EICHLER-SHIMURA COHOMOLOGY GROUPS AND THE IWASAWA MAIN CONJECTURE

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For Aidan

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ABSTRACT

Ohta has given a detailed study of the ordinary part of p-adic Eichler-Shimura cohomology groups (resp., generalized p-adic Eichler-Shimura cohomology groups) from the perspective of p-adic Hodge theory [O1, O2, O3]. Assuming various hypotheses, he is able to use the structure of these groups to give a simple proof of the Iwasawa main conjecture over $\mathbb{Q}$ [O2, O3, O4, O5]. The goal of this thesis is to extend Ohta's arguments with a view towards removing these hypotheses.

1

INTRODUCTION

There are many "main conjectures" in Iwasawa theory, but all have the same salient characteristic, namely, they give a relationship between invariants of arithmetic and analytic objects. In the case of the main conjecture over $\mathbb{Q}$ we get a relationship between an invariant associated to an inverse limit of ideal class groups, an arithmetic object, and a p-adic L-function, an analytic object. Let us begin by considering the arithmetic side of this relationship.

We fix a prime $p \geqslant 5$ throughout. Let θ and ψ be Dirichlet characters (possibly imprimitive) defined modulo M_θ and M_ψ, respectively,

$$\theta : (\mathbb{Z}/M_\theta\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$$
$$\psi : (\mathbb{Z}/M_\psi\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times,$$

such that p does not divide the conductor of ψ, which we denote by f_ψ, and the product of these characters $\theta\psi$ is even. Furthermore, we assume that $M_\theta M_\psi = Np$ or N for some positive integer N prime to p. We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and through this consider θ and ψ as p-adic Dirichlet characters. As with the prime p, the characters θ and ψ will be fixed throughout.

Let μ_{p^n} denote the group of p^nth roots of unity and define

$$\mathbb{Q}(\mu_{p^\infty}) = \bigcup_{n=1}^{\infty} \mathbb{Q}(\mu_{p^n}).$$

The cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, which we denote by $\mathbb{Q}_\infty$, is the unique subextension of $\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}$ satisfying $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \cong \mathbb{Z}_p$. For each $n \geqslant 1$, let $\mathbb{Q}_n \subset \mathbb{Q}_\infty$ denote the finite subextension of $\mathbb{Q}_\infty/\mathbb{Q}$ satisfying $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \cong$

$\mathbb{Z}/p^n\mathbb{Z}$. Let F denote the finite extension of $\mathbb{Q}$ corresponding to the fixed field of $\ker(\theta\omega) \cap \ker(\psi)$, where ω denotes the Teichmüller character. We define the cyclotomic $\mathbb{Z}_p$-extension of F to be $F_\infty = F\mathbb{Q}_\infty$ and set $F_n = F\mathbb{Q}_n$. Let $\mathrm{Cl}(F_n)[p^\infty]$ denote the Sylow p-subgroup of the ideal class group of the field F_n, and set

$$A_\infty = \varprojlim_n \mathrm{Cl}(F_n)[p^\infty],$$

where the projective limit is taken with respect to norm maps. Since the ideal class groups are finite abelian groups, we know that A_∞ is a pro-p abelian group. In fact, we can say more.

By class field theory we know that $\mathrm{Cl}(F_n)[p^\infty] \cong X_n := \mathrm{Gal}(H_n/F_n)$, where H_n is the maximal unramified p-extension of F_n. Moreover, there is a natural action of $\mathrm{Gal}(F_n/\mathbb{Q})$ on both $\mathrm{Cl}(F_n)[p^\infty]$ and X_n, and this action commutes with the aforementioned isomorphism. Specifically, for any $\sigma \in X_n$ and $\tau \in \mathrm{Gal}(F_n/\mathbb{Q})$, we define $\sigma^\tau = \tilde{\tau}\sigma\tilde{\tau}^{-1}$, where $\tilde{\tau}$ is any lift of τ to $\mathrm{Gal}(H_n/\mathbb{Q})$. Furthermore, the fact that $p^2 \nmid M_\theta M_\psi$ implies that $F \cap \mathbb{Q}_\infty = \mathbb{Q}$ [MW, §2]. This allows us to decompose $\mathrm{Gal}(F_n/\mathbb{Q})$ as $\Delta \times \Gamma_n$, where $\Delta = \mathrm{Gal}(F/\mathbb{Q})$ and $\Gamma_n := \mathrm{Gal}(F_n/F)$. This $\Delta \times \Gamma_n$-module structure commutes with norm maps $\mathrm{Cl}(F_{n+1})[p^\infty] \to \mathrm{Cl}(F_n)[p^\infty]$ and restriction maps $X_{n+1} \to X_n$. Letting X_∞ denote the Galois group of the maximal unramified pro-p abelian extension of F_∞, we obtain in the inverse limit an isomorphism of $\mathbb{Z}_p[\Delta][\![\Gamma]\!]$-modules $A_\infty \cong X_\infty$, where $\Gamma := \mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p$. While we will now shift our consideration to the Galois side of this isomorphism, it is important to keep in mind the arithmetic underpinnings of this construction.

For reasons that will be made clear later, we would like to consider a particular eigenspace of X_∞. For any Dirichlet character χ, we let χ_0 denote the unique primitive character associated to it. Set $\xi = (\theta_0\psi_0^{-1})_0$ and $\mathcal{O}_\xi = \mathbb{Z}_p[\xi]$, and define

$$X_{\infty,(\xi\omega)^{-1}} = X_\infty \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\xi,$$

where the homomorphism $\mathbb{Z}_p[\Delta] \to \mathcal{O}_\xi$ is induced by $(\xi\omega)^{-1}$. As an inverse limit of finite $\mathbb{Z}_p[\Gamma_n]$-modules, X_∞ is a finitely generated torsion $\mathbb{Z}_p[\![\Gamma]\!]$-module.

Hence, $X_{\infty,(\xi\omega)^{-1}}$ is a finitely generated torsion $\mathcal{O}_\xi[\![\Gamma]\!]$-module. A ring of the form $\mathcal{O}_\xi[\![\Gamma]\!]$ is called an Iwasawa algebra, and consequently we call $X_{\infty,(\xi\omega)^{-1}}$ an Iwasawa module. Thanks in large part to Serre, the structure of both Iwasawa algebras and their modules is well understood.

Let γ_0 be a topological generator of Γ. For example, take γ_0 corresponding to 1 under the isomorphism $\Gamma \cong \mathbb{Z}_p$. Serre showed that we can identify the Iwasawa algebra $\mathcal{O}_\xi[\![\Gamma]\!]$ with $\Lambda_\xi := \mathcal{O}_\xi[\![X]\!]$ through the continuous $\mathcal{O}_\xi$-linear map induced by $\gamma_0 \mapsto 1+X$. Just as one has a structure theorem for finite abelian groups, one has a similar theorem for finitely generated torsion Iwasawa modules. Specifically, there exists a homomorphism

$$X_{\infty,(\xi\omega)^{-1}} \to \Lambda_\xi/(f_1) \oplus \cdots \oplus \Lambda_\xi/(f_r)$$

with finite kernel and cokernel, where the f_i are non-zero divisors of Λ_ξ. While the f_i are not uniquely determined by $X_{\infty,(\xi\omega)^{-1}}$, their product is. With this in mind, we define the characteristic ideal of $X_{\infty,(\xi\omega)^{-1}}$ to be

$$\mathrm{Char}_{\Lambda_\xi}\left(X_{\infty,(\xi\omega)^{-1}}\right) = (f_1 \cdots f_r).$$

Just as the order of a finite abelian group is the most important invariant of the group, the most important invariant of a finitely generated torsion Iwasawa module is its characteristic ideal.

On the analytic side, Iwasawa has shown that if φ is a Dirichlet character with conductor not divisible by p^2, then there exists a unique element

$$F(X,\varphi) \in \begin{cases} \mathbb{Z}_p[\varphi][\![X]\!] & \varphi \neq \mathbb{1} \\ \frac{1}{X-p}\mathbb{Z}_p[\![X]\!] & \varphi = \mathbb{1}, \end{cases}$$

where $\mathbb{1}$ denotes the trivial character, such that for all $k \geqslant 2$ and characters $\epsilon : 1+p\mathbb{Z}_p \to \overline{\mathbb{Q}}_p^\times$ of p-power order, we have

$$F(\epsilon(u)u^{k-2}-1,\varphi) = L_p(k-2,\varphi\epsilon^{-1}),$$

where $u := 1-p$ and $L_p(s,\varphi\epsilon^{-1})$ denotes the Kubota-Leopoldt p-adic L-function associated to the character $\varphi\epsilon^{-1}$. The Iwasawa main conjecture over $\mathbb{Q}$

then gives us a relationship between the characteristic ideal of $X_{\infty,(\xi\omega)^{-1}}$ and $F(X, \xi\omega^2)$.

Theorem 1.0.1 (Iwasawa main conjecture over $\mathbb{Q}$).

$$\mathrm{Char}_{\Lambda_\xi}\left(X_{\infty,(\xi\omega)^{-1}}\right) = \left(F(X, \xi\omega^2)\right).$$

Despite its name, the main conjecture is actually a theorem. It was first proven by Mazur and Wiles [MW], and has since been proven in even greater generality. In [W2], Wiles simplified the proof in his paper with Mazur while generalizing it to extensions of totally real fields, including the case when $p = 2$. Around the same time, Rubin gave a much simpler proof of the main conjecture over $\mathbb{Q}$ (resp., over imaginary quadratic fields) using a tool from Galois cohomology known as an Euler system [Ru]. More recently, Ohta has given a simple proof in the spirit of Mazur and Wiles [O2, O3, O4, O5]. However, Ohta's argument requires various restrictive hypotheses. Specifically, consider the following list of hypotheses:

(H1) $\psi = \mathbb{1}$,

(H2) $\theta = \omega^i$ with i even,

(H3) $p \nmid \varphi(N)$ (Euler's totient function),

(H4) θ, ψ are non-exceptional: $(\theta\psi^{-1}\omega)(p) \neq 1$,

(H5) The universal ordinary Hecke algebra is Gorenstein.

Then Ohta assumes (H1), (H2) and (H3) in [O2], he assumes only (H1) in [O3], while in [O4] he assumes (H3) and (H4), and in [O5] he assumes (H1) and (H5). The goal of this thesis is to extend Ohta's argument so that these hypotheses are no longer required.

Given that there are already several proofs of the Iwasawa main conjecture over $\mathbb{Q}$, it is natural to ask why one would be interested in generalizing Ohta's proof. Recently, Sharifi [S] conjectured a relationship between X_∞ and the p-adic Eichler-Shimura cohomology groups of modular curves, and the Iwasawa main conjecture over $\mathbb{Q}$ would be but a shadow of this deeper relationship. However, Sharifi's constructions incorporate Ohta's work on the main conjecture, and as such the above hypotheses are assumed. By removing these hypotheses in the context of Ohta's proof of the main conjecture, one hopes to free Sharifi's conjectures of them as well.

OVERVIEW

Let us give a brief overview of how we will go about proving the Iwasawa main conjecture over $\mathbb{Q}$. For reasons that will be made clear in Section 4.2, it suffices to construct an unramified pro-p abelian extension L_∞ of F_∞ satisfying the following conditions:

(1) Δ acts on $\mathrm{Gal}(L_\infty/F_\infty)$ via $(\xi\omega)^{-1}$,

(2) $\mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L_\infty/F_\infty)) = \left(F(X, \xi\omega^2)\right)$.

In fact, as a consequence of the analytic class number formula, the second requirement can be weakened a bit [MW, p. 207]. Specifically, we just need to show

$$\mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L_\infty/F_\infty)) \subseteq \left(F(X, \xi\omega^2)\right).$$

To construct the extension appearing above, we will consider the Galois representation obtained from our p-adic Eichler-Shimura cohomology group of level N. Specifically, by applying the method of Kurihara [Ku] and Harder-Pink [HP] to this representation, we are able to construct a pro-p abelian extension L/F_∞. Without assuming (H1) or (H3) it is possible that this extension is ramified. However, the method of Kurihara and Harder-Pink also allows us to construct an embedding of $\mathrm{Gal}(L/F_\infty)$ into the reduction modulo Eisenstein ideal of a particular lattice of the quotient field of Hida's universal ordinary Hecke algebra. Through this embedding we are able understand the structure of $\mathrm{Gal}(L/F_\infty)$ as an Iwasawa module. We can then use this structure to determine not only which primes can ramify in the extension L/F_∞, but also how this ramification manifests itself in terms of the characteristic ideal of $\mathrm{Gal}(L/F_\infty)$.

The bulk of our effort will be in relating the Iwasawa module structure of the aforementioned lattice to $F(X, \xi\omega^2)$. We do so by showing that Hida's universal ordinary Hecke algebra modulo a particular Eisenstein ideal is isomorphic to $\mathbb{Z}_p[\theta, \psi][[X]]$ modulo the ideal generated by a multiple of $F(X, \xi\omega^2)$. This isomorphism, in combination with the theory of Fitting ideals, allows us to show that the characteristic ideal of $\mathrm{Gal}(L/F_\infty)$ is contained in the ideal generated by $F(X, \xi\omega^2)$. By then considering the ramification of L/F_∞, we can show that the characteristic ideal of the Galois group of the maximal unramified pro-p abelian subextension of L/F_∞ is also contained in the ideal generated by $F(X, \xi\omega^2)$. Let us now give a brief overview of the organization of this thesis.

In Chapter 2, we review the results of classical and Λ-adic modular forms that will be used in subsequent chapters. We begin by giving a brief review of classical modular forms and their Hecke algebras, the primary purpose of which is to fix notation. Having reviewed the requisite results from classical modular forms, we introduce Λ-adic modular forms, the universal Hecke algebras acting on them, and the duality between the two. Finally, we conclude the chapter by constructing Λ-adic Eisenstein series.

In Chapter 3, we determine the image of Eisenstein series under Ohta's Λ-adic residue map. While this image was computed in [O4] when (H3) and (H4) are assumed, we determine this image for Eisenstein series associated to arbitrary characters. This allows us to give a simple proof of the isomorphism between Hida's universal ordinary Hecke algebra modulo Eisenstein ideal and $\mathbb{Z}_p[\theta, \psi][[X]]$ modulo the ideal generated by a multiple of $F(X, \xi\omega^2)$.

Finally, in Chapter 4 we prove the main conjecture. After introducing the p-adic Eichler-Shimura cohomology groups, we use these groups to construct a pro-p abelian extension L/F_∞ by the method of Kurihara [Ku] and Harder-Pink [HP]. We then describe the Iwasawa module structure of $\mathrm{Gal}(L/F_\infty)$, and use this structure to understand the ramification occurring in L/F_∞. With these results in hand, as well as the isomorphism from Chapter 3, we employ the theory of Fitting ideals to prove the Iwasawa main conjecture over $\mathbb{Q}$.

2

MODULAR FORMS

In this chapter, we review the results on classical and Λ-adic modular forms that will be used in subsequent chapters. For brevity, most proofs in this chapter will be omitted in favor of references where one can find them.

2.1 CLASSICAL MODULAR FORMS

Let $\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$ denote the complex upper half plane, and let $\mathrm{GL}_2^+(\mathbb{R})$ denote the group of 2×2 matrices with real entries and positive determinant. Then $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathbb{H}$ by Mobiüs transformations: if $\tau \in \mathbb{H}$ and $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$, then

$$\alpha(\tau) = \frac{a\tau + b}{c\tau + d} \quad \text{where } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let k be an integer and f a complex-valued function on $\mathbb{H}$. Then for $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ as above, we can define a new function on $\mathbb{H}$ by

$$(f|_k\alpha)(\tau) = \det(\alpha)^{k-1}(c\tau + d)^{-k} f(\alpha(\tau)).$$

We will be particularly interested how the functions f and $f|_k\alpha$ compare for α in the modular group $\mathrm{SL}_2(\mathbb{Z})$, or one of its congruence subgroups, which we now define.

For any positive integer M, consider the following subgroups of $SL_2(\mathbb{Z})$:

$$\Gamma_0(M) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{M} \right\}$$
$$\Gamma_1(M) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M) : d \equiv 1 \pmod{M} \right\}$$
$$\Gamma(M) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(M) : b \equiv 0 \pmod{M} \right\}.$$

We say that a subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is a congruence subgroup if $\Gamma(M) \subset \Gamma$ for some positive integer M. The minimal such M is said to be the level of the congruence subgroup Γ.

Let $\Gamma \subset SL_2(\mathbb{Z})$ be a congruence subgroup of level M. Then we know

$$\begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \in \Gamma.$$

Therefore, if $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic function satisfying $f|_k\gamma = f$ for some integer k and all $\gamma \in \Gamma$, we have $f(\tau + M) = f(\tau)$ for all $\tau \in \mathbb{H}$. One can use this fact to show that f has a Laurent expansion

$$f = \sum_{n \in \mathbb{Z}} a_n(f)\, q_M^n \tag{1}$$

for $a_n(f) \in \mathbb{C}$, where $q_M := e^{2\pi i \tau/M}$ (denoted q if $M = 1$) [DS, Section 1.2]. We will refer to the above series as the q_M-expansion (or q-expansion if $M = 1$) of f.

Definition 2.1.1. *Let $\Gamma \subset SL_2(\mathbb{Z})$ be a congruence subgroup of level M. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is said to be a weight k modular form with respect to Γ if:*

(i) $f|_k\gamma = f$ *for all* $\gamma \in \Gamma$,

(ii) $a_n(f|_k\alpha) = 0$ *for all* $n < 0$ *and* $\alpha \in SL_2(\mathbb{Z})$.

We denote the space of weight k modular forms with respect to Γ by $M_k(\Gamma)$. If (ii) holds for $n \leqslant 0$, we say that f is a weight k cusp form with respect to Γ and denote the space of such forms by $S_k(\Gamma)$.

To declutter the notation a bit, we will omit reference to the weight when referencing the action of $GL_2^+(\mathbb{R})$ on modular forms, as it should always be clear from the context.

Our primary focus will be on modular forms with respect to the congruence subgroup $\Gamma = \Gamma_1(M)$ for some positive integer M. By considering q-expansions, for all $k \geqslant 0$ we define

$$M_k(\Gamma)_{\mathbb{Z}} = M_k(\Gamma) \cap \mathbb{Z}[\![q]\!]$$
$$S_k(\Gamma)_{\mathbb{Z}} = S_k(\Gamma) \cap \mathbb{Z}[\![q]\!].$$

It is a well-known result of Shimura that $S_k(\Gamma)$ is spanned as a $\mathbb{C}$-vector space by $S_k(\Gamma)_{\mathbb{Z}}$ for $k \geqslant 2$ [S, Theorem 3.52]. For any ring R we define

$$M_k(\Gamma)_R = M_k(\Gamma)_{\mathbb{Z}} \otimes_{\mathbb{Z}} R \hookrightarrow R[\![q]\!]$$
$$S_k(\Gamma)_R = S_k(\Gamma)_{\mathbb{Z}} \otimes_{\mathbb{Z}} R \hookrightarrow R[\![q]\!].$$

For any Dirichlet character χ and $\alpha \in SL_2(\mathbb{Z})$ with

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we define $\chi(\alpha) = \chi(d)$. A modular form f with respect to Γ is said to have Nebentypus χ if $\Gamma_0(M)$ acts on f through the character χ.

Before moving on let us consider an important example of a modular form called an Eisenstein series. Let φ and χ be primitive Dirichlet characters having conductors f_φ and f_χ, respectively. Furthermore, let us assume that either φ or χ is nontrivial. For any $k \geqslant 2$, integer $t \geqslant 1$, and $\tau \in \mathbb{H}$, we define

$$G_k(\varphi, \chi; t)(\tau) \;=\; \sum_{a=0}^{f_\varphi - 1} \sum_{b=0}^{f_\chi - 1} \sum_{c=0}^{f_\varphi - 1} \varphi(a)\chi^{-1}(b) \left(\sum_{\substack{x \equiv a f_\chi \, (f_\varphi f_\chi) \\ y \equiv b + c f_\chi \, (f_\varphi f_\chi)}} \frac{1}{(ct\tau + d)^k} \right),$$

which is a weight k modular form of level $f_\varphi f_\chi t$ and Nebentypus $\varphi\chi$ [DS, §§4.5, 4.6]. A normalized version of this form will play a central role in this thesis, namely,

$$E_k(\varphi, \chi; t) \;:=\; \frac{(f_\chi)^k (k-1)!}{2^{k+1}(-\pi i)^k} G_k(\varphi, \chi; t).$$

One can show that $E_k(\varphi, \chi; t)$ has the following q-expansion:

$$E_k(\varphi, \chi; t) \;=\; \frac{\chi(0)}{2} L(1-k, \varphi) + \sum_{n=1}^{\infty} \left(\sum_{\substack{d \mid n \\ d > 0}} \varphi(d)\chi\left(\frac{n}{d}\right) d^{k-1} \right) q^{nt},$$

where $L(s, \varphi)$ is the Dirichlet L-function associated to the character φ [DS, §4.5]. We refer to the modular forms $E_k(\varphi, \chi; t)$ as weight k Eisenstein series associated to the characters φ, χ, and t.

2.1.1 *Hecke operators*

Let us fix integers $M > 0$ and $k \geqslant 0$, and set $\Gamma = \Gamma_1(M)$. In this subsection we recall the double coset description of Hecke operators acting on modular forms.

For all primes $\ell \geqslant 1$ (or $\ell = 1$), we define T_ℓ to be the double coset

$$T_\ell \;=\; \Gamma \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \Gamma.$$

Similarly, for all integers d coprime to M we define

$$T_{d,d} = \Gamma\alpha_d \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \Gamma \qquad \langle d \rangle = \Gamma\alpha_d\Gamma,$$

where $\alpha_d \in \Gamma_0(M)$ satisfies

$$\alpha_d \equiv \begin{pmatrix} * & 0 \\ 0 & d \end{pmatrix} \pmod{M}.$$

We can define the product of two double cosets as the formal sum of double cosets in a natural way, and one can show that this product is commutative [DI, §3.1]. With this in mind, we define $\mathbb{T}_M$ to be the commutative algebra over $\mathbb{Z}$ generated by the T_ℓ and $\langle d \rangle$ for all primes $\ell \geqslant 1$ (resp., $\ell = 1$) and integers d coprime to M.

We can make analogous definitions with respect to the adjoints (or adjugates) of the matrices defining the double cosets above. Specifically, for all prime numbers $\ell \geqslant 1$ (as well as $\ell = 1$) and integers d coprime to M we define

$$T_\ell^* = \Gamma \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \Gamma \qquad T_{d,d}^* = \Gamma\alpha_d^\iota \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \Gamma \qquad \langle d \rangle^* = \Gamma\alpha_d^\iota\Gamma,$$

where $\alpha^\iota = \det(\alpha)\alpha^{-1}$ for all $\alpha \in GL_2^+(\mathbb{Q})$. We denote the algebra generated by all T_ℓ^* and $\langle d \rangle^*$ by $\mathbb{T}_M^*$.

We now describe how these algebras act on modular forms. Every double coset $\Gamma\beta\Gamma$ with $\beta \in GL_2^+(\mathbb{Q})$ has a disjoint decomposition

$$\Gamma\beta\Gamma = \coprod_j \Gamma\beta_j$$

for some $\beta_j \in GL_2^+(\mathbb{Q})$. For any $f \in M_k(\Gamma)$ we define

$$f|[\Gamma\beta\Gamma] = \sum_j f|\beta_j.$$

Clearly this action is independent of the decomposition.

Let us give an explicit description of this action with respect to the operators T_ℓ, $T_{d,d}$ and $\langle d\rangle$. We begin by extending the definition of the operators $T_{d,d}$ and $\langle d\rangle$ to all positive integers d by defining $T_{d,d} = 0 = \langle d\rangle$ if $\gcd(d, M) \neq 1$. Since Γ is a normal subgroup of $\Gamma_0(M)$, every element in the double coset $\Gamma\alpha_d\Gamma$ can be written as $\gamma\alpha_d$ for some $\gamma \in \Gamma$. Therefore, for all $f \in M_k(\Gamma)$ we have

$$f|\langle d\rangle \;=\; f|\alpha_d.$$

This in turn implies that $T_{d,d} = d^{k-2}\langle d\rangle$ as operators on $M_k(\Gamma)$. The same identity holds in the case of the adjoint operators $T^*_{d,d}$ and $\langle d\rangle^*$. We will refer to $\langle d\rangle$ and $\langle d\rangle^*$ as diamond operators.

Next, we consider the operator T_ℓ.

Proposition 2.1.2 ([DS], Proposition 5.2.1)**.**

$$\Gamma\begin{pmatrix}1 & 0\\ 0 & \ell\end{pmatrix}\Gamma \;=\; \begin{cases}\coprod_{j=0}^{\ell-1}\Gamma\beta_j & \ell \mid M\\ \coprod_{j=0}^{\ell}\Gamma\beta_j & \ell \nmid M\end{cases}$$

with

$$\beta_j = \begin{pmatrix}1 & j\\ 0 & \ell\end{pmatrix} \quad \textit{and} \quad \beta_\ell = \begin{pmatrix}x & y\\ M & \ell\end{pmatrix}\begin{pmatrix}\ell & 0\\ 0 & 1\end{pmatrix}$$

for $0 \leqslant j \leqslant \ell - 1$, *where* $x\ell - My = 1$.

Using the above disjoint decomposition of the double coset defining T_ℓ, one can give the following description of the action of T_ℓ on modular forms in terms of q-expansions.

Proposition 2.1.3 ([DS], Proposition 5.2.2)**.** *Let* $f \in M_k(\Gamma)$. *Then for all integers* $n \geqslant 0$ *we have*

$$a_n(f|T_\ell) \;=\; a_{n\ell}(f) + \ell^{k-1}a_{n/\ell}(f|\langle\ell\rangle),$$

where $a_{n/\ell} = 0$ *if* $\ell \nmid n$.

While we will deal primarily with the operators T_ℓ, $T_{d,d}$, $\langle d\rangle$, and their adjoints, there are additional operators that we need to consider. For all $e \geqslant 2$ and primes $\ell \geqslant 1$ we define the operator T_{ℓ^e} on $M_k(\Gamma)$ inductively by

$$T_{\ell^e} = T_\ell T_{\ell^{e-1}} + \ell^{k-1}\langle \ell\rangle T_{\ell^{e-2}}.$$

We see that $T_{\ell^e} = T_\ell^e$ if $\ell \mid M$. With these operators in hand, for all integers $n \geqslant 1$ with $n = \prod_i \ell_i^{e_i}$ we define

$$T_n = \prod_i T_{\ell_i^{e_i}}.$$

The operators T_n^* are defined analogously.

