Collaborative Team Evasion Against a Faster Pursuer

by

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Abstract

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In the past decade, the level of autonomy of unmanned vehicles has been rising rapidly from remote-controlled towards fully autonomous. Without human operators on board, teams of autonomous vehicles are the best candidates for high risk applications such as search and rescue after disasters and information gathering in hostile environments. For a team of autonomous vehicles to operate effectively in these scenarios, it must be able to respond promptly to environmental hazards and/or hostile entities. In this dissertation, a collaborative team evasion framework is proposed to maximize the survival time of a team of autonomous vehicles against a faster and more agile hostile agent. The proposed framework is based on an open-loop formulation of the single-pursuer-multiple-evader pursuit-evasion game that is conservative to the evaders and provides guarantees on team survival time in the worst-case scenario. An iterative open-loop approach that repeatedly solves the open-loop problem corresponding to the most current state of the game is developed to relax the conservatism of the open-loop formulation and enhance the survival time performance. Extensions to the framework make it possible to take into account the turning rate constraints of the evaders and uncertainties in the position of the pursuer. Numerical approximations are also proposed to reduced the required computation time. Through extensive simulations, the proposed framework is shown to produce reliable strategies for the evaders that result in significantly longer team survival time than previous work in the literature.

To my family, my wife, and my friends.

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Acknowledgments

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Chapter 1

Introduction

1.1 Background

Due to recent advancements in micro-processors, sensing technologies, and wireless communications, the level of autonomy of unmanned vehicles, such as unmanned aerial vehicles (UAVs), unmanned ground vehicles (UGVs), and unmanned underwater vehicles (UUVs), has been rising rapidly in the past decade from remote-controlled towards fully autonomous. While a single autonomous vehicle is capable of performing interesting tasks such as river tracking [1], interior mapping [2], and cleaning [3], a team of autonomous vehicles can take on even more challenging tasks through communication and collaboration. Developments in collaborative controls have enabled teams of autonomous vehicles to accomplish highly sophisticated tasks such as collaborative surveillance and tracking [4], perimeter and convoy patrol [5, 6], and collaborative grasping and transportation [7]. Without a human operator on board, unmanned vehicles are also the best candidates for high risk applications such as surveillance and control of forest fires [8], search and rescue after disasters [9], and information gathering in hostile environments [10, 11]. For a team of autonomous vehicles to operate effectively in these situations, the team must be able to promptly respond to environmental hazards and/or hostile entities to ensure the survival of its members. However, in some scenarios it might be impossible for all members of the team to survive. For example, when a team encounters a faster and more agile hostile agent aiming to capture every vehicle in the team; in these scenarios the autonomous vehicles need to collaboratively try to delay the capture of the whole team for as long as possible. The focus of this dissertation is to develop a collaborative team evasion framework for a team of autonomous agents to maximize the team survival time against a faster and more agile pursuer.

1.2 Related Works

Pursuit-evasion games, as a form of differential games, have always been an area of rich literature ever since they were proposed by Isaacs in [12]. One of the most famous example is

the Homicidal Chauffeur problem where a faster but less maneuverable pursuer tries to capture a slower but more agile evader in the least amount of time while the evader tries to delay being captured for as long as possible. Early works in this area focused on pursuit-evasion games played between a single pursuer and a single evader. Different versions of the game often varied in the capabilities of the players, the capture condition, and the environment in which the game is played. In [13], properties of the solutions to the Homicidal Chauffeur problem under different speed ratio between the pursuer and the evader and different capture radius were investigated and analyzed. In the Surveillance-Evasion problem proposed in [14], the distance-based capture condition is replaced with a cone-based capture condition. In the Lion-and-Man problem investigated in [15], the pursuer and the evader have equal speed and the game is played in a circular arena with point capture condition. A concise review of these important variants of the Homicidal Chauffeur problem that form the foundation of the field can be found in [16].

With the theoretical advancement achieved through the study of the single-pursuer-single-evader pursuit-evasion games, in recent years there has been an abundance of work on pursuit-evasion games with more than one pursuer and/or evaders. For example, in [17] and [18] the game is played between two pursuers and a single evader. In [19], the cooperative version of the Homicidal Chauffeur problem where multiple pursuers with turning rate constraints aim to capture an evader in an open environment is investigated and a collaborative strategy for the pursuers inspired by the hunting and foraging behavior of fish is proposed. In [20], the game is played between multiple pursuers and a single evader in a closed environment. A cooperative strategy for the pursuers which guarantees the capture of an evader with the same speed is derived in this work through the minimization of the safe-reachable set of evaders. In [21], the lion-and-man problem is augmented with multiple lions and a strategy which guarantees the capture of the evader in complex environments with obstacles using three pursuers is proposed. The approach is later on extended to the multiple-pursuer-multiple-evader scenario in [22].

The pursuit-evasion game where a faster pursuer aims to capture a team of slower evaders in minimum time is referred to as the successive pursuit game. In [23], one of the seminal work on this topic, the complete solution for the 2-evader case was derived. A concise review of the result is provided in Appendix B. While the solution to the 2-evader case provides important insight for the successive pursuit game, the techniques involved are not all generalizable to teams with more than two evaders due to the curse of dimensionality. In [24], the formulation of the successive pursuit game is modified so that the pursuer must declare a specific capture sequence and later commit to it. This modified formulation, referred to as the fixed-sequence formulation, puts the pursuer in a disadvantage and simplifies the solution for the evaders. Similar modification are also investigated in [25] and a numerical method for solving the optimal trajectories of the evaders under a specific capture sequence is proposed in [26].

The fixed-sequence formulation of the successive pursuit problem is closely related to the traveling salesman problem where a salesman has to find the shortest route to visit a

set of cities. A concise review on the development of the traveling salesman problem can be found in [27]. In fact, in the special case where the evaders in the successive pursuit problem are all stationary, the fixed-sequence formulation of the successive pursuit problem is exactly the traveling salesman problem: the pursuer takes the role of the salesman and the evaders takes the role of the cities. In [28], the fixed-sequence formulation is referred to as the Moving-target Traveling Salesman Problem and the one-dimensional special case of the game is solved. It has been shown in [29] that even the one-dimensional variant of the game is NP-hard. The special case of the game where all evaders are moving in the same direction with the same speed in a 2 dimensional space is referred to as the Kinetic Variant of Traveling Salesman Problem. This variant is investigated in [30] and it has been shown that even with 2 evaders, there exists no polynomial time algorithm that can approximate the game with a factor better than 2. In [31], [32] and [33], another variant of the traveling salesman problem is proposed where each city can pick its own location from a given set after knowing the route selected by the salesman; this variant is referred to as the Generalized Traveling Salesman Problem. The fixed-sequence formulation of the successive pursuit game can be interpreted as an extended version of the Generalized Traveling Salesman Problem where the sets of the evaders vary with time. In [34], it is proposed that the optimal headings of the evaders for a specific capture sequence in the fixed-sequence formulation can be derived by solving a one-dimensional root searching problem regardless of the number of evaders in the team. The root searching problem is formulated by exploiting the first-order optimality condition of the optimization problem proposed in [26]. The optimal headings of the evaders for given a specific capture sequence can hence be solved very efficiently and the algorithm scales nicely with the number of evaders. However, to determine the optimal capture sequence for the pursuer, brute-force approaches are used due to the combinatorial nature of the problem and the complexities introduced by moving evaders. Very recently in [35], it is suggested that the solution of the fixed-sequence formulation can be used in an iterative fashion to approximate the optimal behavior for the original successive pursuit problem. However, there is no guarantee on the accuracy of the approximation for either the pursuer or the evaders.

There are also many variants of the fixed-sequence successive pursuit game in the literature. In [36], [37], and [38], a constraint on the turning rate of the pursuer is introduced to the formulation and its effect on the optimal trajectory of the pursuer is investigated. In [39], a variant referred to as the Detection Evasion Problem has the evaders maximize the minimal distance between the pursuer and a specific evader instead of the team survival time. A similar variant where the objective of the evaders is to avoid detection instead of capture can also be found in [40]. This variant is also considered in 3-dimensional space in [41] and [42]. In [43] and [44], the formulation is modified so that the pursuer aims to minimize the distance to a specific evader while pursuing others.

1.3 Motivation

The majority of work in the literature on the single-pursuer-multiple-evader pursuit-evasion problem is pursuer-centric and aims to provide an applicable strategy for the pursuer rather than the evaders. Furthermore, in the works that are not pursuer-centric, such as [23] and [39], most the derived strategies for the evaders are only applicable for teams with 2 or 3 evaders; strategies that are applicable for larger teams are based on heuristics and cannot provide guarantees on the survival time of the team. The goal of this dissertation is to develop an evader-centric framework of the successive pursuit problem to fill in the gaps in the literature on this topic, and to provide applicable strategies with guarantees for a team of evaders to maximize their survival time when facing a faster pursuer.

1.4 Contributions

The main contributions of this dissertation are listed as follows:

- The open-loop formulation of the team evasion problem that is conservative to the evaders. The formulation provides guaranteed team survival time and is the core of the collaborative team evasion framework.

- The derivation of the properties of the open-loop optimal solutions for the pursuer and the evaders that enable the development and implementation of the algorithm that solves for the open-loop optimal joint heading of the evaders to maximize the worst-case survival time of the team.

- The formulation of the iterative open-loop approach for collaborative evasion that relaxes the conservatism of the open-loop approach and improves its effectiveness.

- The derivation of the gradient of the team survival time with respect to the joint heading of the evaders under a specific capture sequence. This leads to various approximations to the iterative open-loop approach that can achieve similar team survival time with much less computation time.

- The extensions to the collaborative team evasion framework that enable the framework to handle uncertainties in the position of the pursuer, constraints on the turning rates of the evaders, and accumulative survival time as the objective function of the evaders. These capabilities greatly enhance the applicability of the framework in more realistic team evasion scenarios.

- The evaluation of the effectiveness of the proposed approaches through extensive simulation.

1.5 Dissertation Outline

The outline of each chapter in this dissertation is as follows.

Chapter 1 of this dissertation provides a concise review of pursuit-evasion games with a focus on the single-pursuer-multiple-evader variants of the game. The lack of evader-centric work in the literature on this topic is identified as the main motivation of the work in this dissertation.

In Chapter 2, the closed-loop formulation of the team evasion game is defined. Open-loop formulations of the game with different levels of action resolutions are also proposed in this chapter and their implications discussed.

Chapter 3 is devoted to the open-loop formulation of the team evasion problem conservative to the evaders, which is the core of the collaborative team evasion framework. Important properties of the optimal solution to this open-loop formulation are derived. These properties are then used to convert the infinite dimensional sup-inf problem to a finite dimensional one that can be solved with standard nonlinear optimization techniques. The resulting behavior and the team survival time of the proposed open-loop approach are compared to that of the pursuer-centric approaches in the literature.

The open-loop approach proposed in the previous chapter is expanded to the iterative open-loop approach in Chapter 4. The formulation and implementation of this approach is detailed in this chapter. Various approximations to the iterative open-loop approach are also introduced in this chapter for the purpose of decreasing the computational requirement of the proposed framework. The survival time and computation time performance of the proposed approach and the approximations are evaluated through extensive simulation.

In Chapter 5, several extensions to the collaborative team evasion framework are proposed. These extensions enable the framework to handle uncertainties in the position of the pursuer, turning rate constraints on the evaders, and alternative objective function. Details of the derivations and implementations of these extensions are presented and their effectiveness is demonstrated. These extensions greatly improve the applicability of the framework in more realistic team evasion scenarios.

Chapter 6 concludes the dissertation and provides possible directions for future research.

Chapter 2

The Team Evasion Game

2.1   Introduction

The *team evasion game* is a pursuit-evasion game between one faster pursuer and N evaders which takes place in a two dimensional space $\mathbb{R}^2$. The goal of the pursuer is to capture all the evaders as soon as possible. The evaders, as a team, aim to delay the capture of the whole team for as long as possible. An evader is considered captured if at some point, the pursuer coincides with the evader. Every agent has the ability to control its heading and speed at any given time subjected only to a constraint on its maximum speed. Also, every agent is aware of the position and trajectory of all other agents at all times.

In Section 2.2, the notations and terms that will be used throughout this dissertation are defined. The closed-loop formulation of the team evasion game is defined Section 2.3 and various open-loop formulations of the game are defined in Section 2.4. Section 2.5 concludes this chapter.

2.2   Notations and Definitions

In this section, the notational convention and basic terms used throughout this work are defined and explained. Variables are subscripted according to the agent that it refers to: properties of the pursuer are subscripted with the letter p and variables describing the properties of a single evader are subscripted with the letter e. When there are multiple evaders in a team, they will be identified by numbers. For example, when there are one pursuer and N evaders, the position of the pursuer is denoted by x_p and the positions of the N evaders will be denoted by $x_1, \ldots, x_N$. A joint variable containing several different variables is boldfaced and this convention also applies to the subscripts. The boldfaced $\mathbf{e}$ is used to denoted the set of indexes of all N evaders; more specifically

$$\mathbf{e} = \{1, \ldots, N\}.$$

The joint position of N evaders is denoted by

$$\mathbf{x_e} = (x_1, \ldots, x_N)$$

and the maximum speeds of all the evaders are denoted by

$$\mathbf{v_e} = (v_1, \ldots, v_N).$$

The maximum speed of the pursuer is denoted by v_p, and without lost of generality it is set to 1. For the pursuer to be faster than all evaders, the following conditions are imposed on the maximum speed of the evaders:

$$0 \le v_i < 1, \ i = 1, \ldots, N. \tag{2.1}$$

The trajectory of an agent is denoted by $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^2$, which is a function that maps time to the position of the agent at the given time; $x(t) \in \mathbb{R}^2$ is the position of an agent at time t. Following the conventions of the subscripts, the trajectory of the pursuer is denoted by $x_p(\cdot)$ and the trajectory of evader i is denoted by $x_i(\cdot)$. The joint trajectory of the pursuer and the evaders is denoted by $\mathbf{x}(\cdot) = (x_p(\cdot), \mathbf{x_e}(\cdot))$ where $\mathbf{x_e}(\cdot) = (x_1(\cdot), \ldots, x_N(\cdot))$ is the joint trajectory of all evaders.

The heading and speed of an agent at time t is represented by a 2-dimensional vector $u(t) \in \mathbb{R}^2$ referred to as the *action vector*, or simply the *action*, of the agent at time t. The direction of the vector $u(t)$ represents the heading of the agent at time t, and the magnitude of $u(t)$ represents the ratio of the velocity of the agent at time t to the maximum velocity of the agent. For example, $\|u_i(t)\| = 0.6$ indicates that agent i is traveling at 60 percent of its maximum speed at time t. The time trajectory of the action vector of an agent, denoted by $u(\cdot) : \mathbb{R}^+ \to \mathbb{R}^2$, is defined as a function that maps a specific time to an action vector. This action trajectory is referred to as the *control*, or *control function*, of the agent. In a similar fashion to the trajectories of agents, the control for the pursuer is denoted by $u_p(\cdot)$ and the joint control of the evaders is denoted by

$$\mathbf{u_e}(\cdot) = (u_1(\cdot), \ldots, u_N(\cdot)).$$

The set of admissible controls for an agent that has full control of its heading and cannot exceed a finite maximum speed is denoted by

$$\mathcal{U} = \{u(\cdot) \mid \|u(t)\| \le 1, t \in [0, \infty)\}. \tag{2.2}$$

The set of admissible joint control for N evaders is defined as

$$\mathcal{U}^N = \{(u_1(\cdot), \ldots, u_N(\cdot)) | u_i(\cdot) \in \mathcal{U} \text{ for } i = 1, \ldots, N\}. \tag{2.3}$$

Note that since the magnitude of the action vector represents the ratio of the current speed to the maximum speed instead of the actual speed, agents with different speeds share the same admissible control set.

In the team evasion game, the dynamics of the agents are decoupled from each other and can be specified by the following differential equations:

$$\dot{x}_i(t) = v_i u_i(t), \ i = p, 1, \ldots, N. \tag{2.4}$$

For convenience, the following notation is used to denote the resulting trajectory of a given control.

Definition 2.1 Resulting trajectory of a given control
For an agent starting at x^0 with maximum speed v, the resulting trajectory by using the control $u(\cdot)$ is denoted by

$$x(\cdot) = traj(x^0, v, u(\cdot))$$

and the trajectory satisfies

$$x(t) = x^0 + \int_0^t vu(\tau)d\tau, \ for \ t \geq 0 \tag{2.5}$$

The same notation is used for the function that maps joint initial position, joint maximum speed, and joint control to the resulting joint trajectory. For example, given $\mathbf{x_e^0}$ as the joint initial position, $\mathbf{v_e}$ as the joint maximum speed, and $\mathbf{u_e}(\cdot)$ as the joint control for N evaders, the resulting joint trajectory of the team is

$$\mathbf{x_e}(\cdot) = traj(\mathbf{x_e^0}, \mathbf{v_e}, \mathbf{u_e}(\cdot))$$

which satisfies

$$x_i(t) = traj(x_i^0, v_i, u_i(\cdot)), i = 1, \ldots, N.$$

An evader is considered to be captured when its distance to the pursuer is zero. This is referred to as the point capture condition. Given the initial positions, maximum speeds, and controls of a pursuer and an evader, the survival time, or capture time of the evader is defined as follows.

Definition 2.2 Evader survival time
Given x_p^0 and x_e^0 as the initial position of the pursuer and the evader, v_p and v_e as their maximum speeds, and $u_p(\cdot)$ and $u_e(\cdot)$ as their controls, the resulting survival time of the evader is defined as

$$\Gamma(x_p^0, x_e^0, u_p(\cdot), u_e(\cdot)) = \inf\{t | x_p(t) = x_e(t)\},$$

where

$$x_p(\cdot) = traj(x_p^0, v_p, u_p(\cdot)) \ and \ x_e(\cdot) = traj(x_e^0, v_e, u_e(\cdot)).$$

Note that the maximum speed of the agents do affect the resulting survival time, since they are constant properties of the agents they are omitted from the arguments of Γ for notational simplicity. It is possible that of the given initial conditions and controls, there

there exist no such t that satisfies the condition $x_p(t) = x_e(t)$; in this case the survival time of the evader is infinity according to the definition of infimum of an empty set. Defining $\tau = \Gamma(x_p^0, x_e^0, u_p(\cdot), u_e(\cdot))$ as the survival time of the evader, then Definition 2.2 implies that when τ is finite, the following is true due to the point capture condition.

$$x_p(\tau) = x_e(\tau). \tag{2.6}$$

The objective function of the team evasion game is the survival time of the evader that survive the longest. In other words, it is the capture time of the evader that is captured last. This time is referred to as the *team survival time.*

Definition 2.3 Team survival time
Given x_p^0 and $\mathbf{x_e^0}$ as the initial position of a pursuer and N evaders, v_p and $\mathbf{v_e}$ as their maximum speeds, and $u_p(\cdot)$ and $\mathbf{u_e}(\cdot) = (u_1(\cdot), \ldots, u_N(\cdot))$ as their controls, the team survival time is defined as

$$\Gamma(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)) = \sup_{i \in \{1, \ldots, N\}} \Gamma(x_p^0, x_i^0, u_p(\cdot), u_i(\cdot)). \tag{2.7}$$

It is worth emphasizing again that the maximum speeds of the agents do affect the team survival time but they are omitted from the arguments of Γ for notational simplicity.

2.3 Closed-loop Formulation of the Team Evasion Game

With the necessary notation and basic elements defined in the previous section, the team evasion game and its closed-loop formulation are presented in detail in this section.

The team evasion game is played between two players: the pursuer and the team of evaders. Note that although the evaders are separated agents, the team acts as a single player in the game. The goal of the pursuer is to minimize the team survival time Γ defined in Definition 2.3 by capturing all evaders as soon as possible; the goal of the team of evaders is to delay the capture of the whole team for as long as possible. There are various formulations of the game that differ in what information are available to each player, in what order the players play, and what kinds of actions are admissible. The closed-loop formulation, which is the most general one, is presented here.

In the closed-loop team evasion game, the pursuer and the team of evaders have full access to the action history of their opponents and are allowed to react to each other instantaneously. Due to the simple motion dynamics defined in Eq. (2.4), the action history of an agent can be understood as the past trajectory of the agent. At every instant, the pursuer determines its action based on the current layout and the past trajectories of the evaders. The evaders also

select their joint actions based on the current layout and the past trajectory of the pursuer. A game with this specific information pattern can be formulated as a *feedback strategy game* where the first player of the game selects a *policy* and the second player of the game selects a control in response to the policy.

A policy defines how a player reacts to different controls of its opponent. More specifically, a policy maps the control of the opponent to a control of the player. In the context of the closed-loop team evasion game, a policy of the pursuer maps the joint control of evaders to a control of the pursuer and is denoted by

$$\pi_p(\cdot) : \mathcal{U}^N \to \mathcal{U}.$$

A policy of the evaders maps the control of the pursuer to the joint control of evaders and is denoted by

$$\pi_{\mathbf{e}}(\cdot) : \mathcal{U} \to \mathcal{U}^N.$$

For example, given that the pursuer adopts a specific control $u_p(\cdot)$, the resulting joint control of the evaders using policy $\pi_{\mathbf{e}}(\cdot)$ will be $\mathbf{u_e}(\cdot) = \pi_{\mathbf{e}}(u_p(\cdot))$.

Note that by definition, a control contains all actions of an agent from the beginning to the end of the game. Hence, a policy defined as above does not automatically respect the *causality rule*, which states that one cannot obtain information that is available only in the future. More specifically in the context of the team evasion game, a player can only determine its action based on the previous actions of its opponent, not the future ones. To enforce the causality rule in the closed-loop formulation of the team evasion game, the policies that are admissible for the first player has to be limited to the class of *non-anticipating policies* defined as follows.

Definition 2.4 Non-anticipating policy
A policy $\pi(\cdot)$ is non-anticipating if for any given controls $u(\cdot)$ and $u'(\cdot)$ such that $u(\tau) = u'(\tau)$ for $\tau \in [0, t]$, the resulting controls of the policy, $u_\pi(\cdot) = \pi(u(\cdot))$ and $u'_\pi(\cdot) = \pi(u(\cdot))$, satisfy the following condition:

$$u_\pi(\tau) = u'_\pi(\tau) \text{ for all } \tau \in [0, t].$$

In other words, given two controls that are the same until time τ, the resulting controls from a non-anticipating policy have to also be the same until time τ.

Denoting the set of all non-anticipating policy by Π_{NA}, the closed-loop formulation of the team evasion game is defined as follows:

Definition 2.5 Closed-loop team evasion game
Given x_p^0 and $\mathbf{x_e^0}$ as the initial positions of the pursuer and the evaders, the closed-loop team evasion game is defined as

$$\sup_{\pi_{\mathbf{e}}(\cdot) \in \Pi_{\mathrm{NA}}} \inf_{u_p(\cdot) \in \mathcal{U}} \Gamma(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \pi_{\mathbf{e}}(u_p(\cdot))). \tag{2.8}$$

In this formulation, the team of evaders plays first by selecting a non-anticipating policy $\pi_{\mathbf{e}}(\cdot) \in \Pi_{\mathrm{NA}}$, and the pursuer plays second by selecting an admissible control $u_p(\cdot) \in \mathcal{U}$. The objective function of the game is the team survival time $\Gamma(x_p^0, \mathbf{x_e}, u_p(\cdot), \pi_{\mathbf{e}}(u_p))$, where $u_p(\cdot)$ is the selected control of the pursuer and $\pi_e(u_p(\cdot))$ is the joint control resulting from the policy of the evaders. Note that since the $\inf_{u_p(\cdot) \in \mathcal{U}}$ lies inside the $\sup_{\pi_{\mathbf{e}}(\cdot) \in \Pi_{\mathrm{NA}}}$, the pursuer determines its control knowing the exact policy selected by the evaders. Hence, the pursuer can always select the control that will result in the minimum possible team survival time in response to the policy selected by the evaders. On the other hand, the evaders have to select a policy without knowing the exact control of the pursuer since the evaders play first in this formulation.

However, being the first player in this closed-loop formulation does not prevent the team of evaders from reacting to different controls of the pursuer differently. By encoding the policy with different joint controls in response to different controls of the pursuer, the team can react differently when different controls are selected by the pursuer. For this reason, the formulation defined in Definition 2.5 is referred to as the *closed-loop formulation* of the team evasion game.

In general, the player who plays later in a game has an advantage. The sup-inf structure of Eq. (2.8) implies that the pursuer has an advantage over the team of evaders. The resulting team survival time is hence bias towards the minimizing player and is called the *lower value* of the game. The game can also be formulated in an inf-sup structure as

$$\inf_{\pi_p(\cdot) \in \Pi_{\mathrm{NA}}} \sup_{\mathbf{u_e}(\cdot) \in \mathcal{U}^N} \Gamma(x_p^0, \mathbf{x_e^0}, \pi_p(\mathbf{u_e}(\cdot)), \mathbf{u_e}(\cdot)). \tag{2.9}$$

where the maximizing player has an advantage. Under this structure, the pursuer plays first by picking a policy and the evaders play later by picking a joint control for the team. The resulting value of this formulation is the *upper value* of the game.

It has been shown in [12] that for a differential game that satisfies Isaac's condition and has continuous value, the lower and upper values coincide. For the closed-loop team evasion game, the Isaac's condition is satisfied due to the decoupled dynamics of the players. The value of the game, defined as the team survival time is indeed continuous with respect to the initial layout. Hence, formulations in Eq. (2.8) and Eq. (2.9) are equivalent. The sup-inf structure in Eq. (2.8) will be used as the standard closed-loop formulation of the team evasion game in this work.

Like most other differential games, the main challenges of the closed-loop team evasion problem come from the exceedingly large dimensions of the admissible sets of the players. The admissible control set $\mathcal{U}$ for the pursuer in Eq. (2.8) contains all possible controls for the pursuer and is infinite dimensional. The set of all non-anticipating policies Π_{NA} is even larger since an element of the set, which is a policy, has to encode responses to all possible admissible controls.

2.4   Open-loop Formulations of the Team Evasion Game

For a team with 2 evaders, the closed-loop team evasion problem can be solved by dynamic programming as shown in [23]. A concise review of the results is presented in Appendix B. While the solution derived through dynamic programming for the 2-evader case provides important insights to the team evasion problem, the method suffers severely from the *curse of dimensionality*. The complexity of the problem depends on the number of possible states which grows exponentially with the number of evaders. For a team of 3 evaders, the state space is 6 dimensional and is intractable for most of the numerical solvers currently available. To derive a computationally efficient control algorithm for the evaders in the team evasion game, modifications and simplifying assumptions have to be made. The open-loop framework for pursuit-evasion games is recently proposed in [45] and is designed to bypass the curse of dimensionality by simplifying the structure of closed-loop games. The application of the open-loop framework in reach-avoid games can be found in [46]. In this section, various open-loop formulations of the team evasion game are proposed and discussed.

The main difference between the open-loop formulations and the closed-loop formulations of the team evasion game is in the decision variable of the players. Formulations of the team evasion game can be categorized by the decision variables of the players and the order in which the players play. For example, in closed-loop formulation of the team evasion game proposed in Definition 2.5, the team of evaders plays first by picking a policy and the pursuer plays second by picking a control. Hence the closed-loop formulation can be categorized as a policy-control formulation that is conservative to the evaders. Four different types of open-loop formulations for the team evasion game are presented in the following sections.

2.4.1   Control-Control Formulations

In a control-control formulation of the team evasion game, the decision variables for both the first and the second player are control functions. A control-control formulation is open-loop in nature in that the first player has to select its control for the game without knowing the control of the second player. This makes it impossible for the first player to react to the second player. The control of the first player is available to the second player when the second player makes its decision. This allows the second player to better exploit the actions of the

first player and hence put the first player in a disadvantage. Hence, the formulation is said to be conservative to the first player.

2.4.1.1   Control-Control Formulation Conservative to the Pursuer

By setting the pursuer as the first player of the game, the control-control formulation for the team evasion game conservative to the pursuer is defined as

$$\inf_{u_p(\cdot)\in\mathcal{U}} \sup_{\mathbf{u_e}(\cdot)\in\mathcal{U}^N} \Gamma(x_p^0, \mathbf{x}_e^0, u_p(\cdot), \mathbf{u_e}(\cdot)). \tag{2.10}$$

Note that the $\sup_{\mathbf{u_e}(\cdot)\in\mathcal{U}^N}$ for the team of evaders is inside the $\inf_{u_p(\cdot)\in\mathcal{U}}$ for the pursuer. This implies that the evaders know the exact control selected by the pursuer when they select their joint control. In this case, even though the evaders are slower than the pursuer, with the knowledge of the control of the pursuer they can avoid capture indefinitely in $\mathbb{R}^2$. In other words, the pursuer will not be able to capture the evaders at all if its control is disclosed to the evaders beforehand. Hence, the resulting optimal team survival time of this formulation of the game is always infinite except for the special case where the pursuer and all evaders start at the same position.

It is worth noting that if a different capture condition is used, Eq. (2.11) can take on a finite value for initial conditions that are not the special case mentioned above. For example, if the capture condition is relaxed so that a capture can happen as long as the pursuer is within a certain non-zero distance to the evader, then there exist some initial conditions where the pursuer can capture the evaders in finite time in this open-loop formulation. However, since the capture condition of the team evasion game is defined to be point capture, this open-loop formulation is too conservative to the pursuer and does not provide any applicable strategies for the pursuer or the evaders.

2.4.1.2   Control-Control Formulation Conservative to the Evaders

By setting the team of evaders as the first player of the game, the control-control formulation of the team evasion game conservative to the evaders is defined as:

$$\sup_{\mathbf{u_e}(\cdot)\in\mathcal{U}^N} \inf_{u_p(\cdot)\in\mathcal{U}} \Gamma(x_p^0, \mathbf{x_e}^0, u_p(\cdot), \mathbf{u_e}(\cdot)). \tag{2.11}$$

The team of evaders select their joint control as the first player before knowing the control of the pursuer. The pursuer, as the second player, solves the optimal control problem

$$\inf_{u_p(\cdot)\in\mathcal{U}} \Gamma(x_p^0, \mathbf{x_e}^0, u_p(\cdot), \mathbf{u_e}(\cdot)) \tag{2.12}$$

to minimize the team survival time given the initial condition and the joint control of the evaders $\mathbf{u_e}(\cdot)$.

This formulation encodes a worst-case mentality for the evaders since they assume that the pursuer will always act optimally against their joint control by applying the optimizer of the optimal control problem defined in Eq. (2.12). Under this worst-case mentality, the evaders assume that they can only achieve the minimum possible survival time given the selected joint control. Planning for the worst case is the key characteristic of the open-loop formulation. By considering the worst case, the resulting team survival time in Eq. (2.11) is a guaranteed lower-bound of the team survival time for the evaders; this implies that by following the optimizer of Eq. (2.11), the evaders can achieve at least the guaranteed team survival time no matter what the pursuer does. Due to these characteristics, the open-loop formulation conservative to the evaders forms the foundation of the open-loop approach to collaborative team evasion. However, solving Eq. (2.11) is a challenging task in that even with a given joint control of the evaders, the optimal control problem faced by the pursuer in Eq. (2.12) is an infinity dimensional optimization problem. The sequence-control formulation of the team evasion game, which will be presented in the next section, will provide more insights to the problem faced by the pursuer and the evaders.

2.4.2 Sequence-Control Formulations

Recall that the control-control formulation conservative for the pursuer in Eq. (2.10) was overly conservative to the pursuer in that the evaders are always able to survival indefinitely. In this section, the sequence-control formulation of the team evasion game is proposed as an alternative open-loop formulation which relaxes the conservatism towards the pursuer.

The set of admissible capture sequences for N evaders is defined as follows.

Definition 2.6 Set of admissible capture sequences for N evaders
The set of all admissible capture sequence for N evaders is defined as:

$$\mathcal{S}_N = \{(s_1, \ldots, s_N)|s_i \in \mathbb{N}^+, s_i \neq s_j \text{ for } i \neq j, \sup_i s_i = N\}. \tag{2.13}$$

The set $\mathcal{S}_N$ contains all possible permutations of $\{1, \ldots, N\}$. Each element of the set is a capture sequence that contains the indexes of the N evaders in the order of the captures and is denoted by $\mathbf{s} = (s_1, \ldots, s_N) \in \mathcal{S}_N$. For example, a capture sequence $\mathbf{s} = (3, 1, 2)$ indicates that evader 3 is captured first, followed by evader 1 and then evader 2. In this case $s_1 = 3$, $s_2 = 1$, and $s_3 = 2$.

The minimum team survival time given a specific joint control of the evaders under a specific capture sequence is defined as follows:

Definition 2.7   Minimum Team Survival Time Given Capture Sequence

Given x_p^0 and $\mathbf{x_e^0}$ as the initial position of the pursuer and the evader, v_p and $\mathbf{v_e}$ as their maximum speeds, and $\mathbf{u_e}$ as the joint control for the evaders, the minimum possible time it takes the pursuer to capture all evaders according to a specific sequence $\mathbf{s} = (s_1, \ldots, s_N)$ is:

$$\Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)) = \inf_{u_p(\cdot) \in \mathcal{U}} \Gamma(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)) \tag{2.14a}$$

$$s.t. \quad \Gamma(x_p^0, x_{s_i}^0, u_p(\cdot), u_{s_i}(\cdot)) \leq \Gamma(x_p^0, x_{s_j}^0, u_p(\cdot), u_{s_j}(\cdot)) \tag{2.14b}$$

$$\text{for all } i < j, \text{ for } j = 1, \ldots, N$$

The constraints listed in Eq. (2.14b) state that the capture time of the i-th evader in the capture sequence has to be smaller than or equal to the capture time of the j-th evader in the capture sequence if j is larger than i. These constraints ensure that only the pursuer controls that result in the specified capture sequence are considered admissible. Note that the team survival time under a specific sequence, denoted by $\Gamma_{\mathbf{s}}^{\star}$, is no longer a function of the pursuer's control $u_p(\cdot)$ in that it is assumed that the pursuer picks the optimal control against the joint control of the evaders to achieve the minimum team survival time under the capture sequence. Note that this minimum team survival time under a capture sequence is closely related to, but not the same as, the minimum team survival defined previously in Eq. (2.12). The difference between the two is that in Eq. (2.12), the pursuer can capture the evaders in any order, but in Eq. (2.14) the pursuer has to capture the evaders according to the order specified by $\mathbf{s}$.

2.4.2.1   Sequence-Control Formulation Conservative to the Pursuer

Given x_p^0 and $\mathbf{x_e^0}$ as the initial positions of the pursuer and the evaders, the sequence-control formulation of team evasion game conservative to the pursuer is

$$\inf_{\mathbf{s} \in \mathcal{S}^N} \sup_{\mathbf{u_e}(\cdot) \in \mathcal{U}^N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)). \tag{2.15}$$

As indicated by the inf-sup structure, in this formulation the pursuer plays first by selecting a capture sequence; the evaders then select their joint control to maximize the team survival time with the knowledge of the sequence selected by the pursuer. Although this formulation is still conservative towards the pursuer, it is less so than the control-control formulation conservative to the pursuer defined in Eq. (2.10). In this formulation, after the capture sequence and the joint control of the evaders have been selected, the pursuer is allowed to then pick the optimal control which captures the evaders according to the selected capture sequence in minimum possible time. However, note that the resulting team survival time does not serve as an upper-bound of the value of the closed-loop game defined in Definition 2.5. This is because that the resulting optimal pursuer control from Eq. (2.14) relies on knowing the future joint control of the evaders, which is not available in the closed-loop setting due to causality. In other words, the team capture time in Eq. (2.15) might only be achievable by a

pursuer that utilizes an anticipating policy. The evaders, on the other hand, have to solve the optimal control problem

$$\sup_{\mathbf{u_e}(\cdot)\in\mathcal{U}^N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)) \tag{2.16}$$

to maximize the minimum possible capture time given the specified capture sequence $\mathbf{s}$.

The sequence-control formulation of the team evasion game conservative to the pursuer is equivalent to the fixed-sequence formulation of the successive pursuit game addressed in [25], [26], and [34]. A concise review for these works is provided in Appendix B.

2.4.2.2 Sequence-Control Formulation Conservative to the Evaders

Given x_p^0 and $\mathbf{x_e^0}$ as the initial positions of the pursuer and the evaders, the sequence-control team evasion game conservative to the evaders is defined as

$$\sup_{\mathbf{u_e}(\cdot)\in\mathcal{U}^N} \inf_{\mathbf{s}\in\mathcal{S}^N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)). \tag{2.17}$$

In this formulation, the evaders play first by choosing their joint control. The pursuer, as the second player of the game, then selects a capture sequence and the corresponding optimal control for that sequence with the knowledge of the joint control of evaders.

The following remark addresses the relationship between the sequence-control formulation conservative to the evaders and the control-control formulation conservative to the evaders defined in Eq. (2.11).

Remark 2.1
Given x_p^0 and $\mathbf{x_e^0}$ as the initial position of the pursuer and N evaders, v_p and $\mathbf{v_e}$ as their maximum speeds with $v_p > v_i$ for $i = 1, \ldots, N$, and $\mathbf{u_e}(\cdot) \in \mathcal{U}^N$ as the joint control for the evaders,

$$\inf_{u_p(\cdot)\in\mathcal{U}} \Gamma(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)) = \inf_{\mathbf{s}\in\mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)), \tag{2.18}$$

and hence

$$\sup_{\mathbf{u_e}(\cdot)\in\mathcal{U}^N} \inf_{u_p(\cdot)\in\mathcal{U}} \Gamma(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)) \equiv \sup_{\mathbf{u_e}(\cdot)\in\mathcal{U}^N} \inf_{\mathbf{s}\in\mathcal{S}^N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)). \tag{2.19}$$

The control-control formulation, and sequence-control formulation of the team evasion game are equivalent when formulated conservative to the evaders. The sequence-control formulation will be used as the main open-loop formulation for the evaders in that it provides more insights to the structure of the optimal control problem the evaders have to solve. The solutions and implications of this formulation will be discussed in detail in Chapter 3.

2.5 Conclusion

The closed-loop formulation of the team evasion game is proposed in this chapter. While the closed-loop formulation best captures the essence of the team evasion game, it is extremely hard to solve even for a team with 3 evaders. Two types of open-loop formulations for the game are also proposed in this chapter: the control-control formulation where the decision variables of the pursuer and the evaders are control functions and the sequence-control formulation where the pursuer selects a capture sequence instead of a control function. Each type of formulation can be formulated either to the pursuer or to the evaders. The control-control formulation conservative to the pursuer serves no particular use to the pursuer or the evaders in that the evaders can always survive indefinitely in this formulation. The sequence-control formulation conservative to the pursuer is equivalent to the fixed-sequence formulation of the successive pursuit game in the literature and is pursuer-centric. Finally, the control-control formulation and the sequence-control formulation conservative to the evaders are shown to be equivalent; the sequence-control formulation will serve as the core of the collaborative team evasion framework and will be the focus of the next chapter.

Chapter 3

Open-loop Approach for Collaborative Team Evasion

3.1 Introduction

A lot of research has been done on the sequence-control formulation of the team evasion game conservative to the pursuer proposed in Chapter 2. The solution, while providing the pursuer with a method to select the optimal capture sequence in a conservative way, does not provide the evaders with an applicable strategy in the team evasion scenarios. The focus of this chapter is to derive the optimal solution to the sequence-control formulation conservative to the evaders, which will provide the evaders with a collaborative strategy to maximize the team survival time when facing a faster pursuer.

This chapter is organized as follows. In Section 3.2, the analytical properties of the optimal solution to the open-loop team evasion problem are derived. Section 3.3 describes the algorithm used to compute the optimal solution to the team evasion problem. The resulting behavior and performance of the proposed open-loop approach for collaborative team evasion are presented and discussed in Section 3.4. Finally, Section 3.5 concludes this chapter.

3.2 Properties of the Open-loop Optimal Controls

The proposed open-loop collaborative team evasion framework relies on solving the sequence-control team evasion problem conservative to the evaders defined in the previous chapter as

$$\sup_{\mathbf{u_e}(\cdot) \in \mathcal{U}^N} \inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma^\star_{\mathbf{s}}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)). \tag{3.1}$$

Note that the admissible set for the evaders is infinite dimensional since it contains all possible joint controls for the team of evaders. Although not directly solvable in this form, the

optimization problem can be converted into a form that can be solved much more efficiently through exploitation of some properties of the optimizer of the definitions.

Definition 3.1 Set of admissible constant-heading-maximum-speed controls
The set of all controls with constant heading and maximum speed is defined as

$$\mathcal{U}_\theta = \{u(\cdot) \mid u(t) = (\cos\theta, \sin\theta) \text{ for } t \geq 0\}, \tag{3.2}$$

where θ is referred to as the heading of the agent.

Definition 3.2 Set of admissible constant-heading-maximum-speed joint controls
The set of all joint controls for N evaders with constant heading and maximum speed is defined as

$$\mathcal{U}_\Theta^N = \{(u_1(\cdot), \ldots, u_N(\cdot)) \mid u_i(\cdot) \in \mathcal{U}_{\theta_i} \text{ for } i = 1, \ldots, N\}, \tag{3.3}$$

where $\Theta = (\theta_1, \ldots, \theta_N)$ is referred to as the joint heading of agents.

