k - 3$. Then the graph $G = G_1(H_1)\widehat{\odot}G_2(H_2)\widehat{\odot}\cdots\widehat{\odot}G_m(H_m)$ is strong k-i-bicritical if and only if $k = i(G_1) = i(G_2) = \cdots = i(G_m)$ and for every pair of independent vertices $\{x, y\} \subseteq V(G)$, $\{x, y\}$ is contained in some $V(G_j)$, $1 \leq j \leq m$, such that $i(G_j - \{x, y\}) = k - 2$.*

Proposition 3.62. *The graph $G[H]$ is strong i-bicritical if and only if for every independent pair of vertices $\{u, v\} \subseteq V(G)$ there exists an i-set S of G such that $\{u, v\} \subseteq S$ and H is strong i-bicritical with $i(H) \geq 3$, then .*

Proof. Consider $v_1 = (g_1, h_1) \in V(G[H])$ and $v_2 = (g_2, h_2) \in V(G[H])$ such that $v_1v_2 \notin E(G[H])$.

Case 1: $g_1 = g_2 = g$.

That is, h_1 and h_2 are in the same copy of H in $G[H]$. Let S_1 be an i-set of G such that $g \in S_1$, let S_g be an i-set of $H - \{h_1, h_2\}$, and let S_2 be an i-set of H. Then $D = \{(g, h) : h \in S_g\} \cup \{(x, y) : x \in S_1 - \{g\}, y \in S_2\}$ is an independent dominating set of $G[H] - \{v_1, v_2\}$. But $i(G[H] - \{v_1, v_2\}) \leq |D| = i(H) - 2 + (i(G) - 1)(i(H)) = i(G)i(H) - 2 = i(G[H]) - 2$.

Case 2: $g_1 \neq g_2$ and so $g_1g_2 \notin E(G)$.

Let S_1 be an i-set of G such that $\{g_1, g_2\} \subseteq S_1$, let S_{g_1} be an i-set of $H - h_1$, let S_{g_2} be an i-set of $H - h_2$, and let S_2 be an i-set of H. Then $D = \{(g_1, h) : h \in S_{g_1}\} \cup \{(g_2, h) : h \in S_{g_2}\} \cup \{(x, y) : x \in S_1 - \{g_1, g_2\}, y \in S_2\}$ is an independent dominating set of $G[H] - \{v_1, v_2\}$. But then $i(G[H] - \{v_1, v_2\}) \leq |D| = i(H) - 1 + i(H) - 1 + (i(G) - 2)(i(H)) = i(G)i(H) - 2 = i(G[H]) - 2$.

Thus in all cases $G[H]$ is strong i-bicritical.

For the converse suppose that $G[H]$ is strong i-bicritical with $x, y \in V(G[H])$ with $x = (g_1, h_1)$ and $y = (g_2, h_2)$ and $g_1g_2 \notin E(G)$. Then $xy \notin E(G[H])$. Let S be an i-set of $G[H]$ with $\{x, y\} \subseteq S$. Let $S' = \{u \in V(G) : \exists v \in V(G)$ with $(\mathrm{u}, \mathrm{v}) \in \mathrm{S}\}$. Thus S' is an i-set of G and $\{g_1, g_2\} \subseteq S'$.

Suppose that H is not strong i-bicritical and let $h_1, h_2 \in V(H)$ such that $i(H - \{h_1, h_2\}) \geq i(H) - 1$. Let $x = (g, h_1)$ and $y = (g, h_2)$ for some $g \in V(G)$. Let S be an i-set of $G[H] - \{x, y\}$ and let $S' = \{u \in V(G) : \exists v \in V(H)$ with $(\mathrm{u}, \mathrm{v}) \in \mathrm{S}\}$. Then S' is an independent dominating set of G if $i(H) \geq 3$. For each $u \in V(G)$, let $S_u = \{v \in V(H) : (u, v) \in S\}$. If $S_g = \emptyset$, then S is an independent dominating set of G, a contradiction. If $S_g \neq \emptyset$, then $|S_g| \geq i(H - \{h_1, h_2\}) \geq i(H) - 1$ and for all other $S_u \neq \emptyset$, $|S_u| \geq i(H)$. Thus $|S| \geq (i(G) - 1)i(H) + i(H) - 1 = i(G)i(H) - 1$, a contradiction. Therefore H is strong i-bicritical. □

Notice then that if G is strong i-bicritical, $i(G)$ and the vertex-connectivity can differ by an arbitrary amount. Consider $G \cong C_n[K_{3,3}]$, where n is such that for every pair of independent vertices $\{u, v\} \subseteq V(C_n)$, there exists an i-set S of C_n such that $\{u, v\} \subseteq S$. Then by Proposition 3.62, G is strong i-bicritical. Notice that G is 12-connected and $i(G) = 3\lceil \frac{n}{3} \rceil$. Since a strong i-bicritical graph is also an i-bicritical graph, we have the same observation for i-bicritical graphs: if G is i-bicritical, then $i(G)$ and the vertex-connectivity can differ by an arbitrary amount.

3.3 Summary and Directions for Future Work

This chapter studied i-bicritical graphs and strong i-bicritical graphs. Examples of i-bicritical graphs were provided in Section 3.1, including the complete bipartite graphs $K_{n,n}$ and $K_{n,n+1}$ where $n \geq 3$ and the generalized Petersen graph $G(7, 2)$. Constructions seen earlier in Chapter 2 were revisited in the context of creating i-bicritical graphs, including the disjoint union, the join, the coalescence, the joined coalescence, and the wreath product. Necessary and sufficient conditions for producing i-bicritical graphs were presented for the disjoint union, the join, and the joined coalescence constructions, and sufficient conditions for producing i-bicritical graphs were presented for the coalescence and the wreath product. Additionally, the expansion of G via v construction was presented as a way to construct graphs that are i-bicritical but not i-critical. While the expansion via v construction was already known for i-bicritical graphs the sufficient conditions were extended. The end of Section 3.1 briefly investigated i-bicritical graphs with end-vertices, and showed that each vertex v in an i-bicritical graph is adjacent to at most one end-vertex. Using this result, an earlier proof claiming that there are no i-bicritical trees was corrected.

While Section 3.1 concentrated on i-bicritical graphs, Section 3.2 focused on strong i-bicritical graphs. Here the strong 3-i-bicritical graphs were characterized and it was shown that strong i-bicritical graphs are both i-critical and i-bicritical, but not every i-critical graph is strong i-bicritical and not every i-bicritical graph is strong i-bicritical. One important structural property of strong i-bicritical graphs was discovered, that a strong i-bicritical graph does not contain a cut-vertex. In particular, this showed that the coalescence construction is not a valid way to create strong i-bicritical graphs. Strong i-bicritical graphs with a 2-vertex-cut were studied and in particular it was shown that if G is strong i-bicritical with a 2-vertex-cut $\{u, v\}$ then $uv \in E(G)$. Many of the same constructions from before were investigated, including the disjoint union, the join, the joined coalescence, and the wreath product. Necessary and sufficient conditions to construct strong i-bicritical graphs were presented for the disjoint union, the join, the joined coalescence, and the wreath product.

We close this chapter with a collection of open questions:

1. Does there exist an i-bicritical graph that has a block, which is not an end-block,

isomorphic to K_2?

2. Corollary 1.22 of Chapter 1 states that no tree is i-critical and Corollary 3.41 states that no tree is i-bicritical. Hence the number of edges in an i-critical graph or an i-bicritical graph is at least as large as $|V(G)|$. For a fixed k, find the maximum number of edges for a k-i-critical graph of order n. Likewise, find the maximum number of edges for a k-i-bicritical graph of order n.

4

Total Domination Critical and Bicritical Graphs

4.1 Known Results on γ_t-Critical Graphs and γ_t-Bicritical Graphs

Recall that a graph is total domination critical, or γ_t-critical, if $\gamma_t(G-v) < \gamma_t(G)$ for every $v \in V(G)$ such that $G - v$ has no isolated vertices. Total domination critical graphs were first defined and studied by Goddard, Haynes, Henning, and van der Merwe [23]. The main result of this paper was an upper bound on the diameter of k-γ_t-critical graphs. This result, and other results concerning the diameter of critical graphs, will be discussed in Chapter 5. Before that, we should discuss more introductory results regarding γ_t-critical graphs.

Proposition 4.1. [23] *A cycle C_n is γ_t-critical if and only if $n \equiv 1, 2 \pmod 4$.*

Observation 4.2. [23] *If G is a γ_t-critical graph, then $\gamma_t(G - v) = \gamma_t(G) - 1$ for every $v \in V(G)$ such that $G - v$ has no isolated vertices. Furthermore, a γ_t-set of $G - v$ contains no neighbour of v.*

Observation 4.3. [23] *If a graph G has nonadjacent vertices u and v where $G - v$ has no isolated vertices and $N(u) \subseteq N(v)$, then G is not γ_t-critical.*

As we have seen for γ-critical and i-critical graphs, there are lower bounds on $\gamma_t(G - v)$ and $\gamma_t(G - S)$ for $S \subseteq V(G)$.

Observation 4.4. [23] *For any graph G with $v \in V(G)$ such that $\delta(G - v) \geq 1$, $\gamma_t(G - v) \geq \gamma_t(G) - 1$.*

Proposition 4.5. *For any graph connected graph G and vertices $S \subseteq V(G)$ with $|S| = k$ and such that $\delta(G - S) \geq 1$, $\gamma_t(G - S) \geq \gamma_t(G) - k$.*

Proof. Consider a γ_t-set D of $G - S$. If D is not a total dominating set of G, then it is possible to add a vertex $x \in N[S]$ that is adjacent to a vertex not dominated by D to create a new set D'. If D' is not a total dominating set of G, then it is possible to add a vertex $x' \in N[S]$ that is adjacent to a vertex not dominated by D' to create a new set D''. Continuing in this fashion, it is possible to arrive at a total dominating set of G from D by adding at most k vertices in $N[S]$. Therefore $\gamma_t(G) \leq \gamma_t(G - S) + k$. □

In particular, Proposition 4.5 shows that if G is γ_t-bicritical, then $\gamma_t(G) - 2 \leq \gamma_t(G - \{x, y\}) \leq \gamma_t(G) - 1$. This case was shown directly by Jafari Rad [30]. Jafari Rad also studied the extremal case where $\gamma_t(G - \{x, y\}) = \gamma_t(G) - 2$.

Proposition 4.6. [30] *If $\gamma_t(G) - 2 = \gamma_t(G - \{x, y\})$, then*

- *if $xy \notin E(G)$, then $d(x, y) \geq 3$, and*
- *if $xy \in E(G)$, then $N(x) \cap N(y) = \emptyset$.*

In the same publication, Jafari Rad defines a *strong total domination bicritical graph* as one where $\gamma_t(G - \{x, y\}) = \gamma_t(G) - 2$ for any two vertices $\{x, y\} \subseteq V(G)$

such that $G - \{x, y\}$ has no isolated vertex. It was then shown that no graph G with $\delta(G) \geq 2$ is strong γ_t-bicritical. However, according to Proposition 4.6, $\gamma_t(G - \{x, y\}) \geq \gamma_t(G) - 1$ when $d(x, y) = 2$. This implies that in a strong γ_t-bicritical graph, all vertices x and y such that $d(x, y) = 2$ leave isolated vertices when they are removed.

It is easy to see that when deleting the closed neighbourhood of a vertex x we have that $\gamma(G - N[x]) \geq \gamma(G) - 1$ and $i(G - N[x]) \geq i(G) - 1$. The bound for total domination when deleting the closed neighbourhood of a vertex is not quite the same.

Proposition 4.7. *For any graph G with $x \in V(G)$, $\gamma_t(G - N[x]) \geq \gamma_t(G) - 2$.*

Proof. Let D be a γ_t-set of $G - N[x]$ and let $y \in N(x)$. Then $D \cup \{x, y\}$ is a total dominating set of G. Thus $\gamma_t(G) \leq \gamma_t(G - N[x]) + 2$. □

The graph C_6 is an example of a graph where $\gamma_t(C_6 - N[x]) = 2 = \gamma_t(C_6) - 2$. It is easy to show that $\gamma_t(C_6 \cdot C_6) = 6 = \gamma_t(C_6) + \gamma_t(C_6) - 2$ and that $\gamma_t(C_6 \cdot C_6 - v) = 6$, where v is the vertex of identification. This gives an example of a graph that is not γ_t-critical but has blocks that are γ_t-critical. It is also possible to construct a γ_t-critical graph that contains blocks that are not γ_t-critical. An infinite family of such graphs was discovered by Goddard et al. [23], and this family is used in Chapter 5 in the discussion of k-γ_t-critical graphs with maximum diameter. These examples complicate the coalescence construction for γ_t-critical graphs. The coalescence construction for γ_t-critical graphs is studied in detail later in this chapter.

The *corona* of a graph G, denoted $cor(G)$, is the graph obtained by adding a pendant edge to each vertex of G. The corona of C_4 is pictured in Figure 4.2. Notice that if each component of G has order $n \geq 2$, then $\gamma_t(cor(G)) = n$ (and $\gamma_t(cor(G)) = 2$ if $|V(G)| \leq 2$). It is easy to see that if $|V(G)| \geq 3$ and $\delta(G) \geq 2$, then $cor(G)$ is a γ_t-critical graph since the only vertices that are allowed to be removed are the newly added pendant vertices. The converse of this also holds.

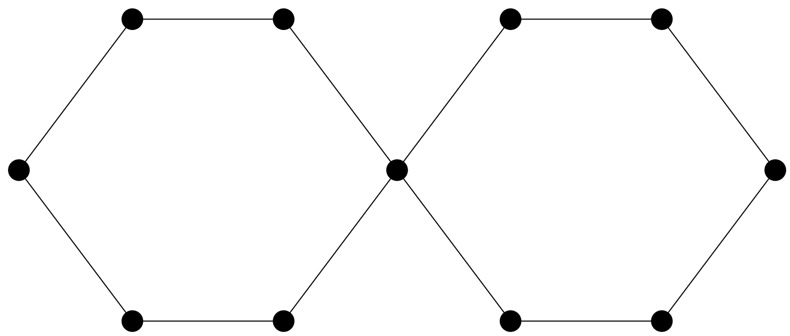

Figure 4.1: The graph $C_6 \cdot C_6$.

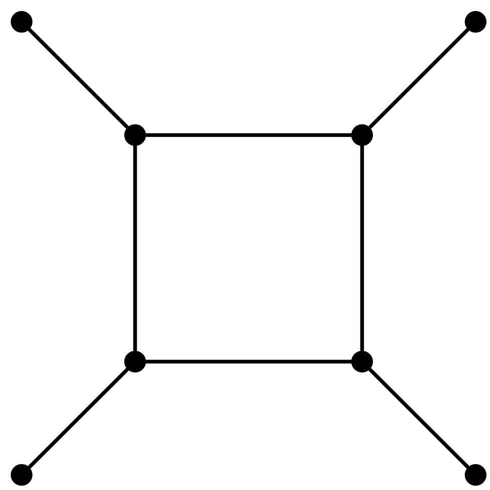

Figure 4.2: The graph $cor(C_4)$.