As with the operator T_ℓ for ℓ a prime, we are able to give an explicit description of the action of T_n on modular forms in terms of q-expansions for all integers $n \geqslant 1$.

Proposition 2.1.4 ([DS], Proposition 5.3.1)**.** *Let $n \geqslant 1$ and $f \in M_k(\Gamma)$. Then for all integers $m \geqslant 0$ we have*

$$a_m(f|T_n) = \sum_{\substack{d \mid \gcd(m,n) \\ d>0}} d^{k-1} a_{mn/d^2}(f|\langle d\rangle),$$

where (m, n) denotes the greatest common divisor of m and n.

Proposition 2.1.5. *Let d, m, and n be positive integers, with d coprime to M. Then*

(i) $T_n T_m = T_m T_n$ *if* $\gcd(m, n) = 1$

(ii) $T_n\langle d\rangle = \langle d\rangle T_n$.

The analogous statements for the adjoint operators hold as well.

It is well known that $M_k(\Gamma)_{\mathbb{Z}}$ and $S_k(\Gamma)_{\mathbb{Z}}$ are stable under the action of $\mathbb{T}_M$ and $\mathbb{T}_M^*$ [H2, Section 1]. With this in mind, we define the weight k, level M Hecke algebra $\mathfrak{H}_k(\Gamma)_{\mathbb{Z}}$ (resp., $\mathfrak{h}_k(\Gamma)_{\mathbb{Z}}$) to be the image of $\mathbb{T}_M$ in $\mathrm{End}_{\mathbb{Z}}(M_k(\Gamma)_{\mathbb{Z}})$ (resp., $\mathrm{End}_{\mathbb{Z}}(S_k(\Gamma)_{\mathbb{Z}})$). Note that this is the $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{Z}}(M_k(\Gamma)_{\mathbb{Z}})$

(resp., $\mathrm{End}_{\mathbb{Z}}(S_k(\Gamma)_{\mathbb{Z}})$) generated by $\{T_n\}$ for all $n \geqslant 1$. For any commutative ring R with unity, we define

$$\begin{aligned} \mathfrak{H}_k(\Gamma)_R &= \mathfrak{H}_k(\Gamma)_{\mathbb{Z}} \otimes_{\mathbb{Z}} R \\ \mathfrak{h}_k(\Gamma)_R &= \mathfrak{h}_k(\Gamma)_{\mathbb{Z}} \otimes_{\mathbb{Z}} R, \end{aligned}$$

with $\mathfrak{H}_k^*(\Gamma)_R$ and $\mathfrak{h}_k^*(\Gamma)_R$ defined analogously with respect to $\mathbb{T}_M^*$.

We remark that our definition of the above Hecke algebras differs slightly from the standard definition when $k = 1$ [DI, Proposition 3.5.1]. However, for our purposes this definition will suffice, as we will only consider modular forms having non-negative weight $k \neq 1$.

2.1.2 *The space of ordinary forms*

Let $k \geqslant 2$ and let $\mathcal{O}$ be the ring of integers of a complete subextension of $\mathbb{C}_p$. Recall the integer N from Chapter 1, and set $N_r = Np^r$ and $\Gamma_r = \Gamma_1(N_r)$ for all $r \geqslant 1$. Rather than consider the whole space $M_k(\Gamma_r)_{\mathcal{O}}$ (resp., $M_k^*(\Gamma_r)_{\mathcal{O}}$), we will often restrict our considerations to the maximal subspace on which the action of Hecke operator T_p (resp., T_p^*) is invertible. We define the ordinary projector e attached to T_p to be the limit

$$e = \lim_{n \to \infty} T_p^{n!} \in \mathfrak{h}_k(\Gamma_r)_{\mathcal{O}},$$

with e^* defined analogously with respect to the operator T_p^*. This operator will play a major role in all subsequent theory.

Definition 2.1.6. *Let* $k \geqslant 2$ *and let* $M = M_k(\Gamma_r)_{\mathcal{O}}$ *(resp.,* $S_k(\Gamma_r)_{\mathcal{O}}$*,* $\mathfrak{H}_k(\Gamma_r)_{\mathcal{O}}$*, or* $\mathfrak{h}_k(\Gamma_r)_{\mathcal{O}}$*). Then we define* $M^{\mathrm{ord}} = eM$*. The spaces with respect to the idempotent* e^* *are defined and denoted analogously.*

2.2 Λ-ADIC MODULAR FORMS

Let $\mathcal{O}$ be as in the previous subsection, and set $\Lambda = \mathcal{O}[\![X]\!]$. In this section we will introduce Λ-adic modular forms.

Let $U_r = 1 + p^r\mathbb{Z}_p$ for all $r \geqslant 1$, and recall the topological generator $u = 1 + p$ of U_1. Denote by $\widehat{U_1/U_r}$ the group of continuous $\overline{\mathbb{Q}}^\times$-valued characters on U_1/U_r, and define

$$\widehat{U}_1 \;=\; \bigcup_{r \geqslant 1} \widehat{U_1/U_r}.$$

For $\epsilon \in \widehat{U}_1$, let $\mathcal{O}[\epsilon]$ denote the ring generated by the values of ϵ over $\mathcal{O}$. We then define

$$M_k(\Gamma_r, \epsilon)_{\mathcal{O}[\epsilon]} \;=\; \{f \in M_k(\Gamma_r)_{\mathcal{O}[\epsilon]} : f|\sigma_\alpha = \epsilon(\alpha)f \text{ for all } \alpha \in U_1\},$$

where $\sigma_\alpha \in \Gamma_1$ is a matrix satisfying

$$\sigma_\alpha \;\equiv\; \begin{pmatrix} \alpha^{-1} & * \\ 0 & \alpha \end{pmatrix} \pmod{p^r}. \tag{2}$$

We define $S_k(\Gamma_r, \epsilon)_{\mathcal{O}[\epsilon]}$ analogously. Note that ϵ is not necessarily the Nebentypus character of $M_k(\Gamma_r, \epsilon)_{\mathcal{O}[\epsilon]}$. Rather, ϵ is the factor of the Nebentypus character whose order and conductor is a power of p. We denote the ordinary subspace of $M_k(\Gamma_r, \epsilon)_{\mathcal{O}[\epsilon]}$ (resp., $S_k(\Gamma_r, \epsilon)_{\mathcal{O}[\epsilon]}$) by $M_k(\Gamma_r, \epsilon)^{\mathrm{ord}}_{\mathcal{O}[\epsilon]}$ (resp., $S_k(\Gamma_r, \epsilon)^{\mathrm{ord}}_{\mathcal{O}[\epsilon]}$).

Definition 2.2.1. *A Λ-adic modular form (resp., cusp form) F of level N is a formal q-expansion*

$$F \;=\; \sum_{n=0}^{\infty} a_n(F)(X)q^n \;\in\; \Lambda[\![q]\!]$$

such that

$$\nu_{k,\epsilon}(F) \;:=\; \sum_{n=0}^{\infty} a_n(F)(\epsilon(u)u^{k-2} - 1)q^n \;\in\; M_k(\Gamma_r, \epsilon)_{\mathcal{O}[\epsilon]}$$

(resp., $S_k(\Gamma_r, \epsilon)_{\mathcal{O}[\epsilon]}$) for all $k \geqslant 2$ and all but finitely many $\epsilon \in \widehat{U}_1$. We denote the space of Λ-adic modular forms (resp., cusp forms) of level N by $M(N)_\Lambda$ (resp.,

$S(N)_\Lambda$). *If* $\nu_{k,\epsilon}(F) \in M_k(\Gamma_r,\epsilon)^{\mathrm{ord}}_{\mathcal{O}[\epsilon]}$ *(resp.,* $S_k(\Gamma_r,\epsilon)^{\mathrm{ord}}_{\mathcal{O}[\epsilon]}$*) for all* k *and* ϵ *as above, we say that* F *is an ordinary* Λ*-adic modular form (resp., cusp form) of level* N*. We denote the space of such forms by* $M(N)^{\mathrm{ord}}_\Lambda$ *(resp.,* $S(N)^{\mathrm{ord}}_\Lambda$*).*

The space of ordinary Λ-adic modular forms has a very nice structure which we now recall.

Proposition 2.2.2. *The* Λ*-modules* $M(N)^{\mathrm{ord}}_\Lambda$ *and* $S(N)^{\mathrm{ord}}_\Lambda$ *are free and finitely generated.*

Proof. This was first proven by Hida [H2, H3] using cohomology, and later by Wiles, who gave a simpler and more compact proof [W1]. Assuming that $M(N)^{\mathrm{ord}}_\Lambda$ and $S(N)^{\mathrm{ord}}_\Lambda$ are finitely generated, we would like to record a new and even simpler proof of their freeness. We will prove this for $M(N)^{\mathrm{ord}}_\Lambda$, with the proof for $S(N)^{\mathrm{ord}}_\Lambda$ being identical.

Let

$$0 \longrightarrow R \longrightarrow \Lambda^d \longrightarrow M(N)^{\mathrm{ord}}_\Lambda \longrightarrow 0$$

be a minimal presentation of $M(N)^{\mathrm{ord}}_\Lambda$ by a free module. Noting that $M(N)^{\mathrm{ord}}_\Lambda$ is Λ-torsion free, as it is a Λ-submodule of $\Lambda[\![q]\!]$, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & \Lambda^d & \longrightarrow & M(N)^{\mathrm{ord}}_\Lambda & \longrightarrow & 0 \\
 & & \downarrow X & & \downarrow X & & \downarrow X & & \\
0 & \longrightarrow & R & \longrightarrow & \Lambda^d & \longrightarrow & M(N)^{\mathrm{ord}}_\Lambda & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & R/XR & & \mathcal{O}^d & & M(N)^{\mathrm{ord}}_\Lambda / X M(N)^{\mathrm{ord}}_\Lambda & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By the snake lemma we get

$$0 \to R/XR \to \mathcal{O}^d \to M(N)^{\mathrm{ord}}_{\Lambda}/XM(N)^{\mathrm{ord}}_{\Lambda} \to 0.$$

Now, recall the specialization map $\nu_{2,\mathbb{1}} : M(N)^{\mathrm{ord}}_{\Lambda} \to \mathcal{O}[\![q]\!]$ defined by

$$\sum_{n \geqslant 0} a_n(F)(X)q^n \mapsto \sum_{n \geqslant 0} a_n(F)(0)q^n.$$

Clearly, $XM(N)^{\mathrm{ord}}_{\Lambda} \subset \ker(\nu_{2,\mathbb{1}})$. Suppose $F \in \ker(\nu_{2,\mathbb{1}})$. Then $X \mid a_n(F)$ for all $n \geqslant 0$. Furthermore, since

$$\nu_{k,\epsilon}(F/X) \;=\; \frac{\nu_{k,\epsilon}(F)}{\epsilon(u)u^{k-2}-1} \;\in\; M_k(\Gamma_r, \epsilon)^{\mathrm{ord}}_{\mathcal{O}[\epsilon]}$$

for all $k \geqslant 2$ and all but finitely many $\epsilon \in \widehat{U}_1$, we see that $F \in XM(N)^{\mathrm{ord}}_{\Lambda}$. Hence, $\ker(\nu_{k,\mathbb{1}}) = XM(N)^{\mathrm{ord}}_{\Lambda}$. This implies that $\nu_{k,\mathbb{1}}$ induces an injection of $M(N)^{\mathrm{ord}}_{\Lambda}/XM(N)^{\mathrm{ord}}_{\Lambda}$ into the torsion-free $\mathcal{O}$-module $\mathcal{O}[\![q]\!]$. Since $\mathcal{O}$ is a principal ideal domain and $M(N)^{\mathrm{ord}}_{\Lambda}/XM(N)^{\mathrm{ord}}_{\Lambda}$ is finitely generated, we know that $M(N)^{\mathrm{ord}}_{\Lambda}/XM(N)^{\mathrm{ord}}_{\Lambda}$ is a free $\mathcal{O}$-module. Because we chose our presentation to be minimal, Nakayama's lemma tells us that $M(N)^{\mathrm{ord}}_{\Lambda}/XM(N)^{\mathrm{ord}}_{\Lambda} \cong \mathcal{O}^d$ which implies $R/XR = 0$. Consequently $R = 0$, and we have $\Lambda^d \cong M(N)^{\mathrm{ord}}_{\Lambda}$. □

Proposition 2.2.3 ([O2] Proposition 2.5.1, [O1] Proposition 2.6.4)**.** *For each* $k \geqslant 2$ *and* $\epsilon \in \widehat{U}_1$*, let* $P_{k,\epsilon} := X - \epsilon(u)u^{k-2} + 1$*. Then*

$$M(N)^{\mathrm{ord}}_{\Lambda}/P_{k,\epsilon}M(N)^{\mathrm{ord}}_{\Lambda} \;\cong\; M_k(\Gamma_r)^{\mathrm{ord}}_{\mathcal{O}[\epsilon]}$$
$$S(N)^{\mathrm{ord}}_{\Lambda}/P_{k,\epsilon}S(N)^{\mathrm{ord}}_{\Lambda} \;\cong\; S_k(\Gamma_r)^{\mathrm{ord}}_{\mathcal{O}[\epsilon]}$$

where r *is determined by* $\ker(\epsilon) = U_r$*.*

Corollary 2.2.4. *Let* $\Lambda_{\mathbb{Z}_p} := \mathbb{Z}_p[\![X]\!]$*. Then*

$$M(N)^{\mathrm{ord}}_{\Lambda} \;\cong\; M(N)^{\mathrm{ord}}_{\Lambda_{\mathbb{Z}_p}} \otimes_{\Lambda_{\mathbb{Z}_p}} \Lambda \tag{3}$$
$$S(N)^{\mathrm{ord}}_{\Lambda} \;\cong\; S(N)^{\mathrm{ord}}_{\Lambda_{\mathbb{Z}_p}} \otimes_{\Lambda_{\mathbb{Z}_p}} \Lambda \tag{4}$$

Proof. We only prove this for $M(N)_\Lambda^{ord}$, as the proof in the case of $S(N)_\Lambda^{ord}$ is identical. By Proposition 2.2.3, we have

$$M(N)_\Lambda^{ord}/P_{k,\mathbb{1}}M(N)_\Lambda^{ord} \cong M_k(\Gamma_1)_{\mathcal{O}}^{ord} = M_k(\Gamma_1)_{\mathbb{Z}_p}^{ord} \otimes_{\mathbb{Z}_p} \mathcal{O}$$
$$\cong (M(N)_{\Lambda_{\mathbb{Z}_p}}^{ord}/P_{k,\mathbb{1}}M(N)_{\Lambda_{\mathbb{Z}_p}}^{ord}) \otimes_{\mathbb{Z}_p} \mathcal{O}.$$

If $\{F_1, \ldots, F_m\}$ is a $\Lambda_{\mathbb{Z}_p}$-basis for $M(N)_{\Lambda_{\mathbb{Z}_p}}^{ord}$, then $\{F_1 \bmod P_{k,\mathbb{1}}, \ldots, F_m \bmod P_{k,\mathbb{1}}\}$ is an $\mathcal{O}$-basis for $M_k(\Gamma_1)_{\mathcal{O}}^{ord}$. Hence, by Nakayama's lemma, $\{F_1, \ldots, F_m\}$ is a Λ-basis for $M(N)_\Lambda^{ord}$.

□

2.2.1 *The universal Hecke algebra*

In this subsection, we recall the construction of Hida's universal Hecke algebra and describe how this algebra acts on Λ-adic modular forms.

For all $k \geqslant 0$ and $r \geqslant 1$, the natural injections

$$M_k(\Gamma_r)_{\mathcal{O}} \hookrightarrow M_k(\Gamma_{r+1})_{\mathcal{O}}$$
$$S_k(\Gamma_r)_{\mathcal{O}} \hookrightarrow S_k(\Gamma_{r+1})_{\mathcal{O}}.$$

commute with the Hecke action. Therefore, if we restrict the operators of $\mathfrak{H}_k(\Gamma_{r+1})$ (resp., $\mathfrak{h}_k(\Gamma_{r+1})$) to the image of $M_k(\Gamma_r)_{\mathcal{O}}$ (resp., $S_k(\Gamma_r)_{\mathcal{O}}$) we obtain surjective $\mathcal{O}$-algebra homomorphisms

$$\begin{aligned} \mathfrak{H}_k(\Gamma_{r+1})_{\mathcal{O}} &\twoheadrightarrow \mathfrak{H}_k(\Gamma_r)_{\mathcal{O}} \\ \mathfrak{h}_k(\Gamma_{r+1})_{\mathcal{O}} &\twoheadrightarrow \mathfrak{h}_k(\Gamma_r)_{\mathcal{O}}. \end{aligned} \tag{5}$$

Definition 2.2.5 ([H3], (1.2))**.** *The universal Hecke algebras of level* N *over* $\mathcal{O}$ *are defined by*

$$\mathfrak{H}_k(N)_{\mathcal{O}} = \varprojlim_r \mathfrak{H}_k(\Gamma_r)_{\mathcal{O}}$$
$$\mathfrak{h}_k(N)_{\mathcal{O}} = \varprojlim_r \mathfrak{h}_k(\Gamma_r)_{\mathcal{O}},$$

where the projective limit is taken with respect to the above restriction maps.

We denote the operators corresponding to the projective limits of T_n, $T_{d,d}$, and $\langle d\rangle$ by the same symbols. In addition, we denote the projective limit of Hida's idempotent by e, and define the ordinary universal Hecke algebras by

$$\begin{aligned}\mathfrak{H}_k(N)^{\mathrm{ord}}_{\mathcal{O}} &= e\mathfrak{H}_k(N)_{\mathcal{O}}\\ \mathfrak{h}_k(N)^{\mathrm{ord}}_{\mathcal{O}} &= e\mathfrak{h}_k(N)_{\mathcal{O}}.\end{aligned}$$

In fact, Hida has shown that the algebras $\mathfrak{H}_k(N)^{\mathrm{ord}}_{\mathcal{O}}$ (resp., $\mathfrak{h}_k(N)^{\mathrm{ord}}_{\mathcal{O}}$) are isomorphic for all $k \geqslant 2$ [H3, Theorem 1.1]. In light of this, we will omit reference to the weight in notation for the universal ordinary Hecke algebras from here on out.

Let

$$\mathbb{Z}_{p,N} = \varprojlim_r \mathbb{Z}/Np^r\mathbb{Z} \cong (\mathbb{Z}/N\mathbb{Z}) \times \mathbb{Z}_p.$$

Then we give $\mathfrak{H}(N)^{\mathrm{ord}}_{\mathcal{O}}$ and $\mathfrak{h}(N)^{\mathrm{ord}}_{\mathcal{O}}$ an $\mathcal{O}[\mathbb{Z}_{p,N}^{\times}]$-algebra structure by letting any positive integer d coprime to Np act as $T_{d,d}$. We remark that this is a twist of the action defined by Hida [H3, remarks after (1.9)] in which any positive integer d coprime to Np acts as $d^2T_{d,d}$. This difference is cosmetic, and ultimately stems from our choice of specialization map $X \mapsto u^{k-2} - 1$ compared to Hida's choice of $X \mapsto u^k - 1$. Since

$$\mathcal{O}[\mathbb{Z}_{p,N}^{\times}] \cong \mathcal{O}[(\mathbb{Z}/Np\mathbb{Z})^{\times}][\![U_1]\!],$$

we can identify $\mathcal{O}[\mathbb{Z}_{p,N}^{\times}]$ with $\mathcal{O}[(\mathbb{Z}/Np\mathbb{Z})^{\times}][\![X]\!]$ by the isomorphism

$$\iota : \mathcal{O}[(\mathbb{Z}/Np\mathbb{Z})^{\times}][\![U_1]\!] \to \mathcal{O}[(\mathbb{Z}/Np\mathbb{Z})^{\times}][\![X]\!] \tag{6}$$

induced by $u \mapsto 1 + X$. Hence, $\mathfrak{H}(N)^{\mathrm{ord}}_{\mathcal{O}}$ and $\mathfrak{h}(N)^{\mathrm{ord}}_{\mathcal{O}}$ are $\mathcal{O}[(\mathbb{Z}/Np\mathbb{Z})^{\times}][\![X]\!]$-modules, and in particular, Λ-modules.

Proposition 2.2.6 ([O2], Thereom 1.5.7)**.** *$\mathfrak{H}(N)^{\mathrm{ord}}_{\mathcal{O}}$ and $\mathfrak{h}(N)^{\mathrm{ord}}_{\mathcal{O}}$ are free and finitely generated Λ-modules.*

For all positive integers M we define

$$w_M = \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}.$$

Since $w_M \Gamma_1(M) w_M^{-1} = \Gamma_1(M)$, the matrix w_M acts on $M_k(\Gamma_1(M))$ and $S_k(\Gamma_1(M))$. Furthermore, this action of interchanges the action of Hecke operators with that of their adjoints, i.e. for all $f \in M_k(\Gamma_1(M))$ (resp., $S_k(\Gamma_1(M))$) we have

$$\begin{aligned} f|w_{N_r}|T_n^* &= f|T_n|w_{N_r} \\ f|w_{N_r}|T_{q,q}^* &= f|T_{q,q}|w_{N_r} \\ f|w_{N_r}|\langle q\rangle^* &= f|\langle q\rangle|w_{N_r}. \end{aligned} \tag{7}$$

Therefore, for all $k \geqslant 2$ and $r \geqslant 1$ we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{H}_k(\Gamma_{r+1})_{\mathcal{O}} & \xrightarrow{\sim} & \mathfrak{H}_k^*(\Gamma_{r+1})_{\mathcal{O}} \\ {\scriptstyle \text{res}}\downarrow & & \downarrow{\scriptstyle \text{res}} \\ \mathfrak{H}_k(\Gamma_r)_{\mathcal{O}} & \xrightarrow{\sim} & \mathfrak{H}_k^*(\Gamma_r)_{\mathcal{O}} \end{array}$$

where the vertical maps are the restriction maps (5). Using this we can construct the adjoint universal Hecke algebras $\mathfrak{H}_k^*(N)_{\mathcal{O}}$, $\mathfrak{h}_k^*(N)_{\mathcal{O}}$, as well as their ordinary projections $\mathfrak{H}^*(N)_{\mathcal{O}}^{\text{ord}}$, $\mathfrak{h}^*(N)_{\mathcal{O}}^{\text{ord}}$.

2.2.2 *A projective system of modular forms*

It is not immediately clear how the universal Hecke algebras act on the space of Λ-adic modular forms. In order to define this action, we recall Ohta's construction of a projective system of modular forms that is isomorphic to the space of Λ-adic modular forms [O1, §2.3], [O2, §2.2]. Not only will this isomorphism allow us to give the space of Λ-adic modular forms a Hecke module structure, it will also be central to our construction of the Λ-adic residue map of Chapter 3. We begin by recalling the trace maps on modular forms, as these will be the maps with which we construct our projective system.

For all $r \geqslant 1$ we have the following inclusions of groups

$$\Gamma_{r+1} \subset \Gamma_r \cap \Gamma_0(p^{r+1}) \subset \Gamma_r,$$

with

$$\Gamma_r \cap \Gamma_0(p^{r+1}) = \coprod_{\alpha \in U_r/U_{r+1}} \sigma_\alpha \Gamma_{r+1} = \coprod_{\alpha \in U_r/U_{r+1}} \Gamma_{r+1} \sigma_\alpha$$

where σ_α is as defined in (2), and

$$\Gamma_r = \coprod_{0 \leqslant j \leqslant p-1} \gamma_j \left(\Gamma_r \cap \Gamma_0(p^{r+1})\right) = \coprod_{0 \leqslant j \leqslant p-1} \left(\Gamma_r \cap \Gamma_0(p^{r+1})\right) \gamma_j$$

for

$$\gamma_j = \begin{pmatrix} 1 & 0 \\ N_r j & 1 \end{pmatrix}.$$

We then define

$$\begin{aligned}
\mathrm{Tr}_r^1 &: M_k(\Gamma_{r+1})_{\mathbb{C}_p} \to M_k(\Gamma_r \cap \Gamma_0(p^{r+1}))_{\mathbb{C}_p} : f \mapsto \sum_{\alpha \in U_r/U_{r+1}} f|\sigma_\alpha \\
\mathrm{Tr}_r^2 &: M_k(\Gamma_r \cap \Gamma_0(p^{r+1}))_{\mathbb{C}_p} \to M_k(\Gamma_r)_{\mathbb{C}_p} : f \mapsto \sum_{0 \leqslant j \leqslant p-1} f|\gamma_j \\
\mathrm{Tr}_r &:= \mathrm{Tr}_r^2 \circ \mathrm{Tr}_r^1 : M_k(\Gamma_{r+1})_{\mathbb{C}_p} \to M_k(\Gamma_r)_{\mathbb{C}_p}
\end{aligned}$$

Using the identities given in (7), one can show by direct calculation that

$$\begin{aligned}
\mathrm{Tr}_r^1(f|w_{N_{r+1}}) &= \mathrm{Tr}_r^1(f)|w_{N_{r+1}} \\
\mathrm{Tr}_r^2(f) &= f|w_{N_{r+1}}|T_p|w_{N_r}^{-1},
\end{aligned}$$

which implies

$$\mathrm{Tr}_r(f) = \sum_{\alpha \in U_r/U_{r+1}} f|w_{N_{r+1}}|\sigma_\alpha|T_p|w_{N_r}^{-1}. \tag{8}$$

With this identity in mind, we define

$$\begin{aligned}
M_k^*(\Gamma_r)_{\mathcal{O}} &= \{f \in M_k(\Gamma_r)_{\mathbb{C}_p} : f|w_{N_r} \in M_k(\Gamma_r)_{\mathcal{O}}\} \\
S_k^*(\Gamma_r)_{\mathcal{O}} &= \{f \in S_k(\Gamma_r)_{\mathbb{C}_p} : f|w_{N_r} \in S_k(\Gamma_r)_{\mathcal{O}}\},
\end{aligned}$$

and we can see that Tr_r maps $M_k^*(\Gamma_{r+1})_{\mathcal{O}}$ (resp., $S_k^*(\Gamma_{r+1})_{\mathcal{O}}$) into $M_k^*(\Gamma_r)_{\mathcal{O}}$ (resp., $S_k^*(\Gamma_r)_{\mathcal{O}}$).