Definition 3.3 Front-Truncated Function
Given a function $f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^N$, the notation $f^{+\tau}(\cdot)$ denotes the front-truncated version of $f(\cdot)$ where the section of $f(\cdot)$ between 0 and τ is truncated. More specifically,

$$f^{+\tau}(t) = f(t + \tau) \text{ for } t \geq 0.$$

The notation of front-truncated function is often applied to trajectories and controls of the agents in this dissertation. For example, given $\mathbf{u_e}(\cdot)$ as the joint control of evaders, the front-truncated version of the joint control is denoted by $\mathbf{u_e}^{+\tau}(\cdot)$ and it describes the joint control of the evaders after time τ.

3.2.1   Open-loop Optimal Pursuer Control

Recall that $\Gamma_\mathbf{s}^\star(x_p^0, \mathbf{x_e}^0, \mathbf{u_e}(\cdot))$ represents the minimum possible team survival time under the capture sequence $\mathbf{s}$ given that x_p^0 and $\mathbf{x_e}$ are the initial positions of the pursuer and the evaders and $\mathbf{u_e}(\cdot)$ is the joint control of the evaders. To be able to solve the optimization problem in Eq. (3.1), an efficient algorithm that is capable of evaluating this minimum possible team survival time under a specific capture sequence is to be developed. Note that the maximum speeds of the pursuer and the evaders are denoted by v_p and $\mathbf{v_e}$ and they satisfy $v_p > v_i, i = 1, \ldots, N$. They are omitted in the arguments of the $\Gamma_\mathbf{s}^\star$ function for notational simplicity.

The following lemma describes an important property of the open-loop optimal pursuer control against a single slower evader with given control.

Lemma 3.1 Open-loop optimal pursuer control against a single evader

Define the optimal open-loop control for a pursuer starting at x_p^0 with maximum speed v_p to capture an evader starting at x_e^0 with maximum speed v_e and control $u_e(\cdot)$ as

$$u_p^\star(\cdot) = \operatorname*{arg\,inf}_{u_p(\cdot) \in \mathcal{U}} \ \Gamma(x_p^0, x_e^0, u_p(\cdot), u_e(\cdot)),$$

then for all $u_e(\cdot) \in \mathcal{U}$,

$$u_p^\star(\cdot) \in \mathcal{U}_\theta.$$
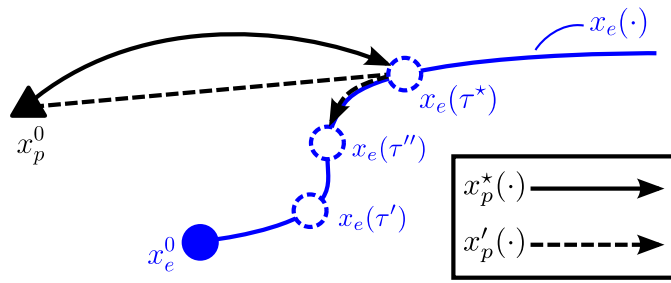
Figure for proof of Lemma 3.1

Proof. Assume for contradiction that there exists a specific $u_e(\cdot) \in \mathcal{U}$ such that the resulting $u_p^\star(\cdot) \notin \mathcal{U}_\theta$, and the resulting minimum capture time is

$$\tau^\star = \Gamma(x_p^0, x_e^0, u_p^\star(\cdot), u_e(\cdot)) = \inf_{u_p(\cdot) \in \mathcal{U}} \Gamma(x_p^0, x_e^0, u_p(\cdot), u_e(\cdot)). \tag{3.4}$$

Define the resulting pursuer trajectory $x_p^\star(\cdot) = traj(x_p^0, v_p, u_p^\star(\cdot))$ and the resulting evader trajectory $x_e(\cdot) = traj(x_e^0, v_e, u_e(\cdot))$. Based on the point capture condition, at capture time $\tau^\star$

$$x_p^\star(\tau^\star) = x_e(\tau^\star). \tag{3.5}$$

Since the shortest distance between two points in $\mathbb{R}^2$ is a straight line, there exist an alternative pursuer control $u_p'(\cdot) \in \mathcal{U}_\theta$ such that the resulting pursuer trajectory, $x_p'(\cdot) = traj(x_p^0, v_p, u_p'(\cdot))$, satisfies

$$x_p'(\tau') = x_e(\tau^\star) \text{ for } \tau' \leq \tau^\star. \tag{3.6}$$

At time τ', the pursuer is at $x_p'(\tau')$ and the evader is at $x_e(\tau')$. Knowing the evader control $u_e(\cdot)$, and hence the resulting trajectory of the evader $x_e(\cdot)$, the pursuer can "backtrack" the resulting trajectory of the evader between $x_e(\tau')$ and $x_e(\tau^\star)$ and capture the evader at $x_e(\tau'')$ where $\tau' < \tau'' < \tau^\star$. Note that with this alternative control, the pursuer can capture the evader at time $\tau'' < \tau^\star$, which contradicts statement that $\tau^\star$ is the minimum possible capture time in Eq. (3.4). Hence, by contradiction, $u_p^\star(\cdot) \in \mathcal{U}_\theta$ for all $u_e(\cdot) \in \mathcal{U}$. ∎

Lemma 3.1 shows that to capture an evader with known control in minimum time, the pursuer must travel in straight line with maximum speed.

The next lemma is necessary for extending Lemma 3.1 to cases with multiple evaders.

Lemma 3.2 Effect of delayed capture on team survival time under a capture sequence

Given $\mathbf{x_e^0}$ as the initial positions of N evaders, $\mathbf{v_e}$ as their maximum speeds, $\mathbf{u_e}(\cdot)$ as their joint controls, and $\mathbf{s} = (s_1, \ldots, s_N)$ as a capture sequence, define $\mathbf{x_e}(\cdot) = traj(\mathbf{x_e^0}, \mathbf{v_e}, \mathbf{u_e})$ as the resulting joint trajectory of the evaders, $\hat{\mathbf{s}} = (s_2, \ldots, s_N)$ as the front-truncated capture sequence, and $\hat{\mathbf{e}} = \{s_2, \ldots, s_N\}$ as the set of evaders with the first evader in the capture sequence removed. Assuming that the maximum speed of the pursuer is faster than all evaders, then for all τ' such that $\tau' > \tau \geq 0$,

$$\tau' + \Gamma_{\hat{\mathbf{s}}}^\star(x_{s_1}(\tau'), \mathbf{x}_{\hat{\mathbf{e}}}(\tau'), \mathbf{u}_{\hat{\mathbf{e}}}^{+\tau'}(\cdot)) > \tau + \Gamma_{\hat{\mathbf{s}}}^\star(x_{s_1}(\tau), \mathbf{x}_{\hat{\mathbf{e}}}(\tau), \mathbf{u}_{\hat{\mathbf{e}}}^{+\tau}(\cdot)) \tag{3.7}$$

(a) $x_p^0 = x_{s_1}(\tau)$

(b) $x_p^0 = x_{s_1}(\tau')$
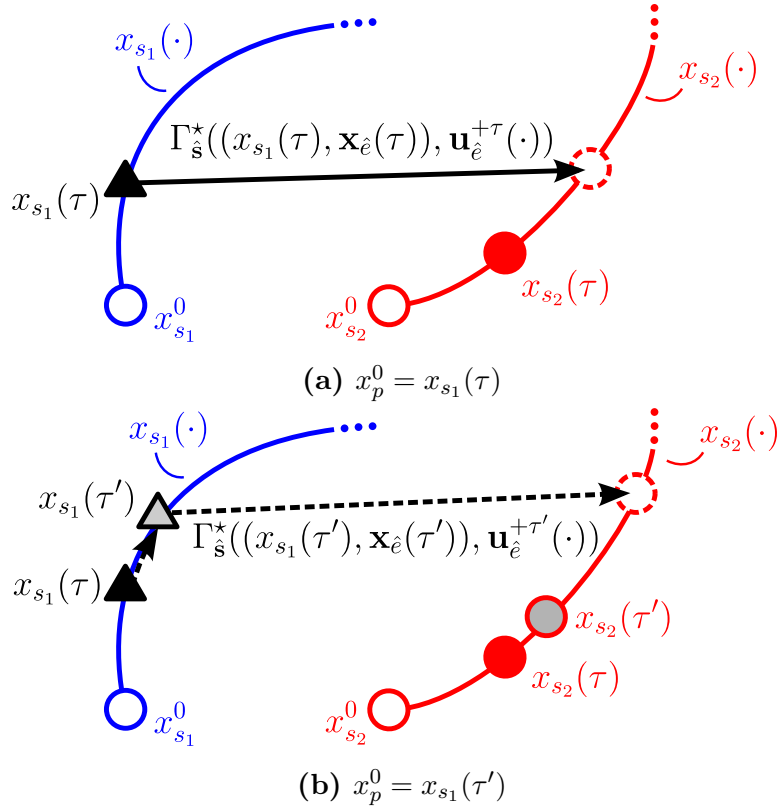
Figure for proof of Lemma 3.2

Proof. Assume for contradiction that there exists a $\tau' > \tau \geq 0$ such that

$$(\tau' - \tau) + \Gamma_{\hat{\mathbf{s}}}^\star(x_{s_1}(\tau'), \mathbf{x}_{\hat{\mathbf{e}}}(\tau'), \mathbf{u}_{\hat{\mathbf{e}}}^{+\tau'}(\cdot)) < \Gamma_{\hat{\mathbf{s}}}^\star(x_{s_1}(\tau), \mathbf{x}_{\hat{\mathbf{e}}}(\tau), \mathbf{u}_{\hat{\mathbf{e}}}^{+\tau}(\cdot)). \tag{3.8}$$

Consider the scenario where the pursuer starts at $x_{s_1}(\tau)$, which is a point on the trajectory of the first evader, and aims to capture the $N-1$ evaders according to the capture sequence $\hat{\mathbf{s}} = (s_2, \ldots, s_N)$. Note that the first evader, indexed by s_1, is omitted from $\hat{\mathbf{s}}$. The $N-1$ evaders start at $\mathbf{x}_{\hat{e}}(\tau)$ with $\mathbf{u}_{\hat{e}}^{+\tau}(\cdot)$ as their joint control. The term, $\Gamma_{\hat{\mathbf{s}}}^{\star}(x_{s_1}(\tau), \mathbf{x}_{\hat{e}}(\tau), \mathbf{u}_{\hat{e}}^{+\tau}(\cdot))$, on the right hand side of Eq. (3.8) is the minimum time it takes a pursuer starting at $x_{s_1}(\tau)$ to capture all $N-1$ evaders according to $\hat{\mathbf{s}}$.

The term on the left hand side of Eq. (3.8) is the resulting team survival time of a specific pursuer control. Starting from $x_{s_1}(\tau)$, the pursuer follows $x_{s_1}(\cdot)$, the future trajectory of evader s_1, with speed v_{s_1} instead of its maximum speed v_p, until it reaches $x_{s_1}(\tau')$ at time τ'. This takes the pursuer $\tau' - \tau$ units of time. At time τ' the $N-1$ evaders are at $\mathbf{x}_{\hat{e}}(\tau')$ and their joint control starting from this point is $\mathbf{u}_{\hat{e}}^{+\tau'}(\cdot)$. The pursuer then follows the optimal pursuer control which captures all the $N-1$ evaders from this configuration with the minimum possible time $\Gamma_{\hat{\mathbf{s}}}^{\star}(x_{s_1}(\tau'), \mathbf{x}_{\hat{e}}(\tau'), \mathbf{u}_{\hat{e}}^{+\tau'}(\cdot))$. In other words, the left hand side of Eq. (3.8) is an achievable team capture time for a pursuer starting at $x_{s_1}(\tau)$ to capture $N-1$ evaders starting at $\mathbf{x}_{\hat{e}}(\tau)$ with joint control $\mathbf{u}_{\hat{e}}^{+\tau}(\cdot)$ according to the capture sequence $\hat{\mathbf{s}}$.

Recall that the right hand side of the equation is the minimum achievable team capture time for such scenario according to the definition. Equation (3.8) contradicts itself by stating that there exists a team capture time that is shorter than the minimum possible team capture time. Through contradiction, Eq. (3.7) is shown to be true for all τ' such that $\tau' > \tau \geq 0$.   ■

Lemma 3.2 has shown that when the joint control of the evaders is known and the capture sequence is given, delaying the capture any of the evaders will always result in a longer team survival time.

Based on Lemma 3.2, the following theorem describes the optimal pursuer control that achieves the minimum possible team survival time given the joint control of the evaders under a specific capture sequence.

Theorem 3.3 Optimal pursuer control to capture a team of evaders with given controls under a capture sequence
Given $\mathbf{x}_{\mathbf{e}}^0$ as the initial position of N evaders, $\mathbf{v}_{\mathbf{e}}$ as their maximum speeds, and $\mathbf{u}_{\mathbf{e}}(\cdot)$ as their joint control, for a pursuer starting at x_p^0 with maximum speed $v_p > v_i$ for $i = 1, \ldots, N$ to capture all evaders according to the capture sequence $\mathbf{s}$ in minimum possible time $\Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \mathbf{u}_{\mathbf{e}}(\cdot))$, the pursuer must always capture the next evader in the capture sequence in minimum possible time.

Proof. The statement is obviously true for $N = 1$. Assuming that the statement is true for teams with $N-1$ evaders, by Lemma 3.2 for the pursuer to achieve $\Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \mathbf{u}_{\mathbf{e}}(\cdot))$, the first evader has to be captured in minimum possible time. By induction, all evaders have to be captured in minimum time according to the capture sequence.   ■

The optimal pursuer control, which always captures the next evader in the capture sequence in minimum possible time, is referred to as the *greedy pursuer control*.

Definition 3.4 Greedy pursuer control
Given $\mathbf{x_e^0}$ as the initial position of N evaders, $\mathbf{v_e}$ as their maximum speeds, $\mathbf{u_e}(\cdot)$ as their joint control, and $\mathbf{s} = (s_1, \ldots, s_N)$ as the capture sequence, assume that at time t the next evader to capture is s_i and define

$$u'_p(\cdot) = \arg\inf_{u'_p(\cdot) \in \mathcal{U}} \Gamma(x_p(t), x_{s_i}(t), u'_p(\cdot), u_{s_i}^{+t}(\cdot)). \tag{3.9}$$

The greedy pursuer control, denoted by $u_p^\star(\cdot)$, satisfies

$$u_p^\star(t) = u'_p(0)$$

for all t such that the next evader to capture is s_i.

Figure 3.2 shows an example of the resulting trajectory of the greedy pursuer control to capture three evaders with given controls in a specific order.
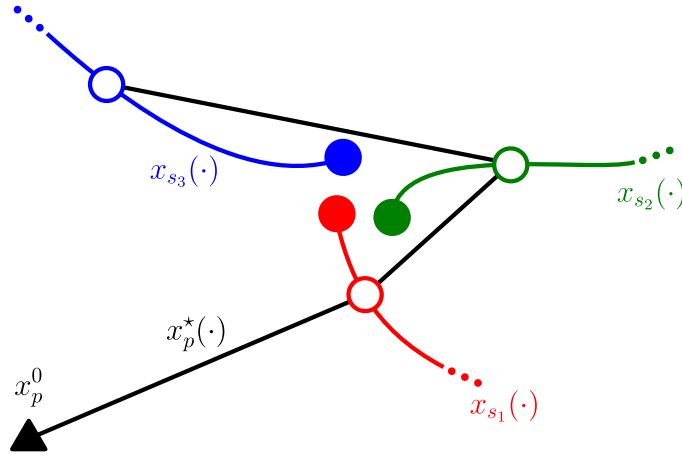
Figure 3.2: Example of the resulting trajectory from greedy pursuer control

In conclusion, in this section it has been shown that to capture a team of slower evaders with joint initial position $\mathbf{x_e^0}$ and known joint control $\mathbf{u_e}(\cdot)$ in a specific capture sequence $\mathbf{s}$ in minimum time, the pursuer, starting at x_p^0 should always capture each evader in minimum time according to the capture sequence. This results in straight-line-maximum-speed pursuer trajectories between the capture points of the evaders, and the resulting team survival time is denoted by $\Gamma_{\mathbf{s}}^\star(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot))$.

3.2.2 Open-loop Optimal Evader Control

In the open-loop formulation for the team evasion problem conservative to the evaders, the evaders have to find the joint control that maximizes the team survival time under the assumption that the pursuer will act optimally against it using the optimal control derived in the previous section. In this section, the properties of the optimal joint control of evaders in the open-loop formulation are derived through a series of lemmas.

Starting with the single evader case, the following lemma reveals the relationship between the control, capture point, and survival time of the evader in an open-loop setting.

Lemma 3.4 Minimum-time capture point and survival time
Given x_p^0 and $\mathbf{x_e^0}$ as the initial positions of a pursuer and an evader, v_p and v_e as their maximum speeds, and $u_e(\cdot) \in \mathcal{U}$ as the evader's control, the minimum possible capture time against an optimal pursuer in the open-loop setting is defined as

$$\tau = \inf_{u_p(\cdot) \in \mathcal{U}} \Gamma(x_p^0, x_e^0, u_p(\cdot), u_e(\cdot)). \tag{3.10}$$

The resulting minimum-time capture point is denoted by $x_e(\tau)$ where $x_e(\cdot) = traj(x_e^0, v_e, u_e(\cdot))$. Given an alternative evader control $u_e'(\cdot) \in \mathcal{U}$, the resulting evader trajectory is denoted by $x_e'(\cdot) = traj(x_e^0, v_e, u_e'(\cdot))$.

For all $u_e'(\cdot) \in \mathcal{U}$ such that the resulting evader trajectory $x_e'(\cdot)$ satisfies

$$x_e'(\tau') = x_e(\tau) \text{ for a } \tau' < \tau, \tag{3.11}$$

the following inequality is true:

$$\inf_{u_p(\cdot) \in \mathcal{U}} \Gamma(x_p^0, x_e^0, u_p(\cdot), u_e'(\cdot)) \geq \tau. \tag{3.12}$$
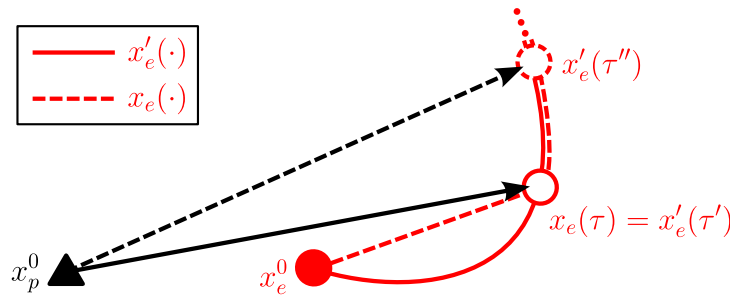
For proof of Lemma 3.4

Proof. Assume for contradiction that there exists a $u_e'(\cdot) \in \mathcal{U}$ with the resulting evader trajectory $x_e'(\cdot) = traj(x_e^0, v_e, u_e'(\cdot))$ passing through $x_e(\tau)$ at time $\tau' \leq \tau$ such that

$$\inf_{u_p(\cdot) \in \mathcal{U}} \Gamma((x_p^0, x_e^0), u_p(\cdot), u_e'(\cdot)) = \tau'' < \tau.$$

This implies that the pursuer can reach $x'_e(\tau'')$ at time τ''. Since $v_p > v_e$, the pursuer can follow the trajectory $x'_e(\cdot)$ further and reach $x'_e(\tau')$ before time τ'. With the condition $x'_e(\tau') = x_e(\tau)$ and $\tau' \leq \tau$, this implies that the pursuer starting at x^0_p can reach $x_e(\tau)$ strictly before time τ. Knowing the evader control, the pursuer will then be able to capture the evader with the original control $u_e(\cdot)$ before time τ by "backtracking" the evader's trajectory. This contradicts the fact that τ is the minimum possible capture time as defined in Eq. (3.10). Hence, by contradiction, Eq. (3.12) is true for all $u'_e(\cdot)$ that satisfies Eq. (3.11). ■

The following theorem makes use of Lemma 3.4 to derive an important property of the optimal joint control of evaders given a specific capture sequence.

Theorem 3.5 Open-loop optimal joint control of the evaders given a capture sequence
Given x^0_p and $\mathbf{x}^0_\mathbf{e}$ as the initial positions of a pursuer and N evaders, v_p and $\mathbf{v_e}$ as their maximum speeds, and $\mathbf{s} = (s_1, \ldots, s_N)$ as the capture sequence, define the optimal joint control for the evaders in the open-loop formulation of team evasion game conservative to the evaders as

$$\mathbf{u^\star_e}(\cdot) = \arg\sup_{\mathbf{u_e}(\cdot) \in \mathcal{U}^N} \Gamma^\star_s(x^0_p, \mathbf{x}^0_\mathbf{e}, \mathbf{u_e}(\cdot)). \tag{3.13}$$

Then

$$\mathbf{u^\star_e}(\cdot) \in \mathcal{U}^N_\Theta. \tag{3.14}$$

Proof. When $N = 1$, the optimal open-loop control for the single evader is constant heading maximum speed and points directly away from the pursuer.

Define $\hat{\mathbf{s}} = (s_2, \ldots, s_N)$ and $\hat{\mathbf{e}} = \{s_2, \ldots, s_N\}$, and assume for induction that $\mathbf{u^\star_{\hat{e}}}(\cdot) \in \mathcal{U}^{N-1}_\Theta$. Consider a joint control for N evaders $\mathbf{u}_e(\cdot) = (u_{s_1}(\cdot), \ldots, u_{s_N}(\cdot))$ and define the resulting minimum capture time of the first evader to be

$$\tau = \inf_{u_p(\cdot) \in \mathcal{U}} \Gamma(x^0_p, x^0_{s_1}, u_p(\cdot), u_{s_1}(\cdot)),$$

and the resulting capture point to be $x_{s_1}(\tau)$ with $x_{s_1}(\cdot) = traj(x^0_{s_1}, v_{s_1}, u_{s_1}(\cdot))$. Assume for contradiction that $u_{s_1}(\cdot)$ is not constant heading and maximum speed between time 0 and time τ, but the joint control $\mathbf{u_e}(\cdot)$ does achieve the maximum team survival time $\sup_{\mathbf{u_e}(\cdot) \in \mathcal{U}^N} \Gamma^\star_\mathbf{s}(x^0_p, \mathbf{x}^0_\mathbf{e}, \mathbf{u_e}(\cdot))$. Consider an alternative control for the first evader that travels in constant heading and maximum speed to reach $x_{s_1}(\tau)$ at a time earlier than τ (this is possible since the original $u_{s_1}(\cdot)$ is not constant heading and maximum speed between time 0 and τ), and then follows the original trajectory $x_{s_1}(\cdot)$. According to Lemma 3.4, since with this alternative control the evader can reach $x_{s_1}(\tau)$ before the original minimum capture time τ, the pursuer can not capture the evader with the alternative control at any time earlier than τ. In other words, with the alternative control, the evader will be capture by the optimal pursuer at $x_{s_1}(\tau')$ with $\tau' > \tau$ instead of $x_{s_1}(\tau)$. According to Lemma 3.2, this will result

in a longer team survival time than the original $\mathbf{u_e}(\cdot)$. However, since the original $\mathbf{u_e}(\cdot)$ is assumed to achieve the maximum possible team survival time, this leads to a contradiction. By proof of contradiction, to achieve the maximum open-loop team survival time, the first pursuer in the sequence has to travel in fixed heading and maximum speed.

By induction, this is true for all N and hence all evaders have to travel in constant headings and maximum speeds to achieve the maximum team survival time in an open-loop setting. ∎

With Theorem 3.5, the optimal joint control for a team of evaders in an open-loop setting can be specified by the joint heading of the team. The heading of evader i is denoted by θ_i and the joint heading of a team with N evaders is denoted by $\Theta_{\mathbf{e}} = (\theta_1, \ldots, \theta_N)$. The following remarks are made to simplify the optimal control problem faced by the evaders in the sequence-control formulation conservative to the pursuer.

Remark 3.6
The optimal control problem for the evaders proposed in Eq. (2.16),

$$\sup_{\mathbf{u_e}(\cdot) \in \mathcal{U}^N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)),$$

is equivalent to

$$\sup_{\mathbf{u_e}(\cdot) \in \mathcal{U}_{\Theta_{\mathbf{e}}}^N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)),$$

where

$$\Theta_{\mathbf{e}} = (\theta_1, \ldots, \theta_N).$$

With a slight abuse on the notation of $\Gamma_{\mathbf{s}}^{\star}$, the optimal control problem for the evaders can be expressed as

$$\sup_{\Theta_{\mathbf{e}}} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}}). \tag{3.15}$$

Note that the optimal control problem in Eq. (2.16) is a variational optimization problem which is infinite dimensional due to the admissible joint control set $\mathcal{U}^N$. However, by exploiting the properties of the optimal control derived through Theorem 3.5, the problem is simplified to the finite dimensional optimal control problem in Eq. (3.15). The admissible set for the joint heading $\Theta_{\mathbf{e}}$ is $\mathbb{R}^N$.

Remark 3.7
The sequence-control formulation conservative to the pursuer,

$$\inf_{\mathbf{s} \in \mathcal{S}_N} \sup_{\mathbf{u_e}(\cdot) \in \mathcal{U}^N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)),$$

as defined in Eq. (2.15) can be reformulated as

$$\inf_{\mathbf{s} \in \mathcal{S}_N} \sup_{\Theta_{\mathbf{e}}} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}}), \tag{3.16}$$

where $\Theta_{\mathbf{e}}$ is the joint heading of the evaders.

While Eq. (3.15) can be solved as a nonlinear optimization problem using techniques such as sequential quadratic programming, it can be solved even more efficiently by the method proposed in [34] and reviewed in B. However, the method proposed in [34] is based on first-order optimality condition which is only applicable for the formulation conservative to the purser, but not the one conservative to the evaders. The inf-sup formulation and the derivation of its optimal solution is presented here to lay the foundation for the following extensions of Theorem 3.5, which are important to the optimal solution of the open-loop formulation conservative to the evaders.

Corollary 3.8
Given x_p^0 and $\mathbf{x_e^0}$ as the initial positions of a pursuer and N evaders, v_p and $\mathbf{v_e}$ as their maximum speeds, define the optimal joint control of evaders for the open-loop team evasion problem conservative to the evaders as

$$\mathbf{u_e^\star}(\cdot) = \arg\sup_{\mathbf{u_e} \in \mathcal{U}^N} \inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_s^\star(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)). \tag{3.17}$$

Then

$$\mathbf{u_e^\star}(\cdot) \in \mathcal{U}_\Theta^N. \tag{3.18}$$

Proof. Assume for contradiction that there exists $\mathbf{u_e}(\cdot) \notin \mathcal{U}_\Theta^N$ such that

$$\inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_{\mathbf{s}}^\star(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)) = \sup_{\mathbf{u_e} \in \mathcal{U}^N} \inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_s^\star(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)). \tag{3.19}$$

Since $\mathbf{u_e}(\cdot) \notin \mathcal{U}_\Theta$, according to Theorem 3.5, for every $\mathbf{s} \in \mathcal{S}_N$ there exists a $\mathbf{u'_e}(\cdot) \in \mathcal{U}_\Theta^N$ such that

$$\Gamma_{\mathbf{s}}^\star(x_p^0, \mathbf{x_e^0}, \mathbf{u'_e}(\cdot)) > \Gamma_{\mathbf{s}}^\star(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)).$$

This implies that there exists $\mathbf{u'_e}(\cdot)$ such that

$$\inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_{\mathbf{s}}^\star(x_p^0, \mathbf{x_e^0}, \mathbf{u'_e}(\cdot)) > \inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_{\mathbf{s}}^\star(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot))$$

and hence contradicts Eq. (3.19). The proof is completed by proof of contradiction. ∎

With this corollary the following remark is made on the optimal control problem the team of evaders faces in the sequence-control open-loop formulation conservative to the evaders.

Remark 3.9

The optimal control problem for the team of evaders in the sequence-control open-loop formulation conservative to the evaders as formulated in Eq. (2.17),

$$\sup_{\mathbf{u_e}(\cdot)\in\mathcal{U}^N} \inf_{\mathbf{s}\in\mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)),$$

is equivalent to

$$\sup_{\mathbf{u_e}(\cdot)\in\mathcal{U}_{\Theta_{\mathbf{e}}}^N} \inf_{\mathbf{s}\in\mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)).$$

With a slight abuse on the notation of $\Gamma_{\mathbf{s}}^{\star}$, the optimal control problem for the evaders can be expressed as

$$\sup_{\Theta_{\mathbf{e}}} \inf_{\mathbf{s}\in\mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}}).$$
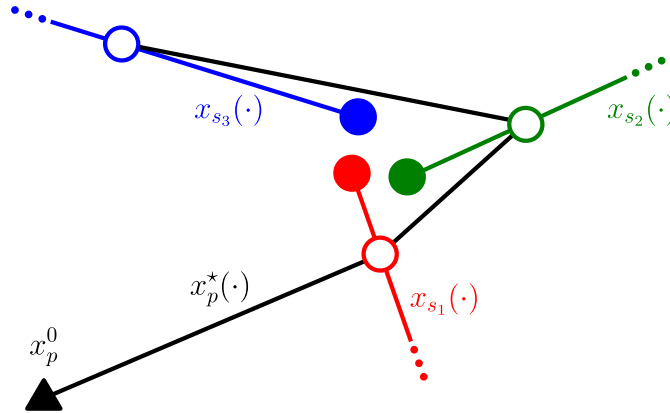
Figure 3.3: Resulting optimal trajectories for the open-loop team evasion problem conservative to the evaders

Figure 3.3 shows the resulting trajectories of the optimal solution to the open-loop team evasion problem conservative to the evaders for a 3-evader layout. The dark triangle is the initial position of the pursuer while the solid colored circles are the initial positions of the evaders. The hollow circles are the capture points of the evaders and the lines are the trajectories of the agents.

3.3   Solutions for the Open-loop Formulations

Recall the open-loop formulation of the team evasion game conservative to the pursuer as defined in Remark 3.7 as

$$\inf_{\mathbf{s}\in\mathcal{S}_N} \sup_{\Theta_{\mathbf{e}}} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}}) \tag{3.20}$$

and the open-loop formulation of the team evasion game conservative to the evaders as defined in Remark 3.9 as

$$\sup_{\Theta_\mathbf{e}} \inf_{\mathbf{s} \in \mathcal{S}_N} \ \Gamma_\mathbf{s}^\star(x_p^0, \mathbf{x_e^0}, \Theta_\mathbf{e}). \tag{3.21}$$

In this section, the algorithms used to solve for the open-loop optimal solutions to these formulations will be presented in detail.

3.3.1   Minimum Survival Time of a Given Evader Control

The following corollary describe how the minimum capture time of an evader with an arbitrarily control can be evaluated based on Lemma 3.1.

Corollary 3.10 Minimum Capture Time of a Single Evader
Given x_p^0 and x_e^0 as the initial positions of a pursuer and an evader, v_p and v_e as their maximum speeds, and $u_e(\cdot) \in \mathcal{U}$ as the control for the evader, the minimum possible capture time is

$$\inf_{u_p(\cdot) \in \mathcal{U}} \Gamma(x_p^0, x_e^0, u_p(\cdot), u_e(\cdot)) = \min\{\tau \mid \tau \geq 0 \text{ and } \left\| x_p^0 - x_e(\tau) \right\|_2 - v_p\tau = 0\} \tag{3.22}$$

where $x_e(\cdot) = traj(x_e^0, v_e, u_e(\cdot))$.
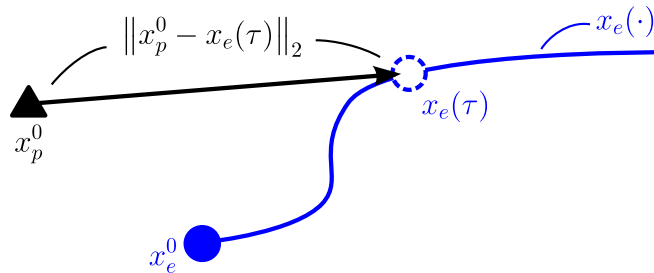
Figure for proof of Corollary 3.10

Proof. The term $\left\| x_p^0 - x_e(\tau) \right\|_2$ in Eq. (3.22) is the distance between the initial position of the pursuer and the position of the evader at time τ. The term $v_p\tau$ is the maximum distance the pursuer can travel in time τ. From Lemma 3.1, the minimum time capture trajectory by the pursuer is always a straight line. Hence, the minimum positive τ such that $\left\| x_p^0 - x_e(\tau) \right\|_2 = v_p\tau$ is the first possible time that the evader can be captured. ∎

The minimum capture time of a single evader with given control can hence be computed by finding the minimum positive root of $\left\| x_p^0 - x_e(\tau) \right\|_2 - v_p\tau = 0$. For cases where $v_p > v_e$, a finite positive root always exist and the root searching can be done in finite time.

When the evaders are traveling with constant headings and maximum speeds, the evaluation of $\inf_{u_p(\cdot)\in\mathcal{U}} \Gamma(x_p^0, x_e^0, u_p(\cdot), u_e(\cdot))$ can be further simplified according to the following corollary.

Corollary 3.11
Given x_p^0 and x_e^0 as the initial positions of a pursuer and an evader, v_p and v_e as their maximum speeds, assuming that the evader is traveling with a constant heading θ with maximum speed, the minimum capture time is

$$\frac{v_e}{v_p^2 - v_e^2}\left(\hat{x}\cdot\hat{e}_\theta + \sqrt{(\hat{x}\cdot\hat{e}_\theta)^2 + (\frac{v_p^2}{v_e^2} - 1)(\hat{x}\cdot\hat{x})}\right), \tag{3.23}$$

where $\hat{x} = x_e - x_p$ is the vector pointing from the pursuer to the evader, and $\hat{e}_\theta = [\cos\theta, \sin\theta]$ is the unit vector point at the direction specified by θ.
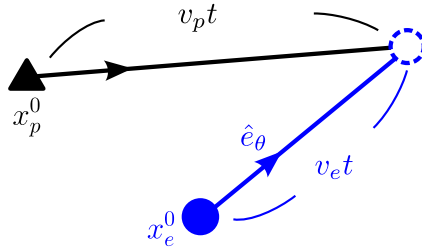
Figure for proof of Corollary 3.11

Proof. At time t, the position of the evader is $x_p^0 + \hat{x} + (v_e t)\hat{e}_\theta$, and the distance from this point to the starting position of the pursuer is $\|\hat{x} + (v_e t)\hat{e}_\theta\|_2$. According to Corollary 3.10, for capture to happen at time t, the following equation must be true:

$$\|\hat{x} + (v_e t)\hat{e}_\theta\|_2 = v_p t.$$

Taking the square of both sides and re-arranging the terms results in the following 2nd order polynomial of t:

$$(v_p^2 - v_e^2)t^2 - 2v_e(\hat{x}\cdot\hat{e}_\theta)t - \hat{x}\cdot\hat{x} = 0. \tag{3.24}$$

The expression in Eq. (3.23) is the closed-form formula for the bigger root of the polynomial. ∎

Note that compared to the case in Corollary 3.10 where the minimum survival time has to be computed through a one dimensional search, with $u_e(\cdot) \in \mathcal{U}_\theta$ the minimum survival time can be evaluated much more efficiently through a closed-form formula.

Algorithm 3.1 Minimum Capture Time of a Capture Sequence Given Joint Control of Evaders: $\Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot))$

1: Given $x_p^0, \mathbf{x_e^0} = (x_1^0, \ldots, x_N^0), \mathbf{s} = (s_1, \ldots, s_N), \mathbf{v_e} = (v_1, \ldots, v_N) < v_p$, and $\mathbf{u_e}(\cdot) = (u_1(\cdot), \ldots, u_N(\cdot))$
2: Initialize $x_p \leftarrow x_p^0, \tau \leftarrow 0$
3: **for** $i = 1, \ldots, N$ **do**
4:    $x_{s_i}(\cdot) \leftarrow traj(x_{s_i}^0, v_{s_i}, u_{s_i}(\cdot))$
5:    $\tau \leftarrow \tau + \inf_{u_p(\cdot) \in \mathcal{U}} \Gamma(x_p, x_{s_i}(\tau), u_p(\cdot), u_{s_i}^{+\tau}(\cdot))$
6:    $x_p \leftarrow x_{s_i}(\tau)$
7: **end for**
8: **return** τ

The minimum team survival time under a specific capture sequence given the joint control of evaders can be evaluated by Algorithm 3.1. The algorithm is initialized with the pursuer on the initial position and the time variable τ set to zero. For the i-th evader in the capture sequence, indexed by s_i, the resulting trajectory $x_{s_i}(\cdot)$ is computed at line 4. The term $\inf_{u_p(\cdot) \in \mathcal{U}} \Gamma(x_p, x_{s_i}(\tau), u_p(\cdot), u_{s_i}^{+\tau}(\cdot))$ in line 5 is the minimum capture time of evader s_i starting from time τ; the pursuer is at the capture position of evader s_{i-1} and the evader is at $x_{s_i}(\tau)$ with its future control being the front-truncated control $u_{s_i}^{+\tau}(\cdot)$. In other words, it is the difference between the survival time of evader s_i and evader s_{i-1}. Its value can be computed according to Corollary 3.10 when $u_e(\cdot) \notin \mathcal{U}_\theta$ and according to Corollary 3.11 when $u_e(\cdot) \in \mathcal{U}_\theta$. The final value of the variable τ is the survival time of the last captured evader in that it is the sum of the difference in survival time of all N evaders. The computation scales linearly with the number of evaders.

3.3.2 Optimal Solution to the Open-loop Formulation Conservative to the Pursuer

In the open-loop formulation of the team evasion game conservative to the pursuer as defined in Eq. (3.20), the pursuer has to select a capture sequence first. The evaders, given the capture sequence of the pursuer, can then choose their joint heading to maximize the team survival time. The optimal control problem faced by the team of evader is

$$\sup_{\Theta_{\mathbf{e}}} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}}). \tag{3.25}$$

This optimization problem can be treated as a unconstrained non-linear optimization problem, and can be solved numerically with standard non-linear optimization techniques such as the *sequential quadratic programming* (SQP) as reviewed in [47]. The dimension of the problem grows linearly with the dimension of the decision variable $\Theta_{\mathbf{e}}$, which is the same as the number of evaders in the team.

As proposed in [34] and reviewed in Chapter B, by exploiting the first order optimality condition of the open-loop formulation conservative to the pursuer, the problem of finding the optimal joint heading of the evaders can be converted to a one dimensional root searching problem. While the root searching approach is much more efficient than the nonlinear optimization approach for a team with large number of evaders, it is not applicable for the open-loop formulation conservative to the evaders. Several examples of the resulting optimal joint heading of the evaders are given in Section 3.4.

In the open-loop formulation conservative to the pursuer as formulated in Eq. (3.20), the pursuer has to pick a capture sequence assuming that the evaders will react to the selected capture sequence with the optimal joint heading. As mentioned in Chapter 1, the pursuer faces a dynamic traveling salesman problem. While algorithms such as branch-and-bound [27] are able to solve the traveling salesman problem of moderate size efficiently, these algorithms all require that the inter-city distances being independent from the route selected by the salesman. For example, given 3 cities labeled as A, B, and C, the time it takes the pursuer to travel from A to B is the same whether or not the route selected by the pursuer is (A, B, C) or (C, A, B). However, this is not true in the open-loop team evasion problem conservative to the pursuer since the distance between the evaders is a function of time. Using the same example, the time it takes the pursuer to capture evader B after evader A is captured, depends on when evader A is captured. In [30] it has been shown that even with only 2 evaders, there exists no polynomial time algorithm that can approximate the game with a factor better than 2. The pursuer has to rely on heuristics or brute-force approaches to derive the optimal capture sequence to capture the evaders. Examples of the optimal capture sequence for the pursuer in this setting are given in Section 3.4.

3.3.3 Optimal Solution to the Open-loop Formulation Conservative to the Evaders

In the open-loop formulation conservative to the pursuer discussed in the previous section, the evaders are facing a relatively straight forward problem in that the capture sequence is known to them. However, in the open-loop formulation conservative to the evaders formulated in Eq. (3.21) as

$$\sup_{\Theta_{\mathbf{e}}} \inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}}),$$

the evaders have to select their joint heading without knowing which capture sequence the pursuer will choose. Hence, as indicated by the $\inf_{s \in \mathcal{S}_N}$, the evaders assume that the pursuer will pick the optimal capture sequence against the selected joint heading to minimize the team survival time.

The decision variable of the first player, the joint heading of the evaders, is a continuous variable, and the decision variable of the second player, the capture sequence, is a discrete one. These are the characteristics of a *minimax* optimization problem. Although the open-loop team evasion problem has the sup-inf structure where the first player is the maximizer and

the second player is the minimizer, it can be easily converted to a inf-sup format by flipping the sign of the objective function. However, to avoid confusion, and also to highlight the fact that the goal of the evaders is to maximize the team survival time, the open-loop team evasion problem will be kept in its sup-inf form.

