Essays in Game Theory on Investment and Social Organization

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Contents

1 Abstract

This dissertation uses cooperative and non-cooperative game theory to examine the role of investment (broadly defined) in social organization. It's composed of three chapters. The first chapter examines bidirectional investment in partnerships and characterizes the stable relationships among the benefits players produce and receive, their costs, and their payoffs. The second chapter extends the model of the first chapter to allow for multilateral matching and investment; it shows that many of the results of the bilateral case remain true in the more general case. The third chapter examines investment in social links to secure future help and characterizes the equilibrium network/linking architecture and welfare.

2 Introduction

This dissertation uses cooperative and non-cooperative game theory to examine the role of investment (broadly defined) in social organization. Following the standard of economics, it is organized into self-contained papers, each of which focuses on a particular domain/aspect of the relationship between investment and organization.

- Chapter 1 – "Matching with Continuous Bidirectional Investment"

  This paper examines situations where players (e.g., interns and firms or men and women) exert costly effort to produce benefits for their partners. In these situations, the benefit a person produces is not exogenous, but rather depends on their own cost of effort, the benefit their partner provides to them, and their own and their partner's outside opportunities with the other players.

  This paper examines how these forces – in particular the cost of effort – influence the benefits players produce for their partners and receive from them, as well as their welfare. To do this, we construct a one-to-one matching game where heterogeneous men and women exert costly effort to make their partners happy. When a man and a woman match, they come to an agreement about the efforts they exert. The man's (woman's) payoff is the benefit produced by the woman's (man's) effort less the cost of his (her) own effort. The solution concept is a stable allocation, i.e., a stable matching and vector of efforts. We find, for instance, that men and women with lower marginal costs of effort choose to provide their partners with higher benefits by exerting more effort; in return, they receive higher benefits from their partners and attain higher payoffs. (These results provide a novel rationalization for the empirical psychological observation that couples match on the basis of conscientiousness – e.g., Rammstedt and Schupp [57]).

  This paper is the first to examine simultaneous bidirectional effort/investment in partnerships and is the first to highlight the role of cost heterogeneity in determining "equilibrium" sorting and investment behavior. While pre-match investment in partnerships is well-studied (e.g., Burdett and Coles [12], Cole et al. [16], and Peters [52]), these papers do not explore the ramifications of cost heterogeneity and they do not allow players' investments to turn on the actual investments of their partners. We also contribute to the classical matching literature (e.g., Hatfield and Milgrom [34] and Roth and Sotomayor [60]) by weakening the key assumption that the set of possible agreements is finite. In addition, several comparative statics results are developed – e.g., we find that as a player's ability increases, the benefits he or she produces increases, as does the benefit he or she receives.

- Chapter 2 – "Three-Sided Matching with Trilateral Investment"

  This paper generalizes the model of Chapter 1 to allow for multiple groups in order to study how heterogeneity in the costs of effort shape matching and investment patterns in multi-sided matching markets – e.g., civic projects like parks, where towns, architects, and construction companies compete for each other's business. Under a weak symmetry condition, we find that stable allocations exist and that many of the conclusions of my job market paper remain true – e.g., lower marginal cost architects and construction companies work harder, and end up working for the towns who pay/invest more.

  Since Alkan [3] showed that three-sided matching games may lack stable allocations, much of the literature on multilateral matching has concerned itself with developing preference restrictions that are sufficient for existence, e.g., Danilov et al. [22]. This paper contributes to this literature by establishing a new existence condition and by leveraging this condition to characterize how players behave in stable allocations.

- Chapter 3 – "Rivalry and Professional Network Formation"

  This chapter examines situations where players must invest in connections to other players in order to secure their help in the future. We focus on a simple case: consulting firms. At a consulting firm like Deloitte, consultants are hired into a common pool. Each partner then trains some subset of these consultants in her area of expertise (i.e., she invests in these consultants) and employs a handful of these trained consultants whenever she gets a project (if they're available). Since consultants can only work on one project at a time and training by one partner does not preclude training by another, the partner's use of consultants is rivalrous.

  In this paper, we examine how this rivalry shapes network structure (i.e., the size and configuration of the set of consultants each partner trains) and welfare. To do this, we build a two-stage network formation game where two partners select subsets of consultants to train, randomly get projects, and then choose which trained consultants to use to complete their projects. We find, for instance, that the partners always hold "minimally overlapping" networks and that their equilibrium interests are "opposed." We also develop comparative statics results for network sizes and welfare by drawing upon the theory of supermodular games. We find, for example, that as one partner's cost of training falls, the size of the other partner's network and her welfare both fall.

This paper contributes to three literatures. The first is the multiple-commons literature – e.g., Ilkilic [36] who studies how rivalry influences players'/cities' consumption of water from the aquifers to which they are connected. This literature only allows players to choose whether they consume the resources to which they are connected. We contribute to it by allowing players to also choose the resources to which they are connected. The second literature examines how buyers and sellers come together and form networks for trade, e.g., Kranton and Minehart [45]. We contribute to it by considering a non-market environment where partners "buy" labor from multiple consultants. The third literature examines co-authorship, e.g., Jackson and Wolinsky [39]. In this literature, there is an implicit rivalry for players' time. We differ in the nature of the rivalry we consider and this difference has substantive implications for the network structure.

Reflecting the fact that each chapter is a complete paper, the definitions, equations, results, and so on are numbered within each chapter; the references are, however, unified at the end of this dissertation.

3 Matching with Continuous Bidirectional Investment

3.1 Introduction

Partners usually exert costly efforts to produce benefits for each other. For instance, when an (unpaid) intern and an employer match, the intern exerts effort to complete tasks that benefit the employer, while the employer exerts effort to train the intern in industry methods and practices. Likewise, following the classic example of Gale and Shapley [31], when man and a woman date, the man exerts effort to benefit his girlfriend (e.g., by cooking her dinners), while the woman exerts effort to benefit her boyfriend. Other examples include mentors and mentees, masters and apprentices (e.g., professors and graduate students), senior managers and organizations (e.g., deans and colleges), buyers and sellers of specialized goods and services (e.g., artists and patrons or lawyers and clients), and so on.

These benefits (and the associated efforts) are not exogenous. Rather, the benefit a person chooses to provide to their partner depends directly on (i) their own cost of effort and (ii) the benefit their partner provides to them. It also depends indirectly on (iii) their own and their partner's outside opportunities with the other men and women, which in turn depend on the benefits and partners chosen by these other players. For instance, if a man has a low cost of effort or receives a high benefit from his partner, then he is better positioned to exert effort on her behalf. Also, if other men desire his partner, then he needs to provide a higher benefit in order to retain her; yet, he would never choose to provide her a benefit so high as to make himself worse off than he could be with some other attainable woman. Analogous logic applies for his partner.

Our goal is to characterize how these forces shape the benefits that people produce and receive when they simultaneously choose their partners and efforts. In particular, we ask two questions. First, how does the benefit a person produces compare to the benefit they receive, i.e., the benefit their partner produces? Second, how does a person's cost of effort influence the benefits they produce and receive?

To answer these questions, we first introduce a general one-to-one matching game, called the General Game, and we prove several results concerning the existence and Pareto optimality of solutions. Subsequently, we develop a novel application of this game, called the Effort Game, in which heterogeneous men and women (interns and employers, etc.) pair with each other and exert costly efforts to benefit their partners, and we answer our main questions. Throughout, we couch the games, results, and discussions in terms of men and women with the understanding that these are only labels for the two sides.

In the General Game, a finite number of men and women pair with each other. When a man and a woman match, they select an agreement that specifies their individual and joint actions from a countable or uncountable set of feasible agreements.[1] The man's and the woman's payoffs are determined by their identities and the agreement they select. Our solution concept is a stable matching and vector of agreements, which we call a "stable allocation." In a stable allocation, (i) each player earns at least the value of being single and (ii) no man and woman can do strictly better by pairing and selecting a new agreement, i.e., no two players "block" the allocation.

We first show that a stable allocation exists when the payoffs are continuous in the agreement and the set of feasible agreements is compact (Proposition 1). The proof illuminates a connection between our game and the class of "Deferred Acceptance" algorithms. We also show that a strongly Pareto optimal stable allocation exists (Proposition 2). Subsequently, we give intuitive sufficient conditions for the existence of and the Pareto optimality of stable allocations with interior agreements (Proposition 3 and 4, respectively).

We next develop the Effort Game. In the Effort Game, when a man and a woman match, they come to an agreement about the effort each exerts. Their efforts produce benefits for each other and are chosen from a compact interval. The man's payoff is the benefit produced by the woman's effort less the cost of his own effort. Likewise, the woman's payoff is the benefit produced by the man's effort less the cost of her own effort. For simplicity, all players have the same benefit production function, which is increasing in effort.[2] Each player is also endowed with a type (e.g., ability) that affects his or her cost of effort.[3]

To address our questions, we focus on stable allocations with interior agreements. Under the natural assumptions that players dislike exerting extremely high effort and dislike exerting effort when their partners don't exert any effort, these interior stable allocations exist, are Pareto optimal, and are the only stable allocations where players are matched (Corollary 1).

As to our first question on how partners' benefits compare, we find that players match based on the benefits they produce in *any* stable allocation with interior agreement.[4] Specifically, a man who produces the l-th highest benefit among men matches to a woman who produces the l-th highest benefit among women (Proposition 5); the analogous result holds

[1] The notion that partners select agreements is natural. Since partners implicitly and explicitly choose the "rules" that govern their individual and joint actions, they effectively select an agreement. Our simplification is that these rules, or at least the important ones, are chosen at the onset of the partnership instead of over its duration.

[2] Our core results, Proposition 5 to 8, hold under weaker assumptions. As we discuss in Section 4, they obtain, for instance, when men have one benefit function and women have a different benefit function. Thus, these results extend to settings where one side pays the other for a service, e.g., lawyers and clients.

[3] Players may directly benefit from their own efforts because there are no monotonicity restrictions on the cost of effort; see Section 4 for details.

[4] There is usually a multiplicity of interior stable allocations. Unless mentioned otherwise, our findings pertain to all interior stable allocation.

for women. Thus, (i) men who produce strictly higher benefits are matched to women who produce strictly higher benefits and (ii) men who produce the same benefit are matched to women who produce the same benefit.

The intuition for this result is that players "compete" with each other for higher benefit partners. To illustrate, suppose two men m and m' both produce the same benefit, but m' has partner w' who produces a strictly higher benefit than the partner of m. Then m finds it best to increase his effort by an arbitrarily small amount in order to produce a slightly larger benefit and "win" w' away from m'. Man m is able to win w' in this fashion because she desires the largest benefit possible. He's willing to win w' because he gains a strictly higher benefit in exchange for a smaller increase in his cost. Thus, m and w' block. It follows that a necessary condition of stability is that men who produce the same benefit are matched to women who produce the same benefit. As the intuition indicates, Proposition 5 depends on the continuity of effort and doesn't obtain with effort is discrete; see Section 5 for details.

Since the benefit production function is common across players, it follows that players match based on the effort they exert (Corollary 2). Interestingly, Rammstedt and Schupp [57] and Watson et al. [69] give empirical support for this prediction: they both find that conscientious people more frequently date other conscientious people, while lazy people more frequently date other lazy people. Our result provides a novel rationalization for this observation.

To answer our second question on how the benefits players produce and receive are shaped by their costs, we suppose that the *marginal cost* of effort is decreasing in type/ability. We find that this is sufficient to ensure that higher ability players produce larger benefits. Specifically, we show that if man m' has a strictly higher type than another man m, then m' produces a benefit at least as large as the benefit produced by m (Proposition 6); the analogous result holds for women. The intuition is that higher ability players can "outcompete" lower ability players because their lower marginal costs allow them to profitably offer slightly higher benefits. Thus, the competition for partners drives them to provide higher benefits. As the intuition suggests, Proposition 6 depends on the continuity of effort and doesn't obtain when effort is discrete; see Section 5 for details.

It follows that higher ability players (i) exert more effort (Corollary 3) and, surprisingly, (ii) receive *higher* benefits from their partners (Corollary 4). In the context of men and women, these corollaries give us an idea of how a particular kind of person may behave and fair in the dating "market." And, in the context of interns and employers, they help us understand how some employers routinely attract outstanding interns (and eventually employees). Stepping back from the model, Proposition 6 and these corollaries also provide guidance on how an employer can, via pre-match investments that reduce its marginal-cost of training, improve the rank-order caliber of intern it obtains (see the discussion after Corollary 4 in Section 5).

Under the additional assumption that all players share a common cost of zero effort, we find that weakly higher ability players have weakly higher payoffs (Proposition 7). This additional assumption ensures that higher ability players have lower costs. Thus, they profitably "imitate" and outcompete weakly lower ability players whenever such players do strictly better. Hence, a necessary condition of stability is that higher ability players earn at least as much as lower ability players. As with our previous results, Proposition 7 depends on the continuity of effort; see Section 5 for details.

Our findings suggest that higher ability players match with other higher ability players. While this need not happen in every interior stable allocation (we discuss why in Section 5), we establish that there is at least one interior stable allocation where it happens (Proposition 8). Intriguingly, Belot and Francesconi [9] and Hitsch et al. [35] find that more educated people date each other.[5] Since more educated people often have a lower cost of effort (e.g., Regan et al. [58]), their findings lend support to this prediction.

We close by examining how decreases in players' opportunity costs of effort, perhaps due to reductions in work/family responsibilities or new technologies, affect the *sizes* of players' decisions and outcomes. We proceed by supposing men and women have a symmetric endowment of types, and we select the unique "symmetric" stable allocation. (This allocation is focal because it treats equals equally and it maximizes social welfare (Lemma 10).) We find that decreases in players' marginal costs of effort, i.e., increases in their types, increase their efforts, the benefits they produce and receive, and their payoffs in this allocation (Proposition 9). The proposition follows from the symmetry of the type endowment, the submodularity of the cost function, and the fact that all players have a common cost of zero effort. Returning to our intern-employer example, this result shows that a firm can increase the *level* of benefit it obtains from its intern (and its payoff) by reducing its marginal cost of training.

The balance of this section discusses the related literature. Subsequently, Section 2.2 describes the General Game and stability, Section 2.3 gives Propositions 1 to 4, Section 2.4 describes the Effort Game and gives Corollary 1, and Section 2.5 presents Propositions 5 to 9 and Corollaries 2 to 4.

3.1.1 Related Literature

Our work makes economic and technical contributions to three literatures. The first literature examines the existence of stable allocations in matching games with agreements. We

[5] Arcidiacono et al. [5] overview the recent empirical literature on mate preferences.

contribute to it via the General Game by weakening many of the traditional assumptions and by giving a new and general existence proof; we also contribute a novel application in the Effort Game. The second literature examines pre-match investment. We contribute to it via the Effort Game by allowing investment/effort to be co-determined with the matching and by characterizing the previously unexplored relationships among the benefits players produce and receive, their costs of effort, and their payoffs. The third literature examines when players match assortatively in their endowed types. We contribute to it via the Effort Game by providing simple conditions on payoffs that guarantee assortative matching in types and by focusing on the relationship between benefits and types, instead of the relationship between matched players' types.

Some background is helpful. There are three seminal games in the matching literature: the "Marriage Game" of Gale and Shapley [31]; the "Assignment Game" as articulated by Demange and Gale [24]; and the "Generalized Marriage Game" as articulated by Hatfield and Milgrom [34].[6] In the Marriage Game, men and women match with each other and receive payoffs based only on the identities of their partners. The Assignment Game generalizes the Marriage Game by allowing men and women to agree to a (real-valued) monetary transfer when they match. Thus, players' payoffs depend on their partners' identities and their transfers; these payoffs are either quasi-linear or weakly monotone in money. Both the Marriage Game and the Assignment Game are special cases of the General Game; see Example 3 (in Section 2). The Generalized Marriage Game extends the Marriage Game by allowing for many-to-one matching – e.g., a firm hiring multiple workers – and by allowing a worker and a firm to select an agreement from a *finite* set of possible agreements when they match. Agreements are more than simple transfers; in addition to salaries, they may specify vacation time, job responsibilities (e.g., teaching loads), and a myriad of factors. Hence, players' payoffs depend on the identities of their matches and their agreement. When firms can only hire one worker, the Generalized Marriage Game is a special case of the General Game.

While one might think that the Effort Game is an Assignment Game, this isn't the case. An Assignment Game only allows each matched couple to agree to a single point in $\mathbb{R}$, whereas the Effort Game allows each matched couple to agree to a point in $\mathbb{R}^2$. Thus, the Effort Game is suitable for examining the benefits that both players produce, while an Assignment Game is not. The Effort Game also makes no assumptions about the monotonicity of payoffs. Furthermore, the Effort Game is not a Generalized Marriage Game since its set of possible agreements is uncountable.

Our existence results, Propositions 1 and 2, are related to the existence results of (i)

[6]See Roth and Sotomayor [60] and Sonmez and Unver [65] for the histories of these games.

Adachi [1], Echenique and Oviedo [25], Fleiner [30], Hatfield and Milgrom [34], and Roth [59], (ii) Crawford and Knoer [21] and Quinzii [56], and (iii) Alkan and Gale [4] and Kaneko [41]. Adachi [1], Echenique and Oviedo [25], Fleiner [30], Hatfield and Milgrom [34], and Roth [59], prove existence for the Generalized Marriage Game by giving various "Deferred Acceptance" algorithms that halt at stable allocations in finite time.[7] Crawford and Knoer [21] prove existence for a quasi-linear Assignment Game by employing a contradiction argument that illuminates a connection between the class of Deferred Acceptance algorithms and the existence of stable allocations in their game. Quinzii [56] generalizes Crawford and Knoer's result by proving existence for a weakly monotone Assignment Game via Scarf's Balancedness Theorem.

Kaneko [41] considers a more abstract, one-to-one matching game between men and women. In his game, each man m and each woman w have a set of achievable payoffs V_{mw} from which they pick some point when they're matched. He shows that if this set satisfies certain conditions (e.g., if $\boldsymbol{v}' \in V_{mw}$, then $\boldsymbol{v} \leq \boldsymbol{v}'$ implies $\boldsymbol{v} \in V_{mw}$), then Scarf's Balancedness Theorem implies a stable allocation exists. Alkan and Gale [4] build on Kaneko's game by assuming each pair's set of achievable payoffs are given by their Pareto frontier. When this frontier is described by a bounded, strictly decreasing, and continuous function that intersects both the horizontal and vertical axes, they describe an algorithm that computes a stable allocation in finite time.

We differ from these studies in our assumptions and in our method of proof. Consequently, our result applies to natural games like Example 2 (in Section 2) where payoffs are non-monotone, Kaneko's [41] technical properties don't hold, and the Pareto frontier doesn't intersect either axis. Also, our method of proof is novel and it illuminates a connection between the class of Deferred Acceptance algorithms and the existence of stable allocations in the General Game. Furthermore, while many of these studies characterize general properties of the stable set (e.g., its connectedness and its lattice structure), none of them consider our main application, the Effort Game, or develop our characterizations.

Speaking of the Effort Game, our findings on how players' benefits (and efforts) compare, Proposition 5 and Corollaries 1 and 2, are closely related to the rich literature on pre-match investment, including (i) Cole et al. [16, 17], Mailath et al. [48], Noldeke and Samuelson [49], Peters [52], and Peters and Siow [53], (ii) Chiappori et al. [14] and Iyigun and Walsh [37], and (iii) Burdett and Coles [13] among others. These studies, with the exception of Burdett

[7] These algorithms require that firms regard workers as "substitutes." While this requirement is naturally satisfied in the General Game (since it's a one-to-one game), it needn't hold in arbitrary many-to-one games and, as a rule, it's restrictive. See, for instance, Kelso and Crawford [43] and Hatfield and Milgrom [34] for discussions of the requirement, and see, for instance, Kominers and Sonmez [44], Hatfield and Kojima [33], and Sonmez and Switzer [64] for ways to weaken it.

and Coles [13] (which we discuss below), consider two-stage games where players first invest in themselves (e.g., go to school and the gym) and then enter a matching game. In Chiappori et al. [14], Cole et al. [16, 17], and Iyigun and Walsh [37], this game is an Assignment Game where each pair receives a joint surplus that they divide with their transfer. Each pair's surplus depends on their investments and endowed characteristics, and each player's payoff is their portion of the surplus less their investment cost. In Peters [52] and Peters and Siow [53], this game is a Marriage Game where each player's payoff depends on their and their partner's investments. Noldeke and Samuelson [49] consider a more general game that's akin to Alkan and Gale's [4] game and encompasses both the Assignment and Marriage Games.

These studies examine the (Pareto) optimality/efficiency of players' investments. Cole et al. [16, 17], Mailath et al. [48], and Peters [52] find that equilibrium investments can be inefficient due to coordination failures, with both a finite and infinite number of players. In contrast, Chiappori et al. [14], Iyigun and Walsh [37], and Peters and Siow [53] find that equilibrium investments are efficient. Noldeke and Samuelson [49] explain these divergent findings by identifying general conditions under which all equilibria are efficient.

Efficiency is not the focus of Chiappori et al. [14] and Iyigun and Walsh [37]. Rather, these papers examine how labor market returns and sex ratios shape players' pre-match educational investments and the divisions of their marital surpluses. Chiappori et al. [14] show that educated men may match with uneducated women in equilibrium when men's returns to education are sufficiently high. Iyigun and Walsh [37] show that, when men are in short supply to women, then they invest less and obtain a greater proportion of the joint surplus in equilibrium than women. In addition, both studies, as well as Cole et al. [16, 17], establish that players who invest more match with each other when each pair's joint surplus is supermodular in their investments.

Burdett and Coles [13] take a different approach. They draw on search theory to build a two-stage, finite game where players first invest in themselves and then enter a matching game with *frictions*, wherein they randomly encounter each other over time and, at each encounter, they may either match and exit or keep searching. Each player's payoff is the quality of their partner, which depends on their partner's investment and endowment, discounted by the time spent searching less the cost of their own investment. The authors show that (i) players with higher qualities usually match with each other in equilibrium and (ii) that equilibrium investments may be inefficiently large.

Our work is complementary to these papers. Like them, we study the effects of investment. However, in the Effort Game, investment/effort is match-specific since it's determined at the time players match, whereas pre-match investments are the same for every possible match. This distinction is both intuitive and economically meaningful since it ensures, for instance, that players' investments are Pareto optimal in every solution, see Corollary 1. More importantly, we explore the relationships among the benefits that players produce and receive, their costs of effort, and their payoffs; none of the papers discussed above characterize these relationships.

To the best of our knowledge, there are no other papers that study simultaneous matching and investment in non-market environments like ours. There are, however, papers that study post match investment with commitment and information problems. (If there are no commitment and information problems then simultaneous investment is the same as post-match investment.) For instance, Kaya and Vereshchagina [42] consider a three-stage "roommates" game with transferable utility where: (i) players pair off, (ii) each pair writes a contract that divides the profit from their joint venture, and (iii) each player decides how much costly and unobservable effort to contribute to the venture. They show that efforts/investments are (Pareto) inefficient, due to the moral hazard problem, and they develop conditions under which players match assortatively in their endowed types/productivities. Our work is complementary because we allow players to verifiably choose and compete on the basis of effort/benefit and, unlike Kaya and Vereshchagina [42], we examine the relationship between benefit and type.

Speaking of assortative matching in endowed types, our finding in this area, Proposition 8, is also related to Becker [8], Chiappori and Reny [15], and Legros and Newman [47] among others.[8] In his seminal paper, Becker [8] uses a quasi-linear Assignment Game to show that players match assortatively when each pair's joint surplus is supermodular in their types. Chiappori and Reny [15] examine how men and women, with heterogeneous risk preferences, match to "share [the] risk" of uncertain future incomes. In their model, a man and woman share this risk by agreeing to a schedule of divisions of their future incomes when they match; the man's payoff is his expected utility-of-wealth under the schedule, likewise for the woman. Their principal finding is that matching in stable allocations is "negative assortative," i.e., that more risk-averse (higher type) men match to less risk-averse (lower type) women and vice versa. Legros and Newman [47] consider a general, non-transferable utility matching game that's akin to the game of Alkan and Gale [4] and nests both Becker's and Chiappori and Reny's games. They develop general conditions on the Pareto frontiers of each man-woman pair that guarantee that matching is either assortative or negative assortative. Our work is complementary to these studies since (i) we develop natural conditions on the primitive

[8] In a related work, Farrell and Scotchmer [28] study when agents with similar productivities form groups in a coalition formation game where coalitions (are forced to) divide their outputs equally. Pycia [54] builds on their paper and examines when players with similar risk preferences form groups given that coalitions divide their outputs according to exogenous sharing rules. In addition, there is a rich matching-with-frictions literature on assortative matching in types – see Smith [63] for an overview.

payoff functions under which higher types match with each other in a game with endogenous investments (and thus enriches the economic understanding of when assortative matching occurs) and (ii) we focus on the relationship between the benefit and type, instead of the relationship between matched players' endowed types.

3.2 Description of the General Game

This section describes the General Game, defines a stable allocation, and discusses several examples.

3.2.1 Environment

There are two finite sets of players, men $\mathcal{M} = \{1, \ldots, M\}$ and women $\mathcal{W} = \{M+1, \ldots, N\}$, with $N > M > 0$. Let $\mathcal{N} = \mathcal{M} \cup \mathcal{W}$. We write m for the m-th man, w for the w-th woman, and i for the i-th player (regardless of gender).

Each player may either be single or may match with a member of the opposite sex. We adopt the convention that a single player is matched to himself or herself. A **matching** is a function that specifies each player's match, i.e., is a $\phi : \mathcal{N} \to \mathcal{N}$ such that: (i) for each man m, $\phi(m) \in \mathcal{W} \cup \{m\}$; (ii) for each woman w, $\phi(w) \in \mathcal{M} \cup \{w\}$; and (iii) for each man m and each woman w, $\phi(m) = w \iff \phi(w) = m$. We say player i is **partnered** if $\phi(i) \neq i$. We write Φ for the finite set of all matchings.

When a man and a woman match, they select an **agreement** $\boldsymbol{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, where $k \geq 1$. Their agreement specifies their individual and joint **actions** x_1, ..., x_k. For instance, x_1 and x_2 may give the number of hours m and w spend at work each week respectively, x_3 may give the number of days m and w spend camping every year, and so on. Also, each single player has an agreement $\boldsymbol{x} \in \mathbb{R}^k$ with himself or herself.

Given a $\phi \in \Phi$, we write $\boldsymbol{x}^i$ for the agreement player i has with either (i) his or her partner or (ii) himself or herself. Thus, $\boldsymbol{x}^i = \boldsymbol{x}^{\phi(i)}$ for each player i.[9] We write $\bar{\boldsymbol{x}} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^M, \boldsymbol{x}^{M+1}, \ldots, \boldsymbol{x}^N)$ for the vector of players' agreements. Notice that $\bar{\boldsymbol{x}} \in A(\phi) = \{(\tilde{\boldsymbol{x}}^1, \ldots, \tilde{\boldsymbol{x}}^N) \in \mathbb{R}^{kN} \mid \tilde{\boldsymbol{x}}^i = \tilde{\boldsymbol{x}}^{\phi(i)} \text{ for all } i \in \mathcal{N}\}$, where $A(\phi)$ is the set of possible agreement vectors for the matching ϕ, and that $A(\phi)$ is a nonempty vector subspace of $\mathbb{R}^{kN}$. An **allocation** is a $(\phi, \bar{\boldsymbol{x}})$ such that $\phi \in \Phi$ and $\bar{\boldsymbol{x}} \in A(\phi)$, i.e., is a matching and a vector of agreements. We write $\mathcal{A} = \{(\phi, \bar{\boldsymbol{x}}) \in \Phi \times \mathbb{R}^{kN} | \bar{\boldsymbol{x}} \in A(\phi)\}$ for the set of allocations.