Definition 2.2.7. *For each $k \geqslant 2$, we set*

$$\mathfrak{M}_k^*(N)_\Lambda = \varprojlim_{r \geqslant 1} M_k^*(\Gamma_r)_{\mathcal{O}}$$
$$\mathfrak{S}_k^*(N)_\Lambda = \varprojlim_{r \geqslant 1} S_k^*(\Gamma_r)_{\mathcal{O}}$$

where the limit is taken with respect to the trace maps Tr_r.

Using the identities of (7) one can see that $M_k^*(\Gamma_r)_{\mathcal{O}}$ and $S_k^*(\Gamma_r)_{\mathcal{O}}$ are stable under the actions of $\mathfrak{H}^*(\Gamma_r)_{\mathcal{O}}$ and $\mathfrak{h}^*(\Gamma_r)_{\mathcal{O}}$, respectively. Furthermore, by (8) we see that this Hecke action commutes with the trace maps. Therefore, $\mathfrak{M}_k^*(N)_\Lambda$ and $\mathfrak{S}_k^*(N)_\Lambda$ are modules over the adjoint universal Hecke algebras $\mathfrak{H}_k^*(N)_{\mathcal{O}}$ and $\mathfrak{h}_k^*(N)_{\mathcal{O}}$, respectively. Because of this we may consider their ordinary projections

$$\mathfrak{M}_k^*(N)_\Lambda^{\mathrm{ord}} = e^*\mathfrak{M}_k^*(N)_\Lambda,$$
$$\mathfrak{S}_k^*(N)_\Lambda^{\mathrm{ord}} = e^*\mathfrak{S}_k^*(N)_\Lambda,$$

where e^* denotes the projective limit of Hida's idempotent with respect to T_p^*.

Theorem 2.2.8 ([O2], Theorem 2.2.3)**.** *For each $k \geqslant 2$, we have isomorphisms of Λ-modules*

$$M(N)_\Lambda^{\mathrm{ord}} \cong \mathfrak{M}_k^*(N)_\Lambda^{\mathrm{ord}}$$
$$S(N)_\Lambda^{\mathrm{ord}} \cong \mathfrak{S}_k^*(N)_\Lambda^{\mathrm{ord}}$$

given by sending $F \in M(N)_\Lambda^{\mathrm{ord}}$ to $(f_r)_{r\geqslant 1} \in \mathfrak{M}_k^*(N)_\Lambda^{\mathrm{ord}}$ defined by

$$f_r = \frac{1}{p^{r-1}}\left(\sum_{\epsilon \in \widehat{U_1/U_r}} \nu_{k,\epsilon}(F)|T_p^{-r}\right)|w_{N_r}^{-1}$$

and sending $(f_r)_{r\geqslant 1} \in \mathfrak{M}_k^*(N)_\Lambda^{\mathrm{ord}}$ to the unique element $F \in M(N)_\Lambda^{\mathrm{ord}}$ satisfying

$$\nu_{k,\epsilon}(F) = \sum_{\alpha \in U_1/U_r} \epsilon(\alpha)(f_r|w_{N_r}|T_p^r|\sigma_\alpha^{-1})$$

for all $\epsilon \in \widehat{U}_1$, where σ_α is as defined in (2).

Through these isomorphisms we can endow $M(N)^{\text{ord}}_{\Lambda}$ and $S(N)^{\text{ord}}_{\Lambda}$ with Hecke module structures. Specifically, for every $F \in M(N)^{\text{ord}}_{\Lambda}$ and all $H \in \mathfrak{H}(N)^{\text{ord}}_{\mathcal{O}}$, we define $F|H$ to be the element of $M(N)^{\text{ord}}_{\Lambda}$ corresponding to $(f_r|H^*)_{r\geqslant 1} \in \mathfrak{M}^*_k(N)^{\text{ord}}_{\Lambda}$ (note that this is independent of the weight k). In particular, this means that for all $F \in M(N)^{\text{ord}}_{\Lambda}$ we have

$$\begin{aligned} \nu_{k,\epsilon}(F|T_n) &= \nu_{k,\epsilon}(F)|T_n \\ \nu_{k,\epsilon}(F|T_{q,q}) &= \nu_{k,\epsilon}(F)|T_{q,q} \end{aligned} \tag{9}$$

for all $k \geqslant 2$ and all but finitely many $\epsilon \in \widehat{U}_1$.

2.2.3 *Λ-adic Eisenstein series*

Set $\mathcal{O}_{\theta,\psi} = \mathbb{Z}_p[\theta,\psi]$ and $\Lambda_{\theta,\psi} = \mathcal{O}_{\theta,\psi}[\![X]\!]$. In this subsection, we will construct $\Lambda_{\theta,\psi}$-adic Eisenstein series associated to the characters θ and ψ introduced in Chapter 1. However, before we do so we want to briefly introduce some of the objects and notation that will be employed in this construction, as they will also come into play in subsequent sections.

First, for any Dirichlet character χ defined modulo M_χ and any integer n, we let χ_n denote the character defined modulo $\operatorname{lcm}(M_\chi, n)$ that is induced by χ.

Next, let $\langle\!\langle \cdot \rangle\!\rangle : \mathbb{Z}_p^\times \to U_1$ be the projection defined by $\langle\!\langle a \rangle\!\rangle = a\omega(a)^{-1}$. While it is more common to denote this projection by $\langle a \rangle$, we have chosen this alternate notation to avoid confusion with diamond operators. Note that for any character $\epsilon \in \widehat{U}_1$ we have $\epsilon(a) = \epsilon(\langle\!\langle a \rangle\!\rangle)$. We define $s : \mathbb{Z}_p^\times \to \mathbb{Z}_p$ to be the group homomorphism given by $\langle\!\langle a \rangle\!\rangle = u^{s(a)}$. Then for all $a \in \mathbb{Z}_p^\times$ we have

$$(1+X)^{s(a)} = \sum_{i=0}^{\infty} \binom{s(a)}{i} X^i \in \Lambda_{\mathbb{Z}_p},$$

and for all $\epsilon \in \widehat{U}_1$ and $k \geqslant 2$, we have

$$\big(1 + (\epsilon(u)u^{k-2} - 1)\big)^{s(a)} = (\epsilon\omega^{2-k})(a)a^{k-2}. \tag{10}$$

Finally, recall from Chapter 1 that if $\varphi \neq \mathbb{1}$ is an even character with conductor not divisible by p^2, there exists a unique element $F(X,\varphi) \in \mathbb{Z}_p[\varphi][\![X]\!]$ satisfying

$$F(\epsilon(u)u^{k-2}-1,\varphi) \;=\; L_p(k-2,\varphi\epsilon^{-1})$$

for all $k \geqslant 2$ and $\epsilon \in \widehat{U}_1$, where $L_p(s,\varphi\epsilon^{-1})$ denotes the Kubota-Leopoldt p-adic L-function associated to the character $\varphi\epsilon^{-1}$. We set $S = u^{-1}(1+X)^{-1}-1$ and define $G(X,\varphi) = F(S,\varphi)$. The power series $G(X,\varphi)$ and $F(X,\varphi)$ will play an important role in subsequent chapters.

We are now ready to define our $\Lambda_{\theta,\psi}$-adic Eisenstein series. Suppose that $(\theta_0,\psi) \neq (\omega^{-2},\mathbb{1})$. Then for all integers $t \geqslant 1$ we define the following formal series in $\Lambda_{\theta,\psi}[\![q]\!]$:

$$\mathcal{E}_{\theta,\psi;t} \;=\; \frac{\psi(0)}{2}G(X,\theta\omega^2) + \sum_{n=1}^{\infty}\left(\sum_{\substack{0<d\mid n\\ p\nmid d}} \theta(d)\psi\left(\frac{n}{d}\right)(1+X)^{s(d)}d\right)q^{tn}. \tag{11}$$

We set $\mathcal{E}_{\theta,\psi} = \mathcal{E}_{\theta,\psi;1}$. It is well known that if $\theta = \theta_0$ and $\psi = \psi_0$ (i.e. θ and ψ are primitive), the above q-expansion is an element of $M(N)^{\mathrm{ord}}_{\Lambda_{\theta,\psi}}$ under certain conditions.

Theorem 2.2.9 ([O2], Theorem-Definition 2.3.10)**.** *Let $t \geqslant 1$ be prime to p. Then the power series $\mathcal{E}_{\theta_0,\psi_0;t}$ is an element of $M(N)^{\mathrm{ord}}_{\Lambda_{\theta,\psi}}$ if the following conditions are satisfied:*

(1) $f_\theta f_\psi t \mid Np$

(2) $(f_\psi, p) = 1$

(3) $(\theta_0\psi_0)(-1) = 1$.

Furthermore, for all $k \geqslant 2$ and $\epsilon \in \widehat{U}_1$ we have

$$\nu_{k,\epsilon}(\mathcal{E}_{\theta_0,\psi_0;t}) \;=\; E_k((\theta_0\epsilon\omega^{2-k})_p,\psi_0;t) \;\in\; M_k(f_\theta f_\psi p^r t/\gcd(f_\theta,p),\epsilon)^{\mathrm{ord}}_{\mathcal{O}_{\theta,\psi}[\epsilon]}$$

having Nebentypus $\theta_0\psi_0\epsilon\omega^{2-k}$, where $\ker(\epsilon) = U_r$.

The $\Lambda_{\theta,\psi}$-adic Eisenstein series $\mathcal{E}_{\theta_0,\psi_0;t}$, as θ_0, ψ_0, and t satisfying the above conditions vary, are $\Lambda_{\theta,\psi}$-linearly independent modulo $S(N)^{\mathrm{ord}}_{\Lambda_{\theta,\psi}}$.

We would like to show that $\mathcal{E}_{\theta,\psi;t}$ can be an element of $M(N)^{\mathrm{ord}}_{\Lambda_{\theta,\psi}}$ when θ and ψ are imprimitive as well. Let D_θ and D_ψ be the largest square-free factors of M_θ and M_ψ, respectively, such that

$$\gcd(D_\theta, f_\theta p) \;=\; 1 \;=\; \gcd(D_\psi, f_\psi).$$

Suppose we have the following factorizations of D_θ and D_ψ,

$$\begin{aligned} D_\theta &= p_1 \cdots p_r \\ D_\psi &= p'_1 \cdots p'_s, \end{aligned}$$

keeping in mind that the sets $\{p_1, \ldots, p_r\}$ and $\{p'_1, \ldots, p'_s\}$ may not be disjoint. For $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$ define

$$\begin{aligned} \theta_i &= \theta_{p_1 \cdots p_i} \\ \psi_j &= \psi_{p'_1 \cdots p'_j}. \end{aligned}$$

Note that $\theta = \theta_r$ and $\psi = \psi_s$.

Proposition 2.2.10. *For all integers* $t \geqslant 1$ *we have*

$$\mathcal{E}_{\theta,\psi;t} \;=\; \sum_{\substack{\alpha | D_\theta \\ \beta | D_\psi}} \alpha\mu(\alpha)\mu(\beta)\theta_0(\alpha)\psi_0(\beta)(1+X)^{s(\alpha)}\mathcal{E}_{\theta_0,\psi_0;\alpha\beta t}$$

where μ *is the Möbius function.*

Proof. We begin by considering the non-constant terms of $\mathcal{E}_{\theta,\psi;t}$. For all $n \geqslant 1$ and $1 \leqslant i \leqslant r$

$$a_{nt}(\mathcal{E}_{\theta_{i-1},\psi;t}) - a_{nt}(\mathcal{E}_{\theta_i,\psi;t}) \;=\; \sum_{\substack{0<d|n \\ p\nmid d \\ p_i | d}} \theta_{i-1}(d)\psi\left(\frac{n}{d}\right)(1+X)^{s(d)}d$$

$$= \begin{cases} \theta_{i-1}(p_i)(1+X)^{s(p_i)}p_i \displaystyle\sum_{\substack{0<d|(n/p_i) \\ p\nmid d}} \theta_{i-1}(d)\psi\left(\frac{n}{dp_i}\right)(1+X)^{s(d)}d & \text{if } p_i \mid n \\ 0 & \text{if } p_i \nmid n \end{cases}$$

$$= \;\theta_0(p_i)(1+X)^{s(p_i)}p_i \cdot a_{nt}(\mathcal{E}_{\theta_{i-1},\psi;p_it}).$$

This gives us the recursive identity

$$a_{nt}(\mathcal{E}_{\theta_i,\psi;t}) = a_{nt}(\mathcal{E}_{\theta_{i-1},\psi;t}) + p_i\mu(p_i)\theta_0(p_i)(1+X)^{s(p_i)}a_{nt}(\mathcal{E}_{\theta_{i-1},\psi;p_it}),$$

from which we obtain

$$a_{nt}(\mathcal{E}_{\theta,\psi;t}) = \sum_{\alpha|D_\theta}\alpha\mu(\alpha)\theta_0(\alpha)(1+X)^{s(\alpha)}a_{nt}(\mathcal{E}_{\theta_0,\psi;\alpha t}).$$

All that remains to be shown is that

$$a_{nt}(\mathcal{E}_{\theta_0,\psi;\alpha t}) = \sum_{\beta|D_\psi}\mu(\beta)\psi(\beta)a_{nt}(\mathcal{E}_{\theta_0,\psi_0;\alpha\beta t}). \quad (12)$$

Note that

$$a_{nt}(\mathcal{E}_{\theta_0,\psi;\alpha t}) = 0 = a_{nt}(\mathcal{E}_{\theta_0,\psi;\alpha\beta t})$$

for all $\beta \mid D_\psi$ if $\alpha \nmid n$. Suppose $\alpha \mid n$ and let $n = m\alpha$. Then for $1 \leqslant j \leqslant s$ we have

$$a_{m\alpha t}(\mathcal{E}_{\theta_0,\psi_{j-1};\alpha t}) - a_{m\alpha t}(\mathcal{E}_{\theta_0,\psi_j;\alpha t}) = \sum_{\substack{0<d|m \\ p\nmid d \\ p_j'|(m/d)}}\theta_0(d)\psi_{j-1}\left(\frac{m}{d}\right)(1+X)^{s(d)}d.$$

$$= \begin{cases} \psi_0(p_j')\displaystyle\sum_{\substack{0<d|(m/p_j') \\ p\nmid d}}\theta_0(d)\psi_{j-1}\left(\frac{m/p_j'}{d}\right)(1+X)^{s(d)}d & \text{if } p_j' \mid m \\ 0 & \text{if } p_j' \nmid m \end{cases}$$

$$= \psi_0(p_j')\cdot a_{m\alpha t}(\mathcal{E}_{\theta_0,\psi_{j-1};\alpha t p_j'}).$$

Applying the same recursive argument as above we obtain (12).

Finally, we want to consider the constant term of $\mathcal{E}_{\theta,\psi;t}$. If $\psi_0 \neq \mathbb{1}$, then $a_0(\mathcal{E}_{\theta,\psi;t}) = 0 = a_0(\mathcal{E}_{\theta_0,\psi_0;t})$. Suppose $\psi_0 = \mathbb{1}$ and note that

$$\sum_{\beta|D_\psi}\mu(\beta) = \begin{cases} 1 & D_\psi = 1 \\ 0 & D_\psi > 1 \end{cases}.$$

Then for all $k \geqslant 2$ we have

$$a_0\big(\nu_{k,\mathbb{1}}(\mathcal{E}_{\theta,\psi;t})\big) = \frac{\psi(0)}{2}L(1-k,\theta_p) = \left(\sum_{\beta|D_\psi}\mu(\beta)\right)\frac{L(1-k,\theta_p)}{2}. \quad (13)$$

Recall that if χ is any Dirichlet character, $L(1-s,\chi)$ has the following Euler product expansion

$$L(1-s,\chi) = \prod_{\ell \text{ prime}} \left(1-\ell^{s-1}\chi(\ell)\right)^{-1}$$

for all $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) > 1$ [Wa, Chapter 4]. Therefore, we may write (13) as

$$= \left(\sum_{\beta|D_\psi} \mu(\beta)\right) \prod_{\ell|D_\theta} (1-\ell^{k-1}\theta_0(\ell)) \frac{L(1-k,(\theta_0)_p)}{2}$$

$$= \left(\sum_{\beta|D_\psi} \mu(\beta)\right) \prod_{\ell|D_\theta} (1-\ell^{k-1}\theta_0(\ell))\, a_0\big(\nu_{k,\mathbb{1}}(\mathcal{E}_{\theta_0,\psi_0;t})\big)$$

$$= \left(\sum_{\beta|D_\psi} \mu(\beta)\right) \left(\sum_{\alpha|D_\theta} \alpha^{k-1}\mu(\alpha)\theta_0(\alpha)a_0\big(\nu_{k,\mathbb{1}}(\mathcal{E}_{\theta_0,\psi_0;t})\big)\right)$$

$$= \sum_{\substack{\alpha|D_\theta \\ \beta|D_\psi}} \alpha^{k-1}\mu(\alpha\beta)\theta_0(\alpha)a_0\big(\nu_{k,\mathbb{1}}(\mathcal{E}_{\theta_0,\psi_0;\alpha\beta t})\big), \tag{14}$$

with the last equality following from the fact that $a_0(\mathcal{E}_{\theta_0,\psi_0;\alpha\beta t}) = a_0(\mathcal{E}_{\theta_0,\psi_0;t})$ for all α, β, and t. For all $k \equiv 2\ (p-1)$ we may write (14) as

$$= \sum_{\substack{\alpha|D_\theta \\ \beta|D_\psi}} \alpha\mu(\alpha\beta)\theta_0(\alpha)(1+(u^{k-2}-1))^{s(\alpha)}a_0\big(\nu_{k,\mathbb{1}}(\mathcal{E}_{\theta_0,\psi_0;\alpha\beta t})\big).$$

Since $a_0(\mathcal{E}_{\theta,\psi;t})$ and

$$\sum_{\substack{\alpha|D_\theta \\ \beta|D_\psi}} \alpha\mu(\alpha\beta)\theta_0(\alpha)(1+X)^{s(\alpha)}a_0(\mathcal{E}_{\theta_0,\psi_0;\alpha\beta t})$$

are equal when evaluated at $X = u^{k-2}-1$ for infinitely many positive integers k, and $|u^{k-2}-1|_p < 1$, the two must be equal as a consequence of the Weierstrass preparation theorem [Wa, Corollary 7.4].

□

Since $M_\theta M_\psi = N$ or Np, $p \nmid M_\psi$, and the product $\theta\psi$ is even, we have the following immediate consequence of the above proposition and Theorem 2.2.9.

Corollary 2.2.11. *Let* t *be a positive integer coprime to* p. *The formal power series* $\mathcal{E}_{\theta,\psi;t}$ *is an element of* $M(Nt)^{\mathrm{ord}}_{\Lambda_{\theta,\psi}}$.

We conclude this subsection by showing that the Eisenstein series $\mathcal{E}_{\theta,\psi}$ is a normalized common eigenform for $\mathfrak{H}(N)_{\mathcal{O}}$.

Proposition 2.2.12.

$$\begin{aligned} \mathcal{E}_{\theta,\psi}|T_{q,q} &= (\theta\psi)(q)(1+X)^{s(q)} \cdot \mathcal{E}_{\theta,\psi} && (q \text{ a positive integer prime to } Np) \\ \mathcal{E}_{\theta,\psi}|T_\ell &= (\theta(\ell)\ell(1+X)^{s(\ell)} + \psi(\ell)) \cdot \mathcal{E}_{\theta,\psi} && (\text{primes } \ell \neq p) \\ \mathcal{E}_{\theta,\psi}|T_p &= \psi_0(p) \cdot \mathcal{E}_{\theta,\psi} \end{aligned}$$

Proof. It is well known that $\mathcal{E}_{\theta_0,\psi_0;t}$ satisfies the above identities for all operators except T_ℓ when $\ell \mid N$ [O4, Lemma 1.4.8]. Therefore, by Proposition 2.2.10 we know that the same holds for $\mathcal{E}_{\theta,\psi}$.

Suppose $\ell \mid N = M_\theta M_\psi / \gcd(p, M_\theta)$. By Proposition 2.2.9 we know that $\nu_{k,\epsilon}(\mathcal{E}_{\theta,\psi})$ has Nebetypus $\theta\psi\epsilon\omega^{2-k}$ for all $k \geqslant 2$ and $\epsilon \in \widehat{U}_1$. This, along with the fact that $\nu_{k,\epsilon}(\mathcal{E}_{\theta,\psi}|T_\ell) = \nu_{k,\epsilon}(\mathcal{E}_{\theta,\psi})|T_\ell$, allows us to explicitly write down the action of T_ℓ on the q-expansion of $\mathcal{E}_{\theta,\psi}$. Specifically, by Proposition 2.1.4, for all $n \geqslant 0$ we have

$$a_n(\mathcal{E}_{\theta,\psi}|T_\ell) = a_{n\ell}(\mathcal{E}_{\theta,\psi}) + (\theta\psi)(\ell)\ell(1+X)^{s(\ell)} a_{n/\ell}(\mathcal{E}_{\theta,\psi}) = a_{n\ell}(\mathcal{E}_{\theta,\psi}),$$

where $a_{n/\ell} = 0$ if $\ell \nmid n$. If $n \geqslant 1$ we have

$$\begin{aligned} a_{n\ell}(\mathcal{E}_{\theta,\psi}) &= \sum_{\substack{d \mid n\ell \\ p \nmid d}} \theta(d)\psi\left(\frac{n\ell}{d}\right)(1+X)^{s(d)} d \\ &= \begin{cases} \psi(\ell) \cdot a_n(\mathcal{E}_{\theta,\psi}) & \ell \mid M_\theta, \ell \nmid M_\psi \\ \theta(\ell)(1+X)^{s(\ell)}\ell \cdot a_n(\mathcal{E}_{\theta,\psi}) & \ell \nmid M_\theta, \ell \mid M_\psi \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Recalling that $a_0(\mathcal{E}_{\theta,\psi}) = 0$ if ψ is not the trivial character, we see that the above holds when $n = 0$ as well.

$\square$

We remark that $\mathcal{E}_{\theta,\psi;t}$ is not necessarily a T_ℓ-eigenform for primes $\ell \mid N$ if $t > 1$. To see why, suppose $\ell \mid t$ with $t' = t/\ell$. Then we have

$$a_{t'}(\mathcal{E}_{\theta,\psi;t}|T_\ell) = a_t(\mathcal{E}_{\theta,\psi;t}) = 1,$$

while $a_{t'}(\mathcal{E}_{\theta,\psi;t}) = 0$.

2.2.4 *Duality*

In this subsection we prove a duality result between Λ-adic modular forms and the universal ordinary Hecke algebra. For any Λ-module M, we set $M_{Q(\Lambda)} = M \otimes_\Lambda Q(\Lambda)$ and denote the Λ-dual of M by $M^\vee$.

Proposition 2.2.13. *Let $\mathcal{A}$ be a free and finitely generated Λ-submodule of $M(N)^{\mathrm{ord}}_\Lambda$ that is stable under the action $\mathfrak{H}(N)^{\mathrm{ord}}_\mathcal{O}$. Define $\mathfrak{H}(\mathcal{A})$ to be the Λ-subalgebra of $\mathrm{End}_\Lambda(\mathcal{A})$ generated by the Hecke operators $\{T_n : n \geqslant 1\}$. Suppose*

$$\{F \in \mathcal{A}_{Q(\Lambda)} : a_n(F) \in \Lambda \textit{ for all } n \geqslant 1\} = \mathcal{A}.$$

Then we have the following isomorphisms of Λ-modules,

$$\mathcal{A} \cong \mathfrak{H}(\mathcal{A})^\vee \quad : F \mapsto a_1(F|\cdot) \tag{15}$$

$$\mathfrak{H}(\mathcal{A}) \cong \mathcal{A}^\vee \quad : H \mapsto a_1(\cdot|H). \tag{16}$$

Proof. Set $\mathcal{A}_2 = \nu_{2,\mathbb{1}}(\mathcal{A}) \subset M_2(\Gamma_1, \mathbb{1})^{\mathrm{ord}}_\mathcal{O}$ and define $\mathfrak{H}(\mathcal{A}_2)$ to be the $\mathcal{O}$-subalgebra of $\mathrm{End}_\mathcal{O}(\mathcal{A}_2)$ generated by the Hecke operators $\{T_n \in \mathfrak{H}_2(\Gamma_1)^{\mathrm{ord}}_\mathcal{O} : n \geqslant 1\}$. Suppose $F \in \mathcal{A}$ satisfies $\nu_{2,\mathbb{1}}(F) = 0$. Then by Proposition 2.2.3 we know $F = XF'$ for some $F' \in M(N)^{\mathrm{ord}}_\Lambda$. However, our assumption that

$$\{F \in \mathcal{A}_{Q(\Lambda)} : a_n(F) \in \Lambda \text{ for all } n \geqslant 1\} = \mathcal{A}$$

implies $F' \in \mathcal{A}$. Hence, the specialization map $\nu_{2,\mathbb{1}} : M(N)^{\text{ord}}_{\Lambda} \to M_2(\Gamma_1, \mathbb{1})^{\text{ord}}_{\mathcal{O}}$ induces an isomorphism $\mathcal{A}/X\mathcal{A} \cong \mathcal{A}_2$.

Let $\mathcal{X} = \mathcal{A}$ or $\mathcal{A}_2$, with $\mathfrak{H}(\mathcal{X})$ denoting the corresponding Hecke algebra. We set

$$R = \begin{cases} \Lambda & \mathcal{X} = \mathcal{A} \\ \mathcal{O} & \mathcal{X} = \mathcal{A}_2 \end{cases}$$

and begin by showing

$$\mathcal{X}_{Q(R)} \cong \text{Hom}_{Q(R)}(\mathfrak{H}(\mathcal{X})_{Q(R)}, Q(R)). \tag{17}$$

Since R is a Noetherian ring and $\mathfrak{H}(\mathcal{X})$ is an R-submodule of the finitely generated R-module $\text{End}_R(\mathcal{X})$, we know that $\mathfrak{H}(\mathcal{X})$ is finitely generated. Because $\mathcal{X}$ is finitely generated by assumption, in order to prove (17) it suffices to show the pairing

$$\mathfrak{H}(\mathcal{X})_{Q(R)} \times \mathcal{X}_{Q(R)} \to Q(R)$$

defined by $(F, H) \mapsto a_1(F|H)$ is non-degenerate.