The open-loop team evasion problem can be treated as an unconstrained non-linear optimization problem with $\inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e}, \Theta_{\mathbf{e}})$ being the objective function and the $\Theta_{\mathbf{e}} \in \mathbb{R}^N$ being the decision variable. Given a tuple of $(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}})$, the objective function returns the team survival time with the evaders captured according to the optimal capture sequence defined as

$$\mathbf{s}^{\star} = \arg\inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e}, \Theta_{\mathbf{e}}).$$

In other words, the objective function is the point-wise minimum over a set of different $\Gamma_{\mathbf{s}}^{\star}$ functions, each function in the set represents the minimum team survival time under a specific capture sequence. The values of $\Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}})$ and $\inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}})$ of a 2-evader layout given different joint heading $\Theta_{\mathbf{e}}$ are visualized in Fig. 3.4 and Fig. 3.6.

By exploiting the minimax structure, the unconstrained sup-inf problem proposed in Eq. (3.21) can be converted to the following constrained optimization problem by introducing a dummy variable z:

$$\sup_{\Theta_{\mathbf{e}}, z} z \tag{3.26a}$$

$$\text{s.t. } z \leq \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}}) \text{ for all } \mathbf{s} \in \mathcal{S}_N. \tag{3.26b}$$

The decision variable of the problem is $[\Theta_{\mathbf{e}}, z]$ which has a dimension of $N + 1$. Note that Eq. (3.26b) encodes one constraint on z and $\Theta_{\mathbf{e}}$ for each of the possible capture sequences in $\mathcal{S}_N$. The objective function z represents the minimum survival time of the team in that it is constrained to be less than or equal to the survival time of all possible capture sequences.

The optimization problem proposed in Eq. (3.26) is linear in the objective function with nonlinear constraints. It can then be converted in to the following optimization problem through the use of Lagrange multiplier.

$$\sup_{\Theta_{\mathbf{e}}, z, \lambda_1, \dots, \lambda_m} z + \sum_{i=1}^{m} \lambda_i (\Gamma_{\mathbf{s}_i}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}}) - z) \tag{3.27}$$

$$\text{s.t. } \lambda_i \geq 0, \text{ for } i = 1, \dots, m \tag{3.28}$$

Note that through these transformations, the minimax problem Eq. (3.21) has been converted to a standard non-linear optimization problem with simple constraints. In this form it can be solved numerically with sequential quadratic programming. Examples of the open-loop optimal joint heading of the evaders and the resulting trajectories are presented and discussed in the next section.

3.4 Results and Discussion

In the open-loop formulation conservative to the pursuer in Eq. (3.20), the pursuer has to declare a capture sequence and commit to it; the evaders can then optimize their headings specifically to the declared capture sequence. The resulting optimal joint heading for the evaders is

$$\Theta_{\mathbf{e}}^{fix\star} = \arg\sup_{\Theta_{\mathbf{e}}} \ \Gamma_{\mathbf{s}^{fix\star}}^{\star}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}}),$$

which is optimal specifically to the capture sequence

$$\mathbf{s}^{fix\star} = \arg\inf_{\mathbf{s}\in\mathcal{S}_N} \ \sup_{\Theta_{\mathbf{e}}} \ \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}}).$$

In the open-loop formulation conservative to the evaders in Eq. (3.21), it is assumed that the evaders have to disclose their joint heading and the pursuer can then choose from all possible capture sequences according to the joint heading of the evaders. The resulting optimal joint heading of the evaders in this formulation is

$$\Theta_{\mathbf{e}}^{\star} = \arg\sup_{\Theta_{\mathbf{e}}} \ \inf_{\mathbf{s}\in\mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}})$$

and the optimal capture sequence is defined as

$$\mathbf{s}^{\star} = \arg\inf_{\mathbf{s}\in\mathcal{S}_N} \ \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}}^{\star}).$$

In this section, the resulting behavior of the optimal solutions of these two formulations are presented and compared.

3.4.1 Resulting Behavior of Different Formulations

A 2-evader layout as shown in Fig. 3.5 is used to illustrate the difference in the resulting behavior of the two formulations. The initial position of the pursuer, denoted by x_p^0, is represented as a black triangle in the figures. The initial positions of the evaders are denoted by $\mathbf{x}_{\mathbf{e}}^0 = (x_r^0, x_b^0)$ and are shown in the figures as the red and blue solid circles respectively. The resulting trajectories of the agents are drawn as colored dashed lines, and the colored hollow circles are the capture positions of the evaders. The initial layout is set to be $x_p^0 = (0,0), x_r^0 = (1,0), x_b^0 = (1.1, -0.1)$. The maximum speeds of the evaders are set to be $1/4$ units per second and that of the pursuer is set to be 1 unit per second.

Conservative to the Pursuer

In the open-loop formulation conservative to the pursuer, the evaders are allowed to pick their joint heading after knowing the exact capture sequence the pursuer is using.

Figure 3.4 shows the value of $\Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}})$, which is the minimum possible team survival time, given all possible joint heading $\Theta_{\mathbf{e}}$ under different capture sequences. The height of

(a) $\mathbf{s} = (r, b)$ **(b)** $\mathbf{s} = (b, r)$

Figure 3.4: Value of $\Gamma^\star_{\mathbf{s}}(x^0_p, \mathbf{x}^0_{\mathbf{e}}, \Theta_{\mathbf{e}})$ over all possible $\Theta_{\mathbf{e}}$ under different capture sequences

(a) $\mathbf{s} = (r, b)$ **(b)** $\mathbf{s} = (b, r)$

Figure 3.5: Resulting optimal trajectories for $\sup_{\Theta_{\mathbf{e}}} \Gamma^\star_{\mathbf{s}}(x^0_p, \mathbf{x}^0_{\mathbf{e}}, \Theta_{\mathbf{e}})$ under different capture sequences.

the surface in each figure represents the team survival time and the optimal team survival time for each capture sequence is marked by a black star on the surface and the contour. The x-axis and y-axis represent the heading of the red and blue evaders respectively and are drawn from 0 to 2π. It can be observed from the figure that by choosing the capture sequence $\mathbf{s} = (r, b)$, the pursuer can achieve lower team capture time against optimal joint heading of the evaders. Hence, in the open-loop formulation conservative to the pursuer as defined in Eq. (3.20), the optimal capture sequence for the pursuer is $\mathbf{s}^{fix\star} = (r, b)$.

Figure 3.5 shows the resulting trajectories of the optimal joint heading for each capture sequence. Just like the geometrical solution described in [23], the last captured evader always moves directly away from the capture point of the evader that is captured before it. It is worth noting again that these joint headings of the evaders are optimal under the condition that the capture sequence is known to the evaders and the pursuer is not allowed to deviate from the capture sequence during the game.

Conservative to the Evaders

In the open-loop formulation conservative to the evaders, the evaders have to determine their joint heading with the assumption that the selected joint heading will be revealed to the pursuer and that the pursuer will then pick the optimal capture sequence against the selected joint heading. With this worst-case mentality, the evaders try to find the joint heading that has the longest minimum possible team survival time. Figure 3.6 shows the value of $\inf_{\mathbf{s}\in\mathcal{S}_N} \Gamma^{\star}_{\mathbf{s}}(x^0_p, \mathbf{x}^0_{\mathbf{e}}, \Theta_{\mathbf{e}})$ with different $\Theta_{\mathbf{e}}$ for the specific layout. Similar to Fig. 3.4, the height of the surface indicates the team survival time given the corresponding joint heading of the evaders; the color of the surface represents the optimal capture sequence against the joint heading. For a joint heading of the evaders in the red region, the optimal capture sequence for the pursuer is $\mathbf{s}^{\star} = (r, b)$; for a joint headings in the blue region, the optimal capture sequence is $\mathbf{s}^{\star} = (b, r)$. For a joint heading that is on the boundary of both regions, the pursuer can pick either of the capture sequence to achieve the same minimum team survival time. Note that the surface in Fig. 3.6 is the point-wise minimum of the surfaces in Fig. 3.4a and Fig. 3.4b.

The optimal joint heading for the evaders which results in the maximum team survival time is marked by black stars both on the surface and on the contour. As shown in Fig. 3.6, for this specific initial condition the optimal joint heading of the evaders happens to be on boundaries of both the blue and the red regions. This indicates that against the open-loop optimal joint heading of evaders, the minimum possible team survival time that can be achieved by the pursuer is the same for both capture sequences. Figure 3.7a and Fig. 3.7b show the resulting minimum time capture trajectories against the optimal joint heading under the two capture sequences. The length of the dashed black lines are the same in these two figures.

Note that the non-uniqueness of the optimal capture sequence is not a general property of the open-loop optimal joint heading of the evaders, but instead a special property of the
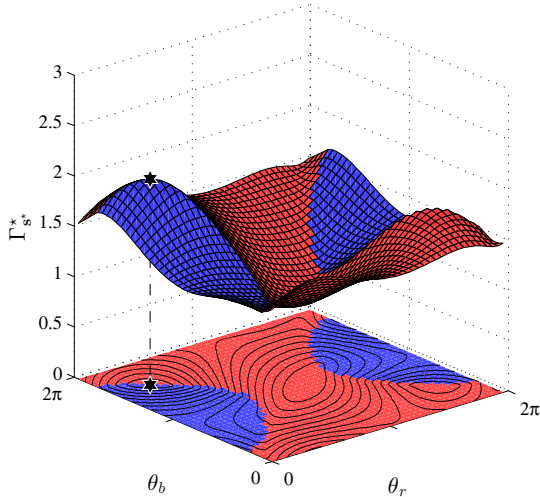
Figure 3.6: The value of $\inf_{\mathbf{s}\in\mathcal{S}_N} \sup_{\Theta_{\mathbf{e}}} \Gamma^{\star}_{\mathbf{s}}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}})$ in a 2-evader case.
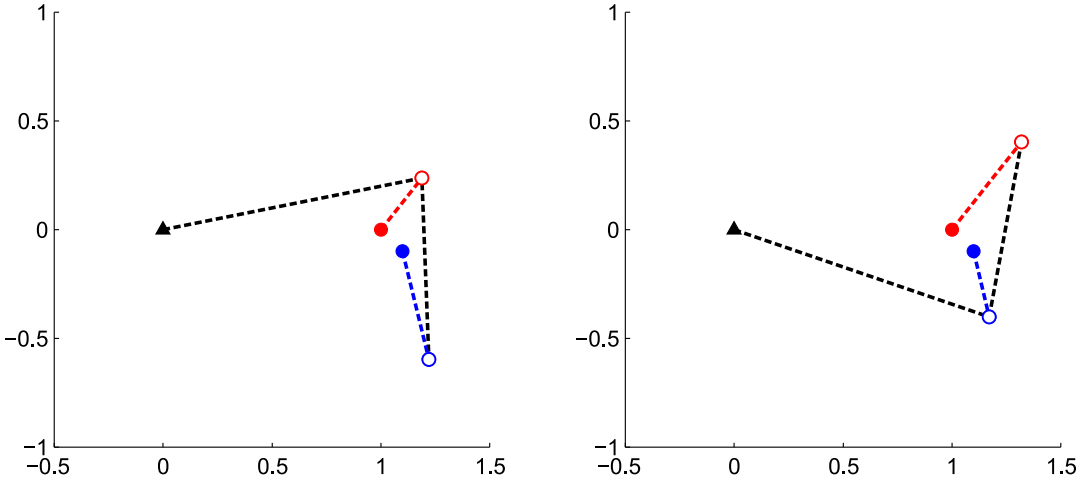
(a) $\mathbf{s}^{\star} = (r, b)$

(b) $\mathbf{s}^{\star} = (b, r)$

Figure 3.7: Resulting optimal trajectories of the open-loop formulation conservative to the evaders.
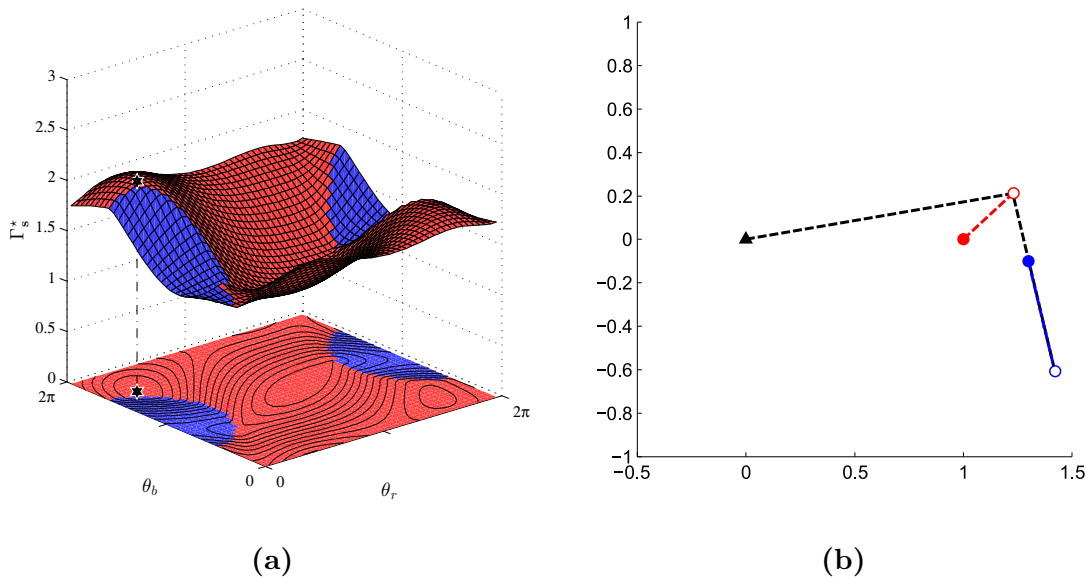
Figure 3.8: The value of $\inf_{\mathbf{s} \in \mathcal{S}_N} \sup_{\Theta_{\mathbf{e}}} \Gamma^{\star}_{\mathbf{s}}(x^0_p, \mathbf{x}^0_{\mathbf{e}}, \Theta_{\mathbf{e}})$ in a 2-evader case and the resulting open-loop optimal trajectories

initial condition. Figure 3.8a shows the value of the open-loop team survival time and its resulting trajectories of a slightly different initial condition where $x_p = (0,0), x_r = (1,0), x_b = (1.3, -0.1)$, and $v_r = v_b = 0.25$. Compared to the case shown in Fig. 3.6, the optimal joint heading for the evaders is inside the red region instead of being on the boundaries of both regions. The optimal capture sequence against the open-loop optimal joint heading of the evaders is unique in this case and the resulting trajectories are shown in Fig. 3.8b.

Teams with More Evaders

Figure 3.9 shows the resulting trajectories of the optimal joint heading of the evaders for a specific layout with 3 evaders given different capture sequences. Figure 3.10 shows the resulting optimal trajectories for a 4-evader layout and a 5-evader layout. As shown by the figures, the optimal joint heading of the evaders for the open-loop formulation conservative to the pursuer share similar characteristics with the two-evader case described previously. The most pronounced characteristic is that the last captured evader is always moving directly away from the capture point of the evader that is captured before it. This is because that the function $\Gamma^{\star}_{\mathbf{s}}(x^0_p, x^0_{\mathbf{e}}, \Theta_{\mathbf{e}})$, which is the minimum possible team survival time under a specific capture sequence, is differentiable under $\Theta_{\mathbf{e}}$ and hence has to satisfy the first-order optimality condition which states that the gradient has to vanish at the optimizer. Since the survival time of the last evader in the sequence is defined as the team survival time, if the last evader is not heading directly away from the capture point of its previous evader, the joint heading can not have a zero gradient vector in that the survival time of the team can obviously be

increased by adjusting the heading of the last evader.

Figure 3.11 shows the trajectories resulting from the open-loop optimal solution to the team evasion problem conservative to the evaders. Note that in all four cases, the optimal capture sequences are not unique. Only the trajectories resulting from one of the optimal capture sequences is shown in each figure. In these examples, the last captured evaders are not traveling directly away from the capture point of their previous evaders.

3.4.2   Difference in Team Survival Time Performance

Most of the works in the literature use the formulation conservative to the pursuer to provide the pursuer with a way to select a reasonable capture sequence; the strategy derived for the evaders with this formulation is not effective in a more realistic team evasion scenario. In a realistic team evasion scenario, it is unlikely that the pursuer will declare its capture sequence to the evaders, neither will it be obligated to follow a specific capture sequence during the pursuit. In this case, the evaders have to use the strategy resulting from the formulation conservative to them.

To quantitatively measure the performance of different approaches, a benchmarking data set is constructed with 500 randomly generated layouts for each of the team sizes: 2, 3, 4, and 5 evaders. In all of the 2000 layouts, the pursuer always starts from the origin $(0, 0)$. The positions of the evaders are sampled randomly from a uniform distribution in a unit square centered at the origin. The vertexes of the square are located at $(-0.5, -0.5), (-0.5, 0.5), (0.5, 0.5)$, and $(0.5, -0.5)$. The maximum speed of the pursuer is set to 1 unit per second and the maximum speeds of the evaders are set to be 0.25 units per second. This benchmarking dataset is used throughout this dissertation.

Given a specific joint heading of the evaders, $\Theta_{\mathbf{e}}$, the minimum possible team survival time against a pursuer that knows the joint heading of the evaders and is not obligated to commit to any specific capture sequence is defined as

$$\inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}}).$$

For each layout, the minimum possible team survival time for two different joint heading of the evaders is measured: one is the joint heading optimal for the formulation conservative to the pursuer, denoted by $\Theta_{\mathbf{e}}^{fix\star}$; the other is the joint heading optimal for the formulation conservative to the evaders, denoted by $\Theta_{\mathbf{e}}^{\star}$. The distribution of the ratio of the resulting team survival time from $\Theta_{\mathbf{e}}^{fix\star}$ to that from $\Theta_{\mathbf{e}}^{\star}$ over the 500 layouts are shown in Figure 3.12. As shown in the figures, this ratio is always smaller than or equal to one. This implies that when facing a pursuer that is not constrained to commit to a specific capture sequence, a team of evader using the open-loop formulation conservative to the evaders will always survive longer than (or at least equally as long as) a team using the formulation conservative to the pursuer. Also, it can be observed that the difference in team survival time performance of the two formulations is more pronounced when the number of evaders in the team increases. While
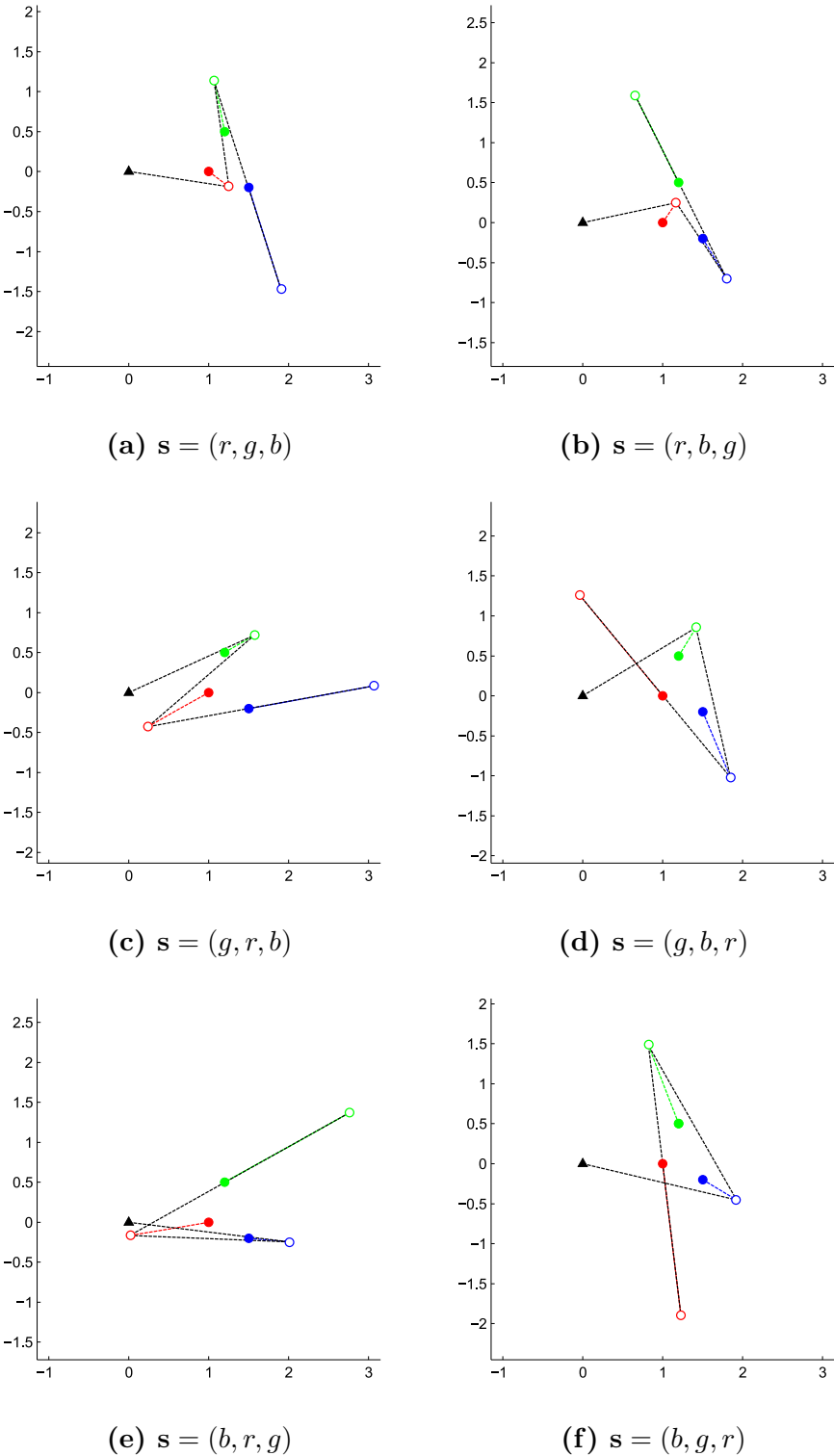
Figure 3.9: Optimal trajectories for the open-loop formulation conservative to the pursuer of a 3-evader layout under different capture sequences

(a) 4 evaders   **(b)** 5 evaders
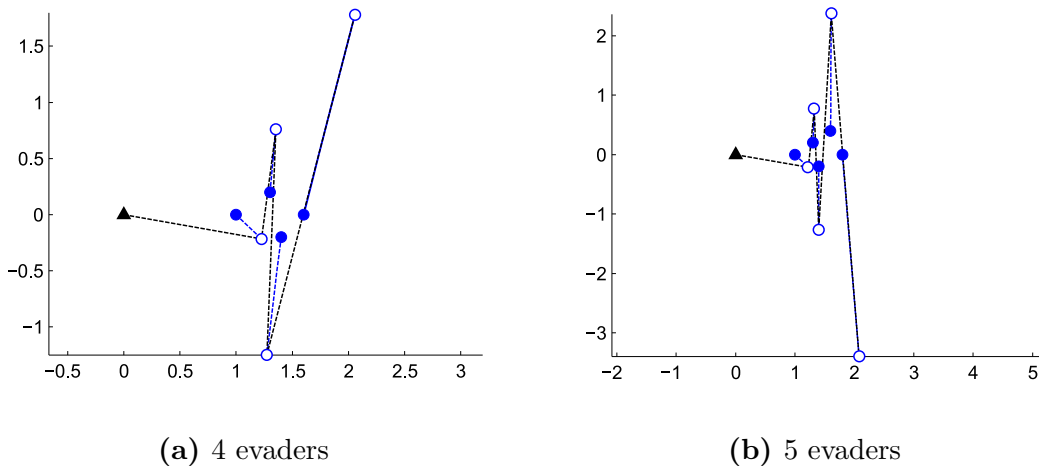
Figure 3.10: Optimal trajectories for the open-loop formulation conservative to the pursuer of a 4-evader and a 5-evader layout

in the 2-evader cases, the survival time ratio is 1 for about 450 out of the 500 layouts, in the 5-evader cases, none of the layouts have a ratio of 1 and a significant amount of the layouts have a ratio below 0.5. This shows that the formulation conservative to the evader is much more suitable than the one conservative to the pursuer for a team with several evaders in a team evasion scenario.

3.5 Conclusion

In this chapter, the properties of the optimal solution to the open-loop formulation of the team evasion problem are derived. It has been shown that for any given joint control of the evaders, the optimal strategy for a faster pursuer to capture the team of evaders in a specific capture sequence is to greedily minimize the survival time of the currently targeted evader. An algorithm is proposed to compute the minimum possible team survival time of a team of evaders under a specific capture sequence with a given joint control. The resulting optimal trajectory of the pursuer connects the capture points of the evaders with straight lines. It has also been shown that the optimal strategy for the evaders in the open-loop setting is to all travel with their maximum speeds with constant headings. With these properties, the optimization problem that the evaders have to solve for the optimal joint control can be converted from an infinity dimensional optimization problem to a finite dimensional optimization problem. The resulting finite dimensional sup-inf optimization problem can be solved through standard nonlinear optimization techniques such as sequential quadratic programming.

The resulting open-loop optimal joint heading of the evaders maximizes the minimum possible team survival time under the assumption that the pursuer will select the optimal
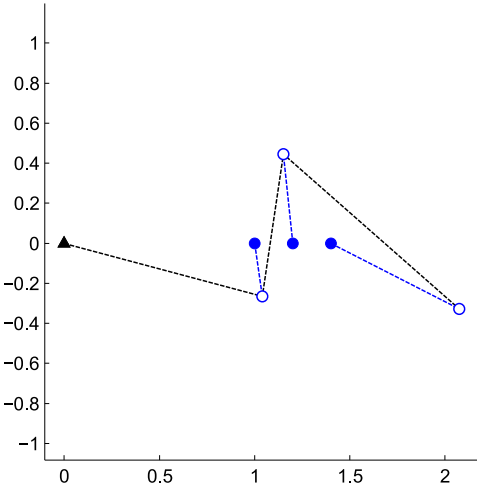
(a) 3 evaders
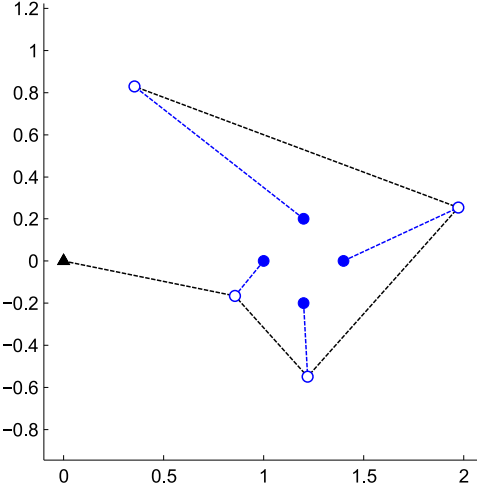
(b) 4 evaders
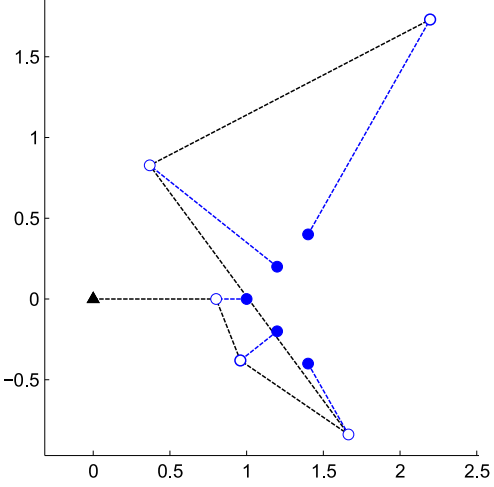
(c) 5 evaders
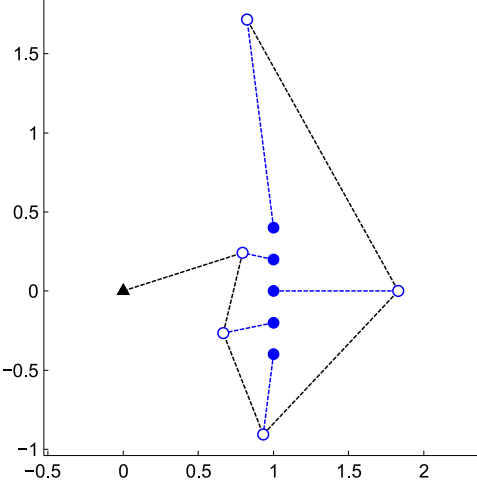
(d) 5 evaders

Figure 3.11: Examples of optimal trajectories of the open-loop formulation conservative to the evaders

(a) 2 evaders

(b) 3 evaders

(c) 4 evaders
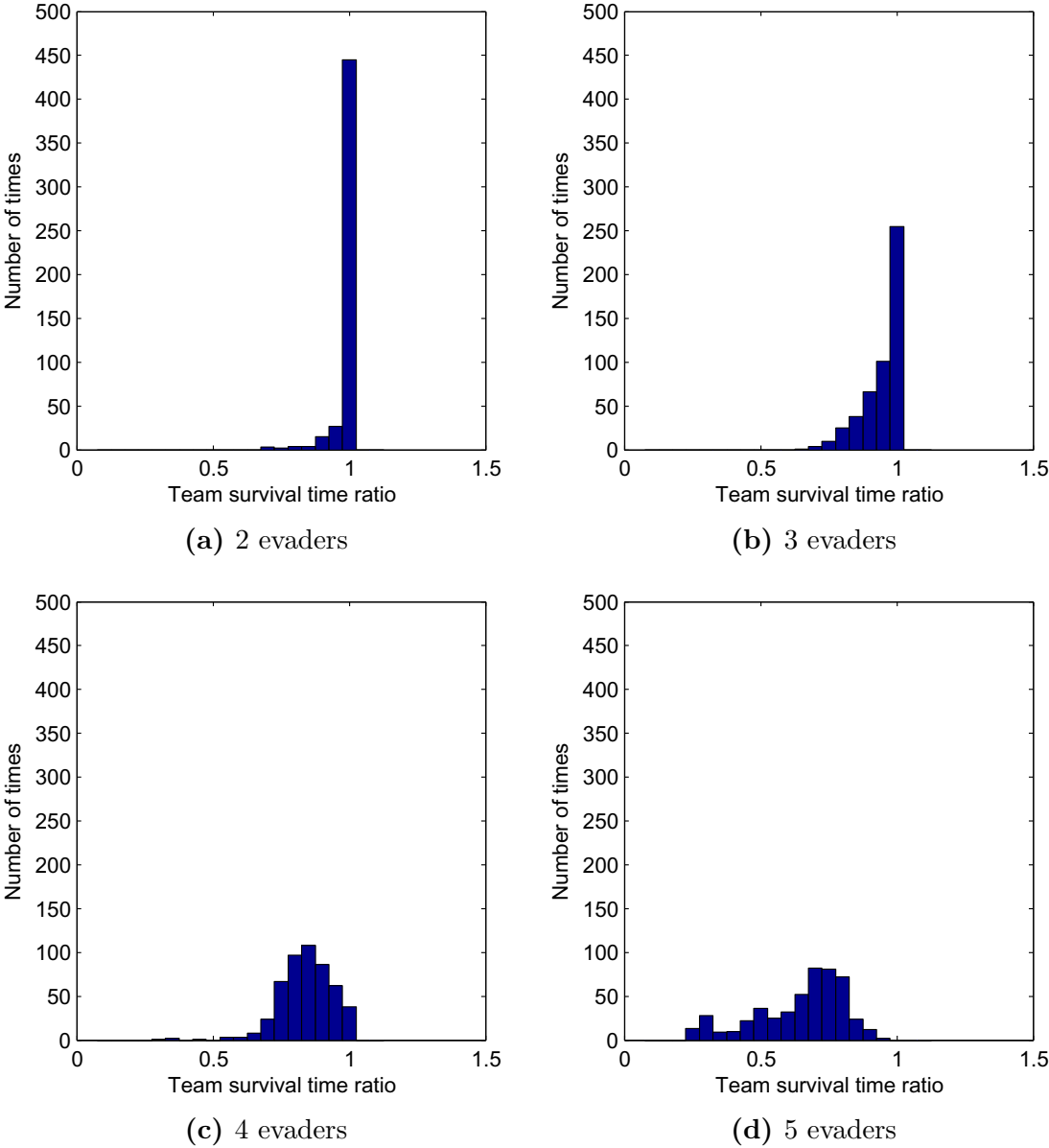
(d) 5 evaders

Figure 3.12: Distributions of ratio of team survival time resulting from the pursuer-centric formulation to that resulting from the evader-centric formulation over 500 layouts for teams with different number of evaders

capture sequence against the joint heading of the evaders. This is different from the pursuer-centric frameworks in the literature where the pursuer has to declare a capture sequence and commit to it before knowing the joint heading of the evaders. As a result of the worst-case mentality of the proposed framework, the resulting open-loop optimal joint heading of the evaders comes with a guaranteed team survival time. The team is guaranteed to survive at least as long as this team survival time by following the optimal joint heading regardless of the action of the pursuer; the evaders can achieve an even higher team survival time if the pursuer acts suboptimally. A team that follows the joint heading derived from the pursuer-centric frameworks performs poorly when facing a pursuer that is not obligated to capture the evaders in any specific capture sequence. This is due to the overly aggressive headings of the evaders being easily exploitable by the pursuer. Through extensive simulations, it has been shown that in a more realistic team evasion scenario, the proposed open-loop collaborative team evasion framework performs much better in terms of team survival time than the pursuer-centric frameworks in the literature.

In conclusion, the proposed open-loop collaborative team evasion framework provides the evaders with a strategy to maximize the minimum possible team survival time when facing a faster pursuer by solving a finite dimensional optimization problem; the conservatism towards the evaders embedded in the formulation results in a guaranteed team survival time for the team which is not provided by any of the previous work in the literature.

Chapter 4

Iterative Open-loop Approach for Collaborative Team Evasion

4.1   Introduction

In the open-loop approach for collaborative team evasion proposed in the previous chapter, the team of evaders determines their joint heading by solving the open-loop formulation of team evasion problem conservative to the evaders. The evaders then follow the open-loop optimal joint heading with respect to the initial condition of the game until they are all captured by the pursuer eventually. In the open-loop approach, the evaders are guaranteed to achieve the derived team survival time and be able to achieve even better performance if the pursuer acts suboptimally.

In this chapter, the iterative open-loop approach is proposed to relax the conservatism of the open-loop approach by adjusting the joint heading of the evaders during the chase using open-loop optimal joint heading with respect to the most current state of the game. The formulation and implementation of the iterative open-loop approach is outlined in Section 4.2. Several approximations to the iterative open-loop approach are proposed in Section 4.3 to reduce the computational requirement. The performance of the iterative open-loop approach and the approximations are presented and discussed in Section 4.4, and the chapter is concluded in Section 4.5.

4.2   Iterative Open-loop Approach

4.2.1   Formulation

The iterative open-loop approach for team evasion is based on the open-loop approach proposed in the previous chapter. The open-loop optimal joint heading for the team evasion problem conservative to the evaders is defined as follows.

Definition 4.1 Open-loop optimal joint heading of the evaders
Given x_p^0 and $\mathbf{x_e^0}$ as the initial positions of a pursuer and N evaders and v_p and $\mathbf{v_e}$ as their maximum speeds where $v_p > \mathbf{v_e}$, the open-loop optimal joint heading is defined as

$$\Theta_{\mathbf{e}}^{\star}(x_p^0, \mathbf{x_e^0}) = \arg\sup_{\Theta_{\mathbf{e}}} \inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}}), \tag{4.1}$$

where $\mathcal{S}_N$ is the set of all possible capture sequences for N evaders and $\Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}})$ is the minimum possible team survival time of the given layout and joint heading of evaders under the condition that the evaders must be captured according to the sequence $\mathbf{s}$.

Note that although the speeds of the pursuer and the evaders do affect the open-loop optimal joint heading $\Theta_{\mathbf{e}}^{\star}$ and the team survival time $\Gamma_{\mathbf{s}}^{\star}$, since they are properties of the agents that stay constant during the game, they are omitted in the arguments for notational simplicity.

The largest lower bound of the open-loop team survival time given a specific initial condition is defined as follows:

Definition 4.2 Largest lower bound of the open-loop team survival time
Given x_p^0 and $\mathbf{x_e^0}$ as the initial position for a pursuer and N evaders and v_p and $\mathbf{v_e}$ as their maximum speeds where $v_p > \mathbf{v_e}$, the largest lower bound of the open-loop team survival time is defined as

$$\Gamma^{ol\star}(x_p^0, \mathbf{x_e^0}) = \sup_{\Theta_{\mathbf{e}}} \inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}}). \tag{4.2}$$

The joint control of a team of evaders using the open-loop approach for collaborative team evasion proposed in the previous chapter can be described as

$$\Theta_{\mathbf{e}}(t) = \Theta_{\mathbf{e}}^{\star}(x_p^0, \mathbf{x_e^0}) \text{ for } t \geq 0, \tag{4.3}$$

where $(x_p^0, \mathbf{x_e^0})$ is the initial layout of the game. In other words, the evaders commit to the open-loop optimal joint heading of the initial layout of the game and do not change their headings once the game starts. With the open-loop approach, the team is guaranteed to survive for at least $\Gamma^{ol\star}(x_p^0, \mathbf{x_e^0})$ against an optimal pursuer that knows the exact joint control of the evaders. Any suboptimal action from the pursuer will result in a longer team survival time than the largest lower bound.

Figure 4.1 shows states of the game at different times during the pursuit where the team of evaders uses the open-loop team evasion approach against an optimal pursuer. The pursuer is marked by a dark triangle and the evaders are marked by solid blue dots. The dashed lines are the open-loop optimal trajectories for the initial condition shown in Fig. 4.1a and the hollow dots are the predicted capture points. Figure 4.1b shows the state of the game when the first evader is captured at $t = 1.21$. The solid lines in this figure are the traversed trajectories of the agents. Note that with the open-loop approach to team evasion, the second evader is still following the open-loop optimal heading with respect to the initial condition

(a) $t = 0$ **(b)** $t = 1.21$
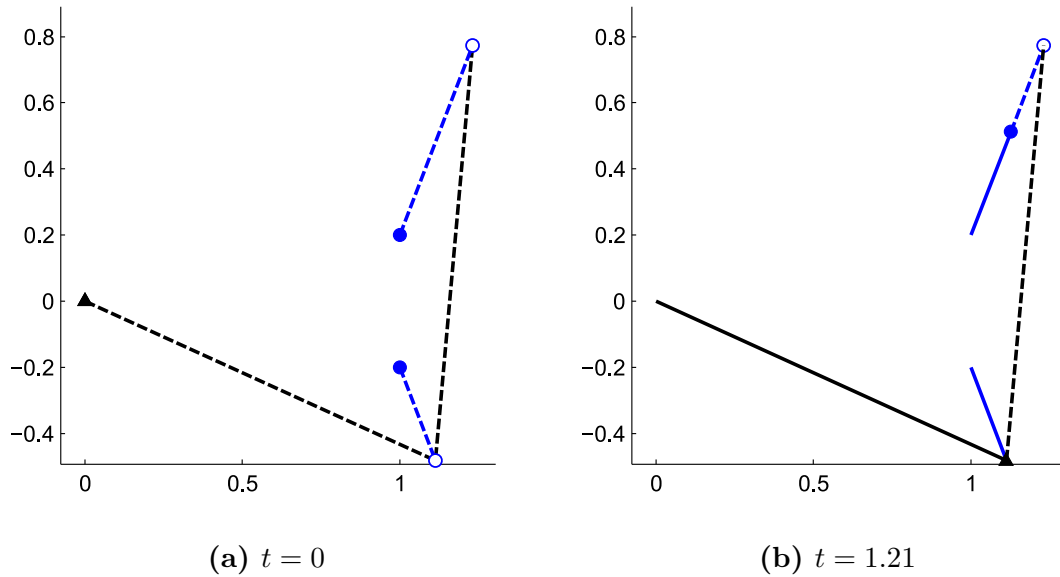
Figure 4.1: Snapshots of a simulation with 2 evaders using the open-loop approach against an optimal pursuer

of the game. However, the team can actually achieve a longer team survival time if the second evader adjust its heading to move directly away from the pursuer. This example demonstrates the conservatism of the open-loop approach. The evaders can further increase the team survival time by adjusting their headings during the game according to the most current layout instead of blindly following the joint heading that was optimal for the initial condition.

The iterative open-loop approach is designed to relax the conservatism of the open-loop approach by allowing the evaders to adjust their headings during the game. With this additional freedom, the evaders can better exploit suboptimal actions of the pursuer and achieve longer team survival time than the open-loop approach. In the iterative open-loop approach, the evaders adjust their joint heading to be the open-loop optimal one with respect to the most current state of the game in a predetermined frequency. The approach can be described as follows.