A player's payoff depends (only) on the identity of his or her match and their agreement. Formally, each man m has a payoff function $u_m : \{\mathcal{W} \cup \{m\}\} \times \mathbb{R}^k \to \mathbb{R}$ over his possible

[9] If player i is partnered, then i and his or her match $\phi(i)$ have *an* agreement $\boldsymbol{x}$, so $\boldsymbol{x}^i = \boldsymbol{x}$ and $\boldsymbol{x}^{\phi(i)} = \boldsymbol{x}$. If player i is single, then he or she as an agreement $\boldsymbol{x}$ and $\phi(i) = i$, so $\boldsymbol{x}^{\phi(i)} = \boldsymbol{x}^i = \boldsymbol{x}$.

matches and agreements. Likewise, each woman w has a payoff function $u_w : \{\mathcal{M} \cup \{w\}\} \times \mathbb{R}^k \to \mathbb{R}$. We normalize the value of being single to zero, i.e., for every $\boldsymbol{x} \in \mathbb{R}^k$, we have $u_i(i, \boldsymbol{x}) = 0$ for each player i. Let $(\phi, \bar{\boldsymbol{x}}) = (\phi, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^i, \ldots, \boldsymbol{x}^N) \in \Phi \times \mathbb{R}^{kN}$, in a slight abuse of notation we write $u_i(\phi, \bar{\boldsymbol{x}})$ for the payoff of player i in $(\phi, \bar{\boldsymbol{x}})$, i.e., $u_i(\phi, \bar{\boldsymbol{x}}) \equiv u_i(\phi(i), \boldsymbol{x}^i)$.

3.2.2 Stable Allocations

The next four definitions develop the idea of a stable allocation. Let $X \subset \mathbb{R}^k$ be the (nonempty) set of "feasible" agreements.[10]

Definition. An allocation $(\phi, \bar{\boldsymbol{x}})$ is *feasible* if agreements are in X, i.e., $\bar{\boldsymbol{x}} \in X^N$.

Definition. An allocation $(\phi, \bar{\boldsymbol{x}})$ is *individually rational* if every player gets at least the value of being single, i.e., $u_i(\phi, \bar{\boldsymbol{x}}) \geq 0$ for each player i.

Definition. A man m and a woman w *block* a $(\phi, \bar{\boldsymbol{x}}) \in \Phi \times \mathbb{R}^{kN}$ if they both obtain strictly higher payoffs by matching with each other at a feasible agreement than they obtain in $(\phi, \bar{\boldsymbol{x}})$, i.e., if there exists an $\boldsymbol{x} \in X$ such that

$$u_m(w, \boldsymbol{x}) > u_m(\phi, \bar{\boldsymbol{x}}) \text{ and } u_w(m, \boldsymbol{x}) > u_w(\phi, \bar{\boldsymbol{x}}).$$

Definition. An allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is *stable* if (i) it is feasible, (ii) individually rational, and (iii) no man and woman block it.

Stable allocations are our solution concept. When an allocation is stable: (i) no player can do strictly better by choosing to be single (per individual rationality) and (ii) no two players can do strictly better by matching with each other and choosing a new agreement instead of following $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ (per no blocking). As in Gale and Shapley [31], we might imagine that a stable allocation is the outcome of a bargaining process where players try to maximize their own payoffs. The rational is that no player can do strictly better by (i) opting out or (ii) by trying to strike a new bargain with some other player j, as j would reject this bargain since it doesn't make him or her strictly better off. After bargaining concludes, we imagine that players match with their agreed upon partners and take their agreed upon actions, and then receive their payoffs.[11]

[10] We take X to be the same for all men and women for notational simplicity. Our results for the General Game readily extend to the case where different man-woman pairs have different sets of feasible agreements.

[11] Generally, players may commit to their agreements because (i) there is community enforcement and players punish each other for defections (e.g., Kandori [40]), (ii) the agreements constitute enforceable contracts (e.g., a written, verbal, or "implied-in-fact" contract), (iii) they have a preference for doing so (e.g., a sense of responsibility/honor), or (iv) they maintain/gain social esteem for doing so. In the case of interns and firms, commitment is natural because of (ii); in fact, firms have been sued for not adequately training

Observe that the set of stable allocations and the core coincide because the payoffs of a matched man and woman only depend on their identities and their agreement. In addition, there are usually many stable allocations. In light of this, we focus on results about the set of stable allocations or specific selections thereof.

3.2.3 Examples

Since the General Game is quite abstract, it's helpful to give a few examples.

Example 1. A Simple Effort Game.

Suppose there are four players, two men and two women, i.e., $\mathcal{M} = \{1, 2\}$ and $\mathcal{W} = \{3, 4\}$. Let each player i have a type $\theta_i \in R$, e.g., innate ability. We set $\theta_1 = \theta_4 = 2$ and $\theta_2 = \theta_3 = 1$. Let $X = [0, 2]^2$ be the set of feasible agreements. When a man m and a woman w are matched (to each other), their payoffs to agreement $(x_1, x_2) \in \mathbb{R}^2$ are

$$u_m(w, x_1, x_2) = x_2 - \frac{(x_1)^2}{\theta_m} - {}^1\!/_8 \text{ and } u_w(m, x_1, x_2) = x_1 - \frac{(x_2)^2}{\theta_w} - {}^1\!/_8.$$

Recall that single players get zero. For $(x_1, x_2) \in X$, we think of x_1 and x_2 as the man's and the woman's efforts respectively. Thus, both players exert costly effort to make each other happy: m exerts effort x_1 to produce a benefit of x_1 for w and incurs a cost of $(x_1)^2/\theta_m + \frac{1}{8}$ for doing so, while w exerts effort x_2 to produce a benefit of x_2 for m and incurs a cost of $(x_2)^2/\theta_w + \frac{1}{8}$ for doing so. (The assumption that each player's effort is in $[0, 2]$ reflects the idea that people can only work so hard as they face time and energy limitations.)

One stable allocation of this game is $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \boldsymbol{x}^{2^\star}, \boldsymbol{x}^{3^\star}, \boldsymbol{x}^{4^\star})$, where $\phi^\star(1) = 4$ and $\boldsymbol{x}^{1^\star} = \boldsymbol{x}^{4^\star} = (1, 1)$, and $\phi^\star(2) = 3$ and $\boldsymbol{x}^{2^\star} = \boldsymbol{x}^{3^\star} = ({}^1\!/_2, {}^1\!/_2)$. That is, man 1 matches with woman 4 and they both exert effort 1, while man 2 matches with woman 3 and they both exert effort ${}^1\!/_2$.

Let's verify $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable. To do this, we need to show that it's feasible, individually rational, and not blocked. Feasibility is automatic since $(1, 1)$ and $({}^1\!/_2, {}^1\!/_2)$ are in X. Individual rationality requires a bit more work. We have $u_1(\phi^\star, \bar{\boldsymbol{x}}^\star) = 1 - {}^1\!/_2(1)^2 - {}^1\!/_8 = {}^3\!/_8$. By analogous calculations $u_2(\phi^\star, \bar{\boldsymbol{x}}^\star) = {}^1\!/_8$, $u_3(\phi^\star, \bar{\boldsymbol{x}}^\star) = {}^1\!/_8$, and $u_4(\phi^\star, \bar{\boldsymbol{x}}^\star) = {}^3\!/_8$. It follows that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is individually rational, i.e., no player can do better by choosing to be single.

We need to make sure that there are no blocking pairs. The definition of blocking makes no assumptions about which players block an allocation: players who are matched may block,

their interns (e.g., the 2013 case of Glatt et al. vs. Fox Searchlight Pictures). In the case of men and women, commitment is less natural, but still justifiable as an approximation: early in their relationships couples behave in ways that indicate their long-term conduct, so we may think of them as selecting goals/an agreement and trying to follow through.

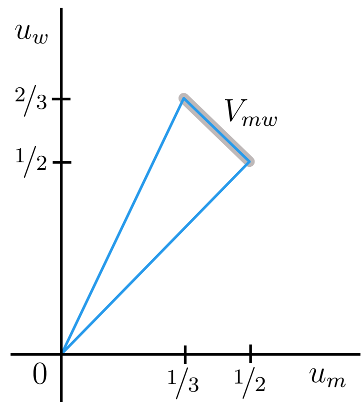


Figure 3.1: Achievable Payoffs in Example 2 for Man m and Woman w

as may players who are not matched. Thus, we need to check four pairs.

Consider man 1 and woman 3. They block if there is a $(x_1, x_2) \in X$ such that $u_1(3, x_1, x_2) > {}^3\!/_8$ and $u_3(1, x_1, x_2) > {}^1\!/_8$, i.e., if $x_2 - \frac{1}{2}(x_1)^2 > \frac{1}{2}$ and $x_1 - (x_2)^2 > \frac{1}{4}$. Since it's readily verified that this system has no real solution, they can't block, i.e., they can't do better by matching and choosing a new agreement instead of following $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Symmetry gives that man 2 and woman 4 can't block either.

Consider man 1 and woman 4. They block if there is a $(x_1, x_2) \in X$ such that $u_1(4, x_1, x_2) > {}^3\!/_8$ and $u_4(1, x_1, x_2) > {}^3\!/_8$, i.e., if $x_2 - \frac{1}{2}(x_1)^2 > \frac{1}{2}$ and $x_1 - \frac{1}{2}(x_2)^2 > \frac{1}{2}$. Since this system has no real solution, they can't block. An analogous argument gives that man 2 and woman 3 can't block. It follows that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable. △

Example 2. Hiking.

Let $\mathcal{M} = \{1, 2\}$, let $\mathcal{W} = \{3, 4\}$, and let $X = [0, 1]$. When a man m and a woman w are matched, their payoffs to agreement $x \in \mathbb{R}$ are

$$u_m(w, x) = \begin{cases} x & \text{if } x \leq \frac{1}{2} \\ 1 - x & \text{if } x > \frac{1}{2} \end{cases} \text{ and } u_w(m, x) = \begin{cases} 2x & \text{if } x \leq \frac{1}{3} \\ 1 - x & \text{if } x > \frac{1}{3}, \end{cases}$$

Recall that single players get zero. For $x \in X$, we think of x as the percentage of their Sundays m and w spend hiking – w only wants to go for a third of the day, while m wants to go for half the day.

Let V_{mw} be the set of payoffs that man m and woman w can achieve when matched, i.e., $V_{mw} = \{(v_m, v_w) \in \mathbb{R}^2 | u_m(w, x) = v_m \text{ and } u_w(m, x) = v_w \text{ for some } x \in X\}$. This set is the blue triangle plotted in Figure 2.1; it's bounded and has an empty interior. The Pareto frontier is the northeastern edge of V_{mw}; it's highlighted in gray in the figure. While the frontier is strictly decreasing, it never intersects the horizontal or vertical axes. Thus, V_{mw}

does not satisfy the assumptions of Alkan and Gale [4] or Kaneko [41].

Nonetheless, there are a continuum of stable allocations. One set is described by $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, where $\phi^\star(1) = 3$, $\phi^\star(2) = 4$, and $\bar{\boldsymbol{x}}^\star = (x, x, x, x)$ with $x \in [\frac{1}{3}, \frac{1}{2}]$. That is, man 1 matches with woman 3, man 2 matches with woman 4, and all players hike for a common proportion of their Sunday x, with $x \in [\frac{1}{3}, \frac{1}{2}]$.

Such an allocation is trivially feasible and individually rational. In such an allocation, (a) man 1 and woman 3 or (b) man 1 and woman 4 can't block as an increase in the duration makes woman 3 or woman 4 worse off and a decrease in the duration makes man 1 worse off. Likewise, (c) man 2 and woman 3 or (d) man 2 and woman 4 can't block. $\triangle$

As previously mentioned, the General Game nests both the Marriage and Assignment Games. Since payoffs in the Marriage Game only depend on the identities of players' matches, we can embed it in the General Game by taking X to be a singleton. The next example illustrates the embedding of a simple, quasi-linear Assignment Game.

Example 3. A Quasi-Linear Assignment Game.

Recall that in a quasi-linear Assignment Game, men and women receive some surplus from matching, which they split via a monetary transfer. For each man m and each woman w, let $f_m(w) \geq 0$ give m's surplus from matching with w and let $f_w(m) \geq 0$ give w's surplus from matching with m. When m and w match, they agree to a transfer $x \in \mathbb{R}$; m earns his surplus less the transfer x and w earns her surplus plus x. Single players earn nothing.

To embed this game in the General Game, first let each man m and each woman w's payoffs to matching with agreement $x \in \mathbb{R}$ be

$$u_m(w, x) = f_m(w) - x \text{ and } u_w(m, x) = f_w(m) + x.$$

(Recall that single player automatically get zero.) Next, let the set of feasible agreements be $X = [-\bar{\alpha}, \bar{\alpha}]$, where $\bar{\alpha} = \max_{(m,w) \in \mathcal{M} \times \mathcal{W}} \{f_m(w) + f_w(m)\}$. (This is without loss, if a man and woman have a transfer that's not in X, then one of them has a negative payout. Thus, in any individually rational – and, by inclusion, any stable allocation – all transfers are in X.[12]) This completes the embedding since it's easily verified that a stable allocation satisfies Demange and Gale's [24] definition of stability. $\triangle$

Remark. While the General Game subsumes both the Marriage and Assignment Games, it doesn't preserve many of their properties – e.g., there is usually no "man-preferred" stable allocation because of indifference over agreements. We discuss this in the Supplement for

[12] Generally, Demange and Gale's [24] they require that, for each man m and each woman w, there is a finite α'_{mw} such that if m and w's transfer is not in $[-\alpha'_{mw}, \alpha'_{mw}]$, then either m or w has a negative payoff. Thus, letting $X = [-\bar{\alpha}', \bar{\alpha}']$, where $\bar{\alpha}' = \max_{m,w}\{\alpha'_{mw}\}$, is without loss.

this chapter.

3.3 Results for the General Game

In this section, we state and prove our main results for the General Game; we develop several additional results in the Supplement for this Chapter. The proofs are deferred to the end of each subsection in order to discuss the results.

3.3.1 Existence of Stable Allocations

In this subsection, we prove that stable allocations exist under the following assumption.

Assumption 1. Compactness and Continuity.
The set of feasible agreements X is compact and, for each man m and each woman w, the payoffs $u_m(w', \boldsymbol{x})$ and $u_w(m', \boldsymbol{x})$ are continuous in $\boldsymbol{x}$ for all women w' and for all men m' respectively.

Proposition 1. Existence of a Stable Allocation.
Let Assumption 1 hold, then a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *exists.*

Proposition 1 gives, for instance, that there is a stable allocation in Examples 1 and 2, in the (Generalized) Marriage Game, and in the Assignment Game since inspection shows that Assumption 1 holds in each. We prove Proposition 1 by contradiction. Specifically, we show that if there is no stable allocation when X is compact, then there is no stable allocation when agreements are restricted to a finite subset of X. This contradicts a well-known result (see Lemma 1 below) that there is always a stable allocation when the set of feasible agreements is finite.[13] The key insight of our proof is that under the contradiction hypothesis and Assumption 1, we can use the Heine-Borel Theorem to ensure the existence of this finite subset of X. We need four preliminary results to make this argument precise. (Also, we remark on ways to weaken Assumption 1 after the proof.)

Lemma 1. Finite Existence.
Let the set of feasible agreements X *be finite, then a stable allocation* $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *exists.*

Results like Lemma 1 are standard in the one-to-one and many-to-one matching literatures when payoffs are such that players can strictly order matches and agreements; for instance, Hatfield and Milgrom's Theorem 3 [34] and Roth's Theorem 1 [59]. Both papers prove existence by giving variants of Gale and Shapley's [31] Deferred Acceptance algorithm

[13] The intuition here is similar to that of Crawford and Knoer [21]; however, our game, assumptions, and formal approach are quite different from theirs.

that find a stable allocation in finite time. The problem is that players in our game may be indifferent. Fortunately, these algorithms can be modified to allow for indifferences via the inclusion of a "tie-breaking" rule – see Roth and Sotomayor's Theorem 2.8 [60] for an example of this kind of modification – so the lemma obtains. (In the Supplement for this chapter, we give a proof of Lemma 1 that's based on tie-breaking.)

Lemma 2. Continuity of $u_i(\phi, \bar{\boldsymbol{x}})$.
Let Assumption 1 hold and let $\phi \in \Phi$, then $u_i(\phi, \bar{\boldsymbol{x}})$ is continuous in $\bar{\boldsymbol{x}}$ for each player i.

Proof. This a direct consequence of Assumption 1. □

To prove Proposition 1, we need to (i) represent the set of feasible and individually rational allocations as a collection of compact sets and (ii) establish that the set of allocations a man and woman block with a given agreement is open. To these ends, we introduce some notation. Let $\phi \in \Phi$ and let

$$F(\phi) = \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} |\, \bar{\boldsymbol{x}} \in X^N \cap A(\phi) \text{ and } u_i(\phi, \bar{\boldsymbol{x}}) \geq 0 \text{ for all } i \in \mathcal{N}\}$$

be the (possibly empty) set of agreement vectors in $\mathbb{R}^{kN}$ such that the pair $(\phi, \bar{\boldsymbol{x}})$ is a feasible and individually rational allocation for each $\bar{\boldsymbol{x}} \in F(\phi)$. Let $\Phi_F = \{\phi \in \Phi | F(\phi) \neq \emptyset\}$ be the set of matchings such that, for each $\phi \in \Phi_F$, there is an $\bar{\boldsymbol{x}}$ so that $(\phi, \bar{\boldsymbol{x}})$ is feasible and individually rational. Clearly, an allocation $(\phi, \bar{\boldsymbol{x}})$ is feasible and individually rational if an only if $\phi \in \Phi_F$ and $\bar{\boldsymbol{x}} \in F(\phi)$.

Lemma 3. Compactness of $F(\phi)$.
Let Assumption 1 hold and let $\phi \in \Phi$, then $F(\phi)$ is compact.

Proof. We take $F(\phi)$ to be nonempty because the empty set is trivially compact. We have that $F(\phi)$ is bounded as X^N is compact. We also have that $F(\phi)$ is closed because (i) X^N and $A(\phi)$ are closed and (ii) the function $u_i(\phi, \bar{\boldsymbol{x}})$ is continuous in $\bar{\boldsymbol{x}}$ for each $i \in \mathcal{N}$ per Lemma 2. □

When a man m and a woman w block a $(\phi, \bar{\boldsymbol{x}}) \in \Phi \times \mathbb{R}^{kN}$ with agreement $\boldsymbol{x} \in X$, we say $(m, w, \boldsymbol{x})$ **blocks** $(\phi, \bar{\boldsymbol{x}})$. Let $C = \mathcal{M} \times \mathcal{W} \times X$. For a $\phi \in \Phi$ and $c = (m, w, \boldsymbol{x}) \in C$, let

$$D_\phi(c) = \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} |\, u_m(w, \boldsymbol{x}) > u_m(\phi, \bar{\boldsymbol{x}}) \text{ and } u_w(m, \boldsymbol{x}) > u_w(\phi, \bar{\boldsymbol{x}})\}$$

be the set of vectors in $\mathbb{R}^{kN}$ such that c blocks the pair $(\phi, \bar{\boldsymbol{x}})$ for each $\bar{\boldsymbol{x}} \in D_\phi(c)$.

Lemma 4. Openness of $D_\phi(c)$.
Let Assumption 1 hold, let $\phi \in \Phi$, and let $c \in C$, then $D_\phi(c)$ is open.

Proof. We take $D_\phi(c)$ to be nonempty since the empty set is trivially open. Let $(m, w, \boldsymbol{x}) = c$. Since $u_m(\phi, \bar{\boldsymbol{x}})$ and $u_w(\phi, \bar{\boldsymbol{x}})$ are continuous in $\bar{\boldsymbol{x}}$ per Lemma 2, we have that $\{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | u_m(\phi, \bar{\boldsymbol{x}}) < u_m(w, \boldsymbol{x})\}$ and $\{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | u_w(\phi, \bar{\boldsymbol{x}}) < u_w(m, \boldsymbol{x})\}$ are open sets. Since $D_\phi(c) = \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | u_m(\phi, \bar{\boldsymbol{x}}) < u_m(w, \boldsymbol{x})\} \cap \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | u_w(\phi, \bar{\boldsymbol{x}}) < u_w(m, \boldsymbol{x})\}$, it follows that $D_\phi(c)$ is open. □

Proof of Proposition 1.[14] Suppose that there is no stable allocation. Since the set of feasible and individually rational allocations is non-empty (because players may always be matched to themselves), every feasible and individually rational allocation is blocked by some man and woman. Let $\phi \in \Phi_F$. Then, for every $\bar{\boldsymbol{x}} \in F(\phi)$, we have that $(\phi, \bar{\boldsymbol{x}})$ is blocked by a $c \in C$, which implies that

$$F(\phi) \subset \cup_{c \in C} D_\phi(c).$$

Since $D_\phi(c)$ is open by Lemma 4, we have $\{D_\phi(c)\}_{c \in C}$ is an open cover of $F(\phi)$. Since $F(\phi)$ is compact by Lemma 3, the Heine-Borel Theorem gives the existence of a finite sub-cover $\{D_\phi(c_{\phi j})\}_{j=1}^{l_\phi}$. Thus, for every $\bar{\boldsymbol{x}} \in F(\phi)$, we have $(\phi, \bar{\boldsymbol{x}})$ is blocked by some element of $\{c_{\phi j}\}_{j=1}^{l_\phi}$. Repeating this argument for all matchings in Φ_F gives a set

$$E = \cup_{\phi \in \Phi_F} \{c_{\phi j}\}_{j=1}^{l_\phi}$$

such that every feasible and individually rational allocation is blocked by an element of E. Since Φ is finite, Φ_F and thus E are finite. Let

$$E_X = \{\boldsymbol{x} \in X \,|\, (m, w, \boldsymbol{x}) \in E \text{ for some } (m, w) \in \mathcal{M} \times \mathcal{W}\}$$

be the set of agreements associated with E; we have that E_X is finite.

To establish the contradiction, suppose that the set of feasible agreements is E_X instead of X. Since E_X is finite, Lemma 1 gives that there is a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. By the definition of stability, (i) $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in \mathcal{A}$, (ii) $\bar{\boldsymbol{x}}^\star \in (E_X)^N$, (iii) $u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq 0$ for each player i, and (iv) there is no $(m, w, \boldsymbol{x}) \in \mathcal{M} \times \mathcal{W} \times E_X$ such that $u_m(w, \boldsymbol{x}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_w(w, \boldsymbol{x}) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Since $(E_X)^N \subset X^N$, we have $\bar{\boldsymbol{x}}^\star \in X^N$, which, in light of (i) and (iii), implies that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is a feasible and individually rational allocation when the set of feasible agreements is X. Thus, the previous paragraph gives that there is a $(m', w', \boldsymbol{x}') \in E$ such that

$$u_{m'}(w', \boldsymbol{x}') > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star) \text{ and } u_{w'}(w', \boldsymbol{x}') > u_{w'}(\phi^\star, \bar{\boldsymbol{x}}^\star).$$

Since $\boldsymbol{x}' \in E_X$, we have that $(m', w', \boldsymbol{x}') \in \mathcal{M} \times \mathcal{W} \times E_X$. Hence, (iv) is contradicted, i.e.,

[14] This proof benefited greatly from discussions with Asaf Plan.

the display equation gives that m' and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ when the set of feasible agreements is E_X, a contradiction of stability. □

Remark. It's possible to weaken a few of our assumptions and still obtain existence. In particular, we may (i) allow different men and women to have different sets of feasible agreements, (ii) allow for heterogeneous values to being single (or remove the option to be single), and (iii) replace continuity with upper-semicontinuity (or even weaker continuity assumptions). We may also generalize the agreement space to be a closed subset of a compact metric space. We cannot, however, easily dispense with the compactness of X. We discuss these issues in the Supplement for this chapter.

3.3.2 Pareto Optimality

In this subsection, we first make the notion of a Pareto optimal allocation precise and then we show that Assumption 1 also ensures the existence of Pareto optimal stable allocations. Such allocations are important for social welfare and have suggested as an alternative solution concept in two-sided matching games (e.g., Sotomayor [67]).

Definition. An allocation $(\phi, \bar{\boldsymbol{x}})$ is *Pareto optimal* if there is no other feasible allocation $(\phi', \bar{\boldsymbol{x}}')$ such that (i) all players do weakly better in $(\phi', \bar{\boldsymbol{x}}')$ than $(\phi, \bar{\boldsymbol{x}})$, i.e., $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi, \bar{\boldsymbol{x}})$ for each player i, and (ii) at least one player does strictly better in $(\phi', \bar{\boldsymbol{x}}')$ than $(\phi, \bar{\boldsymbol{x}})$, i.e., $u_i(\phi', \bar{\boldsymbol{x}}') > u_i(\phi, \bar{\boldsymbol{x}})$ for some player i. If a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is Pareto optimal, we say it is a Pareto stable allocation.

We focus on "strong" Pareto optimality instead of "weak" Pareto optimality since every stable allocation is weakly Pareto optimal and since strong Pareto optimality is more intuitive. (Notice that Pareto optimal allocations won't generally maximize social welfare because payoffs are not quasi-linear.)

Proposition 2. Existence of a Pareto Stable Allocation.
Let Assumption 1 hold, then a Pareto stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exists.

Proposition 2 gives, for instance, that there is a Pareto stable allocation in Examples 1 and 2, in the Marriage Game, and in the Assignment Game. We prove Proposition 2 by showing that the set of stable allocations is "compact" given Assumption 1 (see Lemma 5 below). Thus, there is an allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ that maximizes total welfare on this set (see Lemma 6 below). We then argue that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is Pareto optimal via contradiction. Indeed, if it weren't, then there would be another feasible allocation $(\phi', \bar{\boldsymbol{x}}')$ where all players do weakly better and one player does strictly better. But then, $(\phi', \bar{\boldsymbol{x}}')$ is stable (see Lemma 7 below). Hence, $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ does not maximize total welfare on the set of stable allocations, a contradiction. We develop our three preliminary lemmas before making this argument precise.

Let $\mathcal{S}$ denote the set of stable allocations. We need to represent $\mathcal{S}$ as collection of compact subsets. To these ends, for each $\phi \in \Phi$, let $\mathcal{S}(\phi) = \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | (\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}\}$ be the (possibly empty) set of agreement vectors that yield a stable allocation when paired with ϕ. Let $\Phi_{\mathcal{S}} = \{\phi \in \Phi | \, \mathcal{S}(\phi) \neq \emptyset\}$ be set of matchings which are part of a stable allocation. An allocation $(\phi, \bar{\boldsymbol{x}})$ is stable if and only if $\phi \in \Phi_{\mathcal{S}}$ and $\bar{\boldsymbol{x}} \in \mathcal{S}(\phi)$.

Lemma 5. Compactness of $\mathcal{S}(\phi)$.
Let Assumption 1 hold, then $\mathcal{S}(\phi)$ is compact for each $\phi \in \Phi$.

Proof. It's easily seen that $\mathcal{S}(\phi) = F(\phi) \cap \overline{D(\phi)}$, where $D(\phi) = \cup_{c \in C} D_\phi(c)$ is the set of agreement vectors such that the pair $(\phi, \bar{\boldsymbol{x}})$ is blocked by some man and woman for each $\bar{\boldsymbol{x}} \in D_\phi$, and $\overline{D(\phi)}$ is the complement of $D(\phi)$. Since $F(\phi)$ is closed (per Lemma 3) and $\overline{D(\phi)}$ is closed (since $D(\phi)$ is open because it's the union of open sets by Lemma 4), we have that $\mathcal{S}(\phi)$ is closed. Since $\mathcal{S}(\phi) \subset X^N$, it's also bounded. □

Given a $(\phi, \bar{\boldsymbol{x}}) \in \Phi \times \mathbb{R}^{kN}$, let $T(\phi, \bar{\boldsymbol{x}}) = \Sigma_{i \in \mathcal{N}} u_i(\phi, \bar{\boldsymbol{x}})$ be the total payoff to all players in $(\phi, \bar{\boldsymbol{x}})$.

Lemma 6. Total Welfare and Stable Allocations.
Let Assumption 1 hold, then there is a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ such that $T(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq T(\phi', \bar{\boldsymbol{x}}')$ for every $(\phi', \bar{\boldsymbol{x}}') \in \mathcal{S}$.

Proof. By Proposition 1, we have that $\mathcal{S}$ is nonempty. Thus, $\Phi_{\mathcal{S}}$ is nonempty (and finite), as is $\mathcal{S}(\phi)$ for each $\phi \in \Phi_{\mathcal{S}}$. For each $\phi \in \Phi_{\mathcal{S}}$, we have the existence of an $\bar{\boldsymbol{x}}_\phi \in \mathcal{S}(\phi)$ that solves $\max_{\bar{\boldsymbol{x}} \in \mathcal{S}(\phi)} T(\phi, \bar{\boldsymbol{x}})$ by the Extreme Value Theorem applies since $T(\phi, \bar{\boldsymbol{x}})$ is continuous in $\bar{\boldsymbol{x}}$ by Lemma 2, $\mathcal{S}(\phi)$ is nonempty, and $\mathcal{S}(\phi)$ is compact by Lemma 5. Let $I = \cup_{\phi \in \Phi_{\mathcal{S}}} \{(\phi, \bar{\boldsymbol{x}}_\phi)\}$ and let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ solve $\max_{(\phi, \bar{\boldsymbol{x}}) \in I} T(\phi, \bar{\boldsymbol{x}})$. A solution exists since I is finite.