Let $H \in \mathfrak{H}(\mathcal{X})_{Q(R)}$ and suppose $a_1(F|H) = 0$ for all $F \in \mathcal{X}_{Q(R)}$. Then we have $a_n(F|H) = a_1(F|H|T_n) = a_1(F|T_n|H) = 0$, for all $n \geqslant 1$, which implies $F|H \in Q(R)$. However, by definition there are no non-trivial constant classical modular forms of positive weight (Definition 2.1.1). Therefore, there are no non-trivial constant forms in $\mathcal{X}$, which implies $F|H = 0$. Since F was arbitrary it follows that H is the zero operator.

Next, let $F \in \mathcal{X}_{Q(R)}$ and suppose $a_1(F|H) = 0$ for all $H \in \mathfrak{H}(\mathcal{X})_{Q(R)}$. Then in particular, for all $n \geqslant 1$ we have $a_n(F) = a_1(F|T_n) = 0$, which by the same argument as above implies $F = 0$. Hence, we have proven (17).

We will now use (17) to show $\mathcal{X} \cong \text{Hom}_R(\mathfrak{H}(\mathcal{X}), R)$. By the above result and the fact that $\mathcal{X}$ is R-torsion free, the R-module homomorphism from $\mathcal{X} \to \text{Hom}_R(\mathfrak{H}(\mathcal{X}), R)$ is injective. To show surjectivity, note that any $\varphi \in \text{Hom}_R(\mathfrak{H}(\mathcal{X}), R)$ can be "lifted" to an element $\varphi' \in \text{Hom}_{Q(R)}(\mathfrak{H}(\mathcal{X})_{Q(R)}, Q(R))$ by defining $\varphi'(H) = \varphi(H)$ for all $H \in \mathfrak{H}(\mathcal{X})$ and extending $Q(R)$-linearly. By what was previously shown, we know that there exists an $F \in \mathcal{X}_{Q(R)}$ such that $\varphi'(H) = a_1(F|H)$ for all $H \in \mathfrak{H}(\mathcal{X})_{Q(R)}$. However, this implies that

$$a_n(F) = a_1(F|T_n) = \varphi'(T_n) = \varphi(T_n) \in \Lambda,$$

for all $n \geqslant 1$. Since the constant term of every form in $\mathcal{X}$ is 0 by assumption, we know $F \in \mathcal{X}$ and we have $\mathcal{X} \cong \mathrm{Hom}_R(\mathfrak{H}(\mathcal{X}), R)$. In fact, we can say more when $\mathcal{X} = \mathcal{A}_2$. Because $\mathcal{A}_2$ is a torsion-free $\mathcal{O}$-module, we know that $\mathfrak{H}(\mathcal{A}_2)$ is as well. Therefore, $\mathfrak{H}(\mathcal{A}_2)$ is a finitely generated, torsion-free $\mathcal{O}$-module, and the fact that $\mathcal{O}$ is a principal ideal domain implies $\mathfrak{H}(\mathcal{A}_2)$ is free. Hence,

$$\mathfrak{H}(\mathcal{A}_2) \cong \mathrm{Hom}_{\mathcal{O}}(\mathrm{Hom}_{\mathcal{O}}(\mathfrak{H}(\mathcal{A}_2), \mathcal{O}), \mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{A}_2). \qquad (18)$$

Finally, we will prove (16). Once again, by the above result and the fact that $\mathcal{A}$ is Λ-torsion free, we know the Λ-module homomorphism $\mathfrak{H}(\mathcal{A}) \to \mathcal{A}^\vee \cong \mathfrak{H}(\mathcal{A})^{\vee\vee}$ defined by $H \mapsto a_1(\,\cdot\,|H)$ is injective. Let $\mathcal{Q}$ denote the cokernel of this map. We begin by showing $\mathcal{Q}$ is a finite Λ-module.

Let $\mathfrak{p} \subset \Lambda$ be a height 1 prime ideal. Since $\mathfrak{H}(\mathcal{A})$ is a finitely generated, torsion-free Λ-module, we know that the localization $\mathfrak{H}(\mathcal{A})_\mathfrak{p}$ of $\mathfrak{H}(\mathcal{A})$ with respect to the prime $\mathfrak{p}$ is a finitely generated, torsion-free module over the discrete valuation ring $\Lambda_\mathfrak{p}$. Consequently, $\mathfrak{H}(\mathcal{A})_\mathfrak{p}$ is free and the map

$$\mathfrak{H}(\mathcal{A})_\mathfrak{p} \to \mathrm{Hom}_{\Lambda_\mathfrak{p}}(\mathrm{Hom}_{\Lambda_\mathfrak{p}}(\mathfrak{H}(\mathcal{A})_\mathfrak{p}, \Lambda_\mathfrak{p}), \Lambda_\mathfrak{p})$$

is an isomorphism. Since localization is exact and our height 1 prime ideal $\mathfrak{p} \subset \Lambda$ was arbitrary, we must have $\mathcal{Q}_\mathfrak{p} = 0$ for all height 1 prime ideals $\mathfrak{p} \subset \Lambda$. This means that $\mathcal{Q}$ is a pseudo-null Λ-module, which implies it is finite [NSW, Chapter V, §1].

Now, consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & \mathrm{Tor}_X(\mathcal{Q}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{H}(\mathcal{A}) & \longrightarrow & \mathcal{A}^\vee & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow X & & \downarrow X & & \downarrow X & & \\
0 & \longrightarrow & \mathfrak{H}(\mathcal{A}) & \longrightarrow & \mathcal{A}^\vee & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathfrak{H}(\mathcal{A})/X\mathfrak{H}(\mathcal{A}) & & \mathcal{A}^\vee/X\mathcal{A}^\vee & & \mathcal{Q}/X\mathcal{Q} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & ,
\end{array}
$$

where $\mathrm{Tor}_X(\mathcal{Q}) := \{\alpha \in \mathcal{Q} : X \cdot \alpha = 0\}$. By the snake lemma we obtain the following exact sequence,

$$0 \longrightarrow \mathrm{Tor}_X(\mathcal{Q}) \longrightarrow \mathfrak{H}(\mathcal{A})/X\mathfrak{H}(\mathcal{A}) \longrightarrow \mathcal{A}^\vee/X\mathcal{A}^\vee \longrightarrow \mathcal{Q}/X\mathcal{Q} \longrightarrow 0. \qquad (19)$$

We would like to show that the map $\mathfrak{H}(\mathcal{A})/X\mathfrak{H}(\mathcal{A}) \to \mathcal{A}^\vee/X\mathcal{A}^\vee$ is surjective. Note that this map is defined by $H + X\mathfrak{H}(\mathcal{A}) \mapsto \varphi_H + X\mathcal{A}^\vee$, where $\varphi_H \in \mathcal{A}^\vee$ is defined by $F \mapsto a_1(F|H)$. Since $\mathrm{Tor}_X(\mathcal{Q}) \subset \mathcal{Q}$ is finite, we know that the image of $\mathfrak{H}(\mathcal{A})/X\mathfrak{H}(\mathcal{A})$ in the free $\mathcal{O}$-module $\mathcal{A}^\vee/X\mathcal{A}^\vee$ is the submodule generated by $\{\varphi_n + P_k\mathcal{A}^\vee : n \geqslant 1\}$, where $\varphi_n \in \mathcal{A}^\vee$ is defined by $F \mapsto a_1(F|T_n)$. Therefore, in order to prove our desired surjectivity, it suffices to show that the set $\{\varphi_n + X\mathcal{A}^\vee : n \geqslant 1\}$ generates all of $\mathcal{A}^\vee/X\mathcal{A}^\vee$ as an $\mathcal{O}$-module. To do so, consider the following composition of $\mathcal{O}$-module isomorphisms,

$$\mathcal{A}^\vee/X\mathcal{A}^\vee \;\cong\; \mathcal{A}^\vee \otimes_\Lambda \Lambda/(X) \;\cong\; \mathrm{Hom}_{\Lambda/(X)}(\mathcal{A} \otimes_\Lambda \Lambda/(X), \Lambda/(X))$$

$$\cong\; \mathrm{Hom}_{\Lambda/(X)}(\mathcal{A}/X\mathcal{A}, \Lambda/(X)) \;\cong\; \mathrm{Hom}_{\mathcal{O}}(\mathcal{A}_2, \mathcal{O}) \;\cong\; \mathfrak{H}(\mathcal{A}_2),$$

with the second isomorphism following from the fact that $\mathcal{A}$ is a free and finitely generated Λ-module. In particular, this composition sends $\varphi_n + X\mathcal{A}^\vee$ to $T_n \in \mathfrak{H}(\mathcal{A}_2)$. Since $\mathfrak{H}(\mathcal{A}_2)$ is the free $\mathcal{O}$-module generated by $\{T_n : n \geqslant 1\}$, we know that $\mathcal{A}^\vee/X\mathcal{A}^\vee$ is the free $\mathcal{O}$-module generated by $\{\varphi_n + X\mathcal{A}^\vee : n \geqslant 1\}$.

Finally, because the map $\mathfrak{H}(\mathcal{A})/X\mathfrak{H}(\mathcal{A}) \to \mathcal{A}^\vee/X\mathcal{A}^\vee$ is surjective we have $\mathcal{Q}/X\mathcal{Q} = 0$. Nakayama's lemma then implies $\mathcal{Q} = 0$. $\square$

3

THE Λ-ADIC RESIDUE MAP

Our primary goal in this chapter is to determine the image of Eisenstein series under Ohta's Λ-adic residue map. While this image was determined by Ohta [O4] for Eisenstein series associated to pairs of primitive, non-exceptional characters, we would like to generalize this result to pairs of arbitrary characters.

3.1 CUSPIDAL GROUPS

For a positive integer M, we let $X_1(M)$ denote the modular curve $\Gamma_1(M) \backslash \mathbb{H}^*$, where $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. In this section we will consider the cusps of $X_1(M)$, that is, the $\Gamma_1(M)$-equivalence classes of $\mathbb{Q} \cup \{\infty\}$. We begin by giving an algebraic description of these cusps following Ohta [O4, Section 2.1] and Shimura [Sh, Section 1.6].

We denote the cusps of $X_1(M)$ by C_M, which we identify with $\Gamma_1(M) \backslash \mathbb{P}^1(\mathbb{Q})$. The map $SL_2(\mathbb{Z}) \to \mathbb{P}^1(\mathbb{Q})$ (resp., $GL_2^+(\mathbb{Q}) \to \mathbb{P}^1(\mathbb{Q})$) defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a}{c}$$

induces a bijection

$$\Gamma_1(M) \backslash SL_2(\mathbb{Z}) / \Gamma_\infty \ \to \ \Gamma_1(M) \backslash \mathbb{P}^1(\mathbb{Q})$$
$$\big(\text{resp., } \Gamma_1(M) \backslash GL_2^+(\mathbb{Q}) / \tilde{\Gamma}_\infty \ \to \ \Gamma_1(M) \backslash \mathbb{P}^1(\mathbb{Q})\big)$$

where $\Gamma_\infty \subset SL_2(\mathbb{Z})$ (resp., $\tilde{\Gamma}_\infty \subset GL_2^+(\mathbb{Q})$) is the isotropy subgroup of the cusp at ∞,

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}) \right\}$$

$$\tilde{\Gamma}_\infty := \left\{ \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \in GL_2^+(\mathbb{Q}) \right\}.$$

Let

$$A_M := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in (\mathbb{Z}/M\mathbb{Z})^2 : \gcd(x, y) = 1 \right\} / \sim, \tag{20}$$

where

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim \begin{bmatrix} x' \\ y' \end{bmatrix} \iff \begin{array}{l} x \equiv x' \pmod{\gcd(M, y)} \\ y \equiv y' \pmod{M} \end{array}$$

and denote by

$$\begin{bmatrix} a \\ c \end{bmatrix}_M = \text{class of } \begin{bmatrix} a \\ c \end{bmatrix} \text{ in } A_M.$$

Then we have a natural bijection between $\Gamma_1(M) \backslash SL_2(\mathbb{Z}) / \Gamma_\infty$ (resp., $\Gamma_1(M) \backslash GL_2^+(\mathbb{Q}) / \tilde{\Gamma}_\infty$) and $A_M/\{\pm 1\}$ induced by the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{bmatrix} a \\ c \end{bmatrix}_M \mod \{\pm 1\}.$$

Hence, we can identify C_M with $A_M/\{\pm 1\}$. To make the notation less cumbersome, let

$$\begin{bmatrix} a \\ c \end{bmatrix}'_M = \begin{bmatrix} a \\ c \end{bmatrix}_M \mod \{\pm 1\}.$$

For any two coprime integers M_1 and M_2 satisfying $M = M_1 M_2$, there are bijections

$$\Gamma_1(M) \backslash SL_2(\mathbb{Z}) / \Gamma_\infty \xrightarrow{\sim} A_M/\{\pm 1\} \xrightarrow{\sim} (A_{M_1} \times A_{M_2})/\{\pm 1\}$$

induced by the maps

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{bmatrix} a \\ c \end{bmatrix}_M' \mapsto \left(\begin{bmatrix} a \\ c \end{bmatrix}_{M_1}, \begin{bmatrix} a \\ c \end{bmatrix}_{M_2} \right)'.$$

Unfortunately, this decomposition does not hold with respect to C_M, that is, in general

$$(A_{M_1} \times A_{M_2})/\{\pm 1\} \neq (A_{M_1}/\{\pm 1\}) \times (A_{M_2}/\{\pm 1\}).$$

For this reason we will often work directly with A_M and then reduce modulo $\{\pm 1\}$ to obtain an element of C_M.

Finally, for a ring R, let $R[A_M]$ denote the free R-module generated by A_M. By the decomposition above, we have

$$R[A_M] \cong R[A_{M_1}] \otimes_R R[A_{M_2}].$$

We can then define $R[C_M]$ as the quotient of $R[A_M]$ by the R-submodule generated by the set $\{\mathfrak{a} - (-1)\mathfrak{a} : \mathfrak{a} \in A_M\}$.

3.1.1 *Hecke operators acting on cuspidal groups*

Let $r \geqslant 1$, and once again let $\mathcal{O}$ denote the ring of integers of some complete subfield of $\mathbb{C}_p$. To simplify the notation a bit, let $A_r = A_{N_r}$ and $C_r = C_{N_r}$. In this section and the next, we will consider the action of Hecke operators on $\mathcal{O}[A_r]$ and $\mathcal{O}[C_r]$.

For any $\alpha \in GL_2^+(\mathbb{Q})$, we define the action of the double coset

$$\Gamma_r \alpha \Gamma_r = \coprod_i \Gamma_r \beta_i$$

on $\mathcal{O}[A_r]$ by

$$[\Gamma_r \alpha \Gamma_r] \begin{bmatrix} a \\ c \end{bmatrix}_{N_r} \mapsto \sum_i \beta_i \begin{bmatrix} a \\ c \end{bmatrix}_{N_r}.$$

Taking

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_r)$$

we get the diamond operator $\langle d \rangle$, while taking

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \tag{21}$$

for a prime number ℓ gives us the operator T_ℓ. As in the case of modular forms, the action of the operators T_ℓ and $\langle d \rangle$ commute with one another.

We remark that our notation for the operator defined by (21) differs from that of Ohta in [O2] and [O4], where this operator is denoted by T_ℓ^*. The reason for this difference in notation stems from the fact that Ohta identifies the cuspidal group $\mathcal{O}[C_r]$ with its dual group $\mathrm{Hom}(\mathcal{O}[C_r], \mathcal{O})$ via the perfect pairing

$$\mathcal{O}[C_r] \times \mathcal{O}[C_r] \to \mathcal{O} \; : \; \left(\sum_{c \in C_r} a_c c, \sum_{c \in C_r} b_c c \right) \mapsto \sum_{c \in C_r} a_c b_c.$$

One can show that under this identification the action of adjoint operator T_ℓ^* is given by the double coset defining T_ℓ above [O2, Proposition 3.4.12].

For future reference we want to determine the action of the operator $\langle d \rangle$ explicitly. For $d \in (\mathbb{Z}/N_r\mathbb{Z})^\times$, the action of the diamond operator $\langle d \rangle$ on A_r is given by

$$\langle d \rangle \begin{bmatrix} a \\ c \end{bmatrix}_{N_r} = \begin{bmatrix} d'a \\ dc \end{bmatrix}_{N_r}$$

where d' is an integer such that $dd' \equiv 1 \pmod{N_r}$. From the definition we see that the action of $(\mathbb{Z}/N_r\mathbb{Z})^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/p^r\mathbb{Z})^\times$ via the diamond operator is compatible with the decomposition $A_r = A_N \times A_{p^r}$.

Of particular interest to us will be the operators $\langle \alpha \rangle$ for $\alpha \in U_1/U_r \hookrightarrow \{1\} \times (\mathbb{Z}/p^r\mathbb{Z})^\times \subset (\mathbb{Z}/N_r\mathbb{Z})^\times$. Let $\sigma_\alpha \in \Gamma_1$ be a chosen element satisfying

$$\sigma_\alpha \equiv \begin{pmatrix} \alpha^{-1} & * \\ 0 & \alpha \end{pmatrix} \pmod{p^r}$$

Then $\langle\alpha\rangle\mathfrak{a} = \sigma_\alpha\mathfrak{a}$ for all $\mathfrak{a} \in A_r$.

The above operators induce operators on $\mathcal{O}[C_r]$ via the projection mapping $\mathcal{O}[A_r] \twoheadrightarrow \mathcal{O}[C_r]$, which we will denote by the same symbols.

3.1.2 *Ordinary cuspidal group*

We set $A_r^{\mathrm{ord}} = eA_r$ and $C_r^{\mathrm{ord}} = eC_r$. In this section our goal is to give a description of $\mathcal{O}[A_r^{\mathrm{ord}}]$ and $\mathcal{O}[C_r^{\mathrm{ord}}]$ as $\mathcal{O}[U_1/U_r]$-modules.

Proposition 3.1.1 ([O2] Prop. 4.3.4, [O4] (2.2.3))**.** *Let*

$$D_r = \left\{ \begin{bmatrix} a \\ c \end{bmatrix}_{N_r} \in A_r : p \mid c \right\}.$$

Then $\mathcal{O}[A_r^{\mathrm{ord}}] \cong \mathcal{O}[A_r]/\mathcal{O}[D_r]$.

Consider the set

$$A_r^0 = \left\{ \left(\begin{bmatrix} a \\ c \end{bmatrix}_N , \begin{bmatrix} 0 \\ \omega(c) \end{bmatrix}_{p^r} \right) \in A_r : \begin{array}{l} 0 < c < Np, \ \gcd(c,p) = 1 \\ 0 \leqslant a < \gcd(N,c) \end{array} \right\}.$$

In the next proposition we show that A_r^0 is an $\mathcal{O}[U_1/U_r]$-basis for $\mathcal{O}[A_r^{\mathrm{ord}}]$.

Proposition 3.1.2. $\mathcal{O}[U_1/U_r][A_r^0] = \mathcal{O}[A_r^{\mathrm{ord}}]$.

Proof. It will suffice to show

$$\{\mathfrak{a} \in A_r : e\mathfrak{a} \neq 0\} = \{\sigma_\gamma \mathfrak{a} : \gamma \in U_1/U_r, \ \mathfrak{a} \in A_r^0\}.$$

By Proposition 3.1.1 and the fact that Hida's idempotent e commutes with diamond operators, we know that

$$\{\sigma_\gamma \mathfrak{a} : \gamma \in U_1/U_r, \mathfrak{a} \in A_r^0\} \subset \{\mathfrak{a} \in A_r : e\mathfrak{a} \neq 0\}.$$

So we only need to prove the opposite inclusion.

Let $\mathfrak{a} \in A_r$ with $e\mathfrak{a} \neq 0$. Then once again by Proposition 3.1.1 we know

$$\mathfrak{a} = \begin{bmatrix} a \\ c \end{bmatrix}_{N_r} = \left(\begin{bmatrix} a \\ c \end{bmatrix}_N , \begin{bmatrix} a \\ c \end{bmatrix}_{p^r} \right)$$

with $p \nmid c$. Since $\gcd(c, p) = 1$, we have

$$\begin{bmatrix} a \\ c \end{bmatrix}_{p^r} = \begin{bmatrix} 0 \\ c \end{bmatrix}_{p^r} .$$

Finally, we note that

$$\left(\begin{bmatrix} a \\ c \end{bmatrix}_N , \begin{bmatrix} 0 \\ c \end{bmatrix}_{p^r} \right) = \left(\begin{bmatrix} a \\ c \end{bmatrix}_N , \begin{bmatrix} 0 \\ \langle\langle c \rangle\rangle \omega(c) \end{bmatrix}_{p^r} \right) = \sigma_{\langle\langle c \rangle\rangle} \cdot \left(\begin{bmatrix} a \\ c \end{bmatrix}_N , \begin{bmatrix} 0 \\ \omega(c) \end{bmatrix}_{p^r} \right) .$$

□

We define $C_r^0 = A_r^0 / \{\pm 1\}$. The utility of the above proposition will be realized in the next section.

3.1.3 *Λ-adic cuspidal group*

For all $s \geqslant r \geqslant 1$, the map

$$\begin{bmatrix} a \\ c \end{bmatrix}'_{N_s} \mapsto \begin{bmatrix} a \\ c \end{bmatrix}'_{N_r} . \tag{22}$$

induces a surjection $\mathcal{O}[C_s^{\mathrm{ord}}] = \mathcal{O}[U_1/U_s][C_s^0] \twoheadrightarrow \mathcal{O}[U_1/U_r][C_r^0] = \mathcal{O}[C_r^{\mathrm{ord}}]$. Furthermore, from Subsection 3.1.1 we see that the Hecke action commutes with these surjections. We define the Λ-adic cuspidal group by

$$C(N)_\Lambda^{\mathrm{ord}} = \varprojlim_{r \geqslant 1} \mathcal{O}[C_r^{\mathrm{ord}}] = \varprojlim_{r \geqslant 1} \mathcal{O}[U_1/U_r][C_r^0].$$

From the right hand side of the above definition, we see that $C(N)^{\mathrm{ord}}_{\Lambda}$ is a module over

$$\Lambda = \mathcal{O}[\![X]\!] \cong \varprojlim_{r \geqslant 1} \mathcal{O}[U_1/U_r].$$

3.2 RESIDUES OF Λ-ADIC EISENSTEIN SERIES

Let θ, ψ, and $\xi = (\theta_0\psi_0^{-1})_0$ be as in Chapter 1. For the remainder of the chapter we set $\mathcal{E} = \mathcal{E}_{\theta,\psi}$ and let $\mathcal{O}_\infty$ denote the ring of integers of a complete subfield of $\mathbb{C}_p$ containing *all roots of unity*. We set $\Lambda_\infty = \mathcal{O}_\infty[\![X]\!]$.

In [O4], Ohta constructs the following exact sequence of Hecke modules

$$0 \longrightarrow S(N)^{\mathrm{ord}}_{\Lambda_\infty} \longrightarrow M(N)^{\mathrm{ord}}_{\Lambda_\infty} \xrightarrow{\mathrm{Res}_\Lambda} C(N)^{\mathrm{ord}}_{\Lambda_\infty} \longrightarrow 0, \tag{23}$$

where the map Res_Λ is the Λ-adic residue map, defined explicitly by

$$\mathrm{Res}_\Lambda(F) = \varprojlim_{r \geqslant 1} \left(\frac{1}{p^{r-1}} \sum_{\mathfrak{c} \in C_r} \left(\sum_{\epsilon \in \widehat{U_1/U_r}} \mathrm{Res}_{\mathfrak{c}}\Big(\nu_{2,\epsilon}(F)|T_p^{-r}|w_{N_r}^{-1}\Big) \right) \cdot e\mathfrak{c} \right), \tag{24}$$

where $\mathrm{Res}_{\mathfrak{c}}(f)$ denotes the residue of the differential $\omega_f = f\frac{dq}{q}$ at the cusp $\mathfrak{c}$. Our goal for the remainder of this section is to prove the following Proposition.

Proposition 3.2.1. *Suppose* $(\theta_0, \psi) \neq (\omega^{-2}, \mathbb{1})$. *Then* $\mathrm{Res}_\Lambda(\mathcal{E}) = A \cdot \mathfrak{e}_\infty$ *where*

$$A = A_{\theta,\psi} := \prod_{\substack{\ell | \tilde{f}_\theta f_\psi \\ \ell \nmid f_\xi}} \Big((1+X)^{s(\ell)} - (\xi\omega^2)^{-1}(\ell)\langle\!\langle \ell \rangle\!\rangle^{-2} \Big) G(X, \xi\omega^2),$$

and $\mathfrak{e}_\infty := \mathfrak{e}_\infty^{\theta,\psi} \in C(N)^{\mathrm{ord}}_{\Lambda_{\theta,\psi}}$.

Before moving on, let us determine exactly what proving Proposition 3.2.1 will entail. First we recall that by Proposition 2.2.10 the Eisenstein series $\mathcal{E}$ can be written as

$$\mathcal{E} = \sum_{\substack{\alpha | D_\theta \\ \beta | D_\psi}} \alpha\mu(\alpha)\mu(\beta)\theta_0(\alpha)\psi_0(\beta)(1+X)^{s(\alpha)}\mathcal{E}_{\theta_0,\psi_0;\alpha\beta}.$$

Therefore, in order to prove Proposition 3.2.1 it will suffice to show that when $t \geqslant 1$ is prime to p and satisfies $f_\theta f_\psi t \mid Np$, we have

$$\mathrm{Res}_\Lambda(\mathcal{E}_{\theta_0,\psi_0;t}) \;=\; A \cdot \mathfrak{e}_{\infty,t} \tag{25}$$

for some $\mathfrak{e}_{\infty,t} \in C(N)^{\mathrm{ord}}_{\Lambda_{\theta,\psi}}$.

