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THE DEVELOPMENT OF MATHEMATICS-FOR-TEACHING: THE CASE OF FRACTION MULTIPLICATION

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To the four families that made this work possible: Berkopes, Threlkelds, Alexanders, and Jewels. Without your support, I would have abandoned this dream for others long ago.

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ABSTRACT

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The parallel research traditions of explicit-objective and tacit-emergent vary greatly in how they define, assess, and enable development of teacher mathematical knowledge. Despite these diversities, widespread agreement exists in mathematics education research that a teacher's mathematical knowledge is a key competency of an effective teacher. This research report investigates the nature and development of teacher mathematical knowledge of fraction multiplication defined from a tacit-emergent perspective. Questions about the nature and development of teacher mathematical knowledge for fraction multiplication were investigated in this report at the individual and collective levels. In addition, this research report also investigated the developmental links between these levels. The concept study design and the framework for teacher knowledge used in this report derived from the work of Davis and colleagues (Davis & Simmt, 2006; Davis & Renert, 2014).

The results from this report were multifaceted for both the individual and collective levels of mathematical knowledge. Teachers' individual mathematics-for-teaching (M_4T) knowledge of fraction multiplication developed throughout their participation in the

mathematical environments of the concept study. Furthermore, two types of collective action emerged as proposed links between the collective and individual development of teachers' M_4T knowledge of fraction multiplication. These proposed links, titled *synergistic realizations* and *recursive elaborations* emerged in this report as patterns of mathematical action existent in moments of coaction. Recursive elaboration defines the decision-making mechanism where the collective expands the realm of what is possible for a single mathematical realization. Synergistic realization defines the collective decision action in which all previous realizations are abandoned for one innovation in the mathematical realization of a mathematical concept. A discussion of the implications for defining teachers' mathematical knowledge of fraction multiplication as nested systems of individual and collective knowledge is included in the conclusion of this report.

CHAPTER 1. INTRODUCTION

What does it mean to be a knowledgeable mathematics teacher? This question is significant, as contemporary research has found that a knowledgeable teacher is a core element of teacher effectiveness (Baumert et al., 2010; Grossman & Schoenfeld, 2005; Krauss et al., 2008; Mewborn, 2003; NCTM, 2000). Therefore, we must also ask what it means to be a knowledgeable and an *effective* mathematics teacher—and the answer to these questions is more subjective than one might expect. The ways one characterizes knowledge, learning, and the purposes of mathematics education can have quite an impact on what it means for a mathematics teacher to be both knowledgeable and effective in the classroom (Davis, 2004).

Early research traditions investigating mathematics teacher knowledge and effectiveness were very different in the manner that they defined knowledge and effectiveness (Hill, Rowan, & Ball, 2005). Empirical studies in the *process-product* tradition (Good, Grouws, & Ebmeier, 1983) found that certain *teacher behaviors* positively influenced students' performance on basic skills tasks but not on tasks requiring problem-solving skills. Thus, teacher knowledge was defined by non-content-specific teacher behaviors and effectiveness was tracked by student performance on assessments. Critiques of process-product literature concentrated mainly on the absence of content focus, claiming that the subject matter being taught influenced the findings of these studies (Shulman, 1986). *Production-function* literature characterized teacher expertise as the content knowledge resources that teachers brought with them to the classroom. This research (Harbison & Hanushek, 1992; Mullens, Murnane, & Willet, 1996; Rowan, Chiang, & Miller, 1997) focused on proxy variables for teacher mathematical knowledge: such as courses taken, degrees attained, or scores on certification examinations. Thus, expertise was defined by teacher content preparation rather than on teacher behaviors while effectiveness was still tracked by student outcomes on assessments. Critiques of production-function literature concentrated on the imprecise definition of "teacher knowledge", making way for the emergence of a new paradigm for defining teacher knowledge and effectiveness.

Mathematics education scholars cite Lee Shulman's (1986) AERA presidential address (Ball, Thames, & Phelps, 2008) as the beginning of the new paradigm that weighted teacher behavior equally with teacher resources brought to the classroom. Shulman and his colleagues' work expanded the definition of teacher expertise to include combined facets of process-product and production-function literature. In his address, Shulman defined three domains of knowledge for teacher expertise: *common content knowledge*, *curricular knowledge*, and new and innovative type of teacher knowledge pedagogical *content knowledge* (PCK). Shulman and his colleagues' were also the first to distinguish between the ways teachers must know academic content and the ways ordinary adults know such content. Subsequent empirical work verified these claims in the context of mathematics (Ball, 1990, 1991; Borko et al., 1992). This work prompted a significant conceptual leap forward for research into what it means to be a knowledgeable and effective mathematics teacher. It also provided dramatic new insights into for considering mathematical content knowledge for teaching as specific type of professional knowledge.

For nearly two decades, the promising insights that began with Shulman (1986) for mathematical teacher knowledge and expertise went largely underdeveloped. However, after this brief period of stagnation the early twenty-first century spawned innovation. Some research groups focused on building on the promise of Shulman and colleagues' notion of PCK and then attempted to link that knowledge to teacher effectiveness (Ball & Hill, 2008; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill, et al. 2008; Ma, 1999). Others concentrated on designing models for teacher knowledge, building from theories for learning that provide a groundwork for making claims about the nature and development of such knowledge (Davis & Simmt, 2006; Davis & Renert, 2009a, 2009b, 2014; Simmt, 2011). Despite conflicting theoretical underpinnings, these research traditions managed a key development in the field—widespread agreement amongst scholars that a type of mathematical knowledge specific to teaching existed and that knowledge was part of teacher expertise (Baumert et al., 2010).

Ongoing research now focuses on refining the definition of mathematical knowledge for teaching. Some research agendas—referred to here as *explicit-objective research*—depict mathematics for teaching as explicit knowledge (Ma, 1999; Hill, Rowan, Ball, 2005; Hill, Sleep, Lewis, & Ball, 2008; Izsak, 2008; Izsak & Araujo, 2012; Izsak, Orrill, Cohen, & Brown, 2010). Explicit knowledge defines knowledge as an objective possession of expert teachers and can be assessed by observations, interviews, or teacher performance on written-task-based assessments. In contrast other research agendas—referred to here as *tacit-emergent research*, depicts mathematical knowledge for teaching as largely unconscious, tacit knowledge (Adler & Davis, 2006; Davis & Renert, 2009, 2014; Davis & Simmt, 2003, 2006; Davis, 2011, 2012; Simmt, 2011). The kind of knowledge that is "not a set of skills stored in one's head but rather an emergent phenomenon that is enacted in the context of teaching mathematics" (Simmt, 2011, p. 153). The explicit-objective and tacit-emergent research agendas do agree that teacher knowledge is much more complex than it was characterized in earlier process-product or production-function literature. Where they differ is in how to define, assess, and enable development of mathematical knowledge specific for teaching.

Currently we have better answers than ever before as to what it means to be a knowledgeable and effective mathematics teacher. A better understanding of how teachers learn to become knowledgeable and effective is an intuitive next step—and this is precisely where my research finds its place. Mathematics-for-teaching (M_4T)—the tacit-emergent definition for teacher mathematical knowledge guiding this study—is defined by the work of Brent Davis and his colleagues (Davis & Simmt, 2006; Davis, 2011, 2012; Davis & Renert, 2009, 2014; Simmt, 2011). This type of mathematical knowledge for teaching is described as "collective" (Davis & Simmt, 2003, 2006; Towers & Martin, 2009a), "tacit" (Polanyi, 1966; Davis, 2011, 2012), "complex" (Davis & Renert, 2014; Davis & Simmt, 2006), and "simultaneously biological and cultural" (Davis & Simmt, 2003, 2006). These descriptors are important to my study and will aid in situating it in the other tacit-emergent research.

For the purposes of my research, I focus on how M_4T can be characterized by two different kinds of "collective." The first—as Davis and Simmt (2006) argued—is that

M_4T “always occurs in the contexts that involve others: hence, an awareness of how others might be engaged in productive collectivity is an important aspect” (p. 309). This aspect of teacher knowledge is a more refined evolution of the process-product literature definitions for teacher knowledge. The second type of “collective”—radically different than any other teacher knowledge literature—is that M_4T is depicted as collectively emergent (Davis & Simmt, 2003, 2006; Kieren, Pirie, & Gordon-Calvert, 1999; Martin, Towers, Pirie, 2006; Martin & Towers, 2009a). This conceptualizes M_4T as a collective body of shared cultural knowledge, distributed amongst teachers. Davis and Simmt (2003, 2006) expand upon this notion, finding that a group of teachers working together in some instances can also be characterized as an emergent cognitive unity called a *collective learner.*

Davis and his colleagues have provided insight into answering questions of access, development, and study of this form of collective M_4T through their work with teachers in collective mathematical environments called *concept studies* (Davis, 2008a, 2008b; Davis, 2012; Davis et al., 2009; Davis & Renert, 2009; Davis & Simmt, 2006; Davis & Sumara, 2007, 2008; Simmt, 2011). A concept study combines the collaborative work of lesson *study* (Chokshi & Fernandez, 2004; Fernandez & Yoshida, 2004) with the mathematical disciplinary knowledge focus of *concept* analysis (Usiskin, Peressini, Marchisotto, & Stanley, 2003). To date, preliminary results have shown that the concept study is a collective mathematical environment that “supports the development of robust, flexible individual understandings” (p. 309). Davis and his colleagues propose that development is possible in concept studies of differing mathematical foci, and with differing sample sizes of participants (Davis, 2008a, 2008b; Davis, 2012; Davis et al., 2009; Davis & Renert, 2009; Davis & Simmt, 2006; Davis & Sumara, 2007, 2008; Simmt, 2011).

The work that remains is researching how individual and collective notions of M_4T develop independently or collectively in mathematical environments like a concept study. Martin, Towers, and Pirie (2006) suggest that such research findings would provide potent perspectives on the process of coming to understand mathematics. Equally influential would be an understanding of the collective nature of M_4T and how that links to individual M_4T knowledge development.

1.1 Purpose of the Study

The purpose of this study is to research how teachers' individual and collective M_4T knowledge of fraction multiplication develops in the mathematical learning environment called a concept study. The research questions guiding this study are:

1. How does in-service middle school teachers' M_4T of fraction multiplication develop while collaboratively engaging in a concept study focused on multiplication?
2. How does the collective level of M_4T of fraction multiplication develop through engagement in a concept study focused on multiplication?
3. What links exist between the collective and individual teachers' M_4T of fraction multiplication?

1.2 Significance of the Study

The parallel research traditions of explicit-objective and tacit-emergent perspectives vary greatly in how they define, assess, and enable development of teacher mathematical knowledge. However, there is agreement from both traditions on the need for researchers to continue to better understand what types of mathematics teachers need to know to be an effective teacher (Davis & Renert, 2014; Thames & Ball, 2010). Davis (2012) concluded, “as a research community, mathematics educators are still far from making definitive claims about the relationships between teachers’ profound understandings of mathematics and their students’ mathematical understandings” (p. 19). Significant breakthroughs have been made, but much work is still to be done to continue to unravel the mystery of what it means to be a knowledgeable and effective mathematics teacher.

CHAPTER 2. LITERATURE REVIEW

2.1 Teacher Subject Matter Knowledge: A Comparative Back Drop

Lee Shulman and colleagues' research program, *Knowledge Growth in Teaching,* examined teaching expertise. The research group focused on a teacher's ability to manage students (Brophy & Good, 1986; Gage, 1986; Rosenshine & Stevenson, 1986) and the management of ideas of students within the discourse of a classroom (Shulman, 1987). Shulman (1986) coined the phrase *the missing paradigm* to refer to the content knowledge that this research group proposed as central to teaching. A central contribution of this work was that it shifted the scholarly focus of research on teacher knowledge from general aspects of teaching to the role of content knowledge in the action of teaching.

This new paradigm of thought provided a non-discipline specific model that refined the definition of "content knowledge" to include three categories: "subject matter content knowledge" (SCK), "pedagogical content knowledge" (PCK), and "curricular knowledge" (CK). This model offered the first clear categorizations of teachers' non-discipline specific professional knowledge (Shulman, 1986, p. 9-10). Shulman's (1986) AERA presidential address called on researchers to develop discipline-specific models for "categories of content knowledge in the minds of teachers" (p. 9). Within the context of mathematics education, the explosion of innovation in research produced diverse perspectives on how teacher expertise could be theoretically and physically modeled. The researchers constructing these models interpreted knowledge ranging from “knowledge-as-static to knowledge-as-dynamic, from knowledge-as-Platonic to knowledge-as-embodied, from knowledge-as-established to knowledge-as-emergent” (Davis & Renert, 2009, p. 37).

What follows is a sequential discussion of three contemporary research programs that have elaborated upon Shulman’s (1986) original paradigm for teacher knowledge and expertise. The research-based models are sequenced to represent development from explicit-objective model for mathematical teacher disciplinary knowledge to the tacit-emergent model that guides this study. These terms were explicitly defined in Chapter 1 and should be considered as frames that contain many of the contemporary research programs examining teaching disciplinary knowledge for teaching mathematics.

2.1.1 Mathematical Knowledge for Teaching

Deborah Ball and her colleagues analyzed the tasks of teaching in order to define the mathematical skill requisite for handling these tasks (Thames & Ball, 2010). Their work produced Mathematics Knowledge for Teaching (MKT), an explicit-objective model for teacher knowledge that represents all of the mathematical knowledge important for teaching (Ball, Thames, & Phelps, 2008; Thames & Ball, 2010). Ball and her team found that MKT consisted of distinguishable, discrete domains that each correlated to the different tasks of their unique definition of expert teaching (Thames & Ball, 2010). Ball and colleagues created an elaboration of Shulman’s notions of content, curricular, and pedagogical content knowledge (PCK) within the discipline of mathematics education

(Ball, Thames, & Phelps, 2008). The most recent MKT model, from Thames and Ball (2010), appears below in Figure 2.1.

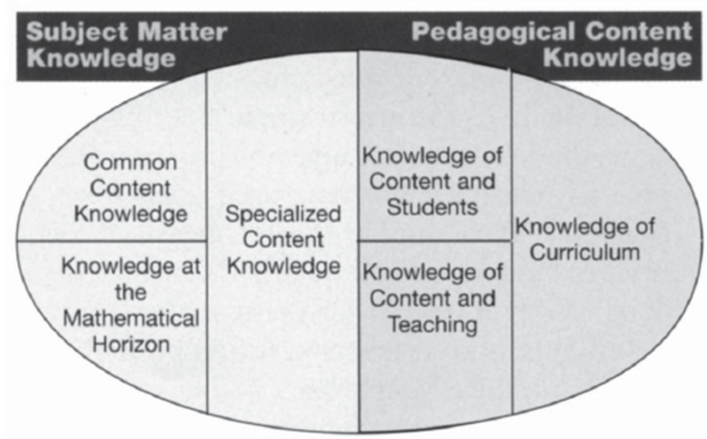

Figure 2.1 The "knowledge egg" categorizations of the MKT model (Ball, 2010, p. 223).

Ball's research group has provided significant insights into the mathematical knowledge requisite of expert teaching. Their work also produced findings (Ball & Bass, 2003; Bass, 2004) that characterized the knowledge for teaching as categorically different from the formal mathematical knowledge necessary for others, including mathematicians.

One goal of research using MKT is to explicitly link specific facets of teacher knowledge to student achievement. The MKT model has been used by researchers to measure teacher quality (Ball & Hill, 2008; Ball, Hill, & Bass, 2005; Hill, et al. 2008), to understand teacher topic specific knowledge of students (Ball, Hill, & Shilling, 2008), to investigate predictors and effect of teacher knowledge on student achievement (Hill, 2010; Hill, Rowan, & Ball, 2005), and to specifically measure teacher knowledge for teaching of fractions (Izsak, 2008; Izsak, Jacobson, & Araujo, 2012; Izsak et al., 2010). In the last 15 years, fine-grain analysis of differing aspects of teacher content knowledge has produced empirical results linking MKT knowledge domains to greater gains in student achievement (Thames & Ball, 2010; Hill, 2010; Hill, et al., 2008). This has been an important movement empirically linking explicit representations of MKT to student performance. These research findings can serve as a lever to explore ways to support, develop, and strengthen the types of mathematical knowledge upon which effective teaching draws.

One interpretation of Ball's work is that it leaves unexplored the notion that teacher knowledge may be conceived as more than just explicit knowledge that is easily measured through assessments, interviews, and observation (Davis & Renert, 2014). As I will later argue, explicit representations of teacher knowledge leave completely unexplored unconscious and potentially tacit teacher knowledge. Tacit knowledge is knowledge that is not necessarily accessible to consciousness and is related to expert webs of associations that activate explicit knowledge. Tacit knowledge is not easily communicated through verbal or written means. As will be later described, tacit knowledge as it applies to teacher disciplinary knowledge is grounded in the work of Polanyi (1966) and other research agendas built from Polanyi's foundational piece (Adler & Davis, 2006; Davis, 2011; Davis & Renert, 2009).

MKT is presented as "distinguishable domains, each defined in relation to the work of teaching" (Thames & Ball, 2010, p. 223). Ball, Thames, and Phelps (2008) recognized categorizing knowledge with distinct boundaries as a shortcoming of their model: "It is not always easy to discern where one of our categories divides from the next, and this affects the precision (or lack thereof) of our definitions" (p. 403). There is little evidence that Ball and colleagues have progressed in attending to this shortcoming or in better understanding the relationship between the discrete domains in the act of teaching. Also noticeably absent from the MKT model is a theory for learning that could ground knowledge claims about how teachers can learn and develop mathematical knowledge important for teaching. Addressing this absence would aid in understanding if MKT should be characterized as "objects or sets of skills stored in one's head" or as "emergent phenomenon that is enacted in the context of teaching" (Simmt, 2010, p. 153).

2.1.2 Profound Understanding of Fundamental Mathematics (PUFM)

Ma (1999) proposed another model for the study of teacher disciplinary knowledge for mathematics: Profound Understanding of Fundamental Mathematics (PUFM). Ma's title for her model offers an explicit picture of her theory for teacher disciplinary knowledge of mathematics. *Fundamental* is defined as having three "related meanings: foundational, primary, and elementary" (p. 116). She argues that, despite advancements in pure and theoretical mathematical research, what is taught in elementary school is still regarded as the foundation for even the newest parts of the discipline. The word *profound* describes the inherent possibilities of Ma's PUFM model. Here, profound means a teacher's "deep, vast, and thorough" understanding of mathematics (p. 120). More importantly, profound in Ma's work means that a teacher must understand mathematics with breadth and depth in order to formulate cognitive interwoven connections among mathematical topics for teaching. A significant claim in Ma's work is conceptualizing mathematical knowledge for teaching as much more than a procedural fluency with mathematics.

The PUFM model physically models knowledge as an interwoven-web of connections, a movement beyond the concern of the MKT model's discrete sub-domains of knowledge. The connections between knowledge categories called *knowledge* *packages* consist of key ideas, sequences for developing the ideas, and concept knots that link related ideas of mathematical concepts. PUFM for the elementary operations mathematics knowledge package is modeled below in Figure 2.2.

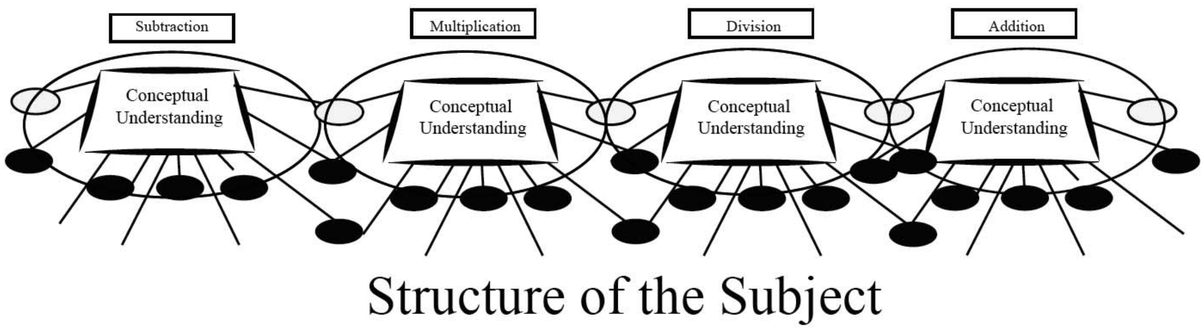

Figure 2.2 Example of PUFM as four interrelated content domains (Ma, 1999, p. 25).

Figure 2.2 as a whole represents what Ma (1999) defines as a knowledge package. The key ideas of this package are the four individual concepts inside the larger ellipses. The smaller interconnected ellipses of varying shades represent basic principles and conceptual knowledge that form a solid conceptual structure of the elementary operations for a teacher. Teachers' mathematical knowledge, according to Ma, is constructed from many of these sorts of knowledge packages. Ma's *concept knot* is the teachers' knowledge, organized in such a way that privileges the interconnectedness of differing knowledge packages and the knots, which act as the knowledge support mechanism for teachers' actions during teaching. Ma found that these packages were diverse in their construction across teacher participants in her empirical work. One interpretation of the diversity is that the connections between types of teacher knowledge are very important, and that the connections are unique for each teacher. Ma's work addresses the concern of distinguishable domains of knowledge by modeling that both the domains and the interconnections that intertwine and intersect the domains should be considered as part of the mathematical knowledge used by a teacher.

Both the MKT and PUFM models advanced the field in research of teacher disciplinary knowledge for mathematics. The shortcoming remains that these models ignore advancements in cognitive science that define knowledge as more than objects or sets of skills stored in teacher's heads. Knowledge can be characterized very differently, such as phenomena that are enacted and simultaneously biologically and environmentally dependent (Lakoff & Nunez, 2000; Davis & Renert, 2014; Simmt, 2011; Varela, Thompson, & Rosch, 1991). Both models also solely assess a teacher's explicit knowledge, ignoring research that has conceptualized expert knowledge as largely tacit and inaccessible to conscious thought (Lakoff & Nunez, 2000; Polanyi, 1966; Davis & Renert, 2014; Varela, Thompson, & Rosch, 1991). Characterizing knowledge as both explicit and tacit has implications for how knowledge is assessed. Enactive knowledge never reaches a steady state but transitions non-linearly between differing levels of refinement, suggesting that development could be privileged over explicit assessment (Kieren & Pirie, 1992).

The work of Ball and colleagues (Ball, Thames, & Phelps, 2008) and Ma (1999) has advanced the field of research concerned with teacher mathematical knowledge. In this report, I am concerned with more than the construction of a model for teacher disciplinary knowledge; I am also concerned with activities that enable mathematics teachers to refine and develop such knowledge. The model that I adopt for these purposes, proposed initially by researchers Brent Davis and Elaine Simmt (2006), capitalizes on qualities of the MKT and PUFM while addressing their shortcomings.

2.1.3 Mathematics-for-Teaching (M_4T)

Davis (2012) describes teacher disciplinary knowledge of mathematics as “vast, intricate, and evolving,” accounting for both explicit and unconscious knowledge. Davis and colleagues’ (Davis & Simmt, 2006; Davis, 2012; Davis & Renert, 2014) have built a model that also characterizes knowledge as enactive, emergent, and embodied. These defining characteristics, as will later be described, necessitate that mathematical knowledge-for-teaching is understood as “a flexible, vibrant category of knowing that is distributed across a body of professionals” (Davis, 2012, p. 3).

Davis and Simmt’s M_4T model was developed through professional development activities with teachers. This focus explains their emphasis on cognitive science innovations in regards to knowledge and development built into their model’s structure. Davis and Simmt (2006) model teacher knowledge in the tacit-emergent form similarly to other researchers (Adler & Davis, 2006; Davis, 2011; Davis & Renert, 2009). The goal of this work is not to focus on assessing teacher knowledge and link it to student achievement. Rather, this research agenda concerns investigating the complexity of teacher knowledge and helping teachers to refine and develop their own conceptions of the mathematics that they teach. Professional development research built using this model (Davis, 2008a, 2008b; Davis, 2012; Davis et al., 2009; Davis & Renert, 2009; Davis & Simmt, 2006; Davis & Sumara, 2007, 2008; Simmt, 2011) has found that teacher mathematical knowledge can be interrogated and develops in professional collective settings called concept studies.

Concept analysis (Lakoff & Nunez, 2000; Leinhardt, Putnam, & Hattrup, 1992; Usiskin, Peressini, Marchisotto, & Stanley, 2003) is a fine-grained interrogation of individual mathematical concepts. Davis and Renert (2013) paraphrased Usiskin et al.'s (2003) description of a concept analysis as examining the historical origins, common usages and applications, and the representations and definitions of a mathematical concept. Therefore, Usiskin' s approach to mathematical study for teachers in his working group's textbook is an example of a concept analysis approach. Another example of this type of approach, found in Leinhardt, Putnam, and Hattrup (1992) describe the purpose of their chapter on number sense as an "attempt to clarify the many dimensions of number sense, and to examine the ways in which number sense develops and is exhibited" by students (p. 3). Lakoff and Nunez (2000) further describe a concept analysis as " a mathematical idea analysis, framed in terms of cognitive mechanisms, of what is required to understand—really understand" a mathematical equation or concept (p. 384). Lesson study (Fernandez & Yoshida, 2004; Stigler & Hiebert, 1999) is defined as a unique activity where teachers plan, observe, and discuss lessons collaboratively with their peer teachers. The key component of lesson study success, as described in Stigler and Hiebert (1999), is that teachers are provided with the opportunities to work collaboratively. A further discussion of concept studies is provided after the discussion of collective and individual M_4T knowledge of fraction multiplication.

Davis and Simmt (2006) intended to create a model for teacher disciplinary knowledge that recognized the complexity of teacher knowledge and oriented mathematical knowledge as emergent, dynamic, nested, and unconscious. The M_4T model that emerged (Davis & Simmt, 2006) utilized the cross-disciplinary branch of scientific inquiry known as complexity science and merged it with enactivism, a theory of cognition built from the work of Varela, Thompson, and Rosch (1991). Complexity science allows categorizations of teacher knowledge similar in content to the MKT and PUFM models, but physically modeled as nested complex systems that render obsolete the discrete modeling limitations of these models. Enactivism allows the recognition that teacher knowledge is predominantly tacit, evolving, dynamic, nested, and simultaneously an individual and collective knowledge construct. What follows is a clarification of enactivism and complexity science to ground the reader's understanding of Davis and colleagues' proposed M_4T model. This discussion is then followed by a description of the M_4T model in broad terms. Finally, a discussion on the model specifically in the context of fraction multiplication is provided.

2.1.3.1 Understanding Complex Systems

Complexity science has emerged in the past 60 years as a branch of scientific inquiry. Complexity science, known less formally as the science of learning systems (Davis & Simmt, 2003), provides an alternative perspective for how the universe is composed. This perspective closely aligns the physical environment with the biological mind and body of those co-creating the environment. Complexity science thought rejects the reductionist notion that the universe is composed of fundamental components that explain the higher order structure that has arisen from them. The universe, instead, is considered to be comprised of embedded complex systems. A complex system is not comprised of simple, discrete parts but rather is a collective of dynamic and similarly complex systems (Davis & Sumara, 2001). A complex system is irreducible, meaning that analyzing its embedded systems (or parts) in no way enables a predictive understanding of its transcendent whole.

Researchers using complexity theory have amassed evidence and assembled a model to help explain the growth of civilizations, weather patterns, brain patterning, and assumptions associated with tacit theories of formal education (Davis & Simmt, 2006). Critics of complexity science argue that the inability of scientists to reduce complex systems to the constituent parts and their interactions for study is a human computational inadequacy rather than a theoretical issue (Wilson, 1975). Further critique comes from reductionist scientists claiming that the whole may not be *just* the sum of the parts, but can also be explained by the sum of the parts and the interaction between those parts (Crick, 1994).

To illustrate why complexity science has gained traction despite thoughtful criticism and a long history of reductionist success, I distinguish complexity science from computationally complicated phenomena. In 1948, Weaver, head of the applied mathematics panel of the US office of scientific research and development, specified a definition of complexity and called for a change from the reductionist understanding of the natural world (Weaver, 1948). Weaver (1948) divided phenomena into three categories: *simple systems, disorganized complex systems,* and *organized complex systems*. He proposed that the first two were complicated mechanical systems that are relatively predictable and dependent upon inputs and outputs. Waldrop (1992) built further vocabulary for these systems and termed them as non-complex or complicated systems. Weaver's (1948) organized complex system and Waldrop's (1992) complex system are phenomena theorized to be fundamentally different from complicated systems. These complex systems are not comprised of simple discrete parts, but rather are theorized to be comprised of a “collective of dynamic and similarly complex systems” (Davis & Sumara, 2001, p. 88).

Waldrop (1992) described three hypothesized distinguishing characteristics of complex systems from the reductionist view of complicated reducible systems. The first characteristic is that complex systems have the capacity to undergo self-organization, so that the capacities of the whole far outreach the capacities of the individual parts. For example, an individual is comprised of nested systems of the brain, heart, and organs and their interactions, but an individual cannot be reduced to a combination of the interactions of these systems. The capacities and potentialities of the individual far outreach the capacities of the individually nested complex systems. Moreover, an individual can do much more than a liver, yet the individual and the liver co-implicate and co-specify each other’s existence. That is, they would not exist nor would their purposes for existence be completely understood without their mutual embedded relationship. Waldrop’s (1992) second distinguishing characteristic of complexity is that these phenomena are adaptive. Here “adaptive” refers to another type of co-specifying relationship. This relationship describes the co-implication of an individual organism and its environment as the two shape and evolve each other. Adaptive also means that a system contains the ability to adapt its structure and to change its operations through the action of operating. It can “learn.”

Waldrop’s (1992) third distinguishing characteristic of complicated versus complexity is that complexity is an emergent phenomenon. The definition of “emergence” is the least agreed upon, but is the most widely used adjective in the varied fields of complexity science (Corning, 2002). Davis and Simmt (2003) described emergence as the mechanism by which systems spontaneously coalesce into an entity that is more than the sum of its parts. Emergence should be understood as defining a phenomenon when novel or coherent structures momentarily arise through the patterns of self-organization. For example, language emerges from the combinations and patterns of letters and words (Corning, 2002). Emergence is a key indicator for complexity that will be utilized in later sections to help describe a "collective learner."

Complexity theorists focus on the co-implicated processes of subjects and environments. Complexity science research should be understood as having the ability to explain profound similarities between diverse natural phenomena. For example, complexity theorists find patterns similar in diverse natural phenomena such as water boiling, ant colonies, and neurological connections. Johnson (2001) explained the nature of complexity as both contextualized and decontextualized, meaning that there are suggested patterns that exist across contextualized cases (Opfer & Pedder, 2011). Looking across cases has produced insight on necessary pre-conditions for differentiating between the emergence of complexity and just interaction of combined forces (Bloom, 2000; Casti, 1994; Corning, 2002; Davis & Simmt, 2003, 2006; Davis & Sumara, 2001; Johnson, 2001; Lewin & Regine, 2000; Opfer & Pedder, 2011; Ricks, 2007).

2.1.3.2 Indicators for Complexity

Davis and Simmt's (2003) proposed pre-conditions or indicators for complexity were defined as *internal diversity, internal redundancy, decentralized control, neighboring interactions,* and *organized chaos*. What follows is a careful description of these indicators for complexity in the context of mathematical learning environments.

After careful description, I portray each pre-condition, including brief notes on how a researcher can claim the existence of or support the development of each individual pre-condition for complexity in the design of a research study.

2.1.3.2.1 Internal Diversity

The ability to adapt and respond to novel mathematical situations is an indicator for the existence or potential for emergence of a complex system. This condition is a result of the lack of ability to predict what will be necessary for a novel mathematical task. Thus, internal diversity as a pre-condition is the amassing of all of the cognitive resources available to operate successfully within the context of the mathematical task. In complexity terms, this potential for adaptability and innovation is defined precisely with the word *intelligence* (Johnson, 2001). Internal diversity can be understood at differing levels of complexity. In the individual teacher context, internal diversity of mathematical knowledge would be the individual's rich resource of diverse components from which to draw to respond appropriately and innovatively to a novel mathematical teaching task. This is different from the collective level of complexity where internal diversity can be understood as the diversity amongst the individuals forming the collective. Each individual brings a variation of intelligence to aid in solving novel mathematical tasks. For example, a teacher with many experiences teaching fraction multiplication for the first time would bring diversity to a group of educators that predominantly only teach fraction multiplication as a remediation topic.

Davis and Simmt (2003) explained that internal diversity of individuals or collectives cannot be "assigned or legislated" (p. 149). The internal diversity of both the collective and the individuals are considered as already present. Consciousness of diversity is not necessary for individuals, as it will emerge through the collaborative interactions of an environment like a concept study (Davis & Renert, 2014). A researcher concerned with the design of mathematical environments that occasion the emergence of complexity can take careful steps to enable the appreciation of diversity, though appreciation of diversity cannot be mandated.

2.1.3.2.2 Internal Redundancy

Davis and Simmt (2003) claim that the ability for individuals to work together often depends upon their similarities rather than their differences. Their claim is that similarities among individuals allow for more familiar interaction in mathematical environments. Internal redundancy of an individual mathematics teacher is the elements of commonality between knowledge systems that allow the teacher to make sense of novel mathematical teaching tasks. Within the context of the collective learner, internal redundancy is commonalities across individuals such as shared vocabularies, teaching experiences, student interactions, and expectations for mathematical and teaching proficiencies (Davis & Simmt, 2003). These offer environmental commonalities that allow individual and collective learners the ability to build an agreed-upon mathematical course of action for overcoming novel mathematical tasks. For example, teachers that teach the same levels of mathematics would have common experiences, or redundancies, about the teaching of fraction multiplication with students. Further this example, teachers could share the experience that students mistakenly establish common denominators when trying to multiply fractions. This redundancy would provide a platform for collective work investigating this shared facet of a common student misconception.

Redundancy is not something that can be mandated by a researcher interested in supporting the emergence of complexity (Davis & Simmt, 2003). The potential for redundancy in a research setting results from the careful selection of participants during the planning phase of a research study. This selection may include considerations of teachers' area of expertise, similar school environments, and similar work environments. Such a selection process may result in commonalities among participants and the potential for redundancy in interaction. The researcher can also foster and support appreciation for redundancy of the collective (Davis & Simmt, 2003). Finally, the researcher can create a context for redundancy by designing opportunities for teachers to share commonalities of experience, expectations, and purposes for the teaching and learning of mathematics.

2.1.3.2.3 Decentralized Control

Davis and Simmt (2003) found that no one entity was in charge or acted as an overseer or director in the teachers' learning community where complexity emerged. Control was decentralized. Similar findings have been reported in other non-education studies involving complex learning systems (Johnson, 2001). Decision-making is dispersed, adaptively and democratically, to the individual components comprising the system (Davis & Simmt, 2003). Some sub-systems may have a greater impact on the system's outcomes, but no one centralized power exists. Within the context of the individual's mathematical knowledge for teaching, decentralized control can be thought of as weighted differences in importance for differing knowledge types in differing environments. For example, a teacher within the context of teaching the concept of divisibility by zero may not privilege graduate level mathematical content knowledge when working with middle school students, yet would perhaps do so when working with peer teachers. Within the context of the collective learner, decision-making is dispersed, adaptively and democratically, to the individual teachers co-acting in the mathematical environment (Varela, 1999; Davis & Simmt, 2003).

Research in professional development literature has shown that decentralized control is an essential element for success. Yoshida and Fernandez (2004) described how the lesson study environment creates a collaborative collective. The collective is achieved through the empowerment of the participating teachers to organize, design, and implement their chosen goals for the lesson study instead of by situating power with the researcher. Chokshi and Fernandez (2004) warn that researchers interested in establishing this type of collaborative environment must carefully consider the stereotypes that exist in the United States' educational system. For example, researchers may need to carefully plan methodologies that will allow for the distribution of power away from the researcher as the location for the final say in any mathematical investigation.

2.1.3.2.4 Organized Randomness

This pre-condition for complexity ensures balance between diversity and redundancy amongst embedded systems. Too much diversity or redundancy can result in the dissolution of a once-complex system (Davis & Simmt, 2003; Ricks, 2007). Complex systems can be understood as being governed by particular boundaries or rules. These boundaries are limits to possible types of activity. These limits channel the activity, providing a context for higher potentialities of innovation (Varela, 1999; Johnson, 2001; Corning, 2002; Davis & Simmt, 2003). In other mathematics education research, this understanding can be illustrated through the use of the metaphor of improvisational action. Martin, Towers, and Pirie (2006) build from improvisational theory (Berliner, 1997; Sawyer, 2000) the understanding that complexity occurs when community structures emerge that focus the action but simultaneously aid the further innovation within the constraints. This is referred to as "the collective striking a groove" (Martin, Towers, & Pirie, 2006). The groove is the agreement by the collective to follow certain rules of action, but within those boundaries innovation is privileged. For example, when investigating the area model for fraction multiplication it would not be useful to begin by translating all written text into German. The boundaries, or organized randomness, provides limits for freedom that eliminates irrelevant action like translating into German. This then provides constraints that activate innovation.

Within the individual mathematical knowledge for teaching context, organized randomness can be understood as the rules that bound the individual's cognition. Davis, Sumara, and Luce-Kapler (2000) described these rules as *liberating constraints*, meaning that the rules of the system set boundaries for the system to operate within, while simultaneously liberating the individual to achieve total freedom of innovation within those bounds. For example, the rules that would govern an individual trying to solve a novel mathematical task would focus as well as channel the individual's productive efforts. Organized randomness channels the efforts of the individual into necessary boundaries, after which total innovation is then privileged. For the collective learner context, Davis and Simmt (2003) found that the teachers collaborating in their research study were operating under the restraints of time, course requirements, available technology, and established produced mathematics. Within the context of these restraints, however, the researchers found that the teachers were able to co-produce an environment rich with possibilities and innovation. A researcher can enable organized randomness by carefully limiting the mathematical focus of the co-created environment while simultaneously supporting complete innovation within those boundaries.

2.1.3.2.5 Neighbor Interactions

Embedded systems with the potential for complexity must "be able to affect one another's activities" (Davis & Simmt, 2003, p. 155). Rather than referring to literal physical location and interaction, this term refers to the interactions of "ideas, hunches, queries and other manners of representations" (p. 156). Within the context of an individual's mathematical knowledge for teaching, this interaction would include the teacher's ability to co-develop action based on cross-embedded system-relevant knowledge, enabling a richer, more innovative, and dynamic approach to novel mathematical teaching tasks. For example, an individual teacher profits from the interactions of knowledge systems that contain knowledge about students' cognition of the concept, best practices for modeling the concept, as well as the curricular level appropriateness of various definitions of the concept. Within the context of the collective mathematical learner, researchers have shown (Cobb, 1999; Rotman, 2000; Davis & Simmt, 2003) that ideas, metaphors, and images of individuals must be actively exposed and scrutinized by others in collaborative mathematical contexts. With the emergence of complexity, this is the mathematical action of the individuals' ideas as they collide and make space for collective development.