Theorem 4.8. [23] *Let G be a connected graph of order at least three with at least one end-vertex. Then G is k-γ_t-critical if and only if $G = cor(H)$ for some connected graph H of order k with $\delta(H) \geq 2$.*

As a consequence of this theorem, something can be said about the γ_t-criticality of trees. Notice the similarity to results seen earlier for γ-criticality and i-criticality.

Corollary 4.9. [23] *No tree is γ_t-critical.*

A graph G is said to be *vertex diameter k-critical* if $\text{diam}(G) = k$ and $\text{diam}(G - v) > k$ for all $v \in V(G)$. This definition is useful in providing a characterization of 3-γ_t-critical graphs.

Theorem 4.10. [23] *A connected graph G is 3-γ_t-critical if and only if $\overline{G}$ is vertex diameter 2-critical or $G = cor(K_3)$.*

Goddard et al. presented a list of open problems and questions in their paper [23].

1. Characterize the 3-γ_t-critical graphs with diameter 3. Does there exist a 4-γ_t-critical graph with diameter 2?

2. Consider the connection between γ-critical and γ_t-critical graphs. For example, $K_3 \square K_3$ is γ-critical but not γ_t-critical. The cycle C_5 is γ_t-critical but not γ-critical. So, which graphs are domination vertex-critical and total domination vertex-critical (or one but not the other)?

3. Determine the maximum diameter of a k-γ_t-critical graph.

4. If G is a γ_t-critical graph of order n, then it can be shown that $n \leq \Delta(G)(\gamma_t(G)-1)+1$. Characterize those graphs achieving equality.

5. Cockayne et al. [14] showed that if G is a connected graph of order $n \geq 2$, then $\gamma_t(G) \leq \max(n - \Delta(G), 2)$. Characterize γ_t-critical graphs G with $\gamma_t(G) = n - \Delta(G)$.

In Chapter 5 an answer to question 3 is provided as an upper bound is given for k-γ_t-critical graphs. It is also shown that this bound is tight when $\gamma_t(G) \equiv 2 \pmod 3$. Later in this chapter, to give a partial response to question 2, a construction is given that creates γ_t-critical graphs that are not γ-critical.

Question 1 posed above was answered by Loizeaux and van der Merwe [34], where they provided a construction of a 4-γ_t-critical graph with diameter 2 and of order $3k+2$, $k \geq 3$. Mojdeh and Jafari Rad made some progress towards an answer to question 4 posted above [38].

Theorem 4.11. [38] *Any γ_t-critical graph G of order $n = \Delta(G)(\gamma_t(G) - 1) + 1$ is regular.*

Corollary 4.12. [38] *A 2-regular graph G is γ_t-critical if and only if $G = C_n$ where $n \equiv 1 \pmod 4$.*

Theorem 4.13. [38] *If there exists a k-γ_t-critical r-regular graph G of order $r(k-1)+1$, then k is odd and r is even.*

Thus there are no γ_t-critical graphs of maximum order which have $\gamma_t(G)$ even.

For a graph G, a total dominating set D is called an *efficient total dominating set* if each vertex in $V(G)$ is totally dominated by exactly one vertex of D, that is, if $|N(x) \cap D| = 1$ for each $x \in V(G)$.

Lemma 4.14. [38] *Let G be a k-γ_t-critical r-regular graph of order $r(k-1)+1$ with k odd. Let $v \in V(G)$ and S be a $\gamma_t(G-v)$-set, then S is an efficient total dominating set for $G - v$.*

Lemma 4.15. [38] *If a r-regular graph G of order $r(k-1)+1$ contains cor(K_4) as an induced subgraph, then G is not k-γ_t-critical.*

These results on γ_t-critical graphs of maximum order yield some results on diameter.

Theorem 4.16. [38] *The diameter of a 3-γ_t-critical r-regular graph of order $2r+1$ is 2.*

Theorem 4.17. [38] *The diameter of a k-γ_t-critical r-regular graph of order $n = r(k-1)+1$ is at least $(3k-5)/2$.*

Of course, a 3-γ_t-critical 2-regular graph with maximum order would have $n = 5$ according to the bound. The cycle C_5 is such a graph. The next possible candidate for a 3-γ_t-critical r-regular graph is for $r = 4$.

It was thought by Mojdeh and Jafari Rad that no 3-γ_t-critical 4-regular graph of order 9 exists, which led to the following conjecture.

Conjecture 4.18. [38] *For $r \geq 6$, there is no 3-γ_t-critical r-regular graph of order $2r+1$.*

However a 3-γ_t-critical 4-regular graph of order 9 was discovered by [32]. This graph is pictured in Figure 4.3.

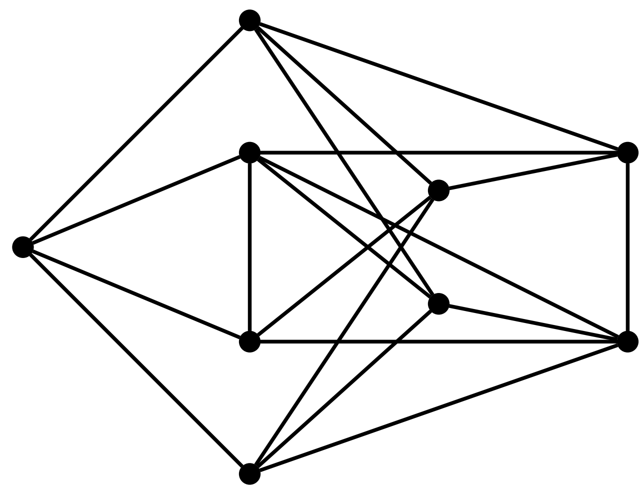

Figure 4.3: A 3-γ_t-critical 4-regular graph of order 9.

Additionally, no γ_t-critical graphs of maximum order were found by Mojdeh and Jafari Rad other than the cycles C_n where $n \equiv 1 \pmod 4$. The existence problem for γ_t-critical graphs of maximum order was left as an open problem.

Wang, Hu, and Li [47] added to the results of Mojdeh and Jafari Rad by defining a family of graphs they called Ψ. The family Ψ is defined as follows:

1. K_1, $K_2 \in \Psi$.

2. Let G be a connected graph with at least 3 vertices. $G \in \Psi$ if and only if both of the following two conditions hold:

   (a) G is a regular graph;

(b) For any $v \in V(G)$, there exists an $A \subseteq V(G) - v$ such that $N(v) \cap A = \emptyset$, $\langle A \rangle$ is 1-regular, $d_A(y) = 1$ for each $y \in V(G) - A - v$.

This family Ψ was then used to give a characterization of γ_t-critical graphs of maximum order.

Theorem 4.19. [47] *Let G be a connected γ_t-critical graph of order n. Then $n = \Delta(G)(\gamma_t(G) - 1) + 1$ if and only if $G \in \Psi$.*

This family Ψ includes the only previously known family of γ_t-critical graphs of maximum order, the cycles C_n where $n \equiv 1 \pmod 4$. The authors used their family Ψ to add one more graph to the collection of known γ_t-critical graphs of maximum order. A *Cayley graph* $G(H; S)$ on a group H is defined to be the digraph with the elements of H as its vertices and edges joining h and hs for all $h \in H$ and $s \in S$. The set S is called the connection set. Wang, Hu, and Li provided the example of the Cayley graph which uses H as the cyclic group Z_{33} with connection set $S = \{6, 27\}$, and note that $G(H; S)$ is in Ψ. Since Cayley graphs are vertex transitive, up to symmetry only one vertex needs to be deleted to verify that $G(H; S)$ is in Ψ, and Wang, Hu, and Li provided the choice of deleting $v = 0$ and stated that $\langle A \rangle = \{(2, 8), (11, 12), (21, 22), (25, 31)\}$ shows that $G(H; S)$ is in fact in the family Ψ.

A *Harary graph* $H_{k,n}$, $2 \leq k < n$, is defined as follows. Place n vertices around a circle, equally spaced. If k is even, $H_{k,n}$ is formed by adding an edge between each vertex and the nearest $k/2$ vertices to it in each direction around the circle. If k is odd and n is even, $H_{k,n}$ is formed by adding an edge between each vertex and the nearest $(k-1)/2$ vertices to it in each direction around the circle and to the vertex directly across the circle from it. If k is odd and n is odd, then $H_{k,n}$ is constructed from $K_{k-1,n}$ by adding edges between vertex i and $i + (n-1)/2$ for $0 \leq i \leq (n-1)/2$. Henning and Jafari Rad [27] made a contribution towards answering question 2 posed above by Goddard et al. by studying the γ_t-criticality of some of these Harary graphs.

Theorem 4.20. [27] *For all integers $l \geq 1$ and $k \geq 2$, the Harary graph $H_{2k+1,2l(2k_1)+2}$ is a $(2k+1)$-connected graph that is $(2l+2)$-γ_t-critical.*

For those Harary graphs the case of $l = 1$ produces a $(2k+1)$-connected graph that is 4-γ_t-critical and has diameter 2 for every $k \geq 2$. It was noted that this family is different from that which was provided by Loizeaux and van der Merwe [34].

Theorem 4.21. [27] *For all integers $l \geq 1$ and $k \geq 2$, the Harary graph $H_{2k,l(3k+1)+1}$ is a $2k$-connected graph that is $(2l+1)$-γ_t-critical.*

In some cases, the above two classes of Harary graphs are not γ-critical, thus providing a class of γ_t-critical graphs that are not γ-critical.

A graph G is γ_t-bicritical if $\gamma_t(G - \{x, y\}) < \gamma_t(G)$ for every subset $\{x, y\}$ such that $G - \{x, y\}$ has no isolated vertices. Jafari Rad [30] was the first to define and study these γ_t-bicritical graphs. As mentioned earlier, he also defined the notion of a strong γ_t-bicritical graph. Only γ_t-bicritical graphs are discussed in this work.

Proposition 4.22. [30] *If G is γ_t-bicritical, then $V^+ = \emptyset$.*

Proposition 4.23. [30] *If G is γ_t-bicritical then G is γ_t-critical or $G - v$ is γ_t-critical for all $v \in V^0$.*

Recall that a cycle C_n is γ_t-critical exactly when $n \equiv 1, 2 \pmod 4$. As it turns out, there is only one cycle that is γ_t-bicritical.

Proposition 4.24. [30] *C_n is γ_t-bicritical if and only if $n = 5$.*

Thus the cycles C_n with $n \equiv 1, 2 \pmod 4$, $n \neq 5$, provide examples of graphs that are γ_t-critical but are not γ_t-bicritical.

The γ_t-critical graphs with end-vertices were characterized by Goddard et al. [23] and γ_t-bicritical graphs with end-vertices were characterized by Jafari Rad [30]. Both characterizations are largely based on taking the corona of a graph. The characterization of γ_t-critical graphs with end-vertices gives the following result.

Corollary 4.25. [30] *No tree is γ_t-bicritical.*

Theorem 4.26. [30] *If a connected graph G with minimum degree at least two is 3-γ_t-bicritical, then either $\overline{G}$ is vertex diameter 2-critical or $\overline{G-v}$ is vertex diameter 2-critical, for some vertex v with $\gamma_t(G-v) = \gamma_t(G)$.*

Proposition 4.27. [30] *If G is a γ_t-bicritical graph of order n, then $n \leq (\gamma_t(G) - 1)\Delta(G) + 2$.*

Proposition 4.28. [30] *If G is a regular γ_t-bicritical graph of order n, then $n \leq (\gamma_t(G) - 1)\Delta(G) + 1$.*

In addition to the above results, Jafari Rad provided constructions for γ_t-bicritical graphs. These constructions are discussed in the next section.

The rest of this chapter is devoted to constructions for γ_t-critical graphs and γ_t-bicritical graphs. In particular, we investigate the coalescence construction for both types of criticality.

4.2 Constructions for γ_t-Critical Graphs and γ_t-Bicritical Graphs

Much like with i-critical and i-bicritical graphs, there are constructions to create γ_t-critical and γ_t-bicritical graphs. We revisit two familiar constructions and present one construction that is specifically useful for total domination.

Observation 4.29. [23] *The graph $G_1 \cup G_2 \cup \cdots \cup G_k$ is γ_t-critical if and only if every G_1, G_2, $\ldots$, G_k is γ_t-critical.*

The following observation corrects an incorrect version stated in [30].

Observation 4.30. [30] *The graph $G_1 \cup G_2 \cup \cdots \cup G_k$ is γ_t-bicritical if and only if every G_1, G_2, …, G_k is γ_t-bicritical and at most one of G_1, G_2, …, G_k is not γ_t-critical.*

We now focus on the coalescence construction, $G \cdot H$.

Proposition 4.31. *For any graphs G and H, $\gamma_t(G) + \gamma_t(H) - 2 \leq \gamma_t(G \cdot_{xy} H) \leq \gamma_t(G) + \gamma_t(H)$.*

Proof. Let D_1 be a γ_t-set of G and D_2 be a γ_t-set of H. Then $D_1 \cup D_2$ is a total dominating set of $G \cdot_{xy} H$ and so $\gamma_t(G \cdot_{xy} H) \leq \gamma_t(G) + \gamma_t(H)$.

Let D be a γ_t-set of $G \cdot_{xy} H$ where v is the vertex of identification. Let $D_1 = D \cap V(G)$ and $D_2 = D \cap V(H)$.

If $v \notin D$ then either D_1 is a total dominating set of G or D_2 is a total dominating set of H. If D_1 is a total dominating set of G, then $|D_1| \geq \gamma_t(G)$. But then D_2 is a total dominating set of $H - y$ and so $|D_2| \geq \gamma_t(H) - 1$. Thus $|D| = |D_1| + |D_2| \geq \gamma_t(G) + \gamma_t(H) - 1$. Likewise $|D| = |D_1| + |D_2| \geq \gamma_t(G) + \gamma_t(H) - 1$ if D_2 is a total dominating set of H.