Having reduced our task to showing (25), let us return to the definition of the Λ-adic residue map. Plugging in $\mathcal{E}_{\theta_0,\psi_0;t}$ for an integer t as above we get

$$\mathrm{Res}_\Lambda(\mathcal{E}_{\theta_0,\psi_0;t}) = \varprojlim_{r\geqslant 1}\left(\frac{\psi_0(p)^{-r}}{p^{r-1}}\sum_{\substack{\mathfrak{c}\in C_r\\ \epsilon\in\widehat{U_1/U_r}}}\mathrm{Res}_\mathfrak{c}\left(E_2((\theta_0\epsilon)_p,\psi_0;t)|w_{N_r}^{-1}\right)\cdot e\mathfrak{c}\right). \tag{26}$$

Here we're using the fact that $E_2((\theta_0\epsilon)_p,\psi_0;t)$ is a T_p-eigenform with eigenvalue $\psi_0(p)$. Next, we note that since

$$w_{N_r}^{-1} \;=\; \begin{pmatrix} 1/t & 0 \\ 0 & 1\end{pmatrix} w_{N_r/t}^{-1},$$

for all $\tau \in \mathbb{H}$ we have

$$\left(E_2((\theta_0\epsilon)_p,\psi_0;t)\,\middle|\begin{pmatrix} 1/t & 0 \\ 0 & 1\end{pmatrix}\right)(\tau) \;=\; \det\begin{pmatrix} 1/t & 0 \\ 0 & 1\end{pmatrix}\cdot E_2((\theta_0\epsilon)_p,\psi_0;t)(\tau/t)$$
$$= \frac{1}{t}\cdot E_2((\theta_0\epsilon)_p,\psi_0).$$

Therefore, (26) becomes

$$\varprojlim_{r\geqslant 1}\left(\frac{\psi_0(p)^{-r}}{tp^{r-1}}\sum_{\substack{\mathfrak{c}\in C_r\\ \epsilon\in\widehat{U_1/U_r}}}\mathrm{Res}_\mathfrak{c}\left(E_2((\theta_0\epsilon)_p,\psi_0)|w_{N_r/t}^{-1}\right)\cdot e\mathfrak{c}\right). \tag{27}$$

Now, the above sum is over those cusps $\mathfrak{c} \in C_r$ such that $e\mathfrak{c} \neq 0$. In Section 3.1.2, we determined an $\mathcal{O}_\infty[U_1/U_r]$-basis C_r^0 for the $\mathcal{O}_\infty$-module generated by such cusps. This basis, along with the fact that the diamond operator commutes with e, allows us to write (27) as

$$\varprojlim_{r\geqslant 1}\left(\frac{\psi_0(p)^{-r}}{tp^{r-1}}\sum_{\substack{\mathfrak{c}\in C_r^0\\ \epsilon\in\widehat{U_1/U_r}\\ \gamma\in U_1/U_r}}\mathrm{Res}_{\mathfrak{c}}\left(E_2((\theta_0\epsilon)_p,\psi_0)|w_{N_r/t}^{-1}|\sigma_\gamma\right)\cdot(\sigma_\gamma\cdot e\mathfrak{c})\right). \tag{28}$$

By the identity $f|w_{N_r/t}^{-1}|\sigma_\gamma = f|\sigma_\gamma^{-1}|w_{N_r/t}^{-1}$ for all $f\in M_2(\Gamma_1(N_r/t))_{\mathbb{C}_p}$, the above becomes

$$\varprojlim_{r\geqslant 1}\left(\frac{\psi_0(p)^{-r}}{tp^{r-1}}\sum_{\substack{\mathfrak{c}\in C_r^0\\ \epsilon\in\widehat{U_1/U_r}\\ \gamma\in U_1/U_r}}\epsilon^{-1}(\gamma)\cdot\mathrm{Res}_{w_{N_r/t}(\mathfrak{c})}\left(E_2((\theta_0\epsilon)_p,\psi_0)\right)\cdot(\sigma_\gamma\cdot e\mathfrak{c})\right), \tag{29}$$

with the last equality following from the facts that $\sigma_\gamma\in\Gamma_1$ and $w_{N_r/t}(\mathfrak{c}) = w_{N_r/t}^{-1}(\mathfrak{c})$ for all $\mathfrak{c}\in C_r$.

So we have further reduced our task to determining the residue of $E_2((\theta_0\epsilon)_p,\psi_0)$ at the cusps $w_{N_r/t}(\mathfrak{c})$ for $\mathfrak{c}\in C_r^0$. The following definition and proposition give us a simple means of computing this.

Definition 3.2.2. *Let $\gamma\in SL_2(\mathbb{Z})$ correspond to the cusp $\mathfrak{c}\in C_r$. The minimal choice of $h>0$ such that*

$$\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}\in\gamma^{-1}\Gamma_r\gamma$$

is called the width of the cusp $\mathfrak{c}$.

Proposition 3.2.3 ([O2], Section 4.5)**.** *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and let $f\in M_2(\Gamma)$. Then $\mathrm{Res}_{\mathfrak{c}}(f) = h_{\mathfrak{c}}\cdot a_0(f|_{\mathfrak{c}})$, where $h_{\mathfrak{c}}$ is the width of the cusp $\mathfrak{c}$ and $a_0(f|_{\mathfrak{c}})$ is the constant term of f at $\mathfrak{c}$.*

Therefore, in order to determine the projective limit (29) defining the image of the Eisenstein series $\mathcal{E}_{\theta_0,\psi_0;t}$ under the Λ-adic residue map, we simply need to compute the constant term of $E_2((\theta_0\epsilon)_p,\psi_0)$ at the cusp $w_{N_r/t}(\mathfrak{c})$ and the width of the cusp $w_{N_r/t}(\mathfrak{c})$ for all $\mathfrak{c}\in C_r^0$. In Subsection 3.2.1 we will determine the former, and in Subsection 3.2.2 the latter. Finally, in Subsection 3.2.3 we will put all of this together to prove Proposition 3.2.1.

3.2.1 *The constant term of Eisenstein series at the cusps*

Let $r \geqslant 1$ and $\epsilon \in \widehat{U}_{1,f}$. We begin by recalling the following results due to Ohta.

Proposition 3.2.4 ([O4], Prop. 2.5.5). *Let $\mathfrak{c} \in C_r$ with*

$$\mathfrak{c} = \begin{bmatrix} a \\ c \end{bmatrix}'_{N_r}.$$

If $f_{\theta_0\epsilon} \mid c$, then the constant term of $E_2(\theta_0\epsilon, \psi_0)$ at the cusp $\mathfrak{c}$ is given by

$$\frac{1}{2}\frac{g((\xi\epsilon)^{-1})}{g((\theta_0\epsilon)^{-1})}\left(\frac{f_{\theta_0\epsilon}}{f_{(\xi\epsilon)^{-1}}}\right)^2 \psi_0\left(-\frac{c}{f_{\theta_0\epsilon}}\right)(\theta_0\epsilon)^{-1}(a) \cdot \left(\prod_{\substack{\ell \mid f_{\theta_0\epsilon}f_\psi \\ \ell\nmid f_{\xi\epsilon}}} (1-(\xi\epsilon)^{-1}(\ell)\ell^{-2})\right) L(-1,\xi\epsilon)$$

where $g(\chi)$ is the Gauss sum of the character χ. If $f_{\theta_0\epsilon} \nmid c$, then the constant term is 0.

Corollary 3.2.5 ([O4], Cor. 2.5.7). *Let $\mathfrak{c} \in C_r$ with*

$$\mathfrak{c} = \begin{bmatrix} a \\ c \end{bmatrix}'_{N_r}$$

and assume that $p \mid c$. Then the constant term of $E_2((\theta_0\epsilon)_p, \psi_0)$ at $\mathfrak{c}$ is equal to the constant term of $E_2(\theta_0\epsilon, \psi_0)$ at $\mathfrak{c}$ multiplied by $1-(\xi\epsilon)(p)p$.

With respect to the above corollary, note that if $p \mid f_{\theta_0\epsilon}$, then $(\theta_0\epsilon)_p = \theta_0\epsilon$ and $(\xi\epsilon)(p) = 0$.

Let $N = \tilde{f}_\theta PQt$, where $\tilde{f}_\theta = f_\theta$ if $p \nmid f_\theta$ and f_θ/p otherwise, and

$$P = \prod_{\substack{\ell \mid f_\psi \\ \ell \text{ prime}}} \ell^{\mathrm{ord}_\ell(N/\tilde{f}_\theta t)}.$$

Note that P is dependent on N, θ_0, ψ_0, and t, and Q is the largest factor of $N/\tilde{f}_\theta t$ prime to f_ψ. Let $\mathfrak{c} \in C_r^0$ with

$$\mathfrak{c} \;=\; \begin{bmatrix} a \\ c \end{bmatrix}'_{N_r} \;=\; \left(\begin{bmatrix} a \\ c \end{bmatrix}_N, \begin{bmatrix} 0 \\ \omega(c) \end{bmatrix}_{p^r} \right)'. \tag{30}$$

By the definition of C_r^0, we know that $0 \leqslant a < \gcd(N, c)$ and $0 < c < N_1$ with $\gcd(c, ap) = 1$. The cusp we're interested in is $w_{N_r/t}(\mathfrak{c})$, which is given by

$$w_{N_r/t}(\mathfrak{c}) \;=\; \begin{bmatrix} -c/\Delta \\ aN_r/\Delta t \end{bmatrix}'_{N_r} \;=\; \begin{bmatrix} -c/\Delta \\ a\tilde{f}_\theta PQp^r/\Delta \end{bmatrix}'_{N_r}$$

where $\Delta := \gcd(aN_r/t, c) = \gcd(\tilde{f}_\theta PQ, c)$. Now, by Proposition 3.2.4, in order for the constant term of the Eisenstein series $E_2(\theta_0\epsilon, \psi_0)$ to be non-zero at the cusp $w_{N_r/t}(\mathfrak{c})$, the following conditions must be satisfied:

(1) $f_{\theta_0\epsilon} \;\mid\; \dfrac{a\tilde{f}_\theta PQp^r}{\Delta}$

(2) $\psi_0\left(\dfrac{a\tilde{f}_\theta PQp^r}{f_{\theta_0\epsilon}\cdot\Delta}\right) \;\neq\; 0.$

(3) $\theta_0^{-1}\left(\dfrac{c}{\Delta}\right) \;\neq\; 0.$

We want to unravel these conditions in order to get characterizations of a, c and Δ.

First we note that $\Delta \mid \gcd(N, c)$, and any divisor of the quotient $\gcd(N, c)/\Delta$ must be a divisor of t. With this in mind, we define

$$d_t \;=\; \gcd(N, c)/\Delta.$$

Then $c = \Delta d_t y$ for some y satisfying $0 < y < N_1/\Delta d_t$ with $\gcd(y, N_1/\Delta d_t) = 1$, while $0 \leqslant a < \Delta d_t$ with $\gcd(a, \Delta d_t y) = 1$.

Next, since $f_{\theta_0\epsilon} = \tilde{f}_\theta p^s$ for some $s \leqslant r$ and $\gcd(c, ap) = 1$, condition (1) is equivalent to $\Delta \mid PQ$. Furthermore, by expanding condition (2),

$$\psi_0(a)\,\psi_0\!\left(\frac{PQ}{\Delta}\right)\psi_0(p^{r-s}) \;\neq\; 0$$

we see that we must also have $P \mid \Delta$ so that $\psi_0(PQ/\Delta) \neq 0$. Hence, $\Delta = Pd_Q$ for some $d_Q \mid Q$. Furthermore, we have $a \neq 0$.

Putting this altogether, we see that if the constant term of $E_2(\theta_0\epsilon, \psi_0)$ is to be non-zero at $w_{N_r/t}(\mathfrak{c})$, the cusp $\mathfrak{c} \in C_r^0$ can be written as

$$\mathfrak{c} = \left(\begin{bmatrix} x \\ d_t d_Q P y \end{bmatrix}_N, \begin{bmatrix} 0 \\ \omega(d_t d_Q P y) \end{bmatrix}_{p^r} \right)' \tag{31}$$

with

(1) $d_t \mid t$ and $d_Q \mid Q$

(2) $x \in (\mathbb{Z}/d_t d_Q P\mathbb{Z})^\times$.

(3) $y \in (\mathbb{Z}/(N_1/d_t d_Q P)\mathbb{Z})^\times$

Having characterized the cusps in C_r^0 at which the constant term of the Eisenstein series $E_2(\theta_0\epsilon, \psi_0)$ is non-trivial, we will now use Proposition 3.2.4 and Corollary 3.2.5 to determine the constant term at these cusps.

Proposition 3.2.6. *Suppose $\theta_0 = \chi\omega^i$, where $f_\chi = \tilde{f}_\theta$. If $\mathfrak{c} \in C_r^0$ is of the form given by (31), then the constant term of $E_2((\theta_0\epsilon)_p, \psi)$ at $w_{N_r/t}(\mathfrak{c})$ is*

$$C \cdot \epsilon\left(\frac{\tilde{f}_\theta}{f_\xi d_t y}\right) \psi_0\left(\frac{xQp^r}{d_Q}\right) \theta_0^{-1}(d_t y) \cdot \left(\prod_{\substack{\ell \mid \tilde{f}_\theta f_\psi \\ \ell \nmid f_\xi}} (1 - (\xi\epsilon\omega^2)^{-1}(\ell)\langle\langle \ell \rangle\rangle^{-2}) \right) L(-1, (\xi\epsilon)_p) \tag{32}$$

where C is a p-adic unit in some finite cyclotomic extension of $\mathbb{Q}_p$ depending only on θ_0 and ψ_0.

Proof. We have

$$w_{N_r/t}(\mathfrak{c}) = \begin{bmatrix} -d_t y \\ x\tilde{f}_\theta Qp^r/d_Q \end{bmatrix}'_{N_r} \tag{33}$$

for some $d_t \mid t$ and $d_Q \mid Q$. By Proposition 3.2.4 the constant coefficient of $E_2(\theta_0\epsilon, \psi_0)$ at the cusp $w_{N_r/t}(\mathfrak{c})$ is given by

$$\frac{1}{2}\frac{g((\xi\epsilon)^{-1})}{g((\theta_0\epsilon)^{-1})}\left(\frac{f_{\theta_0\epsilon}}{f_{\xi\epsilon}}\right)^2\psi_0\left(\frac{-xQp^{r-s}}{d_Q}\right)(\theta_0\epsilon)^{-1}(-d_t y) \cdot \left(\prod_{\substack{\ell \mid f_{(\theta_0\epsilon)^{-1}} f_\psi \\ \ell \nmid f_{(\xi\epsilon)^{-1}}}} (1-(\xi\epsilon)^{-1}(\ell)\ell^{-2})\right) L(-1,\xi\epsilon). \quad (34)$$

With respect to the first term of (34) involving Gauss sums, we note that if χ_1 and χ_2 are two distinct characters such that $\gcd(f_{\chi_1}, f_{\chi_2}) = 1$, then

$$g(\chi_1\chi_2) = \chi_1(f_{\chi_2})\chi_2(f_{\chi_1})g(\chi_1)g(\chi_2) \quad (35)$$

(see [O4], 2.6.8). Since $(\xi\epsilon)^{-1} = (\psi_0\chi^{-1})\cdot(\omega^i\epsilon)^{-1}$ and $\gcd(f_{\psi_0\chi^{-1}}, f_{\omega^i\epsilon}) = 1$, we have

$$\frac{g((\xi\epsilon)^{-1})}{g((\theta_0\epsilon)^{-1})} = \frac{g(\psi_0\chi^{-1})}{g(\chi^{-1})}\cdot\frac{g((\omega^i\epsilon)^{-1})}{g((\omega^i\epsilon)^{-1})}\cdot\frac{(\psi_0\chi^{-1})(p^s)}{\chi^{-1}(p^s)}\cdot\frac{(\omega^i\epsilon)^{-1}\left(f_{\psi_0\chi^{-1}}\right)}{(\omega^i\epsilon)^{-1}\left(\tilde{f}_\theta\right)}$$

$$= \frac{g(\psi_0\chi^{-1})}{g(\chi^{-1})}\cdot\psi_0(p^s)\cdot\omega^i\left(\frac{\tilde{f}_\theta}{f_{\psi_0\chi^{-1}}}\right)\cdot\epsilon\left(\frac{\tilde{f}_\theta}{f_{\psi_0\chi^{-1}}}\right). \quad (36)$$

With respect to the second term of (34) we have

$$\left(\frac{f_{\theta_0\epsilon}}{f_{\xi\epsilon}}\right)^2 = \left(\frac{\tilde{f}_\theta\cdot f_{\omega^i\epsilon}}{f_{\psi_0\chi^{-1}}\cdot f_{\omega^i\epsilon}}\right)^2 = \left(\frac{\tilde{f}_\theta}{f_{\psi_0\chi^{-1}}}\right)^2. \quad (37)$$

Recalling that $\gcd(p, f_\psi) = 1$, we have

$$\psi_0\left(\frac{-xQp^{r-s}}{d_Q}\right)(\theta_0\epsilon)^{-1}(-d_t y) = \psi_0\left(\frac{xQ}{d_Q}\right)\psi_0(p^{r-s})\theta_0^{-1}(d_t y)\epsilon^{-1}(d_t y),$$

we have used the fact that $(\theta_0\psi_0)(-1) = 1 = \epsilon(-1)$. Putting this altogether, we see that the first half of (34) can be written as

$$= C\cdot\psi_0(p^r)\cdot\epsilon\left(\frac{\tilde{f}_\theta}{f_{\psi_0\chi^{-1}}d_t y}\right)\cdot\theta_0^{-1}(d_t y)\cdot\psi_0\left(\frac{xQ}{d_Q}\right) \quad (38)$$

where

$$C := \frac{1}{2}\cdot\frac{g(\psi_0\chi^{-1})}{g(\chi^{-1})}\cdot\omega^i\left(\frac{\tilde{f}_\theta}{f_{\psi_0\chi^{-1}}}\right)\cdot\left(\frac{\tilde{f}_\theta}{f_{\psi_0\chi^{-1}}}\right)^2$$

is a p-adic unit in some finite cyclotomic extension of $\mathbb{Q}_p$ that depends only on θ_0 and ψ_0.

Next, we consider the product occurring in the latter half of (34). Since $f_{\theta_0\epsilon} f_\psi = \tilde{f}_\theta f_\psi p^n$ and $f_{\xi\epsilon} = f_\xi p^n$ for some integer $n \geqslant 0$, we have

$$\left(\prod_{\substack{\ell \mid f_{\theta_0\epsilon} f_\psi \\ \ell \nmid f_{\xi\epsilon}}} (1 - (\xi\epsilon)^{-1}(\ell)\ell^{-2}) \right) = \left(\prod_{\substack{\ell \mid \tilde{f}_\theta f_\psi \\ \ell \nmid f_\xi}} (1 - (\xi\epsilon\omega^2)^{-1}(\ell)\langle\!\langle \ell \rangle\!\rangle^{-2}) \right).$$

Finally, by Corollary 3.2.5 we know that the constant term of $E_2((\theta_0\epsilon)_p, \psi_0)$ at $w_{N_r/t}(\mathfrak{c})$ is $1 - (\xi\epsilon)(p)p$ times the constant term of $E_2(\theta_0\epsilon, \psi_0)$ at $w_{N_r/t}(\mathfrak{c})$. Since

$$(1 - (\xi\epsilon)(p)p)L(-1, \xi\epsilon) = L(-1, (\xi\epsilon)_p),$$

we have the proposition. □

3.2.2 *Width of the cusps*

In this section we would like to determine the width of the cusp $w_{N_r/t}(\mathfrak{c})$ for $\mathfrak{c} \in C_r^0$. Let $\mathfrak{c} \in C_r^0$ be of the form (31), i.e.

$$\mathfrak{c} = \begin{bmatrix} x \\ d_t d_Q P y \end{bmatrix}'_{N_r}$$

for some $d_t \mid t$ and $d_Q \mid Q$. Let $\gamma \in SL_2(\mathbb{Z})$ correspond to $w_{N_r/t}(\mathfrak{c})$, that is,

$$\gamma = \begin{pmatrix} -d_t y & * \\ x\tilde{f}_\theta Q p^r / d_Q & * \end{pmatrix}.$$

Let h be the width of the cusp $w_{N_r/t}(\mathfrak{c})$. Then by definition

$$\begin{pmatrix} 1 - d_t y \left(\frac{x\tilde{f}_\theta Q p^r}{d_Q}\right) h & * \\ -\left(\frac{x\tilde{f}_\theta Q p^r}{d_Q}\right)^2 h & 1 + d_t y \left(\frac{x\tilde{f}_\theta Q p^r}{d_Q}\right) h \end{pmatrix} = \gamma \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \gamma^{-1} \in \Gamma_r. \quad (39)$$

Therefore, we must have

$$d_t y\left(\frac{x\tilde{f}_\theta Q p^r}{d_Q}\right)h \;\equiv\; 0 \pmod{N_r} \qquad \text{and} \qquad \left(\frac{x\tilde{f}_\theta Q p^r}{d_Q}\right)^2 h \;\equiv\; 0 \pmod{N_r}.$$

However, since $\gamma \in SL_2(\mathbb{Z})$ it must be the case that

$$\gcd\left(d_t y, \frac{x\tilde{f}_\theta Q p^r}{d_Q}\right) \;=\; 1,$$

which implies

$$\left(\frac{x\tilde{f}_\theta Q p^r}{d_Q}\right)h \;\equiv\; 0 \pmod{N_r}.$$

The smallest value of h satisfying the above congruence is $h = td_Q P/\gcd(x,t)$, which is a p-adic unit dependent on the cusp $\mathfrak{c}$.

3.2.3 *Proof of Proposition 3.2.1*

In this section we will put everything we have proven thus far together in order to prove Proposition 3.2.1. Recall that it suffices to show

$$\text{Res}_\Lambda(\mathcal{E}_{\theta_0,\psi_0;t}) \;=\; A \cdot \mathfrak{e}_{\infty,t}$$

for some $\mathfrak{e}_{\infty,t} \in C(N)^{\text{ord}}_{\Lambda_\infty}$.

As was shown earlier, the level r component of the projective limit defining $\text{Res}_\Lambda(\mathcal{E}_{\theta_0,\psi_0;t})$ is given by

$$\frac{\psi_0(p)^{-r}}{tp^{r-1}} \sum_{\substack{\mathfrak{c}\in C_r^0 \\ \widehat{\epsilon\in U_1/U_r} \\ \gamma\in U_1/U_r}} \epsilon^{-1}(\gamma)\,\text{Res}_{w_{N_r/t}(\mathfrak{c})}(E_2((\theta_0\epsilon)_p,\psi_0))\,(\sigma_\gamma\cdot \epsilon\mathfrak{c})\,. \tag{40}$$

By Proposition 3.2.3 we know that $\text{Res}_{w_{N_r/t}(\mathfrak{c})}(E_2((\theta_0\epsilon)_p,\psi_0))$ is simply the width of the cusp $w_{N_r/t}(\mathfrak{c})$ times the constant term of the Eisenstein series $E_2((\theta_0\epsilon)_p,\psi_0)$ at that cusp. Furthermore, in Subsection 3.2.1 it was shown that the constant term of $E_2((\theta_0\epsilon)_p, \psi_0)$ at the cusp $w_{N_r/t}(\mathfrak{c})$ for $\mathfrak{c} \in C_r^0$, is zero unless $\mathfrak{c}$ is of the form

$$\mathfrak{c}_{r,d_t,d_Q}^{x,y} := \left(\begin{bmatrix} x \\ d_t d_Q P y \end{bmatrix}_N, \begin{bmatrix} 0 \\ \omega(d_t d_Q P y) \end{bmatrix}_{p^r} \right)'$$

with $d_t \mid t$, $d_Q \mid Q$, $x \in (\mathbb{Z}/d_t d_Q P\mathbb{Z})^\times$, and $y \in (\mathbb{Z}/(N_1/d_t d_Q P)\mathbb{Z})^\times$. In order to shorten notation, for each $d_t \mid t$ and $d_Q \mid Q$, let

$$\mathcal{S}_{d_t,d_Q} = (\mathbb{Z}/d_t d_Q P\mathbb{Z})^\times \times (\mathbb{Z}/(N_1/d_t d_Q P)\mathbb{Z})^\times,$$

then (40) can be written as

$$\frac{\psi_0(p)^{-r}}{tp^{r-1}} \sum_{\substack{d_t|t \\ d_Q|Q}} \sum_{\substack{\epsilon \in \widehat{U_1/U_r} \\ \gamma \in U_1/U_r \\ (x,y) \in \mathcal{S}_{d_t,d_Q}}} \epsilon^{-1}(\gamma) \operatorname{Res}_{w_{N_r/t}\left(\mathfrak{c}_{r,d_t,d_Q}^{x,y}\right)}\big(E_2((\theta_0\epsilon)_p, \psi_0)\big) \left(\sigma_\gamma \cdot \epsilon \mathfrak{c}_{r,d_t,d_Q}^{x,y}\right). \tag{41}$$

By Proposition 3.2.6 we know that the constant term of $E_2((\theta_0\epsilon)_p, \psi_0)$ at the cusp $w_{N_r/t}(\mathfrak{c}_{r,d_t,d_Q}^{x,y})$ is

$$C \cdot \epsilon\left(-\frac{\tilde{f}_\theta d_t y}{f_\xi}\right) \psi_0\left(\frac{xQp^r}{d_Q}\right) \theta_0^{-1}(d_t y) \cdot \left(\prod_{\substack{\ell \mid \tilde{f}_\theta f_\psi \\ \ell \nmid f_\xi}} (1 - (\xi\epsilon\omega^2)^{-1}(\ell)\langle\!\langle \ell \rangle\!\rangle^{-2}) \right) L(-1, (\xi\epsilon)_p),$$

where C is a p-adic unit depending only on θ_0 and ψ_0, while in Section 3.2.2 we showed that the width of the cusp $w_{N_r/t}(\mathfrak{c}^{x,y}_{r,d_t,d_Q})$ is $td_Q P/\gcd(x,t)$. Therefore, (41) can be written as

$$\frac{CP}{p^{r-1}} \sum_{\substack{d_t \mid t \\ d_Q \mid Q}} \sum_{\substack{\epsilon \in \widehat{U_1/U_r} \\ \gamma \in U_1/U_r \\ (x,y) \in \mathcal{S}_{d_t,d_Q}}} \frac{d_Q}{\gcd(x,t)} \, \epsilon^{-1}(\gamma)\, \epsilon\left(-\frac{\tilde{f}_\theta d_t y}{f_\xi}\right) \psi_0\left(\frac{xQ}{d_Q}\right) \theta_0^{-1}(d_t y)$$
$$\cdot \left(\prod_{\substack{\ell \mid \tilde{f}_\theta f_\psi \\ \ell \nmid f_\xi}} (1-(\xi\epsilon\omega^2)^{-1}(\ell)\langle\!\langle \ell \rangle\!\rangle^{-2}) \right) L(-1,(\xi\epsilon)_p) \left(\sigma_\gamma \cdot e\mathfrak{c}^{x,y}_{r,d_t,d_Q}\right). \quad (42)$$