Rotman (2000) described mathematical action as inherently containing the sorts of idea collisions necessary for the emergence of complexity at both the individual and collective levels. Namely, mathematical action is filled with the engagement of action between the self, others, and the cultural body of knowledge produced mathematical systems. As with the other preconditions of complexity, researchers cannot mandate the neighboring interactions through the design of the mathematical environment (Davis & Simmt, 2003). What has been shown to be possible is that the researcher can purposefully lead the ideas of individuals to stumble across one another at a high rate of efficiency in hopes of fostering neighboring interactions.

In summary, it is possible for a researcher to consciously support the creation of mathematical environments that support the emergence of preconditions for complexity. Yet, complexity is synergistic action between embedded systems that cannot be mandated from the top down. The synergistic action is a consistent, negotiated collaborative action between systems that enables and restrains the emergence and sustainment of complexity. All the necessary preconditions can be consciously supported and present in a mathematical environment, and yet for reasons not fully understood complexity may not arise (Davis & Simmt, 2003). Following is an introduction to biological metaphors for cognition that will give the reader further tools for understanding the M_4T model.

2.1.3.3 Enactivism: Biological Cognition of Mathematics in Action

Following the publication of *The Embodied Mind* (Varela, Thompson, & Rosch, 1991), enactivism has grown in popularity as a theory of learning in mathematics education research (Ernest, 2006). Enactivism defines knowledge as adequate or viable action in the world (Proulx, 2004). Knowledge as action is an emergent process, integrating past and present experiences to form new activities (von Foerster, 1972). Perception is considered an active process of categorization made possible by previous interactions with the lived world. This orientation to knowledge renders perception and action inseparable in lived cognition (Varela et al., 1991). Two critical elements of enactivism are the individual cognizing agent and the environment that is co-implicated with that agent. What follows is a description of these elements.

2.1.3.3.1 The Agent

Since knowledge is defined as the adequate or viable action of the agent in the world as assessed by an observer, learning is acquired through experiences that enable the continued viable action of an agent in an environment. The acquisition and categorization of these experiences emerge for an individual agent as recurrent sensorimotor patterns that enable action to be guided by perception (Varela et. al, 1991), meaning that there is a history of interaction between two or more systems. These systems are the agent's body and mind, coupled to its environment. In dynamical-systems mathematical language, "the state variables of one system are parameters of the other system, and vice versa" (Thompson, 2007, p. 45). Learning results in a structural change of the agent. The agent's structure is comprised of his or her own biological structure as well as previous actions in an environment (Kieren et al., 1995). This structure is highly dynamic and easily shaped or molded, "a continuing interplay of biological constitution and socially and historically framed experience" (Davis, Sumara, & Kieren, 1996, p. 154). Every action changes the agent's structure, which explains why an agent may act differently in seemingly identical environments.

It is important to note that enactivism renders the terms "structure" and "organization" of an agent as non-interchangeable. The organization of an agent is the amalgam of particular characteristics that constrain the agent in order for that agent to remain part of that environment or community. For example, a student in a mathematics classroom has a highly dynamic individual structure, but the student must maintain certain characteristics of organization to continue to maintain his or her unique existence as a student in that classroom setting. Changes to organization are relatively static, constrained by the need of the individual to remain a member of a particular community (Lozano, 2005). The structure and organization of an agent defines Maturana and Varela's (1987) structural determinism. The environment does not determine learning by an agent, but rather the structure and organization of the agent orients the effect that an experience with the environment can produce. The structure-determined engagement of a given system with its environment or another system is defined as structural coupling (Maturana & Varela, 1987). For example, the organization of the learning environment of a classroom does not determine the learning of an individual agent (student); it is the structural coupling possible between the student and the classroom environment that determines the type of learning possible. To elaborate this structural coupling a discussion of what constitutes an agent's environment follows.

2.1.3.3.2 The Environment

Learning is not determined by the agent's environment, but it does depend on that environment. An agent's interaction with the environment is dependent on that agent's structure and organization, which allow for the recognition and activation of triggers that motivate action (Proulx, 2004). The agent and the environment are "reciprocal and simultaneous specifications of each other" or co-specifying (Proulx, 2008, p. 21). This is a continual, reciprocal, co-specifying, and co-determining relationship between the agent and the known, defined as *structural coupling*. Enactivism describes the embodiment of cognition as doubly embodied, meaning that both the individual and the environment experience the process of evolution: they are constant triggers that produce adaptations in each other's structure (Kieren et al., 1995). A person's perception is a structure-dependent activity that changes the world of lived experience as the individual's structure continually changes. This is aligned with the work of Merleau-Ponty (1962), where perception is active participation and engagement in the world, not separation from it.

2.1.3.3.3 Knowing is Being is Doing

Enactivist theory for cognition is a radical divergence from the pervasive understanding that knowledge is a possession of the individual to be sought after and accumulated. Varela et al. (1991) stated that learning is the complex interplay between the agent and the environment that also cannot be abstracted from an understanding of self. This orientation necessitates that mathematics education researchers concerned with theorizing about and assessing teacher disciplinary knowledge of mathematics pay attention to both formulated and unformulated mathematical knowledge. Formulated knowledge, or explicit knowledge, is knowledge that is directly available to our consciousness. Unformulated knowledge is “tacit, embodied knowing that we continuously enact as we move through the world” (Davis et al., 1996, p. 155). Most cognition, from this perspective, is actually unconscious, unformulated knowledge. For example, we quickly and effortless navigate the complex rules and guidelines of language, without consciously considering word placement or grammatical appropriateness. What a person says and the actions of language that emerge in this situation could be considered as representations of that person’s unconscious knowledge of language. Specifically when considering mathematical knowledge, Lakoff and Nunez (2000) described unformulated mathematical knowledge as unconscious cognition similar to other types of cognition not largely available to us.

Researching cognition from an enactivist, embodied orientation demands a particular interpretation of the discipline of mathematics and mathematical cognition. A researcher interested in assessing knowledge cannot simply look at conscious representations of formulated knowledge. Embodied cognition necessitates interpreting the preceding actions, “the unformulated exploration, the undirected movement, the unstructured interaction, wherein the body is wholly engaged in mathematical play,” as representations of unconscious mathematical knowledge (Davis et al., 1996, p. 156). This interpretation provides a researcher with a comprehensive set of tools to assess unconscious and conscious teacher disciplinary knowledge of mathematics and to make claims about the development of this type of knowledge. With a fuller understanding of complexity science as a tool to understand the structural coupling between the agent and its environment as part of embodied mathematical cognition, we can now focus on the specifics of Davis and Simmt's (2006) M_4T model for teacher disciplinary knowledge of mathematics.

2.1.3.4 The M_4T Model

Davis and Simmt's (2006) M_4T model builds from the language of complexity science in order to attempt to provide a two-dimensional modeling of co-implicated phenomena. The nested complex systems of the two-dimensional model are an arbitrary representation of a system that is neither two-dimensional nor has visible boundaries between the nested systems. The M_4T model achieves a coupling of the individual system with the environment of that system by placing the individual as a "subsystem to a series of increasingly complex systems (such as a classroom, a school, a neighborhood, a culture, humanity, the biosphere)" (p. 117). The nested complex systems of the M_4T model build from the smaller *knowledge-producing systems*, such as the subjective individual teacher and the immediate collective mathematical environment of that teacher. These knowledge-producing systems are then embedded in the *knowledge-produced* systems relevant to mathematical teacher cognition such as curriculum structures and mathematical objects. The M_4T model is then meant to portray the harmonization and co-evolution of historical action of the discipline of mathematics, represented as the *knowledge-produced* systems, with the emerging *knowledge-producing* of the teacher's own individual cognitive activity.

Figure 2.3 is the representation of the M_4T model. It provides a visual for understanding the proposed competing evolutionary tensions of a teacher's conscious and unconscious mathematical knowledge as four nested complex systems. These systems are labeled subjective understandings, collective understandings, curriculum structures, and mathematical objects (Davis & Simmt, 2006; Davis & Renert, 2009). The time scale located at the left of Figure 2.3 represents Davis and Simmt's attention to competing evolutionary tension of each embedded system. Size of the ellipses in this model is meant to signify the level of embedded complexity as well as the amount of time required to see significant evolutions in the system.

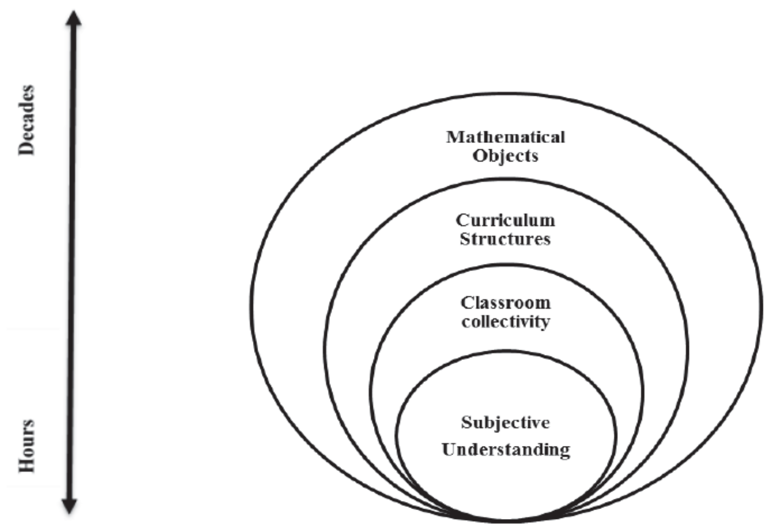

Figure 2.3 Nested complex phenomena of M4T Model.

To enable readers to better understand how a researcher can collect data about a teacher's explicit and tacit mathematical knowledge for teaching, what follows is an elaboration of each complex system of the M4T model.

2.1.3.4.1 Subjective Understanding

This complex system is concerned with the unconscious potentially tacit and explicit representations of the individual teacher's mathematical knowledge. Individual cognition is relatively volatile, meaning it can change and adapt quickly to new mathematical environments and stimuli. The ellipse labeled subjective understanding represents embodied mathematical cognition as a structural coupling of the individual and the social, physical environments that an individual co-creates with others (Davis & Simmt, 2006; Davis & Renert, 2014; Simmt, 2011). The subjective understanding complex system is dynamic on multiple levels, as it represents teachers' harmonization of their own emerging mathematical knowledge with their interpretations of the evolving cognition of student mathematical knowledge. The subjective understanding system evolves as a leveled, non-linear, and recursive phenomenon (Davis & Simmt, 2006; Pirie & Kieren, 1989). The experiences, images, and interpretations are teachers' subjective understanding, as represented by their actions in relevant mathematical environments.

The M_4T model proposes, through its enactivist grounding, that subjective understanding of mathematical concepts are both biological and cultural. This claim is significant, as Davis and Simmt (2006) have included in their model that "mathematical knowing is rooted in our biological structure, framed by bodily experiences, elaborated within social interactions, enabled by cultural tools, and part of an ever-unfolding conversation of humans and the biosphere" (p. 315). Subjective understanding cannot be understood without being coupled with its environment, and for the first time this understanding appears in a mathematics-for-teaching knowledge model. The next level of the model is the first environmental system in which the subjective learner is embedded. This system includes the teacher's knowledge of and participation in the immediate environment where the teacher's mathematical cognition takes place.

2.1.3.4.2 Classroom Collectivity

This embedded system concerns the teacher's knowledge of how to participate in collective mathematical action and knowledge of how best to enable students to productively engage in mathematical action (Davis & Simmt, 2006). The ellipse labeled "classroom collectivity" represents Davis and colleagues' model of the embodied mathematical cognition as a structural coupling of the individual and the social and physical environment within which the individual teacher co-exists (Davis & Simmt, 2006; Davis & Renert, 2014; Simmt, 2011). It also represents pervasive high levels of volatility of this type of mathematical knowledge. Classroom collectivity is described as an inter-subjective complex system because it is meant to model the structural coupling between differing individual cognitions as embedded systems of a collective mathematical environment.

The M_4T model embeds the social context of mathematical cognition with the subjective understanding and the broader systems later described. Simmt (2011) described facets included in the classroom collectivity system drawing from other research (Bowers & Nickerson, 2001; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). These aspects are "(a) the social norms and individual students' beliefs about them, (b) the socio-mathematical norms and individual students' beliefs about them, and (c) the mathematical practices and individual students' mathematical understandings" (Bowers & Nickerson, 2001, p. 4). Cobb (1999) offered examples of social norms from research (Cobb, Yackel, & Wood, 1989) such as norms for "explaining and justifying solutions, attempting to make sense of explanations given by others, indicating agreement or disagreement, and questioning alternatives in situations in which a conflict in interpretations had become apparent" (p. 7). Cobb (1999) further identified socio-mathematical norms that included student activities specific to the mathematics classroom, such as what counts as "differing mathematical solutions, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation" (p. 8). The classroom collectivity and the embedded subjective understanding systems then represent the entirety of the knowledge-producing systems. I will now describe the knowledge-produced systems that are the larger complex systems containing all teachers' subjective understanding systems and all of the embedded social contexts that are relevant to those teachers' mathematical cognition.

2.1.3.4.3 Curriculum Structures

This complex system is concerned with the teacher's knowledge of the shared cultural interpretations of the structure of mathematics for schooling (Davis & Simmt, 2006). The ellipse labeled curriculum structures represents Davis and colleagues' model of the embodied mathematical cognition as a structural coupling of the individual, the social context, and the larger cross-cultural curricular systems within a mathematical environment (Davis & Simmt, 2006; Davis & Renert, 2014; Simmt, 2011). The size of the ellipse represents the relatively static nature of mathematical knowledge at the curriculum level compared to the more volatile collectivity and subjective system levels.

Curriculum structures are made up of "curriculum," including teacher's knowledge of the curriculum resource materials as well as mandated programs of study (Davis & Simmt, 2006; Simmt, 2011). Simmt (2011) defines this as *curriculum-as-planned* and ties it in closely with the definition of intended mathematical curriculum

(Reys, 2006). A second facet of the curriculum structures system is *curriculum-as-lived*. This aspect is defined as the teacher's knowledge of the "thought, action and relationships among the teacher and learners and objects" in the classroom (Simmt, 2011, p. 3). This closely aligns with what Clandinin and Connelly (1992) defined as the enacted mathematical curriculum. The knowledge-produced curriculum structures system and its embedded knowledge-producing systems are finally embedded in the largest complex system of the M_4T model--the mathematical objects system.

2.1.3.4.4 Mathematical Objects

This complex system is a teacher's knowledge of the broad system of the discipline and how it has evolved through the participation of all humanity (Davis & Simmt, 2006). The ellipse labeled "mathematical objects" represents the embodied mathematical cognition as a structural coupling of the individual, social context, curricular structures that bind those social contexts, and the larger discipline of mathematics. The larger discipline of mathematics provides the organization, or boundary, inside which all of these systems structurally couple (Davis & Simmt, 2006; Davis & Renert, 2014). The large size of the ellipse represents the relatively static nature of mathematics as a discipline.

The mathematical objects system is a teacher's knowledge of the history and interconnections between fields of mathematical study as well as his/her orientation to the tools and products of the discipline (Davis & Simmt, 2006). Davis and colleagues' notion of mathematical objects is akin to Ball and Bass' (2009) horizonal knowledge. Horizonal knowledge is defined as an individual teacher's knowledge of how students learning in the present is connected to larger mathematical ideas, research, and computational techniques of the evolving discipline of mathematics. The theoretical difference between Ball's horizonal knowledge and Davis and Simmt's mathematical objects system is only in the recognition by Davis and Simmt of the structural coupling with other systems of the M_4T model. In the next section, a further illustration of the definition of each system using the concept of fraction multiplication occurs. This illustration will enable the reader to understand what type of data can be collected about the nature and development of middle school teachers' M_4T knowledge of fraction multiplication.

2.1.4 Individual M_4T Knowledge of Fraction Multiplication

A description of each complex system of M_4T in the context of the middle school mathematical concept fraction multiplication follows. I chose fraction multiplication as the mathematical context for this research report because multiplication is implicit throughout the middle school curriculum. Multiplication on various number sets has been used in previous concept study literature (Davis, 2008, 2011, 2012; Davis & Renert, 2009, 2012, 2014; Davis & Simmt, 2006).

2.1.4.1 Subjective Understanding of Fraction Multiplication

Subjective understanding includes how one's mathematical knowledge is developed, the conceptual blends of topics, and the images and metaphors that define and connect mathematical topics. These ideas from mathematics education literature provide vocabulary for suggesting what I might consider as subjective understanding of fraction multiplication in a concept study environment.

2.1.4.1.1 Representations of Fraction

Lamon (1999) identified four different representations of fractions: (a) fractions as symbols, (b) part-whole fractions, (c) fractions as rational numbers, and (d) fractions as numbers. Each of these is briefly described in the paragraphs that follow.

Fractions as symbols. A fraction can be understood as a way of writing a pair of numbers. Lamon (1999) stated that this pair of numbers is constructed in the form of $\frac{a}{b}$ where "a" is called the numerator and "b" is called the denominator. So the word "fraction" can be used as a symbol for "writing a number, a notational system, a symbol, two numbers written with a bar between them" (Lamon, 1999, p. 27). For example, $\frac{6}{8}$ is a symbol with multiple interpretations.

Fractions as part-whole. A fraction is also a ratio of parts to whole (Greer, 1992; Lamon, 1999, 2007). This definition refers to the unitizing function of fractions, where the fraction "represents one or more parts of a unit that has been divided into some number of equal-sized parts" (Lamon, 1999, p. 27). This first interpretation of fraction that children learn is taken as the basis for their knowledge of fraction in the curriculum (Lamon, 1999). For example, $\frac{6}{8}$ can be understood as the ratio of six parts out of the total of eight equal parts into which an arbitrary unit is divided.

Fractions as rational numbers. Fractions as rational numbers allows individuals to "talk about wholes as well as pieces of a whole" (Lamon, 1999, p. 28). For example, the subset of rational numbers include $\frac{24}{4}$ which is actually six whole groups of four parts of 24. Rational numbers also include $\frac{1}{4}$ which can only be understood as a one piece of a whole. It is important to understand that fractions and rational numbers are not coterminous. All fractions are not rational numbers, and all fractions do not correspond to different rational numbers.

Fractions as numbers. When considering a fraction as a number, the user is actually referring to the underlying rational number, the number that the symbol of fraction represents (Lamon, 1999). Fractions as a number is a relative term, as it represents the single relative amount of all different fractional amounts of the same rational number. For example, $\frac{6}{8}$ can be understood as a number, which is actually the same as all other variations of $\frac{6}{8}$ such as $\frac{3}{4}$ or $\frac{18}{24}$.

2.1.4.1.2 Representations of Multiplication

From a procedural standpoint, multiplication of fractions is simple. Yet, it is quite complex when considered as a psychological process (Greer, 1992). Greer characterized multiplication as images for integers that included *equal groups, multiplicative comparison, Cartesian product,* and *rectangular area,* each of which are further explained below.

Equal Groups. When the number and size of groups is known in the equal groups' situation, the task is considered a multiplicative situation in which the whole is unknown. The multiplicative situation of equal groups has two different interpretations in research literature: repeated addition and related rates (Greer, 1992; Lamon, 1999). For example, the operation of 3×4 can be interpreted as the repeated addition of groups containing three arbitrary units, four distinct times. This interpretation can be transferred easily to the multiplication of rational numbers and whole numbers, for example a situation like $\frac{1}{3} \times 4$. This would signify the repeated addition of a group size of $\frac{1}{3}$, four distinct times.

Multiplicative Comparison. Multiplicative comparison problems consist of two different sets. One set consists of multiple copies of the other set. This situation can be modeled as *n* times as many as the other set. For example, “Mark saved three times as much as last month” or similarly with fractions, “Mark saved half as much as last month.” In these examples either number can be considered the multiplier. Explicitly applied to fraction multiplication, the multiplicative comparison can also be inverted. Therefore, “Mark saved three times as much as last month” could be interpreted as last month’s saved amount was $\frac{1}{3}$ as much as this month (Greer, 1992; Lamon, 1999).

Cartesian product. In Cartesian product, or combination problems, the formal definition of $m \times n$ defines the distinct number of ordered pairs that can be found when pairing two sets (Greer, 1992). The product consists of pairs of things, one member of the pair taken from each of the given sets. For example, if choosing to match four shirts with three pairs of shorts, the Cartesian product is a pairing of these sets in order to come up with the different number of outfits possible, or twelve total. Greer (1992) stated that this sophisticated way of defining multiplication maintains a symmetry between both numbers wherein either can act as the multiplier or the multiplicand. Cartesian products are possible with fractions, as many types of unit conversions have whole numbers of smaller unit sizes. For example, $\frac{1}{3}$ of an hour can be interpreted as 20 minutes, so a Cartesian product would be possible with these fractional quantities.

Rectangular Area. The area representation of a multiplicative situation can also be described as the product of measures problems (Greer, 1992). Rectangular area is similar to the Cartesian multiplicative situations; there is no distinction between which number is the multiplier and which number is the multiplicand. One important facet that distinguishes this multiplicative situation from others, however, is that the product is literally a different type of unit from the two factors. For example, the area of a rectangle with dimensions 3cm by 4cm would be the product of the lengths, or $3cm \times 4cm = 12\ cm^2$. Explicitly applied to the multiplication of fractions, the measures of lengths of sides of rectangles can be fractional lengths of a whole unit. For example, the area of a rectangle that is $\frac{a}{b}$ high and $\frac{c}{d}$ wide where $b,\ d \neq 0$ would be the products of $\frac{a}{b}$ and $\frac{c}{d}$.

Extending the image of multiplication to rational numbers requires a conceptual leap (Lamon, 1999) to include notions such as *rates, part-whole relationships, fractional areas,* and *products of measures* (Greer, 1992). For the purposes of this report, the operation of multiplication on the set of whole numbers does not have a one-to-one relationship with the operation on the set of rational numbers (Fischbein, Deri, Nello, & Marino, 1985). For example, the notion of repeated addition of equal groups works well for understanding the operation of multiplication on whole numbers like 3×4 but does not transfer well to understanding the operation on rational numbers like $\frac{3}{4} \times \frac{1}{2}$.

2.1.4.1.3 Research on Teacher Knowledge of Fraction Multiplication

Following are examples of *explicit* teachers' knowledge of fraction multiplication that have been catalogued. This literature provides language for characterizing teachers' M_4T knowledge of fraction multiplication.

Mistaking realizations for operations. Several studies have shown that teachers have difficulty differentiating between situations where fraction multiplication should be used and those that require division (Armstrong & Bezuk, 1995; Ball, 1990; Ma, 1999).

For example, Ball (1990) investigated knowledge of division of 35 preservice teachers using task-based interviews. Preservice teachers were asked to represent a situation that would illustrate $1\frac{3}{4} \div \frac{1}{2}$. Only four teachers were able to generate an appropriate representation, 12 generated an inappropriate representation, and 19 were unable to generate any representation. The inappropriate representations all depicted multiplication rather than division. Ball suggested that the teachers' primitive notions of whole number division as making numbers smaller could be the potential source of misconception.

Multiple levels of units. Multi-level unit structures play a role in reasoning about fractions as rational numbers and about operations with fractions. Research on unit structures drawn from conceptual analyses (Behr, Harel, Post, & Lesh, 1992) and teaching experiments with children (Hackenberg, 2010; Olive, 1999; Steffe, 2003; Steffe, 2002; Steffe & Kieren, 1994; Tzur, 2004) have shown that teachers have diverse levels of fluency with unit structures. Steffe first defined reasoning with two levels of units as a child's ability to understand simultaneously a whole number as several separate units and as a single unit or entity. For example, the number five can be thought of as a single unit that is comprised of five units of one. A child's reasoning with three-levels of units is defined as a child who can view a whole number as an entity or unit and as a whole number that can be broken into units of other whole numbers, which can then be broken into units of ones. Izsak (2008) found that teachers' actions were different when posed with tasks asking them to reason with multiple levels of units. One teacher used two levels of units when she taught and responded to students' questions. Her actions, though viable in her environment, limited her ability to teach the mathematical concepts. Another teacher's actions provided evidence of reasoning with three levels of units by using lengths and areas when teaching and responding to students in her classroom environment. Despite this facility, the second teacher's actions were not flexible enough to sufficiently respond to students' reasoning about drawings from the *Connected Mathematics Project* (CMP) material (Lappan, Fey, Fitzgeral, Friel, & Phillips, 2002, 2006). Izsak et al. (2012) confirmed Izsak's (2008) findings that teachers reason with units at different levels of proficiency when teaching fraction multiplication. Izsak et al. also found that a central role of teacher's performance seemed to be the ability to act in environments calling for the differentiation of and flexibility to move between two levels of units and three levels of units.

What follows now is a transition from focusing on teacher's individual subjective knowledge of fraction multiplication, signifying a shift to focusing on a teacher's knowledge of student cognition. A teacher's knowledge of student cognition is considered subjective understanding because it represents the necessary harmonization of teachers' own knowledge with that of their students (Davis & Simmt, 2006; Davis & Renert, 2014).

2.1.4.1.4 Teacher Knowledge of Student Cognition of Fraction Multiplication

To find a teacher's knowledge of student cognition, one must understand ways of categorizing a teacher's labels for mathematical knowledge and learning. Below, three broad contemporary theories of cognition (Greeno, Collins, & Resnick, 1992) with examples provide vocabulary for labeling and tracking development of teacher's knowledge of student cognition of fraction multiplication.

Behaviorism. In the behaviorist view of learning, knowing is a process in which associations, skills, and components of skills are acquired (Even & Tirosh, 2010). Transfer occurs "to the extent that behaviors learned in one situation are utilized in another situation" (Greeno, Collins, & Resnick, 1992, p. 16). Even and Tirosh (2010) outlined the basic tenets of behaviorism when applied directly to teacher's knowledge of student learning. Teachers with a behaviorist perspective of cognition work from the assumption that mathematical mistakes are obstacles to mathematical learning and that student access to peer and teacher mathematical errors should be prevented (Even & Tirosh, 2010). A student's motivation to learn mathematics is a characteristic of the learner, and incentives provide a context that the learner seeks out while constructing new associations and skills (Greeno, Collins, & Resnick, 1992). Behaviorism as a theory for mathematical cognition assumes that the teacher cannot build models for what students are thinking–-and therefore correctness of solutions and responses are valued over student cognition (Even & Tirosh, 2010).

Basic Constructivism. To summarize the diverse uses of constructivism is difficult, though there are broad tenets that can be encapsulated here briefly. Following the tradition of Piaget, constructivism is focused on the characterization of the cognitive development of children (Greeno, Collins, & Resnick, 1992). According to the constructivist tradition, children's knowledge differs significantly from adults' knowledge (Even & Tirosh, 2010). Knowledge, be it in the individual or social context, is ultimately a construction of the individual (Davis & Simmt, 2003). Tenets of constructivism make it clear that the mathematics teacher is able to build a model of the students thinking about mathematical concepts. To that end, teachers who view mathematical cognition from a constructivist perspective work to build a model of student mathematical thinking and then construct learning trajectories to enable the construction of viable mathematical knowledge (Even & Tirosh, 2010).

Embodied Cognition/Enactivism. The situative perspective describes knowledge as being distributed among people and their environments (Greeno, Collins, & Resnick, 1992). More precisely, enactivism describes knowledge as viable action in an environment, in which the individual and the environment are structurally coupled (Proulx, 2004). In mathematical environments knowledge is understood as viable action instead of as a discrete body of knowledge to be collected and categorized (Even & Tirosh, 2010). A teacher who views learning from this perspective promotes learning by enabling students to better participate in shared mathematical activities. Teacher actions are guided by the principle that the teacher is "responsible for prompting differential attention" between ideas and students, and then for "selecting among the options for action and interpretation that arise" in the mathematical environment that enable the advancement of viable mathematical action in the classroom environment (Davis, 2005, p. 87).

2.1.4.2 Classroom Collectivity of Fraction Multiplication

As a *knowledge-producing* system, a teacher's knowledge of the classroom culture—similar to a teacher's knowledge of student mathematical cognitions—is based in part on an understanding of the theories of cognition (Even & Tirosh, 2010). Three broad contemporary theories of cognition (Greeno, Collins, & Resnick, 1992) and the context of teacher knowledge of classroom collectivity in the context of fraction multiplication are found below.

2.1.4.2.1 Behaviorism

A teacher's knowledge of the classroom culture informed by a behaviorist conception of cognition would characterize the classroom as a collection of unique individual learners (Even & Tirosh, 2010). The classroom culture then exists to facilitate the transmission of knowledge and the subsequent accumulation of facts, skills, and procedures relevant to fraction multiplication. This transmission typically occurs through the presentation of procedures to reinforce good facility with the mathematical procedures of fraction multiplication and through non-reinforcement of poor facility with the procedures. Presentation of procedures is often followed by the offering of ample time to practice mathematical skills similar to Mehan's (1979) initiation, response, and evaluation (IRE) pattern or Bower and Nickerson's (2001) elicitation, student response, and teacher elaboration (ERE) pattern.

2.1.4.2.2 Basic Constructivism

Teachers informed by the constructivist theories of cognition value differing types of knowledge such as conceptual, problem solving strategies, procedural, and metacognitive skills (Even & Tirosh, 2010). Consequently, the classroom culture exists to facilitate learning as an active process, not a passive acquisition of knowledge. The teacher views knowledge as something that is defined in the head of a person and believes that students have no alternative but to construct knowledge based on their own experiences (von Glaserfeld, 1995). The constructivist classroom is designed to provide contexts where students can experientially test the viability of their mathematical knowledge.

2.1.4.2.3 Embodied Cognition as Enactivism

Teachers informed by an enactivism theory of cognition attend to multiple levels of collectivity. The classroom community is of the utmost importance to the teacher, and the classroom is considered as a collective learner rather than a collection of learners (Davis, 2005; Davis & Simmt, 2003). The teacher selects from viable options for action and interpretation that arise from emergent collective engagement with complex mathematical tasks about fraction multiplication. He/she also ensures that "diverse interpretive possibilities are present in the classroom" (Davis, 2005, p. 87). Implicit in this description is that a teacher will actively support the five characteristics previously described as the necessary preconditions for complexity.

2.1.4.3 Curriculum Structures of Fraction Multiplication

Curriculum structures is a combination of the teacher's knowledge of the established curricular resources available to sequence and teach fraction multiplication and the teacher's knowledge and use of how teacher, student, and curriculum interact in real-time in the classroom.

2.1.4.3.1 Curriculum-as-Planned (CaP)

The learning trajectories for the teaching of fraction multiplication can build from two different hypotheses in regards to the role of children's intuitive knowledge of rational numbers in the development of rational number knowledge. These two hypotheses are the interference (Behr et al., 1992) and reorganization hypotheses (Steffe

& Olive, 2010). They provide vocabulary needed for characterizing teachers' curriculum structures knowledge of fraction multiplication and their knowledge for sequencing learning.

Interference hypothesis. Teachers informed by the interference hypothesis would be aware that children are first engaged with whole numbers and the operations of addition and subtraction, followed by a basic understanding of the operations of multiplication and division (Lamon, 1999). Teachers reason that the conceptual generalizations children have drawn interfere with the learning of computation with fractions. The assumption is that once operations on fractions are introduced, whole number multiplication may be used to operate on fractional quantities. To complicate matters further, the interference hypothesis also assumes that children come to their formal mathematical schooling with a complex array of pre-established conceptual metaphors for operating on rational numbers (Lamon, 1999, 2007; Johanning, 2008). For example, students learn cultural conceptual metaphors for fraction, such as parts, prior to coming to the classroom. Once in the classroom, grounding conceptual metaphors could interfere with other images for fraction and operations on fraction.

The Rational Number Project (RNP) explored how children's intuitive understandings interfere with the learning of fractions and operations on fractions (Behr et al., 1992; Behr, Harel, Post, & Lesh, 1993; Behr, Lesh, Post, & Silver, 1983; Mack, 1995; Post, Cramer, Behr, Lesh, & Harel, 1993; Lamon, 1999). Results from the RNP show that children learning operations on fraction quantities using metaphors conceptually developed with whole numbers (Mack, 1995; Streefland, 1991; Lamon, 1999) over relied on the part-whole model (Kerslake, 1986) and utilized the iterative act of halving (Pothier & Sawada, 1983) to inform their thinking about the multiplication of fractions. The conclusions and curricular design emergent from the RNP research intentionally separates the teaching of whole and rational numbers in the context of the operation of multiplication to reduce interference (Cramer, Post, & del Mas, 2002). A teacher's knowledge of this sequential design and the implicit reasons for the design based upon student conceptions is part of the teacher's curricular structures knowledge.

Reorganization hypothesis. Kieren (1993) hypothesized that fractions and natural numbers share common images and metaphors but that fraction knowledge is much more than a simple extension of natural numbers realizations. Lamon (1999) also argued that realizations of whole number operations, without significant refinement, are rendered defective when applied to fraction mathematical situations. Steffe and colleagues have posited that any interference of whole numbers with the cognition of rational numbers is a direct result of rational numbers isolation in curricular presentation (Steffe & Olive, 2010). This hypothesis, one of reorganization, concludes that students must revise and reorganize operating with conceptual metaphors of whole numbers in order to build fractional knowledge (Olive, 1999; Olive & Steffe, 2002; Steffe, 2002; Tzur, 2004). Curriculum built to align with the reorganization hypothesis structures the learning of fraction and whole number multiplication to help students explicitly examine the differences between their operations rather than isolating the concepts from each other. A teacher's knowledge of this embedded design, and the implicit reasons for the design based upon student conceptions, is part of the curricular structures knowledge.

Textbook as the curriculum. Tarr et al. (2008) proposed that one method for improving school mathematics programs includes the selection and enactment of curriculum in quality textbooks. This proposal from research identifies the textbook as the de facto curriculum for middle school mathematics teachers (Schmidt, McKnight, & Raizen, 1997; Weiss, Banilower, McMahon, & Smith, 2001; Weiss, Pasley, Smith, Banilower, & Heck, 2003). Moreover, Grouws and Smith (2000) found that nearly three-fourths of eighth grade mathematics teachers responding to the NAEP surveys reported using the textbook on a daily basis. The use of the textbook and teachers' knowledge of textbook design are included as part of the curricular structures knowledge, despite the inability to make explicit links to textbooks and student opportunities to learn since teachers have the freedom to make choices about the use and interpretation of the textbook (Tarr et al., 2008).

2.1.4.3.2 Curriculum-as-Lived (CaL)

Curriculum-as-lived for fraction multiplication is similar to the teachers' knowledge of the classroom collective but is the teachers' knowledge of the common patterns of interaction between themselves, students, and the multiplication of fractions curriculum-as-planned (intended curriculum). An explicit manifestation of curriculum-as-lived is the mathematics teacher's knowledge of the educative appropriateness of various images, analogies, and metaphors in the curriculum-as-planned, for example, a teacher's knowledge of the appropriateness of using the array model for fraction multiplication compared to the area model with fractional lengths. Curriculum-as-lived is part of teachers' professional understanding of students and of the learning trajectory of concepts that enable them to make moment-to-moment decisions in their representations of the teaching of fraction multiplication. For example, teachers decide which of the various models for fraction multiplication are viable at particular times during mathematical investigations.

2.1.4.4 Mathematical Objects of Fraction Multiplication

The mathematical objects of fraction multiplication, a *knowledge-produced system,* includes teachers' knowledge of the historical development of fractions and operations on fractions. Much more than an understanding of an anecdotal history, this knowledge-produced system also represents the teacher's use of the diverse fields of mathematical study to interconnect the common images, metaphors, and analogies of a concept such as fraction multiplication. This system also includes how teachers orient themselves to these questions: What is mathematics? How was it developed? Is mathematics an evolving or static discipline? Ernest (2006) illustrated the significance of such philosophical orientations on teaching. Following a brief tracing of the historical development of fraction and operations on fractions that offers language to describe teacher's knowledge of these historical developments is a discussion of the stages of mathematical development. The stages utilized for this discussion were proposed by Davis and Renert (2010) and provide vocabulary for interpreting teachers' philosophical orientation to the discipline of mathematics.

2.1.4.4.1 Historical Development

Fractions were included in the first written accounts of symbolic systems (Struik, 1987). More complex mathematics, beyond primitive record keeping and counting, dates from written records to somewhere between 5000 and 3000 BC (Eves, 1997). Mathematics developed significantly during this era, spurred on by utility for practical tasks like calendar computation, quantifying and distributing harvest, and other public programs such as tax collection (Struik, 1987). During this era the Egyptian, Sumerian, and Babylonian societies greatly advanced rational number understanding and operations on those rational numbers.

The Egyptian system for whole numbers was not a place value system, while Sumerian and Babylonian systems were (Flegg, 1989). Egyptian reliance on unit fractions complicated representations and computations with fractions. For example, the number $\frac{3}{4}$ would have been written as $\frac{1}{2}+\frac{1}{4}$ and then computations on this number would have been accomplished using the addition of two unit fractions. The Egyptian repetitive representations of whole numbers provided an advantage over the positional system of the Babylonians and Sumerians because the Egyptian representation did not require zero (Cajori, 1894; Flegg, 1989; Kaplan, 2000; Katz, 1993). The Sumerian and Babylonian representations of fractions and whole numbers using the same sexagesimal system was perhaps the most significant difference between these systems of antiquity. The Babylonian and Sumerian place value system theoretically made it possible for unlimited accuracy in calculation (Flegg, 1989). Use of the Egyptian system for representation and computation on fractional quantities lasted into the 1400-1500s AD (Flegg, 1989).

2.1.4.4.2 Orientation to Mathematics

The researcher's five suggested stages for the evolution of mathematics as a discipline throughout human history are described by Davis and Renert (2010). These stages are titled the oral stage, pre-formalist stage, formalist stage, hyper-formalist stage, and post-formalist stage. Davis and Renert (2010) hypothesize that much of "mathematics education today resides in the formalist and pre-formalist conceptions of mathematics" (p. 184), meaning that mathematics is a source of procedures and activities that are part of the everyday occurrences of an individual. Moreover, mathematics is considered to be outside of the observer, discovered by humans through empirical study of their surroundings. This theory contrasts the orientation to mathematics that guides this research project: the post-formalist stage, in which mathematics is considered a "socially-constructed interpretive discourse, rooted in our need to make sense of our environments and to construct our reality. Far from being separate from knowers, mathematical knowledge at this stage is embodied and enacted by both the individual and collective knowers" (Davis & Renert, 2010, p. 183). Davis and Renert (2010) argue that the transitions between the stages of mathematics represent a coherent evolution of increasingly sophisticated knowledge of the discipline of mathematics.