We now establish that $T(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq T(\phi, \bar{\boldsymbol{x}})$ for all $(\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}$. Suppose not, then there is a $(\phi', \bar{\boldsymbol{x}}') \in \mathcal{S}$ with $T(\phi', \bar{\boldsymbol{x}}') > T(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Thus, $\mathcal{S}(\phi') \neq \emptyset$ and $(\phi', \bar{\boldsymbol{x}}_{\phi'}) \in I$. Since, $T(\phi', \bar{\boldsymbol{x}}_{\phi'}) \leq T(\phi^\star, \bar{\boldsymbol{x}}^\star)$, we have $T(\phi', \bar{\boldsymbol{x}}_{\phi'}) < T(\phi', \bar{\boldsymbol{x}}')$, a contradiction of the optimality of $\bar{\boldsymbol{x}}_{\phi'}$ on $\mathcal{S}(\phi')$. □

Remark. It's clear from the proof that there is a stable allocation that maximizes the total payoff of any group of players (e.g., the total payoff of women).

Let $(\phi, \bar{\boldsymbol{x}})$ be a feasible allocation and let

$$P(\phi, \bar{\boldsymbol{x}}) = \{(\phi', \bar{\boldsymbol{x}}') \in \mathcal{A} | \, \bar{\boldsymbol{x}}' \in X^N \text{ and } u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi, \bar{\boldsymbol{x}}) \text{ for all } i \in \mathcal{N}\}$$

be the set of feasible allocations that are at least as good as $(\phi, \bar{\boldsymbol{x}})$. This set is non-empty as $(\phi, \bar{\boldsymbol{x}}) \in P(\phi, \bar{\boldsymbol{x}})$.

Lemma 7. A Representation of Pareto Optimality.
L7.1 : Let $(\phi', \bar{\boldsymbol{x}}')$ be a stable allocation, then every element of $P(\phi', \bar{\boldsymbol{x}}')$ is also a stable allocation.
L7.2 : Let $(\phi', \bar{\boldsymbol{x}}')$ be a stable allocation and let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be a solution to

$$\max_{(\phi,\bar{\boldsymbol{x}}) \in P(\phi',\bar{\boldsymbol{x}}')} T(\phi, \bar{\boldsymbol{x}}).$$

Then $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is a Pareto stable allocation.

Proof. These results are almost obvious. We first establish $L7.1$. Let $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in P(\phi', \bar{\boldsymbol{x}}')$. By construction of $P(\cdot)$, $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is feasible and individually rational, so we only need to show that it can't be blocked. We do this by contradiction. Suppose a man m and a woman w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ with agreement $\boldsymbol{x} \in X$, then $u_m(w, \boldsymbol{x}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_m(\phi', \bar{\boldsymbol{x}}')$ and $u_w(m, \boldsymbol{x}) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_w(\phi', \bar{\boldsymbol{x}}')$ where the weak inequality follows from the definition of $P(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Thus, m and w also block $(\phi', \bar{\boldsymbol{x}}')$, a contradiction.

Now we establish $L7.2$. Since $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in P(\phi', \bar{\boldsymbol{x}}')$, we have $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable by $L7.1$. We only need to show that there is no other feasible allocation $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ such that (i) $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$ and (ii) $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) > u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for some $i \in \mathcal{N}$. We do this by contradiction. Suppose there is such a $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$, then $T(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) > T(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Thus, $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ does not solve the maximization problem, a contradiction. □

Proof of Proposition 2. By Lemma 6, there is a $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in \mathcal{S}$ with $T(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq T(\phi, \bar{\boldsymbol{x}})$ for all $(\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}$. Since $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in P(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and since $P(\phi^\star, \bar{\boldsymbol{x}}^\star) \subset \mathcal{S}$ per Lemma 7, we have that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ solves

$$\max_{(\phi,\bar{\boldsymbol{x}}) \in P(\phi^\star,\bar{\boldsymbol{x}}^\star)} T(\phi, \bar{\boldsymbol{x}}).$$

Thus, Lemma 7 gives that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is a Pareto stable allocation. □

3.3.3 Interior Stable Allocations

In this subsection, we first make the notion of an interior allocation precise and then we give conditions that ensure the existence of interior stable allocations. These allocations are important from a technical perspective because our proofs in Section 5 leverage small increases and decreases in effort, which are impossible at boundary allocations.

For a set $S \subset \mathbb{R}^k$, we write ∂S for the boundary of S and $\text{int}(S)$ for the interior of S.

Definition. An allocation $(\phi, \bar{\boldsymbol{x}}) = (\phi, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^N)$ is *interior* if some player is partnered, i.e., $\phi(i) \neq i$ for some $i \in \mathcal{N}$, and all partnered players have agreements on the interior of X, i.e., $\phi(i) \neq i$ implies $\boldsymbol{x}^i \in \text{int}(X)$ for all $i \in \mathcal{N}$. If a (Pareto) stable allocation is interior, we say it is an interior (Pareto) stable allocation.

The next assumption is sufficient for the existence of interior stable allocations.

Assumption 2. Sufficient Conditions for an Interior Stable Allocation.
The set of feasible agreements X has nonempty interior and:
(i) There is a man m, a woman w, and an agreement $\boldsymbol{x} \in \text{int}(X)$ such that $u_m(w, \boldsymbol{x}) \geq 0$ and $u_w(m, \boldsymbol{x}) \geq 0$.
(ii) For each man m, each woman w, and each $\boldsymbol{x} \in \partial X$, we have at least one of the following:
(a) Either $u_m(w, \boldsymbol{x}) < 0$ or $u_w(m, \boldsymbol{x}) < 0$; or
(b) There is a $\boldsymbol{x}' \in \text{int}(X)$ such that $u_m(w, \boldsymbol{x}') > u_m(w, \boldsymbol{x})$ and $u_w(m, \boldsymbol{x}') > u_w(m, \boldsymbol{x})$.

Part (i) ensures that being partnered is not worse than being single for at least one man and one woman; we refer to this as **agreeability**. Part (ii) gives that each boundary agreement is either (a) very undesirable to the man or the woman and/or (b) payoff dominated by some other feasible agreement. If (a) holds for all men, all women, and all $\boldsymbol{x} \in \partial X$, then we say **intolerability** holds. If (b) holds for all men, all women, and all $\boldsymbol{x} \in \partial X$, then we say that **blockability** holds.

While intolerability and blockability are strong assumptions, they are often reasonable. In fact, we make heavy use of intolerability in the next section because it embeds the intuitive ideas that a player is very unhappy when (i) he or she exerts extreme effort β or (ii) he or she exerts positive effort and his or her match does not; see the continuation of Example 1 below.

Proposition 3. Existence of an Interior Stable Allocation.
Let Assumptions 1 and 2 hold. Then, (i) an interior Pareto stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exists and (ii) any stable allocation where some player is partnered is an interior stable allocation.

Proposition 3 gives, for instance, that there is an interior Pareto stable allocation in Examples 1 and 2 since Assumptions 1 and 2 hold. It's easily seen that Assumption 1 hold in both examples, so we only need to verify that Assumption 2 holds. We do this next by showing that agreeability and intolerability/blockability hold.

Example 1 (Continued). Intolerability.

We want to verify that agreeability and intolerability hold. We have that agreeability holds since $u_1(4, \boldsymbol{x}) = 3/8 > 0$ and $u_4(1, \boldsymbol{x}) = 3/8 > 0$ when $\boldsymbol{x} = (1, 1)$.

Showing intolerability takes a bit more work: for each man m and each woman w, we need to walk the boundary of $[0, 2]^2$ and show that either m or w has a negative payoff.

Focus on m and recall that $u_m(w, x_1, x_2) = x_2 - \frac{1}{\theta_m}(x_1)^2 - {}^1\!/\!_8$. For all $y \in [0, 2]$, we have that (i) $u_m(w, y, 0) = -{}^1\!/\!_8 - \frac{1}{\theta_m}y^2 < 0$, i.e., m gets a negative payout when he exerts effort and w does not, and (ii) $u_m(w, 2, y) = y - {}^1\!/\!_8 - \frac{4}{\theta_m} < 0$ since $\theta_w \in \{1, 2\}$, i.e., m gets a negative payout if he exerts extreme effort. By symmetry, we have (iii) $u_w(m, 0, y) < 0$ for and (iv) $u_w(m, y, 2) < 0$ for all $y \in [0, 2]$. Thus, for every $(x_1, x_2) \in \partial[0, 2]^2$, we have either $u_m(w, x_1, x_2) < 0$ or $u_w(m, x_1, x_2) < 0$. △

Example 2 (Continued). Blockability.

We want to verify that this example satisfies agreeability and blockability. We have that agreeability holds since $u_m(w, x) = 0$ and $u_w(m, x) = 0$ for each man m and each woman w when $x = 0$. We also have that blockability holds as m and w's payoffs are strictly increasing at $x = 0$ and strictly decreasing at $x = 1$. △

We prove Proposition 3 in two steps. First, we use a contradiction argument to establish that any stable allocation with a partnered player is an interior stable allocation. The key insight is that if a matched man and woman have a boundary agreement then Assumption 2 ensures either (i) the agreement isn't individually rational or (ii) they can block. In either case, we obtain the requisite contradiction. In the second step, we leverage this result and a construction argument to show the existence of an interior Pareto stable allocation.

Proof of Proposition 3. We begin by establishing that a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$ where some player is partnered is interior. That is, we show that that $\phi^\star(i) \neq i$ implies $\boldsymbol{x}^{i^\star} \in \text{int}(X)$ for each player i. Without loss, consider a man m with $\phi^\star(m) \neq m$, the argument is analogous for a woman. Let $w = \phi^\star(m)$. We argue by contradiction. If $\boldsymbol{x}^{m^\star} \in \partial X$, then, since Assumption 2 holds, we have: (a) $u_m(w, \boldsymbol{x}^{m^\star}) < 0$ or $u_w(m, \boldsymbol{x}^{m^\star}) < 0$ or (b) there is a $\boldsymbol{x}' \in X$ such that $u_m(w, \boldsymbol{x}') > u_m(w, \boldsymbol{x}^{m^\star})$ and $u_w(m, \boldsymbol{x}') > u_w(m, \boldsymbol{x}^{m^\star})$. If (a), we have that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is not individually rational, a contradiction. If (b), m and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ with agreement $\boldsymbol{x}'$, another contradiction. It follows that $\boldsymbol{x}^{m^\star} \in \text{int}(X)$.

We now show the existence of an interior Pareto stable allocation. By Proposition 2, there is a Pareto stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$. There are two cases (i) some man and woman are matched by $\phi^\star$ or (ii) no man and woman are matched by $\phi^\star$. If case (i), then $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an interior allocation by the above paragraph; so we proceed under (ii).

By Assumption 2, there is a man m, a woman w, and an agreement $\tilde{\boldsymbol{x}} \in \text{int}(X)$ such that m and w both get at least zero at agreement $\tilde{\boldsymbol{x}}$. We construct a candidate interior Pareto stable allocation $(\phi', \bar{\boldsymbol{x}}')$ from $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ by taking m and w, matching them, and assigning them agreement $\tilde{\boldsymbol{x}}$, while leaving everyone else's match and agreement alone. Formally, we construct $(\phi', \bar{\boldsymbol{x}}') = (\phi', \boldsymbol{x}^{1'}, \ldots, \boldsymbol{x}^{N'})$ from $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ as follows. First, set $\phi'(m) = w$, set

$\phi'(w) = m$, set $\boldsymbol{x}^{m'} = \tilde{\boldsymbol{x}}$, and set $\boldsymbol{x}^{w'} = \tilde{\boldsymbol{x}}$. Second, set $\phi'(i) = \phi^\star(i)$ and set $\boldsymbol{x}^{i'} = \boldsymbol{x}^{i\star}$ for all $i \in \mathcal{N}\backslash\{m, w\}$.

Clearly, $(\phi', \bar{\boldsymbol{x}}')$ is an interior allocation. Since $u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) = 0$ for all $i \in \mathcal{N}$, we have weakly increased everyone's payoff, i.e., $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$. Hence, Lemma 7 gives that $(\phi', \bar{\boldsymbol{x}}')$ is also a stable allocation. It remains to prove that $(\phi', \bar{\boldsymbol{x}}')$ is Pareto optimal. To these ends, suppose it were not. Then there is another feasible allocation $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ such that (i) $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) \geq u_i(\phi', \bar{\boldsymbol{x}}')$ for all $i \in \mathcal{N}$ and (ii) $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) > u_i(\phi', \bar{\boldsymbol{x}}')$ for some $i \in \mathcal{N}$. Since $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$, (i) and (ii) imply that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is not Pareto optimal, a contradiction. □

3.3.4 Interior Stable Allocations and Pareto Optimality

In this subsection, we give conditions that guarantee *every* interior stable allocation is Pareto optimal. These conditions tie the last two subsections together and allow us to ensure that every solution of the Effort Game is Pareto optimal.

We need a definition. We write $B_r(\boldsymbol{x})$ for the open ball of radius r in $\mathbb{R}^k$ around $\boldsymbol{x}$, i.e., $B_r(\boldsymbol{x}) = \{\boldsymbol{x}' \in \mathbb{R}^k \mid ||\boldsymbol{x} - \boldsymbol{x}'|| < r\}$.

Definition. For a man m and a woman w, we say that $u_m(w, \boldsymbol{x})$ (or $u_w(m, \boldsymbol{x})$) is *locally nonsatiated in* $\boldsymbol{x}$ if, for each agreement $\boldsymbol{x} \in \mathbb{R}^k$, there is a nearby agreement that m (w) strictly prefers, i.e., there is a $\boldsymbol{x}' \in B_r(\boldsymbol{x})$ such that $u_m(w, \boldsymbol{x}) < u_m(w, \boldsymbol{x}')$ for each $r > 0$ (a $\boldsymbol{x}'' \in B_r(\boldsymbol{x})$ such that $u_w(m, \boldsymbol{x}) < u_w(m, \boldsymbol{x}'')$ for each $r > 0$).

The next assumption gives our sufficient conditions; the proposition follows.

Assumption 3. Sufficient Conditions for Interior Stable Allocations to be Pareto Optimal. The set of feasible agreements X has nonempty interior, intolerability holds, and, for each man m and each woman w, the payoffs $u_m(w', \boldsymbol{x})$ and $u_w(m', \boldsymbol{x})$ are continuous and locally nonsatiated in $\boldsymbol{x}$ for all women w' and for all men m' respectively.

Proposition 4. Interior Stable Allocations are Pareto Optimal.
Let Assumption 3 hold, then every interior stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *is Pareto optimal.*

Proposition 4 gives, for instance, that the stable allocation we found in Example 1 is Pareto optimal. Simply, for each man m and each woman w, we have that $u_m(w, x_1, x_2)$ and $u_w(m, x_1, x_2)$ are strictly increasing in either x_1 or x_2 and so are locally nonsatiated. We prove Proposition 4 by using a "bribery" argument. The idea is that local nonsatiation allows a man (woman) to bribe an indifferent woman (man) to form a blocking pair. Thus, if a stable allocation is not Pareto optimal, the player who does strictly better in the improving allocation bribes his or her match in the improving allocation into blocking the initial stable allocation, a contradiction.

Proof of Proposition 4. We argue by contradiction. Suppose that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an interior stable allocation that's not Pareto optimal. Then there is another feasible allocation $(\phi', \bar{\boldsymbol{x}}') = (\phi', \boldsymbol{x}^{1'}, \ldots, \boldsymbol{x}^{N'})$ with (i) $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$ and (ii) $u_i(\phi', \bar{\boldsymbol{x}}') > u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for some $i \in \mathcal{N}$. Suppose that man m is the player who does strictly better; this is without loss as the analogous argument applies when a woman does strictly better. We'll show that m and some woman block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. But first, we need two preliminary facts.

Fact one: m is matched to some woman under ϕ'. If not, then $u_m(\phi', \bar{\boldsymbol{x}}') = 0$. Since $u_m(\phi', \bar{\boldsymbol{x}}') > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$, we have $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) < 0$, a contradiction of the stability of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. In light of this, let $w = \phi'(m)$.

Fact two: m's agreement $\boldsymbol{x}^{m'}$ is on the interior of X. If not, then $\boldsymbol{x}^{m'} \in \partial X$, so either $u_m(\phi', \bar{\boldsymbol{x}}') = u_m(w, \boldsymbol{x}^{m'}) < 0$ or $u_w(\phi', \bar{\boldsymbol{x}}') = u_w(m, \boldsymbol{x}^{m'}) < 0$ by Assumption 3. Since $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$, either $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) < 0$ or $u_w(\phi^\star, \bar{\boldsymbol{x}}^\star) < 0$, a contradiction as $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable.

We now establish that m and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e., we establish that there is an $\boldsymbol{x} \in X$ such that

$$u_m(w, \boldsymbol{x}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \tag{3.1}$$

$$u_w(m, \boldsymbol{x}) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star). \tag{3.2}$$

Since (i) payoffs are continuous in $\boldsymbol{x}$ by Assumption 3 and (ii) $u_m(w, \boldsymbol{x}^{m'}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$, there is an $r > 0$ such that equation (3.1) holds for all $\boldsymbol{x} \in B_r(\boldsymbol{x}^{m'})$. Since $\boldsymbol{x}^{m'} \in \text{int}(X)$, we take r to be sufficiently small such that $B_r(\boldsymbol{x}^{m'}) \subset X$. Recall that $u_w(m, \boldsymbol{x}^{m'}) = u_w(\phi', \bar{\boldsymbol{x}}') \geq u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Since $u_w(\cdot)$ is locally nonsatiated, there is an $\boldsymbol{x} \in B_r(\boldsymbol{x}^{m'})$ with $u_w(m, \boldsymbol{x}) > u_w(m, \boldsymbol{x}^{m'})$. Thus, equations (3.1) and (3.2) hold at $\boldsymbol{x}$, i.e., m and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction. □

Remark. We cannot meaningfully swap blockability for intolerability in Assumption 3. Simply, blockability and local nonsatiation are mutually exclusive conditions when X is compact and we usually take X to be compact.[15]

[15] To see this, suppose local nonsatiation holds, blockability holds, payoffs are continuous in $\boldsymbol{x}$, and X is compact. Let m be a man, and let w be a woman. The Extreme Value Theorem gives that $\alpha = \arg\max_{\boldsymbol{x} \in X} u_m(w, \boldsymbol{x})$ is nonempty. Since blockability gives that every point in ∂X is payoff dominated by a point in $\text{int}(X)$, we have $\alpha \subset \text{int}(X)$. But then, for each $\boldsymbol{x} \in \alpha$, there is a $B_r(\boldsymbol{x}) \subset X$ for some $r > 0$, so local nonsatiation gives that there is a $\boldsymbol{x}' \in B_r(\boldsymbol{x})$ with $u_m(w, \boldsymbol{x}') > u_m(w, \boldsymbol{x})$. Thus, $\boldsymbol{x} \notin \alpha$, a contradiction.

3.4 Description of the Effort Game

In this section, we describe the Effort Game and show that it has a solution.

3.4.1 Environment

Let $\Theta \subset \mathbb{R}_+$ be a finite set of types ordered in the usual way. We endow each player i with a type θ_i (e.g., innate ability) from Θ, and suppose that types are commonly known. Let $b : \mathbb{R} \to \mathbb{R}$ be a continuous and strictly increasing function. We call $b(\cdot)$ the "benefit" function because if a player exerts effort y, then he or she provides benefit $b(y)$ to his or her match. Let $c : \Theta \times \mathbb{R} \to \mathbb{R}$ be continuous in its second argument. We call $c(\cdot)$ the "cost" function because if a type θ player exerts effort y, he or she incurs cost $c(\theta, y)$. Unlike the benefit function, the cost function may be non-monotone in effort. For simplicity, we take $b(0) \geq 0$ and $c(\theta, 0) \geq 0$ for all types θ. We suppose that, due to time and energy limitations, players can only exert efforts between 0 and β, where $0 < \beta < \infty$. Thus, the set of feasible agreements/efforts is $X = [0, \beta]^2$.

When a man m and a woman w are matched (to each other), their payoffs from agreement $(x_1, x_2) \in \mathbb{R}^2$ are

$$u_m(w, x_1, x_2) = b(x_2) - c(\theta_m, x_1) \text{ and } u_w(m, x_1, x_2) = b(x_1) - c(\theta_w, x_2).$$

Recall that single players get zero. As in Example 1, for $(x_1, x_2) \in X$, we think of x_1 and x_2 as the man's and the woman's efforts respectively. Thus, m and w exert effort to produce a benefit for each other.

It may be the case that a player's effort diminishes his or her payoff, i.e., $c(\theta, y)$ is positive and increasing in y. However, a player's effort often provides him or her with a "personal" benefit, e.g., in cooking a delicious dinner for his girlfriend, a man also cooks himself a delicious dinner. We capture this personal benefit by allowing $c(\theta, y)$ to be non-monotone in y and so be decreasing and even negative.

The benefit function is homogenous for simplicity. Proposition 5 continues to hold when (i) the benefit a player produces depends on his/her identity or (ii) payoffs are *not* additively separable, but are continuous and are strictly increasing in benefit. Propositions 6, 7, and 8 continue to hold when the benefit a player produces depends on his or her (i) sex/side or (ii) type, *provided* this benefit is increasing in his or her type. Thus, our core results extend to environments where one side pays the other for a service, e.g., lawyers and clients.[16] We discuss why each result generalizes in Section 5.

[16]In such a game, the clients' benefit and cost functions are linear in money/effort, while the lawyers' benefit and cost functions are non-linear in effort.

We make a clarifying remark before proceeding.

Remark. Since players prefer matches who produce higher benefits, it may appear as though they have a common preference (over matches and agreements). This is not the case.[17] Instead, for a fixed vector of efforts, each side agrees on a ranking of the opposite side – e.g., the women agree on which men are best, second best, and so on. However, efforts aren't fixed, they're endogenous. Thus, each man effectively chooses his position in the women's ranking by choosing his effort, likewise for women. (His choice depends, of course, on the benefits offered by the other men and women, as well as his own cost.)

3.4.2 Interior Stable Allocations

We focus on interior stable allocation since we often maintain the following assumption.

Assumption 4. Agreeability and Intolerability.
The benefit and cost functions, as well as players' endowed types are such that agreeability and intolerability hold.[18]

This assumption embeds the intuitive ideas that a player is very unhappy when (i) he or she exerts extreme effort β or (ii) he or she exerts positive effort and his or her match does not. The class of benefit functions, cost functions, and endowed types satisfying Assumption 4 is nonempty – e.g., Example 1, where $b(y) = y$ and $c(\theta, y) = {}^{y^2}/_{\theta} + {}^1/_8$, as well as Examples 4 and 5 below. Other examples can be readily constructed.

The next corollary motivates our focus on interior stable allocations.

Corollary 1. Existence and Pareto Optimality of Interior Stable Allocations.
Let Assumption 4 hold, then (i) there is a interior stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$*, (ii) all interior stable allocations are Pareto optimal, and (iii) any stable allocation where a player is partnered is an interior stable allocation.*

Proof. Since $b(y)$ and $c(\theta, y)$ are continuous, $b(y)$ is strictly increasing in y (and, thus, locally nonsatiated), X is compact with nonempty interior, and agreeability and intolerability hold, both Propositions 3 and 4 apply. $\square$

[17] To illustrate, consider two men m and m' who have a choice between (a) woman w and agreement $(x_1, x_2) = ({}^1/_4, {}^9/_{10})$ or (b) woman w' and agreement $(x'_1, x'_2) = ({}^1/_2, 1)$. Also, let $b(y) = y$, let $c(\theta_m, y) = y^2 - y + {}^1/_2$, and let $c(\theta_{m'}, y) = (2y)^2 - y + {}^1/_4$. A bit of algebra shows that m strictly prefers (b) to (a), while m' strictly prefers (a) to (b).

[18] When Assumption 4 holds, players cannot obtain strictly positive payoffs from their own efforts because the cost function cannot be negative (per intolerability). However, players can benefit from their own efforts because intolerability allows for decreasing cost functions. In addition, Assumption 4 is inessential to Propositions 5 to 7, so these results hold even when players obtain positive payoffs directly from their own efforts.

Our second motivation for focusing on interior stable allocations is technical: they allow us to consider the effects of small shifts in effort. Such shifts lie at the heart of our proofs in the next section and are impossible at boundary allocations.

3.5 Results for the Effort Game

In this section, we state and prove our results for the Effort Game. We defer the proofs of the major results until the end of each subsection in order to discuss the results and provide economic intuition for them.

3.5.1 Matching on the Basis of Benefit and Effort

In this subsection, we examine how players match on the basis of their benefits and efforts. To do this, we need some notation. Let $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$ be an interior stable allocation. Let $z_i^\star$ denote player i's effort in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e., $z_i^\star$ is the first component of $\boldsymbol{x}^{i^\star} = (x_1^{i^\star}, x_2^{i^\star})$ when i is a man and $z_i^\star$ is the second component of $\boldsymbol{x}^{i^\star}$ when i is a woman. Consider players who are partnered by $\phi^\star$. Each partnered player i produces a positive benefit $b_i^\star = b(z_i^\star)$. We rank partnered players by the benefits they produce from greatest to least and place them into groups of equivalent benefits.[19] For the men, we label these groups $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$, where $G_1^{\mathbb{M}}$ contains the men who produce the highest benefit, $G_2^{\mathbb{M}}$ contains the men who produce the second highest benefit, and so on. For the women, we label the analogous groups $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{W}}}^{\mathbb{W}}$. We have that $J_{\mathbb{M}}$ and $J_{\mathbb{W}}$ are positive and finite, since at least one man and woman are partnered and since $\mathcal{M}$ and $\mathcal{W}$ are finite. To simplify notation, we suppress the dependence of $\{z_i^\star\}_{i\in\mathcal{N}}$, $\{b_i^\star\}_{i\in\{j\in\mathcal{N}|\phi^\star(j)\neq j\}}$, $\{G_l^{\mathbb{M}}\}_{l=1}^{J_{\mathbb{M}}}$, $\{G_l^{\mathbb{W}}\}_{l=1}^{J_{\mathbb{W}}}$, $J_{\mathbb{M}}$, and $J_{\mathbb{W}}$ on $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

The next proposition examines how players match across benefit groups.

Proposition 5. Benefit Groups and Matching.
Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation and let $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$ and $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{W}}}^{\mathbb{W}}$ be the associated benefit groups. Then, (i) there are an equal number of male and female benefit groups, i.e., $J_{\mathbb{M}} = J_{\mathbb{W}}$, and (ii) a man m is in the l-th benefit group of men if and only if his match $\phi^\star(m)$ is in the l-th benefit group of women, i.e., $m \in G_l^{\mathbb{M}} \iff \phi^\star(m) \in G_l^{\mathbb{W}}$ for all $l \in \{1, \ldots, J_{\mathbb{M}}\}$. The analogous result holds for women.

That is, in any interior stable allocation, (i) men who produce strictly higher benefits are matched to women who produce strictly higher benefits and (ii) men who produce the same

[19] We omit single players because they produce benefits for no one.

benefit are matched to women who produce the same benefit. Hence, we say that matching is "assortative in benefit." We can see this in the stable allocation we found in Example 1.

Example 1 (Continued). Assortative in Benefit.

In the stable allocation we found, man 1 and woman 4 each exert effort 1, while man 2 and woman 3 each exert effort $1/2$. It follows that, $b_1^\star = b_4^\star = 1$ and $b_2^\star = b_3^\star = \frac{1}{2}$. Thus, there are two (singleton) benefit groups on each side: $G_1^{\mathbb{M}} = \{1\}$, $G_2^{\mathbb{M}} = \{2\}$, $G_1^{\mathbb{W}} = \{4\}$, and $G_2^{\mathbb{W}} = \{3\}$. Since man 1 is matched to woman 4 and man 2 is matched to woman 3, the matching is assortative in benefit. △

The intuition behind Proposition 5 is that players "compete" for the best possible match. To illustrate, suppose that there is an interior stable allocation where two partnered men m and m' both produce the same benefit, but m' has match w' who produces more benefit than the match of m. Then, m can increase his effort by an arbitrarily small amount and give w' a higher benefit than she is currently receiving. Since w' desires the highest benefit possible, she'll agree to match with m instead of m'. Man m is willing to increase his effort slightly because, in exchange for an arbitrarily small change in cost, he gains a partner with strictly higher benefit, and so does strictly better. Thus, m and w' block. Hence, a necessary condition of stability is that all men of the same benefit are matched to women of the same benefit. The proof makes this intuition precise.