If $v \in D$ then v is dominated by some other vertex in D, call it w. Without loss of generality, suppose $w \in V(G)$. Then D_1 is a total dominating set of G and so $|D_1| \geq \gamma_t(G)$. If $D_2 \cap N_H(y) \neq \emptyset$, then D_2 is a total dominating set of H and so $|D_2| \geq \gamma_t(H)$. In this case $|D| = |D_1| \cup |D_2| \geq \gamma_t(G) + \gamma_t(H) - 1$. Thus assume $D_2 \cap N_H(y) = \emptyset$. If $D_2 - \{y\}$ is a total dominating set of $H - y$, then $|D_2| \geq \gamma_t(H) - 1 + 1$. Again $|D| \geq \gamma_t(G) + \gamma_t(H) - 1$. Finally, assume $D_2 - \{y\}$ is a total dominating set of $H - N[y]$. Then $|D_2 - \{y\}| \geq \gamma_t(H) - 2$ and $|D| \geq \gamma_t(G) + \gamma_t(H) - 2$. □

Proposition 4.32. *If $\gamma_t(G \cdot_{xy} H) = \gamma_t(G) + \gamma_t(H) - 2$, then $G \cdot_{xy} H$ is not γ_t-critical.*

Proof. Let v be the vertex of identification and consider $G \cdot_{xy} H - v \cong (G-x) \cup (H-y)$. Then $\gamma_t(G \cdot_{xy} H - v) = \gamma_t(G-x) + \gamma_t(H-y) \geq \gamma_t(G) - 1 + \gamma_t(H) - 1 = \gamma_t(G) + \gamma_t(H) - 2 = \gamma_t(G \cdot_{xy} H)$ and so $G \cdot_{xy} H$ is not γ_t-critical. □

Proposition 4.33. *The graph $G \cdot_{xy} H$ has $\gamma_t(G \cdot_{xy} H) = \gamma_t(G) + \gamma_t(H) - 2$ if and only if $\gamma_t(G - N[x]) = \gamma_t(G) - 2$ and there exists a γ_t-set D of H such that $y \in D$, or $\gamma_t(H - N[y]) = \gamma_t(H) - 2$ and there exists a γ_t-set D of G such that $x \in D$.*

Proof. The necessity of this result follows from the proof of Proposition 4.31. Now suppose without loss of generality that $\gamma_t(G - N[x]) = \gamma_t(G) - 2$. Let D_1 be a γ_t-set of $G - N[x]$ and let D_2 be a γ_t-set of H such that $y \in H$. Then $D_1 \cup D_2$ is a total dominating set of $G \cdot_{xy} H$ and $|D_1 \cup D_2| = \gamma_t(G) + \gamma_t(H) - 2$. □

Sufficient conditions for $G \cdot H$ to be γ_t-critical and γ_t-bicritical have been studied.

Proposition 4.34. [23] *If $\gamma_t(G \cdot H) = \gamma_t(G) + \gamma_t(H) - 1$ and G and H are both γ_t-critical with $\delta(G) \geq 2$ and $\delta(H) \geq 2$, then $G \cdot H$ is γ_t-critical.*

Proposition 4.35. [30] *If $\gamma_t(G \cdot H) = \gamma_t(G) + \gamma_t(H) - 1$ and G and H are both γ_t-critical and γ_t-bicritical, then $G \cdot H$ is γ_t-critical and γ_t-bicritical.*

In fact, the conditions in the above proposition can be relaxed while maintaining that $G \cdot H$ is γ_t-bicritical.

Proposition 4.36. *If $\gamma_t(G \cdot_{xy} H) = \gamma_t(G) + \gamma_t(H) - 1$ and both G and H are γ_t-bicritical, and at most one of G and H is not γ_t-critical, and x is γ_t-critical in G and y is γ_t-critical in H, then $G \cdot_{xy} H$ is γ_t-bicritical.*

Proof. Without loss of generality, suppose H is γ_t-critical. Consider $G \cdot_{xy} H - \{u, v\}$. If $\{u, v\} \subseteq V(G)$, then let D_1 be a γ_t-set of $G - \{u, v\}$ and D_2 be a γ_t-set of $H - y$. If $\{u, v\} \subseteq V(H)$, then let D_1 be a γ_t-set of $G - x$ and let D_2 be a γ_t-set of $H - \{u, v\}$.

If, without loss of generality, $u \in V(G-x)$ and $v \in V(H-y)$, then let D_1 be a γ_t-set of $G-\{u,x\}$ and let D_2 be a γ_t-set of $H-v$. Then in all cases $D_1 \cup D_2$ is a total dominating set of $G \cdot_{xy} H - \{u,v\}$ of cardinality at most $\gamma_t(G)+\gamma_t(H)-2$, and so $G \cdot_{xy} H$ is γ_t-bicritical. □

Necessary and sufficient conditions for $G \cdot H$ to be γ_t-critical are given in the following result.

Proposition 4.37. *The graph $G \cdot_{xy} H$ is γ_t-critical if and only if $\gamma_t(G \cdot_{xy} H) = \gamma_t(G)+\gamma_t(H)-1$ and either*

(a) G and H are both γ_t-critical or

(b) x is γ_t-critical in G and y is γ_t-critical in H, there exists a γ_t-set D_1 of G such that $x \in D_1$, there exists a γ_t-set D_2 of H such that $y \in D_2$, and $\gamma_t(G - z_1 - N[x]) = \gamma_t(G)-2$ for any vertex z_1 that is not γ_t-critical in G and $\gamma_t(H - z_2 - N[y]) = \gamma_t(H)-2$ for any vertex z_2 that is not γ_t-critical in H.

Proof. Assume $G \cdot_{xy} H$ is γ_t-critical.

Suppose that $\gamma_t(G \cdot_{xy} H) = \gamma_t(G)+\gamma_t(H)$. Then x is not γ_t-critical in G and y is not γ_t-critical in H. But then $\gamma_t(G \cdot_{xy} H - v) = \gamma_t(G-x)+\gamma_t(H-y) \geq \gamma_t(G)+\gamma_t(H) = \gamma_t(G \cdot_{xy} H)$ and so $G \cdot_{xy} H$ is not γ_t-critical. Thus $\gamma_t(G \cdot_{xy} H) = \gamma_t(G)+\gamma_t(H)-1$ if $G \cdot_{xy} H$ is γ_t-critical.

Suppose without loss of generality that x is γ_t-critical in G but y is not γ_t-critical in H. Then $\gamma_t(G \cdot_{xy} H - v) = \gamma_t(G-x)+\gamma_t(H-y) \geq \gamma_t(G)-1+\gamma_t(H) = \gamma_t(G \cdot_{xy} H)$. Thus x is γ_t-critical in G and y is γ_t-critical in H if $G \cdot_{xy} H$ is γ_t-critical.

If G and H are γ_t-critical, then we are done. Hence suppose without loss of generality that G is not γ_t-critical and let $z \in V(G-x)$ such that $\gamma_t(G-z) \geq \gamma_t(G)$. Let D be a γ_t-set of $G \cdot_{xy} H - z$. Then $|D| = \gamma_t(G \cdot_{xy} H) - 1 = \gamma_t(G)+\gamma_t(H)-2$. Let $D_1 = D \cap V(G)$ and let $D_2 = D \cap V(H)$.

Suppose $v \in D$. Then either $D_1 \cap N_G(x) \neq \emptyset$ or $D_2 \cap N_H(y) \neq \emptyset$. Consider two cases, depending on $D_1 \cap N_G(x)$.

Case 1: $D_1 \cap N_G(x) \neq \emptyset$.

Then D_1 is a total dominating set of $G - z$. Thus $|D_1| \geq \gamma_t(G - z) \geq \gamma_t(G)$. If $D_2 \cap N_H(y) \neq \emptyset$ then D_2 is a total dominating set of H and so $|D_2| \geq \gamma_t(H)$. If $D_2 \cap N_H(y) = \emptyset$ then D_2 is a total dominating set of $H - N[y]$. But $\gamma_t(H - N[y]) \geq \gamma_t(H) - 1$ for otherwise $\gamma_t(G \cdot_{xy} H) = \gamma_t(G) + \gamma_t(H) - 2$. Thus $|D_2| \geq 1 + \gamma_t(H) - 1$. In either case $|D| = |D_1 \cup D_2| \geq \gamma_t(G) + \gamma_t(H) - 1$ and so $G \cdot_{xy} H$ is not γ_t-critical, a contradiction.

Case 2: $D_1 \cap N_G(x) = \emptyset$.

Then $D_2 \cap N_H(y) \neq \emptyset$ and so D_2 is a total dominating set of H. Thus $|D_2| \geq \gamma_t(H)$ and $D_1 - \{x\}$ is a total dominating set of $G - z - N[x]$. But since $|D| = \gamma_t(G) + \gamma_t(H) - 2$, we have that $|D_2| = \gamma_t(H)$ and $\gamma_t(G - z - N[x]) = \gamma_t(G) - 2$.

Now suppose that $v \notin D$. If D_1 dominates v, then D_1 is a total dominating set of $G - z$ and so $|D_1| \geq \gamma_t(G - z) \geq \gamma_t(G)$, and D_2 is a total dominating set of $H - y$ and so $|D_2| \geq \gamma_t(H) - 1$. But then $|D| \geq \gamma_t(G) + \gamma_t(H) - 1$, a contradiction. Hence D_2 dominates v. Then D_2 is a total dominating set of H and so $|D_2| \geq \gamma_t(H)$ and D_1 is a total dominating set of $G - \{x, z\}$. But $\gamma_t(G - \{x, z\}) = \gamma_t((G - z) - x) \geq \gamma_t(G - z) - 1 \geq \gamma_t(G) - 1$. In this case $|D| \geq \gamma_t(G) + \gamma_t(H) - 1$ and so $G \cdot_{xy} H$ is not γ_t-critical, a contradiction.

Therefore either G is γ_t-critical, or x is γ_t-critical in G and $\gamma_t(G - z_1 - N[x]) = \gamma_t(G) - 2$ for every vertex z_1 that is not γ_t-critical in G. Likewise, either H is γ_t-critical or y is γ_t-critical in H and $\gamma_t(G - z_2 - N[y]) = \gamma_t(G) - 2$ for every vertex z_2 that is not γ_t-critical in H.

Conversely, suppose that $\gamma_t(G \cdot_{xy} H) = \gamma_t(G) + \gamma_t(H) - 1$ and either (a) or (b) holds. Consider $G \cdot_{xy} H - z$ for some $z \in V(G \cdot_{xy} H)$.

If $z = v$, then let D_1 be a γ_t-set of $G - x$ and D_2 be a γ_t-set of $H - y$. Then $D_1 \cup D_2$

is a total dominating set of $G \cdot_{xy} H$. Since x is γ_t-critical in G and y is γ_t-critical in H, we have that $\gamma_t(G \cdot_{xy} H - z) \leq |D| = \gamma_t(G) + \gamma_t(H) - 2 < \gamma_t(G \cdot_{xy} H)$.

Suppose without loss of generality that $z \in V(G-x)$. Suppose that z is γ_t-critical in G. Let D_1 be a γ_t-set of $G-z$ and let D_2 be a γ_t-set of $H-y$. Then $D_1 \cup D_2$ is a total dominating set of $G \cdot_{xy} H - z$ and so $\gamma_t(G \cdot_{xy} H) \leq |D| = \gamma_t(G) + \gamma_t(H) - 2 < \gamma_t(G \cdot_{xy} H)$. Suppose that z is not γ_t-critical in G. Let D_1 be a γ_t-set of $G - z - N[x]$ and let D_2 be a γ_t-set of H such that $y \in D_2$. Then $D_1 \cup D_2$ is a total dominating set of $G \cdot_{xy} H - z$ and so $\gamma_t(G \cdot_{xy} H - z) \leq |D| = \gamma_t(G) + \gamma_t(H) - 2 < \gamma_t(G \cdot_{xy} H)$. In any case $G \cdot_{xy} H$ is γ_t-critical. □

Providing necessary conditions for $G \cdot H$ to be γ_t-bicritical is left as an open problem.

Let G be any graph, and let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The Mycielski construction creates a graph $M(G)$ with $V(M(G)) = \{v_1, v_2, \ldots, v_n, u, u_1, u_2, \ldots, u_n\}$ and with $E(M(G)) = E(G) \cup \{uu_k : 1 \leq k \leq n\} \cup \{u_k x : x \in N(v_k), 1 \leq k \leq n\}$. For a positive integer k, the k-th Mycielski graph of G, denoted $M^k(G)$, is defined recursively by $M^0(G) = G$ and $M^{k+1}(G) = M(M^k(G))$ for $k \geq 1$. The Mycielski construction is useful for creating γ_t-critical and γ_t-bicritical graphs.

Proposition 4.38. [30] *For any graph G, $\gamma_t(M^k(G)) = \gamma_t(G) + 1$.*

Proposition 4.39. [30] *Let G be a graph that is both γ_t-critical and γ_t-bicritical. For any positive integer k, $M^k(G)$ is both γ_t-critical and γ_t-bicritical.*

Proposition 4.40. [30] *If G is a γ_t-critical graph, then for any positive integer k, $M^k(G)$ is γ_t-critical.*

4.3 Summary and Directions for Future Work

This chapter studied γ_t-critical graphs and γ_t-bicritical graphs. It was shown that for a γ_t-critical graph G, $\gamma_t(G - N[x]) \geq \gamma_t(G) - 2$ for any $x \in V(G)$. The γ_t-critical graphs which contain a vertex x such that $\gamma_t(G - N[x]) = \gamma_t(G) - 2$ in particular complicate the structure of γ_t-critical graphs when compared to the structure of i-critical graphs. In particular, this property makes it possible to construct graphs where each block of the graph is γ_t-critical but the graph itself is not γ_t-critical (recall that $C_6 \cdot C_6$ is such a graph). Some familiar constructions from Chapters 2 and 3 were revisited for the use of creating γ_t-graphs. A corrected statement of the necessary and sufficient conditions for the disjoint union to be γ_t-critical was provided. The coalescence construction was investigated for the creation of γ_t-critical graphs. It was shown that $\gamma_t(G) + \gamma_t(H) - 2 \leq \gamma_t(G \cdot_{xy} H) \leq \gamma_t(G) + \gamma_t(H)$ and that graphs which have $\gamma_t(G \cdot_{xy} H) = \gamma_t(G) + \gamma_t(H) - 2$ are not γ_t-critical. Additionally, the graphs for which $\gamma_t(G \cdot_{xy} H) = \gamma_t(G) + \gamma_t(H) - 2$ were characterized. Necessary and sufficient conditions for $G \cdot H$ to be γ_t-critical were presented. These conditions were extended from the sufficient conditions already known.

We close with some open questions:

1. Proposition 4.33 gives an example of γ_t-critical graphs G and H for which $G \cdot H$ is not γ_t-critical. The issue arises when there exists a vertex $x \in V(G)$ such that $\gamma_t(G - N[x]) = \gamma_t(G) - 2$ (and likewise there exists a vertex in H with the same property). Characterize the γ_t-critical graphs where the removal of the closed neighbourhood of a vertex decreases the total domination number by two.

2. Goddard et al. [23] provided a construction for γ_t-critical graphs with any desired number of blocks that are not γ_t-critical. Provide other constructions which create γ_t-critical graphs with blocks that are not γ_t-critical and charac-

terize such graphs.

3. Proposition 4.36 gives a sufficient condition for $G \cdot H$ to be γ_t-bicritical. Find necessary conditions for $G \cdot H$ to be γ_t-bicritical.

4. Chapter 3 investigates strong i-bicritical graphs where $i(G - \{u, v\}) = i(G) - 2$ for independent vertices $\{u, v\} \subseteq V(G)$. Earlier in this chapter a strong γ_t-bicritical graph was defined to be a graph G where $\gamma_t(G - \{x, y\}) = \gamma_t(G) - 2$ whenever $G - \{x, y\}$ has no isolated vertices and either $d(x, y) \geq 3$ or $xy \in E(G)$ with $N(x) \cap N(y) = \emptyset$. Investigate properties of such graphs. An example of a graph that fits this definition is C_6.