2.1.4.4.3 Advanced and Horizonal Knowledge

Davis's (2012) findings provided evidence that advanced mathematical knowledge of concepts beyond that of the level of mathematics a teacher is teaching, as part of their M_4T knowledge. This advanced knowledge is characterized as a teacher's knowledge of concepts, areas of study, and research in the broader discipline of mathematics beyond the middle school curriculum. *Horizonal*, used as an adjective for this type of M_4T knowledge, is derived from Ball and Bass' (2009) work, described as "a sense of how the mathematics at play now is related to larger mathematical ideas, structures, and principles" (p. 7). This horizonal knowledge is characterized as mathematical objects knowledge because it is knowledge of the relatively static produced system of mathematics activated while a teacher is working with newly produced mathematics in a mathematical environment.

This concludes what comprises the individual M_4T knowledge of fraction multiplication. What follows is a definition of the collective level of M_4T knowledge of fraction multiplication.

2.1.5 Defining the Collective Learner

Donald (2002) described the development of collective unities, cognitive units formed through interactions among humans. The potentialities of these entities stem from the human capacity to coordinate attentional systems and brain functioning that can provide the support for grander-cognitive activities. Davis (2005b) built on this notion, claiming that collectively humans far outreach the potentialities of the individual in both outputs, links, and memory capacities. The human ability to form a communal cognition is biologically rooted, yet is greatly enabled by advances in language and social conventions (Donald, 2002; Davis, 2005a, 2005b). For example, as Donald relates, the human cognitive ability cannot be fully explained by biology alone as similar biological development in apes has resulted in no equivalent change. Rather, the human cognition has been also determined by culture, "the creative collision between the conscious mind and distributed cultural systems" (Donald, 2002, p. 153). Davis (2005b) takes this one step further, relating the potentiality for communal cognition to classroom mathematical teaching. The existence of multiple individual cognitive unities, combined with a potential for communal cognition, warrants a different outlook on the role of a mathematics teacher and the composition of a mathematics classroom. A mathematics classroom, understood this way, is a "*learner* – not a *collection of learners,* but a *collective learner"* (Davis, 2005b, p. 87). Ricks (2007) described a mathematics classroom, or mathematical environment, as embodying the necessary characteristics to be called a complex system.

Davis left largely unexplored the differentiation between the embedded systems of a collective learner and a collection of learners in a mathematical classroom. Martin, Towers, and Pirie (2006) provide vocabulary for this necessary differentiation and for the labeling of these proposed embedded systems within the complex system of a mathematical learning environment. The individual cognitive systems, or learners, and the potential for emergence of a larger cognitive unity can be described by the differentiation between the terms *interaction* and *coaction* provided by Martin and colleagues. Martin, Towers, and Pirie (2006) defined interaction as the collaboration between individuals in a mathematical context where individual contributions are shared, but do not build into the emergence of a collective cognitive unity. Interaction lacks the entanglement of ideas for a higher order cognitive system to arise. Martin, Towers, and Pirie describe mathematical learning environments where individuals interacted by communicating mathematically what they already know, individuals' mathematical understandings were overlapping and occurring simultaneously, and individuals' mathematical ideas were compatible but never taken-as-shared. No higher order unities emerge in these collaborative environments. In contrast, coaction is defined as moments in a mathematical environment when individuals carry out individual mathematical actions but the total product of that environment is not attributable to any one individual (Corning, 2002; Davis, 2005). For the purposes of this report, the existence of Davis'

(2005b) collective learner should be understood as these moments of mathematical coactions.

This report draws on the work of Martin and colleagues (Martin, Towers, & Pirie, 2006) and Davis and colleagues (Davis & Simmt, 2003, 2006) to define the collective learner as coaction and the emergence complexity. Martin, Towers, and Pirie (2006) provide a definition for coaction, establishing a synergistic movement by a collaborative group whose output is greater than what is possible by any one individual. With complexity science, this can be understood as the emergence of complex learning systems (Davis & Simmt, 2003) blending the two research vocabularies. Therefore, the collective learner in the context of this study will hereafter be defined as moments of coaction and the emergence of complexity. Next, one must determine how a researcher may characterize M_4T of the collective learner and what the development of such knowledge would consist of.

The coaction among individual teachers resulting in the occasional emergence of a collective learner requires recognition that the collective learner could have unique evolving identity and coherence maintaining mechanisms. The coaction and resultant emergence of differing compositions of a collective learner could vary, as the environments of the mathematical context and the participation of the individuals vary. The emergence of a collective learner is a dynamic ongoing process of negotiating and interpreting collective engagement and does not disregard the histories and understandings of the individuals that comprise the collective learner (Davis & Simmt, 2003; Martin, Towers, & Pirie, 2006). Rather, the collective is considered to be these individual's understandings as embedded systems entangled in such a way as to enable the growth of mathematical understanding of the transcendent whole (Davis & Simmt, 2006). Therefore, the M_4T development of the collective learner must be understood in two distinct ways. First, it is defined as the synergistic moments of coaction, where the mathematical output of the collective learner is not attributable to any one individual that comprises it. The description of mechanisms of organization for emergence in the context of a mathematical environment investigating fraction multiplication is the first facet of collective learner's M_4T of fraction multiplication development. The second facet of collectivity is the actual development of *knowledge* of this agent, or as defined by Davis and Simmt (2003) "the interactions and prompts that trigger new possibilities and insights for the collective" (p. 144). What emerges during these moments of coaction is considered the M_4T knowledge of fraction multiplication, and how it develops is the self-regulation or decision-making actions of the collective learner to maintain the coaction. The noticeable emergent activities during moments of coaction that reciprocate with individual action (Cobb, 1999; Kieren, 2000) will be understood as the links between the individual and the collective M_4T development.

After this description of the individual and collective M_4T knowledge of fraction multiplication, it is now possible to describe the mathematical environment to which these embedded systems will be coupled. Following is a fuller description of a concept study mathematical environment, framed by the six emphases that guide the planning and creation of a concept study as a context for researching the development of middle school teachers' M_4T knowledge of fraction multiplication.

2.1.6 Concept Studies: A Context for Collective and Individual Development

Concept studies (Davis & Simmt, 2006) combine "elements from two prominent notions in contemporary mathematics education research previously elaborated: *concept* analysis and *lesson* study" (Davis & Renert, 2009, p. 37). A concept study takes the form of six emphases that emerged from initial pilots (Davis, 2011, 2012; Davis & Renert, 2009, 2013, 2014). The six emphases should be construed as always-present potentialities of the concept study that unfold recursively, guided by the participant co-production of the mathematical environments of the concept study (Davis, 2012, 2013; Davis & Renert, 2014). I use these emphases—which are enacted as implicit emphases throughout a concept study rather than consciously implemented—to frame the developmental work of a concept study.

A teacher's work requires the unpacking of mathematical concepts (Ball, Hill, & Bass, 2005) so that students can gain access to the culturally created thought processes and ideas that the concepts represent. Davis and colleagues take this notion further, as unpacking pries apart a mathematical concept for teaching, while substructing (Davis & Renert, 2014) is how the various parts of the concept unite or conflict to formulate a profound understanding of the concept. These two mathematical actions—that of unpacking and then substructing—are the intended purposes of the implicitly and explicitly applied emphases of this concept study. Each of these emphases will be defined to provide vocabulary for readers to understand a concept study as a context for examining individual and collective development of M_4T knowledge of fraction multiplication. The first three emphases are designed to create distinctions between realizations and the consequences of those realizations for mathematical concepts such as fraction multiplication. Next, the blends emphasis is designed to achieve “meta-level coherences by exploring the deep connections among realizations” (Davis & Renert, 2014, p. 70). Finally, the last two emphases are designed to allow teachers to obtain practical application of their emergent realizations for a mathematical concept like fraction multiplication.

2.1.6.1 Emphasis 1: Realizations

The term realizations is borrowed from Sfard (2008) and is defined as the associations that a learner might draw on and connect in efforts to make sense of a mathematical concept (Davis & Renert, 2014). Examples of possible realizations for a mathematical concept like fraction multiplication include formal definitions, algorithmic knowledge, metaphors, images, computational applications, and physical gestures. As evidenced in other concept study research (Davis, 2011, 2012; Davis & Renert, 2009, 2012, 2014), teachers investigating their own realizations for mathematical concepts provide an environment where development of their knowledge of the concept is possible. Individual’s realizations for mathematical concepts evolve and can be shared by many or can be unique to an individual of a collaborative group. It is important to note “the assertion and assumption here is not that any particular realization is right, wrong, adequate, or insufficient” but rather that the process of making realizations will allow for further development of what teachers know about the concept of fraction multiplication (Davis & Renert, 2013, p. 253).

2.1.6.2 Emphasis 2: Landscapes

Davis and colleagues noted a difference in the utility of various realizations for a concept like fraction multiplication. Some realizations for fraction multiplication remain viable in most mathematical contexts in which teachers encounter the concept, while other realizations are relatively situation-specific or learner-specific. For example, the realization of multiplication as “repeated addition” varies in viability depending on the number sets to which it is applied. Landscapes are the activity of organizing and comparing lists of realizations for a particular mathematical concept like fraction multiplication. This type of activity produces a landscape, defined as a macro-level view (Davis, 2011, 2012; Davis & Renert, 2014) of a mathematical concept. The substructing of participant realizations for fraction multiplication provides a context where the collaborating individuals examine how realizations “hold together and fall apart in different contexts and circumstances” (Davis & Renert, 2014, p. 43). For example, a landscape for fraction multiplication would be a macro-level view of collectively produced and then organized realizations for fraction multiplication. This would then be followed by comparing and contrasting the viability of realizations for fraction multiplication such as parts of parts, fractional areas, and number line shrinking.

2.1.6.3 Emphasis 3: Entailments

Entailments is defined as the tracking and scrutinizing of the consequences of any one realization of a mathematical concept. Davis and Renert (2014) describe each realization of a concept as carrying with it a series of nested consequences, due to the nested and axiomatic system that builds contemporary mathematical curriculum. Davis and Renert describe the entailments emphasis as a process where teachers gain access to fresh and innovative approaches to a concept, enabling them to move beyond just well rehearsed realizations for the concept. For example, the emphasis of entailments allows teachers to interrogate the realization for fraction multiplication as "parts of parts" and how it influences the teacher's understanding of the commutative property of multiplication. The emphasis of entailments can be done through collaboratively scrutinizing the realization for fraction multiplication to determine any consequences for understanding the commutativity of the operation of multiplication on various number sets.

2.1.6.4 Emphasis 4: Blends

For a concept study, the activity of creating conceptual blends and collapsing of diverse realizations of a mathematical concept is defined as the emphasis of blends (Davis & Renert, 2014). The first three emphases aim to create distinctions between realizations and the consequences of those realizations for a mathematical concept like fraction multiplication. The blending emphasis is categorically different: teachers are asked to seek out "meta-level coherences by exploring the deep connections among identified realizations and/or assembling those realizations into a more encompassing interpretation—which, of course, might introduce emergent possibilities" (Davis, 2012, p. 12). For example, the realization for fraction multiplication as "parts of parts" could be blended with the realization of fraction multiplication as the "shrinking of a number line." The blends of these realizations focuses the collaborative effort on defining the deep connections between the two realizations, resulting in potentially new emergent possibilities for the concept. Davis and Renert (2014) describe blends as a deliberate shift in emphasis from “multiple (and potentially disjointed) meanings toward coherent and encompassing definitions” (p. 71).

2.1.6.5 Emphasis 5: Participation

The emphasis of participation is defined as the planned effort to engage teachers with the post-formalist orientation to the discipline of mathematics. In previous concept study research (Davis, 2012; Davis & Renert, 2013), participants benefited from gaining researcher-initiated positive experiences with mathematics as a culturally created body of knowledge. Teachers are “vital participants in the creation of mathematics, principally through the selection of and preferential emphasis given to particular interpretations over others” (Davis & Renert, 2013, p. 251). The emphasis of participation explicitly interrogates this power structure through the concept study emphases, positioning teachers to develop awareness of the complex nature of realizations for even elementary mathematical concepts. Teachers need positive experiences with mathematics and with understanding that no “realization is right, wrong, adequate, or insufficient” (Davis & Renert, 2013, p. 253). The emphasis of participation, implicit in the previous four emphases, explicitly focuses teachers’ attention on their roles in actively creating a concept like fraction multiplication as a shared activity.

2.1.6.6 Emphasis 6: Pedagogical Problem Solving

The final emphasis is an explicit practical link between the concept study environments and the environments of the participant teachers’ classrooms. Every teacher of mathematics has encountered a student asking—why? Such questions are about the nature of the mathematics that students and teachers are co-creating in the classroom. The question of *why* is a constant part of creating new mathematics with novice students, and, from experience, is one of the better parts of teaching mathematics. Often, these questions—for example, “Is a fraction a number?”—are neither innocuous nor straightforward. The various realizations for fraction as shared by Lamon (2007), among others, show that fractions can be considered as numbers dependent upon the mathematical context. Pedagogical problem solving “aims to capitalize on the interpretive potentials that arise on the collective level when individual expertise is drawn together around perplexing problems” of teaching mathematics to middle school students (Davis, 2012, p. 15). Unlike the other emphases that are implicitly tied to the actual cultural activities of teaching mathematics, this emphasis is explicitly tied to the work of teaching and investigates questions that many teachers have seen emerge in their own teaching experience.

To conclude, these six emphases should be considered as implicitly applied throughout the course of a concept study environment. Explicit applications of the emphases can be planned, but Davis and colleagues have found that “causing emergent insights was impossible” (Davis & Renert, 2014, p. 71). The vocabulary mentioned in these emphases is used to describe the emergence of mathematical action during the individual and collective cases of this research report. The emphases will frame the presentation of the collective case, as many of the collective learner developmental actions took place while the participants were engaging in one or more of the emphases environments.

CHAPTER 3. METHODOLOGY

3.1 Participants

John Creswell (2003) suggested that when using qualitative research techniques it is often necessary to make a purposeful selection of both the site and the participants. Previous research (Brown & Cole, 2012; Martin & Towers, 2009; Davis & Renert, 2014) provide direction for choosing context and participants when interested in researching embodied cognition of individual and collective learners. With this advice in mind, I recruited a purposive sample of five mathematics teachers to co-create the mathematical environments of this concept study. All of the chosen teacher- participants were from the same mathematics department in a Midwestern suburban middle school. As part of the recruitment agreement, one administrator from the school staff was also included as a participant. Access to the research site was enabled by an existing professional relationship between my advisor and this administrator.

3.2 Contexts

For our purposes in this study, mathematical knowledge is defined as an "emergent phenomenon that is enacted in the context of teaching mathematics" and mathematical environments (Simmt, 2011, p. 153). Mathematical environments are any moment when mathematics is being produced by individuals or the collective. Seven of the eight concept study sessions took place in the same conference room in the administration office of the Midwestern middle school. The conference room provided white board space, and a large table around which the participants were able to sit comfortably and within easy access to each other. One concept study session was moved to another conference room to accommodate security issues for the statewide standardized testing that took place during the months of the concept study. This conference room was similar to the regularly scheduled room.

The design of this concept study was informed by the work of previous concept study literature (Davis, 2008a, 2008b; Davis, 2012; Davis et al., 2009; Davis & Renert, 2009, 2014; Davis & Simmt, 2006; Davis & Sumara, 2007, 2008; Simmt, 2011) with an eye toward heightening the possibility that the participant teachers might recursively elaborate their knowledge through the emphases of the concept study. The eight meetings were scheduled to meet the demands of the teachers' schedules. The time between meetings was spaced to allow initial analysis of the collected data and time for planning for the next concept study session. The number of meetings was chosen to provide enough time for participants to "confront, analyze, and blend represented ideas, concepts, and beliefs" of multiplication while co-creating the different emphases of the concept study design (Davis & Simmt, 2006, p. 299). The realizations and entailments emphases of fraction multiplication were the only pre-planned top down interventions for the concept study design. As part of these interventions, some literature derived tasks were organized to be used if the mathematical movements of the collective warranted the insertion into the environment to promote concept study emphases. These tasks are listed below in Table 3.1:

Table 3.1 Task bank for Emphases of concept study

Fraction Multiplication Difficulties	Sample Prompt for Concept study	Citations
Confusing fraction multiplication with fraction division	Develop pictures, models, stories, or real world experiences that would represent the situation $1\frac{3}{4} \div \frac{1}{2}$	(Armstrong & Bezuk, 1995; Ball, 1990; Ma, 1999)
Recognizing and reasoning with three levels of units	To what unit does each of the following numbers refer to: $\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$	(Hackenberg, 2010; Olive, 1999; L. Steffe, 2003; L.P. Steffe, 2002; L. P. Steffe & Kieren, 1994; Tzur, 2004)
Can use area to represent fraction multiplication	Which of the following represent $\frac{2}{3} \times \frac{3}{4}$: 	Izsak (2008); Izsak et al. (2012)
Length as a representation for fraction multiplication	Tape Problem: I have one-fourth of a meter of tape. I use one-sixth of my amount of tape to hang a flyer. Draw the amount of tape I used. How much tape did I use (in meters)?	Hackenberg, 2010
Reasoning with the reverse of stated multiplicative situation	Peppermint Stick Problem: A 7-inch peppermint stick is three times the length of another stick. Can you draw a picture of this situation? How long is the other stick in inches?	Hackenberg, 2010; Norton, 2008; Steffe, 2002

Below in Table 3.2 are the emphases tasks that were planned as part of the implementation of the six emphases of the concept study design:

Table 3.2 Description of planned interventions for emphases of concept study.

Planned Emphases Interventions	**Description and Days Planned for Implementation**
Realizations	1. Multiplication: Day 1 2. Number: Day 2 3. Fraction Multiplication: Day 3
Landscapes	1. Landscape for fraction multiplication in the middle school curriculum: Day 2
Blends	1. Multiplication, number, fraction multiplication: Day 4
Entailments	1. Multiplication: Day 5 2. Number: Day 6 3. Fraction Multiplication: Day 7
Participation	1. Mathematics as invented/discovered: Day 1 2. Mathematics as static or evolving: Day 6 3. Mathematical Knowledge for teaching: Day 8
Pedagogical Problem Solving	1. Why can fractions not have zeroes in the denominator? Day 2 2. Comparison of area and array models for modeling fraction multiplication: Day 3 3. Why does the algorithm for fraction multiplication work? Day 8 4. Egyptian fractions: Day 8

All other facets were dependent upon the choices and moves of the unique collective towards investigating the realizations for fraction multiplication as well as the consequences of those realizations.

3.2.1 Researcher Role

This qualitative research study required me to be involved extensively and iteratively with the participants of the study. Locke, Spirduso, and Silverman (2000) suggest that this type of researcher involvement with participants introduces a range of intentional, ethical, and personal issues into the qualitative research process. For this research study, due to the necessary co-creation of the mathematical environments and the theoretical framework that places special emphasis on the role of the observer, it was especially important for me to be rigorous about methods for attempting to remove as much personal bias as possible.

Moschkovich and Brenner (2000) make a distinction between the researcher roles of the *participant-observer* (PO) and the *observing participant* (OP). They define a PO as a researcher takes a role in the social situation under observation in order to experience the events in the manner of the participants. The OP, on the other hand, is a researcher "belonging to a community and observing [his] own activity as well as those of others" (p. 476). In this research study I enacted both the PO and OP roles. I collected data about my own participation as an active member of the collectively created mathematical environment and a necessary planning mechanism for each subsequent concept study session. My intention was not to create a case study of my own understandings, but to acknowledge that I was an active member of the collective and that, as such, I uniquely affected its development.

I cannot overlook the effect of my role as a researcher and facilitator. My position in the collective created circumstances of a centralized locus of power that provoked the participants to seek out my approval and validation for their understandings and opinions. I explicitly attended to this through Davis' (2005) definition of a "complexivist teacher" where my major concern was to be "responsible for prompting differential attention, selecting among the options for action and interpretation that arise in the collective" when collaboratively working on fraction multiplication tasks (p. 87). My intention was to confine my role to provoking high levels of neighboring interactions and to influence the creation of the emphases of the study.

3.2.2 Qualitative Design and Appropriateness

A researcher viewing mathematical activity through the lens of enactivism interprets learning and change as synonymous. Two points make this assumption clear: that perception and action are considered “inseparable in lived cognition” and that “cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided” (Varela, Thompson, & Rosch, 1991, p. 173). This premise relates to cognition in that an individual’s environment is engaged with through the available senses and we learn through adapting to feedback through those senses in a “continual process of co-ordinations of actions and our environment” (Brown & Coles, 2012, p. 221). For example, Brown and Coles (2012) studied teachers’ ability to reflect on their own practice from an enactivist perspective and based claims about their learning on analysis of their ability to make sense of their lived world. Put more precisely, they analyzed teacher’s learning as a change in the teacher’s ability to see more links between their actions and students’ learning. The methodological design for this study, framed by an enactivist perspective of cognition intending to make developmental claims for individuals and collective learners, must account for individual action and collective action in their shared environments. This intention necessitates a building of different types of cases for this study.

I selected an intrinsic case study design in order to gather evidence of the development of the collective learner’s M_4T knowledge of fraction multiplication. Stake (1995) defined the intrinsic case study as a study in which the case itself is the primary interest for developing an understanding of the uniqueness of the case comparative to building theory about how the case represents other cases. This approach is most appropriate for collecting and analyzing data for the collective learner case as it allows for the establishing, tracking, and charting of the M_4T development of fraction multiplication. The weakness of this design is that it is possible that a collective learner will not emerge.

Researchers have defined a multiple case study design as a research methodology that allows the concurrent studying of multiple cases in order to achieve greater insight (Johnson & Christensen, 2008; Stake, 1995). This case study design was most appropriate for studying the M_4T development of fraction multiplication of the individual teachers as it provided the ability to compare and contrast the individual development and better answer the research questions. This was done by creating individual cases for each participant, with similar domains of M_4T knowledge to make cross comparisons and generate evidence based conclusions about.

3.2.3 Data and Artifacts

3.2.3.1 Written Documents

Creswell (2003) described the utility for collecting participants' written documents as data sources. He described this practice as being advantageous because written documents capture the voice of the participants without needing to be transcribed. Further, they are convenient for the researcher as they can be analyzed at any time. The documents collected from the individual and collective learner in this study were realizations and entailments lists created by the collective as well as any handwritten mathematical work created by the individual participants. The Table 3.2 below lists these collectively created lists subdivided as realizations lists, landscapes list, entailments lists, blends lists, and then itemization of individual participant hand-written work. The handwritten documents collected are listed in the table by the day and the participant that created them:

Table 3.3 Written documents collected for data

Written Documents	
Realizations Lists	Day 1: Realizations for Multiplication Day 1: Realizations for Number Day 1: Realization for Fraction Day 3: Realizations for Multiplication Day 5: Realizations for Multiplication
Landscapes List	Day 2: Landscapes for Multiplication in Curriculum Day 3: Landscapes for Multiplication (Tools and Uses)
Entailments Lists	Day 2: Fractions are Numbers Because… Day 2: Fractions are Division Operations Because… Day 3: Entailments for Multiplication (Tools and Uses) Day 5: Entailments for Multiplication (Refine Tools and Uses)
Blends Lists	Day 3: Blends of Multiplication (Basic Multiplication) Day 5: Blends of Multiplication (Refine Basic)
Individual Participant Work	Day 1: David computation work Day 2: Faith computation work Day 5: Charlotte array modeling Day 5: Bailey cancellation of common factors work Day 6: Bailey fraction multiplication work Day 7: Charlotte Egyptian fraction work

3.2.3.2 Audio-video Recordings and Transcripts

Video of mathematical environments has increased in its use and capabilities for mathematics education research in the last decade (Powell, Francisco, & Maher, 2003). Figure 3.1 is a diagram of location and recording area of each of the cameras as well as the position of each of the participants.

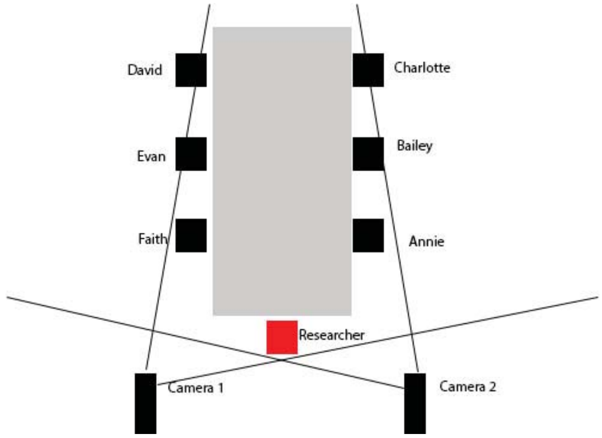

Figure 3.1Camera angle diagram and participant seating

Two videos for each of the eight concept study sessions, totaling 16 videos, were used in the creation of the transcripts for the mathematical activity of the concept study sessions. These line- itemized transcripts were created using a word processing software to transfer spoken mathematical utterances into text form. These texts acted as the main data source for mathematical action and were transferred to coding spreadsheets.

3.2.3.3 <u>Concept Study Reflections</u>

My role as part of the collective necessitated focus and full participation in the movements and choices that emerged in the collective for investigating fraction multiplication. At the conclusion of each day, I audio-recorded a reflective personal journal that I then used to coordinate video and audio data. The transcription of my audio journal was used to ensure that I was actively reflecting on my role as a part of the concept study mathematical environments. It also provided data for making planning choices for subsequent concept study sessions. The reflective audio journals were subdivided into sections titled (a) *overall impression*, (b) *moments of interest*, and (c) *recommendations for future session planning*.

3.2.3.3.1 Moment of Interest Documents

To plan for non-linear development of the collective I created a tracking and planning mechanism that I transcribed and superficially mapped into a preparatory document known as a moment of interest (MOI) document. This document acted as a data source for planning purposes that enabled me to quickly collect key moments from the previous concept study day videotape data and cross-reference it with my reflective journal data. This document became evidence used to plan the subsequent concept study emphases and the selection of interventions to be introduced into the collective mathematical environment. Below, Table 3.3 provides an example of the design of the MOI document that was used to coordinate artifacts and inform planning and later analysis of primary data sources:

Table 3.4 Sample moment of interest (MOI) document

MOI	***Sample Moment of Interest (MOI) Document***	
Researcher Description	When trying to establish "why you teach what you teach" the teachers' response was first that students would ask this question. They did not mention or seem to take any responsibility for the choice of the mathematics that they teach. The given responses indicated the different teachers' frame of mind about how they individually defend what they teach on a daily basis.	
Video Time: 2:00	MOI	Have you ever been asked why you teach what you teach or what it is that you teach?
Researcher Description	This is the first signal by the teachers that mathematics is not a utility for one's life, but actually a way of practicing so that one's brain works better. They used the metaphor of lifting weights for football and doing mathematics for the brain as equal types of operations.	
Video Time: 2:03	MOI	Mathematics is brain practice: no, you will never use this in your life—but it is for practice.

3.2.4 Limitations

Bottorff (1994) states that *density* and *permanence* are two main reasons that videotape has become such a powerful tool for educational researchers. “Density” refers to the ability of videotape data to collect simultaneous, different ongoing behaviors. Density also refers to video’s ability to capture two simultaneous data streams—audio and visual—moment to moment. “Permanence” is a reflection of the ability of the researcher to return again and again to the data source to analyze the moment-to-moment unfolding of mathematical behavior, including the subtle nuances in speech and non-verbal mathematical behavior (Powell, Francisco, & Maher, 2003). Despite its unquestioned power for the purposes of research, videotape data does have limitations such as user error, storage concerns, and data loss (Creswell, 2003). As Pirie (1996) stated, “who we are, where we place the cameras, even the type of microphone that we use governs which data we get and which we will lose” (p. 553). Pirie (1996) went on to suggest means for overcoming these methodological issues for videotape data such as coupling videotape data with created lists and handwritten work by the participants of the study. Hall (2000) furthered these recommendations by suggesting that videotape data be combined with other types of observations such as ethnographic observations, interviews, and teaching experiments—advice I have taken in producing my journal reflections and MOI documents, as well as in my utilizing all written and collectively created lists as data sources to couple with the videotape data.

3.3 Methods of Data Analysis

While the artifacts remained the same, the data analysis techniques differed for analyzing the individual participant M_4T knowledge development and the collective learner case M_4T knowledge development as the goals for analysis of each type of case differed. What follows is a careful description of the data analysis techniques used for the case creations of first the individual case studies followed by the collective learner case study.

3.3.1 Individual Case Data Analysis

The videotape data, video transcripts, MOI documents, audio journal transcripts, and documents produced individually by the participant teachers represented the data available for describing the individual teacher development while co-creating the mathematical environments of this concept study.

3.3.1.1 Written Documents

The lists and handwritten work illustrated in Table 3.1 consist of the artifact source for one form of data for the individual case studies. This included co-produced realization and entailments charts for multiplication and fraction multiplication. Each individual mathematical utterance that went into the creation of the collective lists in Table 3.1 that was attributable to individuals was collected as part of the individual's M_4T knowledge. The handwritten work was coupled with mathematical utterances when possible, or was included as a mathematical action displaying M_4T knowledge of the individual who created the work.

3.3.1.2 Videotape Analysis

The daily concept study session digital video clips and the transcripts of each session served as the primary source of data for analysis. Each participant was given a pseudonym that began with the letters A through F respectively and I used K to indicate my utterances. The letters A through F were distributed counterclockwise to the participants based on their sitting location of the first concept study session.

3.3.1.2.1 Transformation of video utterance to written text

Each two-hour video was transcribed using a word-processing software. Each utterance was transcribed in its entirety, including vocal patterns and non-relevant words such as “um” and “uh” and saved as line itemized transcription data. When the mathematical utterances were in question, the separate videos for each concept study session were used to verify that the transference of utterances was reliable and complete. After the completion of each transcription, I watched the video again with the transcription to verify its completeness and reliability as a data source for the individual participants’ mathematical utterances.

3.3.1.2.2 Preparation of data

Prior to the application of coding schemes, I prepared each video tape and matching transcript data using the protocol described in Table 3.4:

Table 3.5 Method for video transcript data organization

Step	***Method for Video Transcript Data Organization***
1	Segregate each participant’s mathematical and non-mathematical utterances on transcripts using word processing software.
2	Transfer each relevant utterance in its entirety to each participant’s Day 1-8 workbook spreadsheets.
3	Record line numbers for each utterance for tracking purposes and secondary level checks for consistency and inclusivity for all mathematical utterance data sources.
4	Run line totals to ensure that all mathematical and non-mathematical utterances were transferred reliably.

To illustrate how Table 3.4 was applied, the following is a short excerpt of the Day 1 transcript data:

> Researcher: Does anybody else have something to add? So … it … let me ask probably a little bit of a difficult question: If … if we collectively believe that some of the stuff that they're going to do uh, in your curriculum, they may never use in their actual lives, why is it in the curriculum?
>
> Charlotte: Well, not everybody is gonna do the same thing. Some kids might use it, some kids might not.
>
> Researcher: Ok, so, um, how would a kid use something like graphing inequalities someday, for example?
>
> Charlotte: Well, go to Pete's standard answer, what did you say?
>
> David: When?
>
> Evan: Logical [inaudible]
>
> Charlotte: [interrupting] When you can't think of something off the top of your head
>
> David: [interrupting] I thought you were talking about … I talked about this recently?
>
> Charlotte: No, no …

Using the *find* function of the word processor software, each participant's utterances were isolated from the other participant's and a line total was counted. For example, from the above excerpt Charlotte's (C) mathematical utterances were highlighted and then subdivided into an initial Day 1-8 workbook spreadsheet that listed the utterances in chronological order. When using the *find* function on the transcript, I was able to see in the above excerpt that there are four utterances that are attributable to

Charlotte, that total was cross-referenced with the workbook document total to ensure that all mathematical utterances were transferred to the Day 1-8 workbook spreadsheet.

After completion of the artifact transformation into usable data, I created a system for coding and organizing the data so that I could create the individual case narratives for each individual's development of M_4T knowledge of fraction multiplication. The construction of the mini-narratives follows an adapted seven-step process for analyzing videotape data (Powell, Francisco, and Maher, 2003). The adapted seven-step process appears below for reader reference:

1. View videotape data
2. Create MOI documents identifying critical events
3. Transcribe the video in its entirety
4. Code participant mathematical utterances by M_4T framework
5. Apply Coding Scheme I, II, & III
6. Identify and construct a story-line for each complex system
7. Construct the mini-narratives and final full narratives for each case

A fuller description of steps 5, 6, 7 appears below.

3.3.1.2.3 Coding Scheme I

I created an initial coding scheme from the research-based definitions for each of four complex systems of the M_4T model for fraction multiplication. This coding scheme served to find a method for inclusion and exclusion of each mathematical utterance as a data source in the four complex systems of *subjective understanding* (SU), *classroom collectivity* (CC), *curricular structures* (CS), and *mathematical objects* (MO). To ensure consistency, I created a table that started with the broad categories of the four complex systems based on their Chapter 2 definitions. The initial stage of refinement for this table was to code the Day 1 data for all participants and then look for similarities that emerged across participants. These similarities were recorded and then utilized as the first step in the refinement of the coding for inclusion or exclusion. This process was then repeated looking for consistency from the initial coding. As my familiarity grew with the data, I was able to refine the categorizations to finer-grained itemized definitions that enabled me to code and more efficiently isolate mathematical utterances that were specific to the M_4T of fraction multiplication. I adopted a final quick reference table as the coding tool for coding scheme I. Table 3.5 below illustrates this final coding tool of *coding scheme I* for only the Mathematical Objects complex system of the individual teachers M_4T knowledge. A complete table for coding scheme I categorizations can be found in the Appendix:

Table 3.6 Mathematical Objects: Knowledge produced coding scheme I

Mathematical Objects: Knowledge Produced Coding Scheme I	
Historical Development of:	1. Fraction Concept 2. Number Concept 3. Operation of Multiplication 4. Historical References
Orientation of Mathematics: [oral, pre-formalist, formalist, hyper-formalist, post-formalist]	1. What is mathematics? 2. Invented, discovered, or created? 3. Is mathematics a static or dynamic discipline? 4. Connection to natural world
Advanced Mathematical Knowledge	1. References to advanced mathematical study 2. Use of advanced mathematical techniques for understanding multiplication and fraction multiplication
Horizonal Knowledge	1. Connections of middle school mathematical curriculum to the other Pre-K–16 curriculum

Coding scheme I acted as the first level of coding that enabled the transition from all mathematical utterances of participants to a new spreadsheet that isolated only the utterances that were relevant to M_4T knowledge of fraction multiplication. I coded the mathematical utterances sequentially, one complex system at a time for all participants. The differences between utterances that were included and those that were not were based upon the four refined tables for each system that can be found in the Appendix. An example of the coding and organization of coding scheme I is provided in Table 3.6 and is a part of a one participant's day 1-8 document:

Table 3.7 Sample Day 1-8 Document for a participant

Sample Day 1-8 Document								
Mathematical Utterance	**Line #**	**S**	**C**	**CS**	**MO**	**DNA**	**Embedded**	**Count**
Day 2								
Well, I think we try to hit the basic ideas of multiplying now and then … you take, kind of what we said earlier … the rules of math and now we're going to apply it to fractions.	942		1	1	1		3	40
Yeah, it's both [fraction is a number and a division question].	953	1					1	41
It's part of a whole [what a fraction is].	955	1					1	42
We teach it as a number, like its two thirds, it's two out of … if we had a group of three its two out of three. But we also sometimes teach it as an operation because … I hate saying this on camera, but I swear	967	1	1	1	1		4	43

there's a conspiracy with some of our elementary teachers: they don't like fractions so they teach the kids that fraction bar means divide so put it as a decimal. Because it's so much better as a decimal than it is a fraction. When they get here we have to undo that.								
Do you want the middle school answer or the …	975					1	0	44

Column Headings Translation Key
S = Subjective Understanding
C = Classroom Collectivity
CS = Curriculum Structures
MO = Mathematical Objects
DNA = Does Not Apply; No relevant utterance
Embedded = Number of total complex systems per utterance
Count = Running count of total utterances

Figure 3.2 Column headings translation key for sample day 1-8 document

What resulted from this coding was the elaboration of the day 1–8 document for each participant that allowed for the transference of all mathematical utterances into another spreadsheet that organized the mathematical utterances by complex system type. I considered the systems as embedded, therefore one mathematical utterance could be categorized as a data source for more than one of the M_4T knowledge of fraction multiplication systems. Concluding *coding scheme I* resulted in all mathematical utterances, of all participants, coded by the complex system to which they were relevant. The next step for coding was based on the necessity to transfer all of this raw complex system data into a format that was organized and separated by each of the complex systems. This was accomplished by creation and application of *coding scheme II.*

3.3.1.2.4 Coding Scheme II

The purpose of creating and then applying *coding scheme II* was first to better organize the data by complex system for further analysis. Secondly, it was to look for themes within each of the complex systems to begin to have data for building the individual case narratives. I organized this data by transferring all mathematical utterances coded by complex system as well as by their *coding scheme I* codes together into their own spreadsheet. The chronological order was no longer important as the theoretical framework for this research suggests that learning is not chronological, but rather that it depends upon a change in mathematical behaviors around a mathematical concept. I retained line numbers as a reference number that could be searched for as part of the organizational scheme. An example of the organizational structure for the MO system of one participant is provided below in Table 3.7, which shows the organization possible after *coding scheme I* but prior to the application of finer grain codes in *coding scheme II.*

Table 3.8 Mathematical Objects of fraction multiplication UCE document: CS I

Mathematical Objects UCE Document: CS I		
Utterance	**Line #**	**Coding Scheme I**
Orientation to Mathematics		
Counting: math is counting	659	Orientation to Math
In a simplistic way, yes (does my dog do math?)	745	Orientation to Math
Historical Development		
I'll bring my history book in	1167	Historical Development
They'll tune out stuff that is above their head	376	Historical Development

I created the *coding scheme II* codes initially from the *coding scheme I* table of codes. The MO example can be referenced in Table 3.4, which acted as the starting place for creating finer-grained codes for each complex system categorization for M_4T of fraction multiplication. The initial stage of refinement for this table into *coding scheme II* codes was to code the Day 1 data for all participants and then look for similarities that emerged across participants within the subcategories provided by *coding scheme I* codes. I recorded these similarities and then used them as the first step in the refinement of the coding. I then repeated this process, again coding the Day 1 utterances of each participant by the new categorizations of each complex system looking for consistency from the initial *coding scheme II* codes. As my familiarity grew with the data, I was able to further refine the categorizations to finer-grained itemized codes. These codes enabled me to code and more efficiently isolate mathematical utterances that were specific to the *coding scheme I* sub-categories of the complex systems of M_4T of fraction multiplication. I adopted a final quick reference table as the coding tool for *coding scheme II*. Table 3.8 below illustrates this final coding tool of *coding scheme II* for only the Mathematical Objects complex system of the individual teacher's M_4T knowledge. A complete table for *coding scheme II* categorizations can be found in the Appendix:

Table 3.9 Coding Scheme II reference chart

Coding Scheme II Reference Chart	
Mathematical Objects of Fraction Multiplication	1. Historical Development a. Historical Figures b. Concept 2. Orientation to Mathematics a. Define mathematics b. Invented, discovered, or created c. Static or dynamic d. Natural world connection 3. Advanced Knowledge

	a. Name area of study 4. Horizonal Knowledge a. Concept connection

When I obtained the finalized version of the reference chart for *coding scheme II*—illustrated in part in Table 3.7—I then applied *coding scheme II* to each participant's mathematical utterances organized from the *coding scheme I* coding. This application resulted in a new document, a copy of the *coding scheme I* spreadsheet, with the addition of a new column for *coding scheme II* codes. An example of this new iteration of the UCE workbook spreadsheet document is provided in Table 3.9 below:

Table 3.10 Mathematical Objects UCE document: CS II

Mathematical Objects UCE Document: CS II			
Utterance	**Line #**	**CS I**	**CS II**
Orientation to Mathematics			
Counting: math is counting	659	Orientation to Math	Invented, Discovered, or Created
In a simplistic way, yes (does my dog do math?)	745	Orientation to Math	Invented, Discovered, or Created
Historical Development			
I'll bring my history book in	1167	Historical Development	Concept
Really? (reaction to debate about Pythagoras and his theorem)	376	Historical Development	Historical Figures

The application of *coding scheme II* also provided me with a systematic, self-sustaining methodology for removing any utterances that were not actual data for building the individual M_4T case narratives for the individual cases. The newest version of the UCE document provided a highly organized data set for each of the complex systems of each individual participant. Many of the categorizations after *coding scheme II* were still too large, and felt much too broad to build narratives for each participant's individual cases. This issue warranted the creation of *coding scheme III* to further refine the data to make ready for the creation of the individual case narratives.