Definition 4.3 Iterative open-loop approach (iOL) for collaborative team evasion
Given that the evaders adjust their joint heading every Δt starting from $t = 0$, the resulting joint heading control of the evaders is:

$$\Theta_{\mathbf{e}}(t) = \Theta_{\mathbf{e}}^{\star}(x_p(n\Delta t), \mathbf{x_e}(n\Delta t)) \text{ for } n\Delta t \leq t < (n+1)\Delta t, n \in \mathbb{N}^{+}. \tag{4.4}$$

It is worth pointing out that to keep the computation for the iterative open-loop approach tractable, although the evaders can and will change their headings at future updates, they

do not take the future adjustments into account when determining the joint heading for the current update. In other words, at each update the evaders still plan as if they have to commit to the selected joint heading until the end of the game.

4.2.2 Implementation

To implement the iterative open-loop approach, the open-loop optimal joint heading defined in Eq. (4.1) has to be solved at every update. The optimization problem in Eq. (4.1) is a non-linear and non-concave maximization problem and is solved by sequential quadratic programming. Like most of the nonlinear optimization routines, the initialization of the routine affects the resulting optimizer and there is no guarantee that a global optimal will be found. This issue can be addressed by re-sampling, which initializes the optimization routine with several different initial guesses and then selects the best optimizer from the results. However, applying re-sampling at every update will slow down the iterative open-loop approach significantly since the non-linear, non-concave maximization problem has to be solved multiple times at every update. Furthermore, the number of initial guesses required to sufficiently cover the joint heading space to achieve global optimal grows rapidly with the number of evaders in the team.

A technique that is commonly used for receding horizon control problems with nonlinearity is utilized to alleviate a part of the computational burden. At every update, the optimal joint heading of the previous update is used as the initial guess for the optimization routine. With this technique, the optimization problem only has to be solved once at each update. Also, since with a short enough update time the layout does not change much between two consecutive updates, the initial guess is usually very close to the open-loop optimal joint heading of the current layout. For the iterative open-loop approach, this initialization technique also has the benefit of a shorter convergence time in that sequential quadratic programming is especially efficient when initialized closed to an optimizer.

In the simulations used to evaluate the survival time performance of the iterative open-loop approach, for evaders without turning rate constraints, the open-loop optimal joint heading of the initial layout is computed by using the re-sampling technique. After the first update at $t = 0$, the re-sampling process is skipped at the rest of the updates and the open-loop optimal joint heading from the previous update is used to initialize the optimization routine.

4.2.3 Pursuer's Strategy Against the Iterative Open-loop Approach

The team survival time performance of the open-loop approach proposed in the previous chapter is measured by the minimum possible team survival time against an optimal pursuer that knows the exact control of the evaders. The minimum possible team survival time and the optimal pursuer trajectory are simply byproducts of the solution of the sup-inf optimization problem solved by the evaders. However, since the evaders adjust their joint heading at every

update time in the iterative open-loop approach, the resulting team survival time must be measured by simulating the whole game. Furthermore, the optimal pursuer strategy against iterative open-loop evaders is necessary for the simulation. Deriving the optimal trajectory for the pursuer to minimize the team survival time in the iterative open-loop setting is an extremely challenging optimal control problem. The difficulty can be illustrated by the following example.

Consider a scenario where the evaders are using the iterative open-loop approach and only adjust their joint heading twice: first at $t = 0$ and then at $t = t' > 0$. Assume that the pursuer knows that the evaders are using the iterative open-loop approach and the exact update time. In this scenario, given the initial layout $(x_p^0, \mathbf{x_e^0})$, the joint heading of the evaders from $t = 0$ to $t = t'$ is determined as $\Theta_\mathbf{e}^\star(x_p^0, \mathbf{x_e^0})$. Hence, the joint position of the evaders at time t', denoted by $\mathbf{x_e}(t')$, is also determined solely by the initial condition. At time t', the joint heading of the evaders will be adjusted to $\Theta_\mathbf{e}^\star(x_p(t'), \mathbf{x_e}(t'))$. Starting from time t', they can survive for $\Gamma^{ol\star}(x_p(t'), \mathbf{x_e}(t'))$ units of time in the worst case. The resulting minimum possible team survival time starting from $t = 0$ is

$$t' + \Gamma^{ol\star}(x_p(t'), \mathbf{x_e}(t')), \tag{4.5}$$

which is a function of the position of the pursuer at time t'. Denoting the set of all possible position a pursuer starting from x_p^0 at $t = 0$ with maximum speed v_p can reach in t' seconds as

$$\mathbb{D}(x_p^0, v_p t') = \{x \mid \|x - x_p\|_2 \le v_p t'\}, \tag{4.6}$$

the optimal position for the pursuer to be at time t' is

$$\operatorname*{arg\,inf}_{x_p(t') \in \mathbb{D}(x_p^0, v_p t')} \Gamma^{ol\star}(x_p(t'), \mathbf{x_e}(t')). \tag{4.7}$$

For each possible $x_p(t')$ in $\mathbb{D}(x_p^0, v_p t')$, the pursuer has to solve for the open-loop optimal joint heading of the evaders for that specific layout, namely $\Theta^\star(x_p(t'), \mathbf{x_e}(t'))$, to be able to evaluate the open-loop optimal team survival time $\Gamma^{ol\star}(x_p(t'), \mathbf{x_e}(t'))$. Each of these evaluations requires solving an N dimensional optimization problem. Figure 4.2 shows the resulting value of $\Gamma^{ol\star}(x_p(t'), \mathbf{x_e}(t'))$ given different $x_p(t')$ in a 2-evader case. The two colored circles are the position of the evaders at time t' and the dark triangle is the position of the pursuer at $t = 0$. The circular contour contains all possible positions the pursuer can be at time t'. The regions in the circle are colored according to the value of $\Gamma^{ol\star}(x_p(t'), \mathbf{x_e}(t'))$ where a darker color represents a shorter team survival time.

It is worth emphasizing that for each $x_p(t')$, there is a corresponding open-loop optimal joint heading for the evaders. In other words, the resulting value of $\inf_{x_p(t') \in \mathbb{D}(x_p^0, v_p t')} \Gamma^{ol\star}(x_p(t'), \mathbf{x_e}(t'))$ is not the result of one specific open-loop joint heading of the evaders, but instead a family of them. It can be observed from Fig. 4.2 that in general when the pursuer is closer to the evaders the open-loop optimal team survival time is shorter. However, the pursuer being equidistant to the two evaders results in a higher team survival time. The contour shows

that the minimum team survival time is not a convex function of the pursuer's position and that the best pursuer position is not on the point that is closest to one of the evaders. The optimal $x_p(t')$ for the pursuer to minimize the team survival time can only be found by a brute-force approach.
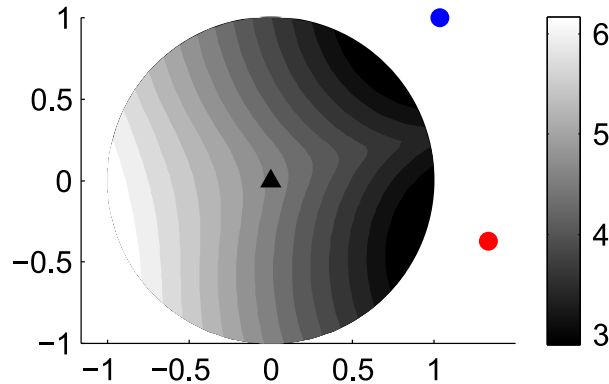
Figure 4.2: Optimal open-loop survival time of a 2-evader team given different pursuer positions

The procedure described above only covers the case where the evaders adjust their joint heading twice. Now consider a case where in addition to $t = 0$ and $t = t'$, the evaders also adjust their joint heading one more time at $t = t'' > t'$. The resulting team survival time is

$$t'' + \Gamma^{ol\star}(x_p(t''), \mathbf{x_e}(t''))$$

and is determined by the positions of the pursuer and the evaders at time t''. The joint position of the evaders at time t' is still determined solely by the initial layout. However, the joint position of the evaders at time t'', denoted by $\mathbf{x_e}(t'')$, is determined by $\Theta^\star(x_p(t'), \mathbf{x_e}(t'))$, which is in turn determined by $x_p(t')$. As of the position of the pursuer at time t'', for each $x_p(t') \in \mathbb{D}(x_p^0, v_p t')$ there is a corresponding $\mathbb{D}(x_p(t'), v_p(t'' - t'))$ describing where the pursuer can be at time t''. To find the optimal position to be at t'', the pursuer must evaluate the resulting minimum possible team survival time given all possible positions in these disks resulting from different $x_p(t')$'s. The following remark can be made regarding the complexity of the optimal control problem faced by the pursuer.

Remark 4.1
The number of N dimensional optimization problems the pursuer has to solve to find the optimal strategy against a team of N evaders using the iterative open-loop approach for team evasion scales exponentially with the number of times the evaders update their joint heading.

Solving for the optimal pursuer strategy against a team of evaders utilizing the iterative open-loop approach for team evasion quickly becomes intractable even for a team with few

evaders and moderate amount of updates. This is beneficial for the evaders in that by updating their joint heading often, it is hard for the pursuer to determine its optimal action. However, this also makes measuring the worst-case team survival time performance of the iterative open-loop approach computationally intractable. Hence, a near optimal greedy strategy for the pursuer is constructed to evaluate the performance of the iterative open-loop approach quantitatively.

Definition 4.4 Greedy open-loop pursuer heading
Given $x_p(t)$ and $\mathbf{x_e}(t)$ as the positions of the pursuer and the evaders at time t and $\Theta_\mathbf{e}(t)$ as the joint heading of the evaders, the greedy open-loop pursuer heading is denoted by

$$\theta_p^{ol\star}(x_p(t), \mathbf{x_e}(t), \Theta_\mathbf{e}(t))$$

and is defined to be the initial heading of the pursuer that can achieve the minimum possible open-loop team survival time

$$\inf_{\mathbf{s}\in\mathcal{S}_N} \Gamma_s^\star(x_p(t), \mathbf{x_e}(t), \Theta_\mathbf{e}(t)).$$

The greedy pursuer strategy against iterative open-loop evaders is defined as

Definition 4.5 Greedy pursuer strategy against iterative open-loop evaders
Given that the evaders update their joint heading every Δt second starting from $t = 0$, the heading of the pursuer resulting from the greedy pursuer strategy is

$$\theta_p(t) = \theta_p^{ol\star}(x_p(n\Delta t), \mathbf{x_e}(n\Delta t), \Theta_\mathbf{e}(n\Delta t)) \text{ for } n\Delta t \leq t < (n+1)\Delta t, n \in \mathbb{N}^+. \tag{4.8}$$

With this strategy, at every update time the pursuer sets its heading to be optimal to the current layout and the current joint heading of the evaders under the assumption that the evaders will stay true to their current joint heading for the rest of the game. For the open-loop optimal joint heading of a specific layout, there often exist multiple optimal capture sequences that can achieve the same minimum possible team survival time in the open-loop setting. Due to the limited numerical resolution of the simulation environment, It is possible for the pursuer to switch between multiple open-loop optimal capture sequences during the chase. However, these capture sequences do not all result in the same team survival time in the iterative open-loop setting. Without actually simulating the rest of the game for each open-loop optimal capture sequences, there is no effective way for the pursuer to distinguish among these candidates of optimal capture sequence. To prevent unwanted switching which hurts the performance of the pursuer significantly, hysteresis is added to the pursuer's strategy so that given several candidates of the optimal capture sequence, the pursuer is encouraged to select the capture sequence that is consistent with the previous update.

Figure 4.3a shows the open-loop optimal trajectories of a layout with 3 evaders. The solid blue circles are the initial position of the evaders and the dark triangle is that of the pursuer.

(a) Open-loop

(b) Iterative open-loop
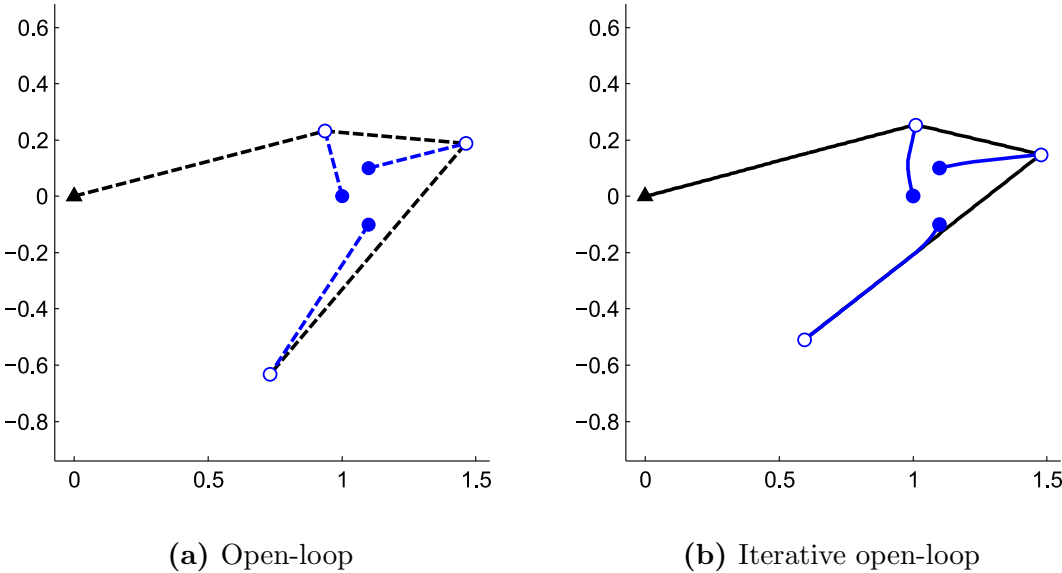
Figure 4.3: Resulting optimal trajectories of the open-loop and iterative open-loop approach for a 3-evader layout

(a) Open-loop

(b) Iterative open-loop
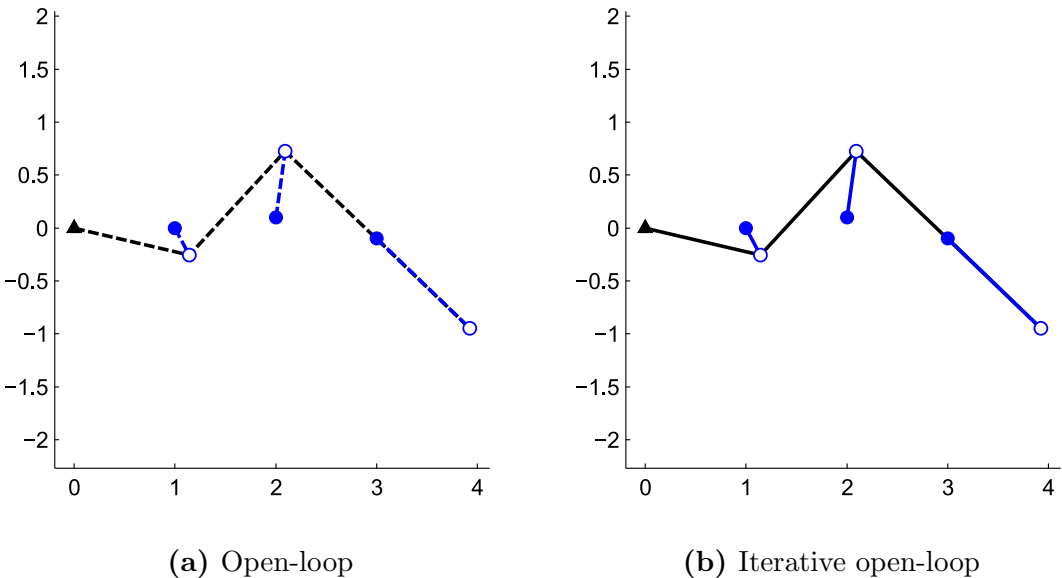
Figure 4.4: Resulting optimal trajectories of the open-loop and iterative open-loop approach for a 3-evader layout

The hollow circles are the open-loop optimal capture positions of the evaders and the dashed lines are the open-loop optimal trajectories of the pursuer and the evaders. Figure 4.3b shows the actual resulting trajectories with the evaders using the iterative open-loop approach and the pursuer using the greedy strategy. The hollow circles are the actual capture position of the evaders and the solid lines are the resulting trajectories of the pursuer and the evaders. The pursuer has a maximum speed of 1 and the evaders' maximum speeds are 0.25. Both the evaders and the pursuer adjust their headings every 0.01 seconds. Note that the resulting trajectories in Fig. 4.3b are curves and are different from the open-loop optimal trajectories of the initial layout shown in Fig. 4.3a. While the open-loop approach predicts an open-loop optimal team survival time of 2.59 seconds for the initial layout, the iterative open-loop approach actually achieves a team survival time of 2.63 seconds. The team survival time ratio of the iterative open-loop approach to that of the open-loop approach is 1.015. In this case, the iterative open-loop approach achieves a slightly better team survival time than the open-loop approach.

Figure 4.4 shows the open-loop and iterative open-loop trajectories of a different 3-evader layout. Note that in Fig. 4.4b the resulting trajectories are straight lines and are exactly the same as the open-loop optimal trajectories of the initial condition shown in Fig. 4.4a. In other words, for the 3-evader layout shown in Fig. 4.4, the iterative open-loop approach perform exactly the same as the open-loop approach in terms of team survival time. The ratio of team survival time is hence 1 and the open-loop approach is actually optimal in the iterative open-loop setting for this layout. The key difference between the layouts shown in Fig. 4.3 and Fig. 4.4 that results in different team survival time ratio is the uniqueness of the optimal capture sequence. In the layout shown in Fig. 4.4, the optimal capture sequence against the open-loop optimal joint heading with respect to the initial layout is unique. Hence, resulting joint heading of the evaders is optimized for one specific capture sequence and stays constant throughout the game against an optimal pursuer. In this case, the iterative open-loop approach can only achieve the same team survival time as the open-loop approach.

4.3 Approximation Approaches

As mentioned in Section 3.3.3, finding the open-loop optimal joint heading for the team of evaders in a given layout requires solving a N dimensional constrained non-liner optimization problem. Although standard nonlinear optimization technique can solve the problem in reasonable time for a small team, it can be time consuming for a team of moderate size. In this section, various approximated formulation of the open-loop team evasion problem are proposed to simplify or bypass the non-linear optimization problem in the interest of computation time.

4.3.1 Gradient of the Open-loop Team Survival Time

The approximation techniques that will be proposed in this section all make use of the gradient vector of the team survival time under a specific capture sequence with respect to the joint heading of the evaders. The definition and derivation of this gradient term is presented here.

The term $\Gamma_{\mathbf{s}}^{\star}(x_p, \mathbf{x_e}, \Theta_{\mathbf{e}})$ is defined in Eq. (2.14) as the minimum possible time it takes a pursuer starting at x_p with maximum speed v_p to capture evaders starting at $\mathbf{x_e} = (x_1, \ldots, x_N)$ with maximum speeds $\mathbf{v_e} = (v_1, \ldots, v_N)$ and constant joint heading $\Theta_{\mathbf{e}} = (\theta_1, \ldots, \theta_N)$ according to the capture sequence $\mathbf{s} = (s_1, \ldots, s_N)$. The following notations are defined to facilitate further discussion regarding the gradient of the team survival time under a specific capture sequence.

Definition 4.6 Minimum survival time of an evader under a capture sequence
Given x_p and $\mathbf{x_e}$ as the positions of one pursuer and N evaders, v_p and $\mathbf{v_e}$ as their maximum speeds, $\Theta_{\mathbf{e}}$ as the joint heading of the evaders, and $\mathbf{s} = (s_1, \ldots, s_N)$ as the capture sequence, the minimum survival time of the j-th evader in the capture sequence against an optimal pursuer that knows the exact control of the evaders is defined as

$$\tau_{s_j} = \Gamma_{\mathbf{s}'}^{\star}(x_p, \mathbf{x_e}, \Theta_{\mathbf{e}}), \ \ where \ \mathbf{s}' = (s_1, \ldots, s_j). \tag{4.9}$$

Following this definition, the minimum survival time difference between the j-th evader and the $(j-1)$-th evader in the capture sequence $\mathbf{s} = (s_1, \ldots, s_N)$ is denoted by $\hat{\tau}_{s_j}$ and defined as

$$\hat{\tau}_{s_j} = \tau_{s_j} - \tau_{s_{j-1}}. \tag{4.10}$$

Note that with this definition the minimum possible survival time of the j-th evader in the capture sequence can be expressed as

$$\tau_{s_j} = \sum_{k=1}^{j} \hat{\tau}_{s_k} \tag{4.11}$$

where $\hat{\tau}_{s_1}$ is defined to be zero. The minimum possible team survival time under a specific capture sequence can be expressed as

$$\Gamma_{\mathbf{s}}^{\star}(x_p, \mathbf{x_e}, \Theta_{\mathbf{e}}) = \sum_{j=1}^{N} \hat{\tau}_{s_j}. \tag{4.12}$$

The gradient of the team survival time with respect to the joint heading of evaders can be written as

$$\nabla_{\Theta_{\mathbf{e}}} \Gamma_{\mathbf{s}}^{\star} = \left[\sum_{j=1}^{N} \frac{\partial \hat{\tau}_{s_j}}{\partial \theta_1}, \sum_{j=1}^{N} \frac{\partial \hat{\tau}_{s_j}}{\partial \theta_2}, \ldots, \sum_{j=1}^{N} \frac{\partial \hat{\tau}_{s_j}}{\partial \theta_N} \right], \tag{4.13}$$

(a) Layout and Optimal Trajectories   **(b)** Gradient Vectors
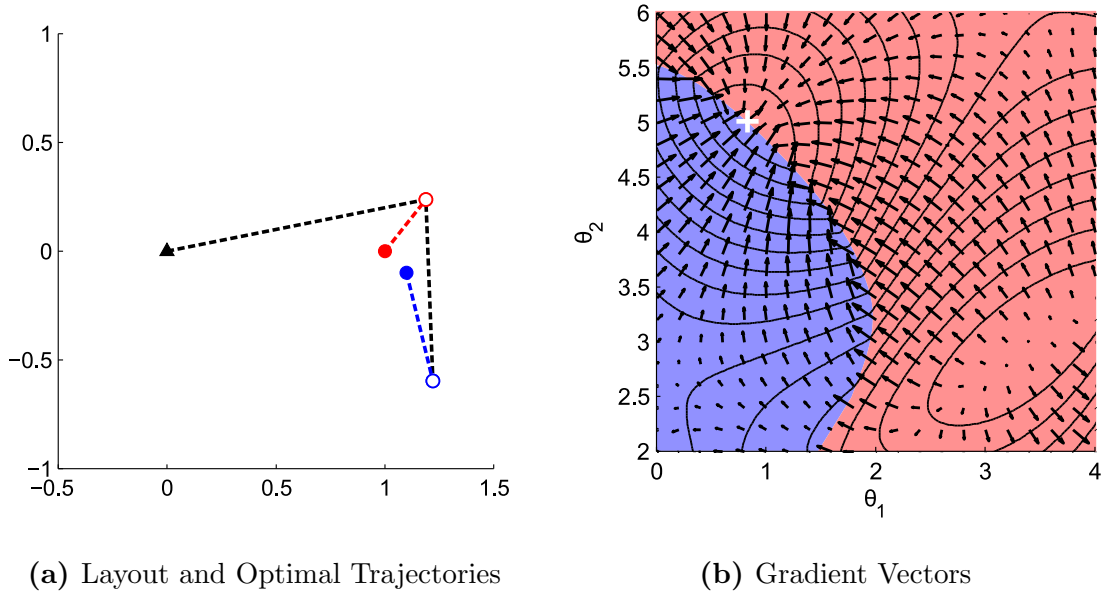
Figure 4.5: The gradient vectors $\nabla_{\Theta_e} \Gamma_{\mathbf{s}^\star}^\star$ given different joint heading of evaders for a 2-evader layout

(a) Layout and Optimal Trajectories   **(b)** Gradient Vectors
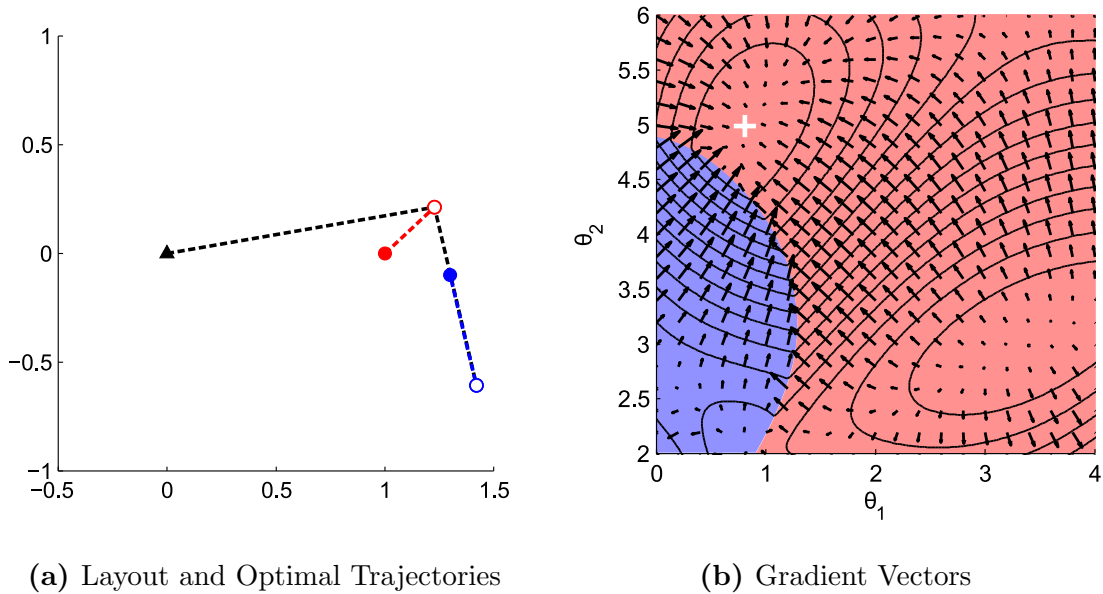
Figure 4.6: The gradient vectors $\nabla_{\Theta_e} \Gamma_{\mathbf{s}^\star}^\star$ given different joint heading of evaders for a 2-evader layout

which is a row vector of length N. The details of the derivation and computation of this gradient term is provided in Appendix A.

Figures 4.5b and 4.6b illustrate the direction and magnitude of the gradient vector given different joint headings of the evaders for the 2-evader layouts shown in Figs. 4.5a and 4.6a respectively. More specifically, the arrows represent the magnitudes and directions of the gradient vectors $\nabla_{\Theta_{\mathbf{e}}} \Gamma^\star_{\mathbf{s}^\star}$ given the joint heading $\Theta_{\mathbf{e}} = (\theta_1, \theta_2)$. The color of the regions in Figs. 4.5b and 4.6b are determined by the optimal capture sequence. In the red regions, the optimal capture sequence $\mathbf{s}^\star$ given the joint heading is to capture the red evader first and then the blue evader; in the blue regions, the optimal capture sequence is to capture the blue evader first and then the red evader. The white cross in each figure indicates the open-loop optimal joint heading for the evaders given the corresponding layouts. The trajectories shown in Figs. 4.5a and 4.6a are the optimal trajectories resulting from the open-loop optimal joint headings.

Note that in Fig. 4.5b the optimal joint heading is located within the red region, which indicates that there exist a unique optimal capture sequence with respect to the optimal joint heading. The gradient vectors point in the general direction of the optimizer and their magnitudes decrease near the optimizer. In Fig. 4.6b, the optimal joint heading is located on the boundary of both the red and blue region. As a result there exists multiple capture sequences that can achieve the minimum possible team survival time against the open-loop optimal joint heading. In this case, the gradient vectors still point in the general direction of the optimizer, however, unlike in Fig. 4.6a, their magnitudes do not vanish near the optimizer. Also, the gradient vectors in this case often point outward near the boundary of the regions. In both of these cases, the gradient vectors behave "nicely" near the optimizer in that a point started nearby will be drawn to the optimizer by following the gradient vector field. These characteristics will be exploited in the gradient-based method, which will be introduced in Section 4.3.3.

4.3.2 Iterative Linear Programming Approach

In the open-loop approach for collaborative team evasion proposed in Section 3.3.3, a team of evaders solves the sup-inf optimization problem

$$\sup_{\Theta_{\mathbf{e}}} \inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma^\star_{\mathbf{s}}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}})$$

for the open-loop optimal joint heading. By introducing a dummy variable z to represent the team survival time as the objective function, the sup-inf optimization problem can be converted to the following constrained optimization problem:

$$\sup_{\Theta_{\mathbf{e}}, z} z \tag{4.14a}$$

$$\text{s.t. } z \leq \Gamma^\star_{\mathbf{s}}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}}) \text{ for all } \mathbf{s} \in \mathcal{S}_N. \tag{4.14b}$$

The constraints listed in Eq. (4.14b) are nonlinear since the function $\Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}})$ is nonlinear in $\Theta_{\mathbf{e}}$. By making use of the gradient of the team survival time with respect to the joint heading of the evaders derived in the previous section, the constraints can be approximated with linear ones. The resulting optimization problem is

$$\sup_{\Theta_{\mathbf{e}}, z} z \tag{4.15a}$$

$$\text{s.t. } z \leq \nabla_{\Theta_{\mathbf{e}}} \Gamma_{\mathbf{s}}^{\star} \bigg|_{(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}}^0)} \cdot (\Theta_{\mathbf{e}} - \Theta_{\mathbf{e}}^0) + \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}}^0), \forall \mathbf{s} \in \mathcal{S}_N \tag{4.15b}$$

$$\left\| \Theta_{\mathbf{e}} - \Theta_{\mathbf{e}}^0 \right\|_{\infty} \leq \Delta \theta, \tag{4.15c}$$

where $\Theta_{\mathbf{e}}^0$ denotes the joint heading of the evaders the constraints are linearized about and the term $\Delta \theta$ is the step size of the updates in joint heading. The right-hand side of the constraints in Eq. (4.15b) are the Taylor expansions to the first order of $\Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}})$ with respect to the joint heading $\Theta_{\mathbf{e}}$ at the specific layout and joint heading $(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}}^0)$. Note that although the Taylor expansion is taken with respect to the joint heading $\Theta_{\mathbf{e}}$, the resulting constraints in Eq. (4.15b) are also linear in the decision variable of the optimization problem $(\Theta_{\mathbf{e}}, z)$. The constraint on the step size of the joint heading defined in Eq. (4.15c) are also linear in terms of $(\Theta_{\mathbf{e}}, z)$. They are added to limit the solution of the optimization problem to a local region of the current joint heading since the linear approximation is only valid locally. The constraints on step size also prevent the solution of the optimization problem from going to infinity. Since both the constraints and the objective function are linear in terms of the decision variables $(\Theta_{\mathbf{e}}, z)$, Eq. (4.15) is a linear program (LP) and can be solved very efficiently by conventional LP techniques such as the simplex method [48].

The iterative linear programming approach for team evasion is implemented similarly to the original iterative open-loop approach. At each update time, a new linear program is formulated with the constraints linearized about $(x_p(n\Delta t), \mathbf{x}_{\mathbf{e}}(n\Delta t), \Theta_{\mathbf{e}}(n\Delta t))$, the most current layout and joint heading of the evaders. The resulting team survival time and computation time performance of the iterative linear programming approach are presented in Section 4.4.

4.3.3 Gradient-Based Approach

The iterative linear programming approach proposed in the previous section replaces the nonlinear optimization problem in the iterative open-loop approach with a linear program. However, to generate the linear program, the gradients of the team survival time for all possible capture sequences have to be computed. Although a linear program with large amount of constraints can be solved fairly efficiently, for a team with moderate amount of evaders the time it takes to compute all the gradient terms in order to construct the constraints quickly surpasses the time it takes to solve the linear program. Furthermore, like any other constrained optimization problem with large amount of constraints, a lot of the constraints are redundant in that ignoring them will not affect the optimizer; only the

constraints that are active at the optimizer will actually effect the solution to the optimization problem. This implies that a lot of the computation spent on constructing the gradient terms are wasted since most of the constraints are not active at the optimizer and do not affect the solution at all. Also, although the iterative linear programming approach is already much more computationally efficient than the original iterative open-loop approach which requires solving a non-linear optimization problem at every update time, solving a LP still requires a certain level of computing power which might not be available in some light weight autonomous vehicles such as micro UAVs. The gradient-based approach is designed to bypass most of the gradient computation and the need of an optimization solver. The approach can be described as follows.

Given that a team of evaders adjusts their joint heading every Δt time starting from $t = 0$, the joint heading resulting from the gradient-based approach is

$$
\Theta_{\mathbf{e}}(t) = \Theta_{\mathbf{e}}(n\Delta t) + \Delta\theta\Delta t \frac{\nabla_{\Theta_{\mathbf{e}}}\Gamma_{\mathbf{s}^\star}^\star\Big|_{(x_p(n\Delta t),\mathbf{x_e}(n\Delta t),\Theta_{\mathbf{e}}(n\Delta t))}}{\left\|\nabla_{\Theta_{\mathbf{e}}}\Gamma_{\mathbf{s}^\star}^\star\Big|_{(x_p(n\Delta t),\mathbf{x_e}(n\Delta t),\Theta_{\mathbf{e}}(n\Delta t))}\right\|_\infty}
$$
$$
\text{for } n\Delta t \le t < (n+1)\Delta t, n \in \mathbb{N}^+ \quad (4.16)
$$

where
$$
\mathbf{s}^\star = \underset{\mathbf{s}\in\mathcal{S}_N}{\arg\inf}\; \Gamma_{\mathbf{s}}^\star(x_p(n\Delta t), \mathbf{x_e}(n\Delta t), \Theta_{\mathbf{e}}(n\Delta t)) \quad (4.17)
$$

is the optimal capture sequence for the layout and joint heading of the evaders at the update time. The term $\Delta\theta$ and the update interval Δt determine the step size of the adjustment in the joint heading space in terms of infinity norm. Note that no evader can change its heading more than the amount of $\Delta\theta\Delta t$ at each update, hence the parameter $\Delta\theta$ can be interpreted as the maximum turning rate of an evader. The joint heading of the evaders is kept constant between updates according to the definition.

Compared to the iterative linear programming approach, in which the gradient term for all possible capture sequences are computed, in the gradient-based approach only the gradient term for the optimal capture sequence $\mathbf{s}^\star$ is computed. Furthermore, no optimization solver is needed for the implementation of the gradient-based approach. These traits improve the computation time performance significantly. The resulting performance in survival time and computation time of the gradient-based approach are presented in Section 4.4.

4.3.4   Constraint Sampling Heuristics

In the gradient-based approach proposed in the previous section, while the gradient term only has to be computed for the optimal capture sequence, the resulting team survival time under all possible capture sequences still has to be evaluated to determine the optimal capture

sequence. Due to the factorial growth of the number of possible capture sequences with the number of evaders in the team, the gradient-based approach at its current state still can not be implemented for a team with a large amount of evaders. The constraint sampling techniques that will be presented in this section are designed to address this issue.

Given x_p and $\mathbf{x_e}$ as the position of the pursuer and the evaders and $\Theta_\mathbf{e}$ as the joint heading of evaders, the optimal capture sequence is defined as

$$\mathbf{s}^\star(x_p, \mathbf{x_e}, \Theta_\mathbf{e}) = \arg\inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_\mathbf{s}^\star(x_p^0, \mathbf{x_e}, \Theta_\mathbf{e}).$$

The task of determining $s^\star$ can be viewed as a search problem on a tree. The root node of the tree is the initial state tuple $(x_p, \mathbf{x_e}, \Theta_\mathbf{e})$ with N evaders. From this node, there are N possible descendant nodes, each representing the resulting state of the game when one of the N evaders is captured. From each of these descendant nodes, there are then $N - 1$ descendant nodes based on which of the $N - 1$ evaders left is captured. The tree grows in this fashion factorially. A transition cost is set for each transition between a parent node and a descendant node as the minimum time it takes the pursuer to capture the evader that is in the parent node but not the descendant node. There are $N!$ final nodes where all the evaders are captured, each path from the root node to the final node represents a specific capture sequence; the sum of transition cost on such path is the minimum possible team survival time under that capture sequence. The heuristics that are used to find the optimal capture sequence are presented as follows.

All-Sequence (All-Seq)

The naive way to find the optimal capture sequence is to evaluate the minimum capture time of all the possible capture sequences. This process can be parallelized by splitting the capture sequences into N groups of $(N - 1)!$ capture sequences according to the first evader in the capture sequence. Each evader is then only in charge of evaluating the capture time of $(N - 1)!$ capture sequences. A randomly assigned evader can then compare the results, each from one of the N groups and determine the optimal capture sequence. With this parallelism the computation time is proportional to $(N - 1)!$ instead of $N!$. However, as the number of evaders increases, this quickly becomes computationally intractable.

For computational tractability with a large team of evaders, some pruning techniques with heuristics are utilized while traversing the tree so that only a subset of the nodes are evaluated. For each state we use the heuristics to pick at most K of the possible descendant nodes to be expanded further. Here K is the branching factor of the heuristic and there will be at most N^K capture sequences evaluated. Among these N^K captures sequences, the one with the minimum capture time will be selected as the approximate optimal capture sequence. Three different heuristics for picking the K descendant are presented as follows.

Stepwise-Distance-K (SD-K)

This heuristic evaluates the distance from the pursuer to the evaders and picks the K closest evaders that are still not captured. Note that when $K = 1$, this heuristic results in a step-wise greedy capture sequence where the pursuer always captures the closest evader first. With this heuristic, only the distances between the pursuer and the evaders are taken into account and the headings of the evaders are ignored.

Stepwise-Time-K (ST-K)

This heuristic compares the minimum time it takes for the pursuer to capture each evader that is still alive, and picks the K evaders that can be captured the soonest. Compared to the SD-K, which only makes use of the positions of the evaders, this heuristic takes into account both the positions and the headings of the evaders.

Stepwise-Sum-of-Distance-K (SSD-K)

This heuristic compares the sum of distances from the pursuer to all evaders that are still alive when a capture happens. The evaders that will result in the K largest sum of distance to other evaders when captured by the pursuer are selected. This heuristic not only takes into account the positions and headings of the evaders at the current state, but also their positions when the next capture happens.

With these heuristics, the number of capture sequences that has to be evaluated in the gradient-based approach drops from $N!$ to K^N. Although it is still exponential in terms of the number of evaders in the team, with a small value of K, it does allowed the approach to be applied to teams with larger number of evaders without requiring too much computation time.

4.4 Results and Discussion

The survival time and computational time of the different approaches proposed in this chapter are evaluated and discussed in detail in this section. The same benchmarking dataset with 500 randomly generated layouts for teams with 2, 3, 4, and 5 as mentioned in the previous chapters is used in the simulations. In all simulations, the pursuer uses the greedy open-loop strategy as described in Section 4.2.3. The update interval Δt is set to be 0.01 seconds. Although in the original formulation, an evader is only captured when its distance to the pursuer is zero, due to the limited numerical resolution of the simulation environment, the capture distance is set to 0.004 units. The simulation environment and the algorithms are implemented in MATLAB with the functions that are time-critical converted to C code in the

form of `.mex` files with the help of MATLAB Coder[1]. The simulations are run on a laptop with Intel® Core™ i5-4300 CPU and 8 gigabytes of RAM.

4.4.1 Team Survival Time

The performance of each proposed approach in terms of the team survival time are presented and discussed in detail in this section.

Iterative Open-loop Approach

The open-loop approach to team evasion proposed in Chapter 3 maximizes the minimum possible team survival time for the evaders with respect to the initial condition of the game under the assumption that they cannot change their headings afterwards. The iterative open-loop approach proposed in this chapter relaxes the conservatism of the open-loop approach by setting the joint heading of the evader to the open-loop optimal joint heading with respect to the most current state of the game at each update time. With the iterative open-loop method, it is possible for the evaders to achieve a longer team survival time that is higher than the one predicted by the open-loop approach according to the initial condition of the game.