5. In Chapter 1 it was mentioned that there are γ-critical graphs where $\gamma(G) = i(G)$. Notice that C_{10} is a graph that is γ-critical with $\gamma(C_{10}) = \gamma_t(C_{10})$. Investigate the γ-critical graphs which have $\gamma(G) = \gamma_t(G)$. Are there graphs which are both γ-critical and γ_t-critical and have $\gamma(G) = \gamma_t(G)$?

6. Propositions 4.27 and 4.28 give upper bounds on the order of a γ_t-bicritical graph. Characterize the k-γ_t-bicritical graphs of maximum order.

7. Corollary 2.5 of Chapter 2 shows that for any graph G and for all $k \geq 3$, there exists a k-i-critical graph H such that G is an induced subgraph of H, while Corollary 2.6 shows that for any graph G and for all $k \geq 4$, there exists a k-i-bicritical graph H such that G is an induced subgraph of H, and Corollary 2.13 shows that for any graph G and for all $k \geq 5$ there exists a strong k-i-bicritical graph H such that G is an induced subgraph of H. Is there a value k_0 such that for any graph G and all $k \geq k_0$ there exists a γ_t-critical graph H such that G is an induced subgraph of H? Likewise, is there such a value for γ_t-bicritical graphs?

8. [23] Characterize the 3-γ_t-critical graphs with diameter 3.

9. [23] Consider the connection between γ-critical and γ_t-critical graphs. For example, $K_3 \Box K_3$ is γ-critical but not γ_t-critical. The cycle C_5 is γ_t-critical but not γ-critical. So, which graphs are domination vertex-critical and total domination vertex-critical (or one but not the other)?

10. [23] If G is a γ_t-critical graph of order n, then it can be shown that $n \leq \Delta(G)(\gamma_t(G) - 1) + 1$. Characterize those graphs achieving equality.

11. [23] Cockayne et al. [14] showed that if G is a connected graph of order $n \geq 2$, then $\gamma_t(G) \leq \max(n - \Delta(G), 2)$. Characterize γ_t-critical graphs G with $\gamma_t(G) = n - \Delta(G)$.

5

The Diameter of Vertex-Critical Graphs

An active area in the study of domination criticality concerns results involving diameter. Results on domination vertex-criticality and diameter first appeared in work by Brigham et al. [9] where the authors conjectured that if G is k-γ-vertex-critical, then $\text{diam}(G) \leq 2(k-1)$. Fulman et al. [21] provided a proof for this conjecture and showed that the bound is sharp.

Theorem 5.1. [21] *The diameter of a* γ*-critical graph* G *satisfies* $\text{diam}(G) \leq 2(\gamma(G)-1)$ *for* $\gamma(G) \geq 2$.

It was noted [21] that the graph created by replacing each of uv on the path P_n by a 4-cycle u, u', v, v', where u' and v' are the new vertices, is an n-γ-critical graph of diameter $2(n-1)$.

The problem of characterizing γ-critical graphs which reach equality in the bound of Theorem 5.1 was investigated [21].

Proposition 5.2. [21] *A graph* G *with diameter four is* 3-γ*-critical if and only if it has two blocks, each of which is* 2-γ*-critical.*

Proposition 5.3. [21] *A graph G with diameter six is 4-γ-critical if and only if it has three blocks, two of which are end-blocks and all of which are 2-γ-critical.*

These characterizations were later generalized by Ao in her Master's thesis.

Lemma 5.4. [7] *For $n \geq 3$, if G is connected and n-γ-critical with* $\operatorname{diam}(G) = 2(n-1)$*, then G is not a block.*

Let $G_1, G_2, \ldots, G_{n-1}$ be 2-γ-critical graphs. Define a coalescence of these graphs, denoted by $G_1 \cdot_\diamond G_2 \cdot_\diamond \cdots \cdot_\diamond G_{n-1}$, inductively as follows:

(i) $G_1 \cdot_\diamond G_2$ is any coalescence of G_1 and G_2.

(ii) For $k \geq 2$, $G_1 \cdot_\diamond G_2 \cdot_\diamond \cdots \cdot_\diamond G_k \cdot_\diamond G_{k+1} = (G_1 \cdot_\diamond G_2 \cdot_\diamond \cdots \cdot_\diamond G_k) \cdot_{uv} G_{k+1}$, where u is the unique vertex of G_k which is non-adjacent to the vertex of identification of $(G_1 \cdot_\diamond G_2 \cdot_\diamond \cdots \cdot_\diamond G_{k-1})$ and G_k, and v is any vertex of G_{k+1}.

Theorem 5.5. [7] *For $n \geq 3$, a graph G is n-γ-critical with* $\operatorname{diam}(G) = 2(n-1)$ *if and only if $G = G_1 \cdot_\diamond G_2 \cdot_\diamond \cdots \cdot_\diamond G_{n-1}$, where G_j is 2-γ-critical for $1 \leq j \leq n-1$.*

Paired domination vertex-critical graphs were investigated by Henning and Mynhardt [28] who provided a construction that shows that for every even $k \geq 4$ there exists a k-γ_{pr}-vertex-critical graph with diameter equal to $3(k-2)/2$. Edwards and Hou [19] used the method of Fulman et al. [21] to verify that if G is k-γ_{pr}-vertex-critical, then $\operatorname{diam}(G) = 3(k-2)/2$. This same method is used in this chapter to find upper bounds on the diameter of i-critical graphs, γ_t-critical graphs, and strong i-bicritical graphs.

5.1 The Diameter of i-Critical Graphs

Recall that in Chapter 2 it was shown that the graph $G \cdot H$ is i-critical if and only if both G and H are i-critical. Furthermore, $i(G \cdot H) = i(G) + i(H) - 1$ if $G \cdot H$ is i-critical.

Theorem 5.6. *If G is i-critical, then* $\operatorname{diam}(G) \leq 2(i(G)-1)$.

Proof. Let G be an i-critical graph with diameter d, and let x be a vertex of maximum eccentricity. We define the *level sets* $X_0, X_1, \ldots, X_d$ by $X_j = \{y \in V(G) : d(x,y) = j\}$, $0 \leq j \leq d$. For $0 \leq j \leq d$, the set U_j is defined by $U_j = X_0 \cup X_1 \cup \ldots \cup X_j$, and let $\langle U_j \rangle$ be the graph induced by U_j.

From Observation 1.16, the only 2-i-critical graphs are K_{2n} less a perfect matching, thus we assume $i \geq 3$ for the remainder of the proof. Let D be any i-set of G. We say that $\langle U_j \rangle$ is D-*sufficient* if $j \leq 2(|D \cap U_j| - 1)$. If $G = \langle U_d \rangle$ is D-sufficient for some i-set D, then $d \leq 2(i(G)-1)$. Let D_x be an i-set of $G-x$ and let $D_x^x = D_x \cup \{x\}$. Notice that D_x^x is an i-set of G and $\langle U_2 \rangle$ is D_x^x-sufficient. Let m be the maximum value of j such that $\langle U_j \rangle$ is D-sufficient. If $m = d$, we are finished, so suppose $m < d$. Then for all $j > m$, $\langle U_j \rangle$ is not D-sufficient. Notice that the value of m may differ for different choices of D, so consider an i-set D that maximizes the value of m.

Suppose $m = 2t+1$ for some $t \in \mathbb{Z}$. Since $\langle U_m \rangle$ is D-sufficient and $\langle U_{m+1} \rangle$ is not D-sufficient, we have that $|D \cap U_m| \geq t+2$, but $|D \cap U_{m+1}| < t+2$, a contradiction. Therefore $m = 2t$ for some $t \in \mathbb{Z}$.

Since $\langle U_m \rangle$ is D-sufficient and $\langle U_{m+1} \rangle$ is not D-sufficient, we have that $|D \cap U_m| \geq t+1$ and $|D \cap U_{m+1}| < (2t+1)/2 + 1$; thus $|D \cap U_m| = t+1$ and $D \cap X_{m+1} = \emptyset$.

Suppose that $d > m+1$. If $D \cap X_{m+2} \neq \emptyset$, then $|D \cap U_{m+2}| \geq 1 + (1 + (m/2)) = 1 + (m+2)/2$, contradicting the maximality of m. Hence, $D \cap X_{m+2} = \emptyset$ and we have $d > m+2$ since we need to dominate X_{m+2}. Furthermore, if $|D \cap X_{m+3}| \geq 2$, we have $|D \cap U_{m+3}| \geq 2 + [1 + (m/2)] = 1 + (m+4)/2 > 1 + (m+3)/2$, again contradicting the maximality of m. Thus $D \cap X_{m+2} = \emptyset$, $D \cap X_{m+3} = \{w\}$, and $D \cap X_{m+4} = \emptyset$ (if X_{m+4} exists), and so w dominates X_{m+2} and X_{m+3}.

Now consider D_w, an i-set of $G - w$ and let $D_w^w = D_w \cup \{w\}$. Notice that $D_w \cap (X_{m+2} \cup X_{m+3}) = \emptyset$. If $|D_w^w \cap U_{m+1}| > |D \cap U_{m+1}|$, then $\langle U_{m+1} \rangle$ is D_w^w-sufficient, a contradiction of the maximality of m. If $|D_w^w \cap U_{m+1}| < |D \cap U_{m+1}|$, then

$(D_w^w \cap U_{m+1}) \cup (D - U_{m+1})$ is an independent dominating set of G with cardinality less than $i(G)$, a contradiction. Thus $|D_w^w \cap U_{m+1}| = |D \cap U_{m+1}| = t+1$. Since $D_w \cap (X_{m+2} \cup X_{m+3}) = \emptyset$, $|D \cap X_{m+4}| \geq 1$ to dominate the vertices in X_{m+3}. But then $|D_w^w \cap U_{m+4}| \geq t+1+2 = t+3$, which means that $\langle U_{m+4} \rangle$ is D_w^w-sufficient, a contradiction of the maximality of m. Thus it follows that either $d \leq 2(i(G)-1)$ or $d = m+1 = 2t+1$. In particular, the theorem is true for all i-critical graphs of even diameter.

Now suppose $d = m+1 = 2t+1$. Then $G \cdot_{xx} G$ is i-critical with diameter equal to $2d$ and $i(G \cdot_{xx} G) = 2i(G) - 1$ and so $2d \leq 2(2i(G)-2)$. Therefore $d \leq 2(i(G)-1)$ as desired.

□

It is now shown that the bound in Theorem 5.6 is sharp. Notice that the cycle on four vertices, C_4, is 2-i-critical with diameter 2, and that C_4 is a graph which reaches equality for the bound in Theorem 5.6. Now $\text{diam}(C_4 \cdot C_4) = 4$, $i(C_4 \cdot C_4) = 3$ by Theorem 2.19, and so $\text{diam}(C_4 \cdot C_4)$ also reaches equality in the bound. This construction can be continued by identifying a vertex of maximum eccentricity in $C_4 \cdot C_4$ with any vertex in C_4. The resulting graph has diameter 6 and independent domination number 4, again achieving equality in the bound. Thus by creating a chain of 4-cycles where the identified vertices are independent, we have an infinite family of graphs that reach equality in Theorem 5.6. In fact, these graphs are also γ-critical and they reach equality in the diameter bound for γ-critical graphs as stated in Theorem 5.1.

5.2 The Diameter of γ_t-Critical Graphs

In a graph G, an *end-vertex* is a vertex of degree one. Notice that if G has an isolated vertex, $\gamma_t(G)$ is undefined. Goddard et al. [23] characterized the γ_t-critical graphs G

with end-vertices. This characterization yields a result on diameter.

Proposition 5.7. [23] *If G is a connected k-γ_t-critical graph with at least one end-vertex, then* $\operatorname{diam}(G) \leq k$ *if* $k \in \{3,4\}$ *and* $\operatorname{diam}(G) \leq k-1$ *if* $k \geq 5$*, and these bounds are sharp.*

Thus in what follows, we consider γ_t-critical graphs without end-vertices.

Observation 5.8. [23] *If G is a γ_t-critical graph without end-vertices, then $\gamma_t(G-v) = \gamma_t(G) - 1$ for every $v \in V(G)$. Furthermore, a γ_t-set of $G - v$ contains no neighbour of v.*

We now consider the maximum diameter of k-γ_t-critical graphs.

Proposition 5.9. [23] *The diameter of a k-γ_t-critical graph G is at most $2k-3$.*

Theorem 5.10. [23] *For all $k \equiv 2 \pmod 3$, there exists a k-γ_t-critical graph of diameter $(5k-7)/3$.*

The graphs from Theorem 5.10 are constructed as follows: Let F be the graph obtained from $P_4 \cup \overline{P_4}$ by adding all edges between P_4 and $\overline{P_4}$ except for the perfect matching between corresponding vertices, and then adding a vertex x adjacent to every vertex of P_4 and a vertex y adjacent to every vertex of $\overline{P_4}$. Thus F is a 3-γ_t-critical graph with diameter 3. Let Q be the graph obtained from two copies of F, call them F_1 and F_2, by deleting y from F_1 and x from F_2 and adding all edges between the the four neighbours of y in F_1 and the four neighbours of x in F_2. Notice that $\gamma_t(Q) = 4$, $\operatorname{diam}(Q) = 5$, and that Q is not γ_t-critical. Let FQ^nF be the graph $F \cdot_{yx} Q \cdot_{yx} Q \cdot_{yx} \cdots \cdot_{yx} Q \cdot_{yx} F$ with n copies of Q. From Theorem 5.10, $\gamma_t(FQ^nF) = 3n+5$, $\operatorname{diam}(FQ^nF) = 5n+6$, and FQ^nF is γ_t-critical for every $n \geq 1$. The graph F is pictured in Figure 5.1, the graph Q is pictured in Figure 5.2, and the the graph FQF is pictured in Figure 5.3.

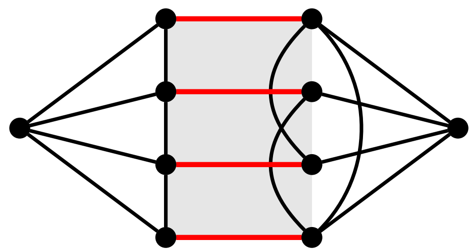

Figure 5.1: The graph F.

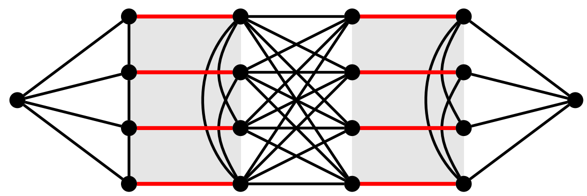

Figure 5.2: The graph Q.

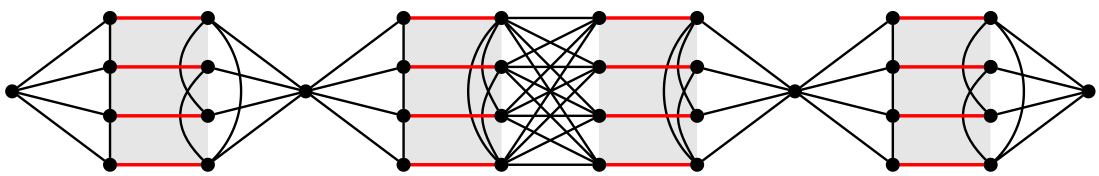

Figure 5.3: The graph FQF.