Since the intuition (and thus the proof) do not depend on players having the same benefit function and do not exploit the additive nature of payoffs, the proposition continues to hold when (i) the benefits players produce depend on their identities or (ii) players' payoffs are non-separable, but are continuous and strictly increasing in the benefits they receive. Interestingly, the proposition also obtains when the benefit function is non-monotone and cost is strictly increasing since we can slightly reduce the effort of w' instead of increasing the effort of m. Unfortunately, it fails when both benefit and cost are non-monotone as then a small change in effort may not make a player strictly better off.

As the intuition implies, the continuity of effort is essential to Proposition 5. If effort is discrete, then the smallest effort increment may be so large that the benefit m gains from matching with w' is dissipated by the increase in his cost from his additional effort. Hence, the proposition may fail. For example, when $\mathcal{M} = \{1, 2\}$, $\mathcal{W} = \{3, 4\}$, $b(y) = 1 + y$, $c(\theta, y) = y/\theta$, $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 1$, and the set of feasible agreements is $\{0, 1\}^2$, then it's stable for man 1 and woman 3 to match with agreement $(0, 1)$ and for man 2 and woman 3 to match with agreement $(1, 1)$. The men clearly produce different benefits and yet are matched to women who produce the same benefit. Nevertheless, a "weak version" of the proposition holds when X is discrete: in any stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, the match of any man in $G_j^{\mathbb{M}}$ produces a weakly higher benefit than the match of any man in $G_{j+1}^{\mathbb{M}}$; likewise for women.

Since the benefit function is increasing, an implication of Proposition 5 is that players in higher benefit groups exert more effort than those in lower benefit groups. In particular, men in $G_1^{\mathbb{M}}$ exert the most effort among men, men in $G_2^{\mathbb{M}}$ exert the second most effort among men, and, in general, men in $G_l^{\mathbb{M}}$ exert the l-th most effort among men. Analogously, women in $G_l^{\mathbb{M}}$ exert the l-th most effort among women. The next corollary formalizes this result.

Corollary 2. Effort and Matching.
Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation. Then, a man m exerts the l-th most effort among men if and only if his match $\phi^\star(m)$ exerts the l-th most effort among women for all $l \in \{1, \dots, J_{\mathbb{M}}\}$. The analogous result holds for women.

Proof. Obvious and omitted. □

The corollary gives that, in any interior stable allocation, the hardest working players match, as do the second hardest working players, and so forth. This is easily seen in the interior stable allocation we found in Example 1.

We prove Proposition 5 by induction. First, we use a competition argument to establish that men in the first benefit group match with women in the first benefit group and vice-versa. Subsequently, we use induction to show that the analogous result holds for the second benefit groups, the third benefit groups, and so on.

Proof of Proposition 5. Let $\{z_i^\star\}_{i \in \mathcal{N}}$ be the associated efforts of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and let $\{b_i^\star\}_{i \in \{j | \phi^\star(j) \neq j\}}$ be the associated benefits.

Let $\underline{J} = \min\{J_{\mathbb{M}}, J_{\mathbb{W}}\}$ and $\overline{J} = \max\{J_{\mathbb{M}}, J_{\mathbb{W}}\}$. We show below that, for each $1 \leq j \leq \underline{J}$, we have $m \in G_l^{\mathbb{M}} \iff \phi^\star(m) \in G_l^{\mathbb{W}}$ for all $l \in \{1, \dots, j\}$. We refer to this as the "induction" result since we prove it by induction on j: we first establish that it holds when $j = 1$; subsequently, we show that if it holds at $j - 1$, then it also holds at j when $j > 1$.

The first part of the proposition follows from the induction result. Simply, if $\underline{J} = J_{\mathbb{M}} < J_{\mathbb{W}} = \overline{J}$, then the women in group $J_{\mathbb{M}} + 1$ are matched to men *not* in groups 1 to $J_{\mathbb{M}}$, a contradiction as these groups contain all partnered men. Analogously, $\underline{J} = J_{\mathbb{W}} < J_{\mathbb{M}} = \overline{J}$ leads to a contradiction. Thus, we have $\underline{J} = J_{\mathbb{M}} = J_{\mathbb{W}} = \overline{J}$. The second part of the proposition also follows from the induction result, take $j = J_{\mathbb{M}}$.

Let $j = 1$. There are two cases: (i) $\overline{J} = 1$ and (ii) $\overline{J} > 1$. If case (i), then $\underline{J} = 1$ and the induction result is trivially true. Thus, we proceed under case (ii).

We establish that $m \in G_1^{\mathbb{M}}$ implies $\phi^\star(m) \in G_1^{\mathbb{W}}$. We do this by contradiction. Let $m \in G_1^{\mathbb{M}}$ and suppose that $w = \phi^\star(m)$ is in $G_{l'}^{\mathbb{W}}$ with $l' > 1$. Let w' be an arbitrary woman in $G_1^{\mathbb{W}}$ and let $m' = \phi^\star(w')$. We have (a) that m' may or may not be in $G_1^{\mathbb{M}}$ and (b) that $b_{w'}^\star - b_w^\star > 0$. We now establish that m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

Man m and woman w' block with agreement $(x_1, z^\star_{w'})$ if there is an $x_1 \in [0, \beta]$ such that

$$u_{w'}(m, x_1, z^\star_{w'}) > u_{w'}(\phi^\star, \bar{\boldsymbol{x}}^\star) \tag{3.3}$$

$$u_m(w', x_1, z^\star_{w'}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star). \tag{3.4}$$

Since $z^\star_{w'}$ is the same on both sides of equation (3.3), the equation reduces to $b(x_1) > b^\star_{m'}$. Since w' gives m benefit $b^\star_{w'}$ at $(x_1, z^\star_{w'})$, equation (3.4) reduces to $b^\star_{w'} - b^\star_w > c(\theta_m, x_1) - c(\theta_m, z^\star_m)$. To summarize, m and w' block if there is an x_1 in $[0, \beta]$ such that

$$b(x_1) > b^\star_{m'} \tag{3.5}$$

$$b^\star_{w'} - b^\star_w > c(\theta_m, x_1) - c(\theta_m, z^\star_m). \tag{3.6}$$

There are two sub-cases. If $m' \notin G^{\mathbb{M}}_1$, then $b(z^\star_m) > b^\star_{m'}$. Thus, $x_1 = z^\star_m$ satisfies equations (3.5) and (3.6) as $b^\star_{w'} - b^\star_w > 0$. Since $z^\star_m \in (0, \beta)$, m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction. If $m' \in G^{\mathbb{M}}_1$, consider $x_1 = z^\star_m + \delta$, with $\delta > 0$. Since $b(z^\star_m) = b^\star_{m'}$ and $b(y)$ is strictly increasing in y, we have that $z^\star_m + \delta$ satisfies equation (3.5). Since $c(\theta, y)$ is continuous in y and since $b^\star_{w'} - b^\star_w > 0$, we also have that $z^\star_m + \delta$ satisfies equation (3.6) for δ sufficiently small. Since $z^\star_m \in (0, \beta)$, we can shrink δ such that $z^\star_m + \delta \in [0, \beta]$. Thus, m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction.

The analogous argument gives that $w \in G^{\mathbb{M}}_1$ implies $\phi^\star(w) \in G^{\mathbb{M}}_1$. Since $\phi^\star(m) = w$ if and only if $\phi^\star(w) = m$, we have $m \in G^{\mathbb{M}}_l \iff \phi^\star(m) \in G^{\mathbb{W}}_l$ for all $l \in \{1\}$, i.e., the induction result is true when $j = 1$.

Let $1 < j \leq \underline{J}$ and assume that $m \in G^{\mathbb{M}}_l \iff \phi^\star(m) \in G^{\mathbb{W}}_l$ for all $l \in \{1, \ldots, j-1\}$. By the induction hypothesis, it suffices to show that $m \in G^{\mathbb{M}}_j \iff \phi^\star(m) \in G^{\mathbb{W}}_j$ to prove the induction result. Again, there are two cases: (i) $j = \overline{J}$ or (ii) $j < \overline{J}$. If case (i), then we're done. Simply, when $m \in G^{\mathbb{M}}_j$, the induction hypothesis gives he is *not* matched to a woman in a lower-indexed group. Since there is no higher index group of women, we necessarily have that $\phi^\star(m) \in G^{\mathbb{W}}_j$. Analogously, if $w \in G^{\mathbb{W}}_j$, then $\phi^\star(m) \in G^{\mathbb{M}}_j$. It follows that $m \in G^{\mathbb{M}}_j \iff \phi^\star(m) \in G^{\mathbb{W}}_j$. Thus, we proceed under case (ii).

We establish that $m \in G^{\mathbb{M}}_j$ implies $\phi^\star(m) \in G^{\mathbb{W}}_j$. As before, we do this by contradiction. Let $m \in G^{\mathbb{M}}_j$ and suppose that $w = \phi^\star(m)$ is in $G^{w}_{l'}$ with $l' \neq j$. The induction hypothesis gives that $l' > j$. Let w' be an arbitrary woman in $G^{\mathbb{W}}_j$ and let $m' = \phi^\star(w')$. Again, m' may or may not be in $G^{\mathbb{M}}_j$; but, the induction hypothesis gives that $m' \in G^{\mathbb{M}}_{l'}$ with $l' \geq j$. We (again) establish that m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

Repeating the same argument as above gives that m and w' block with agreement $(x_1, z^\star_{w'})$

if there is an $x_1 \in [0, \beta]$ such that

$$b(x_1) > b^\star_{m'} \tag{3.7}$$

$$b^\star_{w'} - b^\star_w > c(\theta_m, x_1) - c(\theta_m, z^\star_m). \tag{3.8}$$

If $m' \notin G^{\mathbb{M}}_j$, then $x_1 = z^\star_m$ satisfies equations (3.7) and (3.8) is in $[0, \beta]$. If $m' \in G^{\mathbb{M}}_j$, consider $x_1 = z^\star_m + \delta$, with $\delta > 0$. Since m and m' are in the same benefit group and $b(y)$ is strictly increasing, we have that $z^\star_m + \delta$ satisfies equation (3.7). Since $c(\theta, y)$ is continuous in y and since $b^\star_{w'} - b^\star_w > 0$, we have that equation (3.8) holds for δ sufficiently small. Since $z^\star_m \in (0, \beta)$, we can shrink δ such that $z^\star_m + \delta \in [0, \beta]$. In either case, m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction.

The analogous argument gives that $w \in G^{\mathbb{W}}_j$ implies $\phi^\star(w) \in G^{\mathbb{M}}_j$. It follows that $m \in G^{\mathbb{M}}_j \iff \phi^\star(m) \in G^{\mathbb{W}}_j$. $\square$

3.5.2 The Relationships Among Benefits, Efforts, and Types

In this subsection, we link types to costs with the next assumption and then we characterize the relationships among the benefits players produce and receive, their efforts, and their types. We also discuss the rank-order effects of changes in types.

Assumption 5. Cost and Type.
Higher type players have strictly lower incremental costs of effort, i.e., $c(\theta, y') - c(\theta, y)$ is strictly decreasing in θ for all $y < y'$.

That is, $c(\cdot)$ is strictly submodular, as is the case when the *marginal* cost of effort is strictly decreasing in type. The next proposition examines the relationship between type and benefit produced.

Proposition 6. Benefit Produced and Type.
Let Assumption 5 hold, let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, and let $G^{\mathbb{M}}_1, \ldots, G^{\mathbb{M}}_{J_{\mathbb{M}}}$ and $G^{\mathbb{W}}_1, \ldots, G^{\mathbb{W}}_{J_{\mathbb{M}}}$ be the associated benefit groups. Then, strictly higher type men produce weakly higher benefits, i.e., if two men m and m' are partnered, then $\theta_m < \theta_{m'}$ implies $m \in G^{\mathbb{M}}_l$ and $m' \in G^{\mathbb{M}}_j$ with $j \leq l$. The analogous result holds for women.[20]

Observe that (i) this result applies to every interior stable allocation and (ii) implies

[20] We give and discuss a stronger result in the Supplement for this chapter: strictly higher type men produce *strictly* higher benefits, i.e., $\theta_m < \theta_{m'}$ implies $m \in G^{\mathbb{M}}_l$ and $m' \in G^{\mathbb{M}}_j$ with $j < l$, when (i) Assumption 5 holds, (ii) $b(y)$ and $c(\theta, y)$ are continuously differentiable, and (iii) $\partial b(y)/\partial y > 0$ and $\partial c(\theta,y)/\partial y > 0$ for all $y \in [0, \beta]$.

that players in higher benefit groups have weakly higher types.[21] We can easily see this relationship in the stable allocation we found in Example 1 since the marginal cost $^{2y}/_{\theta}$ is strictly decreasing in θ.

Example 1 (Continued). Benefit and Type.

Recall (i) that man 1 and woman 4 are the high types, with $\theta_1 = \theta_4 = 2$, (ii) that man 2 and woman 3 are the low types, with $\theta_2 = \theta_3 = 1$, and (iii) $b_1^\star = b_4^\star = 1$ and $b_2^\star = b_3^\star = {}^1/_2$. Thus, the higher types produce higher benefits. The higher types also receive higher benefits from their partners (since 1 and 4 are match and 2 and 3 are matched) and exert more effort (since $b(y) = y$). △

The intuition for the proposition is that higher types can "outcompete" lower types because their lower marginal costs allow them to profitably offer slightly higher benefits. Thus, the competition for matches drives them to offer higher benefits. To be more precise, suppose there is an interior stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, with benefits $\{b_i^\star\}_{i\in\{j|\phi^\star(j)\neq j\}}$ and efforts $\{z_i^\star\}_{i\in\mathcal{N}}$, where a lower type man m produces a greater benefit than a higher type man m', i.e., where $b_{m'}^\star < b_m^\star$. Let w be the match of m and let w' be the match of m'. Since the benefit function is the same for both men, w' does strictly better with m whenever he exerts more effort than m'. Thus, stability requires that m does not do better with w' at such an effort, implying $b_w^\star - c(\theta_m, z_m^\star) \geq b_{w'}^\star - c(\theta_m, z_{m'}^\star)$. Since m' has a strictly lower marginal cost than m and since $z_{m'}^\star < z_m^\star$ (because $b_{m'}^\star < b_m^\star$), we have $b_w^\star - b_{w'}^\star > c(\theta_{m'}, z_m^\star) - c(\theta_{m'}, z_{m'}^\star)$. That is, m' does strictly better by offering w agreement $(z_m^\star, z_w^\star)$ then he does in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Hence, he's willing to outcompete m by exerting slightly more effort and offering a slightly higher benefit to win w, implying that m' and w block. It follows that a necessary condition of stability is that higher type men produce higher benefits. The proof makes this intuition precise.

Since the intuition (and thus the proof) do not depend on men *and* women having the same benefit function, the proposition readily generalizes to the case where each side has its own benefit function. In fact, the proposition continues to hold when the benefit a player produces is increasing in his or her type and effort.[22]

As the intuition suggests, the continuity of effort is essential for Proposition 6. When

[21] In particular, for the men, the contraposition of Proposition 6 implies that $\min\{\theta_m | m \in G_l^{\mathbb{M}}\} \geq \max\{\theta_m | m \in G_{l+1}^{\mathbb{M}}\}$ for any $l \in \{1, \ldots, J_{\mathbb{M}} - 1\}$. That is, the lowest type man in the l-th benefit group has a type at least as high as the highest type man in the $l+1$-th benefit group. The analogous result holds for women.

[22] There are two key parts to the intuition: (i) $b_{m'}^\star < b_m^\star$ implies $z_{m'}^\star < z_m^\star$ and (ii) $b(\theta_{m'}, z_m^\star) \geq b(\theta_m, z_m^\star)$, where $b(\theta, y)$ denotes the type-dependent benefit function. Part (i) ensures that m' does strictly better by matching with w and exerting slightly more effort than m, while part (ii) ensures that w does strictly better by matching with m' when he exerts slightly more effort than m. Since both (i) and (ii) are hold when $b(\theta, y)$ is increasing in both arguments, m and w' block and the proposition obtains.

efforts are discrete, lower types may produce *strictly* higher benefits. For example, when $\mathcal{M} = \{1,2\}$, $\mathcal{W} = \{3,4\}$, $b(y) = y$, $c(\theta, y) = {}^{y}\!/\!{}_{\theta}$, $\theta_1 = \theta_3 = 1$ and $\theta_2 = \theta_4 = 2$, and the set of feasible agreements is $\{0, {}^1\!/\!{}_2, 1\}^2$, then it's stable for man 1 and woman 4 to match with agreement $(1,1)$ and for man 2 and woman 3 to match with agreement $({}^1\!/\!{}_2, {}^1\!/\!{}_2)$. Herein, man 2 produces a lower benefit despite having a higher type.

Also essential for Proposition 6 is the fact that m' has a strictly higher type than m. If m and m' have the type, then m' may not have incentive to outcompete m – formally, we cannot ensure $b^\star_w - c(\theta_m, z^\star_m) \geq b^\star_{w'} - c(\theta_m, z^\star_{m'})$ implies $b^\star_w - b^\star_{w'} > c(\theta_{m'}, z^\star_m) - c(\theta_{m'}, z^\star_{m'})$. Thus, players of the same type may produce different benefits, as the next example illustrates.

Example 4. Benefit and Type for a Variant of Example 1.

Suppose $X = [0,2]^2$. Let $\mathcal{M} = \{1,2\}$ and let $\mathcal{W} = \{3,4\}$. Let $b(y) = y$ and let $c(\theta, y) = \frac{1}{\theta} y^2$. Let $\theta_1 = \theta_2 = \theta_3 = 2$ and let $\theta_4 = 1$. It's readily verified that one stable allocation is for (i) man 1 and woman 3 to match with agreement $(1,1)$ and for (ii) man 2 and woman 4 to match with agreement $(q, \frac{1}{2}(1+q^2)) \approx (0.682, 0.733)$, where $q = ((18 + 2\sqrt{93})^{1/3} - 2(\frac{\sqrt{93}-9}{4})^{1/3}) \cdot 6^{-2/3}$. Although both men are of the same type (and Assumption 5 holds), they clearly produce different benefits. $\triangle$

Two corollaries of Proposition 6 follow. The first concerns the relationship between type and effort, while the second concerns the relationship between type and the benefit received.

Corollary 3. Effort and Type.
Let Assumption 5 hold, let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, and let $\{z^\star_i\}_{i \in \mathcal{N}}$ be the associated efforts. Then, strictly higher type men exert weakly more effort, i.e., if two men m and m' are partnered, then $\theta_m < \theta_{m'}$ implies that $z^\star_m \leq z^\star_{m'}$. The analogous result holds for women.

Proof. This result follows directly from Proposition 6 as $b(y)$ is strictly increasing. $\square$

Corollary 4. Benefit Received and Type.
Let Assumption 5 hold, let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, and let $G^{\mathbb{M}}_1, \ldots, G^{\mathbb{M}}_{J_{\mathbb{M}}}$ and $G^{\mathbb{W}}_1, \ldots, G^{\mathbb{W}}_{J_{\mathbb{M}}}$ be the associated benefit groups. Then, strictly higher type men are matched to women in weakly higher benefit groups, i.e., if two men m and m' are partnered, then $\theta_m < \theta_{m'}$ implies $\phi^\star(m) \in G^{\mathbb{W}}_l$ and $\phi^\star(m') \in G^{\mathbb{W}}_j$ with $j \leq l$. The analogous result holds for women.

Proof. Apply Proposition 5 to Proposition 6. $\square$

Corollaries 3 and 4 show that, in any interior stable allocation, players with lower marginal costs exert higher efforts and, surprisingly, receive higher benefits than players with higher marginal costs. We saw both of these patterns in the interior stable allocation we found in Example 1.

Proposition 6 also allows us to examine the "rank-order" effects of a change in a player's type. For concreteness, suppose that (i) the type of man m increases from θ_m to θ'_m, (ii) all players are partnered, and (iii) each man has unique type and produces a unique benefit.[23] Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation before the increase, with benefits $\{b_i^\star\}_{i\in\mathcal{N}}$, and let $(\phi', \bar{\boldsymbol{x}}')$ be an interior stable allocation after the increase, with benefits $\{b'_i\}_{i\in\mathcal{N}}$. Since m's type increases, his rank-order type weakly increases, i.e., there are more men with lower types $|\{m'|\theta_{m'} \leq \theta'_m\}| \geq |\{m'|\theta_{m'} \leq \theta_m\}|$. Thus, Proposition 6 gives that $|\{m'|b'_{m'} \leq b'_m\}| \geq |\{m'|b^\star_{m'} \leq b^\star_m\}|$, i.e., m produces a higher benefit than more men in $(\phi', \bar{\boldsymbol{x}}')$ than in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, so his rank-order benefit (and rank-order effort) increase. It follows from Proposition 5 that the rank-order benefit m receives also increase. Of course, this tells us nothing about whether the *level* of benefit m produces or receives increases; we'll return to this issue in Proposition 9.

We prove Proposition 6 by formalizing the intuition discussed above.

Proof of Proposition 6. Let $\{z_i^\star\}_{i\in\mathcal{N}}$ be the associated efforts of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $\{b_i^\star\}_{i\in\{j|\phi^\star(j)\neq j\}}$ be the associated benefits. Let m and m' be two partnered men with $\theta_m < \theta_{m'}$. Then, $m \in G_l^{\mathbb{M}}$ for some l and $m' \in G_j^{\mathbb{M}}$ for some j. We need to establish that $j \leq l$. If $l = J_{\mathbb{M}}$, then this is trivially true, so we take $l < J_{\mathbb{M}}$. We argue by contradiction. Suppose that $l < j$, i.e., $b^\star_m > b^\star_{m'}$, which implies $z^\star_m > z^\star_{m'}$. Let $w = \phi^\star(m)$ and $w' = \phi^\star(m')$. Proposition 5 gives that $b^\star_w > b^\star_{w'}$, which implies $z^\star_w > z^\star_{w'}$. We'll show that m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. But first, we need two preliminary facts.

Fact one: m does weakly better in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ than he does by matching with w' at agreement $(z^\star_{m'}, z^\star_{w'})$, i.e.,

$$b^\star_w - b^\star_{w'} \geq c(\theta_m, z^\star_m) - c(\theta_m, z^\star_{m'}). \tag{3.9}$$

Consider a possible match of m and w' with agreement $(x_1, z^\star_{w'})$, where $x_1 \in [0, \beta]$. For $x_1 > z^\star_{m'}$, we have that $u_{w'}(m, x_1, z^\star_{w'}) > u_{w'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$ since $b(x_1) > b^\star_{m'} = b(z^\star_{m'})$. Thus, the stability of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ implies, via the no blocking requirement, that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_m(w', x_1, z^\star_{w'})$. Simplifying gives that $b^\star_w - b^\star_{w'} \geq c(\theta_m, z^\star_m) - c(\theta_m, x_1)$ for $x_1 > z^\star_{m'}$. The desired result follows from the continuity of the cost function in effort.

Fact two: m' does strictly better by matching with w at agreement $(z^\star_m, z^\star_w)$ than he does in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e.,

$$b^\star_w - b^\star_{w'} > c(\theta_{m'}, z^\star_m) - c(\theta_{m'}, z^\star_{m'}). \tag{3.10}$$

[23] Assumption 7 (below) is sufficient to ensure that all players are partnered and the conditions described in Footnote 20 are sufficient to ensure that each man produces a unique benefit when he has a unique type.

Since $z^\star_{m'} < z^\star_m$, Assumption 5 gives that $c(\theta_{m'}, z^\star_m) - c(\theta_{m'}, z^\star_{m'}) < c(\theta_m, z^\star_m) - c(\theta_m, z^\star_{m'})$. The desired result now follows from equation (3.9).

Now we show that m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. They block with agreement $(x_1, z^\star_w)$ if there is an $x_1 \in [0, \beta]$ with $u_w(m', x_1, z^\star_w) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_{m'}(w, x_1, z^\star_w) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e., with

$$b(x_1) > b(z^\star_m) \tag{3.11}$$

$$b^\star_w - b^\star_{w'} > c(\theta_{m'}, x_1) - c(\theta_{m'}, z^\star_{m'}). \tag{3.12}$$

Consider $x_1 = z^\star_m + \delta$, where $\delta > 0$. Since $b(y)$ is strictly increasing, we have that $z^\star_m + \delta$ satisfies equation (3.11). Since equation (3.10) holds, the cost function is continuous in y, and $b^\star_w - b^\star_{w'} > 0$, we can pick δ sufficiently small such that $z^\star_m + \delta$ satisfies (3.12). Since $z^\star_m$ is interior, we can make δ small enough so that $z^\star_m + \delta \in [0, \beta]$. Thus, m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction. □

3.5.3 Payoffs and Types

In this subsection, we develop our results on the relationship between types and payoffs. Corollary 4 tells us that higher types receive higher benefits from their matches. At the same time, Corollary 3 tells us higher types exert more effort, implying they may incur higher costs. Thus, it's unclear whether higher types actually do better than lower types. To resolve this quandary, it suffices to make the following assumption.

Assumption 6. Common Cost of Minimum Effort.
There is a common cost of minimum effort, i.e., $c(\theta, 0) = d$ for all types θ, where $d \geq 0$.[24]

An immediate implication of this assumption and Assumption 5 is the following.

Lemma 8. Higher Types Have Lower Costs.
When Assumptions 5 and 6 hold, cost is strictly decreasing in type, i.e., $c(\theta, y) > c(\theta', y)$ when $\theta < \theta'$ and $y > 0$.

Proof. Obvious and omitted. □

This lemma plays a key role in the next proposition, which examines the relationship between type and payoff.

Proposition 7. Payoff and Type.
Let Assumptions 5 and 6 hold, and let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation. Then, higher type men obtain higher payoffs. In particular, for two men m and m', $\theta_m \leq \theta_{m'}$ implies that

[24] If Assumption 4 holds, then d must be greater than $b(0)$: suppose that man m is matched to woman w and at agreement $(0, 0)$, then $u_m(w, 0, 0) = b(0) - d$, so intolerability requires that $b(0) - d < 0$.

$u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \leq u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, and $\theta_m < \theta_{m'}$ implies that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) < u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$ when both men are partnered. The analogous result holds for women.

This result (i) applies to any interior stable allocation and (ii) gives that men with the same type have the same payoff. The next two examples illustrate the proposition.

Example 1 (Continued). Higher Types have Higher Payoffs.

Since Assumption 6 holds, the Proposition 7 gives that strictly higher types earn strictly more. Indeed, we see this in the stable allocation we found. The high types, man 1 and woman 4, each earn $^3/_8$ $(= 1 - {}^1/_8 - {}^1/_2 \cdot 1^2)$, while the low types, man 2 and woman 3, each earn $^1/_8$ $(= {}^1/_2 - {}^1/_8 - ({}^1/_2)^2)$. △

Example 4 (Continued). Equal Types have Equal Payoffs.

Since Assumption 6 holds, the Proposition 7 gives that both men earn the same. In fact, man 1 earns $^1/_2$ $(= 1 - {}^1/_2)$ and man 2 also earns $^1/_2$ $(= \frac{1}{2}(1 + q^2) - \frac{1}{2}q^2$, where $q \approx 0.733)$ in the stable allocation we found. △

The intuition for Proposition 7 is that higher types can "imitate" and outcompete weakly lower types whenever they do strictly better. To elaborate, suppose there is an interior stable allocation where a lower type man m makes *strictly* more than a (weakly) higher type man m'. Since m' has a lower cost, he earns (weakly) more than m when he receives same benefit as m and exerts the same effort as m. Thus, m' can do strictly better by imitating and outcompeting m, i.e., by offering w, the partner of m, a slightly higher benefit than she's currently receiving. In doing so, he obtains the same benefit as m and incurs (at most) a slightly higher cost, so his payoff increases. Hence, m' and w block. It follows that a necessary condition of stability is that higher types make at least as much as lower types. The proof formalizes this intuition.