3.3.1.2.5 Coding Scheme III

I created the *coding scheme III* codes initially from the *coding scheme I & II* table of codes. The MO example can be referenced in Table 3.4 and Table 3.7, which acted as the starting place for creating finer grained codes for each complex system categorization for M_4T of fraction multiplication. The initial stage of refinement for this table into *coding scheme III* codes was the coding of Day 1 data for all participants and then looking for similarities that emerged across participants within the subcategories provided by *coding scheme I & II* codes. I then recorded these similarities and used them as the first step in the refinement of the coding. Then I repeated the process, again coding the Day 1 utterances of each participant by the new categorizations of each complex system looking for consistency from the initial *coding scheme III* codes. As my familiarity grew with the data, I was able to refine the categorizations to finer-grained itemized codes that enabled me to code and more efficiently isolate mathematical utterances that were specific to the *coding scheme I & II* sub-categories of the complex systems of M_4T of fraction multiplication. I adopted a final quick reference table as the coding tool for *coding scheme III*. Table 3.10 below illustrates this final coding tool of *coding scheme III* for only the Mathematical Objects complex system of the individual teacher's M4T knowledge. A complete table for *coding scheme III* categorizations can be found in the Appendix:

Table 3.11 Coding Scheme III reference chart

Coding Scheme III Reference Chart	
Knowledge-Produced Categorizations	
Mathematical Objects of Fraction Multiplication	1. Historical Development a. Historical Figures b. Concept 2. Orientation to Mathematics a. Oral Stage b. Pre-formalist c. Formalist d. Hyper-formalist e. Post-formalist 3. Advanced Knowledge a. Name area of study 4. Horizonal Knowledge a. Concept connection

The resulting finalized UCE document used for creating the individual case study narratives to build the individual M4T cases for fraction multiplication is a highly organized data set for each of the complex systems of each individual participant development. The transitions between the applications of the three coding schemes is illustrated in the transition between Table 3.7 and 3.9, coupled with 3.11 provided below:

Table 3.12 Mathematical Objects finalized UCE Document

Mathematical Objects Finalized UCE Document				
Utterance	**Line**	**CS I**	**CS II**	**CSIII**
Orientation to Mathematics				
Counting: math is counting	659	Orientation to Math	Invented, Discovered, or Created	Oral or Pre-formalist
In a simplistic way, yes (does my dog do math?)	745	Orientation to Math	Invented, Discovered, or Created	Oral or Pre-formalist
Historical Development				

I'll bring my history book in	1167	Historical Development	Concept	N/A
Really? (reaction to debate about Pythagoras and his theorem)	376	Historical Development	Historical Figures	Pythagoras

I used the finalized UCE documents to create the written narrative of each of the individual cases. I constructed these narratives to illustrate the complexity, nature of, and development of the participant's M_4T knowledge of fraction multiplication.

3.3.2 Constructing Individual Case Study Narratives

3.3.2.1 Storyline for Case Study Narratives

I used the final UCE spreadsheet document for each complex system to construct the storyline for the case study narratives, transferring each *coding scheme III* categorization and all of the relevant utterances for that categorization into a text document as a block of text. The block of text, because of the organizational strategies previously described, was already a story of sorts without any transitions or highlights of development. The transference between storyline, and the mini-narratives was a three-step process that will be described below.

3.3.2.2 Transition for Storyline to Final Narrative

The first step was to choose how the narratives would be presented organizationally in the final individual cases. This choice was made based upon the amount of data available, working from the average least amount of data to the most amount of data. This choice resulted in the case study narratives being organized to represent the knowledge-produced systems followed by the knowledge-producing systems starting with MO and CS, and then transferring to CC and SU complex systems. The second step of the storyline transformation was to take the block data storyline and begin to construct the mini-narrative for each complex system. For example, below is the transition from the storyline block for one participants MO complex system knowledge development from text to the first stage of mini-narrative creation. Here is what a sample storyline block looks like:

> I'll bring my history book in […] Yes, Babylonians [the book of the Bible where Pi is estimated] […] Really? [reaction to debate about Pythagoras and his theorem] […] Oh I tell them there is […] One country discovered it before another […] I swear we've seen this. Okay so 2/3rds is 1/3rd and 1/3rd, but in the Egyptian thing, if I remember right, once you use 1/3rd you can't use it again so I go smaller […] Oh! Is that right? [Egyptian representations] […] So that is how the Egyptians did it?

In contrast, this is what the first iteration of the mini-narrative looks like:

Historical development of fraction multiplication. Faith's image for the historical development of fractions and operations on fraction remained relatively tacit throughout her participation in the co-produced concept study sessions. Her actions warrant the claim that she has an anecdotal understanding of the historical use of Egyptian representation of fraction:

> Faith: I swear we've seen this. Okay so 2/3rds is 1/3rd and 1/3rd, but in the Egyptian thing, if I remember right, once you use 1/3rd you can't use it again so I go smaller. […] So that is how the Egyptians did it?

This knowledge remained relatively inert, despite evidence that Faith had the most robust understandings of the historical development of mathematics in the concept study collective.

The transformation between the two versions of the mini-narratives is organizational and also shows a selection of mathematical utterances that helps to tell the story of M_4T development. The final step was the work with professional editors and my advisor for the refinement of the story of each mini-narrative into the full narrative of the M_4T knowledge of fraction multiplication development.

3.4 Collective Case Study Data Analysis

To build the narrative for the collective case I had to first identify data for the collective learner from the transcript data, systematically differentiate between collaboration and the emergence of a collective learner, and then describe the composition and existence of the collective learner.

3.4.1 Identifying Data for Collective Learner

The initial stage of the collective case development was to code the transcript data for instances of high collaboration around the relevant mathematical topic of fraction multiplication. I did this by using the individual UCE documents to locate moments during each of the eight concept study days where fraction multiplication was an explicit part of the collaborative action of the concept study. I transferred each of these moments from the day 1–8 transcripts, in their entirety, to eight separate documents organized numerically as *day 1–8 moments of high interactions.*

3.4.2 Distinguishing Between Coaction and Interaction

The eight separate documents organized numerically as *day 1–8 moments of high interactions* served as the data for finding instances of coaction and the emergence of the collective learner. I coded each moment of high interaction based upon the research distinctions outlined in Chapter 2 between collaborative groups interacting mathematically and the actual emergence of a collective learner (Davis, 1996; Martin & Towers, 2009, 2010; Martin, Towers, & Pirie, 2000, 2006). I created Table 3.12, below, was created to serve as a quick reference table for coding the moments of high interaction as moments either of coaction or of interaction:

Table 3.13 Characteristics of Coaction and Interaction

Distinguishing Characteristics of Coaction and Interaction	
Characteristic	**Description**
Coaction	• Carried out by individuals • Dependent and contingent upon the actions of the others in the group • Acting with the mathematical ideas and actions of others in a mutual, joint way • Understandings are interactively achieved in discourse and may not be attributable as originating from any particular individual • Not automatic, or trivial • Individuals must make a conscious, continued effort to coordinate their language and activity with respect to shared knowledge • Ideas originally stemming from individual learners are taken up, built on, developed, reworked, and elaborated by others and thus emerge as shared understandings for and across the group • Phenomenon that are not located with any one individual or their contribution

Interaction	• Individuals communicate what they already know • Understandings happen to overlap and their intersection is shared • Sets of only individual understandings occurring simultaneously • Different sets of ideas that are compatible with one another, but never truly shared

The moments of interaction from the *day 1–8 moments of high interaction* documents were coded and transferred to a new set of documents labeled *day 1–8 moments of interaction*. The moments of coaction from the *day 1–8 moments of high interaction* documents were coded and transferred to a new set of documents labeled *day 1–8 moments of coaction* and became the data for the creation of the collective learner M_4T development of fraction multiplication case.

3.4.3 Data Analysis for Collective Learner

The data for the collective learner is the collection of moments of high interaction that were coded as coaction. I considered the moments of coaction to be those in which the collective learner emerged. What follows is each of these stages of the data analysis of the collective learner data.

3.4.3.1 Collective Learner Composition

The initial stage of data analysis for the collective learner was done to establish the physical composition of the collective learner and answer the question, "What individuals were present when the collective learner emerged?" It is important to answer this question because change, even change in physical make-up, can be considered a facet of M_4T knowledge development of the collective learner. I analyzed all instances of coaction, and recorded data for which participants actively created the coaction. I define a participant as contributing *actively* when the participation contributed to the coaction and the emergence of the collective learner and was more than simply an utterance that took place in partnership with the coaction. This type of collective learner analysis followed this three-step protocol:

1. Define the composition of the collective learner by individuals who actively participated with mathematical utterances during the moment of coaction.
2. Label each collective learner by its active participant creators, and then group each collective learner by similar active participant creator categorizations.
3. Analyze the self-regulation of the coaction coding for internal diversity, internal redundancy, and decentralized or centralized locus of control.

I recorded this data in Table 3.13:

Table 3.14 Collective Learner Composition

Collective Learner Composition	
Collective Learner Co-Creators	**% of Total Coactions**
{All Participants}	5%
{B,C,D,E,F,K}	65%
{B,C,D,F,K}	25%
{B,D,E,F,K}	5%

This data served as basis for creating the narrative of collective learner development in regards to the composition of each collective learner and how it developed across the moments of coaction.

3.4.3.2 Differing Compositions of a Collective Learner

The composition data served as an analysis point that allowed me to draw conclusions about the irregular change in composition of the collective learner during moments of coaction. What I noticed was troubling: how can an entity like a collective learner have a differing composition from the individuals that make up the unique collective of this concept study? The language of complexity science helped me to answer this question, but first I had to test the hypothesis that the collective learner was in fact a complex system. This became the next step of the analysis of the collective learner data and the creation of the collective learner case.

3.4.3.3 Verifying Complexity in the Coactions

I coded each instance of coaction for the necessary pre-conditions for complexity, and then examined each for the necessary preconditions of complexity by analyzing the written text in the transcripts and the mathematical environments viewable in the videotape data. I fashioned Table 3.14, below, from this relevant research (Bloom, 2000; Casti, 1994; Corning, 2002; Davis & Simmt, 2003; Johnson, 2001; Lewin & Regine, 2000; Ricks, 2007) as a mechanism for coding each instance of coaction for the existence of the necessary pre-conditions of complexity.

Table 3.15 Preconditions for complexity

Pre-conditions for Complexity	
Pre-condition	**Description**
Internal Diversity	The coaction of the diverse individuals, each bringing a variation of the components necessary to aid the synergy needed to solve novel mathematical tasks. The synergy of the embedded complex systems of the individuals adds diversity and intelligence to the collective learner.
Internal Redundancy	Commonalities across individuals that would allow them to cohere in mathematical situations to overcome novel mathematical tasks.
Decentralized Control	Decision-making is dispersed, adaptively and democratically, to the individual teachers coacting in the mathematical environment.
Organized Randomness	Within the context of the restraints placed on the collective, the collective learner is able to co-produce an environment rich with possibilities and innovation.
Neighbor Interactions	Ideas, metaphors, and images must be given the opportunity to collide with one another in collaborative mathematical contexts. With the emergence of complexity, this is the coaction of the individuals' ideas as they collide and make space for collective development.

I moved each moment of coaction that had all the necessary pre-conditions for complexity to a new document titled *coaction and pre-complexity*. Any instances of coaction that were not coded to have all of the necessary pre-conditions for complexity were moved to a new document titled *coaction without pre-complexity*. This document remained empty, as all instances of coaction had evidence of the pre-conditions for complexity.

Based on the complexity science literature, this level of analysis was insufficient, to actually title moments of coaction as complex systems. What was needed was further analysis to look for what Ricks (2007) defined as categorizations for recognizing complexity: emergence and self-regulation. I built a quick reference key, shown in Table

3.15, from literature (Corning, 2002; Davis & Simmt, 2003; Ricks, 2007) for coding for emergence and self-regulation in the moments of coaction and pre-complexity:

Table 3.16 Coding definitions for Emergence and Self-Regulation

Coding Definitions for Emergence and Self-Regulation	
Term	**Definitions**
Emergence	A collective property of individual systems spontaneously materializing into a system that is more than the sum of its parts when prompted or necessitated by the co-created environment in which the system is embedded. Emergence is a property that belongs to the whole system and not to any single member.
Self-Regulation	A collective property of individual systems that approve and disapprove of other systems embedded in mathematical actions. These types of approval and disapproval are then communicated to other systems through interaction and subsequent mathematical action. Spontaneous input and mathematical discussion is a self-regulating mechanism.

Familiarity with the non-complexity term of coaction, may make readers wonder why this step of the data analysis was necessary. As the research-based definitions of these terms are very similar. The coding was motivated by the lack of existence of previous research based evidence for enabling *coaction*, *collective learner*, and *the emergence of complexity* as synonymous terms as I claim in this study. Therefore, each moment from the coaction and pre-complexity document was then analyzed using the code definitions from Table 3.13. Each moment of coaction and pre-complexity that I coded to have emergence and self-regulation was then transferred to a document titled *coaction and complexity* other moments were transferred to a document titled *coaction and non-complexity*.

3.4.3.4 Collective Learner M_4T of Fraction Multiplication

The data for the collective learner became the moments of coaction identified as *coaction and complexity*. As I will describe later, these documents included all moments of coaction and were analyzed by the *coding scheme I, II, III* analysis methods of the individual cases looking for M_4T knowledge of fraction multiplication. Each moment of coaction was placed into an UCE document, where each mathematical utterance was placed on its own row of the UCE document. Each mathematical utterance was then coded by the *coding scheme I, II, III* analysis methods of the individual cases in order to establish the existence of the complex systems of M_4T within the moments of coaction and complexity.

3.4.4 Linking Collective and Individual M_4T Development

The data utilized for the analysis of the moments of coaction as a link to individual M_4T knowledge of fraction multiplication was all of the individual M_4T of fraction multiplication cases and the moments of coaction defined as the collective learner. The noticeable emergent activities during moments of coaction that reciprocated with individual action (Cobb, 1999; Kieren, 2000) were defined as the links between the individual and the collective M_4T development. The analysis methods were created to establish, if possible, categorizations for the emergent activities of the collective during moments of coaction.

3.4.4.1 Mini-narratives of the Coaction

Each moment of coaction in its entirety began as the data source for the construction of the mini-narrative for the emergent activities during the coaction. For example, below is moment of coaction titled Episode 3, Day 1:

> Bailey: I personally think that everything can be put into repeated addition and grouping. 'Cause this is grouping inside of grouping.
> Researcher: Grouping inside of grouping?
> Bailey: It's the grouping of the 1/4ths, grouping of the 3rds.
> Faith: *Of* means multiply. Can we write that up there?
> Bailey: *Of*! Sure!
> Faith: I try to brainwash them.
> Charlotte: Unless it's a preposition
> Researcher: So why does *of* mean multiply?
> Faith: Groups of. Groups of groups.
> Bailey: It translates to multiplication. Why does *is* mean equal, why does *of* mean multiply? I have no idea.
> Faith: *Is* means equals.
> Researcher: Why?
> Charlotte: [inaudible] Switch the order …
> Faith: Result, your result, your final, your ending, your beginning …
> Evan: Could just be a part …
> Faith: Part over whole. *Of*
> Bailey: That's a fraction.
> Faith: Well then he asked …
> Bailey: Part of the whole.
> Faith: *Of* the whole. *Of* means multiply. [Inaudible] two different ways …
> Bailey: Girls do *is* over *of*. The guys do equations.
> Charlotte: I do part of a whole.
> Evan: I do both because 2/3rd of part …
> Faith: I teach *is* over *of*.
> Researcher: For?
> Faith: Percent. But if they want to do it any way can they. So, we're comfortable with the idea of repeated addition working for fractions?
> Bailey: I am.
> David: it's not [inaudible] repeated addition though. You don't just have repeated addition. It's not 2/3rds or 3/4th over and over. You had to do some work first to get to the repeated addition.
> Bailey: I just cut it. It goes back to counting and repeated addition. Cause that's all I did. I counted …
> Evan: She did some grouping and repeated addition. When you do repeated addition …

> David: That grouping is different than the grouping we were talking about before. Grouping before, we said two times three is two groups of three. That's what we were saying before for grouping.
> Bailey: This is just groups of 12.
> David: If we use that same definition, you're saying 2/3rds grouping of 3/4ths.
> Faith: Ok, what if you do it this way …? Keep your numerators. Two groups of three and we divide, 'cause you said earlier or someone did with the half, is that repeated or is it division. Two times three and then you divide by the product of your denominators. So your still saying two groups of three, divide by denominators of fractions.
> Researcher: So do we need to include something in our definition that has to do with dividing?
> Evan: Well, you could say part of the whole with that one. You have two groups of three for the part, that's going to happen. Three groups of the whole thing.

The focus of the data analysis was to first find all of the introductions of innovations or realizations new to the collective by taking a moment of coaction—like the example above—and numbering the new innovations offered to the collective. For example, the first few lines of the moment of coaction given as an example are provided below:

> Bailey: I personally think that everything can be put into repeated addition and grouping. 'Cause this is grouping inside of grouping. [1]
> Researcher: Grouping inside of grouping?
> Bailey: It's the grouping of the 1/4ths, grouping of the 3rds.
> Faith: *Of* means multiply [2]. Can we write that up there?
> Bailey: *Of*! Sure!
> Faith: I try to brainwash them.
> Charlotte: Unless it's a preposition. [3]
> Researcher: So why does *of* mean multiply?
> Faith: Groups of. Groups of groups. [4]

Once all moments of coaction were coded for the introductions of innovations, the second phase of the analysis was to code for how the collective utilized the introduction of innovations. Utilizing previous research (Davis & Simmt, 2003, 2006; Martin & Towers, 2009a, 2009b, 2010; Martin, Towers, & Pirie, 2009; Stahl, 2006) I focused on whether the innovations were utilized or disregarded and how they were collapsed and built upon. For example, the above mini-excerpt then transferred to the next stage of the coding by applying broad descriptive codes at each moment of introduction of a new innovation:

> Bailey: I personally think that everything can be put into repeated addition and grouping. 'Cause this is grouping inside of grouping. [1]–*Shared*
> Researcher: Grouping inside of grouping? [1]–*Shared*
> Bailey: It's the grouping of the 1/4ths, grouping of the 3rds. [1]–*Shared*
> Faith: *Of* means multiply [2]. Can we write that up there?
> Bailey: *Of*! Sure! [2]–*Shared*
> Faith: I try to brainwash them.
> Charlotte: Unless it's a preposition [3]–*Disregarded*
> Researcher: So why does *of* mean multiply? [2]–*Shared*
> Faith: Groups of. Groups of groups. [4]—*Shared and combined with [1]*

I described the patterns that emerged in short mini-descriptions placed at the bottom of each of the moments of coaction. These mini-descriptions were then all transferred to a new document, in which the descriptions were organized by title but were without the contextual dialogue and the mathematical utterances that prompted the description. These descriptions were compared to each other, looking for themes across the descriptions. These themes were then cross-referenced with collective mathematical research looking for research-based definitions that could be utilized to describe the themes that emerged.

CHAPTER 4. INDIVIDUAL CASE STUDIES

This section consists of six individual cases of the M_4T development of fraction multiplication. Development of M_4T of fraction multiplication is defined as “change” (Brown & Coles, 2012, p. 220) where an individual evolves internally or externally to maintain integration of a diverse set of mathematical elements, relationships, and knowledge. Change is not necessarily an increase toward some artificial level of adequacy for the M_4T knowledge of fraction multiplication; rather, change is evolution in any manner identifiable by an observer.

4.1 M_4T of Fraction Multiplication Complex Systems

Each individual case provides evidence of the complexity of the individual’s M_4T knowledge and the development of that knowledge in action. The individual cases are organized by first the *knowledge-produced* systems: mathematical objects (MO) and curriculum structures (CS). The *knowledge-produced* systems are then followed by the *knowledge-producing* systems: classroom collectivity (CC) and subjective understanding (SU). Prior to the presentation of the individual cases I have placed a quick reference definition of the complex systems that comprise a teacher’s M_4T knowledge of fraction multiplication.

4.1.1 Mathematical Objects of Fraction Multiplication

The mathematical objects of fraction multiplication, a knowledge-produced system, includes teachers' knowledge of the historical development of fractions and operations on fractions. Three sub-systems will guide the discussion of this system in the individual cases: (a) orientation to the discipline of mathematics, (b) historical development of concepts, and (c) advanced and horizonal knowledge.

4.1.2 Curriculum Structures of Fraction Multiplication

Curriculum structures is a combination of the teacher's knowledge of the established curricular resources available to sequence and teach fraction multiplication and the teacher's knowledge and use of how teacher, student, and curriculum interact in real-time in the classroom. The two categories of curriculum structures used to frame this discussion in the individual cases are curriculum-as-planned (CaP) and curriculum-as-lived (CaL).

4.1.3 Classroom Collectivity of Fraction Multiplication

As a *knowledge-producing* system, a teacher's knowledge of the classroom culture—similar to a teacher's knowledge of student mathematical cognitions—is based in part on an understanding of the theories of cognition (Even & Tirosh, 2010). Three broad contemporary theories of cognition (a) behaviorism, (b) constructivism, and (c) enactivism are used to frame this discussion on classroom collectivity of fraction multiplication.

4.1.4 Subjective Understanding of Fraction Multiplication

Subjective understanding includes how one's mathematical knowledge is developed, the conceptual blends of topics, and the images and metaphors that define and connect mathematical topics. The realizations that connect and define the concept of fraction multiplication will be discussed in each individual case. These are realizations of (a) fractions, (b) multiplication, and (c) fraction multiplication. Also included, as subjective understanding of fraction multiplication is student cognition of fraction multiplication.

4.2 Individual Case 1: Faith

Faith teaches middle school mathematics at the suburban Indiana school used for the site of this research study. Her career in education has spanned two decades, predominantly in seventh grade mathematics with a recent move to eighth grade mathematics.

4.2.1 Mathematical Objects of Fraction Multiplication

4.2.1.1 Orientation to Mathematics

Faith's initial realization for mathematics emerged as a formalist stage conception:

> Faith: I keep telling them … I make you think, and all of this [mathematical study][1] makes you think. Ya know, playing the piano, that … you can do it if you're creative. I think. It's more of a creative, ya know, it's a talent. Math, you

[1] Context will be provided, when relevant, to enable full understanding of mathematical utterances

gotta think, and I always look for those kids, you know, you'll have a question you're gonna ask every class, and kinda compare your classes. [Day 1: Line 198]

Here Faith represents mathematics as a distinct body of knowledge with a unique knowledge-producing methodology. Mathematics at this stage is a logical discipline with technologies that allow for calculation and problem solving. Faith's orientation often paralleled the complexity of the mathematical environment, allowing for less sophisticated pre-formalist realizations to emerge where mathematics is a mode of reasoning about unchanging forms:

Faith: Counting, math is about counting. [Day 1: Line 1204]

An iterative pattern of development emerged as Faith's orientation evolved between pre-formalist and formalist conceptions of mathematics. The following compilation of Faith's mathematical utterances shows actions in differing environments that illustrate this non-linear development of conceptions:

Faith: When we discover it [mathematics] [Day 2: Line 57], it's not invented; we're in the process of discovery. [Day 1: Line 1182] […][2] It's [mathematics] always been there, we just discovered it. [Day 2: Line 57] […] Then we need to invent [mathematics] [Day 8: Line 1042]. […] Yes [long division is an invented procedure] [Day 2: Line 1451]. [...] Math would still exist. Yes, in a roundabout way, your animals do math, they know [math] [Day 2: Line 1221].

Faith adapts her orientation to mathematics to fit her interpretation of the relative complexity of the mathematical tasks being investigated.

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[2] Bracketed ellipses signify a transition between different mathematics utterances within a compilation

4.2.1.2 Historical Development of Fraction Multiplication

Faith's realizations for the historical development of fractions and operations on fraction did not develop while co-producing the mathematical environments of the concept study. Her actions suggest that she has seen Egyptian fraction representations, but the mathematical environments of the concept study did not support the development of that knowledge:

> Faith: I swear we've seen this. Okay so 2/3rds is 1/3rd and 1/3rd, but in the Egyptian thing—if I remember right—once you use 1/3rd you can't use it again so I go smaller. … So that is how the Egyptians did it? [Day 5: Line 227]

4.2.1.3 Advanced and Horizonal Knowledge

Faith's knowledge of advanced mathematical techniques and areas of research did not develop while co-producing the mathematical environments of the concept study. The areas of advanced mathematical study that emerged through her interactions with fraction multiplication were those of calculus and linear algebra:

> Faith: Sounds like Calculus. [Day 2: Line 989] […] I think it is. I think the formula [for how to set up an array]—I mean isn't that how we learned it when we were taught Linear Algebra [rows by columns] [Day 5: Line 43]?

The following compilation of mathematical utterances provides evidence that advanced mathematical study did not enable Faith to develop her conceptions of fraction multiplication in the concept study sessions:

> Faith: Yeah, I feel that with all those math classes [from undergraduate study] I had to make sure that it made sense and that I got through them. [Day 2: Line

1561] […] No [higher-level mathematics does not help teach middle school mathematics]. [Day 2: Line 1752]

Faith's actions in the concept study sessions did not allow for claims about her horizonal knowledge.

4.2.2 Curriculum Structures of Fraction Multiplication

4.2.2.1 CaP: Interference

Faith's actions provide some evidence that she implicitly orients her understanding of the planned middle school curriculum to be similar to the interference hypothesis:

> Faith: Yes, that's why kids don't remember how to do fractions [lack of whole number mastery] and they still get frustrated in 7th and 8th grade when they do them again. [Day 2: Line 1249]

Faith's participation in co-creating the mathematical environments of the collective provided no evidence of development of this realization for the design of the middle school curriculum.

4.2.2.2 Remediation

. Faith's actions in the concept study reveal a knowledge of the middle school curriculum that recognizes it as repetitive for the purpose of remediation of concepts:

> Faith: But then it's [repeated presentation of concepts] for those other kids that still don't get it, ya know. There's the group that gets it … I know [a student] to multiply and even those that say multiplying fractions is dividing, then we divide and dah dah dah. But then this [repeated presentation of concept] is for that other

> group that didn’t get it back in 4th grade or 5th. You still get them and now they need something, because they’re not remembering [how to perform the operations].[Day 5:Line 457] […] Um, I taught 7th grade before I taught 8th for almost all of the 20 years and—um—it was like, I don’t feel like I introduced a whole lot of new material. [Day 1:Line 143]

Faith’s participation in co-creating the mathematical environments of the collective provided no evidence of development of this realization for the design of the middle school curriculum.

4.2.2.3 Realizations of Student Development

Faith’s diverse realizations for fraction multiplication emerged throughout the concept study when she explicitly referenced student understanding of fraction multiplication. These realizations included arrays, area, number-line hopping, and measurement. The following compilation of mathematical utterances shows the diversity of her realizations for fraction multiplication in the context of student development:

> Faith: Fraction multiplication, no [array not good representation for students]. But I can’t get her [an elementary student] to memorize multiplication facts [with whole numbers]. So we’re back to arrays. [Day 4: Line 916] […] So that’s where I would … can you draw a rectangle for me? ’Cause this doesn't support the repeated addition, but it supports area. Shade 2/3rds of it—OK new color. Of the 2/3rds, shade 3/4ths of that … it’s area, it’s area again. Yeah, but you see it? The area that’s both shaded both [is the answer to the computation] [Day 1: Line 1044] .

Her actions showed a hierarchy of preference for the area model of fraction multiplication when working with students. This preference is linked to her subjective knowledge of the models of fraction multiplication. For example, Faith's preference for the area model may stem from her confusion that emerged with the design of the array model and its relationship to the area model:

> Faith: Yeah, but when you connect [the dots in an array] you're short. I only have one box when I connect [the dots in a 2x2 array modeling the same situation as the 2x2 area model]. [Day 3: Line 1435]

Evidence was circumstantial for making claims about which system of Faith's M_4T knowledge supported most her preference for the area model.

4.2.2.4 CaL: Common Patterns of Interaction

To Faith, student realizations for fraction multiplication are diverse and that the teaching of fraction multiplication must account for students' realizations for the concept:

> Faith: But then this [physical model] is for that other group. That didn't get it back in 4th grade or 5th grade [Day 5: Line 457] […] You guys are right, there's a group of kids that don't need it and they look at you like, "Why are you doing it this way?" if we approach the full class like this. [Day 5: Line 459]

The development of common patterns of collectivity about the co-created curriculum of Faith's classrooms is difficult to track in only the mathematical environments of the concept study. This is directly related to the concept study design lacking classroom interactions with students.

4.3 Classroom Collectivity of Fraction Multiplication

4.3.1 Collective Mathematics with Students

Similar to the impediments for making claims about Faith's CaL knowledge, it is difficult to do more than infer about her M_4T knowledge of collectivity with students. The following compilation of mathematical utterances provides evidence that Faith characterizes her classroom environment as a collection of individual learners. These individual learners, to Faith, can be grouped based on her models for their mathematical knowledge level:

> Faith: All kids learn differently. [Day 2: Line 1369] […] Do you feel that this is just your lower portion? My upper level kids, I would never let them get away with $\frac{24}{3}$ [instead of 8 as a completed answer]. [Day 4: Line 478] […] It's a question that pops up typically with one kid in one class, you know, a higher level kid. He'll ask and then some other others will (sit up) "What? What? What's he talking about?" And I hate to say "he" … but it tends to be a boy. [Day 2: Line 356]

The collectivity in Faith's classroom is oriented to include collaborative activities, providing multiple opportunities for access to mathematical techniques, and also allowing for the presentation of multiple perspectives:

> Faith: Okay, I might present—I say "Who's ready?" Volunteer, I don't force them. So then you know those that, those first 10% grab it, go to the board. And then we get more examples of what they're doing, and I'm watching, trying to help those that are going to be that last 10% that won't get it. [Day 5: Line 711] […] Ya know this is kind of like what we do to the kids [looking at different

> representations]. Like we take what works for us … our personality, the kids too … you might present it this way or you might present it procedurally and they're going to take what works [for them] [Day 5: Line 845].

Faith's interpretation of the collectivity of her classroom is predominantly a constructivist orientation. Further evidence of this developed as Faith shared entailments of her classroom collectivity with students in the context of error handling:

> Faith: I drill that a lot especially when they're wrong. "How'd you get that?" So then I know how to work backwards. [Day 5: Line 904]. […] Something we've been doing and stressing more—and I don't know if it's because of our morning discussions or what—but when there's a wrong answer we put a lot of problems on the board and we're all staring at everything, and they explain it back to the class. So, when something's wrong—or if I notice a common mistake on a quiz or test—I'm like "This is what kids are getting. Why?" [Day 1: Line 461].

Faith's orientation to the collectivity with students developed little from the constructivist orientation. However, development did occur. The following compilation of mathematical utterances is evidence that Faith has a behaviorist orientation for collectivity with students when the mathematical environments focused specifically on the pedagogical processes best suited for student learning:

> Faith: I feel I spend a lot of time on that lately, just explaining that "Yes, that one expected answer is right. But you also got the right answer and this is why." So these are equivalent. [Day 1: Line 499]. […] Oh I tell them [students] that there is [Day 1: Line 1261]. […] You can show them all the philosophy, all the whys and whatever is behind it, but they—I could show them three ways to do it, but they

> are like, "Well show me my three steps. How do I get to my answer?" [Day 2: Line 1390]. […] But for those kids that like it where it is—like that step, step, step, step.

Faith's actions also provided evidence that Faith defines knowledge as training, brainwashing, and programming of students:

> Faith: No, it lets them—and I think 7th and 8th grade are the first two spots where they can leave it 2/4ths—they can leave it as 6/4s—the elementary is brainwashing them to leave it as 1 and 1/2. [Day 7: Line 129] […] I try to brainwash them [into thinking that "of" means to multiply]. [Day 1: Line 722] […] I mean they have been programmed up till this point to find an integer answer. [Day 8: Line 694]

The non-linear development between the constructivist and behaviorist orientation to student collectivity was not unique to Faith's case. Cognition and knowledge is a construct that, to Faith, develops dependent upon the student and the mathematical environment that the student is situated.

4.3.2 Participation in Collectivity of Concept Study

4.3.2.1 <u>Internal Diversity</u>

Faith's contributions to the collective's intelligence, or range of variation embedded in the collective, developed significantly throughout the concept study sessions. The following compilation of mathematical utterances are actions in differing environments where Faith contributed novel insights to the collective, significantly developing the diversity in the environment:

> Faith: Here's something I thought about this past week because I taught the algebra kids exponential growth and decay. So okay it's growing by 2% or we're losing something by 2% and ya know I'm teaching them 100% whole and I was like, "Well how would you explain to kids that back to this fractions aren't numbers?" Fractions aren't numbers so then how would you explain if it increases by 2% or it decreased by 2%, ya know what I mean? [Day 2: Line 464] […] I'm strong under arrays [as a definition for multiplication]. [Day 3: Line 1382] […] The six that are left [definitions for multiplication] I like: I have no problems with. [Day 3: Line 1477] […] Not everyone has to see it every way. Just see a way that makes sense to you. [Day 6: Line 1099]

4.3.2.2 Internal Redundancy

Similarities are as important as diversity for coherence maintenance of a complex system. Faith often easily agreed with others' mathematical propositions, adding redundancy within the mathematical context of fraction multiplication:

> Faith: The six that are left [definitions for multiplication] I like [Day 3: Line 1477] […] I have no problems with that [Day 3: Line 1477] […] I completely agree. [Day 7: Line 1110]

It is unclear how the development between diversity and redundancy for the collective was modified by Faith. Faith was a contributor to the balance between diversity and redundancy and the coherence maintenance of the collective in an unpredictable and evolutionary manner.

4.3.2.3 Decentralized Control

Faith was integral in balancing the distribution of power in the collective of the concept study. Other participants iteratively shifted the power to me, perhaps due to their preconceived notions of professional development and mathematical learning experiences. Faith often served as a mediator, distributing knowledge to the collective, reiterating other's understandings, and at times confirming their validity for the group:

> Faith: But then, you're going to have the Charlottes that are just not going to warm up to the arrays, then yeah. But, then you're going to have those kids that are only going to like the area so then they need that. I don't think they're all going to get [arrays]. [Day 5: Line 471]

Faith also enabled decentralized control by freely contradicting and challenging propositions and actions presented by me to the collective environment:

> Faith: So then yes: with that definition [agrees with statement after clarification of definition]. [Day 1: Line 526] […] Did we say what a whole number was though? [Day 2: Line 1268] […] See [whole number wasn't defined] and you still used it [vocabulary word whole number]. [Day 2: Line 1270]

4.3.2.4 Organized Randomness

Collectivity with complexity depends upon openness for innovation and novel interpretations of mathematical tasks. Openness does not imply that anything goes within a mathematical environment; rather, the randomness must be organized per the constraints existent. Faith's major contribution to this area emerged only sparingly, but she clearly enabled the collective by reminding them of the mathematical constraints in the evolving environments:

> Faith: So then yes, with that definition [response to the question of whether or not multiplication was formulaic]. [Day 1: Line 526] […] But I feel we're trying to use these words to come up with an agreed-upon definition of multiplication. [Day 3: Line 1359]

4.3.2.5 Neighbor Interactions

Rather than physical interaction, this pre-condition for complexity is the collision of realizations for fraction multiplication. Faith acknowledged the input of others as well as evolved it with her own conceptions:

> Faith: Yes, and that's sequential and logical, and to them that's good. But now there is also a group of kids that are going to have it memorized, multiply, multiply, multiply: and then when we get to fractions they know to multiply. Now, there is a reason that we don't introduce multiplication of fractions till—I don't know fourth or fifth grade—and we don't start in first grade with that? [Day 2: Line 1396] […] But, ya know… for those kids that like it where it is: like that step, step, step, step. [Day 4: Line 187]

4.3.3 Subjective Understanding of Fraction Multiplication

4.3.3.1 Realizations of Fraction

Faith's realizations of number developed with her realizations of fraction throughout the concept study sessions:

Faith: Counting a value a quantity [in answer to what is number?]. [Day 1: Line 1156] […] A value a quantity [in answer to what is number?]. [Day 1: Line 1158] […] Oh, I don't care: [to start] measured amount, value, or quantity. [Day 2: Line 284]

Faith's realization of number was highly dynamic, and was directly related to the development of her realizations for fraction.