It is interesting to note that, unlike with γ-critical and i-critical graphs, not every block of a γ_t-critical graph G needs to be γ_t-critical. In addition, if G is a γ_t-critical graph, then $G \cdot G$ need not be γ_t-critical. Consider $C_6 \cdot C_6$. Notice that $\gamma_t(C_6) = 4$, $\gamma_t(C_6 \cdot C_6) = 6$, but $C_6 \cdot C_6$ is not γ_t-critical (the vertex of identification is not a critical vertex). The exact conditions under which $G_1 \cdot_{xy} G_2$ is γ_t-critical are stated in Proposition 4.37. We now improve the bound given in Proposition 5.9.

Theorem 5.11. *The diameter of a connected γ_t-critical graph G without end-vertices satisfies* $\operatorname{diam}(G) \leq 5(\gamma_t(G) - 1)/3$.

Proof. Let G be a γ_t-critical graph with $\operatorname{diam}(G) = d$, and let x be a vertex of maximum eccentricity. We define the level sets $X_0, X_1, \ldots, X_d$ and the sets $U_0, U_1, \ldots, U_d$ as before.

From Proposition 5.9, we know that if $\gamma_t(G) = 3$, then $d \leq 3$ hence we assume that $\gamma_t(G) \geq 4$ in what follows. Let D by any γ_t-set of G. For $j \geq 1$, we say that $\langle U_j \rangle$ is D-*sufficient* if $j \leq (5|D \cap U_j| - 8)/3$. If $G = \langle U_d \rangle$ is D-sufficient, then $d \leq (5\gamma_t(G) - 8)/3 \leq 5(\gamma_t(G) - 1)/3$.

We first show that there exists a γ_t-set D such that $\langle U_2 \rangle$ is D-sufficient. Let $y \in X_1$ and consider a γ_t-set S of $G - y$. By Observation 5.8, $x \notin S$. Since S totally dominates $G - y$, we have that $|S \cap U_2| \geq 2$. But $S \cup \{x\}$ is a total dominating set of G with cardinality $\gamma_t(G)$ and so let $D = S \cup \{x\}$. Therefore $|D \cap U_2| \geq 3$, and for any γ_t-critical graph G there exists a j and a γ_t-set D such that $\langle U_j \rangle$ is D-sufficient. Let m be the maximum value of j such that $\langle U_j \rangle$ is D-sufficient. If $m = d$, we are finished, so suppose that $m < d$. Then for all $j > m$, $\langle U_j \rangle$ is not D-sufficient. Notice that the value of m may differ for different choices of D. Consider a γ_t-set D that maximizes the value of m.

We first show a restriction on m, modulo 5. We have that $|D \cap U_m| \geq 3m/5 + 8/5$ while $|D \cap U_{m+1}| < 3m/5 + 11/5$. Suppose that $m = 5t + 1$ for some $t \in \mathbb{Z}$. Then $|D \cap U_m| \geq 3t + 3$ and $|D \cap U_{m+1}| < 3t + 3$, a contradiction. Suppose that $m = 5t + 3$ for some $t \in \mathbb{Z}$. Then $|D \cap U_m| \geq 3t + 4$ and $|D \cap U_{m+1}| < 3t + 4$, a contradiction. Therefore we have that $m = 5t$, $m = 5t + 2$, or $m = 5t + 4$ for some $t \in \mathbb{Z}$. Suppose that $m < d - 1$.

If $m = 5t$, then $|D \cap U_m| \geq 3t + 2$ and $|D \cap U_{m+1}| \leq 3t + 2$, and $|D \cap U_{m+2}| \leq 3t + 2$. This implies that $|D \cap U_m| = 3t + 2$, $D \cap X_{m+1} = \emptyset$, and $D \cap X_{m+2} = \emptyset$. In addition, $|D \cap U_{m+3}| \leq 3t + 3$, which implies that $|D \cap X_{m+3}| \leq 1$. Since D is a total dominating set, we have that $|D \cap X_{m+3}| \geq 1$. Let $D \cap X_{m+3} = \{w\}$. But then $|D \cap U_{m+4}| \leq 3t + 3$ and so $D \cap X_{m+4} = \emptyset$, a contradiction to D being a total dominating set.

If $m = 5t + 2$, then $|D \cap U_m| \geq 3t + 3$, $|D \cap U_{m+1}| \leq 3t + 3$, and $|D \cap U_{m+2}| \leq 3t + 3$, which implies that $|D \cap U_m| = 3t + 3$, $D \cap X_{m+1} = \emptyset$, and $D \cap X_{m+2} = \emptyset$. In addition, $|D \cap U_{m+3}| \leq 3t + 4$ which implies that $|D \cap X_{m+3}| \leq 1$. Since D is a total dominating set, we have that $|D \cap X_{m+3}| \geq 1$. Let $D \cap X_{m+3} = \{w\}$.

If $m = 5t + 4$, then $|D \cap U_m| \geq 3t + 4$ and $|D \cap U_{m+1}| \leq 3t + 4$, which implies that $|D \cap U_m| = 3t + 4$ and $D \cap X_{m+1} = \emptyset$. In addition, $|D \cap U_{m+2}| \leq 3t + 5$ and $|D \cap U_{m+3}| \leq 3t+5$, which implies that $|D \cap (X_{m+2} \cup X_{m+3})| \leq 1$. Since D is a total dominating set we can conclude that $D \cap X_{m+2} = \emptyset$ and $D \cap X_{m+3} = \{w\}$.

In all cases, we have that $D \cap X_{m+1} = \emptyset$, $D \cap X_{m+2} = \emptyset$, $D \cap X_{m+3} = \{w\}$, and so w dominates all of X_{m+2}. Consider D_w, a γ_t-set of $G - w$. By Observation 5.8, $D_w \cap X_{m+2} = \emptyset$. Let $y \in X_{m+2}$. Then $D_w^y = D_w \cup \{y\}$ is a γ_t-set of G. In all cases, if $|D_w \cap U_{m+1}| > |D \cap U_{m+1}|$, then $\langle U_{m+1} \rangle$ is D_w^y-sufficient, a contradiction of the maximality of m. If $|D_w \cap U_{m+1}| < |D \cap U_{m+1}|$, then $|D_w - U_{m+1}| \geq |D - U_{m+1}|$ and so $(D_w \cap U_{m+1}) \cup (D - U_{m+1})$ is a total dominating set of G with smaller cardinality than D, a contradiction. Therefore suppose that $|D_w \cap U_{m+1}| = |D \cap U_{m+1}|$. If $m = 5t + 2$, then $|D_w^y \cap U_{m+2}| = 3t + 4$, a contradiction of the maximality of m. If $m = 5t + 4$, then $|D_w^y \cap U_{m+2}| = 3t + 5$. Recall that $D_w \cap X_{m+2} = \emptyset$ and that D_w dominates X_{m+3} in $G - w$. Therefore $|D_w \cap (X_{m+3} \cup X_{m+4} \cup X_{m+5})| \geq 2$, and so $|D_w^y \cap U_{m+5}| \geq 3t + 5 + 2 = 3t + 7$. But then $\langle U_{m+5} \rangle$ is D_w^y-sufficient, a contradiction of the maximality of m. We can thus conclude that $m \geq d - 1$.

We now have that either $d = m$ (and so $d \leq (5\gamma_t(G) - 8)/3$) or that $d = m + 1$ with $m = 5t$, $m = 5t + 2$, or $m = 5t + 4$ (so that $d = 5t + 1$, $d = 5t + 3$, or $d = 5t + 5$). Furthermore, if $d = m + 1$, $m = 5t$ or $m = 5t + 2$ or $m = 5t + 4$, and $\langle U_m \rangle$ is D-sufficient, then $D \cap X_{m+1} = \emptyset$ and so $|D \cap U_m| = \gamma_t(G)$. Hence if $d = m + 1$, the above argument gives $d = m + 1 \leq (5\gamma_t(G) - 8)/3 + 1 = 5(\gamma_t(G) - 1)/3$ as desired. $\square$

The following result is an immediate consequence of Proposition 5.7 and Theorem 5.11.

Corollary 5.12. *The diameter of a connected γ_t-critical graph G satisfies* $\mathrm{diam}(G) \leq 5(\gamma_t(G) - 1)/3$.

Notice that if $\gamma_t(G) \equiv 2 \pmod 3$, then $\lfloor 5(\gamma_t(G)-1)/3 \rfloor = (5\gamma_t(G)-7)/3$. Therefore the graph FQ^nF achieves equality in the bound from Theorem 5.11 for all $n \geq 1$. For $\gamma_t \equiv 1 \pmod 3$, it is straightforward to show that $G = F \cdot_{yx} FQ^nF$ is γ_t-critical with $\gamma_t(G) = 3(n+1)+4$ and $\text{diam}(G) = 5(n+1)+4 = \lfloor 5(\gamma_t(G)-1)/3 \rfloor - 1$. For $\gamma_t(G) \equiv 0 \pmod 3$, it is straightforward to show that $G = F \cdot_{yx} F \cdot_{yx} FQ^nF$ is γ_t-critical with $\gamma_t(G) = 3(n+2)+3$ and $\text{diam}(G) = 5(n+2)+2 = \lfloor 5(\gamma_t(G)-1)/3 \rfloor - 1$.

The diameter of γ_t-bicritical graphs was briefly investigated by Jafari Rad [30].

Theorem 5.13. [30] *If G is a k-γ_t-bicritical graph with at least one end-vertex, then* $\text{diam}(G) \leq k$ *if $k \in \{3,4\}$ and* $\text{diam}(G) \leq 2k-2$ *if $k \geq 5$.*

Theorem 5.14. [30] *Let G be a k-γ_t-bicritical graph with minimum degree at least two. Then* $\text{diam}(G) \leq 2k-3$.

5.3 The Diameter of Strong i-Bicritical Graphs

Theorem 5.15. *If G is strong i-bicritical, then* $\text{diam}(G) \leq 3/2i(G)$.

Proof. Let G be a strong i-bicritical graph with diameter d, and let x be a vertex of maximum eccentricity. We define the level sets X_0, X_1, …, X_d and the sets U_0, U_1, …, U_d as before.

Let D be any i-set of G. We say that $\langle U_j \rangle$ is D-sufficient if $j \leq 3/2|D \cap U_j| - 1$. If $\langle U_d \rangle$ is D-sufficient for some i-set D, then $d \leq 3/2i(G) - 1 < 3/2i(G)$. Let $D_{x_1x_2}$ be an i-set of $G - \{x_1, x_2\}$ where $\{x_1\} = X_0$ and $x_2 \in X_2$. Since G is strong i-bicritical, $D_{x_1x_2} \cup \{x_1, x_2\}$ is an i-set of G, and $2 = 3/2(2) - 1 \leq 3/2|D \cap U_2| - 1$ and so we can say that $\langle U_2 \rangle$ is $D_{x_1x_2} \cup \{x_1, x_2\}$-sufficient. Let m be the maximum value of j so that $\langle U_j \rangle$ is D-sufficient. If $m = d$, we are finished, so suppose $m < d$. Notice that the value of m may differ for different choices of D, so consider an i-set D that maximizes the value of m.

First we obtain a restriction on m, modulo 3. Suppose that $m = 3t + 1$ for some $t \in \mathbb{Z}$. Then $|D \cap U_m| \geq 2t + 2$ and $|D \cap U_{m+1}| \leq 2t + 1$, a contradiction. Therefore we have that $m = 3t$ or $m = 3t + 2$ for some $t \in \mathbb{Z}$.

If $m = 3t$, then $|D \cap U_m| \geq 2t+1$ and $|D \cap U_{m+1}| \leq 2t+1$ and $|D \cap U_{m+2}| \leq 2t+1$ which implies that $|D \cap U_m| = 2t+1$, $D \cap X_{m+1} = \emptyset$, and $D \cap X_{m+2} = \emptyset$. In addition, $|D \cap U_{m+3}| \leq 2t + 2$, which implies that $|D \cap X_{m+3}| \leq 1$. Since D is an independent dominating set, we have that $|D \cap X_{m+3}| \geq 1$. Hence let $D \cap X_{m+3} = \{w\}$. Notice that $\{w\}$ dominates all of X_{m+2}. Let D_w be an i-set of $G - w$. Since G is strong i-bicritical, G is also i-critical and so $|D_w| = |D| - 1$. Since $\{w\}$ dominates all of X_{m+2}, $D_w \cap X_{m+2} = \emptyset$. If $|D_w \cap U_{m+1}| > |D \cap U_{m+1}|$, then $\langle U_{m+1} \rangle$ is $D_w \cup \{w\}$-sufficient, a contradiction of the maximality of m. If $|D_w \cap U_{m+1}| < |D \cap U_{m+1}|$, then $|D_w - U_{m+1}| \geq |D - U_{m+1}|$ and so $(D_w \cap U_{m+1}) \cup (D - U_{m+1})$ is an independent dominating set of G with smaller cardinality than $|D|$, a contradiction. Therefore suppose that $|D_w \cap U_{m+1}| = |D \cap U_{m+1}| = 2t + 1$. Since $D_w \cap X_{m+2} = \emptyset$, $|D_w \cap (X_{m+2} \cup X_{m+3})| \geq 1$ in order to dominate X_{m+3}. If $|D_w \cap (X_{m+2} \cup X_{m+3})| \geq 2$, then $(D_w \cup \{w\}) \cap X_{m+4} \geq 2t + 1 + 1 + 2 = 2t + 4$ and then $\langle U_{m+4} \rangle$ is $D_w \cup \{w\}$-sufficient, a contradiction of the maximality of m. Therefore $|D_w \cap (X_{m+2} \cup X_{m+3})| = 1$. If $D_w \cap X_{m+3} \neq \emptyset$, then $|(D_w \cup \{w\}) \cap U_{m+3}| = 2t+1+1+1 = 2t+3$. But then $\langle U_{m+3} \rangle$ is $D_w \cup \{w\}$-sufficient, a contradiction. Hence $D_w \cap X_{m+3} = \emptyset$, $D_w \cap X_{m+4} = \{y\}$ and y is adjacent to every vertex in $X_{m+3} - \{y\}$ and $wy \notin E(G)$. Let D_{wy} be an i-set of $G - \{w, y\}$. Since G is strong i-bicritical, $|D_{wy}| = |D| - 2$ and $D_{wy} \cup \{w, y\}$ is an i-set of G. Notice that $D_{wy} \cap X_{m+2} = \emptyset$ and $D_{wy} \cap X_{m+3} = \emptyset$. If $|D_{wy} \cap U_{m+1}| > |D \cap U_{m+1}|$, then $\langle U_{m+1} \rangle$ is $D_{wy} \cup \{w, y\}$-sufficient, a contradiction. If $|D_{wy} \cap U_{m+1}| < |D \cap U_{m+1}|$, then $(D_{wy} \cap U_{m+1}) \cup (D - U_{m+1})$ is an independent dominating set of G with cardinality less than $|D|$, a contradiction. Therefore $|D_{wy} \cap U_{m+1}| = |D \cap U_{m+1}| = 2t + 1$. To dominate X_{m+3}, $D \cap X_{m+4} \neq \emptyset$. Then $|(D_{wy} \cup \{w, y\}) \cap U_{m+4}| \geq 2t+1+2+1 = 2t+4$ and so $\langle U_{m+4} \rangle$ is $D_{wy} \cup \{w, y\}$-sufficient, a contradiction. Therefore $d < m + 2$.