Since the intuition (and thus the proof) only require that m' has a lower cost than m, the proposition also holds when (i) higher types have lower fixed costs or (ii) men and women have different benefit functions. The proposition also obtains when the benefit function is increasing type and effort since this does not diminish m's ability (or desire) to imitate and outcompete m.

As the intuition suggests, the continuity of effort is essential for Proposition 7. When efforts are discrete, lower types may have *strictly* higher payoffs. For example, when $\mathcal{M} = \{1, 2\}$, $\mathcal{W} = \{3\}$, $b(y) = y$, $c(\theta, y) = {}^y/_\theta$, $\theta_1 = \theta_3 = 2$ and $\theta_2 = 3/2$, and the set of feasible agreements is $\{0, {}^1/_2\}^2$, it's stable for man 2 to match with woman 3 with agreement $({}^1/_2, {}^1/_2)$, while man 1 is single with an arbitrary agreement. The lower type man earns $^1/_6$ while the higher type man earns 0. In addition, equal types may have unequal earnings – e.g., the previous allocation is still stable when $\theta_2 = 2$.

We prove Proposition 7 in two steps. First, we give a lemma that establishes higher types are partnered whenever lower types are partnered. Subsequently, we use this lemma and the argument we sketched above to prove the proposition.

Lemma 9. Higher Types are Partnered when Lower Types are Partnered.
Let Assumptions 5 and 6 hold, and $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *be an interior stable allocation. Consider two men* m *and* m' *with* $\theta_m < \theta_{m'}$*. If* m *is partnered, then so too is* m'*, i.e.,* $\phi^\star(m) \neq m \implies \phi^\star(m') \neq m'$*. The analogous result holds for women.*

Proof. We prove the lemma via contradiction. Suppose that man m is partnered, while man m' isn't. Let $w = \phi^\star(m)$. Let $b_m^\star$ and $z_m^\star$ be the benefit and effort of m in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, and let $b_w^\star$ and $z_w^\star$ be the benefit and effort of w in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. We'll establish that man m' and woman w block. But first, we need a preliminary result:

$$b_w^\star - c(\theta_{m'}, z_m^\star) > 0. \tag{3.13}$$

By individual rationality, we have that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) = b_w^\star - c(\theta_m, z_m^\star) \geq 0$. Since higher type men have strictly lower costs per Lemma 8, we have $b_w^\star - c(\theta_{m'}, z_m^\star) > b_w^\star - c(\theta_m, z_m^\star) \geq 0$.

Now we establish the contradiction. Consider man m' and woman w with agreement $(x_1, z_w^\star)$. Woman w is willing to match with m' if $u_w(m', x_1, z_w^\star) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and m' is willing to match with w if $u_{m'}(w, x_1, z_w^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Simplifying gives that m' and w block if there is an $x_1 \in [0, \beta]$ such that

$$b(x_1) > b(z_m^\star) \tag{3.14}$$

$$b_{w'}^\star - c(\theta_{m'}, x_1) > 0. \tag{3.15}$$

Consider $x_1 = z_m^\star + \delta$, where $\delta > 0$. Since $b(y)$ strictly increasing, we have that $z_m^\star + \delta$ satisfies equation (3.14). Since equation (3.13) holds and $c(\theta, y)$ is continuous in y, we may pick δ sufficiently small such that $z_m^\star + \delta$ satisfies equation (3.15). Since $z_m^\star$ is interior, we may make δ small so that $z_m^\star + \delta \in [0, \beta]$. Thus, m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction. □

Proof of Proposition 7. Let $\{z_i^\star\}_{i \in \mathcal{N}}$ be the associated efforts of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $\{b_i^\star\}_{i \in \{j | \phi^\star(j) \neq j\}}$ be the associated benefits. Let m and m' be two different men. We first establish that $\theta_m < \theta_{m'}$ implies that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \leq u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, with strict inequality if both men are partnered. Subsequently, we establish that $\theta_m = \theta_{m'}$ implies that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. The proposition follows.

Let $\theta_m < \theta_{m'}$. There are four cases to consider: (i) m and m' are both single, (ii) m is single and m' is partnered, (iii) m is partnered and m' is single, and (iv) both m and m' are partnered. If case (i), then both m and m' earn zero. If case (ii), then m earns zero and m'

earns at least zero as $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is individually rational. Since Lemma 9 gives that case (iii) is impossible, only case (iv) remains.

Consider case (iv). We need to establish that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) < u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. We argue by contradiction. Suppose that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, and let $w = \phi^\star(m)$ and $w' = \phi^\star(m')$. We'll show that m' and w block. To do this, we need a preliminary result:

$$b_w^\star - c(\theta_{m'}, z_m^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star). \tag{3.16}$$

This follows directly from the contraction hypothesis and Lemma 8: $b_w^\star - c(\theta_{m'}, z_m^\star) > b_w^\star - c(\theta_m, z_m^\star) = u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

Now we establish the contradiction. Man m' and woman w block with agreement $(x_1, z_w^\star)$ if there is an $x_1 \in [0, \beta]$ such that $u_w(m', x_1, z_w^\star) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_{m'}(w, x_1, z_w^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e., such that

$$b(x_1) > b(z_m^\star) \tag{3.17}$$

$$b_w^\star - c(\theta_{m'}, x_1) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star). \tag{3.18}$$

Consider $x_1 = z_m^\star + \delta$, with $\delta > 0$. Since $b(y)$ strictly increasing, we have that $z_m^\star + \delta$ satisfies equation (3.17). Since equation (3.16) holds and $c(\theta, y)$ is continuous in y, we may pick δ sufficiently small such that $z_m^\star + \delta$ satisfies equation (3.18). Since $z_m^\star$ is interior, we may make δ small enough so that $z_m^\star + \delta \in [0, \beta]$. Thus, m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction.

We now establish that $\theta_m = \theta_{m'}$ implies $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. We again argue via contradiction. Suppose (without loss) that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. As above, m' and w block with agreement $(x_1, z_w^\star)$ if there is an $x_1 \in [0, \beta]$ with $u_w(m', x_1, z_w^\star) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_{m'}(w, x_1, z_w^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e., with

$$b(x_1) > b(z_m^\star) \tag{3.19}$$

$$b_w^\star - b_{w'}^\star > c(\theta_{m'}, x_1) - c(\theta_{m'}, z_{m'}^\star). \tag{3.20}$$

Since $\theta_m = \theta_{m'}$ and since $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) = b_w^\star - c(\theta_m, z_m^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star) = b_{w'}^\star - c(\theta_{m'}, z_{m'}^\star)$, we have $b_w^\star - b_{w'}^\star > c(\theta_{m'}, z_m^\star) - c(\theta_{m'}, z_{m'}^\star)$. Thus, at $x_1 = z_m^\star + \delta$, where $\delta > 0$, we have that both equations (3.19) and (3.20) are true for δ sufficiently small since $b(y)$ is strictly increasing and $c(\theta, y)$ is continuous in y. Since $z_m^\star$ is interior, we may take δ sufficiently small such that $z_m^\star + \delta \in [0, \beta]$. Thus, m' and w block, a contradiction. □

Remark. As is clear from the proof, the result that men of the same type earn the same payoff is independent of Assumptions 5 and 6.

3.5.4 Assortative Matching in Endowed Types

In this subsection, we show that, as Corollary 4 suggests, there's at least one stable allocation where higher type men match with higher type women. We follow Legros and Newman [47] and say that an allocation $(\phi, \bar{\boldsymbol{x}})$ **exhibits assortative matching in types** if, for any two partnered men m and m' with $\theta_m < \theta_{m'}$, we have that m' matches with a higher type woman than m does, i.e., $\theta_w \leq \theta_{w'}$, where $w = \phi(m)$ and $w' = \phi(m')$.[25]

Proposition 8. Assortative Matching in Types.
Let Assumptions 4 and 5 hold, then there is an interior stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ that exhibits assortative matching in types.

Not all interior stable allocations exhibit assortative matching because players may match in arbitrary ways within their benefit groups. For instance, if there are two men m and m' in $G_l^{\mathbb{M}}$, with $\theta_m < \theta_{m'}$, and two women w and w' in $G_l^{\mathbb{W}}$, with $\theta_w < \theta_{w'}$, it's possible that m is matched to w' and m' is matched to w.[26] That said, all players in the same benefit group *receive* the same benefit. Hence, we can rematch these men and women so that higher types are matched to higher types while preserving players' payoffs and, thus, the stability of the allocation. Accordingly, we obtain Proposition 8.

We prove Proposition 8 by formally developing this rematching procedure and then showing that it functions as desired by applying Propositions 5 and 6. In the process, we'll see that it only makes use of the additive nature of payoffs. Hence, it continues to function (and Proposition 8 obtains) when (i) men and women have different benefit functions or (ii) the benefit function is increasing in type and effort.

Proof of Proposition 8. By Corollary 1, there is an interior stable allocation $(\phi', \bar{\boldsymbol{x}}')$. We proceed in three steps. First, we describe the rematching procedure that constructs a new allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ from $(\phi', \bar{\boldsymbol{x}}')$. Second, we establish that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an interior stable allocation. Third, we show that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exhibits assortative matching in types. Throughout, let $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$ and $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{W}}$ be the benefit groups associated with $(\phi', \bar{\boldsymbol{x}}')$, and let z_i' and b_i' denote player i's effort and benefit in $(\phi', \bar{\boldsymbol{x}}')$. Also, let $z_i^\star$ and $b_i^\star$ denote player i's effort and benefit in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

First, we construct $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$ from $(\phi', \bar{\boldsymbol{x}}') = (\phi', \boldsymbol{x}^{1'}, \ldots, \boldsymbol{x}^{N})$ as follows.

[25] Defining assortative matching is terms of men is without loss. If $(\phi, \bar{\boldsymbol{x}})$ exhibits assortative matching and if $\theta_w < \theta_{w'}$ for two women w and w', then it's readily verified that $\theta_{\phi^\star(w)} \leq \theta_{\phi^\star(w')}$.

[26] If, however, the conditions of Footnote 20 hold, then each benefit group contains only a single kind of player. Thus, *every* interior stable allocation exhibits assortative matching in types. We develop this result in the Supplement for this chapter.

1. Set $\phi^\star$ so that single players remain single and partnered players are matched in descending order of their types in their benefit groups. For each $i \in \mathcal{N}$ with $\phi'(i) = i$, set $\phi^\star(i) = i$. For each $l \in \{1, \ldots, J_{\mathbb{M}}\}$,

   (a) List the men in $G_l^{\mathbb{M}}$ in descending order of their types (breaking ties randomly) and label them m_1, m_2, …, $m_{|G_l^{\mathbb{M}}|}$. So m_1 is the highest type man in $G_l^{\mathbb{M}}$, m_2 is the second highest type man, and so on. Likewise, list the women in $G_l^{\mathbb{W}}$ in descending order (breaking ties randomly) of their types and label them w_1, …, $w_{|G_l^{\mathbb{M}}|}$. (Recall that $|G_l^{\mathbb{M}}| = |G_l^{\mathbb{W}}|$ by Proposition 5.)

   (b) Set $\phi^\star$ such that the j-th man and j-th woman on each list are matched, i.e., such that $\phi^\star(m_j) = w_j$ and $\phi^\star(w_j) = m_j$ for $j \in \{1, \ldots, |G_l^{\mathbb{M}}|\}$.

2. Set $\bar{\boldsymbol{x}}^\star$ so that players exert the same efforts. If $\phi^\star(i) = i$, then set $\boldsymbol{x}^{i^\star} = (z_i', 0)$ when i is a man and $\boldsymbol{x}^{i^\star} = (0, z_i')$ when i is a woman. If $\phi^\star(i) \neq i$, then set $\boldsymbol{x}^{i^\star} = (z_i', z_{\phi^\star(i)}')$ when i is a man and $\boldsymbol{x}^{i^\star} = (z_{\phi^\star(i)}', z_i')$ when i is a woman. (Thus, $z_i^\star = z_i'$ for all $i \in \mathcal{N}$ as desired.)

This concludes the construction. Notice (i) that the same players are partnered under ϕ' and $\phi^\star$ by step (1) and (ii) that $\boldsymbol{x}^{i^\star} = \boldsymbol{x}^{\phi^\star(i)^\star}$ for each player i by step (2).

Second, we establish that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an interior stable allocation. Since $\boldsymbol{x}^{i^\star} = \boldsymbol{x}^{\phi^\star(i)^\star}$ for each player i, $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an allocation. It's also interior because z_i' is interior for each partnered player i. Hence, we only need establish that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable. To these ends, observe that $u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_i(\phi', \bar{\boldsymbol{x}}')$ for all $i \in \mathcal{N}$. (This is trivial if i is single, so we take i to be partnered. By step (2), i's effort is unchanged, so his or her costs are unchanged. Since $\phi^\star(i)$ and $\phi'(i)$ are both in the same benefit group in $(\phi', \bar{\boldsymbol{x}}')$ by step (1) and since we keep their efforts constant by step (2), the benefit i receives is unchanged. Thus, i earns the same payoff in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $(\phi', \bar{\boldsymbol{x}}')$.) It follows that $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in P(\phi', \bar{\boldsymbol{x}}')$, so Lemma 7 implies that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable.

Third, we establish that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exhibits assortative matching in types. Let m and m' be partnered men such that $\theta_m < \theta_{m'}$. Proposition 6 gives that there are two cases: (i) $b_m^\star = b_{m'}^\star$ or (ii) $b_m^\star < b_{m'}^\star$. Let $w = \phi^\star(m)$ and $w' = \phi^\star(m')$. If case (i), then in step (1), m' occupies an earlier position in the list than m. Thus, m' is matched to a higher type woman than m, i.e., $\theta_w \leq \theta_{w'}$. If case (ii), we establish that $\theta_w \leq \theta_{w'}$ via contradiction. Suppose $\theta_w > \theta_{w'}$, then Proposition 6 gives that $b_w^\star \geq b_{w'}^\star$. Yet, $b_m^\star < b_{m'}^\star$ implies that $b_w^\star < b_{w'}^\star$ by Proposition 5, a contradiction. □

Remark. It's clear from the proof that every interior stable allocation is payoff equivalent to an interior stable allocation that exhibits assortative matching in types. Thus, Proposition 3 of Legros and Newman [47] implies that the Pareto frontier of each couple satisfies generalized increasing differences, which is their sufficient condition on the Pareto frontier for assortative matching in types. In light of this, one can think of our result as giving natural conditions on the primitive payoffs that imply Legros and Newman's general sufficient condition.

3.5.5 Comparative Statics of Benefits, Efforts, and Payoffs

In this subsection, we first make the idea of a "symmetric stable allocation" precise, and we prove several properties of this allocation (e.g., its existence and uniqueness). Then we examine how increases in players' types affect their efforts, benefits, and payoffs in this allocation. We make the following assumption.

Assumption 7. Universal Strict Agreeability and Universal Intolerability.
There are an equal number of men and women, i.e., $M = N/2$. For all θ and θ' in Θ, we've:
(i) $b(x_2) - c(\theta, x_1) > 0$ and $b(x_1) - c(\theta', x_2) > 0$ for some $(x_1, x_2) \in X$.
(ii) $b(x_2) - c(\theta, x_1) < 0$ or $b(x_1) - c(\theta', x_2) < 0$ for each $(x_1, x_2) \in \partial X$.

Part (i) guarantees any type of man and any type of woman can find an agreement that makes them strictly better off than if they're single. Part (ii) guarantees that intolerability holds. Thus, Assumption 7 implies Assumption 4.

Let $\{\theta_i\}_{i\in\mathcal{N}}$ denote the endowment of players' types.

Definition. We say that the type endowment $\{\theta_i\}_{i\in\mathcal{N}}$ is *symmetric* when:
(i) No two men have the same type and no two women have the same type, i.e., $\theta_m \neq \theta_{m'}$ for any two men m and m' and $\theta_w \neq \theta'_w$ for any two women w and w'.
(ii) Men and women are endowed with the same types, i.e., $\cup_{i\in\mathcal{M}}\{\theta_i\} = \cup_{i\in\mathcal{W}}\{\theta_i\}$.

For instance, the type endowment in Example 1 is symmetric: the endowment is $\{\theta_1, \theta_2, \theta_3, \theta_4\} = \{2, 1, 1, 2\}$, and we have $\{\theta_i\}_{i\in\mathcal{M}} = \{\theta_1, \theta_2\} = \{2, 1\}$ and $\{\theta_i\}_{i\in\mathcal{W}} = \{\theta_3, \theta_4\} = \{1, 2\}$.

Next, we describe the construction of a "symmetric stable allocation."

Construction. Symmetric Stable Allocation.
Let Assumption 7 hold and let the type endowment $\{\theta_i\}_{i\in\mathcal{N}}$ be symmetric. Label the men m_1, m_2, ..., m_M in descending order of their types; so, m_1 is the highest type man, m_2 is the second highest type man, and so on. Likewise, label the women w_1, w_2, ..., w_M in descending order of their types.

We construct a **symmetric stable allocation** $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = (\phi^\dagger, \boldsymbol{x}^{1^\dagger}, \ldots, \boldsymbol{x}^{N^\dagger})$ as follows. Set $\phi^\dagger$ such that men and women with the same rank-order type are matched, i.e., such that $\phi^\dagger(m_l) = w_l$ and $\phi^\dagger(w_l) = m_l$ for each $l \in \{1, \ldots, M\}$. (Thus, every player is partnered.)

Set $\bar{\boldsymbol{x}}^\dagger$ such that $\boldsymbol{x}^{m_l^\dagger} = \boldsymbol{x}^{w_l^\dagger} = (x_l^\dagger, x_l^\dagger)$ where

$$x_l^\dagger = \max\{\arg\max_{y\in[0,\beta]} b(y) - c(\theta_{m_l}, y)\}, \tag{3.21}$$

for each $l \in \{1, \dots, M\}$, i.e., m_l and w_l exert the same effort and this effort is the largest solution of $\max_{y\in[0,\beta]} b(y) - c(\theta_{m_l}, y)$. ○

The next lemma gives a few properties of the symmetric stable allocation. It shows, for instance, that the symmetric stable allocation is actually stable. Also, we say an allocation $(\phi', \bar{\boldsymbol{x}}')$ is **welfare maximizing** if it maximizes $T(\phi, \bar{\boldsymbol{x}})$ on the set of feasible allocations.

Lemma 10. Properties of the Symmetric Stable Allocation.
Let Assumption 7 hold and let the endowment of types $\{\theta_i\}_{i\in\mathcal{N}}$ be symmetric. Then (i) a symmetric stable allocation $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ exists and is unique (i.e., the procedure described above only produces one allocation), (ii) $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is an interior Pareto stable allocation, and (iii) $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is welfare maximizing. In addition, for each player i, we have $\theta_i = \theta_{\phi^\dagger(i)}$ and

$$u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = \max_{y\in[0,\beta]} b(y) - c(\theta_i, y). \tag{3.22}$$

One may verify, for instance, that the stable allocation given in Example 1 is a symmetric stable allocation. The lemma's proof exploits the symmetry of the type endowment and the construction of the symmetric stable allocation to show that payoffs are given by equation (3.22). Everything else follow from this and Assumption 7 because players' payoffs are additively separable in benefit and cost.[27]

We develop comparative statics for the symmetric stable allocation because it's focal: it maximizes welfare and it treats equals equally in that matched players have the same type, exert the same effort, and earn the same payoffs. The next proposition gives these comparative statics. We write $u_i(\phi, \bar{\boldsymbol{x}}, \theta_i)$ to emphasize the dependence of i's payoff in the allocation $(\phi, \bar{\boldsymbol{x}})$ on her type θ_i.

Proposition 9. Comparative Statics of the Symmetric Stable Allocation.
Let Assumptions 5, 6, and 7 hold. Let $\{\theta_i\}_{i\in\mathcal{N}}$ and $\{\theta_i'\}_{i\in\mathcal{N}}$ be symmetric endowments of types such that $\theta_i' \geq \theta_i$ for each player i. Let $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ be the symmetric stable allocation when types are $\{\theta_i\}_{i\in\mathcal{N}}$, let $(\phi', \bar{\boldsymbol{x}}')$ be the symmetric stable allocation when types are $\{\theta_i'\}_{i\in\mathcal{N}}$, and let $\{z_i^\dagger\}_{i\in\mathcal{N}}$ and $\{z_i'\}_{i\in\mathcal{N}}$ be players' efforts in $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ and $(\phi', \bar{\boldsymbol{x}}')$ respectively. Then, for each player i, as types increase from $\{\theta_i\}_{i\in\mathcal{N}}$ to $\{\theta_i'\}_{i\in\mathcal{N}}$,
(i) The effort i exerts increases, i.e., $z_i^\dagger \leq z_i'$.

[27] In light of Lemma 10, it's natural to wonder if there's always a welfare maximizing stable allocation in the Effort Game. Unfortunately, the answer is no – see the Supplement for this chapter for details.

(ii) The effort exerted by i's match increases, i.e., $z^{\dagger}_{\phi^{\dagger}(i)} \leq z'_{\phi'(i)}$.
(iii) The payoff of i increases, i.e., $u_i(\phi^{\dagger}, \bar{\boldsymbol{x}}^{\dagger}, \theta_i) \leq u_i(\phi', \bar{\boldsymbol{x}}', \theta'_i)$.

Since $b(y)$ is strictly increasing, (i) implies that the benefit player i produces increases and (ii) implies that the benefit i receives from his or her match increases. Thus, if a player's opportunity cost of effort decreases, then his or her effort, benefits produced and received, and payoff increase. The next example illustrates this result.

Example 5. Effort and Welfare Comparative Statics for a Variant of Example 1.

Let $\mathcal{M} = \{1, 2\}$ and $\mathcal{W} = \{3, 4\}$. Let $X = [0, 2]^2$. Let $\Theta = \{1, 2, 2.1\}$. Let $b(y) = y$ and $c(\theta, y) = \frac{1}{\theta} y^2 + {}^1\!/\!_8$. It's readily verified that Assumptions 5, 6, and 7 hold for any assignment of types to players. Let $\theta_1 = \theta_4 = 2$ and $\theta_2 = \theta_3 = 1$ be the initial endowment of types. And let $\theta'_1 = \theta'_3 = 2$ and $\theta'_2 = \theta'_4 = \frac{21}{10}$ be the new endowment of types. Since both endowments are trivially symmetric and since $\theta'_i \geq \theta_i$ for each player i, Proposition 9 gives that all players exert more effort, produce and receive greater benefits, and are better off under the new endowment.

We can see this directly. The symmetric stable allocation under $\{\theta_i\}_{i \in \mathcal{N}}$ is $(\phi^{\dagger}, \bar{\boldsymbol{x}}^{\dagger}) = (\phi^{\dagger}, \boldsymbol{x}^{1^{\dagger}}, \boldsymbol{x}^{2^{\dagger}}, \boldsymbol{x}^{3^{\dagger}}, \boldsymbol{x}^{4^{\dagger}})$, where $\phi^{\dagger}(1) = 4$ and $\boldsymbol{x}^{1^{\dagger}} = \boldsymbol{x}^{4^{\dagger}} = (1, 1)$, and $\phi^{\dagger}(2) = 3$ and $\boldsymbol{x}^{2^{\dagger}} = \boldsymbol{x}^{3^{\dagger}} = (1/2, 1/2)$. In this allocation, 1 and 4 each earn $3/8 = 0.375$, while 2 and 3 each earn $1/8 = 0.125$. And the symmetric stable allocation under $\{\theta'_i\}_{i \in \mathcal{N}}$ is $(\phi', \bar{\boldsymbol{x}}') = (\phi', \boldsymbol{x}^{1'}, \boldsymbol{x}^{2'}, \boldsymbol{x}^{3'}, \boldsymbol{x}^{4'})$, where $\phi'(1) = 3$ and $\boldsymbol{x}^{1'} = \boldsymbol{x}^{3'} = (1, 1)$, and $\phi'(2) = 4$ and $\boldsymbol{x}^{2'} = \boldsymbol{x}^{4'} = (21/20, 21/20)$. In this allocation, 1 and 3 each earn $3/8 = 0.375$, while 2 and 4 each earn $2/5 = 0.4$. The players efforts, benefits, and payoffs are clearly higher in $(\phi', \bar{\boldsymbol{x}}')$ than $(\phi^{\dagger}, \bar{\boldsymbol{x}}^{\dagger})$. △

The intuition for Proposition 9 is that cost is submodular and strictly decreasing in type. Thus, as types increase, equation (3.21) ensures that players' efforts increase and equation (3.22) ensures that their payoffs increase. Assumption 7 and the symmetric type endowment are essential to this result. Without them, an increase in types may make a player *strictly* worse off; see the Supplement for this chapter for details. (In Chapter 2, we show these comparative statics generalize to n-sided games with benefit functions that are increasing and supermodular in type and effort.)

We first prove Lemma 10 by first establishing that payoffs are given by (3.22). We then leverage this fact, Assumption 7, and the method of construction to prove the stated results. Subsequently, we prove Proposition 9 by exploiting the submodularity and type-monotonicity of the cost function.

Proof of Lemma 10. We prove the lemma in five steps. First, we argue existence and uniqueness of the symmetric stable allocation $(\phi^{\dagger}, \bar{\boldsymbol{x}}^{\dagger})$. Second, we establish that players' payoffs are given by equation (3.22). Third, we leverage this equation and Assumption 7 to show that the symmetric stable allocation is actually stable. Fourth, we show that the symmetric stable allocation is interior and is Pareto optimal. Fifth, we establish that the symmetric stable allocation maximizes welfare.

We first establish existence and uniqueness of $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. Since the benefit function and the cost function are continuous in y, $\arg\max_{y\in[0,\beta]} b(y) - c(\theta_{m_l}, y)$ is non-empty and compact by standard arguments. Thus, the it has a unique maximal element in the standard order, so $x_l^\dagger$ of equation (3.21) exists and is unique for each $l \in \{1, \ldots, M\}$. It follows that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ exists and, since $\phi^\dagger$ is uniquely determined by the construction because no two players of the same gender have the same type, that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is unique. It's an allocation since $\phi^\dagger$ is a matching by construction and $\bar{\boldsymbol{x}}^\dagger \in A(\phi^\dagger)$ by construction.

Recall that the men and women are labeled $m_1, \ldots, m_M$ and $w_1, \ldots, w_M$ in descending order of their types. Since the type endowment is symmetric, we have $\theta_{m_1} = \theta_{w_1}$, $\theta_{m_2} = \theta_{w_2}$, $\ldots$, and $\theta_{m_M} = \theta_{w_M}$. That is, man m_1 and woman w_1 have the same type, man m_2 and woman w_2 have the same type, and so on.

Second, we establish that equation (3.22) holds. Suppose that player i is a woman with label w_j, then $\phi^\dagger(i)$'s label is m_j by construction. Thus, the previous paragraph gives $\theta_i = \theta_{w_j} = \theta_{m_j} = \theta_{\phi^\dagger(i)}$. Hence,

$$u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = b(x_j^\dagger) - c(\theta_i, x_j^\dagger) = b(x_j^\dagger) - c(\theta_{m_j}, x_j^\dagger) = \max_{y\in[0,\beta]} b(y) - c(\theta_{m_j}, y) = \max_{y\in[0,\beta]} b(y) - c(\theta_i, y),$$

where the third equality is due to optimality. Since the analogous argument applies if i is a man, equation (3.22) holds.

Third, we establish that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is stable. By construction, $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is feasible because (i) $\phi^\dagger$ is a matching and (ii) $\bar{\boldsymbol{x}}^\dagger \in X^N$ since $x_l^\dagger \in [0, \beta]$ for all $l \in \{1, \ldots, M\}$. It's also individually rational. To see this, consider man m. Suppose that m's label is m_j, then m is matched to woman w_j and $\theta_m = \theta_{m_j} = \theta_{w_j}$. Consider the maximum of the sum of m and w_j's payoffs,

$$\begin{aligned} &\max_{(x_1,x_2)\in X} b(x_2) - c(\theta_{m_j}, x_1) + b(x_1) - c(\theta_{w_j}, x_2) \\ &\quad = \max_{x_1\in[0,\beta]} b(x_1) - c(\theta_{m_j}, x_1) + \max_{x_2\in[0,\beta]} b(x_2) - c(\theta_{w_j}, x_2) = 2 \max_{y\in[0,\beta]} b(y) - c(\theta_{m_j}, y). \end{aligned} \quad (3.23)$$

(This max exists by standard arguments.) Since $\theta_{m_j} = \theta_{w_j} \in \Theta$, Assumption 7 implies that there is an $(x_1', x_2') \in X$ such that $b(x_2') - c(\theta_{m_j}, x_1') > 0$ and $b(x_1') - c(\theta_{w_j}, x_2') > 0$. Hence, the first line of equation (3.23) is strictly positive, which implies that $u_m(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) > 0$ by equation (3.22). Since the argument is analogous for a woman, we have individual rationality.