The ratio of the team survival time achieved by the iterative open-loop approach to that of the open-loop approach is used to evaluate the effectiveness of the iterative open-loop approach relative to the open-loop approach. Figure 4.7 shows the distributions of the ratio of the team survival time over the 500 layouts in the benchmarking dataset for teams with $2, 3, 4$, and 5 evaders. As shown in the figures, this ratio is always bigger than or equal to one, indicating that the iterative open-loop approach can always achieve at least the same, and often better, team survival time predicted by the open-loop approach. Recall that in Section 4.2, it has been shown that when the optimal capture sequence against the open-loop optimal joint heading of the evaders is unique, the optimal capture sequence remains the same during the game if the pursuer acts optimally. When this happens, the iterative open-loop approach can only perform equally as well as the open-loop approach against an optimal pursuer. Figures 4.7a to 4.7d shows that the number of times that the iterative open-loop approach performs only equally as well as the open-loop approach decreases rapidly when there are more evaders in the team. This indicates that with more evaders, it is more likely for multiple open-loop optimal capture sequences to exist for the open-loop optimal joint heading. As mentioned before, these open-loop optimal capture sequences that can achieve the same optimal team survival time in the open-loop setting, does not perform equally well in the iterative open-loop setting. There is no efficient way for the pursuer to determine the optimal capture sequence in the iterative open-loop setting; the pursuer has to either simulate the whole chase for each possible capture sequence, or rely on heuristics. As a result, having multiple open-loop optimal capture sequence is actually beneficial to the evaders in

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[1]`http://www.mathworks.com/products/matlab-coder/`

(a) 2 evaders

(b) 3 evaders

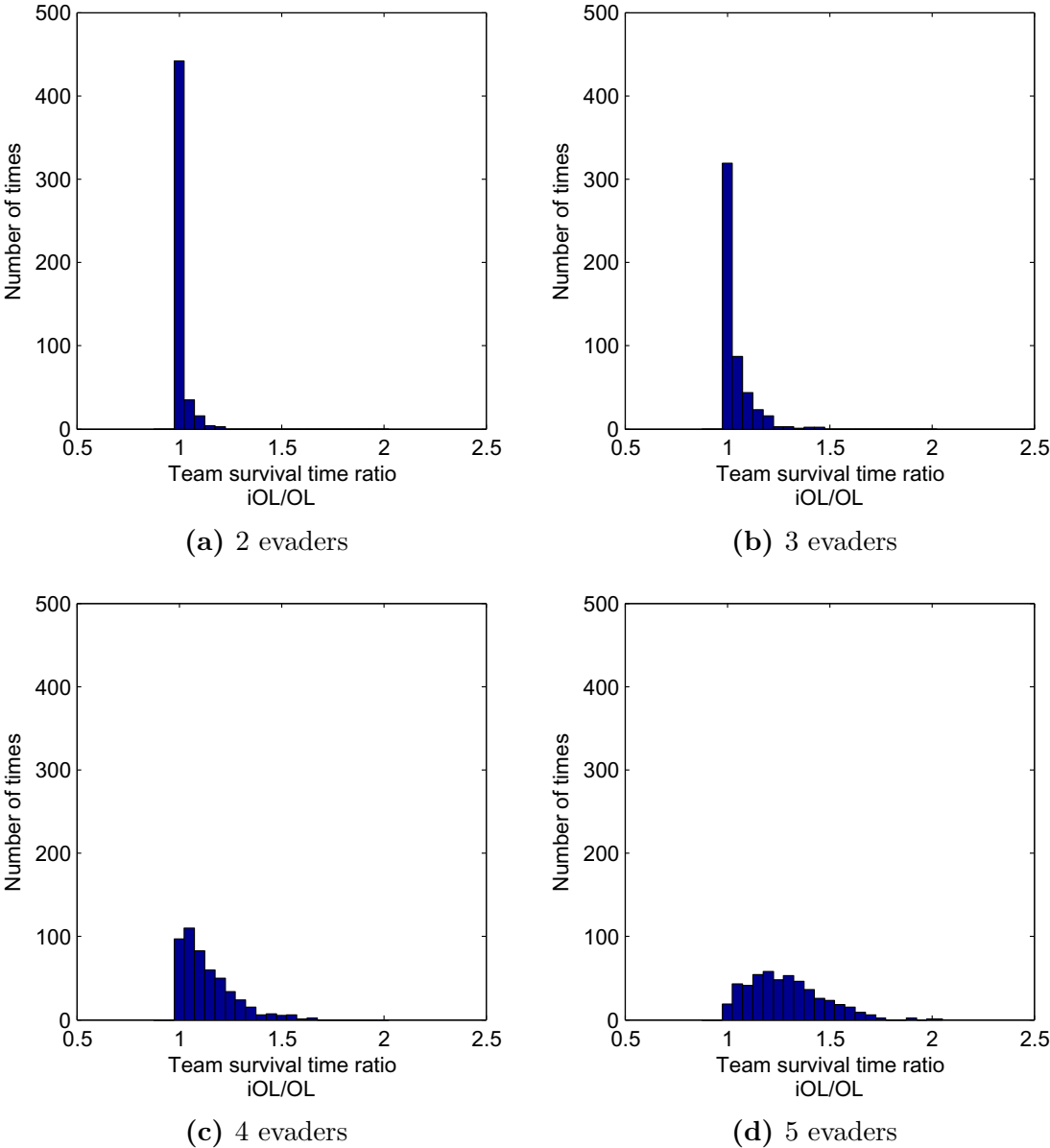**(c)** 4 evaders

(d) 5 evaders

Figure 4.7: Distribution of the team survival time ratio of the iterative open-loop approach to the open-loop approach over 500 layouts for teams with different number of evaders

the iterative open-loop setting in that it makes it harder for the pursuer to determine the optimal capture sequence.

It can also be observed from Figs. 4.7a to 4.7d that not only does the iterative open-loop approach perform better than the open-loop approach more often when there are more evaders, it also achieves a higher ratio of team survival time on average. This shows that a team with more evaders benefits more from the relaxed conservatism of the iterative open-loop approach than a smaller team.

Iterative Linear Programming Approach

The iterative linear programming (iLP) approach approximates the iterative open-loop approach by replacing the constrained nonlinear optimization problem that has to be solved at each update with a linear program. The performance of the iterative linear programming approach relative to the original iterative open-loop approach is measured by the ratio of the resulting team survival time. Figure 4.8 shows the distribution of the ratio over the benchmarking set for teams with $2, 3, 4,$ and 5 evaders. In all simulations, the step size of the iLP approach in the joint heading space, denoted by $\Delta\theta$ in Eq. (4.15c), is set to be π (rad/s), which means that at each update the maximum change in heading for each evader is $\pi \times \Delta t$ (rad) where Δt is the update interval which is set to 0.01.

In Figs. 4.8a to 4.8d, the ratios of team survival time are all very close to one, indicating that the iterative linear programming approach performs similarly as the iterative open-loop approach. This shows that the linearized constraints in Eq. (4.15b) and the limitation on the step size in the joint heading space in Eq. (4.15c) successfully approximate the behavior resulting from the nonlinear optimization problem in the original iterative open-loop approach. The survival time performance of the iterative linear programming approach does degrade slightly when there are more evaders in the team as shown in Figs. 4.8a to 4.8d The very few cases in each of the team size where the iterative linear programming approach outperforms the iterative open-loop approach are caused by the sub-optimality of the greedy pursuer strategy which is used to measure the team survival time of both approaches.

Gradient-Based Approach

The gradient-based approach (GB) approximates the iterative open-loop approach by updating the joint heading of the evaders in the direction of the gradient of the team survival time under the optimal capture sequence. Figure 4.9 shows the team survival time ratio of the gradient-based approach to that of the original iterative open-loop approach. The step size in the joint heading space, denoted by $\Delta\theta$, is set to the same value π (rad/s) as in the iterative linear programming approach. Compared to the performance in team survival time of the iterative linear programming approach shown in Fig. 4.8, the gradient-based approach perform slightly worse than the iterative linear programming approach in terms of the number of times it achieves a team survival time ratio of 1. Also, the ratio of team survival time

(a) 2 evaders

(b) 3 evaders

(c) 4 evaders

(d) 5 evaders

Figure 4.8: Distribution of the team survival time ratio of the iterative linear programming approach (iLP) to that of the iterative open-loop approach (iOL) over 500 layouts for teams with different number of evaders

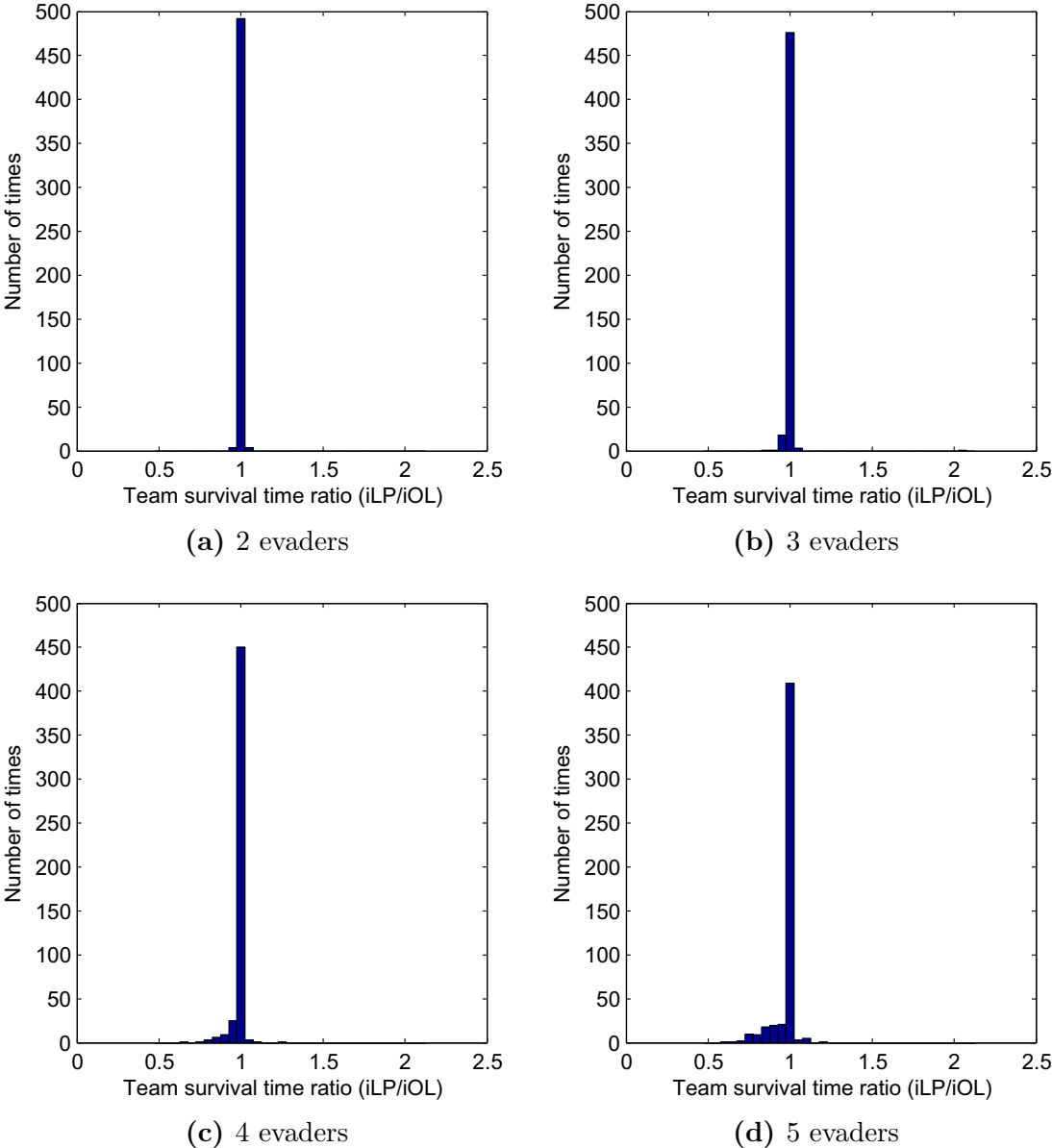**(a)** 2 evaders

(b) 3 evaders
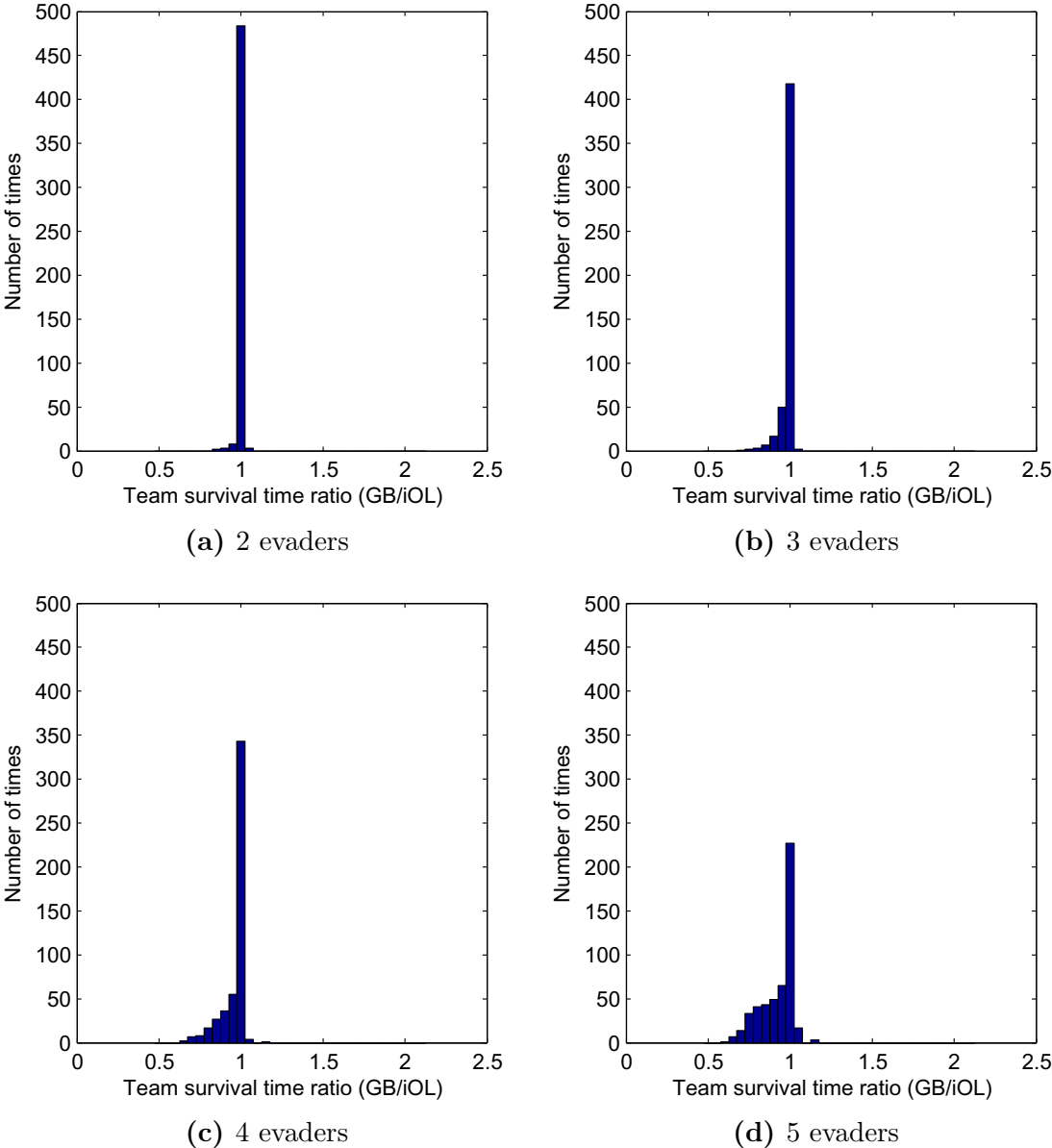
(c) 4 evaders

(d) 5 evaders

Figure 4.9: Distribution of the team survival time ratio of the gradient-based approach (GB) to that of the iterative open-loop approach (iOL) over 500 layouts for teams with different number of evaders

degrades more severely than the iterative linear programming approach when there are more evaders in the team.

The slight drop in survival time performance compared to the iterative linear programming approach is to be expected in that the gradient-based approach can be viewed a simplified version of the iterative linear programming approach. While the iterative linear programming approach makes use of the gradient information of the team survival time under all possible capture sequences, the gradient-based approach only makes use of the gradient information of that of the optimal capture sequence. Actually, if all the constraints of the iterative linear programming approach listed in Eq. (4.15b) are ignored except the one for the optimal capture sequence, the iterative linear programming approach is exactly the same as the gradient-based approach. Although the gradient-based performs slightly worse than the iterative liner programming approach in terms of the team survival time, the main benefit of the approach lies in its much shorter computation time, which will be presented in Section 4.4.2.

Gradient-Based Approach with Constraint Sampling

The constraint sampling heuristics proposed in Section 4.3.4 are designed to further decrease the computation required for the gradient-based approach by evaluating only a subset of the capture sequences when determining the optimal one. Their impact on the team survival time are evaluated on the benchmarking dataset and discussed here.

Figure 4.10 shows the average ratio of the team survival time of the gradient-based approach with different constraint sampling heuristics to that of the iterative open-loop approach. The bars labeled All-Seq are the averaged team survival time ratios when all possible capture sequences are taken into account during the selection of the optimal capture sequence. The bars labeled SD-K, ST-K, and SSD-K are the team survival time performance of the Stepwise-Distance-K, Stepwise-Time-K, and Stepwise-Sum-of-Distance-K heuristics respectively with K being their branching factor. For example, SC-2 represents the heuristics where only the 2 closest evaders to the pursuer are considered during each step of the tree search as described in Section 4.3.4.

For the 2-evader cases, the gradient-based approach performs almost the same as the iterative open-loop method. The performance degrades slightly as the number of evaders increases. The All-Seq method, which is the gradient-based approach without constraint sampling, achieves about 90% of the survival time achieved by the iterative open-loop method in the 5-evader layouts. As for the performance of the constraint sampling heuristics with a branching factor of 1, none of the heuristics performs as well as the brute-force All-Seq method. However, with a branching factor of 2, all three heuristics perform very similarly to the brute-force approach.

The survival time performance of the gradient-based approach are affected by the initial joint heading of the evaders. The results in Fig. 4.10a are gathered from the best-case scenarios where the evaders are initialized with the open-loop optimal joint heading with respect to

the initial layouts. Note that to determining these open-loop optimal joint headings requires solving the constrained nonlinear optimization problem formulated in Eq. (4.14). This is in a way in conflict with the emphasis on short computation time and ease of implementation of the gradient-based approach. Hence, a different set of simulations were run on the benchmarking dataset where the initial headings of the evaders are set to be pointing away from the pursuer instead of the open-loop optimal headings of the layout. The results are shown in Figure 4.10b. As expected, since the evaders are not initialized optimally, the performance in team survival time degrades slightly compared to the optimally initialized cases. The trend of performance degradation with increasing number of evaders remains the same. However, the average survival time is still within 20% of that of the iterative open-loop approach even with a team of 5 evaders. Figure 4.10b also shows that regardless of how the evaders are initialized, there is not a significant difference in team survival time performance between the three different heuristics. This implies that none of these is particularly worse than the others. In this case the SD-K heuristics are preferred since they require the least computation.

4.4.2 Computation Time

The iterative linear programming approach and the gradient-based approach provide different levels of approximation and simplification to the original iterative open-loop approach. The constraint sampling heuristics are used to further decrease the computation required for the gradient-based approach. The computation time is measured as the time it takes an algorithm to return the resulting joint heading of the evaders in an update. This computation time is averaged over all updates in the simulations ran on the benchmarking dataset. Note that the computational overheads introduced by the simulating environment, such as the time it takes to evolve the state of the game given the headings of the agents or the time it takes to record the simulation results are not included in the computation time of the algorithms.

For the iterative open-loop approach, the constrained nonlinear optimization problem is solved by the `fminimax` function of the Optimization Toolbox in MATLAB. The computation time of the iterative open-loop approach is used as a baseline to which other approaches are compared. The open-loop optimal joint heading for each initial condition in the benchmarking dataset is precomputed off-line with re-sampling. All approaches are initialized with the open-loop optimal joint heading at the beginning of each simulation.

Iterative Linear Programming Approach

The linear program formulated by the iterative linear programming approach at each update is solved by the `linprog` function of the Optimization Toolbox in MATLAB. The step size parameter $\Delta\theta$ is set to be π (rad/s) and the update time Δt is set to 0.01.

Figure 4.11 shows the ratio of the averaged computation time of the iterative linear programming approach (iLP) to that of the iterative open-loop approach (iOL). The dark circles in the figure represent the mean computation time over all 500 runs and the error bars

represent plus and minus one standard deviation. As shown in the figure, the computation time of the iterative linear programming approach is about 1/5 of that of the iterative open-loop approach. Combined with the almost identical team survival time performance to that of the iterative open-loop approach as shown in Fig. 4.8, the proposed iLP approach is a highly effective approximation. Also, the ratio of computation time shows a gradual trend of decrease when there are more evaders in the team, indicating that the gain in computation time performance of the iterative linear programming approach is more pronounced when there are more evaders in the team.

Gradient-Based Approach

The majority of the computation time of the iterative linear programming approach are spent on formulating the linear program, which requires the evaluation of the gradient of team survival time under all possible capture sequences. The gradient-based approach bypasses these computational burden by evaluating only the gradient of team survival time of the optimal capture sequence. Furthermore, no optimization routine is needed for the implementation of the gradient-based approach. The algorithm is implemented as `.mex` functions which are compiled `C` codes converted from MATLAB functions by the MATLAB Coder.

Figure 4.12 shows the ratio of the averaged computation time of the gradient-based approach (GB) to that of the iterative open-loop approach (iOL) and that of the iterative linear programming approach (iLP). The dark circles in the figures are the mean of the averaged computation time of the 500 simulations and the error bars represent plus and minus one standard deviation of the averaged computation time. In Fig. 4.12a, the mean of the ratio is between 1/150 and 1/200; in Fig. 4.12b the mean of the ratio is around 1/30. As expected, the gradient-based method is much more computationally efficient than the iterative open-loop approach and also more efficient than the iterative linear programming approach.

In terms of team survival time performance, the gradient-based approach is slightly worse than the iterative iLP approach. However, due to its significantly lower demand on computational power and its ease of implementation, the gradient-based approach is ideal to be implemented on teams composed of simpler autonomous vehicles.

Gradient-Based Approach with Constraint Sampling Heuristics

The main bottleneck of scaling the gradient-based approach to teams with large number of evaders is the need to evaluate the open-loop team survival time resulting from all possible capture sequences which grows factorially with the number of evaders in the team. The constraint sampling heuristics are designed to evaluate only a subset of all the possible capture sequences to increase the computation time performance.

Figure 4.13 shows the ratio of the average computation time per step of the gradient-based approach with various constraint sampling heuristics to that of the iterative open-loop approach. With the heuristics, the gradient-based approach is more than 100 times faster than the iterative open-loop approach. For 2 to 4 evaders, the heuristics with a branching factor of 2 takes about as long or even longer to compute than the All-Seq brute-force approach. This is due to the computational overheads induced by the necessary bookkeeping of the tree search process that is not excluded from the computation time. As the number of evaders increases, the effects of these overheads become less and less significant. For 5-evader cases, the heuristics with $K = 2$ outperforms the brute-force All-Seq approach in computational time. Similar to the survival time performance, there is not a significant difference between the computation time performance of the three different heuristics.

4.5 Conclusion

The iterative open-loop approach to collaborative team evasion proposed in this chapter extends the open-loop approach to team evasion by adjusting the joint heading of the evaders at a predefined frequency according to the open-loop optimal joint heading with respect to the most current state of the game. Compared to the open-loop approach proposed in Chapter 3, which is conservative towards the evaders and uses only the open-loop optimal joint heading with respect to the initial condition of the game, the iterative open-loop approach relaxes the conservatism and is capable of achieving even better team survival time. A greedy pursuer strategy against the iterative open-loop evaders is developed to quantitatively evaluate the team survival time performance of the proposed approach. Through extensive simulations, it has been shown that the iterative open-loop approach can indeed enhance the team survival time in most of the cases and the improvement is more pronounced for teams with more evaders.

An algorithm that can compute the gradient vector of the minimum possible team survival time under a specific capture sequence with respect to the joint heading of the evaders is also proposed in this chapter. Several approximations to the iterative open-loop approach, including the iterative linear programming approach and the gradient-based approach, rely on this gradient term to lower the required computation of the original iterative open-loop approach. The iterative linear programming approach approximates the nonlinear optimization problem that has to be solved at each update in the iterative open-loop approach with a linear program and drastically decreases the computation time. The gradient-based approach further decreases the computation time of the iterative linear programming approach through bypassing the need to compute the gradient terms for all possible capture sequences and the need to solve an optimization problem at every update. The constraint sampling heuristics can further decrease the computation time of the gradient-based approach by evaluating only a subset of the capture sequences when determining the optimal one. Simulation results show that all of these approximations trade some performance in team survival time for less computation; the loss in team survival time performance and the gain

in computation time performance are both more prominent for teams with more evaders. Compared to the original iterative open-loop approach, the iterative linear programming approach achieves 99 percent of the team survival time on average while spending only 1/5 the computation time; the gradient based approach achieves around 96 percent of the team survival time on average while spending around 1/150 of the computation time.

In conclusion, the iterative open-loop approach to collaborative team evasion is capable of achieving better team survival time while keeping the guarantees of the open-loop approach. The approximation approaches provide good trade-offs between team survival time and computation time. With the proposed approximation approaches the collaborative team evasion framework can be implemented on autonomous agent teams with lower computation power.
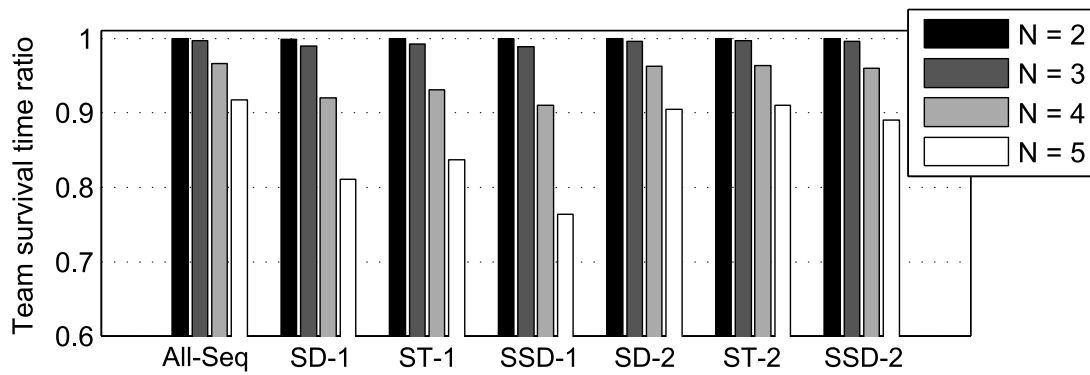
(a) Evaders initialized with open-loop optimal joint heading
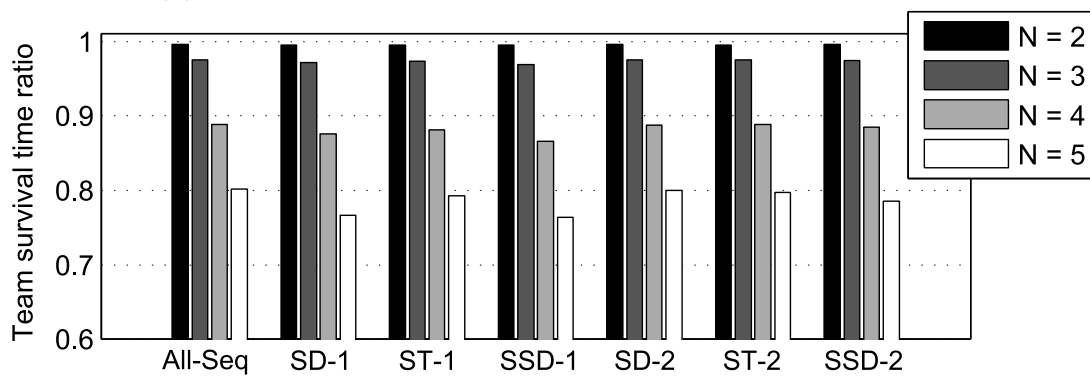
(b) Evader initialized with headings pointing away from the pursuer

Figure 4.10: Averaged ratio of team survival time of different constraint sampling heuristics to that of the iterative open-loop approach
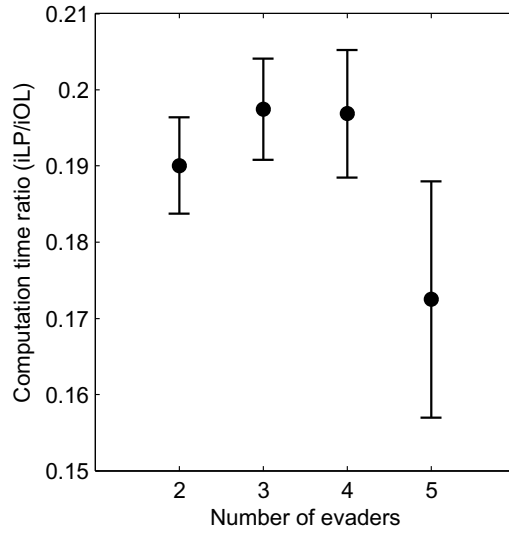
Figure 4.11: Averaged ratio of computation time per step of the iterative linear programming approach (iLP) to that of the iterative open-loop approach (iOL)
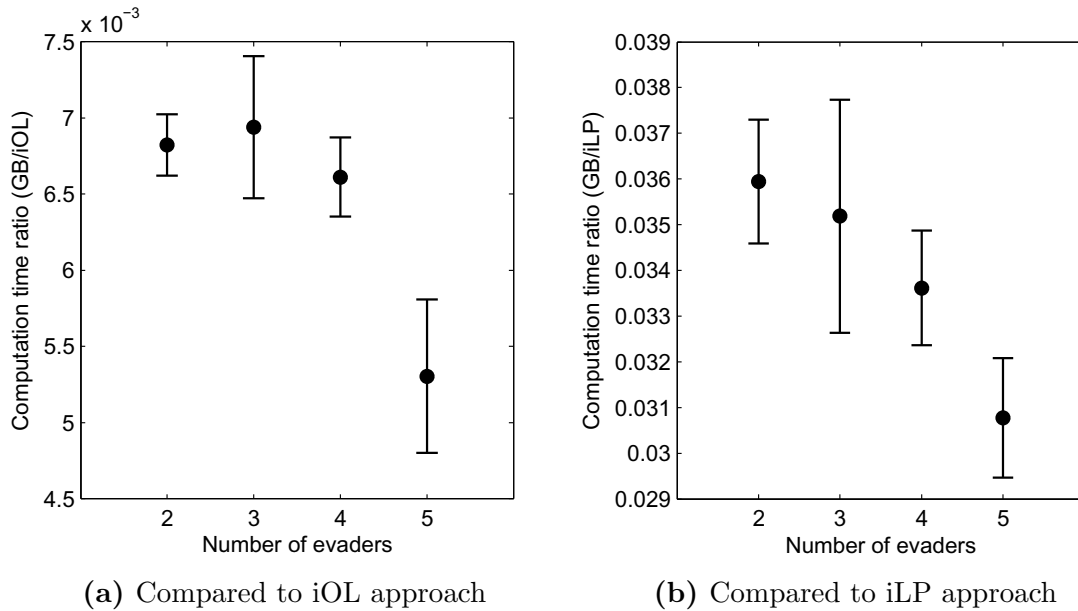
(a) Compared to iOL approach

(b) Compared to iLP approach

Figure 4.12: Averaged ratio of computation time per step of the gradient-based method (GB) to that of the iterative open-loop approach (iOL) and that of the iterative linear programming approach (iLP)
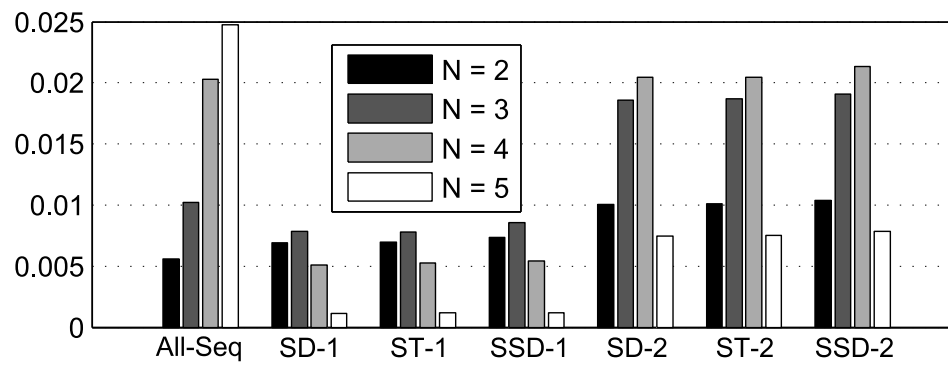
Figure 4.13: Averaged ratio of the computation time per step of the gradient-based approach with different constraint sampling heuristics to that of the iterative open-loop (iOL) approach

Chapter 5

Extensions for Collaborative Team Evasion

5.1 Introduction

The open-loop and iterative open-loop approaches to team evasion proposed in Chapter 3 and Chapter 4 are based on two assumptions that are not necessarily true in realistic team evasion scenarios: the exact position of the pursuer are known to the evaders at all time and the evaders have unlimited turning rates. Also, the objective function that can be handled by the collaborative team evasion framework is limited to the team survival time. In this chapter, several extensions to the collaborative team evasion framework are presented to address these limitations. In Section 5.2, the framework is extended to handle uncertainties in the position of the pursuer. In Section 5.3, modification to the formulations are made to incorporate constraints on the turning rates of the evaders. Section 5.4 explores and evaluates the applicability of the framework to accumulative survival time. Finally, Section 5.5 concludes this chapter.

5.2 Collaborative Evasion Against a Hidden Pursuer

In the open-loop and iterative open-loop approaches for team evasion proposed in the previous chapters, it was assumed that the position of the pursuer is known to the evaders at all time. However, in a more realistic pursuit-evasion scenario, the evaders seldom have perfect information about the position of the pursuer. The evaders have to estimate the position of the pursuer using the information they have gathered with their sensors and uncertainties often arise in the estimation process. In extreme cases, the evaders might not be able to detect the pursuer at all. In this section, the collaborative team evasion framework is extended to enable a team of evaders to maximize the team survival time against a hidden pursuer.

5.2.1 Extensions to the Open-loop Formulation

A team of evaders that cannot detect the pursuer can still obtain some information about the position of the pursuer since when an evader is captured, the pursuer has to be at the capture point. In other words, after losing a teammate to a pursuer, the team of evaders knows the exact position of the pursuer at the capture time. Assuming that the evaders know (or has an estimation of) the maximum speed of the pursuer, the evaders can limit the possible position of the pursuer to a circular disk centered at the previous capture point. The radius of the disk can be determined by the maximum speed of the pursuer and the time that has passed since the previous capture. This disk in which the pursuer can be is referred to as the *pursuer disk*.

Definition 5.1 Notation for a circular disk
Given a point $x \in \mathbb{R}^2$ and a positive length $r \geq 0$, the circular disk centered at x with a radius of r is defined as

$$\mathbb{D}(x,r) = \{x' \mid \|x' - x\|_2 \leq r\}.$$

Definition 5.2 Pursuer disk
Given that at time $t = 0$ a pursuer with maximum speed v_p is located at x_p^0, then the set of all possible positions the pursuer can be at time $t \geq 0$, referred to as the pursuer disk, is

$$\mathbb{D}(x_p^0, v_p \times t). \tag{5.1}$$

Recall the open-loop formulation of the team evasion problem conservative to the evaders defined in Chapter 3 as

$$\sup_{\mathbf{u_e}(\cdot) \in \mathcal{U}^N} \inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)). \tag{5.2}$$

The evaders have to solve a maximization problem with the objective function being

$$\inf_{\mathbf{s} \in \mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)),$$

where x_p^0 is the exact position of the pursuer. Consider the case where the pursuer can be anywhere within a disk $\mathbb{D}(x_p^0, r)$, the open-loop team evasion problem can be defined as follows.

Definition 5.3 Open-loop formulation of team evasion against a pursuer disk
Given a team of evaders with joint position $\mathbf{x_e^0}$ and joint speed $\mathbf{v_e}$, the open-loop team evasion problem against a pursuer with maximum speed v_p that can be anywhere within a disk $\mathbb{D}(x_p^0, r)$ is

$$\sup_{\mathbf{u_e}(\cdot) \in \mathcal{U}^N} \inf_{\mathbf{s} \in \mathcal{S}_N} \inf_{x_p \in \mathbb{D}(x_p^0, r)} \Gamma_{\mathbf{s}}^{\star}(x_p, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)). \tag{5.3}$$

In this extended open-loop formulation of the team evasion game, the evaders have to solve a maximization problem with the objective function being

$$\inf_{\mathbf{s}\in\mathcal{S}_N} \inf_{x_p\in\mathbb{D}(x_p^0,r)} \Gamma_{\mathbf{s}}^{\star}(x_p, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)). \tag{5.4}$$

Note that the inf over all possible capture sequences is put outside of the inf over all possible positions within the pursuer disk. This implies that the pursuer can select its position from the disk after a capture sequence is selected; a different position can be selected for each capture sequence.

To be able to solve the maximization problem for the evaders, an algorithm that can evaluate the value of the objective function given the state of the game and the parameters has to be developed. In this case, with an extra layer of inf in the formulation, the challenge lies in finding the minimum possible team survival time

$$\inf_{x_p\in\mathbb{D}(x_p^0,r)} \Gamma_{\mathbf{s}}^{\star}(x_p, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)) \tag{5.5}$$

where x_p^0 and r are the center and radius of the pursuer disk, $\mathbf{x_e^0}$ and $\mathbf{u_e}(\cdot)$ are the joint position and joint control of the evaders, and $\mathbf{s}$ is a given capture sequence. The following lemma describes how the minimum possible survival time of a single evader against a pursuer that can be anywhere within a disk is derived.

Lemma 5.1 Minimum possible survival time against a pursuer disk
Given a pursuer with maximum speed v_p that can be anywhere within the disk $\mathbb{D}(x_p^0, r)$, the minimum possible time an evader can survival by staring at x_e^0 with maximum speed v_e and following the control $u_e(\cdot)$ is defined as

$$t^{\star} = \inf_{x_p\in\mathbb{D}(x_p^0,r)} \Gamma^{\star}(x_p, x_e^0, u_e(\cdot)). \tag{5.6}$$

Then, when $x_e^0 \notin \mathbb{D}(x_p^0, r)$, $t^{\star}$ satisfies

$$\left\|x_e(t^{\star}) - x_p^0\right\| = v_p \times t^{\star} + r, \tag{5.7}$$

where $x_e(\cdot) = traj(x_e^0, v_e, u_e(\cdot))$ is the resulting trajectory of the evader given the initial condition, maximum speed, and control.

When $x_e^0 \in \mathbb{D}(x_p^0, r)$, $t^{\star} = 0$.

Proof. Denote the position of the pursuer at time t by $x_p(t)$. Since the pursuer has to be at the same position as the evader to capture it and the evader is captured at $t^{\star}$, the following equality is true:

$$x_p(t^{\star}) = x_e(t^{\star}).$$
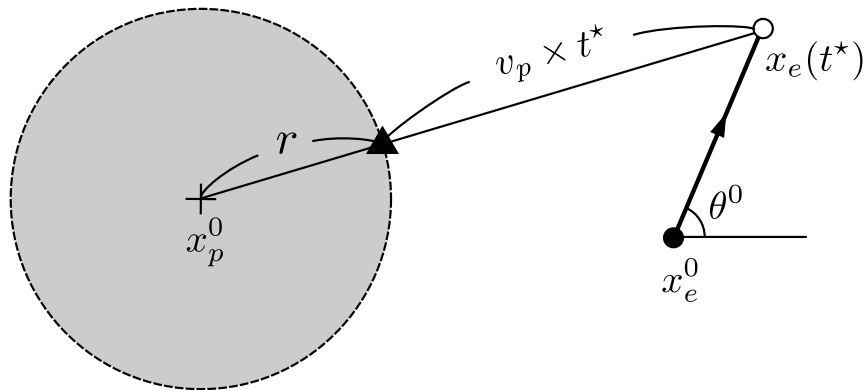
Figure for proof of Lemma 5.1

The minimum distance between $x_e(t^\star)$ and the disk $\mathbb{D}(x_p^0, r)$ is $\left\| x_e(t^\star) - x_p^0 \right\|_2 - r$. This is the distance the pursuer has to traverse in $t^\star$ units of time to capture the evader at $x_e(t^\star)$. Since the maximum speed of the pursuer is v_p, Eq. (5.7) can be derived by rearranging the terms. For cases where $x_e^0 \in \mathbb{D}(x_p^0, r)$, the evader is capture instantaneously since the pursuer can select its starting position $x_p(0)$ to be x_e^0. ∎

The optimal position in the disk for the pursuer to achieve the minimum possible capture time is the point that is closest to the capture point $x_e(t^\star)$. This point is on the boundary of the disk and is co-linear with the center of the disk and the capture point $x_e(t^\star)$.

With the ability to evaluate the minimum possible survival time of a single evader when the pursuer can be anywhere within a circular disk, Algorithm 5.1 describes the evaluation of the minimum possible team survival time given a specific capture sequence and joint control of evaders. The algorithm iterates through the evaders according to the given capture sequence. In each iteration, the variable τ keeps track of the capture time of the previous evader in the sequence. The trajectory of the current evader resulting from the given control is generated in line 4 and the position of the current evader when the previous one is captured is extracted in line 5. The capture time τ is updated in line 6 with $u_{s_i}^{+\tau}(\cdot)$ being the front truncated version of the control of the i-th evader in the sequence. In line 10 and line 11, the radius of the pursuer disk is set to zero and the center of the pursuer disk is set to the capture point of the i-th evader in the capture sequence. The radius of the pursuer disk is only ever non-zero for the first evader in the capture sequence since once an evader is captured, the exact location of the pursuer at the capture time is known to the evaders and there is no further uncertainties regarding the position of an optimal pursuer for the rest of the game.

In Algorithm 5.1 the given joint control of evaders, $\mathbf{u_e}(\cdot)$, can be of any form. In this case, a 1-dimensional search is required to solve for the minimum possible capture time in line 6.