Next, we will rearrange the above sum by grouping those terms that are dependent on r. To once again simplify the notation a bit, let

$$\mathcal{L}_\epsilon = \left(\prod_{\substack{\ell \mid \tilde{f}_\theta f_\psi \\ \ell \nmid f_\xi}} (1-(\xi\epsilon\omega^2)^{-1}(\ell)\langle\!\langle \ell \rangle\!\rangle^{-2}) \right) L(-1,(\xi\epsilon)_p).$$

Then (42) can be written as

$$CP \sum_{\substack{d_t \mid t \\ d_Q \mid Q \\ (x,y) \in \mathcal{S}_{d_t,d_Q}}} \frac{d_Q}{\gcd(x,t)} \, \psi_0\left(\frac{xQ}{d_Q}\right) \theta_0^{-1}(d_t y)$$
$$\cdot \left(\frac{1}{p^{r-1}} \sum_{\substack{\epsilon \in \widehat{U_1/U_r} \\ \gamma \in U_1/U_r}} \epsilon^{-1}(\gamma)\, \epsilon\left(-\frac{\tilde{f}_\theta d_t y}{f_\xi}\right) \mathcal{L}_\epsilon \left(\sigma_\gamma \cdot e\mathfrak{c}^{x,y}_{r,d_t,d_Q}\right) \right). \quad (43)$$

For each $d_t \mid t$, $d_Q \mid Q$, and $(x,y) \in \mathcal{S}_{\theta_0,\psi_0}$, define

$$\mathfrak{c}^{x,y}_{\infty,d_t,d_Q} = \varprojlim_r e\mathfrak{c}^{x,y}_{r,d_t,d_Q} \in C(N)^{\mathrm{ord}}_{\Lambda_\infty}.$$

We will use (43) to show that $\mathrm{Res}_\Lambda(\mathcal{E}_{\theta_0,\psi_0;t})$ is a Λ_∞-linear combination of the $\mathfrak{c}^{x,y}_{\infty,d_t,d_Q}$. To do so, we need to make explicit the Λ_∞-action on the cusps

$\mathfrak{e}^{x,y}_{\infty,d_t,d_Q}$. First we note that by the surjectivity of the Λ-adic residue map, there exists some $F_0 \in M(N)^{\mathrm{ord}}_{\Lambda_\infty}$ such that

$$\mathrm{Res}_\Lambda(F_0) \;=\; \mathfrak{e}^{x,y}_{\infty,d_t,d_Q}.$$

In fact, if we recall the definition of the Λ-adic residue map (24) and note that

$$\frac{1}{p^{r-1}} \sum_{\substack{\epsilon\in\widehat{U_1/U_r}\\ \gamma\in U_1/U_r}} \epsilon^{-1}(\gamma)\left(\sigma_\gamma\cdot e\mathfrak{e}^{x,y}_{r,d_t,d_Q}\right)$$

$$= \frac{1}{p^{r-1}} \sum_{\gamma\in U_1/U_r}\left(\sum_{\epsilon\in\widehat{U_1/U_r}} \epsilon^{-1}(\gamma)\right)\left(\sigma_\gamma\cdot e\mathfrak{e}^{x,y}_{r,d_t,d_Q}\right)$$

$$= \frac{1}{p^{r-1}}\left(\sum_{\epsilon\in\widehat{U_1/U_r}} 1\right) e\mathfrak{e}^{x,y}_{r,d_t,d_Q} \;=\; e\mathfrak{e}^{x,y}_{r,d_t,d_Q}$$

we see that

$$\mathfrak{e}^{x,y}_{\infty,d_t,d_Q} \;=\; \varprojlim_r\left(\frac{1}{p^{r-1}}\sum_{\substack{\epsilon\in\widehat{U_1/U_r}\\ \gamma\in U_1/U_r}} \mathrm{Res}_{\sigma_\gamma\cdot\mathfrak{c}^{x,y}_{r,d_t,d_Q}}(\nu_{2,\epsilon}(F_0)|T_p^{-r}|w_{N_r}^{-1})\left(\sigma_\gamma\cdot e\mathfrak{e}^{x,y}_{r,d_t,d_Q}\right)\right).$$

with

$$\mathrm{Res}_{\sigma_\gamma\cdot\mathfrak{c}^{x,y}_{r,d_t,d_Q}}(\nu_{2,\epsilon}(F_0)|T_p^{-r}|w_{N_r}^{-1}) \;=\; \epsilon^{-1}(\gamma).$$

Characterizing $\mathfrak{e}^{x,y}_{\infty,d_t,d_Q}$ in this way makes it easier to determine the Λ_∞-action. Specifically, if $\lambda\in\Lambda_\infty$ we have

$$\lambda\cdot\mathfrak{e}^{x,y}_{\infty,d_t,d_Q} = \varprojlim_r\left(\frac{1}{p^{r-1}}\sum_{\substack{\epsilon\in\widehat{U_1/U_r}\\ \gamma\in U_1/U_r}} \mathrm{Res}_{\sigma_\gamma\cdot\mathfrak{c}^{x,y}_{r,d_t,d_Q}}(\nu_{2,\epsilon}(\lambda F_0)|T_p^{-r}|w_{N_r}^{-1})\left(\sigma_\gamma\cdot e\mathfrak{e}^{x,y}_{r,d_t,d_Q}\right)\right)$$

$$= \varprojlim_r\left(\frac{1}{p^{r-1}}\sum_{\substack{\epsilon\in\widehat{U_1/U_r}\\ \gamma\in U_1/U_r}} \epsilon^{-1}(\gamma)\cdot\lambda(\epsilon(u)-1)\cdot\left(\sigma_\gamma\cdot e\mathfrak{e}^{x,y}_{r,d_t,d_Q}\right)\right).$$

Therefore, noting that

$$\begin{aligned} A(\epsilon(u)-1) &= \mathcal{L}_\epsilon \\ \epsilon(u)^{s(-\tilde{f}_\theta d_t y/f_\xi)} &= \epsilon\left(-\frac{\tilde{f}_\theta d_t y}{f_\xi}\right) \end{aligned}$$

for all $\epsilon \in \widehat{U}_{1,f}$, we have

$$(1+X)^{s(-\tilde{f}_\theta d_t y/f_\xi)} \cdot A \cdot \mathfrak{e}^{x,y}_{\infty,d_t,d_Q}$$

$$= \varprojlim_r \left(\frac{1}{p^{r-1}} \sum_{\substack{\epsilon \in \widehat{U_1/U_r} \\ \gamma \in U_1/U_r}} \epsilon^{-1}(\gamma)\, \epsilon\left(-\frac{\tilde{f}_\theta d_t y}{f_\xi}\right) \mathcal{L}_\epsilon \left(\sigma_\gamma \cdot e\mathfrak{c}^{x,y}_{r,d_t,d_Q}\right) \right).$$

Putting this together with (43) we see that

$$\mathrm{Res}_\Lambda(\mathcal{E}_{\theta_0,\psi_0;t}) = A \cdot \mathfrak{e}_{\infty,t}$$

where

$$\mathfrak{e}_{\infty,t} := \mathrm{CP} \sum_{\substack{d_t|t \\ d_Q|Q \\ (x,y)\in \mathcal{S}_{d_t,d_Q}}} \frac{d_Q \cdot (1+X)^{s(-\tilde{f}_\theta d_t y/f_\xi)}}{\gcd(x,t)}\, \psi_0\left(\frac{xQ}{d_Q}\right) \theta_0^{-1}(d_t y) \cdot \mathfrak{e}^{x,y}_{\infty,d_t,d_Q}.$$

This completes the proof of Proposition 3.2.1.

3.3 HECKE ALGEBRA MODULO EISENSTEIN IDEAL

We continue to assume that $(\theta_0, \psi) \neq (\omega^{-2}, \mathbb{1})$. Furthermore, to simplify the notation a bit we set

$$\begin{aligned} M &= M(N)^{\mathrm{ord}}_{\Lambda_{\theta,\psi}} & S &= S(N)^{\mathrm{ord}}_{\Lambda_{\theta,\psi}} \\ \mathfrak{H} &= \mathfrak{H}(N)^{\mathrm{ord}}_{\mathcal{O}_{\theta,\psi}} & \mathfrak{h} &= \mathfrak{h}(N)^{\mathrm{ord}}_{\mathcal{O}_{\theta,\psi}}. \end{aligned}$$

Our primary goal in this section will be to prove the following proposition.

Proposition 3.3.1. *Let* I *denote the image of* $\mathrm{Ann}_{\mathfrak{H}}(\mathcal{E})$ *in* $\mathfrak{h}$. *Then we have the following isomorphism of* $\Lambda_{\theta,\psi}$*-algebras*

$$\mathfrak{h}/I \cong \Lambda_{\theta,\psi}/(A),$$

where A *is as in Proposition 3.2.1.*

The form of this result is well known. It was first proven by Wiles [W2] in the case when $\psi = \mathbb{1}$ and θ is primitive and non-exceptional. In [O4], Ohta removed the triviality condition on ψ, proving the result for non-exceptional, primitive pairs of characters. Unfortunately, this proof requires the full force of the Iwasawa main conjecture over $\mathbb{Q}$. Using Katz's p-adic Modular forms, Emerton [E] has given a proof of the above isomorphism in the case when $\psi = \mathbb{1}$ and θ is a nontrivial power of the Teichmüller character. The novelty of our approach lies in its simplicity and generality. Specifically, our proof does not require the Iwasawa main conjecture and makes no restrictions on the characters θ and ψ apart from $(\theta_0, \psi) \neq (\omega^{-2}, \mathbb{1})$.

In his proof, Emerton constructs what he calls the "universal constant term" of the Hecke algebra acting on the space of p-adic modular forms [E, §2]. This is a Hecke operator H_0 satisfying $a_1(f|H_0) = a_0(f)$ for all p-adic modular forms f. We consider a $\Lambda_{\theta,\psi}$-adic analog of this situation. While constructing a Hecke operator satisfying this identity for all of M may not possible, such a construction is possible for a specific free $\Lambda_{\theta,\psi}$-submodule of M when $\psi = \mathbb{1}$. We begin by constructing this submodule.

Proposition 3.3.2. $\Lambda_\infty(\mathfrak{e}_\infty)$ *is a free* Λ_∞*-module.*

Proof. This is equivalent to showing that $\mathfrak{e}_\infty$ is not Λ_∞-torsion. For the sake of contradiction, suppose it is. Then there exists some $\lambda \in \Lambda_\infty$ such that $\mathrm{Res}_\Lambda(\lambda\mathcal{E}) = \lambda A \cdot \mathfrak{e}_\infty = 0$. However, this implies $\lambda\mathcal{E} \in S(N)^{\mathrm{ord}}_{\Lambda_\infty}$, which is a contradiction. $\square$

Proposition 3.3.3 ([O3], Lemma 2.1.1). *The $\Lambda_{\mathbb{Z}_p}$-algebra Λ_∞ is faithfully flat.*

Now, let $\mathcal{P} = \{F \in M : \mathrm{Res}_\Lambda(F) \in \Lambda_{\theta,\psi}(\mathfrak{e}_\infty)\}$. Then by Corollary 2.2.4 and Proposition 3.3.2 we have the following exact sequence

$$0 \longrightarrow S \otimes_{\Lambda_{\theta,\psi}} \Lambda_\infty \longrightarrow \mathcal{P} \otimes_{\Lambda_{\theta,\psi}} \Lambda_\infty \longrightarrow \Lambda_{\theta,\psi}(\mathfrak{e}_\infty) \otimes_{\Lambda_{\theta,\psi}} \Lambda_\infty \longrightarrow 0.$$

Employing Proposition 3.3.3 we have the exactness of

$$0 \longrightarrow S \longrightarrow \mathcal{P} \longrightarrow \Lambda_{\theta,\psi}(\mathfrak{e}_\infty) \longrightarrow 0.$$

We would now like to construct a basis for $\mathcal{P}$.

By Proposition 2.2.2, we know that S is a free and finitely generated $\Lambda_{\theta,\psi}$-module. Let $\{F_1, \ldots, F_m\}$ be a $\Lambda_{\theta,\psi}$-basis for S. By the duality of Proposition 2.2.13, we know that $\mathfrak{h}$ has a $\Lambda_{\theta,\psi}$-basis $\{B_1, \ldots, B_m\}$ such that

$$a_1(F_i|B_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

For each i, let $\mathfrak{B}_i$ be *any* element of $\mathfrak{H}$ that projects to B_i via the natural surjection $\mathfrak{H} \twoheadrightarrow \mathfrak{h}$.

Next, take any $F \in \mathcal{P}$ satisfying $\mathrm{Res}_\Lambda(F) = \mathfrak{e}_\infty$ and define

$$F_0 = F - \sum_{i=1}^{m} a_1(F|\mathfrak{B}_i)F_i \in \mathcal{P}.$$

Note that any two elements of $\mathcal{P}$ mapping to $\mathfrak{e}_\infty$ under the Λ-adic residue map differ by a cusp form, so our definition of F_0 does not depend on the choice of F. It is clear that $\{F_0, F_1, \ldots, F_m\}$ is a $\Lambda_{\theta,\psi}$-basis for $\mathcal{P}$.

Before moving on, let us make several observations about the form F_0. First we note that since $\mathrm{Res}_\Lambda(AF_0 - \mathcal{E}) = 0$, we have

$$F_0 = \frac{\mathcal{E} + F_S}{A} \tag{44}$$

for some $F_S \in S$.

Next we note that by construction

$$a_1(F_0|\mathfrak{B}_i) = a_1(F|\mathfrak{B}_i) - \sum_{j=1}^{m} a_1(F|\mathfrak{B}_i)a_1(F_j|\mathfrak{B}_i) = 0$$

for $1 \leqslant i \leqslant m$. However, we can say even more about the Hecke action on F_0. Recall that Res_Λ is a Hecke module homomorphism. Combining this with the fact that $\mathrm{Res}_\Lambda(\mathcal{E}) = A \cdot \mathfrak{e}_\infty$, we see that

$$\mathfrak{e}_\infty|H = a_1(\mathcal{E}|H) \cdot \mathfrak{e}_\infty$$

for all $H \in \mathfrak{H}$. This implies that

$$F_0|H = a_1(\mathcal{E}|H)F_0 + G_H, \tag{45}$$

for some $G_H \in S$, where the subscript notes the fact that this cusp form may depend on H. So, we see that F_0 is an eigenform modulo S, with eigenvalues agreeing with those of $\mathcal{E}$.

3.3.1 *Case 1:* $\psi \neq \mathbb{1}$

If $\psi \neq \mathbb{1}$, then $a_0(\mathcal{E}) = 0$ and by (44) we see that

$$\mathcal{P}_0 := \left\{ F \in \mathcal{P}_{Q(\Lambda_{\theta,\psi})} : a_n(F) \in \Lambda_{\theta,\psi} \text{ for } n \geqslant 1 \right\} = \mathcal{P}.$$

Therefore, by Proposition 2.2.13 we have $\mathfrak{H}(\mathcal{P}) := \mathfrak{H}/\mathrm{Ann}_{\mathfrak{H}}(\mathcal{P}) \cong \mathrm{Hom}_{\Lambda_{\theta,\psi}}(\mathcal{P}, \Lambda_{\theta,\psi})$. Let $\mathfrak{B}_0 \in \mathfrak{H}(\mathcal{P})$ correspond to F_0 under the above isomorphism. That is, $\mathfrak{B}_0$ satisfies

$$a_1(F_i|\mathfrak{B}_0) = \begin{cases} 1 & i = 0 \\ 0 & i \neq 0 \end{cases}$$

for all i. Note that by (44) we have $a_1(\mathcal{E}|\mathfrak{B}_0) = A$.

Now, consider the map $\Phi : \mathfrak{h} \mapsto \Lambda_{\theta,\psi}/(A)$ defined by

$$\Phi(H) = a_1(F_S|H) \pmod{A}.$$

Let $H \in \mathfrak{h}$. Since $F_S = AF_0 - \mathcal{E}$, if $\tilde{H} \in \mathfrak{H}_X$ is *any* element the projects to H under the natural projection $\mathfrak{H}(\mathcal{P}) \to \mathfrak{h}$, we have

$$a_1(F_S|H) \;=\; a_1(F_S|\tilde{H}) \;=\; Aa_1(F_0|\tilde{H}) - a_1(\mathcal{E}|\tilde{H}).$$

From this we can see that Φ is surjective and $I \subset \ker(\Phi)$.

Suppose $H \in \ker(\Phi)$. Then it must be the case that $a_1(\mathcal{E}|\tilde{H}) \in (A)$ for every $\tilde{H} \in \mathfrak{H}(\mathcal{P})$ projecting to H. Define

$$\tilde{H}_0 \;=\; \tilde{H} - \frac{a_1(\mathcal{E}|\tilde{H})}{a_1(\mathcal{E}|\mathfrak{B}_0)}\mathfrak{B}_0 \;\in\; \mathfrak{H}_{\mathcal{P}}. \tag{46}$$

Then by construction $\mathcal{E}|\tilde{H}_0 = a_1(\mathcal{E}|\tilde{H}_0)\mathcal{E} = 0$. Furthermore, $\tilde{H}_0$ projects to H. Therefore, $H \in I$ and we have $\ker(\Phi) = I$.

3.3.2 *Case 2:* $\psi = \mathbb{1}$

We begin by noting that when $\psi = \mathbb{1}$, we have

$$A \;=\; \prod_{\substack{\ell | \tilde{f}_\theta \\ \ell \nmid f_\theta}} \left((1+X)^{s(\ell)} - (\xi\omega^2)(\ell)\langle\!\langle \ell \rangle\!\rangle^{-2}\right) G(X, \theta_0\omega^2) \;=\; G(X, \theta_0\omega^2) \;=\; 2a_0(\mathcal{E}).$$

So our goal in this case is to show $\mathfrak{h}/I \cong \Lambda_{\theta,\psi}/(G(X,\theta_0\omega^2))$. In this case, we have the following equivalent statements.

Proposition 3.3.4. *The following are equivalent:*

(a) *There exists an* $H \in \mathfrak{H}(\mathcal{P})$ *such that* $a_1(F_0|H) \in \Lambda_{\theta,\psi}^\times$.

(b) *There exists an* $H \in \mathfrak{H}(\mathcal{P})$ *such that* $a_1(F|H) = a_0(F)$ *for all* $F \in \mathcal{P}$.

(c) $\mathrm{Hom}_{\Lambda_{\theta,\psi}}(\mathcal{P}, \Lambda_{\theta,\psi}) \cong \mathfrak{H}(\mathcal{P})$.

(d) $\mathfrak{h}/I \cong \Lambda_{\theta,\psi}/(G(X,\theta_0\omega^2))$.

Proof. We will begin by showing that $(a) - (d)$ are equivalent.

(a) $\Rightarrow$ (b). Let $H \in \mathfrak{H}(\mathcal{P})$ satisfy $a_1(F_0|H) \in \Lambda_{\theta,\psi}^{\times}$. Define

$$H_0 = \frac{1}{2a_1(F_0|H)}\left(H - \sum_{i=1}^{m} a_1(F_i|H)\mathfrak{B}_i\right).$$

By construction we have $a_1(F_i|H_0) = 0 = a_0(F_i)$ for $1 \leqslant i \leqslant m$. Furthermore, we have $a_1(F_0|H_0) = 1/2 = a_0(F_0)$, with the second equality following from (44).

(b) $\Rightarrow$ (c). Suppose $H \in \mathfrak{H}(\mathcal{P})$ satisfies $a_1(F|H) = a_0(F)$ for all $F \in \mathcal{P}$. Define $\mathfrak{B}_0 = 2H$. Then $\mathfrak{B}_0 \in \mathrm{Ann}_{\mathfrak{H}(\mathcal{P})}(S)$, while $a_1(F_0|\mathfrak{B}_0) = 2a_0(F_0) = 1$.

Now, for $0 \leqslant i \leqslant m$ define $\varphi_i \in \mathrm{Hom}_{\Lambda_{\theta,\psi}}(\mathcal{P}, \Lambda_{\theta,\psi})$ by

$$\varphi_i(F_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Then $\{\varphi_0, \ldots, \varphi_m\}$ is a $\Lambda_{\theta,\psi}$-basis for $\mathrm{Hom}_{\Lambda_{\theta,\psi}}(\mathcal{P}, \Lambda_{\theta,\psi})$, and we have a $\Lambda_{\theta,\psi}$-module isomorphism

$$\langle \mathfrak{B}_0, \mathfrak{B}_1, \ldots, \mathfrak{B}_m \rangle_{\Lambda_{\theta,\psi}} \cong \mathrm{Hom}_{\Lambda_{\theta,\psi}}(\mathcal{P}, \Lambda_{\theta,\psi})$$

given by $\mathfrak{B}_i \mapsto a_1(\,\cdot\,|\mathfrak{B}_i) = \varphi_i$. Hence, we just need to show that $\mathfrak{H}(\mathcal{P})$ is isomorphic to $\langle \mathfrak{B}_0, \mathfrak{B}_1, \ldots, \mathfrak{B}_m \rangle_{\Lambda_{\theta,\psi}}$ as a $\Lambda_{\theta,\psi}$-module. Clearly the set $\{\mathfrak{B}_0, \mathfrak{B}_1, \ldots, \mathfrak{B}_m\}$ is $\Lambda_{\theta,\psi}$-linearly independent, so we just need to show that its $\Lambda_{\theta,\psi}$-span is all of $\mathfrak{H}(\mathcal{P})$.

Let $H \in \mathfrak{H}(\mathcal{P})$, and from this H define H_0 as in (46). Our goal is to show $H' = a_1(F_0|H)H_0 \in \langle \mathfrak{B}_0, \mathfrak{B}_1, \ldots, \mathfrak{B}_m \rangle_{\Lambda_{\theta,\psi}}$. To do so, we begin by noting that $H' \in \mathrm{Ann}_{\mathfrak{H}(\mathcal{P})}(S)$, so its action on $\mathcal{P}$ is completely determined by how it acts on F_0. By (45) we have

$$\begin{aligned} a_n(F_0|H') &= a_1(F_0|T_n|H') = a_n(\mathcal{E}_{\theta,\psi})a_1(F_0|H') + a_1(G_{T_n}|H') \\ &= a_n(\mathcal{E}_{\theta,\psi})a_1(F_0|H) \end{aligned}$$

for all $n \geqslant 0$. By the same argument we also have $a_n(F_0|\mathfrak{B}_0) = a_n(\mathcal{E}_{\theta,\psi})$ for all $n \geqslant 0$. Therefore, $H' = a_1(F_0|H_0)\mathfrak{B}_0$.

$(c) \Rightarrow (d)$. This follows by the same argument given in Subsection 3.3.1.

$(d) \Rightarrow (a)$. Recall the map $\Phi : \mathfrak{h} \to \Lambda_{\theta,\psi}/(G(X, \theta_0\omega^2))$ defined by

$$H \mapsto a_1(F_S|H) \pmod{G(X, \theta_0\omega^2)}.$$

where once again $F_S = G(X, \theta_0\omega^2)F_0 - \mathcal{E} \in S$. As was shown in Section 3.3.1, the map Φ is surjective and $I \subset \ker(\Phi)$. Since we know $\mathfrak{h}/I \cong \Lambda_{\theta,\psi}/(G(X, \theta_0\omega^2))$, it must be the case that $\ker(\Phi) = I$.

Let $H \in \mathfrak{h}$ be such that $a_1(F_S|H) \in \Lambda_{\theta,\psi}^\times$. We know such an H exists by the surjectivity of Φ. Then $G(X, \theta_0\omega^2)H \in \ker(\Phi)$, which implies that there exists an $\tilde{H} \in \mathrm{Ann}_{\mathfrak{H}_\mathcal{P}}(\mathcal{E})$ projecting to it. Hence, we have

$$\begin{aligned} a_1(F_0|\tilde{H}) &= \frac{1}{G(X,\theta_0\omega^2)} a_1(\mathcal{E}|\tilde{H} + F_S|\tilde{H}) = \frac{1}{G(X,\theta_0\omega^2)} a_1(F_S|\tilde{H}) \\ &= \frac{1}{G(X,\theta_0\omega^2)} a_1(F_S|G(X,\theta_\omega^2)H) \in \Lambda_{\theta,\psi}^\times. \end{aligned}$$

□

Finally, we will show that there exists an $H \in \mathfrak{H}(\mathcal{P})$ such that $a_1(F_0|H) \in \Lambda_{\theta,\psi}^\times$. Suppose for the sake of contradiction that $a_1(F_0|H) \in \mathfrak{m} := (\pi, X)$ for all $H \in \mathfrak{H}(\mathcal{P})$, where π is a uniformizer of $\mathcal{O}_{\theta,\psi}$. In particular, this implies that $a_n(F_0) = a_1(F_0|T_n) \in \mathfrak{m}$ for all $n \geqslant 1$. Since $a_1(\mathcal{E}|T_p) = 1$, we have $F_0|T_p = F_0 + G_{T_p}$ with $G_{T_p} \in S$. Furthermore, for all $n \geqslant 1$ we have

$$a_1(F_0|T_p|T_n) = a_n(F_0|T_p) = a_n(F_0) + a_n(G_{T_p}),$$

which implies $a_n(G_{T_p}) \in \mathfrak{m}$ by our assumption that $a_1(F_0|H) \in \mathfrak{m}$ for all $H \in \mathfrak{H}(\mathcal{P})$. Let

$$\begin{aligned} f_2 &= \nu_{2,\mathbb{1}}(F_0) \in M_2(Np)^{\mathrm{ord}}_{\mathcal{O}_{\theta,\psi}} \\ f_0 &= a_0(\nu_{2,\mathbb{1}}(F_0)) \in M_0(Np)_{\mathcal{O}_{\theta,\psi}}. \end{aligned}$$

To conclude our proof we will consider the difference between the forms f_0 and f_2. However, since these forms are of differing weights we must first make sense of what we mean by their difference. We remark that the following construction is just a shadow of a much larger theory of p-adic modular forms [H2], but it will suffice for our purposes.