If $m = 3t + 2$, then $|D \cap U_m| \geq 2t + 2$, $|D \cap U_{m+1}| \leq 2t + 2$, $|D \cap U_{m+2}| \leq 2t + 3$, and $|D \cap U_{m+3}| \leq 2t + 3$. Thus $D \cap X_{m+1} = \emptyset$ and $|D \cap (X_{m+2} \cup X_{m+3})| \leq 1$. Thus either $D \cap X_{m+2} = \emptyset$ with $D \cap X_{m+3} = \{w\}$, or $D \cap X_{m+2} = \{w\}$ with $D \cap X_{m+3} = \emptyset$. The argument of $D \cap X_{m+2} = \emptyset$ with $D \cap X_{m+3} = \{w\}$ is similar to the case above where $m = 3t$ and arrives at a contradiction and so $d < m + 2$. Thus suppose $D \cap X_{m+2} = \{w\}$ with $D \cap X_{m+3} = \emptyset$ and so $\{w\}$ dominates all of X_{m+2}. Let D_w be an i-set of $G - w$. Notice that $D_w \cap X_{m+2} = \emptyset$. If $|D_w \cap U_{m+1}| > |D \cap U_{m+1}|$, then $\langle U_{m+1} \rangle$ is $D_w \cup \{w\}$-sufficient, a contradiction. If $|D_w \cap U_{m+1}| < |D \cap U_{m+1}|$, then $(D_w \cap U_{m+1}) \cup (D - U_{m+1})$ is an independent dominating set of G with cardinality smaller than $|D|$, a contradiction. Therefore $|D_w \cap U_{m+1}| = |D \cap U_{m+1}|$. Then $D_w \cap (X_{m+3} \cup X_{m+4}) \neq \emptyset$ in order to dominate X_{m+3}. Suppose $y \in D_w \cap X_{m+3}$. Then $|(D_w \cup \{w\}) \cap U_{m+3}| \geq 2t + 2 + 1 + 1 = 2t + 4$ and so $\langle U_{m+3} \rangle$ is $D_w \cup \{w\}$-sufficient, a contradiction. Therefore $D_w \cap X_{m+3} = \emptyset$ and so $D_w \cap X_{m+4} \neq \emptyset$. Suppose $y \in D_w \cap X_{m+4}$. Consider D_{wy}, an i-set of $G - \{w, y\}$. Since G is strong i-bicritical $|D_{wy}| = |D| - 2$. By similar arguments to before, $|D_{wy} \cap U_{m+1}| = |D \cap U_{m+1}|$. Now $|D_{wy} \cap (X_{m+2} \cup X_{m+3} \cup X_{m+4})| \geq 1$ to dominate X_{m+3}. But then $D_{wy} \cup \{w, y\}$ is an i-set of G and $|(D_{wy} \cup \{w, y\}) \cap U_{m+4}| \geq 2t + 2 + 2 + 1 = 2t + 5$ and so $\langle U_{m+4} \rangle$ is $D_{wy} \cup \{w, y\}$-sufficient, a contradiction. Therefore $d < m + 3$.

Now, if $m = 3t$, we have that $d = m = 3t$ or $d = m + 1 = 3t + 1$. Since $D \cap X_{m+1} = \emptyset$, we have that $i(G) = |D \cap U_m| = 2t + 1$. Then either $d \leq 3/2i(G) - 3/2$ or $d \leq 3/2i(G) - 1/2$. If $m = 3t + 1$, then $d = m = 3t + 1$ and $i(G) = |D \cap U_m| \geq 2t + 2$. Then $d \leq 3/2i(G) - 2$. If $m = 3t + 2$, then $D \cap X_{m+2} = \emptyset$ with $D \cap X_{m+3} = \{w\}$, or $D \cap X_{m+2} = \{w\}$ with $D \cap X_{m+3} = \emptyset$. In the case where $D \cap X_{m+2} = \emptyset$ with $D \cap X_{m+3} = \{w\}$, we have $d = m = 3t + 2$ or $d = m + 1 = 3t + 3$. Then $i(G) = 2t + 2$ and $d \leq 3/2i(G) - 1$ or $d \leq 3/2i(G)$. In the case where $D \cap X_{m+2} = \{w\}$ with $D \cap X_{m+3} = \emptyset$, we have $d = m = 3t + 2$ or $d = m + 1 = 3t + 3$ or $d = m + 2 = 3t + 4$. For $d = m = 3t + 2$ and $d = m + 1 = 3t + 3$, $i(G) = 2t + 2$ and so $d = 3/2i(G) - 1$

or $d = 3/2i(G)$. For $d = m + 2 = 3t + 3$, we have $i(G) = 2t + 2 + 1 = 2t + 3$ and so $3/2i(G) = 3t + 4 + 1/2$ and $d = 3/2i(G) - 1/2$. Thus, in all cases, $d \leq 3/2i(G)$. □

5.4 Summary and Directions for Future Work

In this chapter, upper bounds for the maximum diameter of connected i-critical graphs, connected strong i-bicritical graphs, and connected γ_t-critical graphs were presented. In particular it was shown that if G is an i-critical graph, then $\text{diam}(G) \leq 2(i(G)-1)$. This bound provided a trivial upper bound for the diameter of i-bicritical graphs, namely that if G is i-bicritical, then $\text{diam}(G) \leq 2(i(G) - 1) + 1$. Again, the coalescence construction played an important role in this chapter and here it was used to create i-critical graphs which reach equality in the diameter bound. Section 5.3 showed that if G is strong i-bicritical, then $\text{diam}(G) \leq 3/2i(G)$. It is not known whether this is the best possible bound. Section 5.2 concerned the diameter of γ_t-critical graphs and there it was shown that if G is a γ_t-critical graph without end-vertices, then $\text{diam}(G) \leq 5(\gamma_t(G) - 1)/3$. Previous known results have shown that for all $k \equiv 2 \pmod 3$, there exists a k-γ_t-critical graph of diameter $(5k - 7)/3$, and this reaches equality in the diameter bound for γ_t-critical graphs.

We close with a list of open questions:

1. Graphs for which equality is reached in the upper bounds on the diameter for connected γ-critical graphs, connected i-critical graphs, and connected γ_t-critical graphs are known. Find graphs which reach equality in the upper bounds on the diameter for i-bicritical graphs, and strong i-bicritical graphs, or provide new upper bounds on the diameter for i-bicritical graphs and strong i-bicritical graphs.

2. Find an upper bound on the diameter for γ_t-bicritical graphs.

3. In her Master's thesis, Ao [7] shows that the graphs which reach equality in the

diameter bound of Theorem 5.1 are exactly those graphs for which every block is a 2-γ-critical graph and the blocks are joined so that the underlying block structure is a path (thus maximizing the diameter). Characterize the i-critical graphs of maximum diameter. It is suspected that the class of γ-critical graphs of maximum diameter is equal to the class of i-critical graphs of maximum diameter.

6

The Gamma-Graph of a Tree

The *γ-graph of a graph G*, $G(\gamma) = (V(\gamma), E(\gamma))$, is the graph where the vertex set $V(\gamma)$ is the collection of γ-sets of G. Adjacency between two γ-sets in $G(\gamma)$ can be defined in two different ways:

- Single vertex replacement adjacency model: where γ-set D_1 is adjacent to γ-set D_2 if there exists a vertex $u \in D_1$ and a vertex $v \in D_2$ such that $D_2 = (D_1 - \{u\}) \cup \{v\}$
- Slide adjacency model: where γ-set D_1 is adjacent to γ-set D_2 if there exists a vertex $u \in D_1$ and a vertex $v \in D_2$ such that $D_2 = (D_1 - \{u\}) \cup \{v\}$ and $uv \in E(G)$.

Thus we can think of adjacency between γ-sets D_1 and D_2 in $G(\gamma)$ as a swap of two vertices. In the slide adjacency model, these two vertices must be adjacent in G, hence the γ-graph obtained from the slide adjacency model is a subgraph of the γ-graph obtained in the single vertex replacement adjacency model. The single vertex replacement adjacency model was introduced by Subramanaian and Sridharan

[43] in 2008, and the slide adjacency model was introduced independently by Fricke, Hedetniemi, Hedetniemi, and Hutson [20] in 2011. The single vertex replacement adjacency model was further studied in [33] and [42] and the slide adjacency model was further studied in [16].

The paper of Fricke et al. [20] introduced a series of open questions on γ-graphs:

1. Is $\Delta(T(\gamma)) = O(n)$ for every tree T of order n?
2. Is $\text{diam}(T(\gamma)) = O(n)$ for every tree T of order n?
3. Is $|V(T(\gamma))| \leq 2^{\gamma(T)}$ for every tree T?
4. Which graphs are γ-graphs of trees?
5. Which graphs are γ-graphs? Can you construct a graph H that is not a γ-graph of any graph G?
6. For which graphs G is $G(\gamma) \cong G$?
7. Under what conditions is $G(\gamma)$ a disconnected graph?

6.1 The Maximum Degree of $T(\gamma)$

In the following results we work with rooted trees. A *rooted tree* is a pair (T, c), where T is a tree and $c \in V(T)$ is a special vertex called the *root*. Let (T, c) be a rooted tree. A vertex x is called an *ancestor* of a vertex y if x belongs to the unique path joining y and c. If, in addition, $xy \in E(T)$, then x is a *parent* of y. The terms *descendant* of x and *child* of x, respectively, are used to describe such a vertex y. Note that x is both an ancestor and a descendant of itself. We use T_x to describe the subtree of T induced by the descendants of x, and T_x is rooted at x.

Proposition 6.1. *If D is a γ-set of T and there exist a vertex $x \in D$ and a vertex $y \in V(T)$ such that $D' = (D - \{x\}) \cup \{y\}$ is also a γ-set of T, then $d(x, y) \leq 2$.*

Proof. Let D be a γ-set of T. Say $x \in D$ and $y \in V(T)$ such that $d(x,y) \geq 3$. Suppose $D' = (D - \{x\}) \cup \{y\}$ is a γ-set of T. Root T at the vertex y and let z be the parent of x and let w be the parent of z. (Notice that z and w exist because $d(x,y) \geq 3$.) Since D' dominates T, $D' \cap V(T_z)$ dominates T_x. Notice that $|D' \cap V(T_z)| < |D \cap V(T_z)|$. But then $D'' = (D - V(T_z)) \cup (D' \cap V(T_z))$ is a dominating set of T and $|D''| < |D|$, a contradiction. Therefore $d(x,y) \leq 2$. □

Proposition 6.2. *For a γ-set D of a tree T and a vertex $z \notin D$, there is at most one vertex $v \in D$ such that $(D - \{v\}) \cup \{z\}$ is also a γ-set of T.*

Proof. Let T be a tree and let D be a γ-set of T, and consider a vertex $z \notin D$. Suppose there exists a set $\{x, y\} \subseteq D$ such that $(D - \{x\}) \cup \{z\}$ is a dominating set of T and $(D - \{y\}) \cup \{z\}$ is a dominating set of T. By Proposition 6.1, $d(x,z) \leq 2$ and $d(y,z) \leq 2$.

Suppose $d(x,z) = d(y,z) = 1$. Then x is at most needed to dominate itself and z and y is at most needed to dominate itself and z. But then $D' = (D - \{x,y\}) \cup \{z\}$ is a dominating set of T, and $|D'| < |D|$, a contradiction.

Suppose $d(x,z) = d(y,z) = 2$ and there exists a path xuz and a path yvz, $u \neq v$. Then x is at most needed to dominate u and y is at most needed to dominate v. But then $D' = (D - \{x,y\}) \cup \{z\}$ is a dominating set of T, and $|D'| < |D|$, a contradiction.

Suppose $d(x,z) = d(y,z) = 2$ and there exists a path xvz and a path yvz. Again, x is at most needed to dominate v and y is at most needed to dominate v. But then $D' = (D - \{x,y\}) \cup \{z\}$ is a dominating set of T, and $|D'| < |D|$, a contradiction.

Suppose, without loss of generality, that $d(x,z) = 2$ and $d(y,z) = 1$ and there exists a path xvz. Then x is at most needed to dominate u and y is at most needed to dominate itself and z. But then $D' = (D - \{x,y\}) \cup \{z\}$ is a dominating set of T, and $|D'| < |D|$, a contradiction.

Suppose, without loss of generality, that $d(x,z) = 2$ and $d(y,z) = 1$ and there exists a path xyz. In this case, since $(D - \{x\}) \cup \{z\}$ is a dominating set and y

dominates z, we can say that $D' = D - \{x\}$ is a dominating set of T, a contradiction as $|D'| < |D|$. □

Corollary 6.3. *If* $|V(T)| = n$, *then* $\Delta(T(\gamma)) \leq n - \gamma(T)$.

Notice that the adjacency model used in $T(\gamma)$ to obtain the result of Corollary 6.3 is irrelevant.

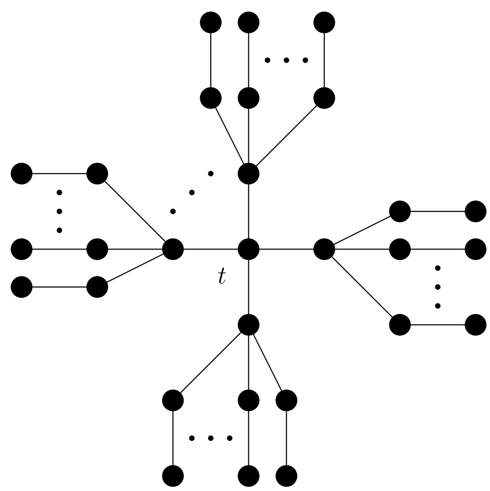

Figure 6.1: An infinite family of useful trees.

Figure 6.1 shows an infinite family of trees which reach equality in the bound from Corollary 6.3. Let the central vertex t have $\deg(t) = a \geq 2$ and suppose that each neighbour v of t has $\deg(v) = b + 1$. Then $n = a + 2ab + 1$ and $\gamma(T) = ab + 1$. Consider the γ-set D comprised of the central vertex t and all the support vertices of T. Then D has degree $a + ab = n - \gamma(T)$ in $T(\gamma)$.

6.2 The Diameter of $T(\gamma)$

Let D be a minimum dominating set of the rooted tree (T, c). We define the *height* of D to be the quantity $ht_T(D) = \sum_{x \in D} d(x, c)$. A γ-set D is called a *highest minimum dominating set* if $ht_T(D) \leq ht_T(F)$ for all γ-sets F of T. We shall show later that every tree T has a unique highest minimum dominating set.