Algorithm 5.1 $\inf_{x_p \in \mathbb{D}(x_p^0, r^0)} \Gamma_{\mathbf{s}}^{\star}(x_p, \mathbf{x_e^0}, \mathbf{u_e}(\cdot))$

1: Given $x_p^0, \mathbf{x_e^0} = (x_1^0, \ldots, x_N^0), \mathbf{s} = (s_1, \ldots, s_N), \mathbf{v_e} = (v_1, \ldots, v_N) < v_p$, and $\mathbf{u_e}(\cdot) = (u_1(\cdot), \ldots, u_N(\cdot))$
2: $\tau \leftarrow 0, x_c \leftarrow x_p^0, r \leftarrow r^0$
3: **for** $i = 1, \ldots, N$ **do**
4: $\quad x_{s_i}(\cdot) \leftarrow traj(x_{s_i}^0, v_{s_i}, u_{s_i}(\cdot))$
5: $\quad x_e \leftarrow x_{s_i}(\tau)$
6: $\quad \tau \leftarrow \tau + \inf_{x_p \in \mathbb{D}(x_c, r)} \Gamma^{\star}(x_p, x_e, u_{s_i}^{+\tau}(\cdot))$
7: $\quad$ **if** $i = N$ **then**
8: $\quad\quad$ **return** τ
9: $\quad$ **else**
10: $\quad\quad r \leftarrow 0$
11: $\quad\quad x_c \leftarrow x_{s_i}(\tau)$
12: $\quad$ **end if**
13: **end for**

However, for the special case where the evaders use only constant headings joint control, e.g. $\mathbf{u_e}(\cdot) \in \mathcal{U}_\theta$, the implementation of line 6 can be simplified through the following lemma.

Lemma 5.2 Minimum possible survival time given a constant-heading-maximum-speed evader control

Following Lemma 5.1, in the special case where $u_e(\cdot) \in \mathcal{U}_\theta$, when $x_e^0 \notin \mathbb{D}(x_p^0, r)$, $t^{\star}$ is the minimum positive root of the 2nd order polynomial of t

$$\left\| x_e^0 + (v_e \times t)e_{\theta_e} - x_p^0 \right\|_2 = v_p \times t + r, \tag{5.8}$$

where $e_{\theta_e} = [\cos\theta_e, \sin\theta_e]^T$.

When $x_e^0 \in \mathbb{D}(x_p^0, r)$, $t^{\star} = 0$.

Proof. The proof follows directly from Lemma 5.1 and the trajectory of the evader in constant heading:

$$x_e(t) = x_e^0 + (v_e \times t)e_{\theta_e}.$$

$\blacksquare$

With this lemma, the evaluation of line 6 can be done very efficiently with the closed-form formula for the roots of a 2nd order polynomial.

With Algorithm 5.1, the value of

$$\inf_{\mathbf{s} \in \mathcal{S}_N} \inf_{x_p \in \mathbb{D}(x_p^0, r)} \Gamma_{\mathbf{s}}^{\star}(x_p, \mathbf{x_e^0}, \mathbf{u_e}(\cdot))$$

can be evaluated by taking the minimum over all possible capture sequence $\mathbf{s} \in \mathcal{S}_N$. It can be shown through the similar procedures described in Section 3.2 that even when facing a pursuer that can be anywhere within in a disk, the optimal joint control for the evaders in the open-loop setting still belongs to the constant headings joint control set $\mathcal{U}_\Theta^N$.

Remark 5.3

The open-loop team evasion problem for the evaders against a pursuer that can be anywhere within a circular disk is defined as

$$\sup_{\Theta_{\mathbf{e}}} \inf_{\mathbf{s} \in \mathcal{S}_N} \inf_{x_p \in \mathbb{D}(x_p^0, r)} \Gamma_{\mathbf{s}}^\star(x_p, \mathbf{x_e}, \Theta_{\mathbf{e}}), \tag{5.9}$$

where $\Theta_{\mathbf{e}}$ is the joint heading of the evaders.

Although the objective function of the maximization problem in Eq. (5.9) is slightly more complicated than that of the original open-loop team evasion problem for the evaders, it still has the minimax structure and can be formulated as a constrained nonlinear optimization problem similar to the one defined in Eq. (3.26). Hence, it can again be solved by sequential quadratic programming.

Figure 5.1 shows examples of the open-loop optimal trajectories given that the pursuer can be anywhere within a disk. The pursuer disk is shown as a gray disk with a hollow triangle at the center and the optimal position of the pursuer is marked by a black triangle. The positions of the evaders are marked with solid blue circles while the predicted captured points are marked by hollow circles. The colored dashed lines are the open-loop optimal trajectories of the pursuer and the evaders resulting from the optimal open-loop joint heading. Note that there often exist multiple optimal capture sequences against the optimal joint heading of the evaders and each of these capture sequence has a corresponding optimal position of the pursuer; for clarity, only the resulting trajectories of one of the optimal capture sequences is shown in the figures.

Figure 5.2 shows the resulting optimal joint headings of the evaders given the same layouts shown in Fig. 5.1 but with much larger pursuer disks that enclose all evaders. In these cases, the first evader in each of the capture sequences is always considered to be instantly captured. For example in Fig. 5.2a, the resulting open-loop optimal joint heading of the evaders results in the evaders moving directly away from each other. This is because that for the capture sequence that captures the lower evader first, the lower evader is considered to be captured instantaneously and the optimal heading for the upper evader is to go directly away from the capture point. The situation is similar for the capture sequence that captures the upper evader first; the resulting optimal heading for the lower evader is to move directly away from the initial position of the upper evader. Similar behaviors can also be observed in Figs. 5.2b to 5.2d.

The proposed extension to the open-loop team evasion framework enables the team of evaders to handle the cases where the pursuer can be anywhere within a circular disk with

(a) 2 evaders

(b) 3 evaders

(c) 4 evaders

(d) 5 evaders

Figure 5.1: Open-loop optimal trajectories against a pursuer that can be anywhere within a circular disk for different layouts

(a) 2 evaders

(b) 3 evaders

(c) 4 evaders

(d) 5 evaders

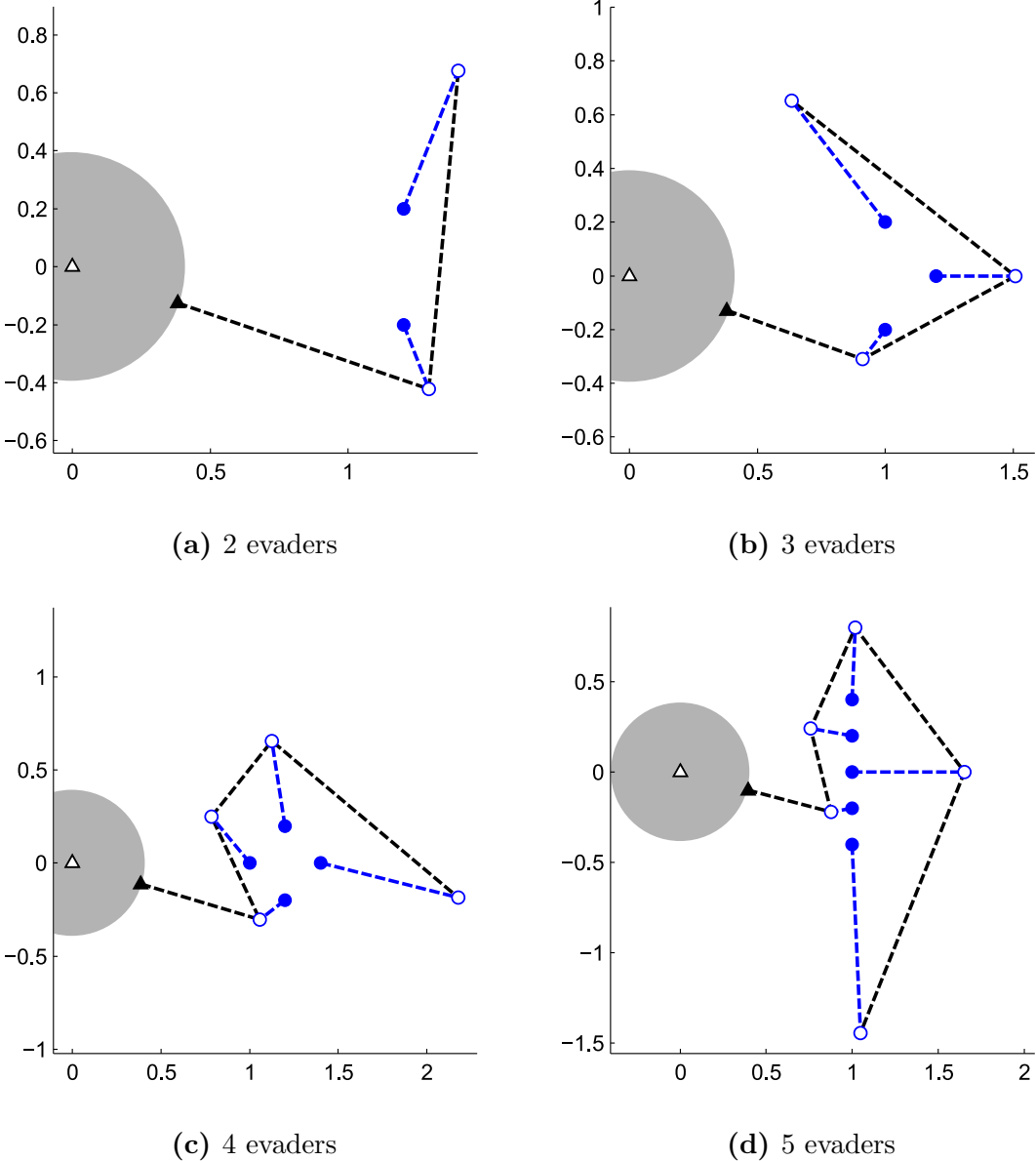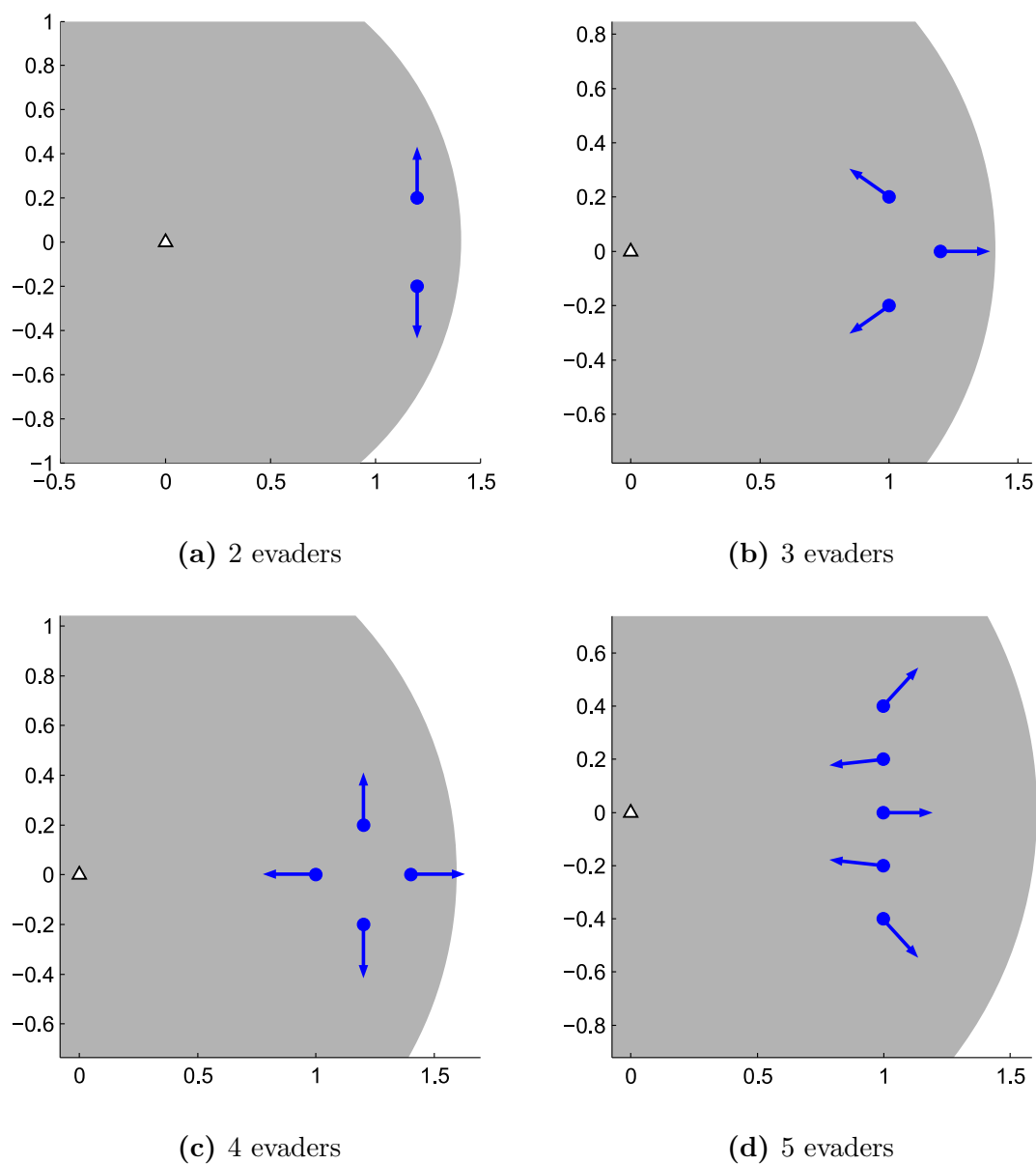**Figure 5.2:** Open-loop optimal trajectories against a pursuer that can be anywhere within a circular disk for different layouts

a known center and radius. The worst-case mentality of the evaders is preserved in this extension and so is the guarantee on the minimum possible team survival time. The resulting behavior, although qualitatively similar to the original case, is more conservative due to the additional advantage granted to the pursuer.

5.2.2 Iterative Open-loop Approach Against a Hidden Pursuer

The extension to the open-loop team evasion framework proposed in the previous section can also be applied to the iterative open-loop approach. In the original iterative open-loop approach for team evasion proposed in Chapter 4, at each update the team of evaders sets the joint heading to the open-loop optimal joint heading with respect to the most current state of the game, which includes the exact location of the pursuer. When facing a hidden pursuer, the exact position of the pursuer is only available to the evaders when a capture happens, hence in the updates between captures the evaders have to keep track of the center and radius of the pursuer disk to determine the corresponding open-loop optimal joint heading. Given that an update happens at time t when the pursuer disk is $\mathbb{D}(x_p^t, r)$ and the joint position of the evaders is $\mathbf{x_e}$, the evaders set their joint heading to

$$\Theta_{\mathbf{e}}^{\star}(\mathbb{D}(x_p^t, r), \mathbf{x_e}) = \arg \sup_{\Theta_{\mathbf{e}}} \inf_{\mathbf{s} \in \mathcal{S}_N} \inf_{x_p \in \mathbb{D}(x_p^t, r)} \Gamma_{\mathbf{s}}^{\star}(x_p, \mathbf{x_e}, \Theta_{\mathbf{e}}). \qquad (5.10)$$

and stay true to this joint heading until the next update. At the next update time $t' > t$, assuming that the joint position of the evaders is $\mathbf{x_e'}$, the radius of the pursuer disk is increased to $r' = r + v_p \times (t' - t)$ and the center of the disk stays at x_p^t. If an evader is captured at x_c in this update, the center of the pursuer disk is set to the capture point x_c and the radius of the disk is set to 0. The rest of the evaders will update their headings to the open-loop optimal joint heading $\Theta_{\mathbf{e}}^{\star}(\mathbb{D}(x_c, 0), \mathbf{x_e'})$. If no evaders are captured at this update, the joint heading of the evaders is set to the open-loop optimal joint heading with respect to the pursuer disk with the increased radius, namely $\Theta_{\mathbf{e}}^{\star}(\mathbb{D}(x_p^t, r'), \mathbf{x_e'})$.

Recall that to measure the team survival time performance of the original iterative open-loop approach, a greedy pursuer strategy was proposed in Section 4.2.3. With the greedy strategy, the pursuer sets its heading to be the open-loop optimal heading against the joint heading of the evaders at every update. The greedy strategy was necessary for measuring the survival time performance of the iterative open-loop approach because computing the optimal pursuer trajectory in the iterative open-loop setting is computationally intractable. However, when facing a hidden pursuer, it is possible to obtain a conservative measurement of the team survival time without specifying a heading control scheme for the pursuer. Since the open-loop optimal joint heading of the evaders is affected only by the pursuer disk and not by the actual position of the pursuer, the heading control for the pursuer does not matter. This conservative measurement can be obtained by modifying the capture condition so that an evader is considered captured when it touches (or is inside of) the pursuer disk. When there are multiple evaders satisfying this condition, the pursuer can only capture one of them.

The resulting team survival time is only affected by the capture sequence selected by the pursuer, not its actual trajectory. The measured team survival time is an under-estimation of the actual minimum possible team survival time that can be achieved by a pursuer that has to pick its heading at each update. The modified capture condition allows the pursuer that might have selected a suboptimal heading in the previous updates to still capture the evader in minimum time possible in the open-loop sense.

Figure 5.3 shows several snapshots taken at different times during a simulation run against a hidden pursuer. In these figures, the pursuer disk is represented by a gray disk with its centered marked by a hollow triangle. The solid blue dots are the current positions of the evaders, and the solid black triangle marks the optimal position of the pursuer. The hollow dots are the predicted optimal capture positions of the evaders given the current open-loop optimal joint headings. The colored dashed lines are the predicted open-loop optimal trajectories of the agents and the solid lines are the past trajectories of the agents. Figure 5.3a shows the initial condition of the game at $t = 0$. Note that there are 3 evaders in the team at this time. This can be interpreted as the beginning of the collaborative evasion for a 4-evader team that starts when one of the 4 evaders is captured at the origin, revealing the exact position of the pursuer to the 3 evaders left. In Fig. 5.3b, the radius of the pursuer disk grows with time. Figure 5.3c shows the moment right before the lowest evader is capture and Fig. 5.3d shows the moment right after the capture happens. Note that after the capture, the center of the pursuer disk is moved to the capture point and the radius of the disk is set to zero. Figure 5.3e shows the moment right before the second evader is captured and Fig. 5.3f shows the state of the game when only one evader is left.

5.2.3 Results and Discussion

To measure the effectiveness of the proposed approach, the survival time performance of the extended iterative open-loop approach against a hidden pursuer is measured on the benchmarking dataset and compared to that of the original iterative open-loop approach against a visible pursuer. Similar to the example shown in Fig. 5.3, at the beginning of each simulation the position of the pursuer is known to the evaders, but during the simulations the evaders can only locate the pursuer when an evader is captured. An evader is considered captured when it is in contact with the pursuer disk; the pursuer is moved to the capture point when a capture happens. The performance in team survival time against a visible pursuer is taken directly from the simulation results in Chapter 4.

Figure 5.4 shows the distribution of the ratio of team survival time against a hidden pursuer to that against a visible pursuer over the 500 layouts in the benchmarking dataset for different number of evaders. According to the result, the team achieves a lower team survival time when facing a hidden pursuer than when facing a visible pursuer. This difference in performance of team survival time comes from the additional conservatism introduced by the extension made for the pursuer disk and the modified capture condition. For the extension, when facing a hidden pursuer, the evaders use the formulation in Eq. (5.9) which comes

(a) $t = 0$

(b) $t = 0.4$

(c) $t = 0.9$

(d) $t = 0.95$

(e) $t = 1.75$

(f) $t = 2.7$
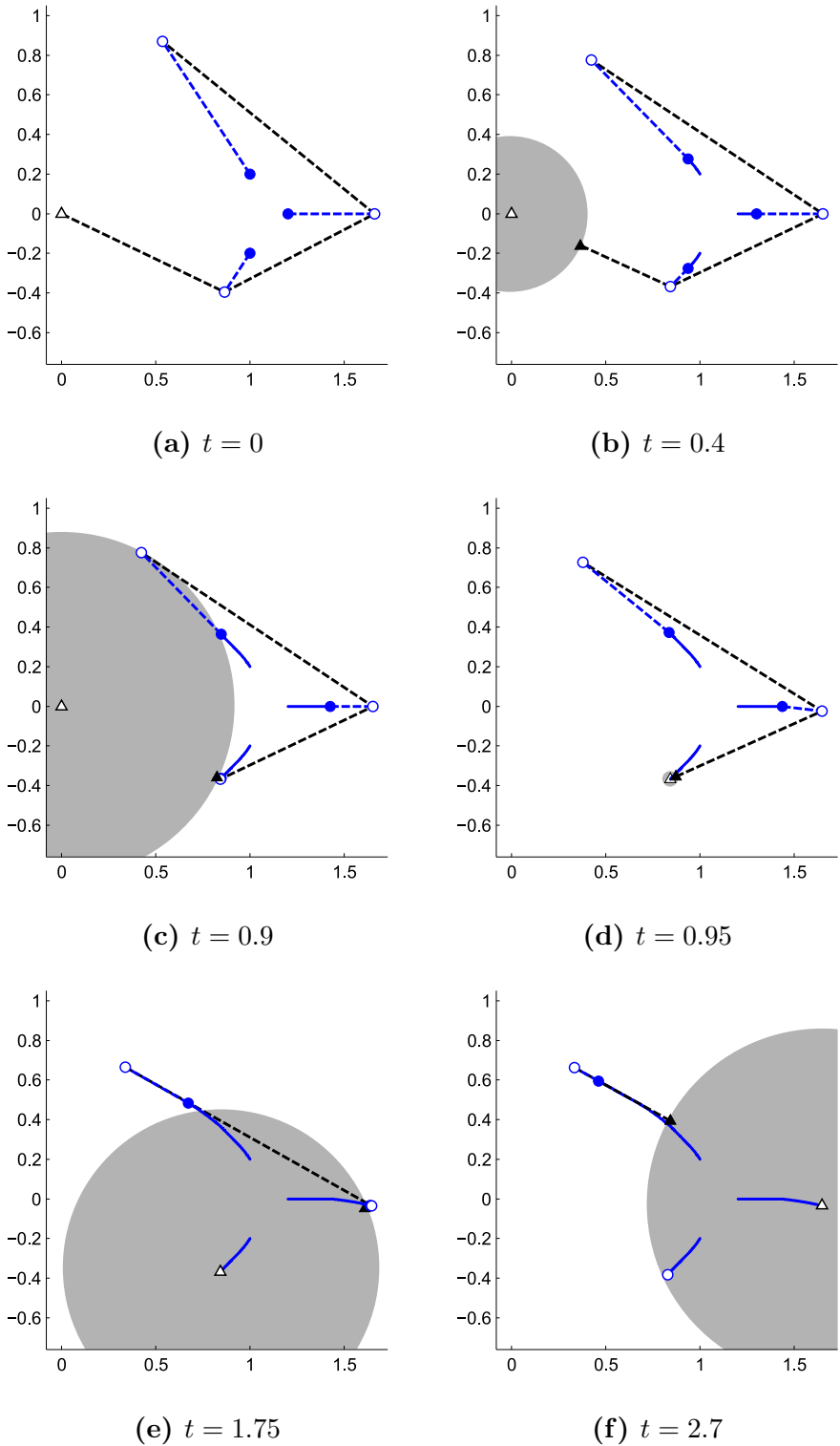
Figure 5.3: Snapshots of a simulation of the iterative open-loop approach against a hidden pursuer

(a) 2 evaders

(b) 3 evaders

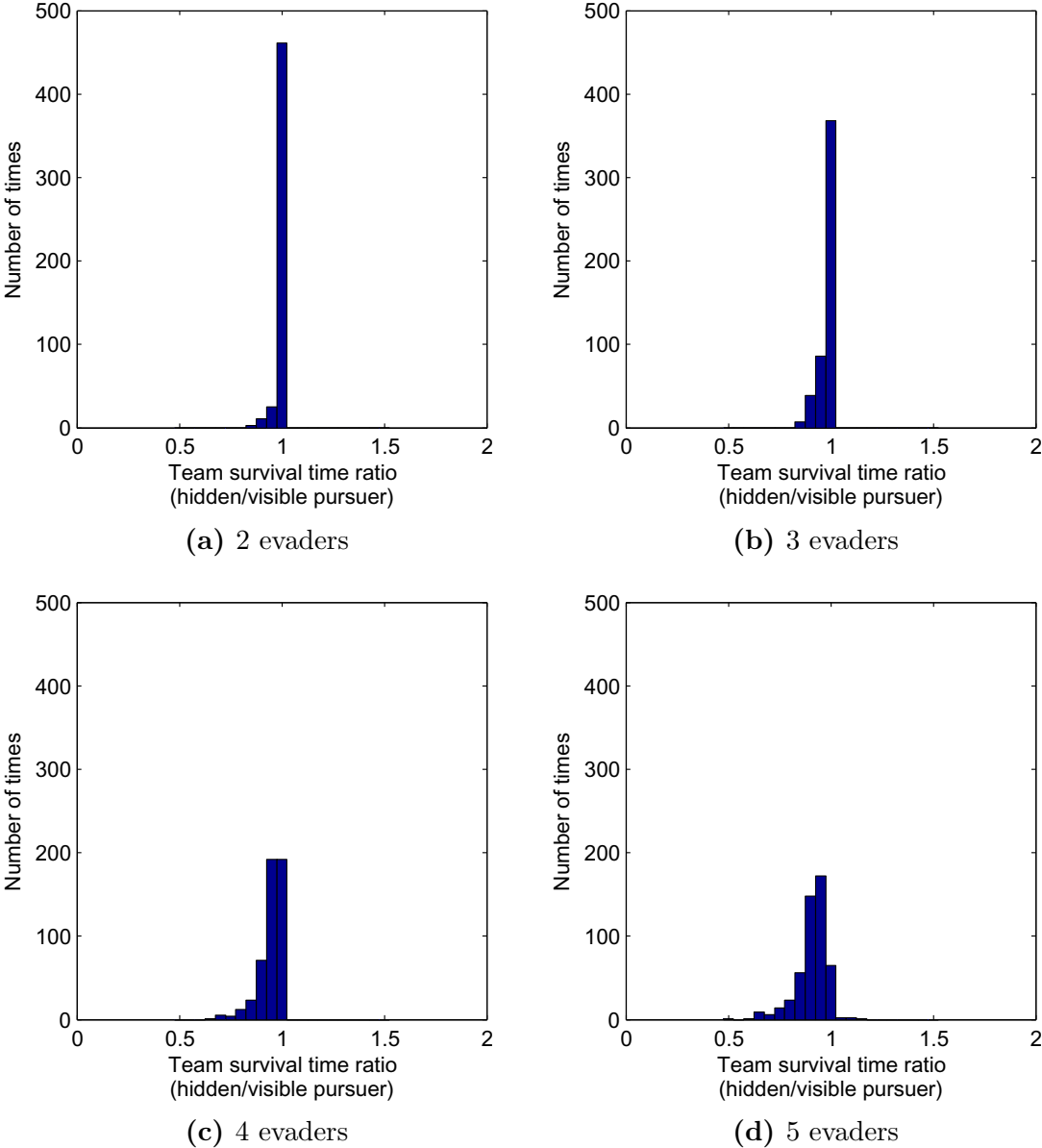**(c)** 4 evaders

(d) 5 evaders

Figure 5.4: Distribution of the ratio of team survival time against a hidden pursuer to that against a visible pursuer over 500 layouts for teams with different number of evaders

with the conservative assumption that the pursuer can be at different positions within the disk when different capture sequences are considered. However, in reality the pursuer can only be at a single position at any given time. This conservative assumption results in a more conservative joint heading of the evaders and hence a shorter team survival time. For the modified capture condition, the team survival time measured by the original capture condition and a greedy pursuer strategy is achievable by an optimal pursuer which knows the joint heading of the evaders at every update; the team survival time measured with the modified capture condition, however, is a lower-bound on the minimum possible team survival time which is some cases might be unachievable even by an optimal pursuer due to causality.

Figures 5.4a to 5.4d show that the survival time performance against a hidden pursuer degrades generally with the increase of the number of evaders in the team. This is to be expected in that when there are more evaders in the team there are more capture sequences to be taken into account by the evaders; the conservatism introduced by assuming that the pursuer can be at different positions when different capture sequences are considered is manifested by the increase in the number of capture sequences. It is worth noting that although the iterative open-loop approach does perform worse when the team is facing a hidden pursuer than when it is facing a visible pursuer, the difference in survival time is small on average. In most of the layouts the ratio of team survival time against a hidden pursuer to that against a visible pursuer is above 0.8. This shows that the proposed approach against a hidden pursuer is effective in such scenarios. The small difference in team survival time performance also supports the statement that the greedy pursuer strategy for heading control proposed in Section 4.2.3 is near optimal in that it often enables a visible pursuer to achieve the minimum possible team survival time achievable only by a hidden pursuer under relaxed capture condition.

The proposed extension to the team evasion framework enables the evaders to maximize their team survival time when the pursuer can be anywhere within a circular disk. By adjusting the radius of the disk according to the maximum speed of the pursuer and updating the center of the disk when a capture happens, a team of evaders can achieve a comparable team survival time when facing a hidden pursuer to when facing a visible pursuer.

5.3 Evaders with Turning Rate Constraints

In the original formulation of the team evasion problem, the evaders are assumed to have unlimited turning rates and hence can change their headings instantaneously. However, most of the autonomous vehicles have limited turning rates due to physical constraints. In this section the collaborative team evasion framework is extended to take into account the constraints on turning rates of the evaders.

5.3.1 Dubins Vehicles and Dubins Path

The idea of a *Dubins vehicle* is proposed by Dubins in his seminal work [49]. A Dubins vehicle is a vehicle that can only travel forward in constant speed and can not exceed a finite maximum turning rate. In the literature for path planning, nonholonomic vehicles such as fixed-wing aircrafts and most of the wheeled ground vehicles are often modeled as Dubins vehicles. The trajectory of a Dubins vehicle is called a *Dubins path.* It is continuous and differentiable at every point and its curvature is upperbounded by a finite value. In this extension to the collaborative team evasion framework, the evaders are modeled as Dubins vehicles and the admissible control set is modified so that the evader can only produce Dubins paths.

Recall the original formulation of the team evasion problem for the evaders defined in Eq. (3.1) as

$$\sup_{\mathbf{u_e}(\cdot) \in \mathcal{U}^N} \inf_{\mathbf{s} \in \mathcal{S}^N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)).$$

The admissible control set for a single evader is defined in Eq. (2.2) as

$$\mathcal{U} = \{u(\cdot) | \|u(t)\| \leq 1, t \in [0, \infty)\}.$$

and the admissible control set for a team of N evaders is defined in Eq. (2.3) as

$$\mathcal{U}^N = \{(u_1(\cdot), \ldots, u_N(\cdot)) | u_i(\cdot) \in \mathcal{U} \text{ for } i = 1, \ldots, N\}$$

The admissible control set for a Dubins evader can be defined as follows.

Definition 5.4 Dubins admissible control set
For a Dubins evader with θ^0 as its current heading and ω as its maximum turning rate, the Dubins admissible control set is defined as

$$\mathcal{D}_{\omega,\theta^0} = \{\theta(\cdot) : \mathbb{R}^+ \to \mathbb{R} \mid \theta(0) = \theta^0, |\dot{\theta}(t)| \leq \omega \text{ for } t > 0\}. \tag{5.11}$$

Note that for a Dubins evader, the control $\theta(\cdot)$ no longer specifies the velocity of the evader since it always travels in maximum speed. $\mathcal{D}_{\omega,\theta^0}$ is a subset of the original admissible control set $\mathcal{U}$.

The admissible joint control set for a team of Dubins evaders is defined as follows.

Definition 5.5 Dubins admissible joint control set
For N Dubins evaders with current joint heading $\Theta^0 = (\theta_1^0, \ldots, \theta_N^0)$ and joint maximum turning rate $\Omega_{\mathbf{e}} = (\omega_1, \ldots, \omega_N)$, the Dubins admissible joint control set is defined as

$$\mathcal{D}_{\Omega,\Theta^0} = \{(\theta_1(\cdot), \ldots, \theta_N(\cdot)) \mid \theta_i(\cdot) \in \mathcal{D}_{\omega_i,\theta_i^0} \text{ for } i = 1, \ldots, N\}. \tag{5.12}$$

The open-loop team evasion problem for Dubins evaders is defined as follows.

Definition 5.6 Open-loop team evasion problem for Dubins evaders

Given x_p^0 and $\mathbf{x_e^0}$ as the positions of the pursuer and evaders and v_p and v_e as their maximum speeds, the open-loop team evasion problem for the team of Dubins evaders that have $\Omega_{\mathbf{e}}$ as their joint maximum turning rate and $\Theta_{\mathbf{e}}^0$ as their current joint heading is

$$\sup_{\Theta_{\mathbf{e}}(\cdot) \in \mathcal{D}_{\Omega_{\mathbf{e}}, \Theta_{\mathbf{e}}^0}} \inf_{\mathbf{s} \in \mathcal{S}^N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}}(\cdot)). \tag{5.13}$$

It is worth noting that to be consistent with the worst-case mentality of the evaders in the open-loop formulation of team evasion problem, no constraints are put on the turning rate of the pursuer. The function $\Gamma_{\mathbf{s}}^{\star}$ returns the minimum possible team survival time against a pursuer that can change its heading instantaneously.

Although the Dubins admissible joint control set is a subset of the general admissible joint control set, the optimization problem in Eq. (5.13) is still an infinite dimensional one. To be able to solve for the open-loop optimal joint control for a team of Dubins evaders, the admissible joint control set has to be further compressed. Consider the following definition of a compressed admissible control set for a Dubins evader.

Definition 5.7 Compressed Dubins admissible control set

For a Dubins evader with current heading θ^0 and maximum turning rate ω, the compressed Dubins admissible control set is defined as

$$\hat{\mathcal{D}}_{\omega, \theta^0} = \{\theta(\cdot) : \mathbb{R}^+ \to \mathbb{R} \mid \theta(0) = \theta^0, \exists t_c \geq 0 \text{ such that}$$
$$|\dot{\theta}(t)| = \omega \text{ for } t \in [0, t_c) \text{ and } \dot{\theta}(t) = 0 \text{ for } t \geq t_c\}. \tag{5.14}$$

Definition 5.8 Compressed Dubins admissible joint control set

For N Dubins evaders with current joint heading $\Theta^0 = (\theta_1^0, \ldots, \theta_N^0)$ and joint maximum turning rate $\Omega_{\mathbf{e}} = (\omega_1, \ldots, \omega_N)$, the compressed Dubins admissible joint control set is defined as

$$\hat{\mathcal{D}}_{\Omega_{\mathbf{e}}, \Theta^0} = \{(\theta_1(\cdot), \ldots, \theta_N(\cdot)) \mid \theta_i(\cdot) \in \hat{\mathcal{D}}_{\omega_i, \theta_i^0} \text{ for } i = 1, \ldots, N\}. \tag{5.15}$$

An evader using a control in this set starts from the initial heading θ^0, turns in maximum turning rate ω in one direction for a finite amount of time t_c, and then travels in constant heading thereafter.

By swapping the admissible joint control set of the evaders $\mathcal{D}_{\Omega_{\mathbf{e}}, \Theta_{\mathbf{e}}^0}$ with the compressed Dubins admissible control set $\hat{\mathcal{D}}_{\Omega_{\mathbf{e}}, \Theta_{\mathbf{e}}^0}$ in Eq. (5.13), the open-loop team evasion problem for Dubins evaders defined in Definition 5.6 can be approximated by

$$\sup_{\Theta_{\mathbf{e}}(\cdot) \in \hat{\mathcal{D}}_{\Omega_{\mathbf{e}}, \Theta_{\mathbf{e}}^0}} \inf_{\mathbf{s} \in \mathcal{S}^N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \Theta_{\mathbf{e}}(\cdot)). \tag{5.16}$$

Note that for a given pair of initial heading θ^0 and maximum turning rate ω, a heading control $\theta(\cdot) \in \hat{\mathbb{D}}_{\omega,\theta^0}$ can be uniquely specified by the direction and duration of the turn. Alternatively, it can also be specified by the direction of the turn and the *goal heading* defined as

$$\phi = \theta(t_s).$$

By specifying a rule to select the appropriate direction of the turn for a given goal heading, a heading control in the compressed Dubins control set can be specified simply by the goal heading in a similar way that a constant heading control can be specified by the heading of an evader. Here the direction of the turn is picked to be the one that can reach the goal heading in minimum amount of time. Figure 5.5a shows a heading control $\theta(\cdot)$ specified by an initial heading θ^0, a maximum turning rate ω, and a goal heading ϕ. The turning direction is selected to be counter-clockwise in this case. Figure 5.5b shows the resulting trajectory of an evader starting at x_e^0 using this heading control.

(a) Heading control   **(b)** Resulting trajectory
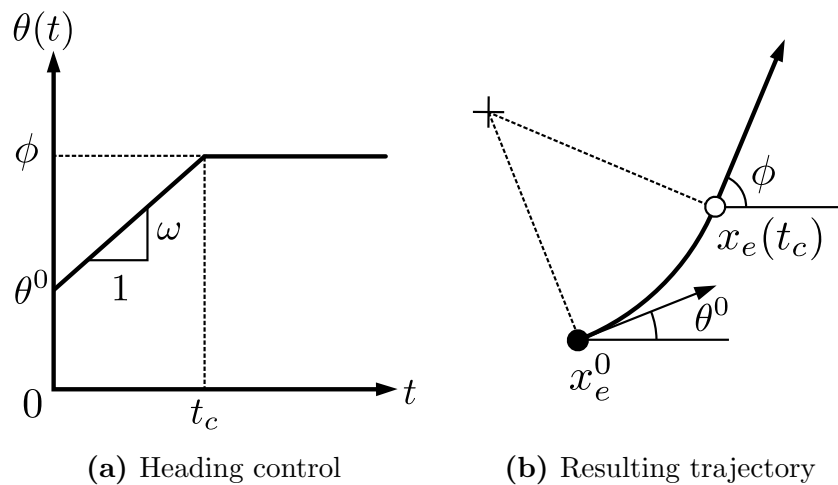
Figure 5.5: Example of heading control in the compressed Dubins admissible set and its resulting trajectory

By parameterizing the Dubins joint heading control for a team of N evaders with the joint goal heading

$$\Phi_{\mathbf{e}} = (\phi_1, \ldots, \phi_N) \in \mathbb{R}^N,$$

the optimization problem in Eq. (5.16) can be written as a finite dimensional sup-inf problem

$$\sup_{\Phi_{\mathbf{e}} \in \mathbb{R}^N} \inf_{s \in \mathcal{S}_N} \Gamma_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}}^0, \Omega_{\mathbf{e}}, \Phi_{\mathbf{e}}). \tag{5.17}$$

5.3.2 Minimum Survival Time on a Dubins Path

Combined with the extension proposed in the previous section which allows the framework to handle a pursuer that can be anywhere within a circular disk, the extended open-loop
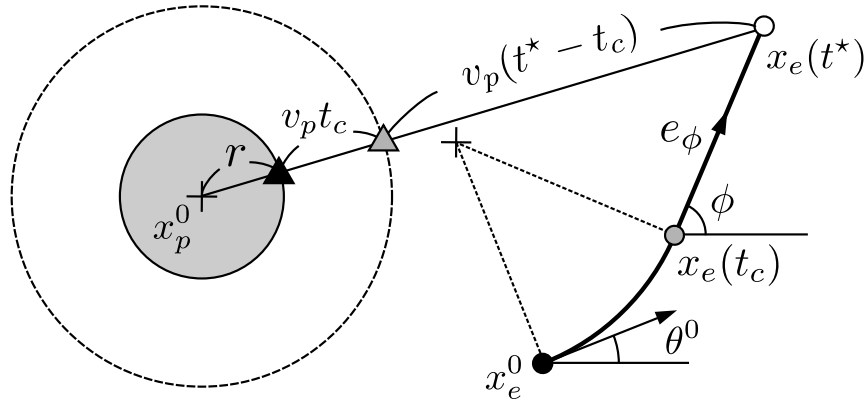
Figure 5.6: Minimum survival time for an evader on a Dubins path against a pursuer that can be anywhere within a circular disk

formulation of the team evasion problem can be written as

$$\sup_{\Phi_{\mathbf{e}} \in \mathbb{R}^N} \inf_{s \in \mathcal{S}_N} \inf_{x_p \in \mathbb{D}(x_p^0, r)} \Gamma_{\mathbf{s}}^{\star}(x_p, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}}^0, \Omega_{\mathbf{e}}, \Phi_{\mathbf{e}}), \tag{5.18}$$

where x_p^0 is the center of the disk and r is the radius of the disk. The function $\Gamma_{\mathbf{s}}^{\star}(x_p, \mathbf{x}_{\mathbf{e}}^0, \Theta_{\mathbf{e}}^0, \Omega_{\mathbf{e}}, \Phi_{\mathbf{e}})$ returns the minimum possible capture time for a pursuer starting at x_p to capture a team of evaders with joint position $\mathbf{x}_{\mathbf{e}}^0$, current joint heading $\Theta_{\mathbf{e}}^0$, maximum joint turning rate $\Omega_{\mathbf{e}}$, and joint goal heading $\Phi_{\mathbf{e}}$ according to the capture sequence $\mathbf{s}$.