By the q-expansion map, we have an embedding,

$$\bigoplus_{k=0}^{\infty} M_k(\Gamma_1)_{\mathcal{O}_{\theta,\psi}} \hookrightarrow \mathcal{O}_{\theta,\psi}[\![q]\!],$$

[H2, §1]. Let us denote the image of this map by M_∞. We consider the difference $f_0 - f_2 \in M_\infty$. Furthermore, it is well known that there is a natural action of T_p that preserves the space M_∞ [H2, §1]. Specifically, if $f \in M_\infty$ with

$$f = \sum_{k=0}^{\infty} f_k \subset \mathcal{O}_{\theta,\psi}[\![q]\!]$$

where $f_k \in M_k(\Gamma_1)_{\mathcal{O}_{\theta,\psi}}$, then

$$f|T_p = \sum_{k=0}^{\infty} f_k|T_p \subset \mathcal{O}_{\theta,\psi}[\![q]\!].$$

Now, by our assumption on the coefficients of F_0, we know that $f_0 \equiv f_2 \pmod{\pi}$. Furthermore, since the Hecke action commutes with the specialization map $\nu_{2,\mathbb{1}}$, we know that $f_2|T_p \equiv f_2 \pmod{\pi}$. Hence,

$$(p-1)f_0 \equiv_\pi pf_0 - f_2 \equiv_\pi f_0|T_p - f_2|T_p \equiv_\pi (f_0 - f_2)|T_p \equiv 0 \pmod{\pi}.$$

However, $(p-1)f_0 = (p-1)/2 \in \mathcal{O}_{\theta,\psi}^\times$, which gives us our contradiction.

4

EICHLER-SHIMURA COHOMOLOGY GROUPS

We begin this chapter by introducing the p-adic Eichler-Shimura cohomology groups and their basic properties. With these cohomology groups in hand, we follow Ohta in employing the method of Kurihara [Ku] and Harder-Pink [HP] to construct an abelian pro-p extension L/F_∞. Finally, we determine the ramification in L/F_∞, which allows us to prove the Iwasawa main conjecture over $\mathbb{Q}$.

We retain the notation of the previous chapters. Let $X_1(N_r)$ denote the canonical model of $\Gamma_r \setminus \mathbb{H}^*$ ($\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$) over $\mathbb{Q}$ in which the cusp at infinity is $\mathbb{Q}$-rational.

Definition 4.0.5. *The* p*-adic Eichler-Shimura cohomology group of level* N *is defined to be*

$$\mathcal{T} = \left(\varprojlim_r H^1_{\text{ét}}(X_1(Np^r) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Z}_p)^{\text{ord}} \right) \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_{\theta,\psi}$$

where $\widehat{\otimes}$ *denotes the completed tensor product and the projective limit is taken with respect to the trace mappings of étale cohomology groups.*

There are natural actions of $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\mathfrak{h}^*$ on $\mathcal{T}$ and these actions commute with one another. Furthermore, it is well known that $\mathcal{T}$ is a free and finitely generated $\Lambda_{\theta,\psi}$-module [O4, §1.2]. For any $\Lambda_{\theta,\psi}$-module M, we once again let $M_{Q(\Lambda_{\theta,\psi})} = M \otimes_{\Lambda_{\theta,\psi}} Q(\Lambda_{\theta,\psi})$.

Proposition 4.0.6 ([O2], Lemma 5.1.2)**.** *$\mathcal{T}_{Q(\Lambda_{\theta,\psi})}$ is a free $\mathfrak{h}^*_{Q(\Lambda_{\theta,\psi})}$-module of rank 2.*

By the above proposition we have a Galois representation

$$\rho : G_{\mathbb{Q}} \to GL_2\left(\mathfrak{h}^*_{Q(\Lambda_{\theta,\psi})}\right).$$

Moreover, one can show that this representation satisfies the Eichler-Shimura relations.

Theorem 4.0.7 ([O2], Theorem 5.1.5)**.** *If ℓ is a prime that does not divide Np and $\Phi_\ell \in G_{\mathbb{Q}}$ is a geometric Frobenius at ℓ, we have*

$$\det(1-\rho(\Phi_\ell)X) = 1 - T^*_\ell X + \ell T^*_{\ell,\ell} X^2.$$

Using the above theorem we can determine $\det(\rho(\sigma))$ for all $\sigma \in G_{\mathbb{Q}}$. Let $\chi_p : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ denote the p-adic cyclotomic character, and recall the map $\iota : \mathcal{O}_{\theta,\psi}[(\mathbb{Z}/Np\mathbb{Z})^\times][\![U_1]\!] \to \mathcal{O}_{\theta,\psi}[(\mathbb{Z}/Np\mathbb{Z})^\times][\![X]\!]$ induced by $u \mapsto 1+X$. Then for all primes $\ell \nmid Np$, we have $\chi_p(\Phi_\ell) = \ell^{-1}$ while $\iota(\ell)$ acts on $\mathfrak{h}^*$ as multiplication by $T^*_{\ell,\ell}$. This implies that

$$\det(\rho(\Phi_\ell)) = \ell T^*_{\ell,\ell} = \chi_p(\Phi_\ell)^{-1}\iota(\chi_p(\Phi_\ell))^{-1}$$

which in turn implies

$$\det(\rho(\sigma)) = \chi_p(\sigma)^{-1}\iota(\chi_p(\sigma)^{-1})$$

for all $\sigma \in G_{\mathbb{Q}}$ by the Čebotarev density theorem.

4.1 CONSTRUCTING ABELIAN PRO-p EXTENSIONS

In this section we employ the method of Kurihara [Ku] and Harder-Pink [HP] to construct an abelian pro-p extension L/F_∞. In Subsection 4.1.2 we will give a characterization of $\mathrm{Gal}(L/F_\infty)$ as an Iwasawa module, and in Subsection 4.1.3 we will use this characterization to determine the ramification in L/F_∞.

4.1.1 The method of Kurihara and Harder-Pink

With respect to our embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ from Chapter 1, we have an inclusion of absolute Galois groups $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow G_{\mathbb{Q}}$ given by $\sigma \mapsto \sigma|_{\overline{\mathbb{Q}}}$. Let $I_p := I_p(\overline{\mathbb{Q}}/\mathbb{Q})$ denote the inertia subgroup of $G_{\mathbb{Q}_p}$ which we regard as a subgroup of $G_{\mathbb{Q}}$ via the above injection.

Proposition 4.1.1 ([O3], Corollary 1.3.8)**.** *We have the following exact sequence of* $\mathfrak{h}^*$*-modules:*

$$0 \longrightarrow \mathcal{T}_+ \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{T}_+ \longrightarrow 0. \tag{47}$$

where we set $\mathcal{T}_+ := \mathcal{T}^{I_p}$*. Furthermore, the action of* $\sigma \in I_p$ *on* $\mathcal{T}/\mathcal{T}_+$ *is given by* $\chi_p(\sigma)^{-1}\iota(\chi_p(\sigma))^{-1}$.

While we would like to construct a splitting of (47) as Hecke modules, it is not known in general, whether such a splitting exists. However, it is well-known that such a splitting exists after localization, and this will suffice for our purposes.

Let $\sigma_0 \in I_p$ be an element satisfying $\sigma_0(\zeta) = \zeta^u = \zeta^{1+p}$ for all primitive p-power roots of unity ζ. We know that such an element exists since there is an element satisfying this property in $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, and the natural restriction map $I_p \to \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ is surjective. We've chosen this element of I_p with its action on $\mathcal{T}/\mathcal{T}_+$ in mind. Specifically, since $\chi_p(\sigma_0) = u$ and $\iota(u) = 1+X$, the action of σ_0 on the quotient $\mathcal{T}/\mathcal{T}_+$ is given by $u^{-1}(1+X)^{-1}$. Recall that $S = u^{-1}(1+X)^{-1} - 1$, and let

$$\mathcal{T}_- = \{x \in \mathcal{T} : \sigma_0 \cdot x = (S+1)x\}.$$

Note that since the action of $G_{\mathbb{Q}_p}$ commutes with the action of $\mathfrak{h}^*$, both $\mathcal{T}_+$ and $\mathcal{T}_-$ are modules over $\mathfrak{h}^*$. For any $\Lambda_{\theta,\psi}$-module M, we let $M_S := M \otimes_{\Lambda_{\theta,\psi}} \Lambda_{\theta,\psi}[S^{-1}]$.

Proposition 4.1.2. *$\mathcal{T}_S = \mathcal{T}_{-,S} \oplus \mathcal{T}_{+,S}$ as $\mathfrak{h}_S^*$-modules.*

Proof. First we will show that $\mathcal{T}_S = \mathcal{T}_{-,S} + \mathcal{T}_{+,S}$. Let $x \in \mathcal{T}_S$. Since σ_0 acts on the quotient $\mathcal{T}/\mathcal{T}_+$ by $S+1$, we know that

$$\sigma_0 \cdot x = (S+1)x + y$$

for some $y \in \mathcal{T}_{+,S}$. Noting that $x + S^{-1}y \in \mathcal{T}_S$ (this is one of the reasons we have localized at S), we have

$$\sigma_0 \cdot (x + S^{-1}y) \ = \ (S+1)x + y + S^{-1}y \ = \ (S+1)(x + S^{-1}y),$$

which implies $x + S^{-1}y \in \mathcal{T}_{-,S}$. Hence, $x \in \mathcal{T}_{-,S} + \mathcal{T}_{+,S}$.

Next, we want to show $\mathcal{T}_{-,S} \cap \mathcal{T}_{+,S} = \{0\}$. Let $x \in \mathcal{T}_{-,S} \cap \mathcal{T}_{+,S}$. Then,

$$(S+1)x \ = \ \sigma_0 \cdot x \ = \ x,$$

which implies $Sx = 0$. Since $\mathcal{T}$ is a free $\Lambda_{\theta,\psi}$-module, we have $x = 0$. □

Proposition 4.1.3 ([O2], Lemma 5.1.3.). *$(\mathcal{T}/\mathcal{T}_+)_{Q(\Lambda_{\theta,\psi})}$ and $\mathcal{T}_{+,Q(\Lambda_{\theta,\psi})}$ are free $\mathfrak{h}^*_{Q(\Lambda_{\theta,\psi})}$-modules of rank 1.*

Fixing $\mathfrak{h}^*_{Q(\Lambda_{\theta,\psi})}$-bases for $\mathcal{T}_{-,Q(\Lambda_{\theta,\psi})}$ and $\mathcal{T}_{+,Q(\Lambda_{\theta,\psi})}$ (in that order), we write

$$\rho(\sigma) \ = \ \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}.$$

Then for all $\sigma \in I_p$ we have,

$$\rho(\sigma) \ = \ \begin{pmatrix} \det(\rho(\sigma)) & 0 \\ c(\sigma) & 1 \end{pmatrix} \tag{48}$$

with

$$\rho(\sigma_0) \ = \ \begin{pmatrix} S+1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Before moving on, we record a result that will be used later in the chapter. Let $\mathcal{B}$ and $\mathcal{C}$ denote the $\mathfrak{h}_S^*$-submodules of $\mathfrak{h}^*_{Q(\Lambda_{\theta,\psi})}$ generated by the sets $\{b(\sigma) : \sigma \in G_\mathbb{Q}\}$ and $\{c(\sigma) : \sigma \in G_\mathbb{Q}\}$, respectively.

Proposition 4.1.4 ([O4], Lemma 3.3.6.). *$\mathcal{B}$ and $\mathcal{C}$ are faithful $\mathfrak{h}_S^*$-modules.*

Let $\mathcal{I}^*$ denote the image of $\mathcal{I} := \mathcal{I}_{\theta,\psi} := \mathrm{Ann}_{\mathfrak{H}}(\mathcal{E})$ under the natural isomorphism induced by $H \mapsto H^*$. Then $\mathcal{I}^*$ is the ideal of $\mathfrak{H}^*$ generated by

$$\begin{aligned} T^*_{q,q} - (\theta_0\psi_0)(q)(1+X)^{s(q)} &\qquad \text{for integers } q \text{ coprime to } Np \\ T^*_\ell - \left(\theta_0(\ell)\ell(1+X)^{s(\ell)} + \psi_0(\ell)\right) &\qquad \text{for primes } \ell \neq p \\ T^*_p - \psi_0(p). & \end{aligned}$$

Let I^* denote the image of $\mathcal{I}^*$ in $\mathfrak{h}^*$. Define the map $\tilde{\rho}$ by

$$\sigma \in G_{\mathbb{Q}} \ \mapsto \begin{pmatrix} \overline{\psi_0(\sigma)\det(\rho(\sigma))} & \overline{b(\sigma)} \\ 0 & \overline{\psi_0(\sigma)^{-1}} \end{pmatrix},$$

where the bar indicates reduction modulo I_S^*, that is, $\overline{\det(\rho(\sigma))} \in \mathfrak{h}_S^*/I_S^*$ while $\overline{b(\sigma)} \in \mathcal{B}/I_S^*\mathcal{B}$. As the next proposition shows, this map is a Galois representation.

Proposition 4.1.5. *For any $\sigma, \tau \in G_{\mathbb{Q}}$ we have $a(\sigma)$, $d(\sigma)$, $b(\sigma)c(\tau) \in \mathfrak{h}_S^*$ with*

$$\begin{aligned} a(\sigma) &\equiv \psi_0(\sigma)\det(\rho(\sigma)) \pmod{I_S^*} \\ d(\sigma) &\equiv \psi_0(\sigma)^{-1} \pmod{I_S^*} \\ b(\sigma)c(\tau) &\equiv 0 \pmod{I_S^*}. \end{aligned}$$

Proof. Let ℓ be a prime that does not divide Np and recall that

$$\begin{aligned} \det(\rho(\Phi_\ell)) &= \ell T^*_{\ell,\ell} \\ \mathrm{tr}(\rho(\Phi_\ell)) &= T^*_\ell, \end{aligned}$$

where once again $\Phi_\ell \in G_{\mathbb{Q}}$ is a geometic Frobenius at ℓ. Then,

$$a(\Phi_\ell) + d(\Phi_\ell) - \psi_0(\Phi_\ell)\det(\rho(\Phi_\ell)) - \psi_0(\Phi_\ell)^{-1}$$

$$= T^*_\ell - \psi_0(\ell)^{-1}\ell T^*_{\ell,\ell} - \psi_0(\ell)$$

$$\begin{aligned} = \ T^*_\ell - \left(\ell\theta_0(\ell)(1+X)^{s(\ell)} + \psi_0(\ell)\right) & \\ - \psi_0(\ell)^{-1}\ell\left(T^*_{\ell,\ell} - (\theta_0\psi_0)(\ell)(1+X)^{s(\ell)}\right) \ &\in \ I_S^*. \end{aligned}$$

Here we're using the fact that $\theta(\ell) = \theta_0(\ell)$ and $\psi(\ell) = \psi_0(\ell)$ for $\ell \nmid Np$. Therefore, by the Čebotarev density theorem we know that for all $\sigma \in G_{\mathbb{Q}}$

$$a(\sigma) + d(\sigma) - \psi_0(\sigma)\det(\rho(\sigma)) - \psi(\sigma)^{-1} \in I_S^*. \quad (49)$$

Let $\sigma \in G_{\mathbb{Q}}$ and $\sigma_0 \in I_p$, where σ_0 is as defined in Subsection 4.1.1. Applying (49) to $\sigma\sigma_0$, and noting that $\psi_0(\sigma_0) = 1$ since $(f_\psi, p) = 1$, we get

$$(S+1)a(\sigma) + d(\sigma) - (S+1)\psi_0(\sigma)\det(\rho(\sigma)) - \psi_0(\sigma)^{-1} \in I_S^*. \quad (50)$$

Subtracting (49) from (50) we get

$$Sa(\sigma) - S\psi(\sigma)\det(\rho(\sigma)) \in I_S^*.$$

Since S is a unit in $\mathfrak{h}_S^*$ (the other reason we needed to invert S), we have the results for $a(\sigma)$ and $d(\sigma)$.

The remaining results follow from the second inclusion and the fact that,

$$d(\sigma\tau) = c(\tau)b(\sigma) + d(\tau)d(\sigma)$$

for all $\sigma, \tau \in G_{\mathbb{Q}}$. □

For future reference, we give a more explicit characterization of $\tilde{\rho}$. Once again let ℓ be a prime that does not divide Np and Φ_ℓ a geometric Frobenius at ℓ. Then we have,

$$\det(\rho(\Phi_\ell)) = \ell T_{\ell,\ell}^* \equiv \langle\langle \ell \rangle\rangle(\theta_0\psi_0\omega)(\ell)(1+X)^{s(\ell)} \pmod{I_S^*}.$$

Noting that

$$\langle\langle \ell \rangle\rangle(\theta_0\psi_0\omega)(\ell)(1+X)^{s(\ell)} = (\theta_0\psi_0\omega)(\Phi_\ell)^{-1}\langle\langle \chi_p(\Phi_\ell)\rangle\rangle^{-1}\iota(\langle\langle \chi_p(\Phi_\ell)\rangle\rangle)^{-1},$$

we once again employ the Čebotarev density theorem to get

$$\det(\rho(\sigma)) \equiv (\theta_0\psi_0\omega)(\sigma)^{-1}\langle\langle \chi_p(\sigma)\rangle\rangle^{-1}\iota(\langle\langle \chi_p(\sigma)\rangle\rangle)^{-1} \pmod{I_S^*} \quad (51)$$

for all $\sigma \in G_{\mathbb{Q}}$. Therefore,

$$\tilde{\rho}(\sigma) = \begin{pmatrix} \overline{(\theta_0\omega)(\sigma)^{-1}} \langle\!\langle \chi_p(\sigma) \rangle\!\rangle^{-1} \iota(\langle\!\langle \chi_p(\sigma) \rangle\!\rangle)^{-1} & \overline{b(\sigma)} \\ 0 & \overline{\psi_0(\sigma)^{-1}} \end{pmatrix} \tag{52}$$

for all $\sigma \in G_{\mathbb{Q}}$.

We will now use the representation $\tilde{\rho}$ to construct our desired extension of F_∞. Set

$$\begin{aligned} F_0 &:= \text{the field corresponding to } \left\{ \sigma \in G_{\mathbb{Q}} : \overline{\det(\rho(\sigma))} = 1 = \psi_0(\sigma) \right\} \\ L_0 &:= \text{the field corresponding to } \ker(\tilde{\rho}), \\ L &:= L_0 F_\infty \end{aligned}$$

Then we have an injection of abelian groups

$$\mathrm{Gal}(L_0/F_0) \hookrightarrow \mathcal{B}/I_S^*\mathcal{B} \;:\; \sigma \mapsto \overline{b(\sigma)}.$$

Clearly L_0/F_0 is abelian, and the fact that $\mathcal{B}/I_S^*\mathcal{B}$ is a finitely generated Λ-module implies that $\mathrm{Gal}(L_0/F_0)$ is pro-p. By (51) we see that

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F_\infty) \subset \left\{ \sigma \in G_{\mathbb{Q}} : \overline{\det(\rho(\sigma))} = 1 = \psi_0(\sigma) \right\} \subset \ker(\theta\omega) \cap \ker(\psi),$$

which implies $F \subseteq F_0 \subseteq F_\infty$. Considering the definition of $\tilde{\rho}$, we see that L_0/F_0 is unramified at p. However, we know F_∞/F is totally ramified at p, which implies $L_0 \cap F_\infty = F_0$. Therefore, $\mathrm{Gal}(L_0/F_0) = \mathrm{Gal}(L_0/L_0 \cap F_\infty) \cong \mathrm{Gal}(L/F_\infty)$, and we see that L/F_∞ is an abelian pro-p extension.

4.1.2 *An isomorphism of the Iwasawa modules*

In addition to showing that $\mathrm{Gal}(L/F_\infty)$ is an abelian pro-p extension, the isomorphism $\mathrm{Gal}(L/F_\infty) \cong \mathrm{Gal}(L_0/F_0)$ also implies that we have an injection

$$\mathrm{Gal}(L/F_\infty) \hookrightarrow \mathcal{B}/I_S^*\mathcal{B}. \tag{53}$$

In this subsection, we will show this injection induces an isomorphism of Iwasawa modules.

Recall from Chapter 1 that $\mathrm{Gal}(F_\infty/\mathbb{Q}) \cong \Delta \times \Gamma$ acts on $\mathrm{Gal}(L/F_\infty)$ by conjugation, and the fact that the extension L/F_∞ is abelian and pro-p implies $\mathrm{Gal}(L/F_\infty)$ is a module over the Iwasawa algebra $\mathbb{Z}_p[\Delta][\![\Gamma]\!]$. We want to identify $\mathbb{Z}_p[\Delta][\![\Gamma]\!]$ with $\mathbb{Z}_p[\Delta][\![X]\!]$ in a particular way. Recall that there is a natural isomorphism $U_1 \cong \mathbb{Z}_p \cong \Gamma = \mathrm{Gal}(F_\infty/F)$. Let $\gamma_0 \in \Gamma$ correspond to $u \in U_1$. Then γ_0 is a topological generator of Γ and $\langle\!\langle \chi_p(\gamma_0) \rangle\!\rangle = u$. We then identify $\mathbb{Z}_p[\Delta][\![\Gamma]\!]$ with $\mathbb{Z}_p[\Delta][\![X]\!]$ by

$$\gamma_0 \longmapsto \langle\!\langle \chi_p(\gamma_0) \rangle\!\rangle = u \overset{\iota}{\longmapsto} 1 + X. \tag{54}$$

In the following proposition we describe how the $\mathbb{Z}_p[\Delta][\![X]\!]$ action on $\mathrm{Gal}(L/F_\infty)$ commutes with the injection (53):

Proposition 4.1.6. *Let $\delta \in \Delta$. Then for all $\sigma \in \mathrm{Gal}(L/F_\infty)$ we have*

$$\begin{aligned} \delta \cdot \sigma &\mapsto (\xi\omega)^{-1}(\delta) \cdot \overline{b(\sigma)} \\ X \cdot \sigma &\mapsto S \cdot \overline{b(\sigma)}. \end{aligned}$$

Proof. Let $\tau \in \mathrm{Gal}(F_\infty/\mathbb{Q})$ with arbitrary lift $\tilde{\tau} \in \mathrm{Gal}(L/\mathbb{Q})$. Then

$$\tau \cdot \sigma \mapsto \overline{b(\tilde{\tau}\sigma\tilde{\tau}^{-1})}$$

for all $\sigma \in \mathrm{Gal}(L/F_\infty)$. Since

$$\tilde{\rho}(\tilde{\tau}\sigma\tilde{\tau}^{-1}) = \begin{pmatrix} 1 & (\xi\omega)(\tau)^{-1}\langle\!\langle \chi_p(\tau)\rangle\!\rangle^{-1}\iota(\langle\!\langle \chi_p(\tau)\rangle\!\rangle)^{-1} \cdot \overline{b(\sigma)} \\ 0 & 1 \end{pmatrix},$$

we see that,

$$\overline{b(\tilde{\tau}\sigma\tilde{\tau}^{-1})} = (\xi\omega)(\tau)^{-1}\langle\!\langle \chi_p(\tau)\rangle\!\rangle^{-1}\iota(\langle\!\langle \chi_p(\tau)\rangle\!\rangle)^{-1} \cdot \overline{b(\sigma)}.$$

Now, if $\delta \in \Delta \times \{1\} \subset \mathrm{Gal}(F_\infty/\mathbb{Q})$ we know $\chi_p(\delta) \in \mu_{p-1}(\mathbb{Z}_p^\times)$ since τ has finite order. This implies $\langle\!\langle \chi_p(\delta)\rangle\!\rangle = 1$, which in turn implies

$$\delta \cdot \sigma \mapsto (\xi\omega)^{-1}(\delta) \cdot \overline{b(\sigma)}.$$

On the other hand, if $\gamma \in \{1\} \times \Gamma \subset \mathrm{Gal}(F_\infty/\mathbb{Q})$ we know $(\xi\omega)(\gamma) = 1$ which implies

$$\gamma \cdot \sigma \;\mapsto\; \langle\!\langle \chi_p(\tau) \rangle\!\rangle^{-1} \iota(\langle\!\langle \chi_p(\tau) \rangle\!\rangle)^{-1} \cdot \overline{b(\sigma)}.$$

By our identification (54), we have

$$X \cdot \sigma \;=\; (\gamma_0 - 1) \cdot \sigma \;\mapsto\; \left(\langle\!\langle \chi_p(\gamma_0) \rangle\!\rangle^{-1} \iota(\langle\!\langle \chi_p(\gamma_0) \rangle\!\rangle)^{-1} - 1 \right) \cdot \overline{b(\sigma)} \;=\; S \cdot \overline{b(\sigma)}$$

□

From the above proposition, we see that $\mathrm{Gal}(L/F_\infty)$ is a $\Lambda_\xi[\![X]\!]$-module on which Δ acts via $(\xi\omega)^{-1}$.

Up to this point in the chapter, we've made no restrictions on the characters θ and ψ. However, consider the case in which $(\theta_0, \psi) = (\omega^{-2}, \mathbb{1})$. In this case $F = \mathbb{Q}(\mu_p)$ and L/F_∞ is an abelian pro-p extension on which Δ acts by ω. It is well known that such an extension must be trivial. Therefore, we will assume that $(\theta_0, \psi) \neq (\omega^{-2}, \mathbb{1})$ for the remainder of the chapter.

With the above proposition in mind, we let $(\mathcal{B}/I_S^*\mathcal{B})^\dagger$ denote the $\Lambda_{\theta,\psi}[X^{-1}]$-module obtained from $\mathcal{B}/I_S^*\mathcal{B}$ by twisting the $\Lambda_{\theta,\psi}[S^{-1}]$-module structure by the involutive $\mathcal{O}$-module automorphism of $\Lambda_{\theta,\psi}$ given by $X \mapsto S$ (i.e. X acts on $(\mathcal{B}/I_S^*\mathcal{B})^\dagger$ as multiplication by S). In doing so, our injection of abelian groups (53) becomes an injection of Λ_ξ-modules. This in turn implies that we have the following injection of $\Lambda_{\theta,\psi}[X^{-1}]$-modules,

$$\mathrm{Gal}(L/F_\infty) \otimes_{\Lambda_\xi} \Lambda_{\theta,\psi}[X^{-1}] \hookrightarrow (\mathcal{B}/I_S^*\mathcal{B})^\dagger.$$

In fact, in the next proposition we show the above map is an isomorphism.

Proposition 4.1.7 ([O3], Lemma 3.3.11.). *The injection (53) induces an isomorphism of $\Lambda_{\theta,\psi}[X^{-1}]$-modules* $\mathrm{Gal}(L/F_\infty) \otimes_{\Lambda_\xi} \Lambda_{\theta,\psi}[X^{-1}] \cong (\mathcal{B}/I_S^*\mathcal{B})^\dagger$.