Proposition 6.4. *Let L be the set of leaves in the tree T and let S be the set of support vertices in T. If D is a highest minimum dominating set, then $S \subseteq D$ and $L \cap D = \emptyset$.*

Proof. Let $x \in L$ and $y \in S$ with $xy \in E(T)$. If $x \in D$, then $y \notin D$. But then $D' = (D - \{x\}) \cup \{y\}$ is a γ-set of T and $ht_T(D') < ht_T(D)$ and so D is not a highest minimum dominating set. Since $|D \cap \{x, y\}| = 1$, this completes the proof. □

Proposition 6.5. *A γ-set D is a highest minimum dominating set if and only if every $x \in D - \{c\}$ has a child $y \in \mathrm{pn}(x, D)$.*

Proof. Suppose D is a highest minimum dominating set and suppose there is a vertex $x \in D - \{c\}$ such that for every child y of x, $y \notin \mathrm{pn}(x, D)$. Let the parent of x be z. If $z \in D$, then $D - \{x\}$ is a dominating set of T, a contradiction. If $z \notin D$, then $D' = (D - \{x\}) \cup \{z\}$ is a dominating set such that $ht_T(D') < ht_T(D)$ and so D is not a highest minimum dominating set, a contradiction.

We now show that if every $x \in D - \{c\}$ has a child $y \in \mathrm{pn}(x, D)$, then D is a highest minimum dominating set. We proceed by induction on n, where $n = |V(T)|$. The base cases for $1 \leq n \leq 5$ are easy to verify.

Thus suppose that $n \geq 6$. Let T be rooted at c, and let D be a γ-set of D such that every $x \in D - \{c\}$ has a child $y \in \mathrm{pn}(x, D)$. Notice that the theorem holds for $T = K_{1,n-1}$ and for any tree T with $\mathrm{diam}(T) = 4$ where c is the central vertex of T. Thus suppose that T is neither $K_{1,n-1}$ nor a tree with $\mathrm{diam}(T) = 4$ where c is the central vertex of T. Let $y \in D$, chosen so that $d(y, c)$ is maximized. Thus y is a support vertex, and the only children of y are leaves. Let x be the parent of y.

Case 1: Suppose that $x \in \mathrm{pn}(y, D)$. Then $x \notin D$. Let z be the parent of x. (Note that z exists because T is not a graph of diameter 4 with $x = c$.) Then $z \notin D$ since $x \in \mathrm{pn}(y, D)$.

First we show that $N(x) = \{y, z\}$. Suppose x has a child v, $v \neq y$. Then either v is a leaf or v has at least one child and all the children of v are leaves. If v is a leaf, then $x \in D$. If v has a child, then $v \in D$. In both cases we have that $x \notin \text{pn}(y, D)$, a contradiction.

Let T_1 be the tree $T_1 = T_x$, and let T_2 be the tree $T_2 = T - T_1$. Then $D_1 = D \cap V(T_1)$ is a γ-set of T_1 and $D_2 = D \cap V(T_2)$ is a γ-set of T_2.

Consider T_1 to be rooted at x and T_2 to be rooted at c. Then by the induction hypothesis, D_1 is a highest minimum dominating set of T_1 and D_2 is a highest minimum dominating set of T_2. Recall that $N(x) = \{y, z\}$. Let S be a highest minimum dominating set of T. Then $y \in S$ and $x \notin S$ (otherwise x would not have a child that is a private neighbour) and $S_2 = S \cap V(T_2)$ is a γ-set of T_2. Thus S_2 is a highest minimum dominating set of T_2. But this implies that $ht_{T_2}(S_2) = ht_{T_2}(D_2)$. Therefore $ht_T(S) = ht_T(D)$ and so D is a highest minimum dominating set of T.

Case 2: Suppose that $x \notin \text{pn}(y, D)$. Let T_1 be the tree $T_1 = T_y$ and let T_2 be the tree $T_2 = T - T_1$. Then $D_1 = D \cap V(T_1)$ is a γ-set of T_1 and $D_2 = D \cap V(T_2)$ is a γ-set of T_2 (since x is not a private neighbour of y). Consider T_1 to be rooted at y and T_2 to be rooted at c. Obviously, $D_1 = \{y\}$ is a highest minimum dominating set of T_1. By induction, D_2 is a highest minimum dominating set of T_2.

Say $x \in D$. Then x has a child that is a leaf. Let S be a highest minimum dominating set of T. Then $x \in S$ and $y \in S$ and $S_2 = S \cap V(T_2)$ is a γ-set of T_2. Therefore $ht_{T_2}(S_2) = ht_{T_2}(D_2)$ and so $ht_T(S) = ht_T(D)$ and thus D is a highest minimum dominating set of T.

Say $x \notin D$.

Case i: Suppose that x has a child w with $w \in D$. Thus w is a support vertex of T and the only children of w are leaves. Let S be a highest minimum dominating set of T. Then $y \in S$ and $w \in S$ and so $S_2 = S \cap V(T_2)$ is a γ-set of T_2. Therefore $ht_{T_2}(S_2) = ht_{T_2}(D_2)$ and so $ht_T(S) = ht_T(D)$ and thus D is a highest minimum dominating set of T.

Case ii: Suppose that x has no child w with $w \in D$. Thus $N(x) = \{y, z\}$. Hence

$z \in D$ and z has a child v, $v \neq x$, such that v is a private neighbour of z. We claim that v has no children. Otherwise v is adjacent to either a leaf or support vertex u where the only children of u are leaves. If v is adjacent to a leaf, then $v \in D$. If v is adjacent to a support vertex u, then $u \in D$. In both cases, $v \notin \text{pn}(z, D)$, a contradiction. Thus $N(v) = \{z\}$. Let S be a highest minimum dominating set of T. Then $y \in S$ and $z \in S$ and so $S_2 = S \cap V(T_2)$ is a γ-set of T_2. Therefore $ht_{T_2}(S_2) = ht_{T_2}(D_2)$ and so $ht_T(S) = ht_T(D)$ and thus D is a highest minimum dominating set of T.

Therefore D is a highest minimum dominating set of T if and only if every vertex $x \in D - \{c\}$ has a child $y \in \text{pn}(x, D)$. □

Proposition 6.6. *Let T be rooted at c. If D is not a highest minimum dominating set, then in $T(\gamma)$ the γ-set D is adjacent to a γ-set D' with $ht_T(D') < ht_T(D)$.*

Proof. If D is not a highest minimum dominating set, then by the previous result there is an $x \in D$, $x \neq c$, such that x has no child y where $y \in \text{pn}(x, D)$. If the parent w of x is in D, then $D - \{x\}$ is a dominating set of T, a contradiction. If $w \notin D$, then $D' = (D - \{x\}) \cup \{w\}$ is a dominating set of T and $ht_T(D') < ht_T(D)$. □

Corollary 6.7. *For any γ-set D of T there is a path in $T(\gamma)$ from D to a highest minimum dominating set D' of T.*

Proposition 6.8. *Let T be a tree rooted at vertex c. Then T has a unique highest minimum dominating set.*

Proof. We proceed by induction on $n = |V(T)|$. Again, the base cases of $1 \leq n \neq 5$ are easy to verify. Suppose $n \geq 6$ and let D be a highest minimum dominating set of T.

Suppose $c \notin D$. Let the children of c be $x_1, x_2, \ldots, x_k$. Then $|D \cap \{x_1, x_2, \ldots, x_k\}| \geq 1$. Label $x_1, x_2, \ldots, x_k$ so that $\{x_1, x_2, \ldots, x_i\} \subseteq D$ and $D \cap \{x_{i+1}, x_{i+2}, \ldots, x_k\} = \emptyset$.

Let T_j be the tree $T_j = T_{x_j}$ and let T'_j be the tree $T'_j = \langle V(T_j) \cup \{c\} \rangle$, $j \in \{1, 2, \ldots, k\}$. Let $D_j = D \cap V(T_j)$.

Then $D_1, D_2, \ldots, D_i$ are γ-sets of $T'_1, T'_2, \ldots, T'_i$ respectively, and $D_{i+1}, D_{i+2}, \ldots, D_k$ are γ-sets of $T_{i+1}, T_{i+2}, \ldots, T_k$ respectively. Consider $T'_1, T'_2, \ldots, T'_i$ all to be rooted at c and $T_{i+1}, T_{i+2}, \ldots, T_k$ to be rooted at $x_{i+1}, x_{i+2}, \ldots, x_k$ respectively. Then $D_1, D_2, \ldots,$ D_k are all highest minimum dominating sets in their respective trees. By induction, these highest minimum dominating sets are unique. Therefore T has only one highest minimum dominating set D with $c \notin D$.

Suppose $c \in D$. Let T_j and T'_j, $j \in \{1, 2, \ldots, k\}$, be defined as before. Let $D_j = D \cap V(T'_j)$, $j \in \{1, 2, \ldots, k\}$. (Notice that $c \in D_j$ for every $j \in \{1, 2, \ldots, k\}$.) Then $D_1, D_2, \ldots, D_k$ are γ-sets of $T'_1, T'_2, \ldots, T'_k$ respectively. Furthermore $D_1, D_2, \ldots, D_k$ are all highest minimum dominating sets of $T'_1, T'_2, \ldots, T'_k$. By induction, these highest minimum dominating sets are unique. Therefore T has only one highest minimum dominating set D with $c \in D$.

Now suppose there are two highest minimum dominating sets where one contains c and one does not contain c. Call them D and S. (And so either $c \in D$ and $c \notin S$, or $c \notin D$ and $c \in S$.) Let $X_i = \{x \in V(T) \mid d(x, c) = i\}$. Say that the greatest distance between c and a leaf is l. Let m be the largest value such that $D \cap X_m \neq S \cap X_m$. Then $D \cap (X_{m+1} \cup X_{m+2} \cup \ldots \cup X_l) = S \cap (X_{m+1} \cup X_{m+2} \cup \ldots \cup X_l)$. Notice that by Proposition 6.1, $m \leq l - 2$. Label D and S so that there is an $x \in X_m$ with $x \in D$ and $x \notin S$. Since D is a highest minimum dominating set, x has a child y such that y is a private neighbour of x. Thus $y \in X_{m+1}$. Let the children of y (if they exist) be $y_1, y_2, \ldots, y_r$. Since $y \in \mathrm{pn}(x, D)$ we have that $\{y, y_1, y_2, \ldots, y_r\} \cap D = \emptyset$. Since $D \cap (X_{m+1} \cup X_{m+2} \cup \ldots X_l) = S \cap (X_{m+1} \cup X_{m+2} \cup \ldots X_l)$, we also have that $\{y, y_1, y_2, \ldots, y_r\} \cap S = \emptyset$. But $x \notin S$ and so S does not dominate y, a contradiction.

Therefore T has a unique highest minimum dominating set. $\square$

The preceding work gives the following result, which was known by Fricke et al. [20].

Corollary 6.9. [20] *For any tree T, the γ-graph $T(\gamma)$ is connected.*

A 2-*packing* of a graph G is a collection of vertices $P \subseteq V(G)$ such that for any two vertices $x, y \in P$, $d(x, y) > 2$. A 2-packing is said to be *maximal* if there is no vertex $v \in V(G)$ such that $P \cup \{v\}$ is also a 2-packing. The 2-*packing number of G* is the cardinality of a maximum 2-packing of G. Meir and Moon [35] showed that for a tree T the 2-packing number of T is equal to $\gamma(T)$. Furthermore, their proof shows that any maximum 2-packing of T can be transformed into a γ-set of T.

Proposition 6.10. *For any tree T,* $\text{diam}(T(\gamma)) \leq 2\gamma(T)$ *in the single vertex replacement adjacency model, and* $\text{diam}(T(\gamma)) \leq 2(2\gamma(T) - s)$ *in the slide adjacency model, where s is the number of support vertices in T.*

Proof. By Meir and Moon [35] T has a maximum 2-packing P with $|P| = \gamma(T)$. For $x, y \in P$ notice that $N[x] \cap N[y] = \emptyset$. Consider D, a γ-set of T. For any $x \in P$ there exists a vertex $v \in D$ such that $x \in N[v]$ since x is dominated by D. Thus since $|P| = |D| = \gamma(T)$ we can consider there to be a one-to-one correspondence between the vertices of P and the vertices of D.

Root T at the vertex c and consider two γ-sets, D and D', of T. Let H be the highest minimum dominating set of T. By Corollary 6.9 we know there is a path from D to H and a path from D' to H in $T(\gamma)$. Joining these two paths together gives an upper bound on $d_{T(\gamma)}(D, D')$.

Consider two γ-sets S and S' of T that are adjacent in $T(\gamma)$. Then $S' = (S - \{x\}) \cup \{y\}$ for some $x \in S$ and some $y \in S'$. Since every vertex $v \in S$ is in $N[z]$ for some $z \in P$ and every vertex $u \in S'$ is in $N[z']$ for some $z' \in P$ and $|P| = \gamma(T)$, each neighbourhood $N[z]$, $z \in P$, contains exactly one vertex from S and exactly one vertex from S'. Therefore if $x \in N[z]$ and $y \in N[z']$, $z, z' \in P$, $z \neq z'$, then z is undominated in S', a contradiction. Hence, to move from S to S' where $SS' \in E(T(\gamma))$ we remove a vertex $x \in S$ and add a vertex $y \in S'$ where $x, y \in N[z]$ for some $z \in P$.

We provide an algorithm that constructs a path from any γ-set D to the highest minimum dominating set H. This provides an upper bound on $d_{T(\gamma)}(D, H)$ and in turn gives an upper bound on $\text{diam}(T(\gamma))$.

The number of vertices in which D and H agree is $|D \cap H|$. We show how to move from D to a γ-set S with $|D \cap H| < |S \cap H|$.

Find a vertex $x \in D$ such that $d(x, c)$ is maximized and $x \notin H$. If x is a leaf then let y be the vertex adjacent to x. Notice that by Proposition 6.4, $y \in H$. Then $S = (D - \{x\}) \cup \{y\}$ is a γ-set of T and $|S \cap H| > |D \cap H|$.

Suppose then that x is not a leaf.

Case 1: Suppose $x \in P$. Let y be the vertex adjacent to x with $d(y, c) < d(x, c)$. Since every vertex $z \in D$ such that $d(z, c) > d(x, c)$ is in H and $x \notin H$, $y \in H$ and $S = (D - \{x\}) \cup \{y\}$ is a γ-set of T. Notice that $|S \cap H| > |D \cap H|$.

Thus suppose that $x \notin P$ and again let y be the vertex adjacent to x with $d(y, c) < d(x, c)$. Then $y \in P$.

Case 2: Suppose $y \in H$. Since every vertex $z \in D$ such that $d(z, c) > d(x, c)$ is in H and $x \notin H$, we can say that $S = (D - \{x\}) \cup \{y\}$ is a γ-set of T. Notice that $|S \cap H| > |D \cap H|$.

Suppose that $y \notin H$. Let v be the vertex adjacent to y such that $d(v, c) < d(y, c)$. Then $v \in H$. Since every vertex $z \in D$ such that $d(z, c) > d(x, c)$ is in H, $S' = (D - \{x\}) \cup \{y\}$ is a γ-set of T and $S = (D - \{x\}) \cup \{z\}$ is a γ-set of T. Notice that $DS' \in E(T(\gamma))$ and that $S'S \in E(T(\gamma))$. Also notice that $|S \cap H| > |D \cap H|$ and that in the single vertex replacement adjacency model $DS \in E(T(\gamma))$.