Faith: Yep [variable is a number]. [Day 2: Line 538] […] Yes [infinity is a number]. [Day 2: Line 807] […] Numbers [finishing other participant's sentence: i.e. operations with numbers are also numbers] [Day 4: Line 611] […] Yes [it's hard to call a variable a number because it is a letter]. [Day 2: Line 583]

For example, her realizations of fraction developed continuously between including and excluding the concept as a number:

Faith: It's a number, it's not a whole number. [Day 1: Line 1080] […] It's [fractions] two integers, not a number. [Day 2: Line 404] […] So it's [fractions] a number. [Day 2: Line 433] […] It's [fractions] a number because we can put it on a number line. [Day 1: Line 806] […] If fractions aren't numbers so then how would you explain if it increases by 2% or it decreases by 2%? [Day 2: Line 464]

4.3.3.2 Realizations of Multiplication

Faith's realization for multiplication as repeated addition was consistent throughout the concept study sessions:

Faith: When we have a whole number [multiplication is repeated addition]. [Day 1: Line 691]

Evidence suggested that this realization of multiplication was consistent, but blended with other realizations for multiplication such as the distributive property and groups-of-groups:

> Faith: Now I also show the distributed property as repeated addition. Which ties in. [Day 1: Line 1002] […] Isn't that repeated addition [the distributive property]? [Day 3: Line 1298] […] Repeated addition grouping, repeated addition, groups-of-groups. [Day 3: Line 1470]

Faith blended her realizations for multiplication, and began creating sub-categories that differentiated between what defined multiplication and what was a use or tool of multiplication:

> Faith: Mhmmm. Good point. [multiplication is a tool used in the distributive property]. [Day 3: Line 1300]

Similarly, area as a realization for multiplication evolved into a tool for multiplication:

> Faith: I can still say it's area: multiplication is area without having that tool though. [Day 3: Line 1470] […] Figure I'd call it a tool though. [Day 2: Line 1193]

It is unclear how Faith differentiated these categorizations for her realizations, as arrays did not evolve this way:

> Faith: I'm strong under arrays [as multiplication]. [Day 3: Line 1382]

The following quote compilation provides evidence that Faith had a less sophisticated understanding of arrays as compared to her other realizations of multiplication:

> Faith: I think officially—isn't it rows times columns? I think that's the way. [Day 1: Line 599] […] No it doesn't [response to the question of whether row x column

> with arrays matters]. [Day 1: Line 601] […] No rows by columns closed rows by columns. [Day 3: Line 1452] […] That's your dimensions [numbers in a multiplication task]. [Day 4: Line 1201] […] Yeah [the dots in an array are arbitrary, you can use other things to build arrays]. [Day 3: Line 1455]

Rather than arrays developing into a tool of multiplication, they were collapsed with the realization of groups-of-groups and were recommended for elimination from the realizations list of the collective:

> Faith: I'm good with getting rid of arrays because we have grouping. It's groups of groups. And that's just the way I see groups of groups. [Day 3: Line 1459]

Counting emerged as another realization for multiplication, but was not blended with groups of groups or tools and uses of multiplication:

> Faith: Can we question mark counting please? [Day 3: Line 1374]

As was volume:

> Faith: Oh—no area I'm keeping volume: I would multiply the easiest ones first. [Day 3: Line 1386] […] Well, because I didn't think volume. Volume is multiplying. [Day 3: Line 1355] […] Because when you start talking about volume—that was more complex because we can't have—everything can't apply. [Day 3: Line 1293]

Faith's actions mirror the blending emphasis of the concept study design. She was looking for coherence for the operation of multiplication as a blend of all the partial fragments of realizations that emerged during the first three emphases of realizations, landscapes, and entailments:

Faith: Can we call all of this [list of words defining multiplication] basic multiplication? [Day 3: Line 1289]

Faith's realizations for multiplication developed considerably while participating in the collective environments of the concept study.

4.3.3.3 Realizations of Fraction Multiplication

Faith's realization for fraction multiplication emerged as embedded realizations with those of number and multiplication. A rectangular model, similar to Faith's realization for multiplication area model, was part of Faith's fraction multiplication realizations. The entailments emphasis was where much of the development for this realization took place for Faith. She began by describing the area model for fraction multiplication as a realization:

Faith: Here, change colors so you can see the overlap. [Day 7: Line 1020][…] You are shading 1/5th of 2/3rds, but you are separating it into parts to make it easier to shade is the way that I see it. [Day 8: Line 438]

Through the entailments emphasis, the collective began to substruct the area model of fraction multiplication and Faith's actions provide evidence that this was difficult for her:

Faith: [The area model] Similar to a Venn diagram, I would say. [Day 7: Line 1063] […] Well, the selecting is what you shade. You shade—there are eight that are not selected, but they are still marked, I just feel leaving that full unequal piece is misleading. [Day 8: Line 408] […] Yeah, because there are nine pieces. [Day 8: Line 411]

The entailments emphasis proved tedious and frustrating for Faith and the other participants. Substructing the realizations for fraction multiplication did provide a context for significant, non-linear development for her realizations of fraction multiplication.

4.3.3.4 Student Cognition of Fraction Multiplication

Similar to other M_4T knowledge that is co-dependent on students' understanding or participation in creating mathematics, it was difficult to chart development of such knowledge in the context of the multiplication of fraction. Utterances were often limited to how Faith understood children's cognition in the context of fraction multiplication, and the environments of this unique concept study offered little activation for evolving this knowledge:

> Faith: Yes [agrees that procedures of mathematics do not teach students how to think]. [Day 2 Line 1527] […] Right [the students have no idea what they are doing during fraction multiplication procedure] [Day 2 Line 1532].

These small excerpts include an orientation to cognition where Faith is able to know what others are thinking. Faith understands student cognition in the context of fraction multiplication as a blend of constructivist and behaviorist notions of cognition. This understanding links well with her notions of collectivity that were also a blend of constructivist and behaviorist notions.

4.4 Individual Case 2: Annie

Annie, an assistant principal at the suburban Indiana school used for the site of this research study, has 20 years of teaching and administrative experience in U.S. public schools. Before her position as an administrator, Annie was a classroom teacher where she gained relevant mathematical content experience for the concept study while teaching elementary level mathematics.

4.4.1 Disclaimer

Annie's participation in the emphases of the concept study was dramatically different from the other participants. Annie troubled the internal coherence maintenance of the preconditions for complexity in the collective. Her position as an administrator was problematic for decentralized control and the redundancy and diversity of the collective. Annie continually removed herself from the collective environment when engaged with mathematical tasks:

> Annie: People that might be a little more outside the math world, the math realm, and may have a little more literacy background [pointing to self]. [Day 3: Line 473] […] I just like listening to what you all think. It's interesting. [Day 3: Line 1196].

It was difficult to make claims about development of Annie's M_4T knowledge of fraction multiplication. Yet, as part of the collective coupled to the co-created mathematical environments, her case remains important to this research study.

4.4.2 Mathematical Objects of Fraction Multiplication

4.4.2.1 Orientation to Mathematics

Annie's initial realization for mathematics emerged as a pre-formalist conception of mathematics:

> Annie: But when you think of teachers teaching math it appears to me a lot of times to be a giver of knowledge. “Here’s how you do the problem. Here’s the rule. Here’s the formula. Here’s the strategy. Go.” [Day 1: Line 473]

The following compilation of mathematical utterances provides evidence that the emphasis of entailments for realizations of mathematics provided a context for little development of Annie’s orientation to the discipline:

> Annie: I think so, yes, about all math that is. I would say all math is formulaic. [Day 1: Line 519] […] Well, that is a way: but still, three times five is 15 regardless of how you got to your answer. So that’s what I mean when I say it’s formulaic. There is an answer. Three times five is always going to be the same answer. Now, I could do multiple different strategies to get that answer, but it is always 15. [Day 1: Line 523] […] Math. I told you, I told you! You guys were expecting an answer [from researcher] because math always has an answer. [Day 1: Line 891]

4.4.2.2 <u>Historical Development of Fraction Multiplication</u>

Annie’s actions in the concept study sessions prevented the ability to make claims about her knowledge of the historical development of fraction multiplication.

4.4.2.3 <u>Advanced and Horizonal Knowledge</u>

Annie’s actions in the concept study sessions prevented the ability to make claims about her knowledge of advanced mathematics and horizonal knowledge.

4.4.3 Curriculum Structures of Fraction Multiplication

4.4.3.1 CaP: Remediation

Annie's actions in the concept study reveal a knowledge of the middle school curriculum that recognizes it as repetitive for the purpose of remediation of concepts:

> Annie: But it is the same in English [as in math]; they reteach parts of speech from Kindergarten, and we reteach speech in 7th and 8th grade. So there are concepts [in math] that are continually retaught and retaught and retaught. Why do you have to do that? [Day 2: Line 1683]

Annie's participation in co-creating the mathematical environments of the collective provided no evidence of development of this realization.

4.4.3.2 CaP: Textbook as Curriculum

Annie's actions in the concept study sessions prevented the ability to make claims about her knowledge of the textbook as mathematical curriculum.

4.4.3.3 CaL: Realizations of Student Development

Annie's actions in the concept study sessions prevented the ability to make claims about her knowledge of student realizations for fraction multiplication.

4.4.3.4 CaL: Common Patterns of Interaction

Annie's actions provided evidence for making claims about her knowledge of the common interactional patterns existing in a mathematics classroom for fraction multiplication. To Annie student realizations of fraction multiplication are often procedural:

Annie: Memorizing your multiplication facts is generally not taught for any sort of pure understanding of the concept [multiplication]. Multiplication is a rote memorization of a process for most kids. [Day 2: Line 1355]

The concept study collective co-created mathematical environments that were entailments of pedagogical realizations for fraction multiplication. In these environments, Annie developed a notion that rendered her previous mathematical utterance problematic:

Annie: Yes [agrees that there is a better way to present fraction multiplication]. [Day 1: Line 1042]

No further evidence emerged to make claims about this development.

4.4.4 Classroom Collectivity of Fraction Multiplication

4.4.4.1 Collective Mathematics with Students

Similar to the impediments for making claims about Annie's and other's CaL knowledge, it is difficult to do more than infer about her M_4T knowledge of collectivity with students. The following compilation of mathematical utterances provides evidence that Annie characterizes a classroom environment as a collection of individual learners where teaching is implicitly referred to as a transference of knowledge:

Annie: Or do you guys probably [Day 2: Line 1693] […] do you think that because they are learning fractions way before they are coming here [Day 2: Line 1701] […] do you think that if kids are taught initially in a more conceptual way and not in a more procedural way [Day 2: Line 1705] […] do you think they would have a better understanding? [Day 2: Line 1712]

Transference of knowledge is a behaviorist orientation to collectivity. Yet, also implicit in Annie's mathematical utterances is the placing of importance on different knowledge types, which is a constructivist orientation to mathematical collectivity. No further evidence emerged to enable claims of development of these orientations to classroom collectivity.

4.4.4.2 Participation of Collectivity in Concept Study

4.4.4.2.1 Internal Diversity

The collective's intelligence was based upon the range of variation embedded in the collective. Annie's physical presence as an administrator significantly impacted the internal diversity of the collective. For example, David directly referenced her presence as an administrator participating in the concept study. Her response to David's utterance appears below:

> Annie: Correct [participants will not be judged]. Feel free to be open. This will not be reflected on your evaluation in any way [laughing]. [Day 1: Line 96]

It is clear that Annie's position in the collective as an administrator impacted the internal dynamics of the system. It is unclear, due to her low mathematical participation, how this impacted the mathematical output of the collective.

4.4.4.2.2 Internal Redundancy

Annie's presence in the collective, as previously mentioned was a reliable stimulus of diversity for the collective. Evidence also suggests that Annie provided collective redundancy through the use of inclusive pronouns such as "our students," “we,” “us,” and “here at our school”. Similar to the findings for diversity, these redundancies were part of the collective social environment rather than mathematical.

4.4.4.2.3 Decentralized Control

While it would seem obvious that Annie’s position as the evaluator of the group would potentially disrupt the locus of power of the collective—distributing it to Annie unequally—this element proved to be the exact opposite of what occurred in mathematical contexts. Annie’s position of self as outside the mathematical culture resulted in little change to the balance of power within the mathematical environments involving fraction multiplication.

4.4.4.2.4 Organized Randomness

The collective had a self-regulating mechanism for balancing the redundancy and diversity that Annie offered to the collective. In non-mathematical environments, power shifted to Annie as an administrator. In mathematical environments, power shifted away from Annie as a non-mathematics expert.

4.4.4.2.5 Neighbor Interactions

Annie’s main position for the collision of ideas was neither interrogation of other’s opinions nor the presentation of her own opinions. Annie’s presence was a coherence environmental mechanism that provided a consistent and reliable locus of power to actively bring participants back to collective participation at intermittent times throughout the eight sessions of the concept study.

4.4.5 Subjective Understanding of Fraction Multiplication

4.4.5.1 Realizations of Fractions

Annie's actions in the concept study sessions prevented the ability to make claims about her realizations for fraction.

4.4.5.2 Realizations of Multiplication

Annie's actions in the concept study sessions prevented the ability to make claims about her realizations for the operation of multiplication.

4.4.5.3 Realizations of Fraction Multiplication

Annie's participation in the entailments emphasis for the realizations of fraction multiplication did provide some evidence of her realizations for fraction multiplication. The mathematical utterance below implicitly references the confusion between the operations of multiplication and division when applied to rational numbers:

> Annie: But I think it is not [2/3 of 60]; I think it looks like division to people who don't teach math. You just see 2/3rds of. [Day 4: Line 310]

This is similar to explicit-objective research that found elementary teacher deficits for this type of knowledge. No evidence emerged to make claims of development for her realizations of fraction multiplication.

4.4.5.4 Student Cognition of Fraction Multiplication

Similar to other M_4T knowledge that is co-dependent on students' understanding or participation in creating mathematics, charting development of such knowledge in the context of the multiplication of fraction was difficult. Annie students acquire associations, skills, and components of skills of fraction multiplication, exhibiting a behaviorist understanding of student cognition:

> Annie: [continuing through interruption] … early, yeah, early at the beginning of multiplication memorizing your multiplication facts is generally not taught by any sort of pure understanding of the concept. Multiplication is a rote memorization of a process for most kids. [Day 2: Line 1355]

Yet, implicit in this utterance is that the teacher can build a model of student cognition, which is inherently a constructivist understanding. The embedded nature of M_4T is evidenced here as the blends of behaviorism and constructivism were also present in other contexts that involved data-collecting about Annie's orientation to cognition of mathematics.

4.5 Individual Case 3: David

David teaches middle school mathematics at the suburban Indiana school used for the site of this research study. David is currently in his fourth year of a middle school teaching career with both seventh and eighth grade level teaching experience.

4.5.1 Mathematical Objects of Fraction Multiplication

4.5.1.1 Orientation to Mathematics

David's initial realization for mathematics was in the formalist stage:

> David: I usually kinda take a … I guess maybe a cop-out easy way out of … you know, this [mathematics] is logical system of problem-solving, of the rules that you have. You know, all these tools and then you can use these tools to solve this problem. How are you going to solve this problem? It might not be a math

> problem that you're dealing with, but it's [mathematics] the thinking. It's the, "Here are my options, here's what I can do, here's what I can't do, here's how I'm going to solve this problem." [Day 1: Line 61]

This is considered the formalist stage because evidence in this utterance suggests that the goal of mathematics is absolute reason. The realization of mathematics as formalist remained relatively constant, as David's mathematical utterances described mathematics as rule-bound and full of formal syntax for representations of truth. A significant development occurred when the collective investigated the entailments of realizations for mathematics:

> David: Math PhDs think about things when you don't need them. [Day 2: Line 1122] […] Yeah [imaginary numbers are intellectual pondering] [Day 2: Line 981].

These mathematical utterances provide evidence of a hyper-formalist realization for mathematics as a discipline completely divorced from the experiential world. Similar to Faith's case, David's realizations for mathematics co-evolved with the level of complexity of the mathematics being investigated. The following compilation of mathematical utterances provides evidence of the non-linear development of David's realizations for mathematics:

> David: The structure of it, I think, quantity [mathematics in the natural world]. It [quantity] was always there [to discover]. [Day 1: Line 1183] […] It's still there once they prove it. It's [mathematical proofs and ideas] always been there. [Day 1: Line 1189] […] I feel like you could argue that everything was invented in general—well, not nature. But I feel like I could argue that everything is

> invented—but you could also argue that the idea of that thing was a possibility because if it wasn't, then we wouldn't have it. So it's been there, but we or the person [who is given credit for discovery] communicated it. [Day 2: Line 1102]

David's realizations for mathematics reached levels of sophistication that were greater than or equal to the others in the collective.

4.5.1.2 Historical Development of Fraction Multiplication

David's actions in the concept study sessions prevented the ability to make claims about his knowledge of the historical development of fraction multiplication.

4.5.1.3 Advanced and Horizonal Knowledge

David has knowledge of advanced mathematical study. His mathematical utterances focused specifically on one course from his undergraduate studies that made a profound impact on him. This course, a geometry course, seemed to connect to facets of his middle school mathematical teaching:

> David: I think my college geometry class … they went through proving. But like, "Here are the theorems that you can use to prove." You know, "Use the theorems from here to construct this and then prove" [Day 2: Line 1583] […] I think that that was … proving all these theorems, I think that helped to a certain extent [with his middle school teaching]. [Day 2: Line 1584]

David recognized his advance mathematical study as helpful for informing his teaching of middle school mathematics:

> David: Yeah [undergraduate math may have implicitly helped him teach middle school]. [Day 2: Line 1590]

This developed as the participants co-created the emphasis of entailments of their realizations for advanced mathematical study. Some levels of mathematics were difficult for David:

> David: I mean there were some things, like, I remember: Linear Algebra, like, was the first time I really … I could do it. I could find the image of the matrix. I had no idea what an image of a matrix was. Like, "What? What does that even mean?" What does … I don't know; I could find it for you. I could find all these things for you, but I didn't really know what I was doing. […] Everything I've heard [in mathematics] is multiplication is established and then use multiplication as an axiom to define division. But I don't know why we do that. [Day 1: Line 327]

No evidence emerged to make claims about how explicitly David's advanced mathematical knowledge informed his teaching of the middle school curriculum. There was also no evidence that emerged to make claims about his horizonal knowledge.

4.5.2 Curriculum Structures of Fraction Multiplication

4.5.2.1 CaP: Interference

David's realizations for the planned middle school curriculum implicitly reference facets of the interference hypothesis. In the compilation of mathematical utterances below, whole number operations are positioned in the middle school curriculum to inform procedural fluency for operations on fractions:

> David: I think it's easier to teach multiplication first because when you first teach multiplication you can use addition really easily—and they [students] learn

> addition first, like counting. Use counting to help them add or adding. [Day 1: Line 777] […] It [whole number multiplication] helps in the procedure [of fraction multiplication]. [Day 2: Line 1389] […] But the process is so—I mean the process is not that much more complicated [fraction multiplication compared to whole number multiplication]. [Day 2: Line 1428]

David's implicit recognition of the advantages of ordering the contemporary curriculum in this manner developed as he co-created entailments of his realizations for the curriculum. While co-creating and entailments environment David's substructure of the middle school curriculum developed his realization for the structure of the curriculum:

> David: Right [the common error with fractions links to students' understanding of the whole number algorithm]. [Day 7: Line 219]

No further evidence emerged to make claims about how this development fully impacted his realizations for the structure of the planned middle school curriculum.

4.5.2.2 CaP: Remediation

David's realizations for the middle school mathematics curriculum provides evidence that he recognizes the curriculum as necessarily repetitive in order for students to comprehend and learn the mathematical concepts effectively:

> David: Right [teachers leave out the complicated concepts in curriculum at certain levels]. [Day 3: Line 348] […] New stuff [curriculum shouldn't have too much new stuff]. [Day 3: Line 351]

No evidence emerged of development of this conception.

4.5.2.3 CaP: Textbook as Curriculum

During the process of landscape building, David and his peers investigated the contemporary curriculum materials, including their textbooks, for representations of fraction multiplication. David's mathematical utterances provide evidence that the textbook is a curricular resource for middle school mathematics teaching that is predominately used to reference the rules and procedures of mathematics:

> David: In the book, they give these rules which is basically, "Here's how you do this." [Day 3: Line 1147] […] Like multiplying fractions gives us a definition there, but it's like how we, how we multiply fraction $\frac{a}{b}$ and $\frac{c}{d}$. Are we counting that [as a definition]? It's not necessarily multiplication, but where in our book does it give a definition of multiplication? [Day 3: Line 1157]

No data emerged that would provide the opportunity to discuss development of these conceptions.

4.5.2.4 CaL: Realizations and Student Development

The images that emerged for fraction multiplication from David's actions were that of arrays, area, volume, line multiplication, repeated addition, line jumping, and metaphors linked to cooking. Little evidence emerged to make inferences about his understanding of the developmental appropriateness of these different realizations for fraction multiplication. Evidence does suggest that the entailments emphasis for the array model developed David's conceptions of the viability for the array model as a pedagogical tool to enable student learning of fraction multiplication:

David: How do you explain to a kid that to get 12 [for the array] you have to multiply the denominators [of the fractions]? [Day 5: Line 1151]

David's actions suggest that learning conceptual models at the middle school level is not developmentally appropriate for students:

David: I think some things need conceptual knowledge [through images], but I think I mentioned this before—or someone did—at this point [middle school] we no longer … like, do we care that they know what four times five means? Or do we just care that they know what four times five is? And if I was a kid, just me personally, if I learn, like, this array stuff and that stuff and then: "Oh by the way the short-cut is just multiply the top and bottom together," I'd be, like, ticked. Like, "What the heck? Why did we do all that?" [Day 5: Line 439]

No further data emerged to make claims about how David's realizations of the models for fraction multiplication developed.

4.5.2.5 CaL: Common Patterns of Interaction

David orients his teachings of the realizations for fraction multiplication in a streamlined manner. A blended realization is preferable for David's teaching of computations with fractions:

David: Well, the reason we do that is because when we get question marks in the denominator then it takes both steps; you know what I mean? And so, instead of teaching … well, you know, if the *x* is in the numerator then it's one step, but if the *x* is in the denominator then I think we just bundle it all and do it every time no matter what/where the x is. [Day 4: Line 533]

Students' desire for a streamlined procedure was a recurrent theme in David's realizations for the lived middle school curriculum. David's actions developed during the entailments emphasis of the common student interactions around fraction multiplication in a middle school classroom. David's utterances provide evidence that he recognizes that procedural fluency of fraction multiplication does not necessarily transfer to student understanding the concept:

> David: Right [students don't connect the procedure for fraction multiplication to parts of parts]. [Day 5: Line 95] […] They just know the procedure [of fraction multiplication]; they really don't know what they are doing. [Day 6: Line 53]

No evidence emerged to make claims about how David attempted to blend the realizations for students desiring procedures and that procedures provide little educational value.

4.5.3 Classroom Collectivity of Fraction Multiplication

4.5.3.1 <u>Collective Mathematics with Students</u>

Similar to the cases of other participants, making more than inferences about David's collective mathematical knowledge of co-created environments with students and the development of such knowledge is difficult. The following compilation of mathematical utterances provides evidence that David's realizations for student cognition of mathematics is characterized as an individual activity where associations, skills, and components of skills are acquired:

> David: I feel like there is a lot of math to be able to tell a student that you can multiply the denominator of one fraction and the numerator of another fraction

> [Day 8: Line 1324] […] I do. Because we make equations, not proportions [how David teaches these tasks compared to his peers]. [Day 5: Line 524] […] I feel like I do 30 example problems every day [when teaching mathematical concepts]. [Day 1: Line 23] […] Right [assigns a lot of homework so that students can overcome nuances in computations of topics]. [Day 5: Line 742] […] Every single time for every situation [algorithms work]. [Day 3: Line 956]

David's realizations for collectivity with students is to avoid error handling. This observation furthers the notion that David orients student cognition by the tenets of behaviorism:

> David: At the same time I feel like if I didn't [perform many examples for students] and I let them come across it, then the next day we're going to spend 3/4 of class because they didn't understand it. They don't get those nuances because we didn't talk about that. [Day 5: Line 755]

David's behaviorist orientation to student cognition develops as David blends his conceptions of knowledge and teaching:

> David: I mean, I don't care if they don't know how to work with mixed numbers: they can leave it improper or change it back. I think just letting them know that if this is a weakness then there is probably a way around it. [Day 7: Line 201]

Evidence here emerges to suggest that David recognizes different knowledge types, a constructivist understanding of collectivity and student cognition. David's participation in the blends of these realizations provides evidence that a certain level of mathematical expertise is necessitated at the middle school level. This level of expertise is more important than the necessity for students to have more than a procedural fluency with fraction multiplication. The compilation of mathematical utterances below provides evidence for this claim:

> David: I'm trying to think, I mean initially back to whenever I learned it [place value] in second grade—I remember, like, manipulatives, where you have hundreds that you break into 10s: you know what I mean. I think initially when you first learn it … but I think after a while you forget what it means and then it becomes a process of doing because you have been at it for so long. You forget that you are breaking things apart and giving it to a 10. [Day 7: Line 232] […] Like multiplying whole numbers—at the beginning we care that they know that two times three is two groups of three. Then eventually, as long as they know that two times three is six then we [teachers] are good. I don't know at what point is an algorithm mindless, or is that okay versus not okay? [Day 4: Line 1107]

This example is a fascinating blend of Enactivist, constructivist, and behaviorist mentalities for collectivity and student cognition. The embedded nature of M_4T renders these compatible, as it seems the environments where mathematics is created impacts how David orients the cognition of that mathematics.

4.5.3.2 Participation in Collectivity of Concept Study

4.5.3.2.1 Internal Diversity

David's contributions significantly added to the intelligence of the collective. David took part in the realizations of fraction multiplication through scrutinizing automated responses of others in regards to their definitions of fraction multiplication:

David: I just told her, I said, "You know he's [the researcher] going to ask what is multiplication right?" What about fractions? That's not repeated addition right? Because it's not repeated addition. You don't just have repeated addition. It's not 2/3rds or 3/4th over and over. You had to do some work first to get to the repeated addition.

David also felt comfortable in his diversity from other participants' mathematical realizations. At times, this diversity created uncomfortable social dynamics within the collective:

David: I feel like you are saying that half is not a number-- then you are, like, in second grade. [Day 2: Line 1316] […] That is honestly what is going on in my head right now. [Day 2: Line 1319] […] I feel like we've changed our definition of grouping multiple times. First grouping was two groups of three' I feel like it's still groups. [Day 1: Line 1060] […] So you're saying if I say it's one gallon it's a quantity. But if I say it's 1/10th of my car tank it's not a quantity? [Day 1: Line 1383]

4.5.3.2.2 Internal Redundancy

David built on his redundancies with others by agreeing with and then adding to other participants' statements. The following mathematical utterances compilation are evidence for moments when David recognized his redundancy with the collective but also added diversity through his additional actions:

David: The procedure is one where it is more time consuming [for students]. There are three: I mean you have to know three operations in order to do the

> procedure, going into it is division, multiplication and then you have to subtract. [Day 2: Line 1436] […] Yeah it would just take a lot longer for students [they can physically divide without other operations]. [Day 2: Line 1446] […] Right it's the time consuming thing that gets me. [Day 2: Line 1449]

4.5.3.2.3 Decentralized Control

David was integral in the constant evolution of the locus of power in the collective. David's diversity with the collective allowed for him to blend his realizations with the collective realizations but permitted a rejection of collective realizations when appropriate. This is evidenced in the following mathematical utterance

> David: I agree that it works, but I feel like it is like telling a kid, "Here is another procedure with a visual. You do a fifth you do 2/3rds and then you—Hey look at the overlap!" But, I don't know—Look, it is like magic: the overlap represents the answer. [Day 8: Line 487]

These actions by David shifted the locus of control away from the momentum of the environment and also from the individuals who were integral in creating that moment. This is a coherence maintaining mechanism that significantly adds to the intelligence of the system.

4.5.3.2.4 Organized Randomness

David's actions served as boundaries, limiting focus that functionally activated mathematical innovations within the realizations, entailments, and blends of the collective activity:

> David: Because if fractions were factors you would—every number would have an infinite number of factors, so there would be no prime numbers. [Day 8: Line 804] […] But you're not adding it over and over. What you're saying is right, but it's not repeated addition. [Day 1: Line 935]

David significantly added to the intelligence of the collective by providing mathematical boundaries for innovation. His actions were often a refocus, or a displacement of, previously accepted realizations

> David: I don't think they need to be exclusive to each other [fractions can be numbers and division questions]. [Day 4: Line 68]

4.5.3.2.5 Neighbor Interactions

David was integral in the collision of ideas. The importance of his contribution is evident in the previously established embedded pre-conditions for complexity. David added diversity, redundancy, and organized randomness through his interactions and collective activity with his peers as they interpreted realization, entailments, and blends of fraction multiplication. His position in the collective necessarily developed as the mathematical environments developed. This is an emergent consistency in the individual cases.

4.5.4 Subjective Understanding of Fraction Multiplication

4.5.4.1 Realizations of Fraction

David's realization of fraction was linked to his realizations for number. The entailments of these realizations was tedious and difficult for David, as he developed between differing conceptions of fractions as number and fractions not as number:

> David: I think so [is a fraction a number?]. [Day 1: Line 1125] […] It doesn't have to be [is a fraction a number?] [Day 5: Line 1173]

These contradictory realizations developed considerably as David co-created the entailments for the realizations of number with his peers in the concept study environments. The following compilation of mathematical utterances provides evidence of this non-linear development:

> David: Then that's like saying a decimal is not a number because we could divide 23 hundredths, 23 divided by a hundred. 23 over 100 could be written as a fraction. [Day 1: Line 1128] […] Well, I think that you could argue that if you wanted [that a decimal isn't a number]. [Day 1: Line 1131] […] You can write it multiple ways. A fraction is one way to represent a number, a decimal: to turn it into that decimal [why do we do operations to make fractions decimals?]. [Day 1: Line 1134] […] That doesn't mean that decimals aren't numbers. [Day 1: Line 1138] […] The parts are numbers [fractions are not numbers]. [Day 1: Line 1393] […] I think it's a number. [Day 1: Line 1341] […] Yeah a fraction is not a whole number but it's a number. [Day 1: Line 1397]

David's development was considerable, as the entailments emphasis for the realizations of fraction introduced irrational quantities for the first time to the collective:

> David: So you don't think rational numbers, irrational numbers, and imaginary numbers are real numbers? [Day 2: Line 394] […] So irrational numbers are not numbers, so π is not a number, some square roots are not numbers? [Day 2: Line 416] […] No, it's an irrational number [square root of two]. [Day 2: Line 419] […] It's a number, we can't measure it. So you're saying only numbers we can

measure on a ruler are numbers? [Day 2: Line 421] […] Yeah I agree it's [the square root of two] measurable. [Day 2: Line 432] […] Yes [square root two is a number] [Day 2: Line 434].

Further entailments blended David's realizations of number with his realizations of operations and their embedded relationship with number:

David: It is a symbol if you're saying that the two thirds isn't a number because it's two divided by three, then you can say twenty is not a number because it's four times five; it's a multiplication problem. [Day 2: Line 438] […] So only one is a number? [Day 2: Line 444] […] If fractions aren't numbers then we can't multiply them, so shut off the cameras and let's get out of here. [Day 2: Line 449] […] Correct [you can't do mathematical operations on anything but numbers]. [Day 2: Line 483]

These blends also prompted David to consider the inclusion of variables as number as evidenced in the compilation of mathematical utterances below:

David: Yes [variables are numbers]. [Day 2: Line 489] […] Because they are [numbers] they represent a quantity. Do we know the quantity? No. But they are a quantity, so yes they [variables] are a number. [Day 2: Line 496] […] They're not written in one of our numbers that we write on paper. But they're numbers—we can add a+a and get 2a. [Day 2: Line 579] […] It has every single property of a number, so why wouldn't we call it a number? We can add it, subtract it, multiply it, divide it—it has all the properties of a number, so why is [it] not a number? Just because it's a letter in our alphabet we don't want to call it a number? [Day 2: Line 570]

David's realization of fractions developed significantly to blend realizations of number, fraction, variable, and irrational quantities. This recursive elaboration was a fascinating example of the embedded nature of M_4T knowledge of fraction multiplication and its relationship with knowledge of other mathematical concepts.

4.5.4.2 Realizations of Fractions

David's realizations for multiplication were tied to the operation's relationship with division:

> David: Everything I've heard is multiplication is established and then we use multiplication to define division. [Day 1: Line 770] […] Yes [all division can be changed to multiplication] [Day 1: Line 765].

The entailments emphasis for the realizations of multiplication developed David's realizations of multiplication to include models as realizations:

> David: I feel like number line jumping, repeated addition, and counting are all kind of the same. One's a visual representation of the repeated addition [Day 3: Line 1330] […] I said repeated addition, number line jumping, and counting were all the same. [Day 3: Line 1340]

The blends of David's realizations for multiplication continued to develop his realization for multiplication to include area. The following mathematical utterances compilation provides evidence for this claim:

> David: I guess what I was thinking goes with that [computation] is the area of a box with the dimensions being the factors. So 2 by 3, which is multiplication [Day 1: Line 602] […] I could give a definition of multiplication, though, as

multiplication is the area of a rectangle whose dimensions are the factors of the multiplication. [Day 1: Line 629] […] I can define multiplication by saying it's the area of a rectangle. [Day 3: Line 1253]

David's realization for multiplication as area was problematic as the entailments emphasis of this realization troubled his comprehension of the realization:

David: But how does that show 2/3rds of a 1/5th? That doesn't show 2/3rds of a 5th. That shows 2/3rds and 1/5th and the overlap, which magically shows us the answer. [Day 8: Line 468] […] I agree that it works, but I feel like it is like telling a kid, "Here is another procedure with a visual. You do a fifth; you do 2/3rds, and then you—Hey, look at the overlap!" But, I don't know, look at the, like, magic: the overlap represents the answer. [Day 8: Line 487]

David's participation in entailments of multiplication developed realizations of volume as an entailment for the area realization:

David: The volume of a rectangular prism with the dimensions if there is three factors to be multiplied. [Day 3: Line 1256] […] Yes [three numbers multiplied together can be represented by volume] [Day 3: Line 1273] […] Yes [volume is a definition of multiplication]. [Day 3: Line 1276]

David's participation in entailments of multiplication developed realizations of the array model—a large focus of investigation during the realizations and entailments of the concept study. David's conception of an array developed significantly as the entailments allowed for considering the various units in the array model for fraction multiplication. This conception is similar to his realizations and blends for area as a realization of multiplication:

> David: I'm fine with that visual representation. I just think it is different from the array that was originally up there. [Day 4: Line 629] […] I think our definition of it [array] has changed. [Day 5: Line 62]

Using the process of entailments to refine, David was able to activate his knowledge of units and apply these actions to conceptualize the unit difference between whole number computations and fraction computations in the array model:

> David: Like, it's a completely different process—it's still using dots, so I don't know why. I feel like we're wanting to say, "Let's use our arrays," because they've seen arrays, but, you know what I mean: we're trying to connect it back to the whole number knowledge, but the process is different than when it's whole numbers. I don't know how much connection there is in between. [Day 5: Line 69] […] So when you have fractions each dot isn't a whole anymore. With whole number arrays each dot represents a whole. With fraction arrays that whole box is a whole. [Day 4: Line 993]

The entailments emphasis of the array model provided enough evidence for David to blend realizations of area and array for multiplication:

> David: They're not different [Day 7: Line 480] […] I said that on day two that they [array and area models] weren't much different. [Day 7: Line 494]

Further entailments of the array, area, and number line jumping realizations for multiplication enabled David categorize these realizations as uses rather than definitions for the operation:

David: Mark it off [remove arrays from the definition of multiplication]. [Day 4: Line 745] […] Then mark off area and number line jumping because those are both tools too. [Day 4: Line 754]

4.5.4.3 Realizations of Fraction Multiplication

The entailments emphasis of multiplication developed a realization of fraction multiplication as “parts of parts”. This was then blended with David’s realizations for multiplication as area:

David: I agree that it helps them come up with the denominator; I’m not disputing that, but what I’m saying is that [modeling parts of parts] doesn’t really represent the original problem. [Day 8: Line 475] […] Yeah [by drawing the line all the way down you do not model the actual task]. [Day : Line 478]

The array model was not a realization for fraction multiplication. The following mathematical utterance compilation provides evidence that David’s limited understanding of the array model is the cause of this:

David: Can’t be done [an array can't model $12 \times \frac{2}{3}$]. [Day 4: Line 591] […] You have to do something to your array [to model the computation]. [Day 4: Line 610] […] What about like 2/3rds times 11? You can’t show that in an array? [Day 4: Line 647] […] Because there’s a remainder [that is why you can't model 2/3rds times 11 with an array]. [Day 4: Line 649]

4.5.4.4 Student Cognition of Fraction Multiplication

David the individual knower is a unique entity, encapsulating his own unique understanding of the mathematics curriculum. Also, repetition and the transference of proper mathematical techniques are the most successful means for teaching mathematics:

> David: I feel like me, personally, I didn't need things repeated. Just solve once, take a snapshot and you're good to go. A lot of kids, that's not going to be the case. I think for some kids repetition, they need to do it. Some kids need to teach it, some kids. I mean, I don't completely buy the whole percentage thing that we see a lot of that you learn this percent of what you hear and this percent of what you see, like the scientific research behind all that. [Day 1: Line 555] […] I think we harp to them about multiplication being across [recognizing fraction multiplication in a word problem with students] [Day 4: Line 307].

The mathematical utterance compilation above is filled with both behaviorist and constructivist realizations for student cognition. The blend of these orientations to student cognition is similar to other embedded systems of David's M_4T knowledge that require realizations for cognition.

4.6 Individual Case 4: Charlotte

Charlotte teaches middle school mathematics at the suburban Indiana school used for the site of this research study. Charlotte is currently in her 12^{th} year of a middle school teaching career with both seventh and eighth grade level teaching experience.