We argue that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ cannot be blocked by contradiction. Suppose that a man m and a woman w block $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. Then, there is an $\boldsymbol{x}' = (x_1', x_2') \in X$ such that $u_m(w, \boldsymbol{x}') > u_m(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ and $u_w(m, \boldsymbol{x}') > u_w(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. Thus,

$$u_m(w, \boldsymbol{x}') + u_w(m, \boldsymbol{x}') > u_m(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) + u_w(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger).$$

Since (i) $u_m(w, \boldsymbol{x}') + u_w(m, \boldsymbol{x}') = b(x_2') - c(\theta_m, x_1') + b(x_1') - c(\theta_w, x_2')$ and since (ii) equation (3.22) gives

$$\begin{aligned} u_m(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) + u_w(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) &= \max_{x_1 \in [0,\beta]} b(x_1) - c(\theta_m, x_1) + \max_{x_2 \in [0,\beta]} b(x_2) - c(\theta_w, x_2) \\ &= \max_{(x_1, x_2) \in X} b(x_2) - c(\theta_m, x_1) + b(x_1) - c(\theta_w, x_2), \end{aligned}$$

we have,

$$b(x_2') - c(\theta_m, x_1') + b(x_1') - c(\theta_w, x_2') > \max_{(x_1, x_2) \in X} b(x_2) - c(\theta_m, x_1) + b(x_1) - c(\theta_w, x_2),$$

which is a contradiction of optimality since (x_1', x_2') is in X. It follows that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is stable.

Fourth, since Assumption 7 implies Assumption 4, Corollary 1 gives that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is interior and Pareto optimal.

Fifth, we establish that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is welfare maximizing. By equation (3.22), $T(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = \sum_{i \in \mathcal{N}} \max_{z_i \in [0,\beta]} b(z_i) - c(\theta_i, z_i)$. Let $(\phi', \bar{\boldsymbol{x}}')$ be a feasible allocation, let $S = \{i \in \mathcal{N} | \phi'(i) \neq i\}$ be the set of partnered players in $(\phi', \bar{\boldsymbol{x}}')$, and let $\{z_i'\}_{i \in \mathcal{N}}$ gives players' efforts in $(\phi', \bar{\boldsymbol{x}}')$. We have

$$T(\phi', \bar{\boldsymbol{x}}') = \sum_{i \in S} b(z'_{\phi'(i)}) - c(\theta_i, z_i') = \sum_{i \in S} b(z_i') - c(\theta_i, z_i').$$

The first equality is due to the fact single players get zero and the second equality follows from rearranging the sum since $b(z_i')$ appears in the term of $\phi'(i)$. Hence,

$$T(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) - T(\phi', \bar{\boldsymbol{x}}') = \sum_{i \in S} (\max_{z_i \in [0,\beta]} \{b(z_i) - c(\theta_i, z_i)\} - (b(z_i') - c(\theta_i, z_i'))) + \sum_{i \in \mathcal{N} \setminus S} \max_{y \in [0,\beta]} b(y) - c(\theta_i, y).$$

The first summand is weakly positive by optimality since $z_i' \in [0, \beta]$ for each $i \in \mathcal{N}$ by the feasibility of $(\phi', \bar{\boldsymbol{x}}')$, and the second summand is positive by Assumption 7. Thus, $T(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) - T(\phi', \bar{\boldsymbol{x}}') \geq 0$. □

Proof of Proposition 9. We need a preliminary fact: $h(\theta) = \max\{\arg\max_{y \in [0,\beta]} b(y) - c(\theta, y)\}$ is non-decreasing. Since $-c(\theta, y)$ is supermodular by Assumption 5 this follows directly from Topkis' Monotonicity Theorem (Theorem 2.8.1 of [68]).

Consider a woman w whose type increases from θ_w to θ_w'. In $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$, she and her match

$\phi^\dagger(w)$ both exert effort $h(\theta_w)$ by construction, while in $(\phi', \bar{\boldsymbol{x}}')$, she and her match $\phi'(w)$ both exert effort $h(\theta'_w)$. Since $h(\theta_w) \leq h(\theta'_w)$, we have that w's effort increases and the effort of her match increases, even though the identity of her match may change. Since the analogous argument holds for each man, parts (i) and (ii) of the proposition follow.

It remains to show that $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger, \theta_i) \leq u_i(\phi', \bar{\boldsymbol{x}}', \theta'_i)$ for each player i. Write

$$u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger, \theta_i) = \max_{y \in [0,\beta]} b(y) - c(\theta_i, y) \leq \max_{y \in [0,\beta]} b(y) - c(\theta'_i, y) = u_i(\phi', \bar{\boldsymbol{x}}', \theta'_i),$$

where the equalities are due to Lemma 10 and the weak inequality is due to Lemma 8. □

4 Matching with Trilateral Investment

4.1 Introduction

When a town decides to take a blighted lot and build a nice park for its residents, it usually works with a landscape architect to design the park, and a construction company to build it. The quality of the park depends (in the abstract) on the efforts exerted by the town, the architect, and the construction company. The architect needs to exert costly effort to generate good designs and communicate them clearly to the construction company. Likewise, the construction company needs to exert costly effort to understand the plans and build the park, and the town needs to exert costly effort to collect and provide funding and to provide a clear vision of what its residents want. Thus, the town's payoff depends on the effort it and its partners exert; likewise for the architect and the construction company. Our goal is to understand how their effort costs shape the efforts they exert and their payoffs in a world where multiple towns, architects, and contractors compete for each other's business.

To these ends, we build a three-sided matching game between towns, architects, and construction companies by extending the "Bilateral Effort Game" of Chapter 1.[28] When a town, architect, and construction company match, they come to an agreement about the effort each exerts. Their efforts produce a benefit for each other. The town's payoff is the benefit produced by the architect and construction company's efforts less the cost of its own effort. Likewise, the architect's payoff is the benefit produced by the town and construction company's efforts less the cost of its own effort, and the construction company's payoff is the benefit produced by the town and architect's efforts less the cost of its own effort. Each player has a type (e.g., ability) that affects the benefit it produces and its cost of effort. We call this game the "Trilateral Effort Game."

Our solution concept is a stable matching and vector of agreements, which we call a stable allocation. In a stable allocation, (i) each player earns at least the value of its outside option and (ii) no town, architect, and construction company can do strictly better by matching and selecting a new agreement, i.e., no three players "block" the allocation.

We show that a (welfare maximizing) stable allocation exists when there are an equal number of towns, architects, and construction companies with the same abilities (Proposition 1). The intuition is that symmetry and the additive nature of payoffs (in benefit and cost) allow us to construct a "symmetric allocation" where players of the same ability are matched

[28] Although we'll frame our discussion in terms of towns, architects, and construction companies, there are many other examples of trilateral investment. For instance, (i) schools, students, and teachers, (ii) workers, firms, and consumers, (iii) men, women, and coaches in mixed-doubles tennis, and so on. Also, all of the results we develop extend to K-sided games, with $K \geq 2$; see the Supplement for this chapter for details.

to each other, exert the same effort, and earn the same payoff.[29] The stability of this allocation, as well as its utilitarian efficiency (and Pareto efficiency), follow from the nature of the construction.

Next, we focus on the structure of symmetric allocations. To examine how abilities effect efforts, benefits, and payoffs, we suppose that ability (i) increases the marginal and absolute benefits players produce and (ii) decreases players' marginal and total costs. We find three results (Proposition 2). First, higher ability players and their partners exert more effort than lower ability players. Second, higher ability players provide larger benefits to their partners and receive larger benefits from their partners. Third, higher ability players earn more.

The intuition for these results is that a player's effort in a symmetric allocation is given by the solution to an optimization problem. Since the marginal benefit is increasing in ability and the marginal cost is decreasing in ability, Topkis' Monotonicity Theorem implies that this solution is increasing in the ability. Thus, higher ability players exert more effort, as do their partners. Since the benefit a player produces is increasing in ability and effort, it follows that higher ability players produce and receive higher benefits. Since the cost of effort is decreasing in ability, the nature of the optimization problem also implies that higher ability players earn more.

We close by developing comparative statics for symmetric allocations. We establish that as players' abilities increase, then their efforts, the benefits they produce and receive, and their welfare increase in every symmetric allocation (Proposition 3). The intuition for this result is analogous to the intuition for Proposition 2.

Propositions 2 and 3 help us understand why some cities are able to routinely attract outstanding partners who help them build exceptional parks. They also provide guidance on how an architect or construction company can, via pre-match investment in their production techniques and technology, improve the caliber of town it obtains and it's payoff.

4.1.1 Related Literature

We contribute to two literatures. The first literature is on multi-sided matching games without agreements. To the best of our knowledge, Alkan [3] is the first to consider such games. He shows, via an example, that they may lack stable allocation. Subsequently, the literature has concerned itself with developing preference restrictions that ensure existence. For instance, Danilov et al. [22] develop a joint lexicographic condition on preferences that's sufficient for existence in three-sided games. This condition is that towns and architects care foremost about each other and then have preferences over construction companies. Boros

[29] There may be more than one symmetric allocation because we must break certain ties in its construction.

et al. [11] study the necessity of this condition, while Eriksson et al. [27] generalize it to establish existence in four-sided games. More broadly, since multi-sided matching games without agreements are special types of coalition formation games, other conditions also give existence – e.g., Alcade and Romero-Medina's [2] collection of conditions, Banerjee et al.'s [61] top-coalitions condition, Pycia's [54] pairwise alignment condition, and Scarf's [62] Balancedness condition.

Our contribution to this literature lies in the nature of our game, our existence argument, and in our focus on how players behave. Our game, to the best of our knowledge, is the first three-sided matching game to allow for agreements. Its structure allows us to show existence via a new, constructive argument that only requires symmetry. This construction, in turn, allows us to develop sharp predictions about player behavior and welfare.

The second literature examines the role of pre-match investments (e.g., getting an education or going to the gym) in shaping players matching decisions and outcomes – e.g., Burdett and Coles [13], Chiappori et al. [14], Cole et al. [16, 17], and Noldeke and Samuelson [49]. While these studies allow players to make investments before they match, we require that players make investments/exert efforts when they match. This difference is intuitive and economically meaningful since it allows players to adjust their efforts in response to their partners' efforts. We also explore the relationships among matched players' benefits, efforts, and costs – none of the above papers examine these relationships.

We close by considering the relationship between the Bilateral and Trilateral Effort Games. The Trilateral Effort Game generalizes the Bilateral Effort Game by allowing for three groups of players, instead of only two, and by allowing the benefit a player produces to depend on her type and effort, instead of only on her effort. Our existence result (Proposition 1) dispenses with the monotonicity assumptions of the Bilateral Effort Game and our structure result (Proposition 2) leverages monotone comparative statics instead of the more complicated proof techniques in [29]. That said, from an economic standpoint, the two games are complements since we make similar economic findings in both games. (For instance, we show in Proposition 7 of [29] that higher types earn more than lower types in every stable allocation, a result reminiscent of Proposition 2 in this paper, and we show in Proposition 9 of [29] that increases in players abilities increase their efforts and their welfare, a result reminiscent of Proposition 3 in this paper.) Thus, the Trilateral Effort Game contributes to our understanding of multi-party matching and investment by moving beyond a two-sided world and showing that many of the lessons of this world are robust to the number of sides involved and to the nature of the benefits players produce.

4.2 The Game

This section describes the Trilateral Effort Game, defines a stable allocation, and gives an example of a stable allocation.

4.2.1 Environment

There are three finite groups of players, towns $\mathcal{T} = \{1, \ldots, T\}$, architects $\mathcal{A} = \{T + 1, \ldots, 2T\}$, and construction companies $\mathcal{C} = \{2T + 1, \ldots, 3T\}$, with $T > 0$. Let $\mathcal{N} = \mathcal{T} \cup \mathcal{A} \cup \mathcal{C}$.[30] We write t for the t-th town, a for the a-th architect, c for the c-th construction company, and i for the i-th player regardless of group.

Each player may be single or may be matched to one member of every other group. For instance, a town may either be single or may be matched with an architect and a construction company. We adopted the convention that player i is single if it's matched to itself. A **matching** is a function that specifies each player's match, i.e., is a $\phi : \mathcal{N} \to \mathcal{N}^2$ such that: (i) for each town t, $\phi(t) \in (\mathcal{A} \times \mathcal{C}) \cup \{(t,t)\}$; (ii) for each architect a, $\phi(a) \in (\mathcal{T} \times \mathcal{C}) \cup \{(a,a)\}$; (iii) for each construction company c, $\phi(c) \in (\mathcal{T} \times \mathcal{A}) \cup \{(c,c)\}$; and (iv) for each town t, architect a, and construction company c, $\phi(t) = (a,c) \iff \phi(a) = (t,c) \iff \phi(c) = (t,a)$. We say player i is **partnered** if $\phi(i) \neq (i,i)$. We write Φ for the finite set of all matchings.

When a town t, architect a, and construction company c match, they select an agreement $\boldsymbol{x} = (x_t, x_a, x_c) \in \mathbb{R}^3$ about the efforts they exert – x_t gives t's effort, x_a gives a's effort, and x_c gives c's effort. Also, each single player has an agreement $\boldsymbol{x} \in \mathbb{R}^3$ with itself. Given a matching ϕ, we write $\boldsymbol{x}^i$ for the agreement player i has with either (i) its partners or (ii) itself. We write $\bar{\boldsymbol{x}} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^N)$ for the vector of players' agreements. For each player i, we have $\boldsymbol{x}^i = \boldsymbol{x}^j = \boldsymbol{x}^k$ for $(j,k) = \phi(i)$. Thus,

$$\bar{\boldsymbol{x}} \in P(\phi) = \{(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^N) | \boldsymbol{x}^i = \boldsymbol{x}^j = \boldsymbol{x}^k \text{ for } (j,k) = \phi(i) \text{ for each player } i\}.$$

We think of $P(\phi)$ as the set of possible agreement vectors for the matching ϕ. An **allocation** is a $(\phi, \bar{\boldsymbol{x}})$ such that $\phi \in \Phi$ and $\bar{\boldsymbol{x}} \in P(\phi)$.

Let $\Theta \subset \mathbb{R}_+$ be a finite set of types. We endow each player i with a type θ_i (e.g., ability) from Θ. Let $\{\theta_i\}_{i \in \mathcal{N}}$ denote the endowment of players' types. Let $b : \Theta \times \mathbb{R} \to \mathbb{R}$ and $d : \Theta \times \mathbb{R} \to \mathbb{R}$ be functions which are continuous in their second arguments. We call $b(\theta, y)$ the "benefit" function and $d(\theta, y)$ the "cost" function.

When a town t, architect a, and construction company c match with agreement (x_t, x_a, x_c), each player produces a benefit of $b(\theta_i, x_i)$ for their partners and incurs a cost $d(\theta_i, x_i)$ for

[30] We consider a three-sided environment only for expositional simplicity. Our results readily extended to K-sided environments; see the Supplement for this chapter for details.

doing so.[31] Thus, t, a, and c's payoffs to matching with each other at agreement (x_t, x_a, x_c) are

$$u_t(a, c, x_t, x_a, x_c) = b(\theta_a, x_a) + b(\theta_c, x_c) - d(\theta_t, x_t)$$
$$u_a(t, c, x_t, x_a, x_c) = b(\theta_t, x_t) + b(\theta_c, x_c) - d(\theta_a, x_a)$$
$$u_c(t, a, x_t, x_a, x_c) = b(\theta_t, x_t) + b(\theta_a, x_a) - d(\theta_c, x_c).$$

We normalize the value of being single to zero, i.e., $u_i(i, i, \boldsymbol{x}) = 0$ for all agreements $\boldsymbol{x}$. In an abuse of notation, for each allocation $(\phi, \bar{\boldsymbol{x}}) = (\phi, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^N)$, we write $u_i(\phi, \bar{\boldsymbol{x}}) \equiv u_i(\phi(i), \boldsymbol{x}^i)$ for player i's payoff in $(\phi, \bar{\boldsymbol{x}})$.

We make a clarifying remark before proceeding.

Remark 1. While all players prefer matches who produce higher benefits, they *do not* have common preferences over matches and agreements. Instead, for a fixed vector of efforts, each side agrees on a ranking of the opposite sides – e.g., the architects and construction companies agree on which towns are best, second best, and so on. However, efforts aren't fixed, they're endogenous. Thus, each town effectively chooses its position in the opposite sides' rankings by its choice of effort. (This choice, of course, depends on the benefits offered by the other towns, architects, and construction companies, as well as its own benefit and cost functions.)

4.2.2 Stable Allocations

The next four definitions develop the idea of a stable allocation. We suppose that, due to time and energy limitations, players can only feasibly exert efforts in $[0, \beta]$, where $0 < \beta < \infty$. Hence, the set of feasible agreements is $X = [0, \beta]^3$.

Definition. An allocation $(\phi, \bar{\boldsymbol{x}})$ is *feasible* if agreements are in X, i.e., $\bar{\boldsymbol{x}} \in X^N$.

Definition. An allocation $(\phi, \bar{\boldsymbol{x}})$ is *individually rational* if every player gets at least the value of being single, i.e., $u_i(\phi, \bar{\boldsymbol{x}}) \geq 0$ for each player i.

Definition. A town t, architect a, and construction company c *block* an allocation $(\phi, \bar{\boldsymbol{x}})$ if they can obtain strictly higher payoffs together than they obtain in $(\phi, \bar{\boldsymbol{x}})$, i.e., if there exists an $\boldsymbol{x} \in X$ such that

$$u_t(a, c, \boldsymbol{x}) > u_t(\phi, \bar{\boldsymbol{x}}), \; u_a(t, c, \boldsymbol{x}) > u_a(\phi, \bar{\boldsymbol{x}}), \text{ and } u_c(t, a, \boldsymbol{x}) > u_c(\phi, \bar{\boldsymbol{x}}).$$

[31] For the time being, we place no monotonicity restrictions the benefit and cost functions.

Definition. An allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is *stable* if (i) it is feasible, (ii) individually rational, and (iii) no town, architect, and construction company block it.

Stable allocations are our solution concept. When an allocation is stable: (i) no player can do strictly better by choosing to be single (per individual rationality) and (ii) no three players can do strictly better by matching with each other and choosing a new agreement instead of following $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ (per no blocking). Observe that the set of stable allocations and the core coincide because the payoffs of a matched town, architect, and construction company only depend on their identities and agreement. In addition, there are usually many stable allocations, when the stable set is non-empty.

To get a feel for the game it's helpful to look at an example.

Example 1. A Simple Game.

Suppose there are two towns, two architects, and two construction companies, i.e., $\mathcal{T} = \{1, 2\}$, $\mathcal{A} = \{3, 4\}$, and $\mathcal{C} = \{5, 6\}$. Let $X = [0, 4]^3$, $b(\theta, y) = y$, and $d(\theta, y) = y^2/\theta$. In addition, let $\theta_1 = \theta_3 = \theta_5 = 1$ and $\theta_2 = \theta_4 = \theta_6 = 2$.

A stable allocation for this game is $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{6^\star})$, where (i) $\phi^\star(1) = (3, 5)$ and $\boldsymbol{x}^{1^\star} = \boldsymbol{x}^{3^\star} = \boldsymbol{x}^{5^\star} = (1, 1, 1)$ and (ii) $\phi^\star(2) = (4, 6)$ and $\boldsymbol{x}^{2^\star} = \boldsymbol{x}^{4^\star} = \boldsymbol{x}^{6^\star} = (2, 2, 2)$. That is, the odd players are matched together and each exerts effort 1, while the even players are matched together and each exerts effort 2.

Let's verify that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable. It's easily seen that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is feasible. It's individually rational because $u_1(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_3(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_5(\phi^\star, \bar{\boldsymbol{x}}^\star) = 2 - 1^2 = 1$ and $u_2(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_4(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_6(\phi^\star, \bar{\boldsymbol{x}}^\star) = 4 - \frac{1}{2}2^2 = 2$. Thus, no player does strictly better by opting out.

To verify stability, we need to check whether any of the eight triples of players block. First, we establish that 1, 3, and 5 can't block. For them to block, there must be an $(x_t, x_a, x_c) \in X$ with $x_a + x_c - (x_t)^2 > 1$, $x_t + x_c - (x_a)^2 > 1$, and $x_t + x_a - (x_c)^2 > 1$. It's readily verified that this system has no solutions in X, implying they can't do strictly better by getting together and selecting a new agreement. An analogous argument gives that 2, 4, and 6 can't block.

Next, consider 1, 3, and 6. For them to block, there must be an $(x_t, x_a, x_c) \in X$ with $x_a + x_c - (x_t)^2 > 1$, $x_t + x_c - (x_a)^2 > 1$, and $x_t + x_a - \frac{1}{2}(x_c)^2 > 2$. Again, it's easily verified that this system has no real solutions. Thus, 1, 3, and 6 cannot block. By symmetry, (i) 1, 4, and 5 and (ii) 2, 3, and 5 can't block.

Finally, consider 1, 4, 6. For them to block, there must be an $(x_t, x_a, x_c) \in X$ with $x_a + x_c - (x_t)^2 > 1$, $x_t + x_c - \frac{1}{2}(x_a)^2 > 2$, and $x_t + x_a - \frac{1}{2}(x_c)^2 > 2$. Again, this system has no real solutions. By symmetry, (i) 2, 3, and 6 and (ii) 2, 4, and 5 can't block. It follows that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable. $\triangle$

4.3 Results

In this section, we state and prove our results. We defer the proofs to the end of each subsection to provide discussion.

4.3.1 Existence of A Stable Allocation

In this subsection, we prove that a least one stable allocation exists under relatively weak symmetry conditions, provided the following assumption holds.

Assumption 1. Universal Agreeability.
For all θ_t, θ_a, and θ_c in Θ, there's an $(x_t, x_a, x_c) \in X$ such that

$$\begin{aligned} &b(\theta_a, x_a) + b(\theta_c, x_c) - d(\theta_t, x_t) > 0, \\ &b(\theta_t, x_t) + b(\theta_c, x_c) - d(\theta_a, x_a) > 0, \text{ and} \\ &b(\theta_t, x_t) + b(\theta_a, x_a) - d(\theta_c, x_c) > 0. \end{aligned}$$

This assumption guarantees that a town, architect, and construction company of any types can find an agreement that makes them strictly better off than if they're single. This type of assumption is standard in the two-sided matching literature and holds, for instance, in Example 1.

The next definition makes the idea of symmetry precise.

Definition. We say that the type endowment $\{\theta_i\}_{i\in\mathcal{N}}$ is *symmetric* when:
(i) Towns, architects, and construction companies are endowed with the same types, i.e.,

$$\cup_{i\in\mathcal{T}}\{\theta_i\} = \cup_{i\in\mathcal{A}}\{\theta_i\} = \cup_{i\in\mathcal{C}}\{\theta_i\}.$$

(ii) There are an equal number of towns, architects, and construction companies with the same type, i.e.,

$$|\{t \in \mathcal{T} | \theta_t = \theta\}| = |\{a \in \mathcal{A} | \theta_a = \theta\}| = |\{c \in \mathcal{C} | \theta_c = \theta\}|$$

for each $\theta \in \Theta$.

For instance, it's readily verified that the type endowment in Example 1 is symmetric.

With this definition in hand, we can give our principle existence result. An allocation $(\phi', \bar{\boldsymbol{x}}')$ is **welfare maximizing** if it maximizes $\sum_{i\in\mathcal{N}} u_i(\phi, \bar{\boldsymbol{x}})$ on the set of feasible allocations. Notice that a welfare maximizing allocation is always Pareto optimal.

Proposition 1. Existence of a Pareto Optimal Stable Allocation.

Let Assumption 1 hold and let the endowment of types $\{\theta_i\}_{i\in\mathcal{N}}$ be symmetric, then a welfare maximizing stable allocation exists.

We'll prove the proposition in two steps. First, we'll exploit the symmetry of the type endowment and the additive nature of payoffs to construct a "symmetric allocation" that treats players equally, i.e., it has all players of the same type match and exert the same effort, implying they earn the same payoff. Second, we'll prove that any symmetric allocation is stable and welfare maximizing (see Lemma 1 below). Let's begin.

Construction. Symmetric Allocation.
Let Assumption 1 hold and let the type endowment $\{\theta_i\}_{i\in\mathcal{N}}$ be symmetric. We begin by ordering the players. First, list the towns in descending order of their types, breaking ties arbitrarily. Label the first town in the list t_1, the second town t_2, and so on down to t_T. Likewise, order the architects and the construction companies on their own lists and assign them the labels $a_1, a_2, \ldots, a_T$ and $c_1, c_2, \ldots, c_T$ respectively.

We construct a **symmetric allocation** $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = (\phi^\star, \boldsymbol{x}^{1^\dagger}, \ldots, \boldsymbol{x}^{N^\dagger})$ as follows. Set $\phi^\dagger$ such that towns, architects, and construction companies with the same rank-order type are matched, i.e.,

$$\phi^\dagger(t_l) = (a_l, c_l),\ \phi^\dagger(a_l) = (t_l, c_l), \text{ and } \phi^\dagger(c_l) = (t_l, a_l),$$

for each $l \in \{1, \ldots, T\}$. Set $\bar{\boldsymbol{x}}^\dagger$ such that $\boldsymbol{x}^{t_l^\dagger} = \boldsymbol{x}^{a_l^\dagger} = \boldsymbol{x}^{c_l^\dagger} = (x_l^\dagger, x_l^\dagger, x_l^\dagger)$ where

$$x_l^\dagger = \max\{\arg\max_{y\in[0,\beta]} 2b(\theta_{t_l}, y) - d(\theta_{t_l}, y)\}, \tag{4.1}$$

for each $l \in \{1, \ldots, T\}$, i.e., t_l, a_l, and c_l exert the same effort and this effort is the largest solution of $\max_{y\in[0,\beta]} 2b(\theta_{t_l}, y) - d(\theta_{t_l}, y)$. ○

Remark 2. Since $b(\theta, y)$ and $d(\theta, y)$ are continuous in y, the set of maximizing arguments in equation (4.1) is non-empty and compact by standard arguments. Hence, this set has maximal element. It follows that a symmetric allocation always exists. In fact, there are usually multiple symmetric allocations because there are usually many ways to break ties when assigning players their labels.

The next lemma gives the properties of a symmetric allocation.

Lemma 1. Properties of a Symmetric Allocation.
Let Assumption 1 hold and let the endowment of types $\{\theta_i\}_{i\in\mathcal{N}}$ be symmetric. Then a symmetric allocation $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is stable and welfare maximizing. In addition, for each player i,

we have $\theta_i = \theta_j = \theta_k$ *for* $(j,k) = \phi^\dagger(i)$ *and*

$$u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = \max_{y\in[0,\beta]} 2b(\theta_i, y) - d(\theta_i, y). \tag{4.2}$$

It follows that the symmetric allocation is Pareto optimal that treats equals equally, i.e., players of the same type match, exert the same effort, produce and receive the same benefit, and earn the same payoff. The stable allocation we found in Example 1 is actually a symmetric allocation: towns, architects, and construction companies with the same rank-order type are matched, the type 1 players' efforts solve $\max_y 2y - y^2$, and the type 2 players' efforts solve $\max_y 2y - {}^1\!/_2 y^2$.

The intuition for the lemma is that the symmetry of types, the additive nature of payoffs, and the construction of the symmetric allocation ensure payoffs are given by equation (4.2). Hence, the allocation is individually rational by Assumption 1. Stability follows from a contradiction argument. If a trio of players block, then each must receive a payoff in excess of equation (4.2). Thus, the sum of their payoffs under the block must exceed the maximum of the sum of their payoffs, an impossibility. Welfare maximization follows directly from equation (4.2). We'll prove the lemma in a moment.[32] However, we first prove Proposition 1.

Proof of Proposition 1. The proposition is an immediate corollary of Lemma 1 and the fact that a symmetric allocation always exists per Remark 2. □

Proof of Lemma 1. We'll prove the lemma in three steps. First, we'll establish that players' payoffs are given by equation (4.2) in a symmetric allocation. Second, we'll leverage this equation and Assumption 1 to show that the symmetric allocation is stable. Third, we'll establish that the symmetric allocation maximizes welfare.