To be able to solve Eq. (5.18) for the optimal joint goal heading of the evaders, an efficient way to evaluate the function has to first be developed. First consider the case for a single evader. Figure 5.6 shows the trajectory of an evader which uses a heading control from the compressed Dubins admissible control set. Note that at $t = 0$ the evader is at the position of the black dot with heading θ^0, and at time $t = t_c$ the evader will reach the position of the gray dot with the goal heading ϕ and then travels in a straight line thereafter. As for the pursuer, it can be anywhere within the gray disk centered at x_p^0 with radius r at time $t = 0$. At $t = t_c$, the radius of the disk will be increased by $v_p \times t_c$, which is the maximum distance the pursuer can cover within that time. Algorithm 5.2 is used to evaluate the minimum possible capture time on such trajectory. In line 2, the trajectory of the evader is generated according to the current state of the evader, its maximum speed and turning rate, and its goal heading. The time t_c in line 3 is the time it takes the evader to reach its goal heading and the x_e' in line 4 is the position of the evader at time t_c. The pursuer with a maximum speed of v_p and is located at x_p^0 when $t = 0$ can be anywhere within the disk $\mathbb{D}(x_p^0, r + v_p \times t_c)$ at time t_c. The computation required to solve for the minimum possible survival time depends on the relationship between $\mathbb{D}(x_p^0, r + v_p \times t_c)$ and x_e'. The two possibilities are as follows: The first possibility is that the evader is not within the pursuer disk at time t_c. Since starting from t_c the evader will travel in a straight line, the minimum time it takes the pursuer to

---

Algorithm 5.2 $\inf_{x_p \in \mathbb{D}(x_p^0, r)} \Gamma^\star(x_p, x_e^0, \theta_e^0, \omega_e, \phi_e)$

1: Given $x_p^0, v_p, x_e^0, v_e, \theta_e^0, \omega_e, \phi_e, r$
2: $x_e(\cdot) \leftarrow traj(x_e^0, \theta_e^0, \omega_e, \theta_e)$
3: $t_c \leftarrow \frac{|\theta_e^0 - \phi|}{\omega_e}$
4: $x_e' \leftarrow x_e(t_c)$
5: **if** $x_e' \notin \mathbb{D}(x_p^0, r + v_p \times t_c)$ **then**
6:     **return** $t_c + \inf_{x_p \in \mathbb{D}(x_p^0, r + v_p \times t_c)} \Gamma^\star(x_p, x_e', \phi)$
7: **else**
8:     **return** $\min\{t \mid \left\|x_p^0 - x_e(t)\right\|_2 = v_p \times t + r, t \geq 0\}$
9: **end if**

---

capture the evader starting from this point in time is $\inf_{x_p \in \mathbb{D}(x_p^0, v_p \times t_c)} \Gamma^\star(x_p, x_e', \phi)$. This can be evaluated by Algorithm 5.1 which involves only the evaluation of a closed-form formula. In this case, the actual minimum survival time of the evader is returned in line 6. The second possibility is that at time t_c the evader is already within the pursuer disk. In this case it is possible for the evader to be capture before time t_c while it is still on the curved section of the trajectory. As described in line 8, the minimum possible capture time for the evader then has to be derived by finding the minimum t that is bigger than or equal to zero and satisfies

$$\left\|x_p^0 - x_e(t)\right\|_2 = v_p \times t + r. \tag{5.19}$$

Such t can be found by solving a one dimensional root searching problem limited to the segment $t \in [0, t_c]$. Although slightly slower than a closed-form formula, this can still be done fairly efficiently.

With the ability to compute the minimum capture time of a single evader against a pursuer that can be anywhere within a disk, the value of $\inf_{x_p \in \mathbb{D}(x_p^0, r)} \Gamma_{\mathbf{s}}^\star(x_p, \mathbf{x_e^0}, \Theta_\mathbf{e}^0, \Omega_\mathbf{e}, \Phi_\mathbf{e})$ for a team of N evaders can be computed by Algorithm 5.3 which is extended from Algorithm 5.1 in Section 5.2. The algorithm is initialized with the initial condition of the game, including the configuration of the pursuer and evaders, their maximum speeds, the maximum turning rates of the evaders, and their initial and goal headings. The $heading(\theta_{s_i}^0, \omega_{s_i}, \phi_{s_i})$ in line 4 returns the heading control function given the initial heading, maximum turning rate, and goal heading of a specific evader as illustrated in Fig. 5.5. The $traj(x_{s_i}^0, v_{s_i}, \theta_{s_i}(\cdot))$ in line 5 returns the resulting trajectory of an evader given its initial position, maximum speed, and heading control. The variable τ keeps track of the time each evader in the capture sequence is captured. In line 6, the position and heading of the i-th evader in the capture sequence when the previous evader is captured is extracted from the resulting trajectory and heading control. In line 7, the minimum time it takes the pursuer to capture the i-th evader in the capture sequence starting from time τ is computed using Algorithm 5.2 and τ is updated with the capture time of the i-th evader. The position of the pursuer is then moved to the minimum time capture position of the i-th evader in line 12. Note that since the pursuer has to be at the same position as an evader to capture it, there is no further uncertainty in the

---

Algorithm 5.3 $\inf_{x_p \in \mathbb{D}(x_p^0, r^0)} \Gamma_{\mathbf{s}}^{\star}(x_p, \mathbf{x_e^0}, \Theta_{\mathbf{e}}^0, \Omega_{\mathbf{e}}, \Phi_{\mathbf{e}})$

1: Given $x_p^0, \mathbf{x_e^0} = (x_1^0, \ldots, x_N^0), \mathbf{s} = (s_1, \ldots, s_N), \mathbf{v_e} = (v_1, \ldots, v_N) < v_p, \Theta_{\mathbf{e}}^0 = (\theta_1, \ldots, \theta_N),$
$\Omega_{\mathbf{e}} = (\omega_1, \ldots, \omega_N),$ and $\Phi_{\mathbf{e}} = (\phi_1, \ldots, \phi_N)$
2: $\tau \leftarrow 0, x_p' \leftarrow x_p^0, r \leftarrow r^0$
3: **for** $i = 1, \ldots, N$ **do**
4: $\quad \theta_{s_i}(\cdot) \leftarrow heading(\theta_{s_i}^0, \omega_{s_i}, \phi_{s_i})$
5: $\quad x_{s_i}(\cdot) \leftarrow traj(x_{s_i}^0, v_{s_i}, \theta_{s_i}(\cdot))$
6: $\quad x_e \leftarrow x_{s_i}(\tau), \theta_e = \theta_{s_i}(\tau)$
7: $\quad \tau \leftarrow \tau + \inf_{x_p \in \mathbb{D}(x_p', r)} \Gamma^{\star}(x_p, x_e, \theta_e, \omega_{s_i}, \phi_{s_i})$
8: $\quad$ **if** $i = N$ **then**
9: $\quad\quad$ **return** τ
10: $\quad$ **else**
11: $\quad\quad r \leftarrow 0$
12: $\quad\quad x_p' \leftarrow x_{s_i}(\tau)$
13: $\quad$ **end if**
14: **end for**

---

pursuer's position after a capture happens. Hence in line 11, the radius of the pursuer disk is set to zero after the first evader in the capture sequence is captured.

With Algorithm 5.2, the optimization problem in Eq. (5.18) can then be solved by exploiting the minimax structure of the problem using the same procedure outlined in Section 3.3. The solution is the open-loop optimal joint goal heading of the evaders that will maximize the worst case team survival time of the team. In the iterative open-loop approach, the open-loop optimal joint heading of the team with respect to the most current state of the game is resolved at every update time. The resulting behavior and performance are presented and discussed in the next section.

5.3.3 Results and Discussion

Resulting Trajectories

Figure 5.7 shows four snapshots taken from a simulation of a team of evaders with limited turning rates utilizing the iterative open-loop approach to evade a faster and hidden pursuer. The positions of the evaders are marked by the blue solid circles and the pursuer disk $\mathbb{D}(x_p^0, r)$ is represented by a gray disk. The center of the disk is marked by a hollow triangle and the optimal position of the pursuer is marked by a dark triangle. The solid lines are the trajectories of the evaders starting from $t = 0$ while the dashed lines represent the planned future trajectories of agents. The hollow circles are the predicted minimum-time capture points of the evaders. Although there often exists multiple optimal capture sequence and each of these capture sequences has a set of predicted trajectories and capture points associated with it, for clarity, only the predicted trajectories and capture points associated with one
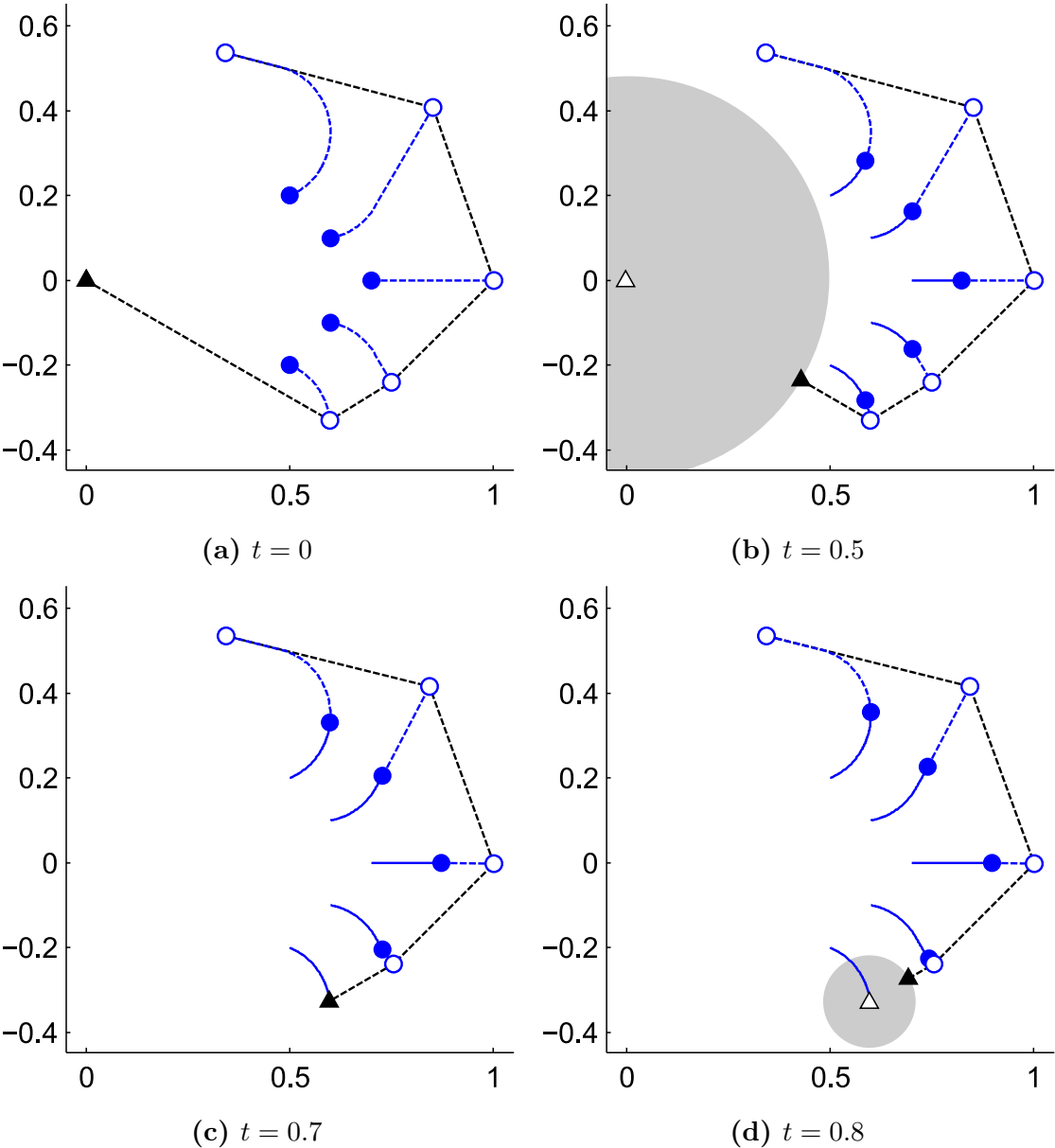
(a) $t = 0$

(b) $t = 0.5$

(c) $t = 0.7$

(d) $t = 0.8$

Figure 5.7: Snapshots of a simulation with Dubins evaders using the iterative open-loop approach against a hidden pursuer

specific optimal capture sequence are shown in the figures. It is worth emphasizing that the actual position and the intended capture sequence of the pursuer are not available to the evaders during the simulations; they are shown in the figures to help visualizing the intention of the agents only.

Figure 5.7a shows the initial condition of the game at $t = 0$ where the pursuer is starting from the origin with its position known to the 5 evaders. It can be interpreted as the time when one of the evaders of a 6-evader team is captured at the origin. Since before the capture there was no way for the evaders to detect the pursuer, the evasion only starts when the first evader is captured. The 5 evaders that are not captured become aware to the presence of the pursuer at time $t = 0$ and start to use the iterative open-loop approach to delay the capture of the whole team with the pursuer disk centered at the origin. Figure 5.7b shows the state of the game 0.5 seconds after the game starts. The uncertainty in the pursuer's position has grown and the pursuer can be anywhere within a circular disk centered at the origin with $r = 0.5$. Figure 5.7c shows the state of the game at $t = 0.7$ when an evader is captured. Note that this again reveals the position of the pursuer to the team of evaders and hence the center of the disk is moved to the capture position and the radius of the disk is set to zero. Figure 5.7d shows the state of the game at $t = 0.8$ which is slightly after the capture at $t = 0.7$. The center of the disk is kept at the capture point of the previously captured evader and the radius of the disk is increased to 0.1. The simulation continues until the last evader is captured. The resulting trajectories of the agents coincide with the predicted trajectories shown in Figure 5.7d.

It is worth noting that in this case the trajectories of the evaders show line symmetry to the x-axis. This is due to the symmetric nature of the initial layout and the fact that there exist multiple capture sequence that can achieve the minimum possible team capture time given the open-loop optimal joint goal heading. For example, although in Fig. 5.7 the pursuer captures the evaders in a counter-clockwise order, it is obvious that the pursuer can achieve the same capture time by capturing the evaders in a clockwise order. The open-loop formulation for team evasion takes all possible capture sequences into account when determining the open-loop optimal joint goal heading for the team. The resulting optimal joint goal heading is often a compromise between multiple capture sequences and is designed to maximize the worst-case team survival time.

Computation Time Performance

As mentioned in Section 5.3.2, in some cases finding the minimum possible survival time for a Dubins path requires a one dimensional root search which is more expensive computationally than evaluating a closed-form formula. To evaluate the impact of the turning rate constraints on the computational time performance of the iterative open-loop approach, initial conditions in the benchmarking dataset are simulated with the maximum turning rates of the evaders set to different values. For each simulation, the computation time is averaged over all updates.
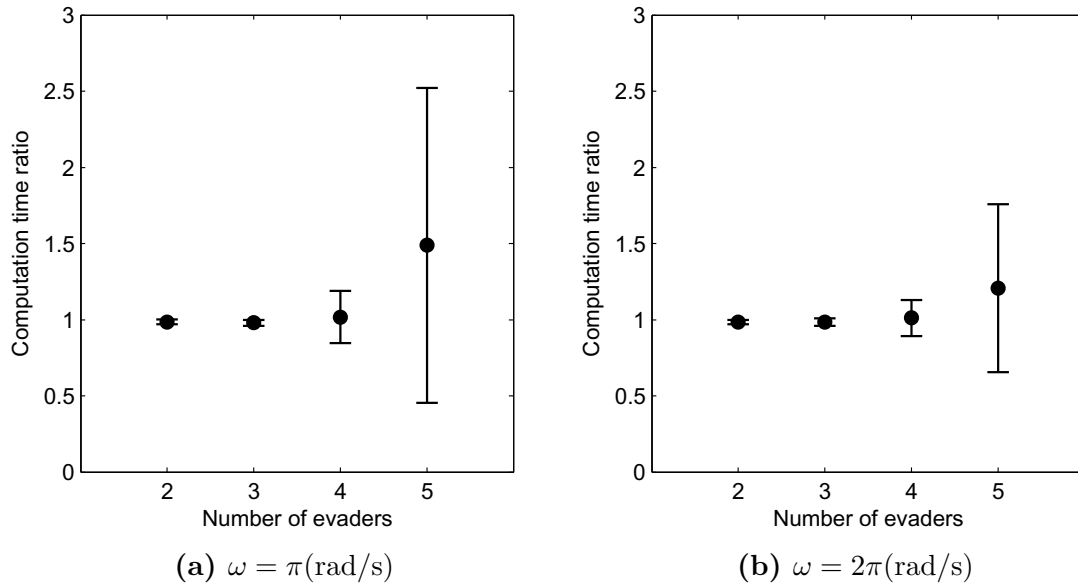
(a) $\omega = \pi(\mathrm{rad/s})$ **(b)** $\omega = 2\pi(\mathrm{rad/s})$

Figure 5.8: Distribution of the ratio of the averaged computation time of the iOL approach with turning rate constraints to that without turning rate constraints over 500 initial conditions with different number of evaders

Figure 5.8 shows the distribution of the ratio of averaged computation time for the iterative open-loop approach with turning rate constraints to that without the turning rate constraints. The black dots are the averaged computation time ratio over the 500 initial conditions in the dataset and the errorbars represent plus and minus one standard deviation of the distribution. In Fig. 5.8a, the maximum turning rates of the evaders are set to π (rad/s) and in Fig. 5.8b the maximum turning rates are set to 2π (rad/s). In both cases, when there are less evaders in the team, the ratio of computation time is almost always 1. This shows that in some cases the iterative open-loop approach with turning rate constraints does not require more computation time than the iterative open-loop approach without turning rate constraints. When there are more evaders in the team, the average computation time is longer for teams with smaller maximum turning rates. This is because that with smaller maximum turning rates, it takes the evaders longer to reach their goal headings. As a result, there are more cases where a capture can happen on the curved sections of the trajectories of the evaders. More specifically, line 8 in Algorithm 5.2 is invoked more often. When it is invoked, determining the survival time of the evader requires a 1 dimensional root searching routine that requires more computation time than the evaluation of a closed-form formula.

The computation time performance of the proposed iterative open-loop approach for evaders with turning rate constraints is comparable to that for evaders without tunning rate constraints. The performance is influenced by both the number of evaders in the team and the maximum turning rates of the evaders.
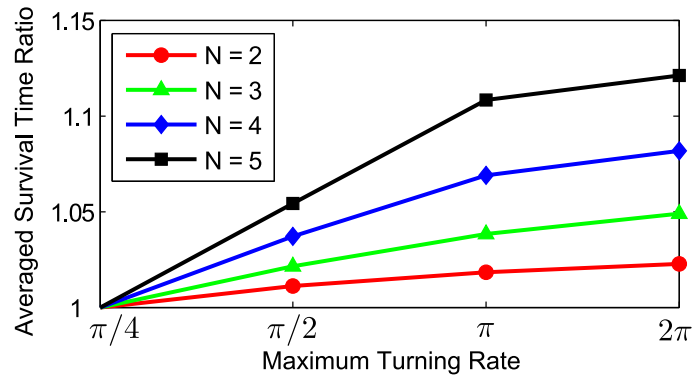
Figure 5.9: Averaged ratio of team survival time of different maximum turning rates to that of the lowest turning rate ($\pi/4$)

Survival Time Performance

The effects of the maximum turning rates of the evaders on the team survival time are evaluated on the benchmarking dataset. Each initial condition in the dataset is simulated 4 times with the maximum turning rates of the evaders set to $\pi/4$, $\pi/2$, π, and 2π radians per second. The team survival time achieved by a team of evaders with $\pi/4$ radians per second as their maximum turning rates is used as the baseline team survival to which other cases are compared. Figure 5.9 shows the averaged ratio of team survival time for teams with different maximum turning rates. The team survival time ratio improves as the maximum turning rates of the evaders increase, however the gain in performance also diminishes gradually. This is to be expected since even with unlimited turning rates the team still can not survive indefinitely and will be captured in finite time. Another important trend shown in the figure is that a team with more evaders benefits more from higher turning rates than a team with fewer evaders.

5.4 Alternative Objective Function for Collaborative Team Evasion

In the proposed open-loop and iterative open-loop approaches for team evasion, the objective of the team of evaders is to maximize the team survival time which is defined to be the survival time of the evader that survives the longest. This objective is a *selfless* one in that each member of the team focuses solely on maximizing the team survival time without any concern to its own survival time. Although the collaborative team evasion framework and its various extensions proposed in this dissertation are originally designed only for this specific objective, they are applicable for different objectives with some minor modifications. In this section, an alternative objective function for collaborative team evasion called *accumulative survival time* is proposed; the resulting behavior of the evaders using this objective function

is evaluated and discussed.

5.4.1 Accumulative Survival Time

In this section, the accumulative survival time is defined and the necessary modifications for the collaborative team evasion framework to use it as the objective are detailed.

The accumulative survival time is defined to be the positively weighted sum of the survival time of all the evaders in the team.

Definition 5.9 Accumulative team survival time
Given x_p^0 as the initial position of the pursuer, $\mathbf{x_e^0} = (x_1^0, \ldots, x_N^0)$ as the initial positions of N evaders, $u_p(\cdot)$ as the control of the pursuer, $\mathbf{u_e}(\cdot) = (u_1(\cdot), \ldots, u_N(\cdot))$ as the joint control of the evaders, and $\mathbf{w} = (w_1, \ldots, w_N)$ as the weightings where $w_i > 0$ for $i = 1, \ldots, N$, the accumulative team survival time is defined as

$$\bar{\Gamma}(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)) = \sum_{i=1}^{N} w_i \Gamma(x_p^0, x_i^0, u_p(\cdot), u_i(\cdot)),$$

where $\Gamma(x_p^0, x_i^0, u_p(\cdot), u_i^0(\cdot))$ is the capture time of evader i as defined in Definition 2.2.

Recall that the original *team survival time* is defined as

$$\Gamma(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)) = \sup_{i \in \{1, \ldots, N\}} \Gamma(x_p^0, x_i^0, u_p(\cdot), u_i(\cdot))$$

which is the maximum of the survival time of all evaders in the team. The accumulative survival time, on the other hand, measures the weighted sum of individual survival times of the evaders in the team. Note that according to the definition, both the team survival time and the accumulative survival time are infinity when the given controls for the pursuer and the evaders do not lead to the capture of all evaders.

Definition 5.10 Accumulative survival time under a capture sequence
Following Definition 5.9, the accumulative survival time under a specific capture sequence $\mathbf{s} = (s_1, \ldots, s_N)$ for N evaders is defined as

$$\bar{\Gamma}_{\mathbf{s}}(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)) =$$
$$\begin{cases} \infty & \text{if } \exists i > j \text{ such that } \Gamma(x_p^0, x_{s_i}^0, u_p(\cdot), u_{s_i}(\cdot)) < \Gamma(x_p^0, x_{s_j}^0, u_p(\cdot), u_{s_j}(\cdot)) \\ \bar{\Gamma}(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)) & \text{otherwise} \end{cases}$$
$$(5.20)$$

Unlike $\bar{\Gamma}(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot))$ in Definition 5.9 which is infinity only when there are evaders left not captured, the $\bar{\Gamma}_{\mathbf{s}}(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot))$ in Definition 5.10 can be infinity also when all the evaders are captured but not according to the capture sequence $\mathbf{s}$.

Definition 5.11 Minimum possible accumulative survival time under a capture sequence

Following Definition 5.10, the minimum possible accumulative survival time for a team of N evaders under a specific capture sequence $\mathbf{s} = (s_1, \ldots, s_N)$ is

$$\bar{\Gamma}_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot)) = \inf_{u_p(\cdot) \in \mathcal{U}} \bar{\Gamma}_{\mathbf{s}}(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)). \tag{5.21}$$

To compute the value of $\bar{\Gamma}_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot))$, an efficient way to determine the minimizer for the optimization problem on the right hand side of Eq. (5.21) is presented in the next section.

5.4.2 Optimal Control with Respect to Accumulative Survival Time

Theorem 3.3 in Chapter 3 states that to minimize the team survival time under a specific capture sequence given the joint control of the evaders, the pursuer should capture each evader in minimum possible time. In other words, greedily minimizing the survival time of the currently targeted evader will minimize the team survival time under a specific capture sequence for the pursuer. The following lemma shows that this is also true when the objective of the pursuer is to minimize the accumulative survival time under a specific capture sequence.

Lemma 5.4 Optimality of the greedy pursuer strategy for accumulative survival time

Given x_p^0 as the initial position of the pursuer, $\mathbf{x_e^0}$ as the initial positions of N evaders, v_p and $\mathbf{v_e}$ as their maximum speeds where $v_p > \mathbf{v_e}$, $\mathbf{u_e}(\cdot)$ as the joint control of the evaders, and $\mathbf{s}$ as the capture sequence, the following equality is always true:

$$\arg\inf_{u_p(\cdot) \in \mathcal{U}} \bar{\Gamma}_{\mathbf{s}}(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)) = \arg\inf_{u_p(\cdot) \in \mathcal{U}} \Gamma_{\mathbf{s}}(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)) \tag{5.22}$$

Proof. Define

$$u_p^{\star}(\cdot) = \arg\inf_{u_p(\cdot) \in \mathcal{U}} \Gamma_{\mathbf{s}}(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)) \tag{5.23}$$

as the optimal pursuer control to minimize the team survival time under capture sequence $\mathbf{s}$. Define

$$\hat{\mathbf{s}}^i = (s_1, \ldots, s_i) \tag{5.24}$$

as the capture sequence that keeps only the first i elements of $\mathbf{s}$. The survival time of the i-th evader in the capture sequence $\mathbf{s}$ given that the pursuer uses the control $u_p(\cdot)$ is

$$\Gamma_{\hat{\mathbf{s}}^i}(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)). \tag{5.25}$$

According to Theorem 3.3, the $u_p^{\star}(\cdot)$ in Eq. (5.23) is also the optimizer for the team survival time under this family of capture sequences, more specifically

$$u_p^{\star}(\cdot) = \arg\inf_{u_p(\cdot) \in \mathcal{U}} \Gamma_{\hat{\mathbf{s}}^i}(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot)) \text{ for } i = 1, \ldots, N. \tag{5.26}$$

The optimizer of a new objective function defined as the positively weighted sum of all $\Gamma_{\hat{\mathbf{s}}^i}$'s is still the same. Hence

$$u_p^\star(\cdot) = \underset{u_p(\cdot) \in \mathcal{U}}{\arg\inf} \sum_{i=1}^N w_i \Gamma_{\hat{\mathbf{s}}^i}(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot))$$

$$= \underset{u_p(\cdot) \in \mathcal{U}}{\arg\inf} \ \bar{\Gamma}_{\mathbf{s}}(x_p^0, \mathbf{x_e^0}, u_p(\cdot), \mathbf{u_e}(\cdot))$$

which completes the proof. ∎

With Lemma 5.4, Algorithm 3.1 can be used to evaluated the minimum possible accumulative survival time of a given layout, joint control of the evaders, and a specific capture sequence with some minor modifications. The modified algorithm is outlined in Algorithm 5.4.

Algorithm 5.4 Minimum accumulative survival time under a capture sequence given joint control of evaders: $\bar{\Gamma}_{\mathbf{s}}^\star(x_p^0, \mathbf{x_e^0}, \mathbf{u_e}(\cdot))$

1: Given $x_p^0, \mathbf{x_e^0} = (x_1^0, \ldots, x_N^0), \mathbf{s} = (s_1, \ldots, s_N), \mathbf{v_e} = (v_1, \ldots, v_N) < v_p, \mathbf{u_e}(\cdot) = (u_1(\cdot), \ldots, u_N(\cdot))$, and $\mathbf{w} = (w_1, \ldots, w_N)$
2: Initialize $x_p \leftarrow x_p^0, \tau \leftarrow 0, \bar{\tau} \leftarrow 0$
3: **for** $i = 1, \ldots, N$ **do**
4: $\quad x_{s_i}(\cdot) \leftarrow traj(x_{s_i}^0, v_{s_i}, u_{s_i}(\cdot))$
5: $\quad \tau \leftarrow \tau + \inf_{u_p(\cdot) \in \mathcal{U}} \Gamma(x_p, x_{s_i}(\tau), u_p(\cdot), u_{s_i}^{+\tau}(\cdot))$
6: $\quad x_p \leftarrow x_{s_i}(\tau)$
7: $\quad \bar{\tau} \leftarrow \bar{\tau} + w_{s_i}\tau$
8: **end for**
9: **return** $\bar{\tau}$

The $u_p^\star(\cdot)$ in Eq. (5.23) is defined as the optimal pursuer control with respect to a specific capture sequence. In the open-loop formulation of collaborative team evasion, the pursuer has the freedom to select any capture sequence. The optimal capture sequence for the pursuer is defined as

$$\mathbf{s}^\star = \underset{\mathbf{s} \in \mathcal{S}_N}{\arg\inf} \ \bar{\Gamma}_{\mathbf{s}}^\star(x_p^0, \mathbf{x_e}, \mathbf{u_e}(\cdot)),$$

which is the capture sequence that will minimize the accumulative team survival time given the specific layout and joint control of the evaders. It is worth noting that Lemma 5.4 does not imply that the optimal capture sequence for team survival time will be the same as the one for accumulative survival time, or that accumulative survival time with different weightings will share the same optimal capture sequence. Given the same layout and joint heading of the evaders, a pursuer might behave differently depending on whether it is aiming to minimize the team survival time or the accumulative survival time.

The open-loop optimal accumulative survival time for the team of evaders is defined as

$$\sup_{\mathbf{u_e}(\cdot) \in \mathcal{U}^N} \inf_{\mathbf{s} \in \mathcal{S}_N} \bar{\Gamma}_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e}^0, \mathbf{u_e}(\cdot)). \tag{5.27}$$

Theorem 3.5 which states that the optimal controls for the evaders in the open-loop formulation are constant heading controls is also true when the evaders are trying to maximize the positively weighted accumulative survival time instead of the team survival time. As a result, the open-loop team evasion problem with the accumulative survival time as the objective can also be simplified as

$$\sup_{\Theta_{\mathbf{e}}} \inf_{\mathbf{s} \in \mathcal{S}_N} \bar{\Gamma}_{\mathbf{s}}^{\star}(x_p^0, \mathbf{x_e}^0, \Theta_{\mathbf{e}}). \tag{5.28}$$

As a finite dimensional minimax problem, it can be converted to a constrained nonlinear optimization problem through the same procedure described in Section 3.3 and solved by sequential quadratic programming. With little or no modifications, the iterative open-loop approach proposed in Chapter 4 and the extensions to the framework proposed in the previous sections of this chapter are all applicable for the team evasion problem with accumulative survival time as the objective function.

5.4.3 Results and Discussion

Resulting Open-loop Optimal Behavior

In this section, the difference in behavior of the evaders when they use accumulative survival time as the objective and when they use team survival time as the objective is examined.

The team survival time as an objective function encourages selfless behavior from the evaders. For an evader that is not the last captured evader of the team, it has no concern of its own survival; the only way it can contribute to the team objective is by "luring" the pursuer to move in certain direction to force the pursuer to traverse a longer distance to reach the last evader. In other words, every evader in the team is trying to maximize the survival time of a specific evader when a specific capture sequence is concerned. The balance between the evaders comes from the fact that there exist multiple optimal capture sequences and the last captured evader might be different in each of them. The accumulative team survival time, on the other hand, encourages a more balanced collaborative behavior. In the case where the survival time of each evader is weighted equally regardless of its order of capture, every second an evader is surviving it contributes to the team objective in two ways: by its own survival time and the survival time it earns for its teammates. As a result, each evader tries to strike a balance between the survival time of itself and the team.

Figure 5.10 shows the resulting open-loop optimal trajectories with respect to different objectives for a 2-evader layout. The dark triangle is the position of the pursuer and the blue solid circles are the initial position of the evaders. The dashed lines are the resulting open-loop optimal trajectories given the layout and the hollow circles are the predicted capture points of

(a) Team survival time

(b) Uniformly weighted accumulative survival time
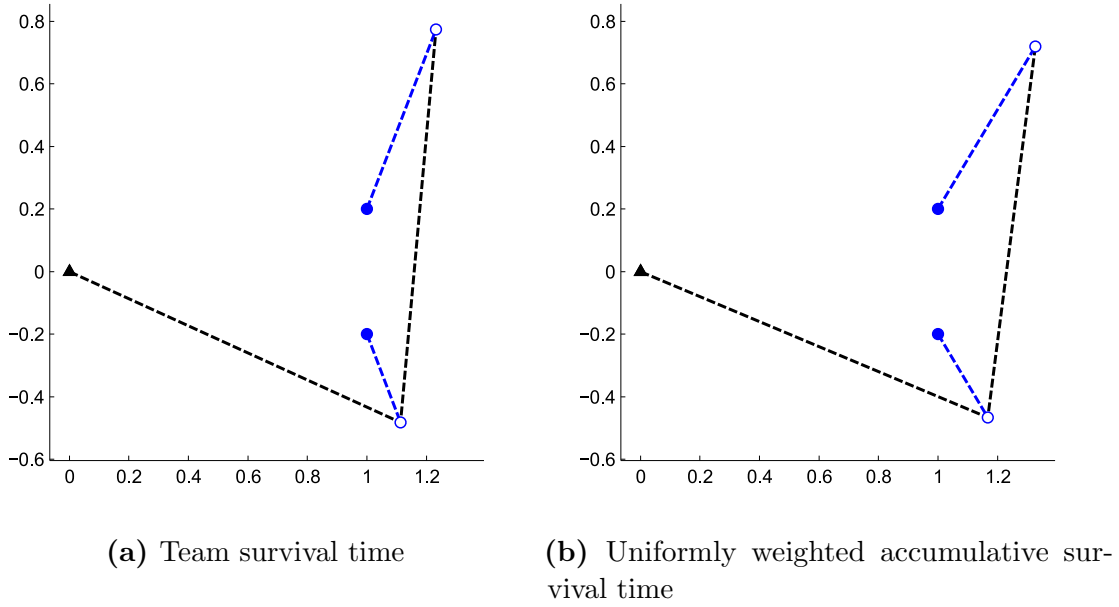
Figure 5.10: Resulting open-loop optimal trajectories of different objectives for a 2-evader layout

(a) Parameters of a 2-evader layout

(b) Split angle of the optimal joint heading with respect to different objectives given distance ratio
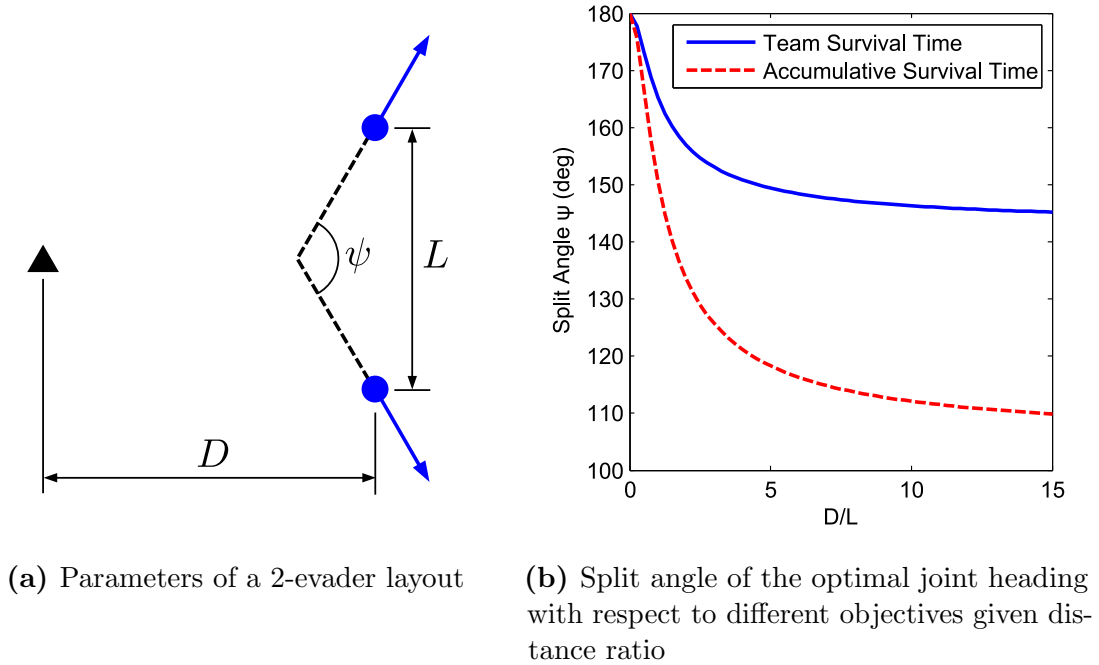
Figure 5.11: The optimal joint heading with respect to the team survival time and accumulative survival time for a family of 2-evader layouts

the evaders. Figure 5.10a shows the resulting trajectories using the team survival time as the objective and Fig. 5.10b shows those using uniformly weighted accumulative survival time as the objective. The evaders using the accumulative team survival time as their objective point slightly more to the right and away from the pursuer. This results in a longer survival time for the evader that is captured first compared to the one shown in Fig. 5.10a. Figure 5.11 shows the difference in the optimal joint heading with respect to the two different objectives for a family of 2-evader layouts. As shown in Fig. 5.11a, the layout is line symmetric to the x-axis. The distance between the evaders is denoted by L and the horizontal distance from the pursuer to the evaders is denoted by D. A layout in this family is characterized by the distance ratio D/L; the larger this ratio is, the farther away the pursuer is from the evaders compared to the distance between the evaders. Due to the symmetric nature of both the layout and the open-loop formulation of team evasion, the headings of the 2 evaders are also always symmetrical to the x-axis. Hence, the optimal joint heading can be specified by the angle between the headings of the evaders which is referred to as the *split angle* and denoted by ψ. A split angle of 0 degree indicates that the evaders are both moving directly to the right; a split angle of 180 degrees indicates that the evaders are moving directly away from each other. Figure 5.11b shows the resulting split angle given different distance ratios and different objectives. The split angle is always smaller when the objective is to maximize the accumulative survival time. The difference between the split angles resulting from the two objectives is more pronounced when the distance ratio is larger.

Figure 5.12 shows the resulting optimal trajectories with respect to different objectives for a 3-evader layout that is symmetrical to the x-axis. Figure 5.13 shows the split angle under different distance ratios resulting from different objectives. Note that for this family of 3-evader layouts, which is symmetrical to the x-axis, the optimal heading of the evader in the middle is always pointing to the right and the optimal headings of the upper and lower evaders are always symmetrical. Hence, the open-loop optimal joint heading can again be specified by the split angle ψ as defined in Fig. 5.13a. Figure 5.13b shows the split angle optimal for different objectives with different distance ratio. Similar to the trend shown in Fig. 5.11b, the split angle optimal for the accumulative survival time is smaller than the split angle optimal for the team survival time and the difference between them increases with the distance ratio. It is worth noting that the split angle optimal for the team survival time is always above 280 degrees regardless of the distance ratio; the headings of the upper and lower evaders are always pointing in the second and third quadrants respectively. However, the split angle optimal for the accumulative survival time can range from 290 degrees to 140 degrees; when the pursuer is far away from the team, the headings of the upper and lower evaders can be pointing in the first and fourth quadrants respectively.