4.6.1 Mathematical Objects of Fraction Multiplication

4.6.1.1 Orientation to Mathematics

Charlotte's realization for mathematics was a situation-specific blend between a formalist and hyper-formalist orientation:

> Charlotte: Well, I think that it depends on the level of math that you are dealing with. Like, when I think of math I don't think about the pondering, high-level math that only people that are in math classes use. I think of math as actually practical and usable [Day 2: Line 1002] […] I don't know when in life I would ever use the square root of -1. Maybe someone out there uses it, I don't, so. [Day 2: Line 926]

Charlotte's realization for mathematics allows for a differentiation between mathematics and middle school mathematics. The emphasis of entailments for realizations of mathematics further developed Charlotte's realizations by substructuring mathematics as "invented" or "discovered". The following mathematical utterances compilation provides evidence of this claim:

> Charlotte: Yes [math is not invented]. [Day 2: Line 1002] […] What does invented mean? Man-made? [Day 1: Line 1181] […] The invented thing. It's [math] not invented—with our definition of invented. [Day 1: Line 1196] […] I think it's [quantity] always there. We're just calling it … and we're putting a name to it [numbers]. [Day 1: Line 1297]

Charlotte's realizations for mathematics continued to develop between hyper-formalist orientations and much less sophisticated realizations when situating mathematics in the middle school classroom context:

> Charlotte: Yeah, I act like I do magic [with mathematics] all the time. [Day 8: Line 527] […] Sometimes I make stuff up, but I think it [the story of Pythagoras] is true. [Day 8: Line 994] […] No, I just sometimes make things up. [Day 8: Line 998] […] It's like Santa Claus [the story of Pythagoras] [Day 8: Line 1038].

The entailments emphasis was especially tedious for Charlotte, as she was continually prompted to substruct her professional knowledge of middle school mathematics and the discipline of mathematics. The following mathematical utterance compilation is evidence of her frustration:

> Charlotte: Why do we care? Why do we care if it is invented or discovered? [Day 2: Line 1473] […] I just don't know why this [invented or discovered debate] is important. [Day 2: Line 1477] […] Yeah, that's philosophy [the invented or discovered debate]; that's not math! [Day 2: Line 1110]

Charlotte's realizations for mathematics developed parallel with environments differentiating between mathematics and middle school mathematics. This was similar to both Faith and David's realizations for mathematics.

4.6.1.2 Historical Development of Fraction Multiplication

Charlotte's actions in the concept study sessions prevented the ability to make claims about her knowledge of the historical development of fraction multiplication.

4.6.1.3 Advanced and Horizonal Knowledge

Charlotte's realizations for advanced mathematical study was limited:

> Charlotte: So when I see these things like in college—because in high school I think it was just plug and chug—but in college I got to think about … somewhere in the world this is useful; somebody uses it, but it is just over my head, and I don't know. [Day 2: Line 1127]

No further evidence emerged to make claims about Charlotte's realizations for advanced mathematical study or her horizonal knowledge realizations.

4.6.2 Curricular Structures of Fraction Multiplication

4.6.2.1 CaP: Interference

Charlotte's realizations for the planned middle school curriculum designed to teach fraction multiplication aligned with the interference hypothesis. The following mathematical utterance compilation provides evidence for this claim:

> Charlotte: What if we said that it's more basic when you're talking about whole numbers? When you throw in integers, and decimals or fractions, it gets more complicated. But, could we agree that if we're just dealing with whole numbers [it is more basic]? [Day 1: Line 904] […] Yes [memorization of multiplication of whole numbers comes before fraction multiplication in curriculum]. [Day 2: Line 1383] […] Yes [the idea of fraction multiplication is complicated]. [Day 2: Line 1260] […] Do fractions complicate things? Yes! [Day 2: Line 1265]

This developed, as the entailments emphasis required the concept study participants to substruct their professional knowledge of multiplication as it operates on differing number sets:

> Charlotte: No [multiplication is not taught specific to different number types]. [Day 1: Line 941] […] Yes [whole number multiplication complicates understanding fraction multiplication]. [Day 2: Line 1260] […] No [whole number multiplication does not transfer to fraction multiplication]. [Day 2: Line 1267] […] So, what, so are you inferring that we should start with parts of numbers? [Day 2: Line 1278]

No further evidence emerged to make claims about Charlotte's developing conception of the realizations for the middle school mathematics curriculum and its alignment with the interference or reorganization hypotheses.

4.6.2.2 CaP: Remediation

Charlotte's realizations for the planned middle school mathematics curriculum as a mechanism for remediation impedes the conceptual teaching of topics:

> Charlotte: Some of them kinda remember how to do these things [fraction multiplication] by multiplying straight across; you know what I'm saying? So you have got to deal with that too. [Day 2: Line 1655] […] I've tried to explain the reasons behind and the whys—that stuff with them—but I usually get the: "Why do we have to think about this? We already know how to do it."—and then so they turn themselves off. [Day 2: Line 1669]

This notion develops through the landscapes emphasis, as the middle school curriculum is not the place for the conceptual teaching of fraction multiplication:

> Charlotte: No. I think where we are failing is the fact that we aren't the ones who should be teaching the conceptual part of this [fraction multiplication], but we are the ones who need to teach like, surface area, conceptually. But I, since it's the first time they've seen it [surface area]—that it makes sense for me to conceptually teach it. [Day 5: Line 558]

These realizations developed concurrently with David's realizations for the middle school curriculum as a place where students should have already automatized fraction multiplication.

4.6.2.3 <u>CaP: Textbook as Curriculum</u>

Charlotte's actions in the concept study sessions prevented the ability to make claims about her knowledge of the textbook as planned curriculum.

4.6.2.4 <u>CaL: Realizations of Student Development</u>

Charlotte, like David, Faith, and others, questioned the viability of the array model as a realization for multiplication suitable for use with students. The following mathematical utterances compilation provides evidence for this claim:

> Charlotte: Kids can do it [model fraction multiplication with an array] if you give them a 12 [initial unit size]. [Day 4: Line 1055] […] Could they do it if you gave them a—11 little pieces of things? [Day 4: Line 1057] […] Well, I don't think the kids are gonna [understand the model] when you have those 30 pieces and you

had to make them into five parts; that is where I think they would struggle: at least without me leading them into it. [Day 7: Line 1232]

Entailments of the array model developed this notion very little. To Charlotte, the area model is the most viable pedagogical realization for multiplication:

Charlotte: I would say that always [area is better than arrays for modeling fraction multiplication]. [Day 8: Line 762] […] Yes [area is better than arrays for modeling fraction multiplication]. [Day 8: Line 764]

This preference is linked to Charlotte's subjective understanding of the area and array models:

Charlotte: Yeah, and I get that. I'm just saying, I don't know, we spend all this time talking about arrays and we don't even use them. Are they going to be helpful? Did they use them in 6th grade? Did they use little dots to do multiplying? [Day 3: Line 101] […] Because they are just pictures of things and they are not helpful [for teaching]. [Day 5: Line 6]

As with other participants, development of M_4T directly linked to collectivity with students was difficult to track because of little alignment between the concept study and the curriculum being taught in the classroom.

4.6.2.5 CaL: Common Patterns of Interaction

Charlotte's actions in the concept study sessions prevented the ability to make claims about her knowledge of the common patterns of interaction between fraction multiplication curriculum and students.

4.6.3 Classroom Collectivity of Fraction Multiplication

4.6.3.1 Collective Mathematics with Students

It is difficult to make claims about development of her knowledge of collectivity with students. Charlotte's realization for her mathematical classroom is a behaviorist orientation that is an environment for providing rules, facts, and skills that can be accumulated efficiently:

> Charlotte: Because we made up rules to help them understand it. When it comes to integers I feel like: so I've tried teaching it with number lines, I've tried teaching it with integer boxes; like, I've tried all these. The best thing is [the teacher] coming up with a rule, making them [the students] memorize it and say, "Here ya go." [Day 3: Line 812] […] I make them chant it [rules for operations on integers]. [Day 3: Line 761]

Implicit in these mathematical utterances are the notion that Charlotte is able to build models of children's mathematical cognition. This developed, to be explicitly represented in her mathematical actions. The following compilation of mathematical utterances provides evidence that Charlotte's realizations for student cognition iterates between constructivist and behaviorist notions:

> Charlotte: Absolutely not. [Students do not think of mathematics philosophically]. [Day 1: Line 1233][…] Cause I feel like it is the only time that I actually get to see what they are thinking and the only time that they actually try to do anything. [Day 8: Line 661] […]I think it's different because they're [students] young and they just know how to repeat the things that they have been told. [Day 1: Line

> 1417] […] Shortcuts no, but teaching them sometimes step-step-step helps starting to get them to think—and helps them to think to the next step. [Day 2: Line 1504] […] That is what I was saying [procedural competence negates conceptual understanding]. [Day 2: Line 1678]

This evolution is similar to other participants that co-produced the concept study environments.

4.6.3.2 Participation of Collectivity in the Concept Study

4.6.3.2.1 Internal Diversity

Charlotte's presence in the collective added diversity for her realizations for mathematics and her position in relation to mathematical research and study:

> Charlotte: I'm not really a math nerd, so I'm not one to argue about it [one as a prime], but if we would like to talk about it more we could [with students]. And so, I tell them [students] that the math community cannot decide if one is prime or not prime. [Day 8: Line 991]

Charlotte was comfortable with her level of diversity and was free to share this diversity amongst the collective:

> Charlotte: Yes [if you don't think fractions are numbers you are a second grader]. [Day 2: Line 1318] […] No, no no… [no name calling]. [Day 2: Line 1322] […] I can see what Bailey is saying, and I see what everybody else is saying, and I'm on this side. But, it all depends on what you call a number, what you call a whole. [Day 1: Line 1385]

4.6.3.2.2 Internal Redundancy

Charlotte's realizations for models of multiplication and fraction multiplication contributed to high levels of redundancy with the other participants of the collective. Similar to other participants, Charlotte's contributions to the redundancy and diversity was directly dependent upon mathematical environment and the mathematical realizations being discussed.

4.6.3.2.3 Decentralized Control

Charlotte, similar to David, was key for the constant evolution of the locus of power in the collective. Charlotte was simultaneously the most forceful in shifting power away from me, but also consistently searched for my approval of collectively produced realizations. The following mathematical utterance compilation provides evidence for these claims:

> Charlotte: Tell me what a number is first [asking researcher], and I'll tell you if I think it's a fraction. [Day 2: Line 241] […] Is there something you want us to say? [Day 1: Line 898] […] What do others say? What else are you looking for? [Day 1: Line 905] […]What's number? I don't know, I don't know. I would not talk about it. I would just be like, "What's wrong with you?" [Day 2: Line 256]

4.6.3.2.4 Organized Randomness

Charlotte's actions served as boundaries for mathematical innovation while providing focus during the realizations, entailments, and blends emphases of the concept study. The following mathematical utterance compilation provides evidence for how

Charlotte actively pursued boundaries of definitions that would allow the collective to operate more efficiently on the mathematical tasks under examination:

> Charlotte: What's the definition of rational? [Day 1: Line 1117] […] What's your definition of a number? [Day 1: Line 1139] […] What does invented mean? Man made? [Day 1: Line 1181] […] I don't have a definition of a number so [Day 2: Line 246] […] I see a problem with that because we haven't agreed with what basic multiplication is. [Day 3: Line 1352] […] I'm stuck on the word "number"; that's where I'm stuck. When did we discuss multiplication as parts of parts? [Day 5: Line 81] […] People just make them up; we don't want to go back to definitions. What's a fraction? What's a definition? What does "investigate" mean? [Day 5: Line 395]

4.6.3.2.5 Neighbor Interactions

Charlotte's high level of participation within the collective mathematical environments often ensured a high level of neighbor interactions around the mathematical topic of fraction multiplication. Charlotte was able to add diversity, redundancy, and especially organized randomness through her interactions and collective activity with her peers.

4.6.4 Subjective Understanding of Fraction Multiplication

4.6.4.1 Realizations of Fraction

Charlotte's realizations for fraction emerged through her entailments of the realizations for number and operations:

> Charlotte: Well, what if you decide all fractions are division problems and you have to do division before you can do multiplication? [Day 1: Line 947] […] A fraction is a number and a fraction is a division. [Day 1: Line 952] […] Yeah [fractions can be both operations of multiplication and division]. [Day 1: Line 1021]

Charlotte continued to develop her realizations through the further entailments of her realizations for number:

> Charlotte: It's the same thing [decimals are the same as fractions]. [Day : Line 958] […] Well, they [decimals] can be written as a fraction; doesn't mean that it is a fraction. [Day 1: Line 1121] […] It's both [a fraction is both a division operation and a number]. [Day 1: Line 964]

Charlotte's realization for fraction continued to develop dynamically as she participated in the tedious entailments emphasis of number and operation. The following mathematical utterance compilation provides evidence of this high rate of volatility for Charlotte's realizations for fraction:

> Charlotte: I don't know any more [she has no definition for number]. [Day 1: Line 1155] […]What a number is I don't know; can I go last? [Day 5: Line 81] […] What if one is the only number? What if a number is only one, base one—just everything is one. [Day 2: Line 268] […] Okay, they invented the whole system around one and what we're doing around it, yes. We're deciding. Because we have different bases. [Day 2: Line 600] […] I agree, which is why I am back to my nothing and a coke can idea, like your two bases are either zero or one. [Day

2: Line 1309] […] No, I don’t want to do that [make numbers not include zero]. [Day 4: Line 135]

No further evidence emerged during the blends emphasis if Charlotte was able to find a coherence to her realizations for fraction.

4.6.4.2 Realizations of Multiplication

Charlotte’s realizations for multiplication emerged as group making or repeated addition:

> Charlotte: So I said—my first answer if you ask me—is repeated addition. […] Yes, that’s why I like my repeated addition for counting slash natural numbers.

This realization developed non-linearly through the entailments emphasis of realizations for multiplication on differing number sets. The following mathematical utterance compilation provides evidence for this claim:

> Charlotte: Now, if you’re multiplying by a fraction, I know that that is different. But, basic multiplication, I decided was repeated addition. [Day 1: Line 579] […] What fraction is a division problem? You’re still doing repeated addition. [Day 1: Line 952] […] No [whole number does not transfer to fraction multiplication]. [Day 1: Line 1268] […] I have nothing else; I’m sorry [nothing to add beyond repeated addition]. [Day 1: Line 655]

The tedious entailments emphasis of the realizations for multiplication developed Charlotte’s realizations to distinguish between what defines multiplication, what models multiplication, and what is a use of multiplication:

> Charlotte: Definition of multiplication? I'm going to go with hardly anything up there [on the list] is a definition. [Day 3: Line 1115] […] If we think that, then we think modeling is a definition of multiplication. What is multiplication? [Day 3: Line 1136] […] I think we can use modeling—the modeling manipulatives can be used to show a definition. [Day 3: Line 1118] […] I think all that's multiplication, all that's using multiplication. [Day 2: Line 1145] […] Well, area is on both sides as a definition and a use. [Day 3: Line 1124] […] So if it shows multiplication, it's showing a use or a definition? [Day 3: Line 1126] […] Is division basic multiplication? [Day 3: Line 1311] […] Is number line jumping modeling or basic multiplication or actual basic multiplication? [Day 3: Line 1418]

No further evidence emerged to better understand her distinguishing characteristic for the realizations of what defines multiplication, what models multiplication, and what is a use of multiplication.

4.6.4.3 Realizations of Fraction Multiplication

Charlotte's realizations for fraction multiplication are embedded with her realizations for unitizing and the array model for fraction multiplication:

> Charlotte: I had a thought about all of that stuff [modeling with an array]. My thought was that when I make my little circle or I make my little square I'm finding 2/3rds of one whole; when you are making your little dots you are finding 2/3rds of 12/12ths; so we aren't really showing $\frac{2}{3} \times \frac{3}{4}$. You are really showing $\frac{12}{12} \times \frac{2}{3} \times \frac{3}{4}$. [Day 7: Line 9] […] Yeah, just draw 12 dots and that is 12/12ths. […]Yeah, but you had to do something when you did two times three, you just

drew dots. Here [with fraction multiplication] you had to have a box or something. So if you have to do some kind of operation [to generate the unit]. [Day 4: Line 597]

Charlotte's realization for fraction multiplication did not develop further through her entailments of the array model. Eventually Charlotte was adamant about moving on from entailments for the array, though she was at ease with the relationship between the area model and fraction multiplication:

Charlotte: It always works [the overlapping area to find the answer to fraction multiplication].

4.6.4.4 Student Cognition of Fraction Multiplication

Charlotte's realizations for student cognition is that students are unique entities, encapsulating their own unique understanding of the mathematics curriculum:

Charlotte: Well, I see kids tune out, and I see kids get excited, I see both happen. But then there are also kids who just don't care, and they will tune out anything. [Day 2: Line 374]

This example suggests a behaviorist view of student cognition in which learning mathematics can be thought of as a characteristic of the individual learner. The following mathematical utterance compilation provides evidence that this realization remained static for part of the concept study environment:

Charlotte: Because there are kids that don't understand and can't visualize these things, but you know what they can do? They can multiple and divide, add and subtract whole numbers—so they can do the procedure [and still get the correct

> answer]. [Day 4: Line 464] […] They don't understand the why, but once they get the answer some of them say, "Okay I see why that answer makes sense. [Day 2: Line 1515] […] Like if we are talking about multiplying fractions and the shortcuts that we teach to them to get the answers then, no. That … those steps have nothing to do with [understanding the concept]. [Day 2: Line 1517] […]
>
> Correct [doing the procedure is devoid of understanding]. [Day 2: Line 1531]

Implicit in these statements are realizations for student cognition as a constructivist mentality for building models of student cognition and directing mathematical teaching based upon those models. This development is similar to other realizations of Charlotte that embed realizations for cognition and realizations for mathematics.

4.7 Individual Case 5: Evan

Evan teaches middle school mathematics at the suburban Indiana school used for the site of this research study. Evan is currently in his 11th year of a middle school teaching career with experience teaching all middle school levels.

4.7.1 Mathematical Objects of Fraction Multiplication

4.7.1.1 Orientation to Mathematics

Evan's initial realizations for mathematics was at the formalist Stage, where mathematics is regarded as the model for intelligent thought and reason:

> Evan: And I think we throw things in [to the curriculum] 'cause it's one … it's a logical process—with math it seems like it's a logical process to build up to the other stuff. [Day 1: Line 181]

Evan’s realizations for mathematics, similar to other participants, developed according to the complexity of the mathematics being co-created in the concept study environments. Evan’s realizations developed between formalist and pre-formalist conceptions of mathematics. The following mathematical utterance compilation provides evidence to support this claim:

> Evan: We’re not arming them to understand all the Cartesian stuff behind it [graphs] … but we’re [teachers] arming them to be able to read a graph because there’s graphs in newspapers, magazines, TV. [Day 4: Line 258] [...] They’re able to understand it and pull some information, from it like she said, and just see it in a different light. [Day 4: Line 300] […] Yes, it [mathematics] is both [applied science and intellectual pursuit]. [Day 6: Line 967]

The final utterance in the above compilation hints at a hyper-formalist realization for mathematics. Evan’s orientation to mathematics was thus similar to David’s and others, where a distinction is made between realizations of the mathematics that he teaches and realizations of the discipline of mathematics.

4.7.1.2 Historical Development of Fraction Multiplication

Evan’s actions in the concept study sessions prevented the ability to make claims about his knowledge of the historical development of fraction multiplication.

4.7.1.3 Advanced and Horizonal Knowledge

Evan’s realizations for advanced mathematical study and techniques were limited to anecdotal awareness of Euclidean and Non-Euclidean geometries as well as calculus. The following mathematical utterances provide evidence that these realizations are not utilized for his development of realizations for fraction multiplication nor for his teaching of middle school mathematics:

> Evan: Just assume I'm going to forget all that [Euclidean and Non-Euclidean Geometries]. [Day 6: Line 494] […] It seems like so long ago [calculus topics]. It's been so long since I've done calculus. [Day 6: Line 565]

4.7.2 Curriculum Structures of Fraction Multiplication

4.7.2.1 CaP: Interference

Evan's realizations for the planned middle school curriculum is constructed by tenets of interference hypothesis:

> Evan: Well I think we try to hit the basic ideas of multiplying now [whole number] and then you take … kind of what we said earlier, the rules of math and now we're going to apply it to fractions. [Day 1: Line 942] […] New … some of that stuff is conceptually new to them; you want to get the basics down [whole number] and get good at some of the basics the [Day 2: Line 352] […] Teaching the easier stuff … building blocks and up as we go along. [Day 2: Line 355]

Evan's realizations for planned middle school curriculum developed through the entailments emphasis of multiplication on whole and fractional quantities:

> Evan: Yes [whole number operations complicate the learning of fraction operations]. [Day 2: Line 355]

This development was non-linear, as later realizations returned to an interference hypothesis orientation to the planned middle school curriculum:

> Evan: I think that some kids … going off what she just did and what she said a while ago: that kids that truly understand the whole numbers get fractions when it is presented. I think that is part of the reason why kids don't get fractions is that they haven't gotten the whole number system thing down. We get kids that are coming up here in the middle school that still can't actually multiply whole numbers. [Day 2: Line 1277]

No further evidence emerged of development of David's realization for the planned middle school curriculum design.

4.7.2.2 CaP: Remediation

Evan's realizations for the planned middle school curriculum acknowledges that repetition is meant to remediate procedural fluency for fraction multiplication:

> Evan: So my thought … I'm kind of with him on that—by eighth grade, by golly, I don't even know if I could show you how it works anymore. I'm okay as long as I know the procedure. [Day 5: Line 453] […] You said it's about a finger-pointing cycle. I don't think it's so much that as times I feel like I'm getting pressure to cover everything. And I don't want to spend a week on something I think we could've covered in two days. So something else is either getting thrown out the window or something else is not going to get hit as hard as it should've been. And so things that should've been conceptually reached back in 4th or 5th grade [Day 5: Line 673] […] our thoughts would be we've got to move on because we have more to do. [Day 5: Line 673]

No further evidence emerged to make claims about development of this realization.

4.7.2.3 CaP: Textbook as Curriculum

The textbook is a source of curriculum in Evan's classroom:

> Evan: In our book or in our curriculum I guess [Day 3: Line 190] [...] We kinda pull decimals and fraction into one big grouping of rational numbers, so we mention it as multiplying rational numbers. [Day 6: Line 337] […] Rational numbers are in the beginning of the whole chapter dedicated to fractions and decimals. [Day 3: Line 346]

The textbook is a part of the curriculum that Evan is free to scrutinize and adapt for the teaching of middle school mathematics:

> Evan: The fractions and decimals almost can't be mentioned all together [pedagogically] as shown by the book. [Day 6: Line 337]

No further evidence emerged to make claims about the development of this realization for the planned curriculum in Evan's classroom.

4.7.2.4 Realizations and Student Development

Evan's developing subjective knowledge of arrays, similar to his co-participants, prompted him to conclude that they were not viable models for working with middle school students:

> Evan: I don't ever remember using them [arrays] myself so I mean [Day 5: Line 126] […] I don't hardly ever use them to show anyone [students] anything. [Day 5: Line 130] […] Well, when we did the array, we were marking the entire array not just … well, I kind of thought that the last time [using an array] when we

weren't marking everything off that that might be confusing for a kid to look at. [Day 8: Line 414]

This, unlike Evan's peers, developed significantly through Evan's participation in the entailments emphasis of fraction multiplication. For example, after the tedious entailments emphasis of the final concept study session Evan's realizations for the array model and student development was significantly different:

Evan: But I think taking them [students] through the arrays has its values. [Day 5: Line 465] […] I was … just think of all those kids [his students] from the last 10 years. I don't know … I just think some of them might respond to the dots. [Day 5: Line 467]

Evan's development towards the array model for multiplication was the most dramatic of the participants. The environments co-produced with the other participants only had this type of impact on Evan, despite these co-participants creating the environments that initiated this change for Evan.

4.7.2.5 CaL: Common Patterns of Interaction

It is difficult to do more than infer about Evan's understanding of the lived curriculum with students, without having the opportunity to observe his classroom environments with students directly. Fractions are not well regarded by the students who co-create his classrooms:

Evan: It's [fractions] the F word in math. [Day 3: Line 236]

This realization seems linked to Evan's realizations students' previous mathematics education prior to coming to the middle school level:

> Evan: I swear there's a conspiracy with some of our elementary teachers; they don't like fractions so they teach the kids that fraction bar means divide so put it as a decimal. Because it's so much better as a decimal than it is a fraction [sarcasm]. [Day 1: Line 967]

No further evidence emerged to make claims about the development of this realization for the lived curriculum in Evan's classroom.

4.7.3 Classroom Collectivity of Fraction Multiplication

4.7.3.1 Collective Mathematics with Students

Similar to the constraints for making claims about Evan's understanding of the lived curriculum with students, it is difficult to make more than inferences about the nature of Evan's M_4T of collectivity with students. Evan's realizations for mathematical collectivity with students orients mathematical activity as an individual activity where associations, skills, and components of skills are acquired:

> Evan: Where we have had to multiply and add the fractions in the same problem, it was more beneficial for them to reduce their product before adding so that they could make nicer simpler common or nicer common denominators from the product ... because without that, we had some kids just turn it off. [Day 7: Line 162]

This is a behaviorist conception of collectivity and student cognition of mathematics. Evan's classroom is designed to facilitate the repetitive practice of mathematical techniques. Implicit in this notion though is the understanding that models of students' cognition can be built, similar to Charlotte's blended notions of collectivity. Evan valued different types of mathematical knowledge, but privileged one over the other for the teaching of middle school mathematics. What mathematics was privileged as part of Evan's realizations for collectivity with students developed significantly from his participation in the concept study environments:

> Evan: Just kind of showing it [mathematics] a little more conceptually, you know, like where things came from, how things were derived, and not just saying: "Here is the formula; just plug it in." Um … you know, beyond just showing them the procedure. [Day 8: Line 107]

No further evidence emerged to make claims about this development for Evan's realizations of collective with students.

4.7.3.2 Participation of Collectivity in the Concept Study

4.7.3.2.1 Internal Diversity

Evan's presence in the collective added to the intelligence of the collective. His coherence-maintaining mechanism for the system was essential as he often pursued propositions by others to make sure that they were mathematically viable:

> Evan: I'd need to check and see if it works [non-traditional algorithm]. [Day 7: Line 567]

This thoughtful pursuance of diversity often allowed the concept study collective time to fully consider the propositions brought forth to the mathematical environments.

4.7.3.2.2 Internal Redundancy

Evan's participation in the diverse mathematical environments of the collective around the topic of fraction multiplication also provided key redundancies for the coherence of the system. He often brought the group back together when the diversity was to strong for internal coherence:

> Evan: And, so I, I think I'm kinda torn. I'm kinda like David and Charlotte where I think if it is unique enough and it might stand out enough where I would be like, "Ooh," and I'm going to work it out and make sure it works. [Day 8: Line 707]

4.7.3.2.3 Decentralized Control

Evan was integral in the constant evolution of the locus of power in the collective. His contributions were different from others as he did not commandeer power, nor did he distribute power to others. This sort of passive participation does seem to evolve the aggregate locus of power.

4.7.3.2.4 Organized Randomness

Evan was a key component of the collective for setting boundaries for the collective and then leaving the innovation to the others in the co-created mathematical environments. For example, the concept study environment was interrogating the use of the word "representation" as a synonym for "variable," during a blends emphases for realizations of number and fraction. Evan aided this discussion by providing a boundary for the word usage and then freed up the innovation that could exist within that boundary:

> Evan: I don't mean whole new, but ya know, just look in the dictionary. It's got one, two, three, four, five different meanings. It's because words stand for different things. [Day 2: Line 550]

Evan's realizations set the boundaries that then subsequently allowed for innovation around the topic of discussion. This boundary-setting was a frequent occurrence for Evan as a coherence maintaining mechanism for the unique collective.

4.7.3.2.5 Neighbor Interactions

Evan's low participation warrants the claim that he was not always a full participant in the collision of ideas within the collective. This should not be construed as downgrading his importance to the collective dynamic, as it is true that when he did contribute to the collective his actions were thoughtful and influential.

4.7.4 Subjective Understanding of Fraction Multiplication

4.7.4.1 Realizations of Fractions

Evan's realizations for fraction emerged during the entailments emphases for number. The following mathematical utterance compilation provides evidence for the development of this entailments of number for Evan:

> Evan: Yeah, like she said, when a kid said it's a symbol that represents a measured amount for a quantity [Day 2: Line 289] […] I mean beyond that I don't know what else to give it [definition of number]. [Day 2: Line 295] […] Zero is a number [Day 2: Line 192] […] They [variables] represent a value, so yeah [they are a number]. [Day 2: Line 502]

After the entailments emphasis for number, Evan's realizations for fraction emerged as an "operation" and as a "part of a whole":

> Evan: Yeah, it's both [fraction is a number and a division question]. [Day 1: Line 953] […] It's part of a whole [what a fraction is]. [Day 1: Line 955] […] We

teach it as a number, like it's two thirds, it's two out of … if we had a group of three it's two out of three. [Day 1: Line 967] […] It [fractions] is a quantity; you can put it on a number line [Day 1: Line 967] […] It [fractions] could have variables. [Day 2: Line 502]

No further evidence emerged to discuss the development of Evan's realizations for fraction.

4.7.4.2 Realizations of Multiplication

Evan's realizations of multiplication emerged as repeated:

Evan: I mean, you could use repeated addition. The arithmetic is for patterns and shows multiplication. [Day 3: Line 1205]

The entailments emphasis for the realization of repeated addition developed Evan's realizations for multiplication significantly. The following mathematical utterance compilation provides evidence to support this claim:

Evan: Not when it's fractions, it's not [response to repeated addition making larger]. [Day 1: Line 690] […] Sidetrack, but look at that first one we put up there, it depends on how you want to look at the three; yes, it gets smaller. But if you're looking at the 1/2, yes it gets bigger. [Day 1: Line 694] […] No, because they all should just keep getting bigger [response to the question whether a user chooses which number gets bigger in whole number multiplication]. [Day 1: Line 699] […] That's what I was thinking. You're something smaller than one being multiplied by something smaller than one; you're going to have it come out even smaller than it already is. [Day 5: Line 218]

Similar to other participants, the entailments of the operation of multiplication provided development in Evan's realization for multiplication. Evan was able to distinguish between tools, uses, and definitions of multiplication as a part of his realizations for the operation:

> Evan: Yeah [tools should be defined as uses of multiplication]. [Day 3: Line 1244] […] Is that just a technique of how to find it [area], or is it just actually defining multiplication? [Day 3: Line 1143] […] I don't think a formula is defining [multiplication]. I think they just use it. [Day 4: Line 275] […] I'm thinking that's why we brought up counting [models involve the counting of parts to evaluate the solution]. [Day 2: Line 578]

No further evidence emerged to make claims about the development of Evan's realizations for the operation of multiplication.

4.7.4.3 Realizations of Fraction Multiplication

Evidence emerged that differentiated Evan's realizations for fraction multiplication from others in the collective. Part of Evan's realization for fraction multiplication is related to probability tasks:

> Evan: $\frac{2}{5} \times \frac{3}{7}$ could mean in one drawer that you have two black socks out of five black socks and in the other drawer you have three black socks out of seven black socks—so you have 6/35 probability of getting matching socks. [Day 7: Line 646]

Correctness and inclusiveness are not of importance in the process of realizations and entailments co-created by the concept study collective. Interestingly, none of Evan's peers challenged the mathematical validity of this realization for fraction multiplication.

The array model was a realization for fraction multiplication that emerged for Evan in the blends emphasis of the concept study environments:

> Evan: I mean, if you were worried about making an array for that [fraction multiplication task]: the bottom tells you how many rows and columns you need, the top tells you how many you should have at the end. Count the dots. [Day 4: Line 949] […] Like, if the first fraction tells me columns and the second fraction is telling me rows … so I would draw three columns across and four rows down and fill in all my rows or my dots and then the two tells me ... okay eliminate it down to two columns. [Day 4: Line 952]

Evan's realization for arrays and fraction multiplication developed significantly while participating in the entailments emphasis of the array realization. The following compilation of mathematical utterances provides evidence of this development:

> Evan: So we have to start always with the big fraction? [Day 4: Line 1096] […] No, because I can't use nine [response to peer proposition that multiples of three are valid unit sizes to model the task]. [Day 4: Line 952] […] I tried, but I couldn't come up with it [another unit number other than 12]. [Day 4: Line 828] […] So I went three across … so I said, "Here's my two of three"—so it happened to be there's four in there. I'm just going to circle three of them. [Day 4: Line 818 […] Right, we are kinda doing the common denominator thing of three times four, which will make the common one of twelfths; but I guess my wheels are still spinning on why the numerators give you the number of pieces, I guess. [Day 7: Line 543] […] Does it have something to do with prime factorization? [Day 7: Line 697] […] I'm stuck on this [hands up in the air]. [Day 7: Line 722]

Evan privileged the area model over the array model for fraction multiplication. Little evidence emerged to explain the preference. No further evidence emerged of development of Evan's realizations for fraction multiplication.

4.7.4.4 Student Cognition of Fraction Multiplication

Evan is able to build models of his students' cognition, a clearly constructivist realization of student cognition:

> Evan: They could've … on one of our questions today I suppose it said 75% of the box was filled with this … but our kids are going to look at 75% and they're going to think not 3/4ths; they're going to think point 75. [Day 5: Line 506]

Similar to other realizations for cognition, Evan's realization of student cognition for fraction multiplication developed iteratively between a constructivist and a behaviorist mentality:

> Evan: Honestly, as long as they [students] know how to punch it [fraction numbers] into a calculator they don't care. [Day 5: Line 601] […] Shortcut [students prefer shortcut to understanding what they are doing]. [Day 5: Line 1067] […] Simpler common or nicer common denominators from the product ... because we had some kids just turn it off [if concept is too difficult]. [Day 7: Line 130]

Similar to previous findings, Evan seems to reduce the complexity of his realizations for cognition and mathematics when moving between discussions of mathematics and mathematics in the middle school classroom.

4.8 Individual Case 6: Bailey

Currently in her eighth year of her middle school mathematics teaching career with experience at all middle school grade levels, Bailey teaches in the middle school mathematics department of the suburban Indiana school used for the site of this research study.

4.8.1 Mathematical Objects of Fraction Multiplication

4.8.1.1 Orientation to Mathematics

Bailey's realizations for mathematics emerged in the pre-formalist stage where mathematics is regarded to be outside of the mathematical knower:

> Bailey: And if it's created, then I agree that it's invented. Inside of me I can't agree though …[Day 1: Line 1278]

Bailey's orientation to mathematics was significantly different from the others who co-created the environments of the concept study with her. She seemed intimately connected to the mathematical content. Her realizations for mathematics developed significantly through the entailments emphasis of the realizations for mathematics.

> Bailey: Oh, I like that statement: that's like gold right there [math can be an intellectual pursuit rather than a description of the physical world] [Day 2: Line 847] […] Because you were talking about intellectual versus physical. [Day 2: Line 854] […] That's kind of how I saw it: intellectual world versus physical world, but our physical world is translated through our intellect. [Day 2: Line 856]

The level of sophistication that can be categorized in the above statements is as dynamic as the statements themselves. Bailey's realizations for mathematics developed quickly between complex orientations and much less sophisticated realizations seemingly without provocation.

4.8.1.2 Historical Development of Fraction Multiplication

Bailey's actions in the concept study sessions prevented the ability to make claims about her knowledge of the historical development of fraction multiplication.

4.8.1.3 Advanced and Horizonal Knowledge

Bailey's realizations for advanced mathematical study emerged referencing levels such as linear algebra, chaos theory, calculus, and statistical analysis. The following mathematical utterance compilation provides evidence for these advanced topic realizations:

> Bailey: That's a linear algebra thing; I do know that [arrays]. [Day 1: Line 588] […] Because it's chaos theory. It's chaotic. [Day 4: Line 4] […] I was stuck on the whole statistic thing; I don't trust statistics anyway. Because … I mean, you can find any statistic to back you up. [Day 4: Line 282] […] It is calculus. [Day 1: Line 990]

Bailey used her calculus training, at least arbitrarily, to help her investigate the pedagogical problem-solving of having a zero in the denominator of a fraction:

> Bailey: Well, I just remember learning in calculus you can't do a derivative of zero divided by zero or infinity divided by infinity; it's the same idea. [Day 2: Line 814] […] Does $\frac{1}{x}$ as x approaches infinity equal zero? [Day 1: Line 988]

Bailey's horizonal knowledge was activated during her realization for fractions that involved her advanced mathematical training in calculus. The following mathematical utterance compilation provides evidence to support this claim:

> Bailey: I feel like in 7th grade that this is the end for them, but we know that where, you know, like calculus and that sort of thing. Since we know where the end is we can kind of bring them along and say that we know that: "This isn't the end for you, but we know what the end looks like." We know here's the next step to review. Maybe. That's why we [teachers] have to learn these upper levels: because that is the end of the mathematics for them. I mean some of these regular 7th graders are never going to see Calculus four, but I could make some connection. [Day 2: Line 1757] […] And I feel like if we can do the upper levels, then we're challenging ourselves on that level so that we should be able to do, ya know, the 7th through 12th grade. Now, I feel like I can't teach that 12th grade math trigonometry and calculus 'cause I'm not at that level and some of these students will jump way above me. Um … and, I feel like they are better at math and they are able to think at that level and I'm not. So I wouldn't teach 12th grade probably. [Day 1: Line 401]

These mathematical actions were significant, as Bailey was the only participant to articulate sentiments of the interconnectivity of mathematical study and the direct usefulness of advanced mathematical study for her work as a middle school teacher.

4.8.2 Curriculum Structures of Fraction Multiplication

4.8.2.1 CaP: Interference or Reorganization

Bailey's actions in the concept study sessions prevented the ability to make claims about her knowledge of the planned middle school curriculum and its design.

4.8.2.2 CaP: Remediation

Bailey's actions in the concept study sessions prevented the ability to make claims about her knowledge of the planned middle school curriculum and its place in terms of remediation.

4.8.2.3 CaP: Textbook as Curriculum

Bailey's actions in the concept study sessions prevented the ability to make claims about her knowledge of the historical development of fraction multiplication.

4.8.2.4 CaL: Realizations and Student Development

Bailey's actions in the concept study sessions prevented the ability to make claims about her knowledge of the appropriateness of realizations for fraction multiplication and for student development.

4.8.2.5 CaL: Common Patterns of Interaction

Students react negatively in Bailey's classroom environments to operations on fractions, or any other substantial shift in the curriculum that is more computationally difficult than the operations on whole numbers:

> Bailey: And I think that sometimes the students are like, "Why can't we just do adding and subtracting and multiplying and dividing? 'Cause that's what I

> [students] know how to do."… and it's when we [teachers] … when I'm challenging a student to do that harder thing that's new to them and they're not used to it yet, that's when they're [students] like, "Why are we having to do this?" [Day 1: Line 134]

Bailey's actions also warrant the claim that mathematical study is not highly regarded by her students:

> Bailey: Yeah, like I was playing a game from there [mathisfun.com] and the kids were like, "Math is not fun." [Day 8: Line 939]

No further evidence emerged to make claims about the common interaction patterns around realizations for fraction multiplication in Bailey's classroom.