The above cases show that we can always move from a γ-set D to a γ-set S with $|S \cap H| > |D \cap H|$. In the single vertex replacement adjacency model we can move directly from D to S and so there is a path in $T(\gamma)$ from D to H where each move in the path is a move from a γ-set to another γ-set that increases the number of vertices that agrees with H. Thus there are at most $\gamma(T)$ moves needed to convert a γ-set D to the highest minimum dominating set H. In the single vertex replacement adjacency model then for any two γ-sets D and D', $d_{T(\gamma)}(D, D') \leq 2\gamma(T)$ and hence $\text{diam}(T(\gamma)) \leq 2\gamma(T)$.

Thus consider the slide adjacency model. Let s be the number of support vertices in T. To move from a γ-set D to the highest minimum dominating set H, at most s leaves would be swapped out of D to include the s support vertices that are in H. Now for each vertex of D that is not a leaf, there may be two swaps needed to change D into a γ-set S' and then into a γ-set S with $DS' \in E(T(\gamma))$, $S'S \in E(T(\gamma))$, and $|S \cap H| > |D \cap H|$ as outlined in Case 2 above. Thus there are at most $2(\gamma(T) - s) + s = 2\gamma(T) - s$ moves needed to go between a γ-set D and the highest minimum dominating set H. In the slide adjacency model then for any two γ-sets D and D', $d_{T(\gamma)}(D, D') \leq 2(2\gamma(T) - s)$ and hence $\text{diam}(T(\gamma)) \leq 2(2\gamma(T) - s)$.

□

6.3 The Order of $T(\gamma)$

Consider a tree T from the infinite family in Figure 6.1. A brief calculation shows that T has

$$(a+1)\binom{a}{0}(2^b-1)^a + 2\binom{a}{1}(2^b-1)^{a-1} + \binom{a}{2}(2^b-1)^{a-2} + \ldots + \binom{a}{a}(2^b-1)^0$$

which equals

$$a2^b(2^b-1)^{a-1} + 2^{ab}$$

γ-sets. These values can be derived by counting based on which leaves are in the γ-set, or counting based on which vertex dominates t in the γ-set. Now $2^{\gamma(T)} = 2^{ab+1} = 2^{ab} + 2^{ab}$ and so if $a2^b(2^b-1)^{a-1} > 2^{ab}$, a negative answer to question 3 posed by Fricke et al. [20] can be given. Thus consider the inequality

$$\log_2[a2^b(2^b-1)^{a-1}] > \log_2[2^{ab}].$$

This can equivalently be expressed as

$$\log_2[a] + (a-1)\log_2[2^b-1] + b > ab$$

which in turn can be written as

$$\log_2[a] + (a-1)\log_2[2^b-1] > (a-1)b.$$

Dividing by $(a-1)$ and rearranging gives

$$\frac{\log_2[a]}{a-1} > b - \log_2[2^b-1]$$

or

$$\frac{\log_2[a]}{a-1} > \log_2\left[1 + \frac{1}{2^b-1}\right].$$

Since the value of b can be chosen so that $\log_2\left[1 + \frac{1}{2^b-1}\right]$ is arbitrarily close to zero, for any fixed value of a there exists a value of b for which this inequality holds. Thus there are infinitely many trees T which have more than $2^{\gamma(T)}$ minimum dominating sets.

For any tree T (T is not necessarily in this infinite family) a straightforward proof by induction on $\gamma(T)$ shows that T has at most $3^{\gamma(T)}$ γ-sets. This bound can be improved though, and the proof follows.

Theorem 6.11. *Any forest F has at most $((1+\sqrt{13})/2)^{\gamma(F)}$ γ-sets.*

Proof. For ease of notation, say that $b = ((1+\sqrt{13})/2)$. Let $T_1, T_2, \ldots, T_k$ be the components of F. Notice that $\gamma(F) = \gamma(T_1) + \gamma(T_2) + \ldots + \gamma(T_k)$. Let T be a component of F with maximum order. Consider T to be rooted at vertex c and let l be a leaf at maximum distance from c and let x be the parent of l. Let y be the parent of x.

Let P be a maximum 2-packing of T. By Meir and Moon [35], we know that $|P| = \gamma(T)$ and that this 2-packing can be transformed into a γ-set of T.

For each vertex $v \in P$, let B_v be the ball of radius one around v. That is, $B_v = N[v]$. Notice that $x \in B_v$ for some $v \in P$, otherwise this implies that $\{x, l\} \cap P = \emptyset$. But then $P \cup \{l\}$ would be a 2-packing with greater cardinality, a contradiction. We proceed by strong induction on $\gamma(F)$. It is easy to check that the result holds for $\gamma(F) = 1$ and $\gamma(F) = 2$.

Suppose x is adjacent to at least two leaves. Then x is in every γ-set of F. Consider a γ-set D of F. If $y \in pn(x, D)$ then $D - \{x\}$ is a minimum dominating set of $F' = F - (T_x \cup \{y\})$. In this case $\gamma(F') = \gamma(F) - 1$ and so by induction F' has at most $b^{\gamma(F)-1}$ minimum dominating sets. Thus F has at most $b^{\gamma(F)-1}$ γ-sets D where $y \in pn(x, D)$. If $y \notin pn(x, D)$ then $D - \{x\}$ is a minimum dominating set of $F' = F - T_x$. In this case $\gamma(F') = \gamma(F) - 1$ and so by induction F' has at most $b^{\gamma(F)-1}$ minimum dominating sets. Thus F has at most $b^{\gamma(F)-1}$ γ-sets D where $y \notin pn(x, D)$. In total then, F has at most $b^{\gamma(F)-1} + b^{\gamma(F)-1} = 2b^{\gamma(F)-1} < b^{\gamma(F)}$ γ-sets.

Thus suppose that x is adjacent to only one leaf, l. That is, suppose that $\deg(x) = 2$.

Consider any γ-set D of F and any maximum 2-packing P of T. Notice that each ball B_v obtained from P contains exactly one vertex of D. Otherwise, if $B_v \cap D = \emptyset$, v is not dominated. Thus we proceed by considering the following three cases for P: $y \in P$, $x \in P$, or $l \in P$. Recall that $\deg(x) = 2$.

Case 1: Suppose that $y \in P$. Then for any γ-set D of F, x is the vertex from B_y that is contained in D as l must be dominated. This implies that $\deg(y) = 2$, for otherwise the children of y are not dominated. Consider the forest $F' = F - (B_y \cup \{l\})$. Notice that $\gamma(F') = \gamma(F) - 1$ and so by induction F' has at most $b^{\gamma(F)-1}$ minimum dominating sets. Hence F has at most $b^{\gamma(F)-1}$ γ-sets.

Case 2: Suppose that $x \in P$. Notice that for any γ-set D of F, either x or l is in D. Hence y is not adjacent to any leaves, for otherwise these leaves would not be dominated. (This is since $y \in B_x$ and $y \notin D$ and for any leaf v adjacent to y $v \notin B_u$ for any vertex $u \in V(T)$. Hence v is not dominated.) First suppose that $l \in D$. Consider the forest $F' = F - T_x$. Notice that $\gamma(F') = \gamma(F) - 1$ and that $D - \{l\}$ is a minimum dominating set of F'. By induction, F' has at most $b^{\gamma(F)-1}$ minimum dominating sets. Therefore there are at most $b^{\gamma(F)-1}$ γ-sets D of F with $l \in D$.

Now suppose that $x \in D$. Consider the forest $F' = F - B_x$. Suppose $\deg(y) = t+2$. Since y is not adjacent to any leaves, we can say that F' is comprised of a forest F'' and t copies of K_2. Notice that $\gamma(F') = \gamma(F) - 1$ and that $\gamma(F'') = \gamma(F) - (t+1)$ and that $D - \{x\}$ is a minimum dominating set of F'. By induction, F'' has at most $b^{\gamma(F)-t-1}$ minimum dominating sets and the t copies of K_2 together have 2^t minimum dominating sets. Thus F' has at most $2^t b^{\gamma(F)-t-1} < b^{\gamma(F)-1}$ minimum dominating sets. Therefore there are at most $b^{\gamma(F)-1} + b^{\gamma(F)-1} = 2b^{\gamma(F)-1} < b^{\gamma(F)}$ γ-sets of F.

Case 3: Finally suppose that $l \in P$. Consider D, a γ-set of F. If y is adjacent to a leaf v, then either $y \in D$ or $v \in D$. In either case, $D - \{x, l\}$ is a minimum dominating set of $F' = F - \{x, l\}$. By induction F' has at most $b^{\gamma(F)-1}$ minimum dominating sets and so F has at most $2b^{\gamma(F)-1} < b^{\gamma(F)}$ γ-sets. Thus suppose that y is not adjacent to any leaves. Let $\deg(y) = t + 1$ $(t \geq 1)$.

There are three possible cases:

(i) $y \in D$,

(ii) $y \notin D$ and at least one child of y is in D, or

(iii) $y \notin D$ and no children of y are in D.

Case (i): Suppose that $y \in D$. Consider the forest $F' = F - B_y$. Notice that $\gamma(F') = \gamma(F) - 1$ and that $D - \{y\}$ is a minimum dominating set of F'. By induction F' has at most $2^t b^{\gamma(F)-t-1}$ minimum dominating sets. Thus F has at most $2^t b^{\gamma(F)-t-1}$ γ-sets D with $y \in D$.

Case (ii): Suppose that $y \notin D$ and that at least one child of y is in D. Consider the forest $F' = F - T_y$. Notice that $\gamma(F') = \gamma(F) - t$ and that $D - (D \cap V(T_y))$ is a minimum dominating set of F' and that $D \cap V(T_y)$ is a minimum dominating set of T_y. By induction F' has at most $b^{\gamma(F)-t}$ minimum dominating sets. Now T_y has $2^t - 1$ minimum dominating sets that do not contain y. In total then, F has at most $(2^t - 1)b^{\gamma(F)-t}$ minimum dominating sets D with $y \notin D$ and at least one child of y in D.

Case (iii): Suppose that $y \notin D$ and that no children of y are in D. Therefore there is a vertex $w \in D$ that dominates y. Consider the forest $F' = F - (B_w \cup T_y)$. Notice that $\gamma(F') = \gamma(F) - t - 1$ and that $D - V(B_w \cup T_y)$ is a minimum dominating set of F'. Also notice that $D \cap V(B_w \cup T_y)$ is a minimum dominating set of $\langle B_w \cup T_y \rangle$. By induction F' has at most $b^{\gamma(F)-t-1}$ minimum dominating sets. Now $\langle B_w \cup T_y \rangle$ has one minimum dominating set that contains w and no children of y. Thus F has at most $b^{\gamma(F)-t-1}$ γ-sets D with $y \notin D$, $w \in D$, and no children of y in D.

Considering these three cases together, we see that F has at most $2^t b^{\gamma(F)-t-1} + (2^t - 1)b^{\gamma(F)-t} + b^{\gamma(F)-t-1}$ γ-sets. Now $2^t b^{\gamma(F)-t-1} + (2^t - 1)b^{\gamma(F)-t} + b^{\gamma(F)-t-1} = b^{\gamma(F)-t-1}[2^t + 1 + b(2^t - 1)]$. Thus if this value is at most $b^{\gamma(F)}$, the proof is complete.

From the desired inequality $2^t + 1 + b(2^t - 1) \leq b^{t+1}$ we obtain the inequality

$$\frac{1}{b}\left(\frac{2}{b}\right)^t + \frac{1}{b^{t+1}} + \frac{2^t - 1}{b^t} \leq 1.$$

Notice that since $b = ((1+\sqrt{13})/2) > 2$, the function

$$f(t) = \frac{1}{b}\left(\frac{2}{b}\right)^t + \frac{1}{b^{t+1}} + \frac{2^t - 1}{b^t}$$

is a decreasing function. That is, $f(t) > f(t+1)$ for $t \geq 1$. Therefore $f(t)$ is maximized for $t = 1$. By evaluating $2^t + 1 + b(2^t - 1) \leq b^{t+1}$ at $t = 1$, we see that $0 \leq b^2 - b - 3$ and that $b \geq ((1+\sqrt{13})/2)$ satisfies this inequality. Since we are using $b = ((1+\sqrt{13})/2)$, we have shown that our desired inequality $2^t b^{\gamma(F)-t-1} + (2^t - 1)b^{\gamma(F)-t} + b^{\gamma(F)-t-1} \leq b^{\gamma(F)}$ and the proof is complete. □

Corollary 6.12. *Any tree T has at most $((1+\sqrt{13})/2)^{\gamma(T)}$ γ-sets.*

Corollary 6.13. *For any tree T, $|V(T(\gamma))| \leq ((1+\sqrt{13})/2)^{\gamma(T)}$.*

6.4 Summary and Directions for Future Work

In this final chapter, the gamma graphs of trees were studied. Particularly, three questions posed by Fricke et al. [20] were answered. It was shown that for a tree T, if $|V(T)| = n$, then $\Delta(T(\gamma)) \leq n - \gamma(T)$ and this result holds for both the slide adjacency model and the single vertex replacement adjacency model. Results on the maximum diameter of $T(\gamma)$ were also provided: for any tree T, $\text{diam}(T(\gamma)) \leq 2\gamma(T)$ in the single vertex replacement adjacency model, and $\text{diam}(T(\gamma)) \leq 2(2\gamma(T) - s)$ in the slide adjacency model, where s is the number of support vertices in T. An upper bound for the number of γ-sets of a tree was provided by showing that for any tree T, $|V(T(\gamma))| \leq ((1+\sqrt{13})/2)^{\gamma(T)}$. This bound is independent of the choice of adjacency model. Figure 6.1 gave an example of an infinite set of graphs which reach equality in the bound for $\Delta(T(\gamma))$ in both adjacency models.

To conclude, one final list of open problems follows:

1. Upper bounds for $\Delta(G(\gamma))$, $\text{diam}(G(\gamma))$, and the order of $G(\gamma)$ were presented

in this chapter when G is a tree. Find upper bounds for $\Delta(G(\gamma))$, $\mathrm{diam}(G(\gamma))$, and the order of $G(\gamma)$ when G is not a tree.

2. Characterize trees which meet equality in the bounds for $\Delta(T(\gamma))$, $\mathrm{diam}(T(\gamma))$, and the order of $T(\gamma)$.

3. Figure 6.1 gives an example of a tree for which $T(\gamma)$ has more than $2^{\gamma(T)}$ vertices, thus giving a negative answer to question 3 posed by Fricke et al. [20]. However, this tree does not reach equality in the bound for the order of $T(\gamma)$ given in Corollary 6.13. Find trees which reach equality in this bound or find a better bound for $|V(T(\gamma))|$.

4. [20] Which graphs are γ-graphs of trees?

5. [20] Which graphs are γ-graphs? Can you construct a graph H that is not a γ-graph of any graph G?

6. [20] For which graphs G is $G(\gamma) \cong G$?

7. [20] Under what conditions is $G(\gamma)$ a disconnected graph?

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