Let $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ be a symmetric allocation and let $t_1, \ldots, t_T$, $a_1, \ldots, a_T$, and $c_1, \ldots, c_T$ be the labels that were assigned to players in the construction of $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. Since the type endowment is symmetric, we have $\theta_{t_1} = \theta_{a_1} = \theta_{c_1}$, $\theta_{t_2} = \theta_{a_2} = \theta_{c_2}$, ..., and $\theta_{t_T} = \theta_{a_T} = \theta_{c_T}$. That is, all matched players have the same types.

We first establish that equation (4.2) holds. Suppose player i is an architect. Since i is partnered (recall that all players are partnered by $\phi^\dagger$), let $(t,c) = \phi^\dagger(i)$. Suppose that i's label is a_k, then t's label is t_k and c's label is c_k by construction. Since the type endowment

[32] The intuition for Lemma 1 is similar to the intuition for Lemma 10 in Chapter 1. The proof, however, differs in that we must account for the third side of the market.

is symmetric, we have $\theta_i = \theta_{a_k} = \theta_{t_k} = \theta_{c_k}$, that is, $\theta_i = \theta_j = \theta_k$ for $(j,k) = \phi^\dagger(i)$. Thus,

$$\begin{aligned}
u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) &= b(\theta_{t_k}, x_k^\dagger) + b(\theta_{c_k}, x_k^\dagger) - d(\theta_i, x_k^\dagger) \\
&= 2b(\theta_{t_k}, x_k^\dagger) - d(\theta_{t_k}, x_k^\dagger) \\
&= \max_{y\in[0,\beta]} 2b(\theta_{t_k}, y) - d(\theta_{t_k}, y) \\
&= \max_{y\in[0,\beta]} 2b(\theta_i, y) - d(\theta_i, y),
\end{aligned}$$

where the third line is due to optimality. Since an analogous argument applies if i is a town or a construction company, equation (4.2) holds.

Second, we establish that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is stable. By construction, $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is feasible because $x_l^\dagger \in [0,\beta]$ for all $l \in \{1,\dots,T\}$. It's also individually rational. To see this, consider town t; the argument is analogous for an architect or a construction company. Suppose that t's label is t_k, then t is matched to architect a_k and construction company c_k by $\phi^\dagger$ and we have $\theta_t = \theta_{a_k} = \theta_{c_k}$. Consider maximum the sum of t, a_k, and c_k's payoffs,

$$\begin{aligned}
&\max_{\boldsymbol{x}\in X} u_t(a_k, c_k, \boldsymbol{x}) + u_{a_k}(t, c_k, \boldsymbol{x}) + u_{c_k}(t, a_k, \boldsymbol{x}) \\
&\quad = \max_{(x_t,x_a,x_c)\in X} 2(b(\theta_t, x_t) + b(\theta_{a_k}, x_a) + b(\theta_{c_k}, x_c)) - d(\theta_t, x_t) - d(\theta_{a_k}, x_a) - d(\theta_{c_k}, x_c) \\
&= \max_{x_t\in[0,\beta]} 2b(\theta_t, x_t) - d(\theta_t, x_t) + \max_{x_a\in[0,\beta]} 2b(\theta_a, x_a) - d(\theta_a, x_a) + \max_{x_c\in[0,\beta]} 2b(\theta_c, x_c) - d(\theta_c, x_c) \\
&\qquad = 3 \max_{x_t\in[0,\beta]} 2b(\theta_t, x_t) - d(\theta_t, x_t).
\end{aligned}$$

It's easily seen that Assumption 1 gives that the second line is strictly positive. Hence, $\max_{y\in[0,\beta]} 2b(\theta_t, y) - d(\theta_t, y) > 0$, implying $u_t(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) > 0$ by equation (4.2).

We need to show that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ cannot be blocked. We argue by contradiction. Suppose there's a town t, architect a, construction company c, and an agreement $\boldsymbol{x}' = (x_t', x_a', x_c') \in X$ such that $u_t(a, c, \boldsymbol{x}') > u_t(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$, $u_a(t, c, \boldsymbol{x}') > u_a(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$, and $u_c(a, t, \boldsymbol{x}') > u_c(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. Then,

$$u_t(a, c, \boldsymbol{x}') + u_a(t, c, \boldsymbol{x}') + u_c(a, t, \boldsymbol{x}') > u_t(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) + u_a(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) + u_c(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger).$$

Since (i) $u_t(a, c, \boldsymbol{x}') + u_a(t, c, \boldsymbol{x}') + u_c(a, t, \boldsymbol{x}') = 2(b(\theta_t, x_t') + b(\theta_a, x_a') + b(\theta_c, x_c')) - d(\theta_t, x_t') -$

$d(\theta_a, x'_a) - d(\theta_c, x'_c)$ and since (ii) equation (4.2) gives,

$$\begin{aligned}
&u_t(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) + u_a(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) + u_c(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) \\
&= \max_{x_t \in [0,\beta]} 2b(\theta_t, x_t) - d(\theta_t, x_t) + \max_{x_a \in [0,\beta]} 2b(\theta_a, x_a) - d(\theta_a, x_a) + \max_{x_c \in [0,\beta]} 2b(\theta_c, x_c) - d(\theta_c, x_c) \\
&\qquad = \max_{(x_t,x_a,x_c) \in X} 2(b(\theta_t, x_t) + b(\theta_a, x_a) + b(\theta_c, x_c)) - d(\theta_t, x_t) - d(\theta_a, x_a) - d(\theta_c, x_c),
\end{aligned}$$

we have

$$\begin{aligned}
&2(b(\theta_t, x'_t) + b(\theta_a, x'_a) + b(\theta_c, x'_c)) - d(\theta_t, x'_t) - d(\theta_a, x'_a) - d(\theta_c, x'_c) \\
&\qquad > \max_{(x_t,x_a,x_c) \in X} 2(b(\theta_t, x_t) + b(\theta_a, x_a) + b(\theta_c, x_c)) - d(\theta_t, x_t) - d(\theta_a, x_a) - d(\theta_c, x_c),
\end{aligned}$$

which is a contradiction of optimality because (x'_t, x'_a, x'_c) is in X. It follows that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is stable.

Third, we establish that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is welfare maximizing. By equation (4.2),

$$\sum_{i \in \mathcal{N}} u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = \sum_{i \in \mathcal{N}} \max_{z_i \in [0,\beta]} 2b(z_i) - d(\theta_i, z_i).$$

Let $(\phi', \bar{\boldsymbol{x}}') = (\phi', \boldsymbol{x}^{1'}, \ldots, \boldsymbol{x}^{N'})$ be another feasible allocation, let $S = \{i \in \mathcal{N} | \phi(i) \neq (i,i)\}$ be the set of partnered players, and let z'_i give the effort of player i in $(\phi', \bar{\boldsymbol{x}}')$ (i.e., z'_i is the first component of $\boldsymbol{x}^{i'} = (x'_t, x'_a, x'_c)$ when i is a town, z'_i is the second component of $\boldsymbol{x}^{i'}$ when i is an architect, and z'_i is the third component of $\boldsymbol{x}^{i'}$ when i is a construction company). We have,

$$\sum_{i \in \mathcal{N}} u_i(\phi', \bar{\boldsymbol{x}}') = \sum_{i \in S} (\sum_{(j,k) = \phi(i)} b(\theta_j, z'_j) + b(\theta_k, z'_k)) - d(\theta_i, z'_i) = \sum_{i \in S} 2b(\theta_i, z'_i) - d(\theta_i, z'_i).$$

The first equality is due to the fact single players earn zero and the second equality follows from reorganizing the sum as $b(\theta_i, z'_i)$ appears in j's and k's terms for $(j,k) = \phi(i)$. Thus,

$$\begin{aligned}
\sum_{i \in \mathcal{N}} u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) - \sum_{i \in \mathcal{N}} u_i(\phi', \bar{\boldsymbol{x}}') &= \sum_{i \in S} (\max_{z_i \in [0,\beta]} \{2b(\theta_i, z_i) - d(\theta_i, z_i)\} - (2b(\theta_i, z'_i) - d(\theta_i, z'_i))) \\
&\quad + \sum_{i \in \mathcal{N} \setminus S} \max_{z_i \in [0,\beta]} \{2b(\theta_i, z_i) - d(\theta_i, z_i)\}.
\end{aligned}$$

Since the first summand is weakly positive by optimality since $z'_i \in [0, \beta]$ for each player i by the feasibility of $(\phi', \bar{\boldsymbol{x}}')$, and the second summand is weakly positive by Assumption 1. Hence, $\sum_{i \in \mathcal{N}} u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) \geq \sum_{i \in \mathcal{N}} u_i(\phi', \bar{\boldsymbol{x}}')$. $\square$

Remark 3. While the conclusions of Lemma 1 aren't robust to asymmetric type endowments, there may still be a stable allocation when the type endowment is asymmetric. We discuss this the Supplement for this chapter.

4.3.2 Structure of Symmetric Allocations

In this subsection, we relate players' efforts and benefits in symmetric allocations to their types with the following assumption.

Assumption 2. Submodular Cost with Common Fixed Cost.
The benefit function $b(\theta, y)$ is increasing and supermodular, while the cost function $d(\theta, y)$ is submodular and decreasing in type.[33]

When the benefit and cost functions are differentiable, supermodularity gives that the marginal benefit is increasing in type and submodularity gives that marginal cost is decreasing in type. The next proposition is the main result of this subsection.

Proposition 2. Structure of the Symmetric Allocation.
Let Assumptions 1 and 2 hold, and let the endowment of types $\{\theta_i\}_{i\in\mathcal{N}}$ be symmetric. Then, in a symmetric allocation $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$, (i) higher types exert more effort and produce greater benefits for their partners than lower types, (ii) higher types receive greater benefits from their partners than lower types, and (iii) higher types earn more than lower types.

The intuition for parts (i) and (ii) is that benefit is supermodular and cost is submodular. Thus, equation (4.1) ensures that higher types exert more effort. Since the benefit function is increasing in type and effort, higher types produce higher benefits. The intuition for part (iii) is that the cost function is decreasing in type. Thus, $\max_y 2b(\theta, y) - d(\theta, y)$ is increasing in type, so higher types earn more by equation (4.2).

Proof. Since benefit is supermodular and cost is submodular, Theorem 2.8.1 of Topkis [68] gives that $h(\theta) = \max\{\arg\max_{y\in[0,\beta]} 2b(\theta, y) - d(\theta, y)\}$ is non-decreasing in θ. Let i and j be two players such that $\theta_i < \theta_j$, then $h(\theta_i) \leq h(\theta_j)$. Since i and her partners exert effort $h(\theta_i)$ and j and her partners exert effort $h(\theta_j)$ by construction, we have the effort parts of (i) and (ii). As to the benefit parts, i and her partners produce benefit $b(\theta_i, h(\theta_i))$, while j and her partners produce benefit $b(\theta_j, h(\theta_j))$. Since the benefit function is increasing, $b(\theta_i, h(\theta_i)) \leq b(\theta_j, h(\theta_j))$.

[33] We do not require that cost is increasing in effort. In fact, it may be decreasing and turn negative. In this way, we allow players to benefit form their own efforts – as might be the case for an architect who works hard to design a nice park near her office so that she has a pleasant place to eat lunch.

As to part (iii), Lemma 1 gives

$$u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = \max_{y\in[0,\beta]} 2b(\theta_i, y) - d(\theta_i, y) \leq \max_{y\in[0,\beta]} 2b(\theta_j, y) - d(\theta_j, y) = u_j(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger),$$

where the inequality follows from the fact that $2b(\theta_i, y) - d(\theta_i, y) \leq 2b(\theta_j, y) - d(\theta_j, y)$ for all $y \in [0, \beta]$ because benefit is increasing in θ and cost is decreasing in θ. □

4.3.3 Comparative Statics of Symmetric Allocations

In this subsection, we develop the following comparative statics result for symmetric allocations. Let $\mathcal{S}(\{\theta_i\}_{i\in\mathcal{N}})$ denote the set of symmetric allocations when the endowment of types is $\{\theta_i\}_{i\in\mathcal{N}}$.

Proposition 3. Comparative Statics of Symmetric Stable Allocations.
Let Assumptions 1 and 2 hold. Let $\{\theta_i\}_{i\in\mathcal{N}}$ and $\{\theta'_i\}_{i\in\mathcal{N}}$ be symmetric endowments of types such that $\theta'_i \geq \theta_i$ for each player i. Then, for each player i and any two symmetric allocations $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) \in \mathcal{S}(\{\theta_i\}_{i\in\mathcal{N}})$ and $(\phi', \bar{\boldsymbol{x}}') \in \mathcal{S}(\{\theta'_i\}_{i\in\mathcal{N}})$,
(i) The effort i exerts is higher in $(\phi', \bar{\boldsymbol{x}}')$ than in $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$.
(ii) The effort exerted by i's matches in $(\phi', \bar{\boldsymbol{x}}')$ is higher than the effort exerted by i's matches in $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$.
(iii) The payoff of i is higher in $(\phi', \bar{\boldsymbol{x}}')$ than in $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$.

That is, as players' types increase, the (i) effort they exert, (ii) the effort their matches exert, and (iii) their welfare increases in every symmetric allocation. Since $b(\theta, y)$ is increasing, (i) and (ii) imply the benefits player i produces and receives increase. The intuition for Proposition 3 is the same as with Proposition 2.

Proof. Without loss, consider an architect a whose type increases from θ_a to θ'_a. In $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$, a and her partners each exert effort $h(\theta_a)$, while in $(\phi', \bar{\boldsymbol{x}}')$, a and her partners each exert effort $h(\theta'_a)$. Since $h(\theta_i) \leq h(\theta'_i)$, a and her partners in $(\phi', \bar{\boldsymbol{x}}')$ exert more effort than a and her partners in $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. (Notice that the identities of a's partners may change as types increase.) To establish that a's payoff increases, write

$$u_a(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger, \theta_a) = \max_{y\in[0,\beta]} 2b(\theta_a, y) - d(\theta_a, y) \leq \max_{y\in[0,\beta]} 2b(\theta'_a, y) - d(\theta'_a, y) = u_a(\phi', \bar{\boldsymbol{x}}', \theta'_a),$$

where the weak inequality follows from the fact benefit rises and cost falls as type increases, and we use $u_i(\phi, \bar{\boldsymbol{x}}, \theta_i)$ to empathize the dependence of i's payoff in $(\phi, \bar{\boldsymbol{x}})$ on her type θ_i. An analogous argument holds for each town and construction company. □

5 Rivalry and Professional Network Formation

5.1 Introduction

Professional networks play a key, internal role at Deloitte Consulting:[34] they are the primary means by which partners obtain consultants to help them complete their projects. Deloitte hires consultants into a pool – any partner may request the help of a consultant and each consultant is expected to provide help, unless she is currently working on another project. When a partner gets a project (from a client), she must decide which consultants in this pool to employ. Since consulting usually requires close collaboration, a partner usually to employs consultants that she likes and has built a good, productive working relationship with, i.e., a partner usually employs consultants from her professional network.

Once a consultant takes on a partner's project, she's usually unavailable to help other partners complete their projects – projects typically involve daylong meetings and other activities at clients' offices. Thus, the partners' employment of consultants is rivalrous (at least in the short term), implying the partners may exert a negative externality on each other. To illustrate, consider two partners A and B. If A employs a large number of consultants who are part of her network and part of B's network, then B's pool of available, in-network consultants is smaller. This reduces B's ability to complete certain projects, and diminishes her earnings – at many consulting firms, including Deloitte, a partner's pay is partially based on the revenue she brings in from her completed projects and her expenses, including the cost of the consultants she employs on the projects she undertakes. In contrast, A is in a better position to complete her project and do well.

Since partners choose their professional networks, our goal is to understand how this rivalry shapes their network formation decisions and, ultimately, their welfare. To these ends, we build a stylized, two-stage game of network formation and rivalry.[35] In our game, there are two partners, A and B, as well as a finite number of consultants, indexed 1 to N. In the first stage, both partners form their professional networks. It's costly for a partner to include a consultant in her network as she must invest effort (and money) to develop a good, productive working relationship with the consultant. For instance, she often needs to invest effort to mentor the consultant in her production techniques/technology so that the consultant may be a capable assistant.[36]

[34] My thanks to a West-Coast based Deloitte consultant for conversations about the inner workings of the firm in the fall of 2013. Also, for those who may not know, Deloitte's partners are the partial owners of the firm. An important part of their job is finding clients with open projects.

[35] We wish to emphasize that, while our game is inspired by Deloitte (and other consulting firms), it is not a complete model of how these firms work.

[36] While the consultant also invests costly effort to build the working relationship, we'll assume, for sim-

In the second stage, the partners sequentially receive projects to complete. The projects and order in which they are awarded are determined by nature. Both projects are received in a short succession, so the first is not completed before the second is received. Projects are of heterogeneous difficulty and each requires a different number of consultants to complete – some, like an evaluation of a small call-center's effectiveness may only require one or two consultants, while others, like an efficiency review of a complex supply chain, may require a dozen or more consultants.

Once a partner receives her project, she employs consultants in her network to help her complete it. She may employ any subset of in-network consultants she wants (including the empty set), save that she cannot employ consultants who are already working on an ongoing project. If the partner manages to employ enough consultants, then she completes her project and earns a positive reward, from which she pays her labor and networking costs. If a partner does not employ enough consultants, she fails to complete the project and earns nothing, but is still liable for her labor and networking costs, if any.

For simplicity, we don't model consultants' behavior. Instead, we assume that the consultants (i) always agree to be in a partner's network and (ii) always agree to be employed on a partner's project, provided they aren't working on another project. These assumptions reflect the expectations consulting companies have of their employees.

Our solution concept is a pure strategy Nash equilibrium.[37] We begin by establishing that an equilibrium exists (Proposition 1). To do this, we use "simple strategies" and an "Auxiliary Game." Partner A's (B's) simple strategy of size n tells her to network with the n lowest (highest) indexed consultants and to employ lower (higher) indexed consultants before higher (lower) indexed consultants. When the partners follow simple strategies, their payoffs are submodular in their networks' sizes, i.e., the sizes of their simple strategies, because of the rivalrous use of consultants. We exploit this fact by constructing an Auxiliary Game where partners use simple strategies, A chooses her network's size, and B chooses the "negative" of her network's size. Since the order of B's choice is inverted, the Auxiliary Game is a two-player supermodular game and so has a pure strategy Nash equilibrium. Subsequently, we show that each equilibrium of the Auxiliary Game induces an equilibrium of the full game; the key insight is that a simple strategy is a best reply to a simple strategy. We call these induced equilibria "simple equilibria."

In equilibrium, we find that both partners' networks are "minimally overlapping" (Proposition 2). That is, the partners try to have networks that share as few consultants as possible.

plicity, that the firm pays the consultant enough to cover such costs.

[37]We focus on pure strategy equilibrium because typical refinements, like subgame perfection, add little economic insight – see Section 2.

The intuition is that by networking with different sets of consultants, the partners minimize the chance that their access to needed consultants will be blocked. An implication of Proposition 2 is that consultants are often in only one partner's network. This prediction finds qualitative support at Deloitte where consultants usually work with a small group of partners, which implies that the partners' networks don't overlap too much.

Two properties play a key role in our subsequent analysis: "employment lists" and "employment efficiency." An equilibrium has employment lists if the first partner who gets a project always uses a ranked list to determine the identities of the consultants she employs: if she employs l consultants, then she employs the first l consultants on her list. This property captures the intuitive idea that each partner uses an address book or other (mental) list. An equilibrium is employment efficient if the first partner to get a project always employs consultants who are exclusively in her network before she employs consultants who are in both her network and the other partner's network. This ensures that the second partner to get a project has a larger pool of available, in-network consultants than she would if the first partner employed consultants in an arbitrary manner. Thus, the second partner is better able to complete projects and has higher earnings.

We focus on equilibria that have employment lists and are employment efficient because these properties are intuitive, especially as both partners work at the same firm, and these equilibria are "robust" – see Section 4. We call these equilibria "ELEE equilibria." Since simple equilibria are ELEE equilibria (Lemma 5), the existence of ELEE equilibria is assured. We then establish that each ELEE equilibrium is payoff equivalent to a simple equilibrium (Proposition 3). The intuition is that, when employment lists and employment efficiency hold, then the equilibrium behavior of the partners is analogous to their behavior in a permutation of a simple equilibrium.

We next establish that the partners' equilibrium interests are opposed. That is, there's an ELEE equilibrium where A does best and B does worst and vice versa (Proposition 4). The intuition for these results is that rivalry (i) causes the Auxiliary Game to be supermodular and (ii) ensures that a partner's optimal payoff is always *weakly* decreasing in the size of the other partner's network. Since (i), there's a there's a maximal (minimal) equilibrium where A holds her largest (smallest) equilibrium network and B holds her smallest (largest) equilibrium network. Since (ii), the maximal (minimal) equilibrium is most (least) preferred by A and least (most) preferred by B. The desired result then follows from Proposition 3. We also establish that if each partner's optimal payoff is *strictly* decreasing in the size of the other partner's network, then A does best in any equilibrium where B does worst and vice versa (Proposition 4). The insight is that strict monotonicity ensures that the maximal (minimal) equilibrium is *the* best (worst) one for A and *the* worst (best) one for B.

We also develop welfare comparative statics for the ELEE equilibria where the partners' interests are opposed. For concreteness, we suppose that A's reward to completing a project weakly increases and that her cost of networking weakly decreases, while B's reward and cost are held constant. We then compare A and B's payoffs and network sizes before and after the shift in equilibria where their interests are opposed. We show that A does better and holds a larger network, while B does worse and holds a smaller network (Proposition 5). This result follow naturally from Proposition 3 and weak monotonicity as the changes in A's reward and cost cause her best response in the Auxiliary Game to increase.

In light of the opposition of interests, it's natural to wonder if one partner actually does better than the other. We establish that A earns more and has a larger network than B in any ELEE equilibria where she does best and B does worst, provided (i) A and B have the same chance of getting a project, (ii) A has a weakly lower networking cost than B, and (iii) A receives weakly more for completing a project than B (Proposition 6). This result follows naturally from Proposition 5 and the insight that, when A has B's reward and cost functions, then she holds a larger network and does weakly better than B in the maximal equilibrium of the Auxiliary Game.

We next turn our attention to efficiency. Unfortunately, there need not be an (ex-ante) efficient equilibrium; we show this via an example. In the example, we establish that one partner always has a strict incentive to defect from the unique efficient (joint) strategy by over-investing in her network, i.e., by holding a larger network than is socially optimal. This over-investment increases the total networking cost and generates a large negative externality on the other partner. As a result, it decreases social welfare and implies that the efficient strategy isn't an equilibrium. This leads us to investigate why an efficient strategy wouldn't be an equilibrium. We find, at least for efficient simple strategies, that the reason is over-investment (Proposition 7). This result follows from weak monotonicity and the fact that simple strategies are best replies to simple strategies.

We conclude by examining the effects of salesmanship on the partners' equilibrium networks and welfare. One partner, say A, may be more skilled at selling projects and, as a result, may obtain projects more frequently. We model this by allowing A to move first in the second stage with greater probability than B. We find that increases in the probability that A moves first increase A's payoff and network size and decrease B's payoff and network size in ELEE equilibria where the partners' interests are opposed (Proposition 8). The intuition is that it's better to move first in the second stage because then one is not subject to less rivalry. Hence, as the probability A moves first increases, A's and B's best responses in the Auxiliary Game increase. From this fact, one may apply Proposition 3 and monotonicity to deduce the result.

5.1.1 Related Literature

Our work makes contributions to two literatures. The first literature on trading-on-networks examines how buyers and sellers come together and trade single items – e.g., Kranton and Minehart [45] and Condorelli and Galeotti [18]. We extend this literature by allowing partners to "buy" labor from multiple consultants and examining how the partners' rivalrous use of consultants affects their networks and welfare. The second literature on multiple common pool resources examines how rivalry affects players' consumption of multiple natural resources – e.g., Ilkilic [36]. In this literature, players choose whether to consume a resource to which they have access. We enrich this literature by also allowing players/partners to choose which resources/consultants they may access.

Kranton and Minehart [45] initiated the trading-on-networks literature with their two-stage game between unit-demand buyers and unit-supply sellers. In the first stage, buyers make costly links/investments with specific sellers that allow them to trade in the second stage. In the second stage, the sellers conduct ascending price auctions with the buyers to whom they're linked. Since players' payoffs are quasi-linear in prices, Kranton and Minehart are able to establish the existence of a pure strategy, efficient equilibrium.[38]

In a related paper, Condorelli and Galeotti [18] develop a two-stage game to examine trade through intermediaries. In the first stage, traders form the links over which trade occurs. In the second stage, one trader is endowed with a single object and traders engage in resell (according to a specified procedure) until the item reaches a trader who would rather keep it than resell it. Each trader has unit demand and quasi-linear preferences. Condorelli and Galeotti find that an equilibrium is generally inefficient because traders from fewer links than is socially optimal because traders fail to internalize the impact an additional link has on the probability the object finds its way to the trader who desires it most.

Our work is complementary to these papers. Like them, we consider a game where players first form links then "transact" over these links. However, we allow partners to tap multiple consultants, we examine the effects of rivalry, and we focus on non-market environments. These differences are economically meaningful. For instance, the papers mentioned above find that equilibria are efficient or are inefficient only due to under-investment; in contrast, we find that equilibria may be inefficient *only* because of over-investment.

The trading-on-networks literature is a subset of the network formation literature. This extensive literature examines the considerations that shape social and economic networks

[38] In a related work, Corominas-Bosch [20] examines an alternating-offer bargaining game between buyers and sellers and characterizes how the exogenously given topology of the network affects the equilibrium terms of trade.

and the effects that these networks have on socio-economic behavior.[39] Our contribution to this literature is to explore how rivalry shapes networks and welfare. Also falling within this literature is Jackson and Wolinsky's [39] Co-Author Game. In this game, a group of players forms links with each other to co-author papers. Each player benefits from having a link/co-author but is harmed when any of her co-authors have links to other players as these co-authors limited time is divided across more projects. Thus, there's a rivalry for players' time. We differ in the nature of the rivalry we explore and this difference has substantive implications for network structure. For instance, Jackson and Wolinsky find that players form disjoint, fully interconnected clusters, implying players' networks (i.e., neighborhoods) are either entirely overlapping or disjoint. This contrasts with our minimally overlapping result.

A common pool resource is one with rivalrous consumption. In their survey, Ostrom et al. [51] give several examples including ground water (as one city's extraction diminishes the amount of water left for other cities), fisheries, and grazing (i.e., the commons problem).[40] Ilkilic [36] considers a game where cities have different degrees of access to water sources (e.g., cities A and B share an aquifer 1, while cities B, C, and D share aquifer 2) and decide how much water to extract from each source they can access to. He takes the map between cities and water sources as given and characterizes the equilibrium extraction from each water source when cities move simultaneously and have quasi-linear payoffs. He finds that sources that are more "central" are more heavily used and are often over-depleted relative to the social optimum. Our work is complementary because we allow cities to choose the water sources they have access to; though we constrain cities to all or nothing use of their sources so our game doesn't nest Ilkilic's game.

More broadly, our work is related to the team formation literature – e.g., Bolle [10] and Lappas et al. [46]. In these models, a principal seeks to hire heterogeneously skilled agents to help her complete a project. In Bolle's early model, these agents are assumed to work equally well with each other. Lappas et al. enrich Bolle's framework by (i) allowing certain agents to be friends and work well with each other and (ii) allowing other agents to enemies and work poorly with each other. While team formation plays a key role in our environment, we also include a rivalry between principals that is absent in both of these papers.

Our work is also related to the cooperative matching with externalities literature – e.g.,

[39]See Granovetter [32] and Jackson [38] for overviews and discussion of the role of networks and network formation in economics.

[40]Rivalrous consumption of a resource is also a concern in the club goods literature, where it's known as "congestion" – see Cornes and Sandler [19] for an overview. However, this literature usually leaves the processes behind congestion un-modeled – e.g., Cornes and Sandler frequently regard the "rate of congestion" as exogenous.