4.8.3 Classroom Collectivity of Fraction Multiplication

4.8.3.1 Collective Mathematics with Students

Without observing Bailey's actions in the co-created mathematical environments of her classroom with students, it is difficult to do more than infer about her collective mathematical knowledge of these environments. She orients mathematical learning as an individual activity in which associations, skills, and components of skills are acquired and in direct relation to the teacher's ability to transfer knowledge:

> Bailey: Especially trying to learn math by yourself. I was in high school and my teacher was horrible, and I can remember crying over my textbooks trying to learn math. I think that's probably why I wasn't as strong. The teacher didn't even help me. All I knew is what I learned and what I could figure out because trying to learn math is different, trying to teach yourself in math is different than trying to

> teach yourself English or history or science, which is a little bit like math. You have to try and figure out what did they do from this step to this step, and you have to read between the lines a lot more in math than you have to do in language or history. [Day 1: Line 452]

These realizations suggest a behaviorist realization for collectivity with students in Bailey's classroom. This conception developed as Bailey related that she piloted the collectivity from the concept study environments in her classroom with students:

> Bailey: That kinda thing, you know … so I or it didn't take long, and it kinda took it [conversation about fractions], and then I took it and I ran with it in two different directions at the same time, based on what the students were giving me—and I was like, "Okay, this is how we are going to do it." So it was kinda cool to, you know, think that way. [Day 8: Line 123]

This development represents a realization closer to an enactivist realization for collectivity with students. This continued to develop, as later Bailey continued to represent her class as a collective cognizing agents:

> Bailey: Yeah, it depends on kind of how they go about doing things ... I mean each class is different. I can think of several examples recently where one class can take me in a completely different direction and I'm thinking in my head, "Man I kinda want to reteach that other class because I like how this class is thinking." [Day 7: Line 190]

Bailey's perspective of the classroom as a single collective learner developed to include subsystems of collective learners based on her models for their mathematical abilities. This perspective is a blend of constructivism and enactivism, similar to the multi-leveled blend inherent in the M_4T model. However, despite these complex conceptions of collectivity in the classroom Bailey's realizations for students learning mathematics seemed to always return to a transitional-type environments where students should learn through repetition.

4.8.3.2 Participation of Collectivity in the Concept Study

4.8.3.2.1 Internal Diversity

Bailey's was in direct opposition to other participants and was even subjected to ridicule for her realizations of number and fraction. Her conception of fractions as division rather than fractions as number brought significant diversity to many of the concept study environments:

> Bailey: I agree with it, but I don't. There's still ...when we get into the debate of fractions that doesn't fit there. I think of it as the integers system. You know, when you think of your integers, your integers are all whole values. So, yes, I can agree with those words describing numbers, but I still [not with fractions] …[Day 2: Line 390]

Bailey's mathematical knowledge added diversity and intelligence to the group. At times, her concept-specific high levels of diversity prevented the collaborative coaction of the collective and the emergence of the collective learner.

4.8.3.2.2 Internal Redundancy

Redundancies emerged through Bailey's participation in the diverse mathematical environments of the collective around the topic of fraction multiplication. For example, she was able to internalize others' conceptions during investigations of pedagogical problem-solving and to agree with them:

> Bailey: Because it is the unit, I think, in our idea of factor, 1 is the unit—so if we throw the possibility of fractions out there your unit could be anything. But if we establish the unit as one, then a factor, a prime number, one is the unit itself. It's the definition of … it's in the definition, so it can't be defined by it. [Day 8: Line 964] […] Yes, according to that definition. [Day 8: Line 960]

This statement is notable, as embedded systems that often add continuous diversity can also aid in the coherence maintenance by occasionally providing needed redundancy.

4.8.3.2.3 Decentralized Control

Bailey was integral in the constant evolution of the locus of power in the collective. At times she, like other participants, shifted the power to me, preventing the necessary pre-condition for complexity:

> Bailey: Dr. Math hasn't told us [what a number is]. [Day 4: Line 103]

Her active participation in the decentralized control pre-condition developed though, as at other times she shifted the locus of power to herself by directly questioning the viability of the actions of others:

> Bailey: You're gonna use the word "multiply" in your definition of multiplication? [Day 3: Line 1259]

Also, Bailey was equally as likely to shift the power to others through question-prompting when she was unable to find viable action on her own:

> Bailey: How would you represent 11 as an array? Would you use a five and six?

The locus of power seemed to shift when the redundancies and diversities were in high volume for Bailey. This is of interest as her flexibility for shifting power potentially aided in her contributions to coherence mechanism for complexity of the collective.

4.8.3.2.4 Organized Randomness

Bailey's engagement in the emphasis of pedagogical problem solving is an example of how she crafted boundaries within which innovation could occur:

> Bailey: Because it is the unit, I think. In our idea of factor, one is the unit so if we throw the possibility of fractions out there, your unit could be anything. But if we establish the unit as one then a factor, a prime number one is the unit itself—it's the definition of … it's in the definition so it can't be defined by it. [Day 8: Line 964]

4.8.3.2.5 Neighbor Interactions

Bailey was also integral in the collision of ideas. Her importance is evident in the previously established embedded pre-conditions for complexity. She was able to add diversity, redundancy, and organized randomness through her interactions and collective activity with her peers as the interpreted realization, entailments, and blends of fraction multiplication.

4.8.4 Subjective Understanding of Fraction Multiplication

4.8.4.1 Realizations of Fractions

Bailey's participation in the entailments emphasis for the realization of number was where her realization of fractions first emerged. The following mathematical utterance compilation provides evidence of her realizations for number:

> Bailey: It gives us a number value or amount … That's what I'm saying. I'm saying the whole numbers are the numbers; that's what I see it as. [Day 1: Line 1374] […] I explained it to my students yesterday about we have all of the numbers, debatably including fractions, that go closer and closer and closer to zero … we never, we never … well, we approach zero, but we never actually obtain zero. So I explain to them that it is the transition. [Day 2: Line 182] […] I don't know if it [zero] is a number necessarily, but I say that it is the transition between positive and negative numbers. [Day 2: Line 184] […] But zero has its own rules where all the other numbers have their rules. [Day 2: Line 884]

The entailments for realizations of number was tedious for Bailey, resulting in frustration and considerable development for her realization of number:

> Bailey: I don't know what a number is! ... Do we know what a number is yet?

Once a realization of number was rendered viable, the blends emphasis rendered her realization for fraction as problematic:

> Bailey: I agree with it, but I don't. [Day 2: Line 390] […] There's still ... when we get into the debate of fractions that doesn't fit there. I think of it as the integers system; you know, when you think of your integers, your integers are all whole values. So, yes, I can agree with those words describing numbers, but I still [don't know about fractions]. [Day 2: Line 397] […]That's a fraction … a part of the whole. [Day 8: Line 626] […] A decimal is a fraction. [Day 1: Line 960] […] A fraction is a ratio of two integers. [Day 1: Line 966]

Further blends emphasis for number and fraction developed Bailey's realizations for both:

> Bailey: ’Cause fraction is division because there is division in the fraction. That line [the fraction bar] is division. [Day 1: Line 949] […] They are a division problem. Because it’s a ratio of integers. That’s what the definition of a fraction is. [Day 1: Line 957] […] It’s a division problem. It’s always been a division problem because we use division to get rid of fractions. [Day 1: Line 1136] […] I don’t think fractions are numbers. [Day 1: Line 1073] […] You just said that you have to do something to find the half. Meaning that half is not a number. [Day 2: Line 1291] […] No, it’s not a number. Why? If you have to do something to find a half, then it’s not a number. [Day 2: Line 1295]

Bailey’s realizations for fraction remained relatively consistent, while her realization of number and the blend with the realization of fraction was highly dynamic throughout the concept study.

4.8.4.2 Realizations of Multiplication

Bailey’s realization for multiplication involved “grouping”, “repeated addition”, “of,” and the “distributive property”:

> Bailey: I think of grouping. [Day 1: Line 577] […] So counting and repeated addition. [Day 1: Line 638] […] Multiplication of two digits is FOIL. [Day 1: Line 895] […] Yes [FOIL is the distributive property]. [Day 1: Line 897] […]
>
> Groups of? It [the word “of”] translates to multiplication. [Day 1: Line 726]

Bailey’s participation in the entailments emphasis for realizations of multiplication developed her realizations of multiplication to distinguish between what defined multiplication and the uses of multiplication:

> Bailey: We use multiplication in area. I don't think that's what multiplication actually is. I think we use it in area so we associate them. But I don't think that's what multiplication is. I think that multiplication is the first two [on the list].

Through the blends emphasis for realizations of multiplication, Bailey was able to collapse all realizations for multiplication into "repeated addition" or "grouping":

> Bailey: I personally think that everything [all realizations of multiplication] can be put into repeated addition and grouping. [Day 1: Line 716]

No further evidence of development emerged of Bailey's realization of multiplication.

4.8.4.3 Realizations of Fraction Multiplication

Bailey's realizations for multiplication as repeated addition emerged with her realizations of fraction multiplication:

> Bailey: Fractions can be repeated addition, somehow. [Day 1: Line 663] […] Parts of parts, so you are adding parts. [Day 1: Line 665] […] I still think it's repeated addition. [Day 1: Line 931] […] Division is multiplying by a fraction. [Day 1: Line 767 […] Division is just multiplication of a fraction. [Day 1: Line 770] […] Yeah [you are dividing first and then adding up the parts]. [Day 1: Line 668]

This developed during a blends emphasis for fraction multiplication. Bailey defended her realization of fraction multiplication as getting smaller by blending her realizations of fraction multiplication as division:

> Bailey: Yeah, there's more division because there's two division problems in that one, 2/3rds by 3/4ths; there's two divisions and there's only one multiplication.

> So, division outranks the multiplication. [Day 5: Line 202] […] Yeah [the computation isn't really a multiplication question, it is a whole bunch of divisions with one multiplication computation]. [Day 5: Line 202] […] Fraction multiplication is two divisions with one multiplication. [Day 1: Line 638]

Bailey's realizations for fraction multiplication included the area and array model. Bailey had difficulty finding viable action with the area model during blending emphasis of realization for fraction multiplication:

> Bailey: I was never presented with this [arrays]. First time I ever saw this [arrays] was when I was taking a masters class like three years ago. That was the first time that I ever saw fraction multiplication presented pictorially [Day 4: Line 653] […] How would you represent 11 as an array? Would you do five and six? […] Because 11 is a prime number. [Day 4: Line 650] […] No it's not 11 [the unit required]. I'm just saying … could I do a pyramid as an array? [Day 4: Line 660] […] If I divide the dots I can [model 11 with an array]. [Day 4: Line 699] […] What is the definition of a dot? [in an array]. [Day 4: Line 704] […] You have to pick your arrays [unit] strategically apparently. [Day 4: Line 834] […] We can't divide up dots, so we have to make sure we have enough dots. [Day 5: Line 415] […] I mean I can do it with 35 dots, but that's because I know five times seven is 35. [Day 4: Line 1064]

The area model as a realization for fraction multiplication emerged as a much more viable realization than the array model:

> Bailey: When I first started doing these, that is how I did it. I did vertically one fraction and then horizontally the other fraction. [Day 7: Line 490] […] When I

> first started figuring out multiplication of fractions that is how I did it. [Day 7: Line 499] […] Vertical is one fraction of the whole, and then horizontal was the other fraction, and then you have the overlap. I then started focusing on the overlap. [Day 7: Line 501] […] That is kinda how I see it … yeah, because a fifth of the whole and 2/3rds of the whole and then fifth done horizontally and 2/3rds done vertically. [Day 7: Line 484]

Entailments of the realization of the area model proved difficult and tedious for Bailey as she was asked to substruct her understanding of the overlap as part of the realization:

> Bailey: You're overlapping, you're grouping the areas … they are overlapping right here. They are overlapping so they are grouped together. [Day 7: Line 1067] […] Hmmm yeah [agreeing that the overlap of the area model is the same as the Venn diagram]. [Day 7: Line 1064] […] No, if I didn't see, like, that this was cut into four pieces and I was selecting three out of those four. I mean, that is really what I'm doing over here—without the overlap I'm just, you know, if this is cut into four pieces I'm picking three out of those four and then I'm going to duplicate it. That is easier for me to see than to look at the whole and then to look at just that part. [Day 7: Line 1067] […] If I would have seen this for the first time and I didn't see the fraction that you guys were talking about, I would have been lost. [Day 7: Line 1094]

No further evidence emerged of development of Bailey's realizations for fraction multiplication.

4.8.4.4 Student Cognition of Fraction Multiplication

To Bailey individual knowers are unique entities, encapsulating their own unique understandings of the mathematics curriculum:

> Bailey: I'm gonna object to this … object to learning this because it's not easy, like, adding and subtracting, multiplying and dividing. I just wanna stay where I'm at. And we're challenging them to go that one step further to learn more to challenge their minds. [Day 3: Line 725]

This realization for student cognition is a behaviorist orientation, where learning mathematics is a characteristic of the learner. This develops to a constructivist orientation where Bailey's actions warrant the claim that she can build models of her students' cognition. These models, to Bailey, are then used to adapt her teaching of the mathematical concepts to meet the needs of her students:

> Bailey: Depending on their level—I mean the lower level kids are just gonna be out there. You know, they're gonna listen to that high level kid because it's the high level kids and they're like … well, the kids all know in each class who gets the good grades and who understands the material. They all know that. They are gonna perk up a little bit, but when it goes over their head they just kinda zone out so they might not be able to participate, I think, in that conversation [defining what is number]. [Day 2: Line 364] […] So I've actually taught it that way before: it only makes sense to about 3/4 of the class. 'Cause then I teach the shortcut and they're like, "Aahh yes!" [Day 3: Line 1222] […] Girls do "is over of"; The guys do equations. [Day 1: Line 737]

The blend of the constructivist and behaviorist realizations for student cognition continued to develop throughout the remainder of the concept study sessions. This was similar to the other participants of the concept study.

CHAPTER 5. COLLECTIVE CASE

The six emphases of a concept study—here a vehicle for activating teacher's potentially tacit or unconscious M_4T knowledge of fraction multiplication— (Davis, 2011, 2012; Davis & Renert, 2009, 2013, 2014) emerged by design and by participant action. What follows is use of the emphases as a frame for presenting evidence to support the claim that M_4T knowledge of fraction multiplication is a distributed collective knowledge. This is followed by a discussion of evidence to support the claim that M_4T knowledge of fraction multiplication is dually collective.

5.1 Emphasis 1: Realizations

The initial realizations emphasis was described to explicitly focus the collective's effort on the generation of realizations for multiplication. Davis (2012) stated that well-rehearsed or automated realizations can often impede other interpretive potentialities. This was consistent with the realization for multiplication as "repeated addition" in this concept study. Interestingly, a realization chart exercise multiplication sparked the generation of realizations for both number and fraction by the collective. These collective moments occurred several times during the spontaneous emergence of the collective learner. The following realizations chart for multiplication, fraction, and number in

Figure 4.1 emerged after considerable debate. I chose the model in Figure 4.1 to illustrate the collective's portrayal of realizations for the three concepts. As evidenced in the actions of the collective, the realizations for number and fraction have implications and are embedded with the realizations for multiplication. This list is similar to lists generated in other concept study literature (Davis, 2012; Davis & Renert, 2009, 2014; Davis & Simmt, 2006).

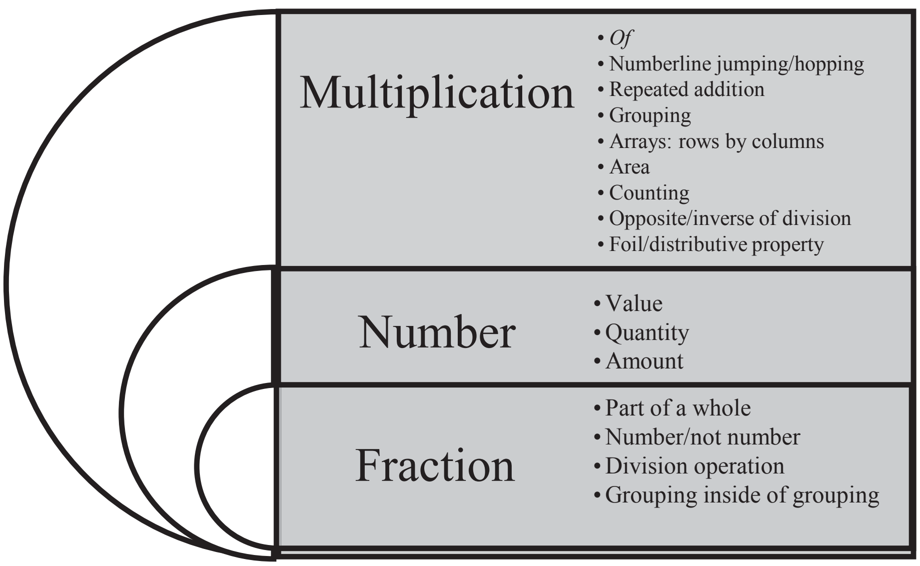

Figure 5.1 Teacher Realizations of the Concepts of Multiplication, Number, and Fraction.

The collective persisted in spontaneous episodes of blending realizations for multiplication, looking for a coherent blend of all realizations. After exhausting their inputs, the participants began to ask questions like "What do others say?" and "Is there something missing from the list that you want us to say?" It should be noted that this mentality hampered the necessary pre-condition of decentralized power, often acting as a mechanism that dissolved the coherence of the collective learner.

5.2 Emphasis 2: Landscapes

The landscapes emphasis emerged as a planned intervention for the collective during the third concept study session. The collective was asked to interrogate their middle school curriculum and build a landscape for the realizations of multiplication that were in the middle school mathematics curriculum they teach. Figure 4.2 below represents the landscape for multiplication created by the collective as part of this emphasis activity. There is no specific order to this visual, it is meant to be a broad landscape, for the concept of multiplication that emerged from the unique collective of this concept study.

Tools and Uses of Multiplication: Realizations of Multiplication on Relevant Number Sets	Patterns	Integer Rules	Properties *of*	Defining Formulas		
	Order of Operations	Solving Equations	Prime Factorizations	Recipricols		
	Rational Number Constructs	Decimals	Fractions	Percent Change		
	Exponents	Square Roots	Scientific Notation	Equivalent Fractions		
	Pythagorean Theorem	Graphing and Slope	Rates	Ratios		
	Measurement Conversions	Proportions	Similar Figures	Scaling		
	Area	Volume	Mark-up & Discount	Polynomials	Foil	Factoring
	Probability	Permutation/Combination	Factorials	Perpendicular Lines		
	Systems of Equations	Exponential Growth/Decay	Completing the Square	Rational Expressions		
	Greatest and Least Common Factors	Measurements of Central Tendancy				

Figure 5.2 Teacher Landscapes for Tools and Uses of Multiplication

The landscape-building for the realizations of multiplication and the spontaneous emergence of the collective learner produced a new collective realizations chart for multiplication that appears below in Figure 5.3. There was no hierarchy to the chart; rather it is simply a listing of collectively agreed-upon realizations.

Defining *Basic* Multiplic-ation	Repeated Addition
	Grouping/Arrays
	Area
	Of
	Number Line Jumping
	Distributive Property
	Division
	Parts-of-Parts

Figure 5.3 Refined Realizations Chart for Multiplication

5.3 Emphasis 3: Entailments

The emphasis on entailments first emerged as a spontaneous co-production of the collective during the second concept study session. The emergence of the collective learner paralleled the emergence of the entailments emphasis for the realizations of "fractions as numbers" and "fractions as an operation of division." The entailments chart in Figure 5.4 below diagrams the substructuring of the two separate realizations for fractions by both individuals and the collective learner. There is no sequential movement in Figure 5.4, but rather the model is built to show how the entailments of a realization is a tedious activity of prying apart collapsed structures of mathematical concepts.

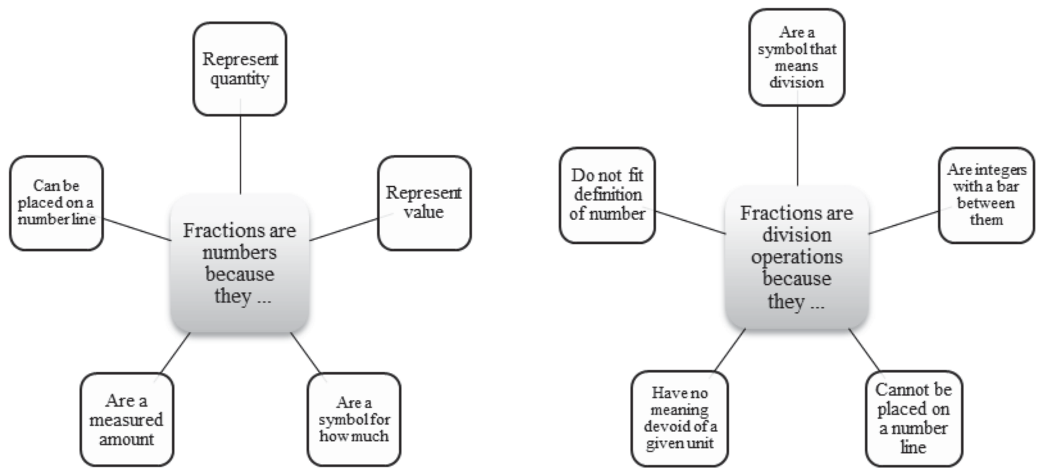

Figure 5.4 Entailments Chart Mapping the Fractions as Numbers Debate

Similar to other concept study research (Davis & Renert, 2014) the entailments emphasis was very “tedious and frustrating” for the unique collective of this concept study (p. 67) and was the contentious environment of this concept study sessions. This debate sparked so much controversy within the collective that the mathematical ability and maturity of some of the concept study participants were brought into question by other participants. The volatility of these environments both hampered and encouraged the emergence of the collective learner.

5.4 Emphasis 4: Blends

The collective participation and creation of the first three emphases was frustrating for the collective. This is not surprising as the teachers were asked to interrogate their professional expertise about the topics they teach. The blends emphasis was less frustrating and emerged spontaneously throughout the concept study sessions as the collective sought coherence for the concepts under study. The emergent collective goal was to collapse the realizations of multiplication into one realization that remained viable across the landscape of the middle school mathematics curriculum. The list in Figure 5.5 represents the most condensed realizations list for multiplication produced by the collective.

Defining *not just basic* Multiplication	Repeated Addition
	Grouping/Arrays
	Area and Volume
	Of

Figure 5.5 Blends of Realizations for Multiplication

While the emphasis on blends was less troubling for the collective, the collective outcome of the blends was the most troubling of all collective action as the collective was unable to find a viable realization for multiplication that blended all of their realizations for the mathematical concept. Widespread, distributed concern stemmed from the perceived pedagogical consequences of mathematical concepts behaving differently in different contexts. The participants wondered aloud about the possibilities of alleviating common misconceptions of fractions and operations on fractions if multiplication is presented to students as a synergistic realization rather than as disjointed realizations.

5.5 Emphasis 5: Participation

The blends emphasis spontaneously created an environment wherein the collective participated in cultural creation of mathematics. There were also explicit researcher-planned emphasis of participation in order to provide mathematical contexts where the collective scrutinized the nature of mathematics as a discipline. The explicit introduction of the emphasis of participation resulted in an ongoing collective interrogation of answers to the questions: “What is mathematics?” and “Is mathematics invented or discovered?” This research-generated emphasis of participation resulted in the emergence of the collective learner through an iterative entailments discussion. The realizations for the discipline of mathematics as invented or as discovered were given equal consideration by the collective. Figure 5.6, below, models how the argument pulled apart the realizations for mathematics, scrutinizing the very nature of the discipline.

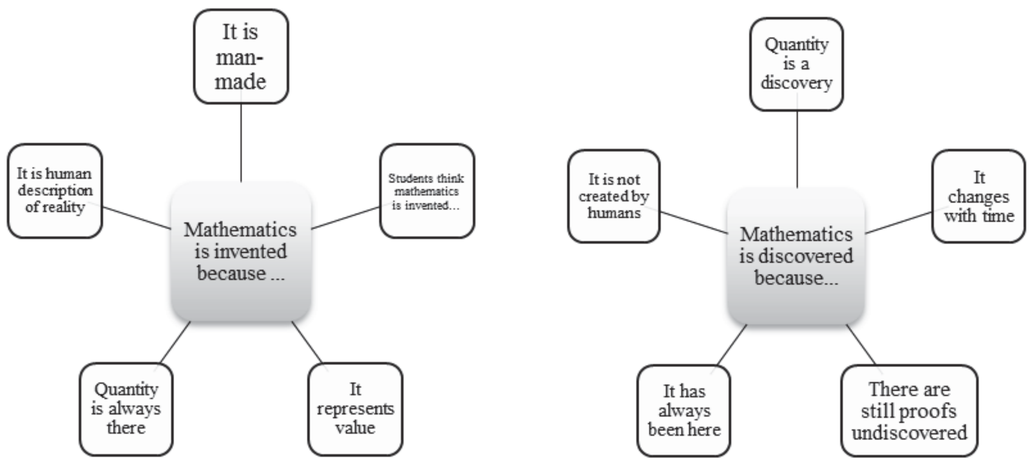

Figure 5.6 Participation of Culturally-created Mathematics through Entailments

5.6 Emphasis 6: Pedagogical Problem Solving

Pedagogical problem-solving “aims to capitalize on the interpretive potentials that arise on the collective level when individual expertise is drawn together around perplexing problems” of teaching mathematics to middle school students (Davis, 2012, p. 15). For this study, the emergence of pedagogical problem-solving came as co-production of the collective rather than as a planned intervention. One participant, Charlotte–for reasons discussed in her individual case–found the entailments emphasis of a concept study particularly tedious and frustrating. She, in her words, “want[s] to have some closure" on evolving realizations before shifting focus. The pedagogical problem-solving questions she generated for collective elaboration are listed below in Figure 5.7.

Pedagogical Problem-Solving Questions	1. How do you use fraction multiplication outside of baking?
	2. Why is slope represented as rise over run?
	3. What is a real-life example where you would subract a negative number?
	4. Can an array show why you can cancel out common factors of fraction multiplication computations like 2/3 x 3/4?

Figure 5.7 Pedagogical Problem-solving questions from Charlotte

The concept study participants dedicated the remainder of that two-hour session to the collective answering of Charlotte's pedagogical problem-solving questions, during which time the collective learner emerged several times.

These six emphases provide evidence that M_4T knowledge of fraction multiplication is better understood as a dually collective and distributed knowledge. This knowledge is highly dynamic while also simultaneously shared and at times individualized. This finding correlates with findings that M_4T knowledge is "more than a set of fundamentals that can be identified, catalogued, transmitted, and tested" (Davis & Renert, 2014, p. 119).

What follows now is a further discussion of the collective level of M_4T knowledge of fraction multiplication and its development. My discussion begins with the distinguishing characteristics between moments of interaction and coaction and is followed by a discussion of the collective learner and what the development of M_4T knowledge of fraction multiplication entails for the collective cognitive unity.

5.7 Finding the Collective Learner

For the purposes of this study, I define the collective learner as moments of coaction and the emergence of complexity. Moments of high collaboration were isolated in the data by searching for collective moments that exhibited the entanglement of ideas around the concept of fraction multiplication. What follows is a discussion of the distinguishing characteristics between moments of coaction and the emergence of the collective learner versus moments of interaction and collective collaboration.

5.7.1 Collective Interaction

Collective interaction is defined as collective action that lacks the entanglement of ideas that would allow a higher order cognitive system (collective learner) to arise. What follows is a description of the distinguishing characteristics for moments of interaction.

5.7.1.1 Lacking Pre-conditions for Complexity

Maintaining the pre-conditions for complexity is ultimately the coherence-maintaining mechanism of the collective. The collective decision as to what to do with mathematical innovations offered to the mathematical environment is directly related to the emergence of interaction instead of coaction. For example, notice in the following moment of interaction how the collective chooses to accept innovations offered to the environment as they respond to the task of representing 12 and 10 as division operations:

> Researcher: Okay what did we use?
>
> Faith: 36 divided by three, bring down the multiplication sign, and then 100 divided by 10.
>
> Researcher: Evan, did you use something similar?

Evan: I used 36 divided by 3 and 20 divided by 2.

Bailey: I used 36 divided by 3, then I used 30 divided by 3.

David: 24 divided by 2 and 30 divided by 3.

Charlotte: 24 divided by 2 and 100 divided by 10.

For reasons not fully understood, the collective chose to investigate this task independently, without attempting to collectively scrutinize individual contributions. There were many such moments. I claim that this type of interaction provides evidence that the necessary preconditions for complexity are a key indicator for distinguishing coaction from interaction. For example, looking again at the above excerpt, there is an obvious balance issue for the redundancy and diversity of the collective. The cause is unclear, but it may be related to the ease of the task or the way the task was introduced to the collective—which was in a manner that decentralized the control to the researcher. It is also clear in the above excerpt that there are no neighboring interactions as the ideas presented by individuals do not necessary collide. There were many instances similar to this excerpt wherein the preconditions for complexity were lacking and so there was no emergence of the collective learner. It should be noted that there is still potential for development of M_4T knowledge of fraction multiplication in these interactions, just not on the collective level.

5.7.1.2 Interaction without Coaction

Lacking the pre-conditions for complexity was not the only distinguishing characteristic between interaction and coaction evidenced in this research study. Evidence supports Martin and Towers' (2009a) claim that collaborative learning (interaction) can occur without the emergence of mutual joint action (coaction). For example, look at the following emphasis on blends in which the collective attempted to blend realizations of fraction as number, fraction as multiplication, and fraction as division through collective action:

> Evan: Didn't we just say a fraction is number and dividing all in one?
>
> Researcher: And multiplication, you said.
>
> Bailey: I don't think fractions are numbers.
>
> Researcher: You don't think they are numbers?
>
> Bailey: They are a division problem. Because it's a ratio of integers. That's what the definition of a fractions is.
>
> Researcher: What do you guys think about that? Is a fraction a number?
>
> David: It doesn't have to be …
>
> Evan: It could have variables.
>
> Researcher: Is it a number?
>
> Faith: It's a number, it's not a whole number.

This moment of interaction provides evidence of "reciprocal, complementary collaboration without the requirement to be mutually building on the just offered action" (Martin & Towers, 2009a, p. 633). This example supports the claim that M_4T knowledge of fraction multiplication can develop at the collective level, as shared distributive insights, without the emergence of a collective learner. This type of collective-level development is related to the collective decision-making for how to accept mathematical innovations offered to the collective. Martin and Towers (2009a) had similar findings, claiming that the potential for coaction is in part dependent upon the collective's ability to accept and interpret a mathematical innovation

5.7.2 Collective Coaction

If reciprocal, complementary collaboration and M_4T knowledge of fraction multiplication development is possible without coaction, what is it that makes coaction possible? Martin and Towers (2009a) claimed that a particular type of synergy from the collective aided the acceptance of an individual mathematical innovation and the emergence of coaction on that innovation. Sawyer (2000) described this acceptance in the context of improvisational performance as a group accepting an innovation and "working with it, building on it, making it their own" (p. 92). Martin and Towers (2009a) transferred this notion to coaction in mathematical contexts. In the language offered by complexity science, coaction is dependent upon the preconditions for complexity, or the intelligence of the collective, that support how much flexibility the collective has for managing a new mathematical innovation. The following excerpt is an example of coaction that took place while the collective built a realizations chart for multiplication during the second concept study session. It illustrates that the innovation of "grouping" and the collective's readiness to accept the innovation of "grouping" was key to how the innovation became part of the emerging performance. This example also provides an example where I myself was part of the synergistic movement consistently re-distributing the locus of power to the collective:

Charlotte: So I said my first answer, to David, if you ask me, is repeated addition.

Researcher: Ok, so can we write that down as a first level [on realizations list]?

Charlotte: Now, if you're multiplying by a fraction, I know that that is different. But basic multiplication I decided was repeated addition.

Researcher: Ok, so repeated addition is something we could start with. What's other—I'm talking when do you see it in your curriculum? What does it do? What's a definition of it [multiplication]? What can you tell students? Is there one definition that works for all numbers? I want to generate a list that we can be comfortable with as far as what you see and what you think multiplication is.

Bailey: I think of grouping.

Researcher: Ok, so any words that we would associate with it, like grouping, I think we should put up there.

Faith: Group, set—would that be the same as grouping?

Researcher: Depends on what her interpretation of grouping is.

Bailey: Groups of?

Faith: I wouldn't have thought of grouping in my level, but I tutor a third grader—

Charlotte: Grouping, isn't that repeated addition though?

The realization of "grouping" emerged and was quickly accepted and acted upon by the collective. This acceptance represents coaction and the emergence of the collective learner. What is significant is that the collective level image of grouping, as analyzed by an observer, is not necessarily the realization for each individual contribution. The collective learner realization, and the subsequent distributed realization for grouping, is not attributable to any one individual. It is a higher order cognitive unity, a nested distributed insight, with much more potential for subsequent action than the individual realizations of grouping.

The next excerpt of coaction illustrates how coaction can also be driven by the collective, without prompting from myself:

> Faith: Well because then you are going to have to have the kid that says 2/8ths is 2/8ths of 2/3rds
>
> Charlotte: Well I understand how they are going to misread it, I'm just saying how to we teach it so that the part that you haven't
>
> David: Right, because you aren't taking $1/5^{th}$ of the entire thing you are taking 1/5ths of the 2/3rds. Yeah, sorry
>
> Charlotte: Do you understand what I'm asking? Why would we tell the kids to divide up that bottom box? Cause we are finding $1/5^{th}$ of the 2/3rds not $1/5^{th}$ of the original whole
>
> Faith: You are shading $1/5^{th}$ of 2/3rds but you are separating it into parts to make it easier to shade is the way that I see it
>
> Bailey: Why am I not taking the $1/5^{th}$ of the whole? Like I understand that multiplication means of, but if I were to switch them around I would be taking 1/5ths of the whole and then?
>
> Charlotte: and then taking 2/3rds of the 1/5 right?
>
> Bailey: Yeah, that is right.
>
> Charlotte: So it would be the same thing, you would have a 5^{th} of the box empty… if it was switched around.

Researcher: Charlotte do you want to draw that? Not Charlotte I mean Bailey since you are closest to the board.

Bailey: So… let's see [drawing on board]

Evan: You are saying the 1/5th first and then using the 2/3rds second

Bailey: Yeah like this.

Researcher: The argument is that you will still have some blank space that you really don't have to talk about necessarily correct?

Similar to the first excerpt, the collective learner realization for modeling fraction multiplication as area, is not attributable to any one individual. It is a higher order cognitive unity, a nested distributed insight, with much more potential for subsequent action than the individual realizations of the area model.

5.8 Collective Level Development: Collective Learner

What follows is a discussion of the types of M_4T of fraction multiplication development that were found during moments of coaction and the emergence of the collective learner.

5.8.1 Adapting Phenomena

The collective learner's M_4T of fraction multiplication development is defined as change. One finding was that during moments of coaction the collective learner emerged as different combinations of the individuals participating in the mathematical environments of the concept study. Table 5.1 below shows the differing combinations of the collective learner for all 20 phenomena coded as coaction from this concept study:

Table 5.1 Collective Learner Composition

Collective Learner Composition	
Collective Learner Co-Creators	**% of Total Coactions**
{All Participants}	5%
{B,C,D,E,F,K}	65%
{B,C,D,F,K}	25%
{B,D,E,F,K}	5%

In complexity science language, this finding provides evidence that the collective learner adapted its physical structure in relation to the mathematical environment that it was co-creating. The collective learner, due to its embedded systems as pre-conditions for complexity, was flexible enough to engage with mathematical tasks of the concept study differently. During moments of coaction, the collective learner balanced its inherent intelligence and limitations towards working on differing mathematical tasks.

Focusing on the 11 instances of coaction that involved the active participation of all participants (with the exception of Annie) revealed new insights for how the collective learner developed its physical structure. Moments of coaction modeled completely different self-regulative structures despite the outward appearance of consistency in its physical structure. I attribute this variance in structure to the fluctuating levels of mathematical expertise, motivation, redundancy, and diversity that enabled and restrained the decision-making of the collective. In complexity terms, this variance transfers well to the understanding that "nested forms have many intermediate layers of organization, all of which influence (both enabling and constraining) one another" (Davis & Simmt, 2006, p. 296). To conclude, the development of the collective learner is a self-regulative process, one that is complex in nature. The system is coupled to its environment, and the collective learner developed its structure to accept innovations towards successfully completing the mathematical tasks of the concept study.

5.8.2 Collective Learner M_4T Development

Coaction and the emergence of a collective learner as defined in this research study, has been shown to emerge in mathematical environments with both students (Martin & Towers, 2009a; Martin, Towers & Pirie, 2006) and teachers (Davis & Simmt, 2003, 2006). Analyzing this literature supports the claim that the focus of the mathematical tasks of that environment link to the types of mathematical knowledge developed during the moments of coaction. For example, the following excerpt from Martin and Towers (2009a, p. 5-6) is of students' coaction around the mathematical topic of triangle definitions:

S: It's all coming back to me

H: I don't remember scalene or isosceles

S: Isosceles is this, okay? (drawing) where two are equal?

M: Yeah

S: Equilateral is when they're all equal?

H: Hm hm

S: And scalene is?

M: They're all wonky?

H? This must be scalene

S: OK

H: When it has one, one SSS … (pause)

M: One longer?

S: Isos., eq. and scale. So the scale none of them are equal?

Analyzing this except using the techniques of this study concludes that the students are only drawing upon subjective knowledge of triangles. This example supports the claim that students are users of mathematics (Ball & Bass, 2003) while investigating mathematical tasks. This is different from teachers' use of mathematics since teachers must unpack and substruct mathematical concepts for the teaching of these concepts (Ball, Thames, & Phelps, 2008; Ma, 1999; Davis & Renert, 2014). The following excerpt is an example of a moment of coaction where teachers utilize different types of M_4T of fraction multiplication knowledge. The excerpt begins as coaction develops to enable the collective learner to compare the area and array models for multiplication of fractions:

David: But area is a way of explaining it too.

Evan: David's point is, though, aren't arrays and area … aren't they the same or about the same?

Faith: No, because you know it goes back to this 3rd grader I had because … okay she does her dots, but she can't memorize it. So she does dots all the time and so I wanted to do this and it complicates it because I want to go from my arrays—connect my arrays—and I have area but it's not.. uh no. Because an array is our area rows by columns.