Accumulative Survival Time Performance of the Iterative Open-loop Approach

Similar to the original open-loop approach for team evasion with respect to team survival time, the one with respect to accumulative survival time can also be applied iteratively to relax

(a) Team survival time

(b) Uniformly weighted accumulative survival time
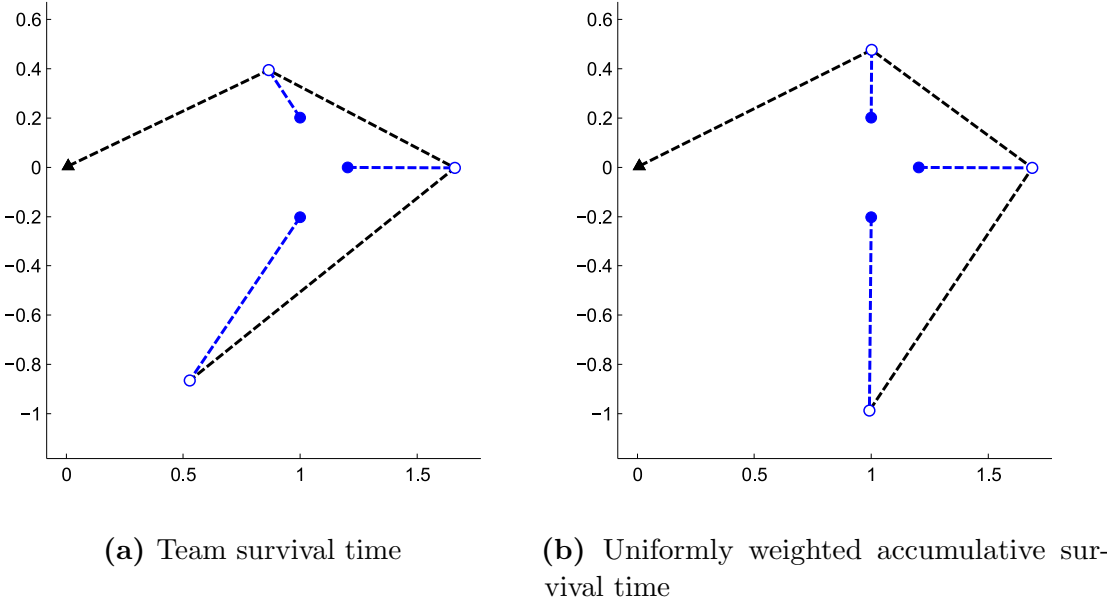
Figure 5.12: Resulting open-loop optimal trajectories of different objectives for a 3-evader layout

(a) Parameters of a 3-evader layout

(b) Split angle of the optimal joint heading with respect to different objectives given distance ratio
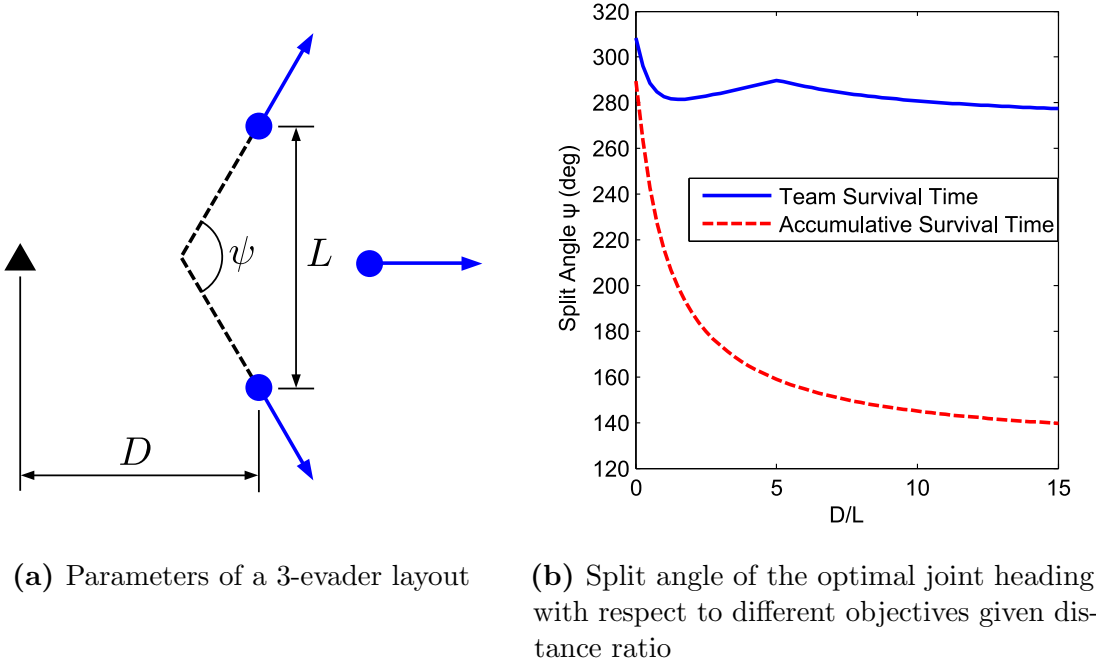
Figure 5.13: The optimal joint heading with respect to the team survival time and accumulative survival time for a family of 3-evader layouts

the conservatism of the open-loop formulation. To evaluate the performance of the iterative open-loop approach for team evasion with accumulative survival time as the objective, a reasonable baseline approach has to be developed. Consider a naive approach to team evasion where each evader tries only to maximize its own survival time by heading directly away from the pursuer at every update. This selfish approach serves well as the baseline approach to which the proposed collaborative approach is compared in that it highlights the performance that can be gained through collaboration.

Figure 5.14 shows the distribution of ratios of accumulative survival time of the collaborative iterative open-loop approach to that of the selfish approach for teams with different number of evaders. From Figs. 5.14a to 5.14d, the benefit in terms of the accumulative survival time is more pronounced when there are more evaders in the team. For a team with 2 evaders, in a lot of the initial conditions the selfish approach performs as well as the collaborative approach. For a team with 5 evaders, the collaborative approach constantly outperforms the selfish approach by a factor of 1.4. These results show the importance of collaboration in the team evasion scenario.

5.5 Conclusion

Three extensions to the collaborative team evasion framework are proposed in this chapter. They enable the framework to handle uncertainties in pursuer's position, evaders with limited turning rates, and the family of accumulative survival time as the objective function.

To handle uncertainties in the pursuer's position, the original open-loop formulation for the team evasion problem conservative to the evaders is augmented with an extra layer of minimization which allows the pursuer to select its position from a circular disk given the joint control of the evaders and the capture sequence. With the proposed algorithm, the augmented open-loop optimization problem can be solved iteratively using the most current state of the game as in the original iterative open-loop approach. By adjusting the center and radius of the pursuer disk at each update time according to the previously known position of the pursuer and its maximum speed, the evaders can efficiently extend their team survival time against a hidden pursuer. The effectiveness of the proposed extension is evaluated through simulations, and the results show that the framework is capable of losing as little as 5 percent of team survival time when facing an optimal hidden pursuer instead of a visible one.

To handle evaders with turning rate constraints, the admissible control set of the evaders is replaced with the compressed Dubins admissible control set which contains all controls that starts by turning with maximum turning rate and then travels with constant heading thereafter. An algorithm is developed to evaluate the minimum possible survival time of an evader with such control. This algorithm requires only the evaluation of a closed-form formula when the evader is captured on the constant heading section of its trajectory; it requires a one dimensional root searching routine when the evader is captured on the curved

(a) 2 evaders

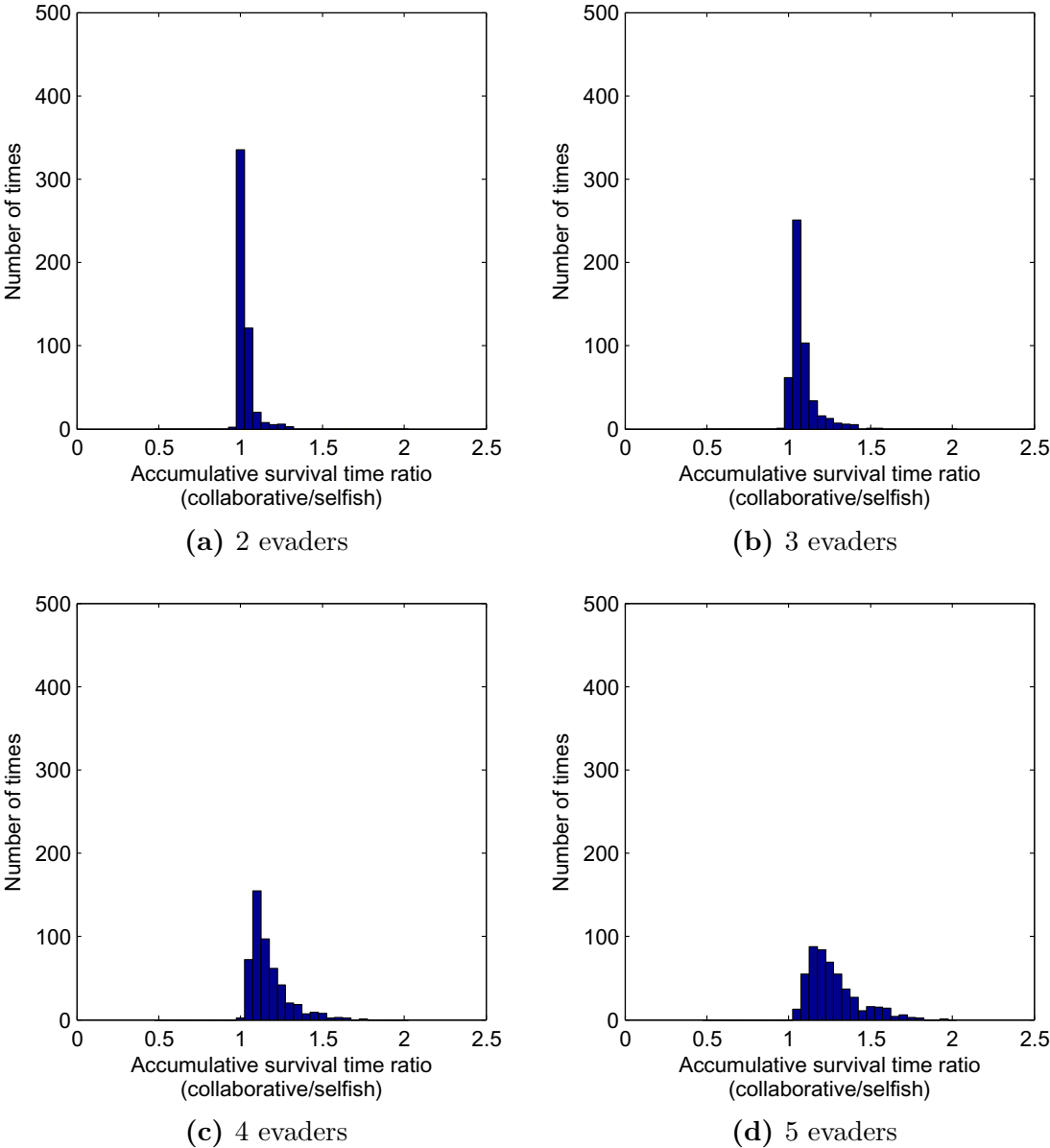**(b)** 3 evaders

(c) 4 evaders

(d) 5 evaders

Figure 5.14: Distribution of ratio of accumulative survival time of the collaborative iterative open-loop approach to that of the selfish approach for teams with different number of evaders

section of its trajectory. The simulation results show that a team with more evaders benefits more from higher turning rates than a team with fewer evaders.

The accumulative survival time is defined as the weighted sum of the survival times of the evaders in the team. It is shown that the optimal pursuer trajectory to minimize a positively weighted accumulative survival time of the team under a specific capture sequence is identical to the trajectory that minimizes the team survival time under the same capture sequence. Hence, the previously proposed algorithms for evaluating the minimum possible team survival time can also be used to evaluate the minimum possible accumulative survival time with minor modification. The uniformly weighted accumulative survival time is used as an example to demonstrate the different behaviors of the team resulting from different objectives of team evasion. The simulations results show that the collaborative team evasion framework with uniformly weighted accumulative survival time as its objective performs much better than a naive non-collaborative approach.

While the open-loop and iterative open-loop approaches proposed in Chapter 3 and Chapter 4 are derived under simple assumptions such as unlimited turning rates and perfect sensing, the extensions proposed in this chapter demonstrate the flexibility of the collaborative team evasion framework and its applicability in realistic team evasion scenarios.

Chapter 6

Conclusions

This dissertation proposes the collaborative team evasion framework as an evader-centric solution to the single-pursuer-multiple-evader pursuit-evasion game where a team of evaders aims to delay the capture of the whole team by a faster pursuer – the team evasion game. Compared to most of the work in the literature on the team evasion game that uses pursuer-centric formulations and results in overly aggressive strategies for the evaders, the proposed collaborative team evasion framework is conservative to the evaders and produces reliable strategies for the team with guaranteed team survival time. The proposed approximations to the framework lower the requirements on computational power through different levels of abstraction and simplification and allow the framework to be implemented on simpler platforms. The various extensions proposed in this dissertation improve the applicability of the collaborative team evasion framework in more realistic team evasion scenarios by enabling the framework to handle uncertainties in the position of the pursuer, constraints on the turning rates of the evaders, and to use accumulative survival time as the team objective.

6.1   Summary

Chapter 1 of this dissertation reviews the history and the current state of the literature on the single-pursuer-multiple-evader pursuit-evasion games. The lack of evader-centric formulations in the literature is identified as the main motivation for this dissertation.

The team evasion problem is defined in Chapter 2 along with its closed-loop formulation, control-control open-loop formulations, and sequence-control open-loop formulations. The properties and implications of these formulations are presented and discussed.

Chapter 3 focuses on the open-loop formulation of the team evasion game conservative to the evaders. In this formulation the joint control of the evaders is made available to the pursuer and the pursuer is assumed to act optimally against the selected joint control of the evaders. It is shown that under a given capture sequence the optimal strategy for the pursuer is to greedily minimize the survival time of the currently targeted evader, and the open-loop

optimal trajectories of the evaders are always straight lines. The minimum possible team survival time of a given capture sequence and joint heading can be evaluated efficiently by exploiting these properties of the optimal open-loop trajectories. The open-loop optimal joint heading of the evaders can then be obtained by solving a finite dimensional sup-inf optimization problem. The performance of the proposed open-loop approach for collaborative team evasion is evaluated through extensive simulations and is shown to outperform the pursuer-centric frameworks in the literature in terms of team survival time.

In Chapter 4 the derivation and implementation of the iterative open-loop approach is presented in detail. The iterative open-loop approach relaxes the conservatism of the open-loop approach by re-solving the open-loop problem at a per-determined frequency using the most current state of the game. The simulation results show that the iteratively open-loop approach consistently outperforms the open-loop approach in terms of team survival time. Several approximations to the framework are also proposed in this chapter, including the iterative linear programming approach and the gradient-based approach with constraint sampling heuristics. The iterative linear programming approach linearizes the nonlinear optimization problem in the original iterative open-loop approach by making use of the gradient of the team survival time with respect to the joint heading of evaders. The gradient-based approach further decreases the requirement on computational power by only considering the gradient of the team survival time under the optimal capture sequence. The proposed approximations are capable of achieving similar team survival time as the original iterative open-loop approach with much less computation time.

In Chapter 5, the collaborative team evasion framework is augmented with several extensions to improve its applicability in more realistic team evasion scenarios. Three extensions are proposed in this chapter. The first extension enables the team to effectively delay capture by a hidden pursuer as long as the maximum speed of the pursuer is know or can be estimated. With this extension, the framework can achieve similar team survival time when facing a hidden pursuer and when facing a visible pursuer. The second extension allows evaders with limited turning rates to plan their evasion accordingly. The impact of the turning rate constraints are shown to be more significant when there are more evaders in a team. Finally, the third extension improves the flexibility of the collaborative team evasion framework by allowing the team to maximize the weighted sum of survival times of its members instead of the team survival time.

Finally, Chapter 6 concludes this dissertation and provides possible directions for future research.

6.2   Future Work

Several aspects of the collaborative team evasion framework can be improved and expanded through further research.

Scaling to larger evader team

In its current centralized form, the proposed collaborative team evasion framework cannot be implemented efficiently for teams with large number of evaders due to the factorial growth of the possible capture sequence with the number of evaders: for a team with N evaders, there exist $N!$ possible capture sequences. Through distributive optimization techniques, such as the proportional-integral distributive optimization proposed in [50], it is possible to develop a distributive version of the collaborative team evasion framework where each evader only has to solve a smaller open-loop team evasion problem involving its neighbors. The accumulative team survival time will be the more appropriate objective for the team in this distributed version of the framework in that it allows the evaders to adjust the weighting of its own survival time according to its connectivity.

Expected Survival Time and Pursuer Modeling

By assuming that the pursuer will always act optimally by picking the best capture sequence and corresponding control against the joint heading of the evaders, the current formulation of the collaborative team evasion aims to maximize the worst-case survival time of the team. However, since selecting the optimal capture sequence to capture a team of moving evaders is a very challenging problem, it is very likely that in a realistic team evasion scenario the pursuer will not be able to select the optimal capture sequence consistently and will exhibit some bias in its selection. Modifying the framework so that the evaders aim to maximize the expected value of the survival time instead of the worst-case survival time can potentially better exploit the suboptimality of the pursuer in a more realistic setting. Tools such as Bayesian inference can be used to infer the intention of the pursuer based on its previous actions which might be sufficient for the evaders to determine the next target of the pursuer. The main challenge is to obtain the probability distribution of how likely the pursuer is to select each capture sequence given the current state of the game.

Alternative Objectives

In the proposed collaborative team evasion framework, the only goal of the team is to survive as long as possible. An interesting direction for future research is to explore other possible objectives for team evasion such as ensuring a specific member of the team can reach certain locations before being captured.

Appendix A

Gradient of the Team Survival Time

Recall the gradient of the team survival time with respect to the joint heading of evaders under a specific capture sequence defined in Section 4.3.1 as

$$\nabla_{\Theta_{\mathbf{e}}} \Gamma_{\mathbf{s}}^{\star} = \left[\sum_{j=1}^{N} \frac{\partial \hat{\tau}_{s_j}}{\partial \theta_1}, \sum_{j=1}^{N} \frac{\partial \hat{\tau}_{s_j}}{\partial \theta_2}, \dots, \sum_{j=1}^{N} \frac{\partial \hat{\tau}_{s_j}}{\partial \theta_N} \right], \tag{A.1}$$

where

$$\hat{\tau}_{s_j} = \tau_{s_j} - \tau_{s_{j-1}} \tag{A.2}$$

and

$$\tau_{s_j} = \Gamma_{\mathbf{s}'}^{\star}(x_p, \mathbf{x_e}, \Theta_{\mathbf{e}}) \text{ with } \mathbf{s}' = (s_1, \dots, s_j). \tag{A.3}$$

The partial derivative $\frac{\partial \hat{\tau}_{s_j}}{\partial \theta_i}$ represents how the change of heading of evader i affects the survival time of the j-th evader in the capture sequence. Note that here the subscript i is used to refer to the indexes of the evaders, and the subscript j is used to refer to the order of an evader in the capture sequence. To clarify the relationship between the two, *captureOrder* is defined as follows.

Definition A.1 Capture Order of an Evader in a Sequence
The capture order of evader i in a capture sequence $\mathbf{s} = (s_1, \dots, s_N)$ is defined as

$$captureOrder(i) = \{ j \mid s_j = i \} \tag{A.4}$$

To compute the partial derivatives, recall Corollary 3.11 which can be written as

$$\hat{\tau}_{s_j} = \frac{v_{s_j}}{v_p^2 - v_{s_j}^2} \left(\hat{x}_{s_j} \cdot \hat{e}_{\theta_{s_j}} + \sqrt{(\hat{x}_{s_j} \cdot \hat{e}_{\theta_{s_j}})^2 + (\frac{v_p^2}{v_{s_j}^2} - 1)(\hat{x}_{s_j} \cdot \hat{x}_{s_j})} \right), \tag{A.5}$$

where

$$\hat{x}_{s_j} = x_{s_j}(\tau_{s_{j-1}}) - x_p(\tau_{s_{j-1}}) \tag{A.6}$$

is the vector pointing from the pursuer to the j-th evader in the capture sequence when the $(j-1)$-th evader is captured and

$$\hat{e}_{\theta_{s_j}} = (\cos \theta_{s_j}, \sin \theta_{s_j}) \tag{A.7}$$

is the unit vector pointing at the heading direction of the j-th evader in the capture sequence. The value of $\frac{\partial \hat{\tau}_{s_j}}{\partial \theta_i}$ depends on the relationship between j and $captureOrder(i)$. Since the heading of an evader cannot affect the survival time of the evaders preceding it in the capture sequence,

$$\frac{\partial \hat{\tau}_{s_j}}{\partial \theta_i} = 0 \text{ for } j < captureOrder(i). \tag{A.8}$$

For $j \geq captureOrder(i)$, the partial derivatives follow the general form:

$$\frac{\partial \hat{\tau}_{s_j}}{\partial \theta_i} = \frac{v_{s_j}}{v_p^2 - v_{s_j}^2} \left\{ A_{i,j} + \frac{1}{B_j} \left((\hat{x}_{s_j} \cdot \hat{e}_{\theta_{s_j}}) A_{i,j} + (\frac{v_p^2}{v_{s_j}^2} - 1)(\hat{x}_{s_j} \cdot \frac{\partial \hat{x}_{s_j}}{\partial \theta_i}) \right) \right\}, \tag{A.9}$$

where

$$B_j = \sqrt{(\hat{x}_{s_j} \cdot \hat{e}_{\theta_{s_j}})^2 + (\frac{v_p^2}{v_{s_j}^2} - 1)(\hat{x}_{s_j} \cdot \hat{x}_{s_j})} \tag{A.10}$$

and the terms $A_{i,j}$ and $\frac{\partial \hat{x}_{s_j}}{\partial \theta_i}$ takes different forms depends on how much larger j is to $captureOrder(i)$.

For $j = captureOrder(i)$,

$$A_{i,j} = \hat{x}_{s_j} \cdot \frac{\partial \hat{e}_{\theta_{s_j}}}{\partial \theta_{s_j}} \tag{A.11}$$

$$\frac{\partial \hat{x}_{s_j}}{\partial \theta_i} = (v_{s_j} \sum_{k=1}^{j-1} \hat{\tau}_{s_k}) \frac{\partial \hat{e}_{\theta_{s_j}}}{\partial \theta_{s_j}}. \tag{A.12}$$

For $j = captureOrder(i) + 1$,

$$A_{i,j} = \frac{\partial \hat{x}_{s_j}}{\partial \theta_i} \cdot \hat{e}_{\theta_{s_j}} \tag{A.13}$$

$$\frac{\partial \hat{x}_{s_j}}{\partial \theta_i} = \frac{\partial \hat{\tau}_{s_{j-1}}}{\partial \theta_i} (v_{s_j} \hat{e}_{\theta_{s_j}} - v_{s_{j-1}} \hat{e}_{\theta_{s_{j-1}}}) - (v_{s_{j-1}} \sum_{k=1}^{j-1} \hat{\tau}_{s_k}) \frac{\partial \hat{e}_{\theta_{s_{j-1}}}}{\partial \theta_{s_{j-1}}}. \tag{A.14}$$

For $j > captureOrder(i) + 1$,

$$A_{i,j} = \frac{\partial \hat{x}_{s_j}}{\partial \theta_i} \cdot \hat{e}_{\theta_{s_j}} \tag{A.15}$$

$$\frac{\partial \hat{x}_{s_j}}{\partial \theta_i} = (v_{s_j} \hat{e}_{\theta_{s_j}} - v_{s_{j-1}} \hat{e}_{\theta_{s_{j-1}}}) \sum_{k=captureOrder(i)}^{j-1} \frac{\partial \hat{\tau}_{s_k}}{\partial \theta_i}. \tag{A.16}$$

Note that the value of $\frac{\partial \hat{\tau}_{s_j}}{\partial \theta_i}$ depends on the value of $\hat{\tau}_{s_k}$ and $\frac{\partial \hat{\tau}_{s_k}}{\partial \theta_i}$ for all $k < j$. Hence, to compute the gradient vector $\nabla_{\Theta_e} \Gamma_s^\star$ of a specific tuple $(x_p, \mathbf{x_e}, \Theta_e)$ under a specific capture sequence $\mathbf{s} = (s_1, \ldots, s_N)$, the survival time of each evader has to first be computed using Algorithm 3.1. Then, the value of the $\frac{\partial \hat{\tau}_{s_j}}{\partial \theta_i}$'s can be computed in a sequential fashion starting with $i = s_1$ and $j = 1$ in that the survival time of the first evader in the capture sequence depends only on the layout and its own heading. The partial derivatives of $i = s_1, j = 2, \ldots, N$ can then be computed sequentially with increasing j. With the value of $\frac{\partial \hat{\tau}_{s_j}}{\partial \theta_i}$ for $i = s_1, j = 1, \ldots, N$, the value of the partial derivatives of $i = s_2$ can then be computed sequentially with $j = 2, \ldots, N$ in a similar fashion. The value of all the partial derivatives can be computed by following this procedure. The amount of computation scales polynomially with the number of evaders.

Appendix B

Solutions for 2-evader Case and Pursuer-centric Formulations

In this appendix, previous work on the successive pursuit problem proposed is reviewed concisely. The solution to the closed-loop formulation of the team evasion game for teams with 2 evaders proposed in [23] is presented in Section B.1 and the general solution to the pursuer-centric formulation of the game proposed in [34] is presented in Section B.2.

B.1   Point Capture of Two Evaders in Succession

The closed-loop team evasion problem is referred to as the *successive pursuit problem* by Breakwell et al. in [23] since the pursuer has to capture all evaders in succession. Although only applicable for a 2-evader team, it is one of the few work in the literature where the closed-loop formulation of the team evasion problem is solved exactly without modification or simplification to the formulation. The solution approach and the characteristics of the solution motivate the open-loop team evasion framework proposed in this dissertation and hence are presented in this section.

The optimal solution to the closed-loop team evasion game with two evaders can be put into one of the two categories: the *geometric solutions* and the *non-geometric solutions*.

Geometrical Solution

The geometric solution is based on the simplifying assumption that the pursuer has to commit to a specific capture sequence that is known to the evaders. While this is not the complete solution to the closed-loop team evasion problem, it is optimal for some initial conditions.

For the case where there is only one evader, all possible capture points of the evader are within the *Apollonius disk* defined as following:
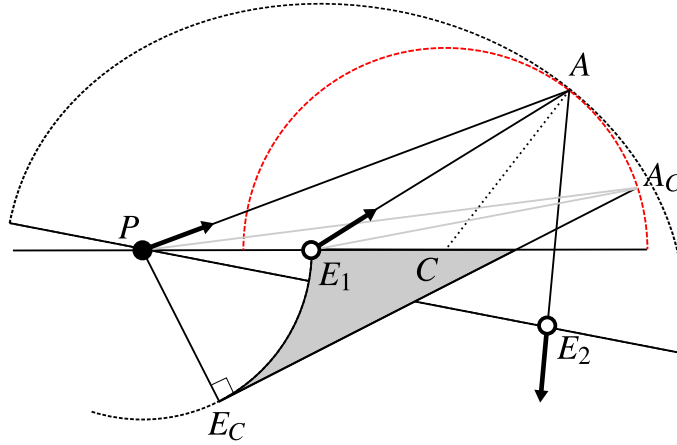
Figure B.1: Geometrical solution for a 2 evader team (Edited from [23])

Definition B.1 Apollonius Disk
For an evader starting at x_e^0 with a maximum speed of v_e and a pursuer starting at x_p^0 with a maximum speed of v_p, the Apollonius disk of this pair of pursuer and evader is

$$Apo(x_p^0, x_e^0, v_p, v_e) = \{x \mid \frac{\|x - x_e^0\|_2}{\|x - x_p^0\|_2} \leq \frac{v_e}{v_p}\}. \tag{B.1}$$

The boundary of the disk, denoted by $\partial Apo(x_p^0, x_e^0, v_p, v_e)$, is referred to as the *Apollonius circle*. For any point on the circle, the ratio of the distance to the evader and that to the pursuer is v_e/v_p. The evader can reach any point inside the circle before the pursuer by traveling with maximum speed in a constant heading; it can reach points on the circle at the same time as an optimal pursuer. In the team evasion problem with a single evader, the evader will always pick the point on the circle that is farthest away from the pursuer to maximize its survival time. By moving directly away from the pursuer, the evader can achieve the maximum survival time

$$\tau = \frac{\|x_p^0 - x_e^0\|_2}{(v_p - v_e)}, \tag{B.2}$$

which is the initial distance between the pursuer and the evader divided by the rate at which the distance shrinks. When the maximum speeds of the pursuer and the evader are given, the optimal survival time is directly proportional to the initial distance between the pursuer and the evader.

For a team with two evaders, the optimal strategy of the evaders is more complicated. Consider the case where the pursuer has to capture the evaders according to a specific capture sequence. Assuming that the first evader is capture at time τ_1 and the capture point is denoted by $x_1(\tau_1)$, the optimal strategy for the second evader to maximize its survival time,

which by definition is the team survival time, is to maximize its distance from $x_1(\tau_1)$ by moving directly away from it. It is worth emphasizing that this strategy for the second evader is optimal under the condition that the pursuer must capture the evaders according to the capture sequence. Recall that the first evader can select its capture point to be anywhere on its Apollonious by moving directly towards the point with constant heading and maximum speed. The following remark describes the resulting team survival time given that the first evader is captured on a specific point on the Apollonious circle and that the second evader moves directly away from this capture point starting from the beginning of the game.

Remark B.1 Maximum survival time of the second evader given a specific capture point of the first evader

Given x_p^0, x_1^0, and x_2^0 as the initial positions of the pursuer and two evaders, v_p and v_e as their maximum speeds and assuming that the first evader is captured by the pursuer on $x_1(\tau_1) \in \partial Apo(x_1^0, x_p^0, v_e, v_p)$ at time τ_1, the maximum survival time for evader 2 against the optimal pursuer is

$$\tau_2 = \frac{\left\|x_p^0 - x_1(\tau_1)\right\|_2 + \left\|x_1(\tau_1) - x_2^0\right\|_2}{v_p - v_e}. \tag{B.3}$$

Note that the survival time of the second evader, given that it acts optimally, depends on the capture point selected by the first evader. Different capture points on the Apollonius disk will result in different optimal trajectories for the second evader and hence different team survival times. The capture point that will result in the maximum possible team survival time must satisfy the conditions stated in the following remark:

Remark B.2 Geometrical condition for the optimal capture point of the first evader

Denote an ellipse with its foci located at x_p^0 and x_1^0 and passes through $x_1(\tau_1)$ by $Elli(x_p^0, x_2^0, x_1(\tau_1))$. Under the same assumptions in Remark B.1, the capture point of the first evader, denoted by $x_1(\tau_1)$, results in maximum possible survival time for the second evader starting at x_2^0 if and only if $\partial Elli(x_p^0, x_2^0, x_1(\tau))$ intersects $\partial Apo(x_1^0, x_p^0, v_e, v_p)$ only at $x_1(\tau_1)$.

Note that for all $x \in Elli(x_p^0, x_2^0, x_1(\tau))$,

$$\left\|x - x_p^0\right\|_2 + \left\|x - x_2^0\right\|_2 \leq \left\|x_1(\tau_1) - x_p^0\right\|_2 + \left\|x_1(\tau_1) - x_2^0\right\|_2. \tag{B.4}$$

As shown Fig. B.1, when $x_1(\tau_1)$ is the only point on the Apollonius circle that touches the boundary of the ellipse, the Apollonius disk is completely enclosed by the ellipse and hence every other possible capture points of the first evader will result in a shorter team survival time compared to $x_1(\tau_1)$. An important geometrical property of this tangent point, denoted by A in Fig. B.1, is that the line $\overline{CA}$ where C is the center of the Apollonius circle bisects $\angle PAE_2$.

Although this geometrical problem has no closed-form solution, it can be formulated as a root searching problem as described in [34] which will be reviewed in the next section.

With the optimal capture point determined, the optimal control for the first evader is to move toward this optimal capture point with constant heading and maximum speed. For the second evader, the optimal control is to move directly away from the capture point of the first evader with constant heading and maximum speed.

The geometrical solution is only optimal under the condition that the pursuer is committed to capture the evaders according to the capture sequence. It is optimal to the closed-loop formulation of the game only when the specified capture sequence stays optimal for the whole duration of the game. Using the layout shown in Fig. B.1 as an example, if the initial position of the second evader is within the region colored in gray, the second evader will become the closer evader to the pursuer when following the geometrical solution and the geometrical solution will no longer be optimal. For these initial conditions, the optimal solutions for the pursuer and the evaders are non-geometrical and have to be solved differently.
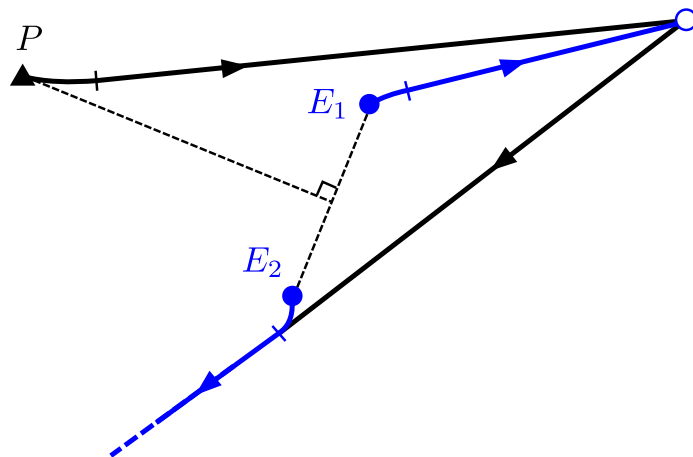
Non-geometrical Solution

Figure B.2: Example of non-geometrical solution for a team of 2 evaders against a faster pursuer (Edited from [23])

The team survival time of the closed-loop team evasion game, denoted by V, is a function of the joint position of the agents. For a 2-evader team, the value can also be denoted as a function of the position vectors of the pursuer and the two evader as $V(x_p(t), x_1(t), x_2(t))$. An action vector, denoted by $u_i(t)$, represents the instantaneous heading of agent i in the form of a unit vector pointing at the traveling direction of the agent. The derivative of the position vector of an agent at time t is determined by its action vector according to

$$\frac{dx_i}{dt}\bigg|_t = v_i u_i(t). \tag{B.5}$$

The main equation of Isaacs for the successive pursuit problem with 2 evaders is:

$$\min_{u_p} \max_{u_1,u_2} \left\{ \frac{\partial V}{\partial x_p} \cdot v_p u_p + \sum_{i=1}^{2} \frac{\partial V}{\partial x_i} \cdot v_i u_i \right\} + 1 = 0. \tag{B.6}$$

Since the dynamics of the agents are disjointed, the optimal action vector of the pursuer should parallel $\frac{\partial V}{\partial x_p}$ in opposite direction, and the optimal action vector for evader i should parallel $\frac{\partial V}{\partial x_i}$. By substituting the optimal action vectors, Eq. (B.6) becomes

$$1 - v_p \left| \frac{\partial V}{\partial x_p} \right| + \sum_{i=1}^{2} v_i \left| \frac{\partial V}{\partial x_i} \right| = 0. \tag{B.7}$$

The value of the game, although originally defined as a function of the position vectors of the pursuer and the 2 evaders denoted by $V(x_p, x_1, x_2)$, can also be defined as $V(\bar{x}_1, \bar{x}_2)$, where $\bar{x}_i = x_i - x_p$ for $i = 1, 2$ are the relative position vectors pointing from the pursuer to the 2 evaders respectively. The partial derivatives of V with respect to the pursuer position vector x_p is

$$\frac{\partial V}{\partial x_p} = \frac{\partial V}{\partial \bar{x}_1} \frac{\partial \bar{x}_1}{\partial x_p} + \frac{\partial V}{\partial \bar{x}_2} \frac{\partial \bar{x}_2}{\partial x_p} \tag{B.8}$$

according to the chain rule. Since $\frac{\partial \bar{x}_i}{\partial x_p} = -1$ and $\frac{\partial V}{\partial \bar{x}_i} = \frac{\partial V}{\partial x_i}$ for $i = 1, 2$, the previous equation can be rewritten as

$$\frac{\partial V}{\partial x_p} + \frac{\partial V}{\partial x_1} + \frac{\partial V}{\partial x_2} = 0. \tag{B.9}$$

This equation governs the heading directions of the pursuer and the evaders during the game.

Unlike the geometrical solution where the optimal trajectories of the agents are all straight lines, in the non-geometrical solution the optimal trajectories are composed of straight lines and curves. The curved phase of the game starts when the two evaders are equidistant to the pursuer. More specifically, when

$$\|\bar{x}_1\| = \|\bar{x}_2\|. \tag{B.10}$$

The curved phase ends when the relative configuration of the game reaches the point represented by the point E_C in Fig. B.1. By denoting the headings of the agents through a measurement relative to a fixed direction in the coordinate system and combining Eq. (B.7), Eq. (B.9), and Eq. (B.10), the resulting optimal trajectories can be derived through backward integration of a system of ODEs detailed in [23]. Figure B.2 shows an example of the non-geometrical solution to the team evasion problem. The initial position of the pursuer is marked by a triangle and the initial positions of the two evaders are marked by solid circles. Note that the evaders are equidistant to the pursuer at the beginning of the game and stay so in the curved sections of their trajectories. The end of the curved sections are marked by a bar on the trajectories of the agents. After reaching these points, the agents then travel in straight lines as in the case of the geometrical solution. In this example, the pursuer decides

to capture evader E_1 before evader E_2. The resulting capture point of the first evader is marked by a hollow circle in the figure. The second evader heads directly away from the capture point of the first evaders at the end of its curved section.

B.2   Solution to the Pursuer-centric Formulation

In the previous section, it has been shown that when the pursuer has to capture two evaders according to a specific capture sequence, the optimal headings of the evaders have to satisfy certain geometrical conditions. In [34], this result is extended to the general case of N evaders through the first-order optimality conditions. The derivation is reviewed concisely in this section.

Denote the initial position of evader i by x_i, its capture point by y_i, and its maximum speed by $\beta_i < 1$ for $i = 1, \ldots, N$. Without lost of generality, the speed of the pursuer is defined to be one and the capture sequence is such that the evaders are captured according to their indexes. The survival time of the i-th evader in the capture sequence is

$$T_i = \frac{1}{\beta_i} \|y_i - x_i\|, \tag{B.11}$$

which is the distance between the starting point and the capture point of the i-th evader divided by its maximum speed. Define the following functions

$$G_i(y_i, y_{i-1}) = \frac{1}{\beta_i} \|y_i - x_i\| - \frac{1}{\beta_{i-1}} \|y_{i-1} - x_{i-1}\| - \|y_i - y_{i-1}\| \tag{B.12}$$

for $i = 1, \ldots, N$ and also $x_0 = y_0$ and $\frac{1}{\beta_0} = 0$. These functions compute the difference between the time the i-th evader can survival after the $(i-1)$-th evader is captured and the time it takes the pursuer to travel from the capture point of the $(i-1)$-th evader to that of the i-th evader; for an optimal pursuer, this difference should always be zero.

To maximize the team survival time, the evaders solve the following constrained optimization problem:

$$\max_{y_1, \ldots, y_N} T_N \tag{B.13}$$

$$\text{s.t. } G_i(y_i, y_{i-1}) = 0, \text{for } i = 1, \ldots, N. \tag{B.14}$$

As pointed out in [34], the Lagrange multiplier rule states that for a local maximum of the constrained optimization problem, denoted by $(y_1^\star, \ldots, y_N^\star)$, there exists a nonzero vector $(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$ such that the partial derivatives of the function

$$f(y_1, \ldots, y_N, \lambda_1, \ldots, \lambda_N) = \|y_N - x_N\| + \sum_{i=1}^{N} \lambda_i G_i(y_i, y_{i-1}) \tag{B.15}$$

with respect to each y_i for $i = 1, \ldots, N$ at the local maximum are zero. More specifically,

$$\frac{\partial f(y_1^\star, \ldots, y_N^\star, \lambda_1, \ldots, \lambda_N)}{\partial y_i} = 0 \text{ for } i = 1, \ldots, N. \tag{B.16}$$

By denoting the unit vector pointing at the heading direction of the pursuer between the capture time of the $(i-1)$-th evader and the i-th evader by e_p^i, and the unit vector pointing at the heading direction of the i-th evader by e_i, Eq. (B.16) can be rearranged as

$$e_p^{i+1} = e_p^i + (1 - \frac{\lambda_i}{\lambda_{i+1}}) w_i \tag{B.17}$$

where

$$w_i = \frac{1}{\beta_i} e_i - e_p^i. \tag{B.18}$$

Combined with the fact that e_p^i is a unit vector for all i's, the following equation describes the relationship between e_p^{i+1} and e_p^i:

$$e_p^{i+1} = e_p^i - 2 \langle e_p^i, \frac{w_i}{\|w_i\|} \rangle \frac{w_i}{\|w_i\|}. \tag{B.19}$$

In other words, given e_i, β_i, and e_p^i, which are the heading of the i-th evader, the maximum speed of the i-th evader, and the optimal heading for the pursuer to capture the i-th evader, the heading of the pursuer to capture the next evader in the capture sequence that will satisfy the first-order optimality condition is uniquely determined. Subsequently, given the derived e_p^{i+1}, the heading for the $(i+1)$-th evader that will satisfies the constraints listed in Eq. (B.14) is also uniquely determined. As a result, given the initial positions of the pursuer and the evaders, their maximum speeds, and a heading for the first evader in the capture sequence, the headings of the rest of the evaders that satisfy the first-order optimality conditions can be derived by repeatedly applying Eq. (B.19). However, the resulting joint heading, which is generated according to a specific heading for the first evader, is only a candidate for the optimal joint heading of evaders. The optimality of the resulting joint heading has to be verified by testing whether or not the heading of the last evader is pointing directly away from the pursuer when the $(N-1)$-th evader is captured. More specifically, by denoting the heading of the first evader by θ_1 and defining the function

$$F(\theta_1) = \langle e_p^N, e_N \rangle, \tag{B.20}$$

the optimal joint heading for the team of evaders under a specific capture sequence can be solved by finding the value of θ_1 which satisfies $F(\theta_1) = 0$. In conclusion, the optimal joint heading of the evaders under a specific capture sequence can be converted to a one dimensional root searching problem and can be solved efficiently even for a team with many evaders.

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