Proof. By the comments preceding the proposition, it suffices to show that our $\Lambda_{\theta,\psi}[X^{-1}]$-module homomorphism is surjective. Recall that

$$\rho(\sigma_0) \;=\; \begin{pmatrix} S+1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then for any $\sigma \in G_{\mathbb{Q}}$ we have

$$\tilde{\rho}(\sigma_0 \sigma \sigma_0^{-1} \sigma^{-1}) = \begin{pmatrix} 1 & S \cdot \overline{b(\sigma)} \\ 0 & 1 \end{pmatrix},$$

which implies $\sigma_0 \sigma \sigma_0^{-1} \sigma^{-1} \mapsto X \cdot \overline{b(\sigma)}$. Since X is a unit in $\Lambda_{\theta,\psi}[X^{-1}]$, to prove surjectivity we just need to show that $(\mathcal{B}/I_S^*\mathcal{B})^{\dagger}$ is generated by the set $\{\overline{b(\sigma)} : \sigma \in G_{\mathbb{Q}}\}$ as a $\Lambda_{\theta,\psi}[X^{-1}]$-module. However, we know that $\mathcal{B}/I_S^*\mathcal{B}$ is generated by the set $\{\overline{b(\sigma)} : \sigma \in G_{\mathbb{Q}}\}$ as a $\mathfrak{h}_S^*/I_S^*$-module, and by Proposition 3.3.1 we have $\mathfrak{h}_S^*/I_S^* \cong \Lambda_{\theta,\psi}[S^{-1}]/(A)$. Consequently, $\mathcal{B}/I_S^*\mathcal{B}$ is generated by the set $\{\overline{b(\sigma)} : \sigma \in G_{\mathbb{Q}}\}$ as a $\Lambda_{\theta,\psi}[S^{-1}]$-module, which implies $(\mathcal{B}/I_S^*\mathcal{B})^{\dagger}$ is generated by $\{\overline{b(\sigma)} : \sigma \in G_{\mathbb{Q}}\}$ as a $\Lambda_{\theta,\psi}[X^{-1}]$-module.

□

4.1.3 *Ramification in* L/F_∞

In this subsection, we will use the structure of $\mathrm{Gal}(L/F_\infty)$ as a Λ_ξ-module to characterize the ramification occurring in L/F_∞.

Let $\ell \neq p$ be an arbitrary prime. It is well known that the prime ℓ will not split completely in the cyclotomic $\mathbb{Z}_p$-extension F_∞/F. Hence, there are only finitely many primes $\mathfrak{l}_1, \ldots, \mathfrak{l}_m$ of F_∞ lying above ℓ. Consider the subgroup $G_\ell \subset \mathrm{Gal}(L/F_\infty)$ generated by the inertia subgroups $I_{\mathfrak{l}_i}$ for $1 \leqslant i \leqslant m$. Let us call the corresponding fixed field K_ℓ. Then K_ℓ/F_∞ is the maximal subextension of L/F_∞ in which all of the $\mathfrak{l}_i$ are unramified.

Of primary interest to us will be the group $G_\ell = \mathrm{Gal}(L/K_\ell)$. We will now use class field theory to obtain a useful characterization of this group. Recall that F_n/F is the finite extension of F satisfying $\mathrm{Gal}(F_n/F) \cong \mathbb{Z}/p^n\mathbb{Z}$. Then for each $n \geqslant 1$, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{I}_{F_{n+1}} & \xrightarrow{\Phi_{F_{n+1}}} & G_{F_{n+1}}^{ab} \\ \downarrow & & \downarrow \\ \mathbb{I}_{F_n} & \xrightarrow{\Phi_{F_n}} & G_{F_n}^{ab} \end{array}$$

where Φ_{F_n} is the global reciprocity map between the idelé group of F_n and the Galois group of the maximal abelian extension of F_n, with the left vertical arrow being the norm map, and the right vertical arrow being restriction. Taking the projective limit of these maps, we obtain a surjective homomorphism

$$\Phi_{F_\infty} : \varprojlim_n \mathbb{I}_{F_n} \twoheadrightarrow \mathrm{Gal}(L/K_\ell) \subset G^{ab}_{F_\infty}.$$

Since $\mathrm{Gal}(L/K_\ell)$ is the subgroup of $\mathrm{Gal}(L/F_\infty)$ generated by the inertia subgroups $I_{\mathfrak{l}}$ for primes $\mathfrak{l} \mid \ell$, we know that the above map is surjective when restricted to

$$\varprojlim_n \prod_{\mathfrak{l}|\ell} \mathcal{O}^\times_{F_{n,\mathfrak{l}}} \twoheadrightarrow \mathrm{Gal}(L/K_\ell),$$

where $F_{n,\mathfrak{l}}$ denotes the completion of F_n with respect to $\mathfrak{l}$ and $\mathcal{O}_{F_{n,\mathfrak{l}}}$ is its ring of integers. Furthermore, since $\mathrm{Gal}(L/K_\ell)$ is a subgroup of the pro-p group $\mathrm{Gal}(L/F_\infty)$, we know that the principal units must lie in the kernel of the above surjection. Therefore, letting $\mathfrak{k}_{F_{n,\mathfrak{l}}}$ denote the residue field of $\mathcal{O}_{F_{n,\mathfrak{l}}}$ we have a surjection

$$\varprojlim_n \prod_{\mathfrak{l}|\ell} \mathfrak{k}^\times_{F_{n,\mathfrak{l}}} \twoheadrightarrow \mathrm{Gal}(L/K_\ell). \tag{55}$$

Lemma 4.1.8 ([O4], Lemma A.2.1)**.** *The Galois group* $\mathrm{Gal}(L/K_\ell)$ *is a cyclic* Λ_ξ*-module annihilated by* $b_\ell(X) := (1+X)^{s(\ell)} - (\xi\omega)(\ell)\ell$.

Proof. Since inertia at ℓ in $\mathrm{Gal}(F_\infty/\mathbb{Q})$ acts trivially on $\varprojlim_n \prod_{\mathfrak{l}|\ell} \mathfrak{k}^\times_{F_{n,\mathfrak{l}}}$, it must also act trivially on $\mathrm{Gal}(L/K_\ell)$ by (55). Hence, the action of $\mathrm{Gal}(F_\infty/\mathbb{Q})$ on $\mathrm{Gal}(L/K_\ell)$ is unramified.

Let Φ_ℓ denote the geometric Frobenius of $\mathrm{Gal}(F_\infty/\mathbb{Q})$. We will prove the result by considering the action of Φ_ℓ on $\mathrm{Gal}(L/F_\infty)$ from two different perspectives. First we consider the action of Φ_ℓ on $\mathrm{Gal}(L/F_\infty)$ via Proposition 4.1.6. We know that the action of Φ_ℓ on the image of $\mathrm{Gal}(L/K_\ell)$ in $\mathcal{B}/I^*_S\mathcal{B}$ is given by

$$(\xi\omega)^{-1}(\Phi_\ell) \cdot \langle\langle \chi_p(\Phi_\ell) \rangle\rangle \cdot \iota(\langle\langle \chi_p(\Phi_\ell) \rangle\rangle) = (\xi\omega)(\ell) u^{s(\ell)} (1+X)^{s(\ell)}.$$

After making the change of variable $X \mapsto S$, we see that action of Φ_ℓ on $\mathrm{Gal}(L/K_\ell)$ is given by

$$(\xi\omega)(\ell)u^{s(\ell)}(1+u^{-1}(1+X)^{-1}-1)^{s(\ell)} = (\xi\omega)(\ell)(1+X)^{-s(\ell)}.$$

On the other hand, the action of Φ_ℓ on $\varprojlim_n \prod_{\mathfrak{l}|\ell} \mathfrak{k}_{n,\mathfrak{l}}^\times$ is given by

$$\Phi_\ell \cdot \varprojlim_n (\dots, a_{\mathfrak{l}}, \dots)_{\mathfrak{l}|\ell} = \varprojlim_n \left(\dots, \Phi_\ell\left(a_{\Phi_\ell^{-1}(\mathfrak{l})}\right), \dots\right)_{\mathfrak{l}|\ell}$$
$$= \varprojlim_n \left(\dots, a_{\mathfrak{l}}^{1/\ell}, \dots\right)_{\mathfrak{l}|\ell}.$$

This implies that Φ_ℓ acts as multiplication by ℓ^{-1} on $\mathrm{Gal}(L/K_\ell)$ via (55). Putting these two actions together, we see that

$$\ell^{-1} - (\xi\omega)(\ell)(1+X)^{-s(\ell)} = \ell^{-1}(1+X)^{-s(\ell)}\left((1+X)^{s(\ell)} - (\xi\omega)(\ell)\ell\right)$$

annihilates $\mathrm{Gal}(L/K_\ell)$.

Finally, we'll show that $\mathrm{Gal}(L/K_\ell)$ is a cyclic Λ_ξ-module. Let $\mathfrak{l}_1, \dots, \mathfrak{l}_m$ be the primes of F_∞ lying above ℓ. For sufficiently large n, we know that that there are m distinct primes $\mathfrak{l}_{i,n}$ in F_n lying above ℓ. That is, for large enough n the primes above ℓ in F_n are unramified and inert. Hence, $\varprojlim_n \mathfrak{k}_{F_n,\mathfrak{l}_{i,n}}^\times$ is a pro-cyclic group. Since $\mathrm{Gal}(F_\infty/\mathbb{Q})$ simply permutes the prime above ℓ transitively, we know that $\mathrm{Gal}(L/K_\ell)$ is a cyclic $\mathcal{O}_\xi[\![\Delta \times \Gamma]\!]$-module, and consequently a cyclic Λ_ξ-module. □

Lemma 4.1.9. *If $\ell \nmid N$ or $(\xi\omega^2)(\ell)$ is not a p-power root of unity, then ℓ is unramified in L/F_∞.*

Proof. Recall that $\tilde{\rho}$ is unramified outside of Np, and $\overline{b(\sigma)} = 0$ for $\sigma \in I_p$. Therefore, the injectivity of the map $\mathrm{Gal}(L/F_\infty) \hookrightarrow B/I_S^*B$ given by $\sigma \mapsto \overline{b(\sigma)}$ implies that L/F_∞ is unramified outside of N.

On the other hand,

$$b_\ell(X) := (1+X)^{s(\ell)} - (\xi\omega)(\ell)\ell = (1+X)^{s(\ell)} - (\xi\omega^2)(\ell)\langle\!\langle \ell \rangle\!\rangle,$$

if $\ell \neq p$. If $(\xi\omega^2)(\ell)$ is not a p-power root of unity

$$|(\xi\omega^2)(\ell) - 1|_p = 1,$$

which implies the power series $b_\ell(X)$ is a unit. Lemma 4.1.8 then implies that ℓ is unramified in L/F_∞. □

4.2 CHARACTERISTIC IDEAL AND THE IWASAWA MAIN CONJECTURE

In order to prove the Iwasawa main conjecture and determine $\mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L/F_\infty))$, we will employ the theory of Fitting ideals. Let us quickly recall the definition and basic properties of these ideals [MW, Appendix].

Definition 4.2.1. *Let R be a commutative ring and M an R-module of finite presentation. Take any presentation of M,*

$$R^m \xrightarrow{\varphi} R^n \longrightarrow M \longrightarrow 0.$$

The Fitting ideal $\mathrm{Fitt}_R(M)$ is defined to be the ideal of R generated by all $n \times n$ minors of φ. This ideal is independent of the choice of presentation.

In fact, what we've described above is the 0^{th} Fitting ideal. One can define higher Fitting ideals (i.e. the i^{th} Fitting ideal for all integers $i \geqslant 0$), but for our purposes the 0^{th} Fitting ideal will suffice. The reason we are considering Fitting ideals is due to their close relationship with characteristic ideals of Iwasawa modules. Specifically, if M is a Λ-module, then $\mathrm{Char}_\Lambda(M)$ is the unique principal ideal of Λ such that $\mathrm{Char}_\Lambda(M)/\mathrm{Fitt}_\Lambda(M)$ is finite.

Next we recall the results on Fitting ideals that we will need.

Proposition 4.2.2 ([MW], Appendix)**.** *For any finitely generated R-module M, the following hold:*

(1) *If $M \twoheadrightarrow M'$ is a surjection of R-modules, then $\mathrm{Fitt}_R(M) \subseteq \mathrm{Fitt}_R(M')$.*

(2) *If M is a faithful R-module, then $\mathrm{Fitt}_R(M) = 0$.*

(3) *For any* R*-algebra* R',

$$\mathrm{Fitt}_{R'}(M \otimes_R R') = \mathrm{Fitt}_R(M) \cdot R'.$$

(4) *If* M *is a direct sum of cyclic* R*-modules, i.e.* $M = R/\mathfrak{a}_1 \times \cdots \times R/\mathfrak{a}_t$*, then*

$$\mathrm{Fitt}_R(M) = \mathfrak{a}_1 \cdot \cdots \cdot \mathfrak{a}_t.$$

We are now ready to prove the main result of this section, which is the Iwasawa main conjecture over $\mathbb{Q}$. As mentioned in Chapter 1, the main conjecture was originally proven by Mazur and Wiles [MW]. The following proof is based on the method of Ohta [O3].

Theorem 4.2.3 (The Iwasawa main conjecture over $\mathbb{Q}$)**.** *We have the following equality of ideals*

$$\mathrm{Char}_{\Lambda_\xi}\left(X_{\infty,(\xi\omega)^{-1}}\right) = \left(F(X, \xi\omega^2)\right).$$

Proof. As was explained in Chapter 1, it is a consequence of the analytic class number formula that in order to prove the above equality one need only show

$$\mathrm{Char}_{\Lambda_\xi}\left(X_{\infty,(\xi\omega)^{-1}}\right) \subseteq (F(X, \xi\omega^2)).$$

In order to make the proof of this inclusion easier to follow, we will make two claims from which the inclusion will be a straightforward consequence. Once this is done, we will go about proving these claims.

Let L^{un}/F_∞ be the maximal unramified subextension of L/F_∞, and denote the image of A under the isomorphism $X \mapsto S$ by $\tilde{A}$,

$$\tilde{A} := \left(\prod_{\substack{\ell \mid \tilde{f}_\theta f_\psi \\ \ell \nmid f_\xi}} ((1+X)^{s(\ell)} - (\xi\omega)(\ell)\,\ell) \right) \cdot F(X, \xi\omega^2).$$

Then

Claim 1: $\left(\prod_{\substack{\ell \mid N \\ \ell \nmid f_\xi}} b_\ell(X)\right) \cdot \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L^{un}/F_\infty)) \subseteq \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L/F_\infty))$.

Claim 2: $X^m \cdot \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L/F_\infty)) \subseteq (\tilde{A})$ for some integer $m \geqslant 0$.

Putting these two claims together, we get the following inclusion

$$X^m \cdot \left(\prod_{\substack{\ell \mid N \\ \ell \nmid \tilde{f}_\theta f_\psi}} b_\ell(X)\right) \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L^{un}/F_\infty)) \subseteq (F(X, \xi\omega^2)). \tag{56}$$

We know that L^{un}/F_∞ is an unramified pro-p abelian extension on which Δ acts via $(\xi\omega)^{-1}$, which implies $\mathrm{Gal}(L^{un}/F_\infty)$ is a quotient of $X_{\infty,(\xi\omega)^{-1}}$. Hence, we have the following inclusion of characteristic ideals

$$\mathrm{Char}_{\Lambda_\xi}\left(X_{\infty,(\xi\omega)^{-1}}\right) \subseteq \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L^{un}/F_\infty)). \tag{57}$$

[Wa, Proposition 15.22]. Putting this together with (56) we get

$$X^m \cdot \left(\prod_{\substack{\ell \mid N \\ \ell \nmid \tilde{f}_\theta f_\psi}} b_\ell(X)\right) \mathrm{Char}_{\Lambda_\xi}\left(X_{\infty,(\xi\omega)^{-1}}\right) \subseteq (F(X, \xi\omega^2)). \tag{58}$$

A well-known result of Ferrero-Greenberg tells us that the power of X dividing the generator of $\mathrm{Char}_{\Lambda_\xi}(X_{\infty,(\xi\omega)^{-1}})$ is equal to that dividing $F(X, \xi\omega^2)$ [FG, Section 4]. In particular, the power of X dividing $F(X, \xi\omega^2)$ is 1 if the pair (θ_0, ψ_0) is exceptional, and 0 otherwise. This, in combination with the fact that $X \nmid b_\ell(X)$ for all primes $\ell \mid N$, implies that $m = 0$.

It will now suffice to show $\gcd(b_\ell(X), F(X, \xi\omega^2)) = 1$. Recall from the proof of Lemma 4.1.9 that $b_\ell(X)$ is a unit if $(\xi\omega^2)(\ell)$ is not a p-power root of unity. Suppose $(\xi\omega^2)(\ell)$ is a p-power root of unity. Then any root of $b_\ell(X) = (1+X)^{s(\ell)} - (\xi\omega^2)(\ell)\langle\langle \ell \rangle\rangle$ must be of the form $u\zeta - 1$ where ζ satisfies $\zeta^{s(\ell)} = (\xi\omega^2)(\ell)$ (here we're using the fact that $u^{s(\ell)} = \langle\langle \ell \rangle\rangle$). Clearly ζ is a root of unity. By the same argument referenced above, we know that if ζ is not a p-power root of unity, then $u\zeta - 1$ is a unit, albeit possibly in some finite extension of $\mathcal{O}_\xi$. However, this would imply that the minimal polynomial of $u\zeta - 1$ in $\mathcal{O}_\xi[X]$ is a unit in Λ_ξ. Thus, we may assume that ζ is a p-power root of unity. Evaluating $F(X, \xi\omega^2) = G(S, \xi\omega^2)$ at $u\zeta - 1$, we have

$$G(u^{-1}(1 + u\zeta - 1)^{-1} - 1, \xi\omega^2) = G(u^{-2}\zeta^{-1} - 1) = L_p(1, \xi\epsilon\omega^2),$$

where ϵ is the character on $1 + p\mathbb{Z}_p$ having p-power order and satisfying $\epsilon(u) = \zeta^{-1}$. However, it is well known that $L_p(1, \xi\epsilon\omega^2) \neq 0$ [Wa, §5.5]. Hence, $\gcd(b_\ell(X), F(X, \xi\omega^2)) = 1$.

Proof of Claim 1: Let ℓ be a prime dividing N that does not divide f_ξ. By Lemma 4.1.9, this is a necessary condition for the prime ℓ to ramify in L/F_∞. Consider the following exact sequence of Λ_ξ-modules,

$$0 \to \mathrm{Gal}(L/K_\ell) \to \mathrm{Gal}(L/F_\infty) \to \mathrm{Gal}(K_\ell/F_\infty) \to 0.$$

By Lemma 4.1.8, we know that

$$b_\ell(X) \cdot \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(K_\ell/F_\infty)) \subseteq \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L/F_\infty)).$$

Now, suppose $\ell' \neq \ell$ is another prime dividing N that does not divide f_ξ. Then we have the exact sequence

$$0 \to \mathrm{Gal}(K_\ell/(K_\ell \cap K_{\ell'})) \to \mathrm{Gal}(K_\ell/F_\infty) \to \mathrm{Gal}((K_\ell \cap K_{\ell'})/F_\infty) \to 0.$$

Since $\mathrm{Gal}(K_\ell/(K_\ell \cap K_{\ell'})) \cong \mathrm{Gal}(K_\ell K_{\ell'}/K_{\ell'})$ with the latter being a quotient of $\mathrm{Gal}(L/K_{\ell'})$, Lemma 4.1.8 tells us

$$b_{\ell'}(X) \cdot \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}((K_\ell \cap K_{\ell'})/F_\infty)) \subseteq \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(K_\ell/F_\infty)).$$

Letting L^{ur}/F_∞ denote the maximal unramified subextension of L/F_∞ and repeating the above argument, we get

$$\left(\prod_{\substack{\ell \mid N \\ \ell \nmid f_\xi}} b_\ell(X)\right) \cdot \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L^{ur}/F_\infty)) \subseteq \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L/F_\infty)).$$

Proof of Claim 2: By Proposition 4.1.4, we know that $\mathcal{B}$ is a faithful $\mathfrak{h}_S^*$-module. Therefore, by Proposition 4.2.2 (2) and (3), we have

$$\mathrm{Fitt}_{\mathfrak{h}_S^*/I_S^*}(\mathcal{B}/I_S^*\mathcal{B}) \;=\; 0.$$

Now, we know that $\mathfrak{h}_S^*/I_S^* \cong \Lambda_{\theta,\psi}[S^{-1}]/(A)$ as $\Lambda_{\theta,\psi}[S^{-1}]$-modules by Proposition 3.3.1, so applying Proposition 4.2.2 (3) once more, we get

$$\mathrm{Fitt}_{\Lambda_{\theta,\psi}[S^{-1}]}(\mathcal{B}/I_S^*\mathcal{B}) \bmod A \;=\; \mathrm{Fitt}_{\mathfrak{h}_S^*/I_S^*}(B/I_S^*B) \;=\; 0.$$

This in turn implies

$$\mathrm{Fitt}_{\Lambda_{\theta,\psi}[X^{-1}]}((\mathcal{B}/I_S^*\mathcal{B})^{\dagger}) \;\subseteq\; (\tilde{A}).$$

By the isomorphism of Proposition 4.1.7 we have

$$\mathrm{Fitt}_{\Lambda_{\theta,\psi}[X^{-1}]}(\mathrm{Gal}(L/F_\infty) \otimes_{\Lambda_\xi} \Lambda_{\theta,\psi}[X^{-1}]) \;\subseteq\; (\tilde{A}).$$

Now, we know that $\mathrm{Gal}(L/F_\infty)$ is a torsion Λ_ξ-module since we have an injection of Λ_ξ-modules $\mathrm{Gal}(L/F_\infty) \hookrightarrow \mathcal{B}/I_S^*\mathcal{B}$, and the latter is annihilated by $A \in \Lambda_\xi$. Suppose $\mathrm{Gal}(L/F_\infty)$ is pseudo-isomorphic to

$$\bigoplus_{i=1}^{t} \Lambda_\xi/(P_i(X)^{e_i}),$$

where the $P_i(X)$ are height one prime ideals of Λ_ξ. Tensoring with $\Lambda_{\theta,\psi}[X^{-1}]$ will kill any finite Λ_ξ-modules, so we have

$$\mathrm{Gal}(L/F_\infty) \otimes_{\Lambda_\xi} \Lambda_{\theta,\psi}[X^{-1}] \;\cong\; \bigoplus_{i=1}^{t} \Lambda_{\theta,\psi}[X^{-1}]/(P_i(X)^{e_i}). \tag{59}$$

Therefore,

$$\mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L/F_\infty)) \cdot \Lambda_{\theta,\psi}[X^{-1}] \;=\; \mathrm{Char}_{\Lambda_{\theta,\psi}[X^{-1}]}\big(\mathrm{Gal}(L/F_\infty) \otimes_{\Lambda_\xi} \Lambda_{\theta,\psi}[X^{-1}]\big)$$

$$= \left(\prod_{i=1}^{t} P_i(X)^{e_i}\right) \cdot \Lambda_{\theta,\psi}[X^{-1}] \;=\; \mathrm{Fitt}_{\Lambda_{\theta,\psi}[X^{-1}]}(\mathrm{Gal}(L/F_\infty) \otimes_{\Lambda_\xi} \Lambda_{\theta,\psi}[X^{-1}]) \;\subseteq\; (\tilde{A}),$$

with the last equality following from (59) and Proposition 4.2.2 (3). Hence

$$X^m \cdot \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L/F_\infty)) \;\subseteq\; (\tilde{A})$$

for some integer $m \geqslant 0$. $\square$

Corollary 4.2.4. *Let* $\tilde{A}_0 = \tilde{A}/X$ *if the pair* (θ_0, ψ_0) *is exceptional, with* $\tilde{A}_0 = \tilde{A}$ *otherwise. Then* $\mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L/F_\infty)) = (\tilde{A}_0)$.

Proof. By Theorem 4.2.3 and its proof, we have the following inclusion

$$\left(\prod_{\substack{\ell \mid N \\ \ell \nmid f_\theta f_\psi}} b_\ell(X)\right) \cdot (\tilde{A}) \subseteq \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L/F_\infty)). \tag{60}$$

Recall that $\mathcal{B}$ is the $\mathfrak{h}_S^*$-submodule of $\mathfrak{h}^*_{Q(\Lambda_{\theta,\psi})}$ generated by $\{b(\sigma) : \sigma \in G_Q\}$. Therefore, the fact that $\mathfrak{h}^*_{Q(\Lambda_{\theta,\psi})}$ is a free and finitely generated $Q(\Lambda_{\theta,\psi})$-module implies that $\mathcal{B}$ is a finitely generated $\mathfrak{h}_S^*$-module. Hence, we have a surjection

$$\left(\Lambda_{\theta,\psi}[S^{-1}]/(A)\right)^n \cong (\mathfrak{h}_S^*/I_S^*)^n \twoheadrightarrow \mathcal{B}/I_S^*\mathcal{B},$$

which implies

$$(\tilde{A})^n \subseteq \mathrm{Char}_{\Lambda_{\theta,\psi}[X^{-1}]}((\mathcal{B}/I_S^*\mathcal{B})^\dagger) \subseteq \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L/F_\infty)) \otimes_{\Lambda_\xi} \Lambda_{\theta,\psi}[X^{-1}].$$

From the injection

$$\mathrm{Gal}(L/F_\infty) \hookrightarrow (\mathcal{B}/I_S^*\mathcal{B})^\dagger,$$

we see that there are no elements of $\mathrm{Gal}(L/F_\infty)$ annihilated by X. Therefore, $X \nmid \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L/F_\infty))$ and the above inclusion implies

$$(\tilde{A})^n \subseteq \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L/F_\infty)). \tag{61}$$

In the remarks preceding the proof of Claim 1, it was shown that

$$\gcd(b_\ell(X), F(X, \xi\omega^2)) = 1.$$

Therefore, by (60) and (61) we have

$$(\tilde{A}) \subseteq \mathrm{Char}_{\Lambda_\xi}(\mathrm{Gal}(L/F_\infty)).$$

Combining this with the result of Ferrero-Greenberg and Claim 2 from the proof of Theorem 4.2.3 we obtain the desired result. □

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