David: Don't you have area in boxes?

Faith: Yeah, but when you connect you're short. I only have one box when I connect.

Evan: Well you're not connecting the dots in an array …

Faith: See that [pointing to the drawing by Bailey]?!

Evan: Yeah, how are those different?

Faith: Oh, because in the top one my vertices are on corners. See the difference?

Evan: Six boxes.

Bailey: You still have six square units. Six dots in the squares.

Faith: Yeah, again I'm trying to take it back to where the kids think, the kids …

Evan: She's thinking when I connect the dots you don't connect the dots.

Charlotte: Why would you connect the array dots?

Faith: That's why I'm saying they're different. We gotta keep array.

The teachers in the excerpt enacted both knowledge-produced and knowledge-producing systems towards completing the emergent mathematical tasks of the concept study. This coaction involved the utilization of the collective learner's joint subjective understanding, knowledge of student cognition, and classroom collectivity knowledge about realizations for multiplication. It provides evidence that teachers' mathematical knowledge is intimately tied to their teaching of that knowledge. The following moment of coaction provides further evidence of this claim and supports the claim that the mathematical task does not have to be teaching related to invoke knowledge about teaching:

Faith: Are all those parts? Yeah, I thought so … all those parts, they're numbers.

Bailey: There's your repeated addition and we're getting into multiplication again. If you're adding up parts, the sum of the parts …

David: The parts are numbers.

Bailey: I would say a fraction is a part and that's division because you're dividing into parts.

Charlotte: I can see what Bailey is saying and I see what everybody else is saying, and I'm on this side. But it all depends on what you call a number, what you call whole.

David: Yeah a fraction is not a whole number, but it's a number

Faith: Above and below a whole? Should that be figured in? I mean, think about how students think about these things.

Researcher: This is your …

David: Real numbers.

Evan: Using the gas tank example: nobody speaks when they're driving, oh I've got 4 gallons of gas. Unless their car has been physically changed on the visual monitor, it saying you have 4 gallons of gas remaining. Most people it's that little gauge and we split it with the little lines and we call it halves and fourths and eighths. You can guess in between …

Bailey: You split it you divide it. I still …

Notice how Faith introduces student cognition as a mechanism for formulating her mathematical innovation for the task. The investigation of this concept did not involve considering the classroom teaching of the concept, yet it emerged in the mathematical environment as part of the collective's M_4T knowledge. These findings support research (Ball & Bass, 2003; Davis, 2011; Davis & Simmt, 2006; Ma, 1999) that teachers need a different type of understanding of mathematics in order to understand, substruct, unpack, and develop their mathematical knowledge for teaching.

CHAPTER 6. LINKING INDIVIDUAL AND COLLECTIVE DEVELOPMENT

I define the collective learner decisions, later described, as the link between the collective learner development and the individual teacher participants' M_4T development. Sawyer (2003) suggested, in the context of an improvisational performance, that when an innovation is offered to a collective there are three possible decisions for action: (a) accepting the innovation and building upon it, (b) rejecting the innovation and continuing as if it did not occur, and (c) partially accepting the innovation to build on only one aspect. Martin and Towers (2009a) identified these same decision-making options in mathematical contexts with students. This research study adds to these claims, finding that in moments of coaction the collective decision types are most closely related to (a) and (c) as discussed by Sawyer (2003). As I will describe below, the collective learner's decision mechanism is claimed as a link between the collective and the individual development of M_4T knowledge of fraction multiplication. What follows are definitions of these two collective learner decisions as they emerged from the data analysis of the moments of coaction and a defense of these decisions as the links between individual and collective learner development.

6.1 Collective Decision: Recursive Elaboration of Realizations

The first decision—defined here as *recursive elaboration*–describes a collective decision action in which partial fragments of realizations produced by individuals are elaborated into a more flexible and adaptable collective realization. Previous research provides language for describing this type of decision-making in both the individual and collective learner contexts. For example, Pirie and Kieren (1994) described individual's mathematical development as the mechanism of the interweaving of fragments of images. Martin and Towers (2009a) applied individual mathematical development to moments of coaction, where the collective takes differing fragments from individuals and coalesces them into a collective realization. My research takes this notion further, defining this moment as a link between individual and collective M_4T knowledge of fraction multiplication development.

Prior to offering a sample of this collective decision making mechanism as a link between individual and collective M_4T development I will defend my choice to name it *recursive elaboration*. Previous research uses the conceptual metaphor (Lakoff & Johnson, 1980) of interweaving to describe the collective production of an image that "may not be attributable as originating from any particular individual" (Stahl, 2006, p. 349). This metaphor characterizes collective output as a combined image produced through collective action. As has already been shown in the collective case, the collective image is a transcendent output of the collective learner, but should not be construed as the image that all individuals share. In an effort to overcome this language barrier for the metaphor of interweaving, I offer *recursive elaboration* as an alternative metaphor. This term, borrowed as a metaphor from fractal geometry, defines the collective output as a "successive iteration of an idea" (Davis & Simmt, 2006, p. 308). The decision-making mechanism of the collective operating on individual contributions then is an expansion of the realm of what is possible for individual realizations during collective coaction, not a transcendent shared realization.

The following moment of coaction illustrates recursive elaboration as a link between collective and individual M_4T knowledge of fraction multiplication development. This moment of coaction came during the pedagogical problem-solving emphasis investigating the utility of the area model teaching fraction multiplication:

> Charlotte: I don't understand how to teach what you are saying. So I see 2/3rds and I see thirds and we select the top two. Basically we are finding 2/3rds of one whole. Now we have to find 1/5th of 2/3rds, so if we are finding 1/5th of 2/3rds why are we even going to find or put anything in that bottom blank box [of the area model unit division]?
>
> Faith: Well because then you are going to have to have the kid that says 2/8ths is 2/8ths of 2/3rds.
>
> Charlotte: Well I understand how they are going to misread it … I'm just saying how do we teach it so that the part that you haven't used is …
>
> David: Right, because you aren't taking 1/5th of the entire thing you are taking 1/5th of the 2/3rds. Yeah, sorry.
>
> Charlotte: Do you understand what I'm asking? Why would we tell the kids to divide up that bottom box? 'Cause we are finding 1/5th of the 2/3rds not 1/5th of the original whole?

Faith: You are shading 1/5th of 2/3rds, but you are separating it into parts to make it easier to shade is the way that I see it.

Bailey: Why am I not taking the 1/5th of the whole? Like, I understand that multiplication means , *of* but if I were to switch them around I would be taking 1/5th of the whole and then …

Charlotte: … and then taking 2/3rds of the 1/5th.

Bailey: Yeah.

Recursive elaboration is defined to signify the expanding potentialities of the realization for the mathematical concept produced during the moment of coaction. In the moment above, the collective learner realization for the area model is not a combined image for all participants; rather it is a collective image that co-creates part of the environment that each individual is coupled with. The collective learner realization above does not imply a coalescing or an interweaving, but rather an increase in the intellectual capacity of the collective learner through the heightening of the redundancies and diversities of the individuals.

6.1.1.1 Recursive Elaboration as a Link

The emergence of a collective learner illustrates a co-dependence on the individual innovations offered to the collective and the collective learner's decision-making mechanism for what to do with those innovations. Evidence supports the conclusion that this action is a type of complex interplay, a link, between individual and collective M_4T cognition. To further illustrate recursive elaboration as a link between individual and collective M_4T development I will use Faith's individual case and the already provided excerpt of coaction.

Faith's M_4T knowledge of the area model for fraction multiplication was activated through her co-production of the mathematical environments of the concept study. What emerged were Faith's developing realizations of area for modeling the computation of fraction multiplication. As described in Faith's individual case, her M_4T knowledge of the area model developed throughout the concept study sessions. The coaction above shows one moment of her individual development, a link between individual and collective development. Prior to this coaction, there is no evidence of Faith interrogating the viability of the area model. After the coaction, Faith's actions show that area was now considered as a confusing representation that might prohibit student development of fraction multiplication understanding. Her confusion developed—or was activated—as she took part in the coaction of the collective learner. The moment of coaction provided a mathematical environment that afforded the opportunity for individuals, like Faith, to develop their own realizations for fractions and subsequently their M_4T knowledge of fraction multiplication. The realization in the moment of coaction is a recursive elaboration, rather than one shared, collective image. The recursive elaboration of the collective learner realization expanded the realm of potentialities for both Faith and the collective learner. This was done by exposing new potentialities for the same image, or a mathematical environment that offered refinement capabilities. This expansion is one link between collective M_4T development of fraction multiplication and individual M_4T development of fraction multiplication.

6.2 Collective Decision: Synergistic Realization

A second decision—defined here as synergistic realization—describes a collective decision action in which all previous realizations are abandoned for one innovation. Previous research provides language for describing this type of decision-making for a collective as the collective building of the *better idea* (Martin & Towers, 2009a; Martin & Towers, 2009b; Martin & Towers, 2010). This decision-making is done through the collective "being responsive to tiny cues from other players" and when an individual offers an innovation, "everyone else drops their own ideas and immediately joins in working on the better idea" (Martin & Towers, 2009a, p. 15). My analysis of the collective learner provided evidence that, during some moments of coaction, realizations offered by individuals replicate this action. Towers and Martin (2009b) referred to this collective action as an occasion in which the better idea can "occasion the growth of collective understanding" or the development of M_4T of the collective learner (p. 45). My research takes this notion further, defining this moment as a link between individual and collective M_4T knowledge of fraction multiplication development.

Prior to offering a sample of this collective decision-making mechanism as a link between individual and collective M_4T development I defend my choice for naming it as a synergistic realization. Previous research (Martin & Towers, 2009a; Martin & Towers, 2009b; Martin & Towers, 2010) frames the "better idea" collective decision-making mechanism as a mechanism for collective development. The "better idea" notion though is incomplete, as evidence from the recursive elaboration data suggests that individuals coacting on the "better idea" can actually interpret the realization as the "better idea" very differently. Similar to the recursive elaboration, this collective decision is an expansion of the possibilities for individual realizations. The expansion is nuanced though, as it is of one concept focused realization. Defining the collective decision making mechanism as a *synergistic realization* would better characterize this type of collective decision. In the language of complexity science, the synergistic realization acts as a serious limiter for the organized randomness in the system. The word *realization* was chosen to signify focus on one concept realization, as it is interpreted by the individuals comprising the collective learner.

The following moment of coaction illustrates a synergistic realization as a link between collective and individual M_4T knowledge of fraction multiplication development. This moment of coaction came during a realization emphasis where participants were investigating multiplication as repeated addition. The number line model had not previously been considered. It was immediately adopted as the innovation of choice by the collective learner to model the expression $3 \times \frac{1}{2}$. This excerpt picks up just after the synergistic realization decision to pursue the number line model as a valid realization for fraction multiplication as repeated addition. This excerpt also provides evidence of how my role can influence a moment of coaction:

> Charlotte: Now, your three one-half times … I need to see it as a number line.
>
> David: Find the midpoint of the hop.
>
> Charlotte: Yep, okay … I think I got it.
>
> Faith: One-half of the hop.
>
> Bailey: So you do [drawing on the board]
>
> Faith: Yes, zero, one, two, three …
>
> Bailey: I need to see it, sorry [continues drawing following Faith's cues].

Researcher: Don't apologize. If you need to see something at any particular moment just ask.

Faith: Ok, so start at zero [directing Bailey to draw].

Bailey: Start at zero and hop?

Faith: No. He [David] said a half of a hop of three. Half of three.

Researcher: So show me how you can physically add three one-half times repeatedly.

Faith: Add … jump all the way to three, that's one half of three: dot, dot, dot, half of it.

Bailey: Half of it? So you're talking parabolas there?

Faith: Oh sure I guess …

Synergistic realization encapsulates the moment-to-moment collective decision to pursue one type of realization by the collective learner to successfully complete a mathematical task. The initiation of the synergistic realization by Faith activated the collective learner to investigate number-line hopping as a model for the multiplication of fractions as repeated addition. Each individual began to talk about, take up and operate on the synergistic realization of number-line hopping. Bailey's interpretation of hopping on the number line was very different from others as her actions warrant the claim that she was interpreting parabolas as part of the model in some way. Faith's original proposition of number line hopping co-evolved with the developing environment and was subsequently shared but in no way should be misconstrued as the exact same realization.

6.2.1.1 Synergistic Realization as a Link

The emergence of a collective learner illustrates a co-dependence on the individual innovations offered to the collective and the collective learner's decision-making mechanism for what to do with those innovations. This action is a type of complex interplay, a link, between individual and collective M_4T cognition. To further illustrate a synergistic realization as a link between individual and collective M_4T development-- I utilize Bailey's individual case and the already-provided excerpt of coaction.

What emerged as crucial in this case was the recognition that the synergistic realization as an innovation seemed to activate individuals' M_4T knowledge in different ways. This activation emerged as expanding the realm of potentialities for each individual. The synergistic realization in no way signifies that all of the individuals in the collective acted similarly in this mathematical environment. In fact, their actions were significantly different based upon their own unique interpretations of the synergistic realization. The following illustrates Bailey's actions in this moment of coaction acting on the synergistic realization differently by mentioning parabolas as part of her realization:

> Faith: Add … jump all the way to three, that's one half of three: dot, dot, dot, half of it.
>
> Bailey: Half of it? So you're talking parabolas there?
>
> Faith: Oh sure I guess …

Bailey's introduction of diversity developed the opportunity for intelligence of the collective learner and subsequently the collective learner expanded the potentialities for the realization that Bailey was exposed to. This is a link between collective M_4T development of fraction multiplication and individual M_4T development of fraction multiplication.

CHAPTER 7. RESULTS, IMPLICATIONS, AND CONCLUSIONS

7.1 Individual M_4T and its Development

This study has added to existing research of teacher knowledge by extending the understanding of how middle school teachers' M_4T of fraction multiplication develops while collaboratively engaging in a concept study focused on multiplication. What follows is a discussion of patterns of development that emerged across the individual M_4T knowledge of fraction multiplication cases.

7.1.1 Knowledge-Produced Systems

7.1.1.1 Mathematical Objects

The differing timescales for how the mathematical objects knowledge develops is one finding of this research report. The stability of an individual's knowledge of mathematical objects, as represented in the individual cases, is as dynamic as their knowledge of other M_4T knowledge of fraction multiplication systems. Evidence to support this claim is that mathematical utterances can be coded as evidence for different systems of knowledge from the M_4T knowledge of fraction multiplication model. To illustrate, consider Evan's statement:

> Evan: And I think we throw things in [to the curriculum] 'cause it's one, it's a logical process, with math it seems like it's a logical process to build up to the other stuff. The excerpt, now coded for the four systems of the M_4T model, illustrates this claim:
>
> Evan: And I think we throw things in [to the curriculum] [CS, CC] 'cause it's one, it's a logical process [MO], with math it seems like it's a logical process to build up to the other stuff [SU, CS].

How could the M_4T systems change on different time scales, if the mathematical action is nested across systems? To move beyond this question, we can consider the difference between *knowledge* and *knowing* implicit in the theory of learning the grounds the M_4T model. Davis and Renert (2014) describe knowledge and knowing as "inseparable, co-implicated phenomena" (p. 90). The M_4T model allows for tracking the two co-implicated phenomena by providing the relative time-scale as part of the model. Mathematical objects knowledge is a collective *knowledge*, one that is distributed and co-created by humanity over the course of the last 5000 or more years. This was represented in the individual cases as the "mathematics discipline" discussed as an entity having a history of development and discovery. Mathematical objects *knowing,* was represented in the individual cases, as the highly volatile *knowing* of the collective mathematical objects *knowledge*. This is a finding for this research study as it provides empirical evidence to support the embedded system modeling of teacher knowledge on differing time scales.

The second finding for mathematical objects development of the individual middle school teachers' M_4T knowledge of fraction multiplication were the patterns of emergence of mathematical utterances for the nested systems of mathematical objects.

For example, the individual's *knowing* about the historical knowledge of fraction multiplication lacked any form of sophistication. The only historical knowing data emerged as anecdotal utterances and could not be included as evidence in the individual cases. To illustrate, consider Charlotte's f mathematical utterance:

> Charlotte: I embellish my story and I tell them about Pythagoras and I tell them that he was a Greek dude that use to sit around his house where, you know they didn't have social media back then so... I was like, you know, and what did they talk about? I'm like oh stars, and religion, and math… so in every class period it embellishes just a little bit more, what they talk about when the sit around this little square.

There was no evidence of development of this type of mathematical objects knowledge by any of the participants. Similarly little evidence emerged to make conjectures about the depth and development of the individual participant's advanced mathematical knowledge or horizonal knowledge. This allows me to conclude that this unique concept study seemed to be unreliable for developing the middle school teachers' historical and horizonal knowledge.

The teachers' realizations for mathematics as a discipline, a part of their mathematical objects system, developed significantly different from the other facets of MO. At the onset of the concept study, most of the individual teachers were observed to have a formalist conception of mathematics. This is unsurprising as previous concept study literature (Davis & Renert, 2009, 2014; Davis & Simmt, 2006) has found that initial responses to the emphasis of realizations is often automatized and "so well-rehearsed that they may eclipse other interpretive possibilities" (Davis & Renert, 2013, p.

253). To illustrate a formalist, automatized conception, consider David's mathematical utterance that took place on the first day of the concept study:

> David: It might not be a math problem that you're dealing with, but it's [mathematics] the thinking. It's the, "Here are my options, here's what I can do, here's what I can't do, here's how I'm going to solve this problem."

When the teachers were co-creating mathematical environments, specifically about the mathematics that they teach, their mathematical utterances often remained at the formalist or pre-formalist stages of mathematics. This was despite evidence suggesting that the teachers had more sophisticated realizations for mathematics. To illustrate, we return to David's later utterances about the mathematics that he teaches:

> David: I usually kinda take a … I guess maybe a cop-out easy way out of … you know, this [mathematics] is logical system of problem-solving, of the rules that you have.

Which contrasts with his more sophisticated understanding of research mathematics as intellectual pondering later described:

> David: Math PhDs think about things when you don't need them. […] Yeah [imaginary numbers are intellectual pondering].

The development of realizations for mathematics as a discipline was a non-linear evolution between differing levels of sophistication for all participants.

7.1.1.2 <u>Curricular Structures</u>

Limited evidence was collected of this knowledge structure likely due to the limited data collected about student interactions with teacher, curriculum, and classroom collective. This is posited to be due to the design and structure of a concept study mathematical environment. The only generalizations possible were that individual teacher's utterances and actions were consistent with the interference hypothesis for fraction multiplication curriculum structure. Little development of this was evident in the individual teacher cases.

7.1.2 Knowledge-Producing Systems

7.1.2.1 Classroom Collectivity

The classroom collectivity system was static in terms of development across the individual cases. Again, limited evidence of this system could be collected since student interactions with teacher and content are not part of the concept study design. For example, Bailey's mathematical utterance illustrates the type of data that I could analyze for classroom collectivity:

> Bailey: The teacher didn't even help me. All I knew is what I learned and what I could figure out because trying to learn math is different, trying to teach yourself in math is different than trying to teach yourself English or history or science, which is a little bit like math.

Notice that this is Bailey's memory of her own mathematical learning experiences that that substructures her knowledge of her own teaching and learning of mathematics. These are facets of her classroom collectivity knowledge, but the concept study environment does not provide mathematical contexts that would readily influence the development of this type of conception.

The data collected about the contributions of each individual to the collective environments of the concept study was much more informative for making conclusions about the development of this knowledge. The language of complexity science is useful for describing the developmental patterns that emerged. The mathematical environments of this concept study supported the evolution of contributions of the individual participants towards the balance of the pre-conditions for complexity. For example, in some mathematical environments Charlotte provided internal redundancy as part of the collective often when discussing classroom collectivity:

> Charlotte: Yes, there is designated homework time at the end of class […] Yep, the rest of time is spent on practice problems […] Yes, I agree, especially when it is the second class, I can say: you know, the other class thought about it this way.

However, in other collective environments Charlotte contributed significant amounts of diversity when discussing classroom collectivity with students:

> Charlotte: I mean I totally understand why it is equal, but no because of what we [middle school teachers] have been telling them up till this point […] No, I think that's something that I try to teach the kids, but others probably don't [connecting curriculum to other classes].

The concept study collective was highly dynamic in its collectivity structure, changing moment-to-moment while the teachers collectively engaged with mathematical tasks. Differing levels of mathematical expertise, as well as environmental dependent pedagogical expertise were both causes for developments in the individual teacher's contributions to the collectivity of the concept study. These can be characterized as shifting socio-mathematical norms of the collective. To illustrate this, consider David's utterance:

> David: I feel like if you are saying that half is not a number then you are like in second grade. That is honestly what is going on in my head right now.

The other cause was non-mathematically specific social norms. These norms played a part in the developing collectivity while the teachers were engaged with mathematics as well as other non-mathematical topics.

7.1.2.2 Subjective Understanding

The knowledge producing system of subjective understanding was the most dynamic system of the individual M_4T knowledge of fraction multiplication. This is posited to be attributable to the design of the six emphases of the concept study. Each of the participants' subjective understanding developed. Some participants demonstrated profound shifts in their understanding of the operator multiplication and its application to various number sets. For example, Bailey's realizations of multiplication were initially:

> Bailey: I think of grouping. […] So counting and repeated addition. […] Multiplication of two digits is FOIL. […] Yes [FOIL is the distributive property]. […] Groups of? It [the word "of"] translates to multiplication.

These realizations for multiplication iterated in complexity, developing to include other realizations such as area, but also condensing through the emphasis of entailments:

> Bailey: We use multiplication in area […] I don't think that's [area] what multiplication actually is. I think we use it in area so we associate them. But I don't think that's what multiplication is. I think that multiplication is the first two

[on the list] […] I personally think that everything [all realizations of multiplication] can be put into repeated addition and grouping.

This then developed further when considering fractions and the operation of multiplication on fractions:

Bailey: Fractions can be repeated addition, somehow. […] Parts of parts, so you are adding parts. […] I still think its repeated addition. […] Division is multiplying by a fraction. […] Division is just multiplication of a fraction. […] Yeah [you are dividing first and then adding up the parts].

This type of evolution was not unique to Bailey's subjective understanding of fraction multiplication, as represented in the individual cases.

What also emerged from the data was the obvious evolving complexity of realizations for fraction multiplication and the embeddedness of M_4T knowledge of fraction multiplication. The emphasis of entailments provided mathematical environments where realizations that appeared early in a teacher's individual case evolved to mean something significantly different in later sessions. This is related to the collapsing of realizations explicit in the emphasis of entailments and blends. For example, Bailey's utterances above show that grouping became some significantly different, evolving to encompass all of the other realizations for fraction multiplication:

Bailey: I personally think that everything [all realizations of multiplication] can be put into repeated addition and grouping.

7.2 Individual M_4T Development Implication and Conclusions

The individual case studies of this research document highlight the complex, embedded, and highly dynamic nature of an individual's M_4T knowledge. The volatile nature of the teacher participants' knowledge was due, in part, to the concept study environment, which was designed specifically to activate teachers' unconscious and explicit knowledge. Through collective action, the teachers were able to develop their M_4T knowledge of fraction multiplication. The claim is not that the development was a movement beyond their teachers' current understandings. Rather the claim is that development is a change in their mathematical actions around fraction multiplication resulting from a collective engagement around fraction multiplication.

This research is evidence for arguing that contemporary measures that examine only explicit teacher knowledge are potentially inadequate for fully assessing teacher's M_4T knowledge. The deficit model utilized by contemporary mathematics education literature depicts teacher's M_4T of fraction multiplication as lackluster when engaged in mathematical environments that involve various realizations of fraction multiplication. My research illustrates that teachers knowledge of fraction multiplication varies as they engage in differing mathematical situations. However, this description seems entirely inadequate as a means to describe fully what teachers *know* about fraction multiplication. The evidence of the diverse realizations produced during the concept study sessions, as described in the individual M_4T cases, shows a much more complex and sophisticated teacher knowledge. Teachers M_4T knowledge for fraction multiplication is better defined as an emergent representation of knowledge of fraction multiplication by the individual teachers instead of an explicit knowledge representation.

Evidence of this research study provides the means to claim a further distinction from explicit-objective research and tacit-emergent research on teachers' mathematical knowledge for teaching. Davis and Renert (2014) stated that Ball's MKT work (Ball, Thames, & Phelps, 2008) and Davis' M_4T work (Davis & Simmt, 20066; Davis & Renert, 2014) as congruous in certain respects. For example, Davis and Renert (2014) stated that "we see our *categories of knowledge* (mathematical objects and curricular structures) as paralleling their [Ball's] *subject matter knowledge* and our *categories of knowing* (subjective understanding and classroom collectivity) as paralleling their *pedagogical content knowledge.* The distinction between the two can be reaffirmed through the use of the results of this research report. *Knowing* and *knowledge* should be considered as inseparable emergent phenomena and part of a teachers mathematical knowledge for teaching. For example, mathematical objects knowledge is evolving at a much slower pace than mathematical objects knowing can evolve on the personal level. Yet, the knowledge and knowing are both being activated in a classroom while the teacher is teaching. A teacher must not only harmonize this type of knowledge and knowing within themselves, but they must also harmonize that within all of the others (students) with whom they share a mathematical environment with. Professional categories of knowledge and knowing then "are perhaps better portrayed as nested phenomena than as neighboring regions" like Ball and colleagues MKT work (Davis & Renert, 2014, p. 92).

Concept studies have anecdotally been shown in the past to have a profound impact on developing teachers' M_4T knowledge. Evidence from my research report suggests that the concept study design to investigate multiplication and fraction multiplication provides a context for profound developments in M_4T knowledge of middle school teachers. The emphases of the concept study design are a viable vehicle for co-constructing mathematical environments where development of M_4T is possible. More research is needed to better understand the emphases as vehicles that can be explicitly applied rather than as an implicit environmental construction in M_4T developmental contexts.

7.3 Collective Level M_4T and its Development

Two facets of M_4T of fraction multiplication can be considered at the collective level. M_4T of fraction multiplication can be considered as distributed collective knowledge, and M_4T of fraction multiplication emerges as part of a higher order cognitive unity defined as a collective learner. Understanding the collective level of M_4T of fraction multiplication as dually collective, has implications for discussing what development was evidenced in the concept study of this research. What follows is a discussion of the two levels of collectivity of M_4T of fraction multiplication followed by the implications and conclusions for considering the development of collective M_4T in this way.

7.3.1 Collectively Distributed Knowledge

In the collective case I provided evidence that supports Davis and colleagues' claim that M_4T knowledge is more than a static set of insights that can be identified, assessed, and transmitted (Davis & Renert, 2014; Davis & Simmt, 2006). For example, in the moments of coaction and interaction shared in the collective case, the M_4T of fraction multiplication that emerged was necessarily coupled to the environment. Realizations such as “number-line hopping” for fraction multiplication, or “grouping” for multiplication emerged in the same mathematical contexts but were not interpreted similarly by the teachers. The insight was distributed, but the individual, collective, and the mathematical environment had an impact on the realization. Here M_4T of fraction multiplication is distributed rather than concentrated in individual teachers.

7.3.2 Collective Learner as a Complex System

This research extends the notion of coaction and the emergence of a collective learner as a higher order cognitive unity. Previous research defined coaction as “the notion of acting with the ideas and actions of others in a mutual joint way” (Towers & Martin, 2009a, p. 44). This research claims that coaction is the emergence of complexity in the form of a higher order cognitive unity called a collective learner. A collective learner as a complex system has implications for how to study, assess, and utilize the conception of a collective learner for mathematics education research and teaching. The implications for defining a collective learner as a complex system that emerged from this research study are described below.

7.4 Collective Level M_4T Development Implications and Conclusions

M_4T has been modeled as nested complex systems of knowledge produced and knowledge producing systems. Davis and Renert (2014) state that the knowledge produced systems parallel, in some ways, the subject matter knowledge of Ball’s work (Ball, Thames, & Phelps, 2008) while the knowledge producing systems parallel, in some ways, to the pedagogical content knowledge in Ball’s work. The major difference offered by Davis and Renert (2014) is the theoretical difference between the ways the two research agendas define knowledge and knowing. As I have already discussed, Ball's work defines knowledge as a possession of an individual knower, while Davis and colleagues work defines knowledge and knowing as action and a structural coupling between the individual knower and the environment that is co-created by that knower. The results of this research study take this distinction one-step further. As Davis and Renert (2014) state, M_4T knowledge is concerned with the "myriad of ways that humans engage with mathematics" (p. 93), which includes the "individual, social, institutional, and cultural dimensions of the generation of mathematical meanings" (p. 93). M_4T knowledge of fraction multiplication as a shared, distributed professional teacher knowledge is very different from a group of teachers' knowledge of a set of unchanging mathematical facts. The teacher's M_4T knowledge of fraction multiplication is a complex interplay between knowledge produced and knowledge producing systems coupled with the environment in which it is produced. It is also a complex interplay between knowledge of mathematics and knowledge of how that mathematics is produced in a mathematical learning environment.

To consider a collective learner as a complex system has implications for the development of M_4T knowledge. In terms of practicality, it may be most productive to consider broader learning systems beyond the individual when concerned with developing M_4T knowledge. Davis and colleagues have claimed that focusing on grander learning systems beyond the individual can be done by focusing on the establishment of a classroom collective (Davis & Simmt, 2006; Davis & Renert, 2014; Davis, Sumara, & Luce-Kapler, 2000). This claim is similar to Towers and colleagues' claims that thinking about a classroom is a collective process (Towers & Martin, 2009a; Towers & Martin, 2009b). This research study takes these claims one-step further and suggests that certain pre-conditions of the environment can promote the emergence of a collective learner and have significant developmental potential for individual and collective M_4T knowledge of fraction multiplication. Further research can investigate how the emphases of a concept study environment can influence the emergence of a collective learner.

7.5 Linking Collective and Individual M_4T Development

This research suggests that there are two decision-making mechanisms of the collective that have been defined as links between the collective learner and individual development of M_4T knowledge of fraction multiplication. This suggests that there are links that exist between the collective and individual teachers' M_4T of fraction multiplication. What follows is a discussion of the implications and conclusions possible from this research report for the defined links-- synergistic realization and recursive elaboration links.

7.5.1 Implications for Recursive Elaboration

Recursive elaboration is defined as a decision-making mechanism of the collective that expands individual and collective M_4T knowledge of fraction multiplication development. Recursive elaboration describes a collective decision action in which partial fragments of realizations produced by individuals are elaborated into a more flexible and adaptable collective emergent realization. Moreover, findings from this research support the claim that M_4T is a collective level, distributed knowledge much too vast and complex to be possessed by any one individual. Recursive elaboration could be a mechanism that supports individual access to the distributed body of M_4T knowledge.

Further research could provide insights into how to foster collective activity that support recursive elaborations of individual realizations. The emphases of a concept study may be of use here. Further research can also provide insight for how much control a researcher or teacher can have over the emergence of recursive elaborations by the collective learner.

7.5.2 Implications for Synergistic Realization

Synergistic realization is defined as a decision-making mechanism of the collective that expands individual and collective M4T knowledge of fraction multiplication development. Realizations singularity is the collective decision action in which all previous realizations are abandoned for one innovation. Similar to recursive elaboration, synergistic realization is a mechanism that supports individual access to the distributed body of M4T knowledge. The realizations singularity is a collective coaction where one M4T realization is taken up and distributed widely to the nested individual systems of the collective learner. Unlike recursive elaboration, the synergistic realization can come from one individual. This finding is useful for professional development and classroom teaching of mathematics as it suggests that a resident expert could introduce the singularity. Further research should be done to better understand the synergistic realization and whether it can be introduced intentionally by a resident expert without upsetting the decentralized control pre-condition.

7.6 Conclusions

Mathematics education research has better answers now than ever before for what it means to be a knowledgeable and effective mathematics teachers. There is widespread agreement that teachers must have a sophisticated content knowledge of mathematics, knowledge of effective pedagogical techniques for mathematics, and also knowledge of the cognition of mathematics. There is still much work to be done for understanding the nuances of this knowledge. Parallel to this research agenda is the need for researchers to consider how best to influence the development of our understanding of the teacher specific knowledge domain. Concept studies have proven to be an effective mathematical environment for influencing the development of both individual and collective M_4T knowledge of fraction multiplication. The results of the research add to these understandings, providing insights for how individual and collective notions of M_4T develops independently and collectively in a concept study environment. Significant questions remain for this work. Can recursive elaboration and synergistic realizations, as links between individual and collective cognition, be influenced by researchers? Are there particular emphases of the concept study that are especially impactful for individual or collective M_4T development? These questions and more remain as the field co-creates parallel research agendas for further understanding mathematical teacher knowledge and expertise and then understanding how to help develop these types of teacher expertise.

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APPENDICES

Appendix A Quick Reference Tables for Coding Scheme I

Table A 1 Mathematical Objects Coding Scheme I

Mathematical Objects: Knowledge Produced Coding Scheme I	
Historical Development of:	5. Fraction Concept 6. Number Concept 7. Operation of Multiplication 8. Historical References
Orientation of Mathematics: [oral, pre-formalist, formalist, hyper-formalist, post-formalist]	5. What is mathematics? 6. Invented, discovered, or created? 7. Is mathematics a static or dynamic discipline? 8. Connection to natural world
Advanced Mathematical Knowledge	3. References to advanced mathematical study 4. Use of advanced mathematical techniques for understanding multiplication and fraction multiplication
Horizonal Knowledge	2. Connections of middle school mathematical curriculum to the other Pre-K–16 curriculum

Table A 2 Curricular Structures Coding Scheme I

Curricular Structures: Knowledge Produced Coding Scheme I	
Curriculum-as-Planned	1. Intended mathematics curriculum for fraction multiplication a. Textbook b. Design of curriculum to teach fraction
Curriculum-as-Lived	1. Enacted mathematics curriculum for fraction multiplication a. Collectivity with students b. Appropriateness of curriculum

Table A 3 Classroom Collectivity Coding Scheme I

Classroom Collectivity Knowledge Producing Coding Scheme I	
Collectivity with Students	1. Social Norms 2. Socio-mathematical norms
Collectivity in Concept Study	1. Social Norms 2. Socio-mathematical norms

Table A 4 Subjective Understanding Coding Scheme I

Subjective Understanding Knowledge Producing Coding Scheme I	
Realizations for fraction multiplication	1. Realizations for number 2. Realizations for multiplication 3. Realizations for fraction multiplication
Students realizations for fraction multiplication	1. Realizations for number 2. Realizations for multiplication 3. Realizations for fraction multiplication

Appendix B <u>Quick Reference Coding Scheme II</u>

Table B 1 Condensed reference chart for Coding Scheme II

Coding Scheme II Reference Chart	

Mathematical Objects of Fraction Multiplication	5. Historical Development
	    a. Name concept
	6. Orientation to Mathematics
	    a. Define mathematics
	    b. Invented, discovered, or created
	    c. Static or dynamic
	    d. Natural world connection
	7. Advanced Knowledge
	    a. Name area of study
	8. Horizonal Knowledge
	    a. Concept connection

Classroom Collectivity of Fraction Multiplication
• **Collective mathematics with students**
• **Collective mathematics in concept study**
	Social Norms :
	1. Systems of rights and obligations
	2. Criteria of value
	3. Explanations and justification solutions
	4. Sense making of other's solutions
	5. Agreement and disagreement
	6. Actions for resolving conflict
	Socio-mathematical Norms:
	1. Types of mathematical solutions
	    a. Different
	    b. Sophisticated
	    c. Acceptable
	    d. Efficient
	2. Mathematical Practices of the collective
	3. Accepted without justification

Curricular Structures of Fraction Multiplication	**Curriculum-as-lived (CaL) Enacted Curriculum:** 1. Interaction Patterns (student, teacher, fraction multiplication) 2. Appropriateness for images, analogies, metaphors that interconnect and animate **Curriculum-as-planned (CaP) Intended Curriculum:** 1. Interference/Reorganization 2. Concept Remediation 3. Other learning trajectory of fraction multiplication 4. Textbook as curriculum

Subjective Understanding of Fraction Multiplication	1. Orientation to mathematical cognition
	2. Conceptual blends of topics, images, metaphors of FM
	3. Realizations
	    a. Fractions
	    b. Multiplication
	    c. Fraction Multiplication

Appendix C Quick Reference Coding Scheme III

Table C 1 Condensed reference chart for Coding Scheme III

<table>
<tr><td colspan="3">Coding Scheme III Reference Chart</td></tr>
<tr><td colspan="3">Knowledge-Produced Categorizations</td></tr>
<tr><td rowspan="2">Mathematical Objects of Fraction Multiplication</td><td>1. Historical Development
a. Name concept</td><td>2. Advanced Knowledge
a. Broad field</td></tr>
<tr><td>3. Orientation to Mathematics
a. Oral Stage
b. Pre-formalist
c. Formalist
d. Hyper-formalist
e. Post-formalist</td><td>4. Horizonal Knowledge
a. Concept connection</td></tr>
<tr><td>Curricular Structures of Fraction Multiplication</td><td colspan="2">Curriculum-as-lived (CaL) Enacted Curriculum:
1. Interaction Patterns (student, teacher, fraction multiplication)
2. Appropriateness for images, analogies, metaphors that interconnect and animate
Curriculum-as-planned (CaP) Intended Curriculum:
1. Interference/Reorganization
2. Concept Remediation
3. Other learning trajectory of fraction multiplication
4. Textbook as curriculum</td></tr>
<tr><td colspan="3">Knowledge-Producing Systems</td></tr>
<tr><td>Classroom Collectivity of Fraction Multiplication</td><td colspan="2">Collective Mathematics with Students :
1. Behaviorism
2. Basic Constructivism
3. Enactivism
Collective Mathematics in Concept Study (Pre-conditions of complexity):
1. Internal Diversity
2. Internal Redundancy
3. Decentralized Control
4. Organized Randomness
5. Neighbor Interactions</td></tr>
<tr><td>Subjective Understanding of Fraction Multiplication</td><td>1. Realizations
a. Fractions
b. Multiplication
c. Fraction Multiplication</td><td>2. Knowledge of Student Cognition
a. Behaviorism
b. Basic Constructivism
c. Enactivism</td></tr>
</table>

VITA

VITA

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Taught Algebra II 1/2
Tutor 9-12 Mathematics

Mathematics Teacher
[2006-2008] Hill Country Middle School, Austin, TX
Taught Algebra I
Taught Pre-Algebra
TAKS Prep Course

Mathematics Teacher
[2004-2006] Mount Carmel High School, Chicago, IL
Taught College Prep Algebra and Geometry
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