Bando [7] and Pycia and Yenmez [55]. In these models, a worker may take jobs with multiple firms and each firm may hire multiple workers. Each player's payoff depends on the identities of everyone's partners and, thus, allows for many kinds of externalities. Both papers use the pairwise-stable set as a solution concept and give different conditions to guarantee that this set is non-empty and has various welfare properties. We complement this literature by considering an externality these models cannot capture: rivalry. In general, rivalry depends not only on the map between partners and consultants, but also on the partners decisions about which consultants to employ and, thus, on their projects and the order in which they get them. For a given map and given employment strategies, low difficulty projects may lead to no or little rivalry, while very difficult projects may cause significant negative externalities.

5.2 The Game

This section describes our environment, our solution concept, and gives an example.

5.2.1 Environment

We consider a two-stage game between two partners, A and B. There is a finite set of consultants $\mathcal{C} = \{1, \ldots, N\}$, where $N \geq 1$. Let i denote an arbitrary partner. For simplicity, we don't model consultant behavior. Instead, we assume that all consultants (i) always agree to be in a partner's network and (ii) always agree to be employed on a partner's project, provided they aren't working on another project.

In the first stage, both partners simultaneously form professional networks. That is, they pick subsets of $\mathcal{C}$ with whom to build friendly and productive working relationships. Let $\mathcal{N}_i \subset \mathcal{C}$ denote partner i's selection.[41] It's costly for i to form a network as she must invest effort (and money) to develop a good, productive working relationship with each consultant in her network. Let $c_i : \mathbb{N} \to \mathbb{R}_+$ such that $c_i(0) = 0$. Partner i's cost of holding the network $\mathcal{N} \subset \mathcal{C}$ is given by $c_i(|\mathcal{N}|)$,[42] where $|\cdot|$ denotes the cardinality of a set and $|\emptyset| = 0$.

In the second stage, the partners sequentially get their projects and employ consultants. Initially no consultant is employed by either partner. At the start of the stage, nature picks a partner to get her project first – both partners have a $1/2$ chance of getting their project first. Once the identity of the first partner is realized, nature draws a project for her. Then this partner employs consultants from her network. Subsequently, nature draws a project

[41] In the language of the network formation literature (e.g., Bala and Goyal [6]), the partners form "directed links" to consultants.

[42] Sometimes, we'll assume that partner i has a constant marginal cost of networking $\kappa_i \geq 0$, so $c_i(|\mathcal{N}|) = \kappa_i |\mathcal{N}|$. This constant marginal cost assumption is common in the literature – e.g., Bala and Goyal [6] and Jackson [38].

for the second partner. Then the second partner employs consultants from her network who aren't already employed by the first partner. Both projects are received in short succession (e.g., on the same day), so the first is not completed by the time the second is awarded. The stage ends with both partners receiving their payoffs.

Let X denote the non-empty and finite set of projects that either partner may receive. For, each partner i, let $P_i : X \to [0,1]$ such that $\sum_{x \in X} P_i(x) = 1$ be the probability that nature draws project $x \in X$ for i. These draws are independent across partners and the order in which they move in the second stage. Let $d : X \to \mathbb{N}_{++}$ give the difficulty of a project, i.e., the minimum number of consultants needed to complete it – some projects are more complicated than other and require more help. Notice that every project requires at least one consultant.

For each partner i, let $r_i : X \to \mathbb{R}_+$ give i's reward to completing the project. Thus, if partner i gets project x, she earns $r_i(x)$ if she employs at least $d(x)$ consultants and she earns 0 if she employs less than $d(x)$ consultants. This production technology reflects two facts. The first is that clients only pay for completed projects, i.e., there's no residual value to partially-completed projects.[43] The second is that most consultants are of high ability (e.g., Deloitte often hires college graduates in the top of their classes). Thus, they're usually capable assistants who can typically help a partner in whatever ways the partner needs, provided they've developed a good, working relationship with the partner.[44] Hence, in the abstract, in-network consultants are homogenous from a partner's perspective.

When partner i gets her project first, she observes the network she chose in the first stage $\mathcal{N}_i$ and her project x_i. Subsequently, she decides which in-network consultants to employ, if any. (Note that i may always choose to forgo a project by not employing any consultants.) A behavioral strategy for i is a $\sigma_{1i} : \mathbb{P}(\mathcal{C}) \times X \to \mathbb{P}(\mathcal{C})$ such that $\sigma_{1i}(\mathcal{N}, x) \subset \mathcal{N}$ for all $(\mathcal{N}, x) \in \mathbb{P}(\mathcal{C}) \times X$. Thus, i's strategy σ_{1i} takes her observations $(\mathcal{N}_i, x_i)$ and returns a (possibly empty) list of consultants $\sigma_{1i}(\mathcal{N}_i, x_i)$ in $\mathcal{N}_i$ who she employs. Partner i pays an exogenously fixed and finite amount $w \geq 0$ for each consultant she employs – at many consulting firms a partner's compensation is linked to the cost of the consultants she employs in the projects she attempts, w is our way of modeling these costs.[45] Thus, her total labor

[43] In the Supplement we examine an alternative production technology where there's residual value. Our results are robust to this extension.

[44] To digress, at Deloitte (and other consulting firms) consultants actually have heterogeneous strengths, implying that some are better at providing certain kinds of help – e.g., some are better interviewers and others are better data analysts. That said, because of their high general skills, consultants are good enough to do most of the work that a partner could request of them, provided they've been mentored in the partner's technology (which occurs in the course of building a good, productive working relationship.)

[45] We assume w is the same for both partners for simplicity. In the Supplement, we relax this assumption and show that our results continue to hold.

cost is $w\,|\sigma_{1i}(\mathcal{N}_i, x_i)|$.[46]

When partner i gets her project second, she observes her network $\mathcal{N}_i$, the set of consultants employed by the other partner $\mathcal{T}$, and her project x_i. Since the consultants in $\mathcal{T}$ are part of an ongoing project, they are unavailable, i.e., i cannot employ them. Subsequently, i decides which available in-network consultants to employ, if any. A behavioral strategy for i is a $\sigma_{2i} : \mathbb{P}(\mathcal{C})^2 \times X \to \mathbb{P}(\mathcal{C})$ such that $\sigma_{2i}(\mathcal{N}, \mathcal{N}', x) \subset \mathcal{N} \backslash \mathcal{N}'$ for all $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$.[47] Thus, i's strategy σ_{2i} takes her observations $(\mathcal{N}_i, \mathcal{T}, x_i)$ and returns a list of consultants $\sigma_{2i}(\mathcal{N}_i, \mathcal{T}, x_i)$ in $\mathcal{N}_i \backslash \mathcal{T}$ who she employs. As before, i has to pay w for each consultant she employs, so her total labor cost is $w\,|\sigma_{2i}(\mathcal{N}_i, \mathcal{T}, x_i)|$.

A strategy for partner i is $\boldsymbol{s}_i = (\mathcal{N}_i, \sigma_{1i}, \sigma_{2i})$, i.e., is a complete specification of her network selection and her two behavioral strategies. Let $\boldsymbol{S}_i$ denote i's finite set of all possible strategies. Let $\boldsymbol{s} = (\boldsymbol{s}_A, \boldsymbol{s}_B)$ denote a vector of strategies denote the vector of the partners' strategies, and let $\boldsymbol{S} = \boldsymbol{S}_A \times \boldsymbol{S}_B$ be the joint strategy space.

Each partner's (ex-post) payoff is her reward less her labor and networking costs. To fix ideas, let i be a partner and let $-i$ be the other partner. Let $\boldsymbol{s} = (\mathcal{N}_i, \sigma_{1i}, \sigma_{2i}, N_{-i}, \sigma_{1-i}, \sigma_{2-i}) \in \boldsymbol{S}$ and let x_i and x_{-i} be their projects. Suppose both partners follow $\boldsymbol{s}$. Then i's ex-post payoff when she gets her project first is

$$u_{1i}(\boldsymbol{s}, x_i, x_{-i}) = r_i(x_i)\mathbb{I}(|\sigma_{1i}(\mathcal{N}_i, x_i)| \geq d(x_i)) - w\,|\sigma_{1i}(\mathcal{N}_i, x_i)| - c_i(|\mathcal{N}_i|),$$

where $\mathbb{I}(\cdot)$ is an indicator function that's equal to one if $|\sigma_{1i}(\mathcal{N}_i, x_i)| \geq d(x_i)$, i.e., if i employs enough consultants to complete her project, and is equal to zero else. When i gets her project second, $-i$ employs $\sigma_{1-i}(\mathcal{N}_{-i}, x_{-i})$. Thus, i's ex-post payoff is

$$\begin{aligned} u_{2i}(\boldsymbol{s}, x_i, x_{-i}) = {} & r_i(x_i)\mathbb{I}(|\sigma_{2i}(\mathcal{N}_i, \sigma_{1-i}(\mathcal{N}_{-i}, x_{-i}), x_i)| \geq d(x_i)) \\ & - w\,|\sigma_{2i}(\mathcal{N}_i, \sigma_{1-i}(\mathcal{N}_{-i}, x_{-i}), x_i)| - c_i(|\mathcal{N}_i|), \end{aligned}$$

where $\mathbb{I}(\cdot)$ is an indicator function that's equal to one if $|\sigma_{2i}(\mathcal{N}_i, \sigma_{1-i}(\mathcal{N}_{-i}, x_{-i}), x_i)| \geq d(x_i)$, i.e., if i employs enough consultants to complete her project after seeing $-i$ employ $\sigma_{1-i}(\mathcal{N}_{-i}, x_{-i})$, and is equal to zero else.

Two observations are in order. First, once i gets her project her ex-post payoff only depends on the size of her network and the number of consultant that she employs. Second, i may always get a payoff of zero by choosing the empty network.

[46] Assuming a constant marginal cost of labor is without loss. Our results continue to hold for any total labor cost function that is increasing in the number of consultants employed.

[47] To avoid extensive notation, we define $\emptyset \backslash \emptyset = \emptyset$.

We assume that players have expected utility, so partner i's ex-ante payoff is

$$U_i(\boldsymbol{s}) = \sum_{(x_i, x_{-i}) \in X^2} (u_{1i}(\boldsymbol{s}, x_i, x_{-i}) + u_{2i}(\boldsymbol{s}, x_i, x_{-i}))P_i(x_i)P_{-i}(x_{-i})/2.$$

We'll occasionally write $U_i(\boldsymbol{s})$ as $U_i(\boldsymbol{s}, r_i, c_i)$ when we wish to emphasize the dependence of i's ex-ante payoff on her reward and cost functions.

Throughout the rest of the paper, we maintain the following simplifying assumption.

Assumption 1. Project Completion is (Weakly) Good.
For all $x \in X$, we have $r_i(x) - w\, d(x) \geq 0$ for each partner i.

The assumption guarantees that both partners have a non-negative payoff to employing the number of consultants that are needed to complete a project. While this simplifies the statements and proofs of our results, it's not essential to them.

5.2.2 Solution Concept

Our solution concept is a pure strategy Nash equilibrium.

Definition. An *equilibrium* is a (pure) strategy vector $\boldsymbol{s}^\star = (\boldsymbol{s}_A^\star, \boldsymbol{s}_B^\star)$ such that

$$U_A(\boldsymbol{s}^\star) \geq U_A(\boldsymbol{s}_A, \boldsymbol{s}_B^\star) \text{ for all } \boldsymbol{s}_A \in \boldsymbol{S}_A \text{ and}$$
$$U_B(\boldsymbol{s}^\star) \geq U_B(\boldsymbol{s}_A^\star, \boldsymbol{s}_B) \text{ for all } \boldsymbol{s}_B \in \boldsymbol{S}_B.$$

Let $\boldsymbol{E}$ denote the set of equilibria.

Notice that when $\boldsymbol{E}$ is non-empty, then it's finite and is usually non-singular because (i) the definition of equilibrium places little restriction on what can happen off of the equilibrium path and (ii) the consultants' identities may always be permuted. We focus on (pure strategy) equilibria because typical refinements, like subgame perfection, add no economic insight. The reason is simple: after she selects her network, a partner never learns of the network selected by her rival. Thus, the only subgame is the game as a whole and so the sets of subgame perfect equilibria and pure strategy equilibria coincide. That said, we show in the Supplement that there is an equilibrium where both partners always (i.e., both on-path and off-path) behave optimally in the second stage and so don't make "non-credible threats."

5.2.3 An Example

It's useful to work an example.

Example 1. A Simple Symmetric Example.

Let $\mathcal{C} = \{1, 2\}$ be the set of consultants and let $X = \{x_1, x_2\}$ be the set of projects. Let $d(x_1) = 1$ and $d(x_2) = 2$ be the difficulties of the two projects. Let $r_i(x_1) = 2$ and $r_i(x_2) = 5$ for $i \in \{A, B\}$ be the rewards to both projects. Let $w = 1$ and suppose both partners have a constant marginal networking cost of $1/2$. Also, let both projects be equally likely, i.e., $P_i(x_1) = P_i(x_2) = 1/2$ for $i \in \{A, B\}$.

An equilibrium is $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star, \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star)$, where $\mathcal{N}_A^\star = \mathcal{N}_B^\star = \{1, 2\}$,

$$\sigma_{1A}^\star(\mathcal{N}, x) = \begin{cases} \{1\} & \text{if } \mathcal{N} = \mathcal{N}_A^\star \text{ and } x = x_1 \\ \{1, 2\} & \text{if } \mathcal{N} = \mathcal{N}_A^\star \text{ and } x = x_2 \\ \emptyset & \text{else} \end{cases}$$

$$\sigma_{1B}^\star(\mathcal{N}, x) = \begin{cases} \{2\} & \text{if } \mathcal{N} = \mathcal{N}_B^\star \text{ and } x = x_1 \\ \{1, 2\} & \text{if } \mathcal{N} = \mathcal{N}_B^\star \text{ and } x = x_2 \\ \emptyset & \text{else,} \end{cases}$$

$$\sigma_{2A}^\star(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{1\} & \text{if } \mathcal{N} = N_A^\star,\ x = x_1, \text{ and } \mathcal{N}' = \{2\} \\ \emptyset & \text{else,} \end{cases}$$

$$\sigma_{2B}^\star(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{2\} & \text{if } \mathcal{N} = N_B^\star,\ x = x_1, \text{ and } \mathcal{N}' = \{1\} \\ \emptyset & \text{else,} \end{cases}$$

where $\mathcal{N}$ and $\mathcal{N}'$ are subsets of $\mathcal{C}$ and x is a project.

Let's verify that $\boldsymbol{s}^\star$ is an equilibrium. We'll first establish that it's best for A to follow $\boldsymbol{s}^\star$ when B does by considering different networks for A. If A holds network $\emptyset$, then her payoff is 0.

If A's network is $\{2\}$, then her (ex-ante) payoff is $-1/4$ when she behaves optimally in the second stage. Let x_A be A's project. If she gets her project first, then it's best for her to employ consultant 2 if $x_A = x_1$ and for her not to employ consultant 2 if $x_A = x_2$. Hence, her expected payoff is $1/2$ when she moves first. If she gets her project x_A second, then A cannot employ consultant 2 as B always employs 2. Thus, her expected payoff is 0 when she moves second. Hence, her (ex-ante) payoff from network $\{2\}$ is $1/2(1/2) - 1/2 = -1/4$.

If A's network is $\{1\}$, her (ex-ante) payoff is $-1/8$ when she behaves optimally in the second stage. If she gets her project first, then it's best for her to employ consultant 1 if $x_A = x_1$ and for her not to employ consultant 1 if $x_A = x_2$. Hence, her expected payoff is $1/2$ when she moves first. If she gets her project second, then it's best for her to employ consultant 1 if $x_A = x_1$ and B employs only consultant 2, otherwise it's best for her to employ no consultants. Since B employs consultant 2 if and only if she gets project x_1, A's expected payoff is $1\,{}^1\!/\!_2\,{}^1\!/\!_2 = {}^1\!/\!_4$ when she moves second. Thus, her (ex-ante) payoff from network $\{1\}$ is ${}^1\!/\!_2({}^1\!/\!_2 + {}^1\!/\!_4) - {}^1\!/\!_2 = -{}^1\!/\!_8$.

If A's network is $\{1, 2\}$, her (ex-ante) payoff is ${}^1\!/\!_4$ when she behaves optimally in the second stage. If she gets her project first, then it's best for her to employ consultant 1 if $x_A = x_1$ and for her to employ both consultants if $x_A = x_2$. Notice that $\sigma^\star_{1A}$ makes exactly this recommendation to A: $\sigma^\star_{1A}(\{1, 2\}, x_1) = \{1\}$ and $\sigma^\star_{1A}(\{1, 2\}, x_2) = \{1, 2\}$. It follows that her expected payoff is ${}^1\!/\!_2\,1 + {}^1\!/\!_2\,3 = 2$ when she moves first. If she gets her project second, then it's best for her to employ consultant 1 if $x_A = x_1$ and B employs consultant 2, otherwise it's best for her to employ no consultants. Notice that $\sigma^\star_{2A}$ makes exactly this recommendation to A: $\sigma^\star_{2A}(\{1, 2\}, \{2\}, x_1) = \{1\}$, $\sigma^\star_{2A}(\{1, 2\}, \{2\}, x_2) = \emptyset$, and $\sigma^\star_{2A}(\{1, 2\}, \{1, 2\}, x) = \emptyset$ for all $x \in X$. Hence, her expected payoff is $1\,{}^1\!/\!_2\,{}^1\!/\!_2 = {}^1\!/\!_4$ when she moves second. Thus, her (ex-ante) payoff from network $\{1, 2\}$ is ${}^1\!/\!_2(2 + {}^1\!/\!_4) - 1 = {}^1\!/\!_8$.

It follows that it's best for A to hold network $N^\star_A = \{1, 2\}$ and follow $\sigma^\star_{1A}$ and $\sigma^\star_{2A}$ when B plays according to equilibrium. Since an analogous argument gives that it's best for B to follow $\boldsymbol{s}^\star$ when A does, we have that $\boldsymbol{s}^\star$ is an equilibrium. $\triangle$

5.3 Results

We now establish the main results.

5.3.1 Equilibrium Existence

Our goal in this subsection is to establish the following proposition.

Proposition 1. Existence of an Equilibrium.
The set of equilibria $\boldsymbol{E}$ *is non-empty.*

We'll prove the proposition in three steps. First, we'll introduce "simple strategies." For each partner, a simple strategy takes an integer and returns a network and behavioral strategies. The integer is the size of the network. Second, we'll consider an "Auxiliary Game" where the partners play simple strategies and choose the "sizes" of their networks. We'll show that this is a supermodular game and so has a pure strategy Nash equilibrium. Third, we'll show that each equilibrium of the Auxiliary Game induces an equilibrium of the original game. The key insight behind our approach is that when one partner plays a simple strategy, the other partner does best by also playing a simple strategy.

Let's develop simple strategies. A **simple strategy for** $\boldsymbol{A}$ of size $n \in \{0, \ldots, N\}$ is a tuple $\tilde{\boldsymbol{s}}_A(n) = (\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1A}, \tilde{\sigma}_{2A})$ such that, for every $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$, we have

$$\tilde{\mathcal{N}}_A = \{1, \dots, n\}$$

$$\tilde{\sigma}_{1A}(\mathcal{N}, x) = \begin{cases} \{1, \dots, d(x)\} & \text{if } d(x) \leq |\mathcal{N}| \text{ and } \mathcal{N} = \tilde{\mathcal{N}}_A \\ \emptyset & \text{else,} \end{cases}$$

$$\tilde{\sigma}_{2A}(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{1, \dots, d(x)\} & \begin{array}{r} \text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|,\ \mathcal{N}' \subset \{d(x)+1, \dots, N\}, \\ \text{and } \mathcal{N} = \tilde{\mathcal{N}}_A \end{array} \\ \emptyset & \text{else.} \end{cases}$$

In this strategy,[48] A includes the *first* n consultants in her network and, when she gets a project x, she employs the *first* $d(x)$ consultants, provided these consultants aren't employed by B and there are $d(x)$ consultants in her network.

A **simple strategy for $\boldsymbol{B}$** of size $n \in \{0, \dots, N\}$ is a tuple $\tilde{\boldsymbol{s}}_B(n) = (\tilde{\mathcal{N}}_B, \tilde{\sigma}_{1B}, \tilde{\sigma}_{2B})$ such that, for every $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$, we have

$$\tilde{\mathcal{N}}_B = \{N+1-n, \dots, N\}$$

$$\tilde{\sigma}_{1B}(\mathcal{N}, x) = \begin{cases} \{N+1-d(x), \dots, N\} & \text{if } d(x) \leq |\mathcal{N}| \text{ and } \mathcal{N} = \tilde{\mathcal{N}}_B \\ \emptyset & \text{else,} \end{cases}$$

$$\tilde{\sigma}_{2B}(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{N+1-d(x), \dots, N\} & \begin{array}{r} \text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|,\ \mathcal{N}' \subset \{1, \dots, N-d(x_B)\}, \\ \text{and } \mathcal{N} = \tilde{\mathcal{N}}_B \end{array} \\ \emptyset & \text{else.} \end{cases}$$

In this strategy, B includes the *last* n consultants in her network and, when she gets a project x, she employs the *last* $d(x)$ consultants, provided these consultants aren't employed by A and there are $d(x)$ consultants in her network.

We need one more piece of notation. For each $(n_A, n_B) \in \{0, \dots, N\}^2$, let $\tilde{\boldsymbol{s}}(n_A, n_B) = (\tilde{\boldsymbol{s}}_A(n_A), \tilde{\boldsymbol{s}}_B(n_B))$ denote a vector of simple strategies for both players.

In the **Auxiliary Game**, A chooses $z_A \in \{0, \dots, N\}$ and B chooses $z_B \in \{0, \dots, N\}$. Then both partners play the original game according to $\tilde{\boldsymbol{s}}(z_A, N - z_B)$ and so get payoffs $U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ and $U_B(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ respectively. Thus, A picks the size of her simple strategy (i.e., the size of her network) and B picks the "negative" of the size of her simple strategy (i.e., the negative of the size of her network). An **equilibrium of the Auxiliary Game** is a $(z_A^\star, z_B^\star) \in \{0, \dots, N\}^2$ such that (i) $U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B^\star))$ for

[48] It's readily verified that $\tilde{\boldsymbol{s}}_A(n) \in \boldsymbol{S}_A$ for every $n \in \{0, \dots, N\}$.

all $z_A \in \{0, \dots, N\}$ and $U_B(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_B(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B))$ for all $z_B \in \{0, \dots, N\}$. Let $\boldsymbol{F}$ be the set of equilibria of this game.

Lemma 1. Supermodularity.
Let $(z_A, z_B) \in \{0, \dots, N\}^2$*, we have that* $U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ *and* $U_B(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ *are supermodular in* (z_A, z_B)*.*

Proof. The proof is given in Section 4.4. □

To see the intuition behind the lemma, focus on A. When z_B decreases, i.e., $N - z_B$ increases, B adds consultants to her network. This allows her to take on more difficult projects and so reduces the probability that A may employ any "shared" consultant, i.e., a consultant who's in both her network and in B's network. Since A has more shared consultants when z_A is larger, the reduction in z_B decreases A's payoff faster when z_A is larger; hence supermodularity as a result of rivalry.

It follows that the Auxiliary Game is a two-player supermodular game. Thus, we have the following result.

Lemma 2. Equilibria of the Auxiliary Game.
The set of equilibria of the Auxiliary Game $\boldsymbol{F}$ *is a non-empty, complete lattice.*

Proof. Since (i) the joint strategy space of the Auxiliary Game $\{0, \dots, N\}^2$ is trivially a compact and complete (sub-)lattice and (ii) the payoff functions $U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ and $U_B(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ are supermodular and continuous in (z_A, z_B) on $\{0, \dots, N\}^2$, this result follows from Theorem 4.2.1 of Topkis [68]. □

It remains to show that each of these equilibria induces an equilibrium of the original game. The next lemma does this.

Lemma 3. Inducement of Equilibria.
Let $(z_A^\star, z_B^\star)$ *be an equilibrium of the Auxiliary Game, then* $\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)$ *is an equilibrium of the original game, i.e.,* $(z_A^\star, z_B^\star) \in \boldsymbol{F}$ *implies that* $\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) \in \boldsymbol{E}$*.*

Proof. The proof is given in Section 4.4. □

The intuition behind this result is the intuition behind Proposition 1: when one partner, say B, plays a simple strategy, then A does best by also playing a simple strategy. This is because the (ex-ante) probability that B employs a consultant is monotone increasing in the index of the consultant: B employs consultant N with the highest probability, consultant $N - 1$ with the second highest probability, and so on. Since A wants to network with consultants she'll be able to employ with high probability, she does best by networking with the lowest indexed consultants. Thus, when B follows $\tilde{\boldsymbol{s}}_B(N - z_B^\star)$, then the best A can do is play $\tilde{\boldsymbol{s}}_A(n)$ for some $n \in \{0, \ldots, N\}$. Since $(z_A^\star, z_B^\star)$ is an equilibrium point of the Auxiliary Game, we necessarily have that A does best by setting $n = z_A^\star$. Since an analogous argument holds for B, it follows that $\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) \in \boldsymbol{E}$.

Proof of Proposition 1. By Lemma 2, there is an equilibrium $(z_A^\star, z_B^\star)$ of the Auxiliary Game. By Lemma 3, $\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) \in \boldsymbol{E}$. □

Let $\boldsymbol{E}_S$ denote the set of equilibria that are induced by the equilibria of the Auxiliary Game, i.e., let

$$\boldsymbol{E}_S = \{\boldsymbol{s} \in \boldsymbol{S} | \boldsymbol{s} = \tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) \text{ for some } (z_A^\star, z_B^\star) \in \boldsymbol{F}\}.$$

We refer to E_S as the set of **simple equilibria**. Observe that each element of $\boldsymbol{F}$ corresponds to a unique element in $\boldsymbol{E}_S$: $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star, \ldots) \in \boldsymbol{E}_S$ if and only if $(|\mathcal{N}_A^\star|, N - |\mathcal{N}_B^\star|) \in \boldsymbol{F}$. (We'll make heavy use of this fact in Section 5.)

5.3.2 General Property: Minimally Overlapping Networks

In this subsection, our goal is to establish that the partners' equilibrium networks share as few consultants "as possible." The next definition formalizes this idea.

Definition. Let $\mathcal{N}_A$ and $\mathcal{N}_B$ be networks for A and B. We say that $\mathcal{N}_A$ and $\mathcal{N}_B$ are *minimally overlapping* if there does not exists a $\mathcal{N}_A' \subset \mathcal{C}$ and a $\mathcal{N}_B' \subset \mathcal{C}$, such that $|\mathcal{N}_A| = |\mathcal{N}_A'|$, $|\mathcal{N}_B| = |\mathcal{N}_B'|$, and $|\mathcal{N}_A' \cap \mathcal{N}_B'| < |\mathcal{N}_A \cap \mathcal{N}_B|$.

That is, A and B's networks are minimally overlapping if there does not exist a way to shrink the number of consultants in both networks without changing the size of A's network or B's network. For instance, when $\mathcal{C} = \{1, 2, 3, 4\}$, the networks $\mathcal{N}_A = \{1, 2, 3\}$ and $\mathcal{N}_B = \{2, 3\}$ aren't minimally overlapping, while the networks $\mathcal{N}_A = \{1, 2, 3\}$ and $\mathcal{N}_B = \{3, 4\}$ are minimally overlapping.

We'll prove that all equilibrium networks are minimally overlapping when the following technical assumption holds.

Assumption 2. Strictly Positive Costs and Rewards and Heterogeneous Project Difficulty. We have $w > 0$ and, for each $x \in X$, we have $P_i(x)(r_i(x) - w\, d(x)) > 0$ for each partner i. Additionally, for each $n \in \{1, \ldots, N\}$, there is a project $x \in X$ such that $d(x) = n$.

The first part of the assumption gives that labor costs are strictly positive, that the reward to completing a project is strictly positive after accounting for labor costs, and that all projects occur with strictly positive probability. This ensures that each partners' optimal behavior in the second stage is essentially unique. The second part of the assumption requires that projects are of sufficiently heterogeneous difficulty. This guarantees, for instance, that a partner always finds occasion to employ two-thirds of her network.

Proposition 2. Equilibrium Networks are Minimally Overlapping.
Let Assumption 2 hold and let $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star, \ldots) \in \boldsymbol{E}$*, then* $\mathcal{N}_A^\star$ *and* $\mathcal{N}_B^\star$ *are minimally overlapping.*

The proposition implies, for instance, that if A and B hold small networks, i.e., if $|\mathcal{N}_A^\star| + |\mathcal{N}_B^\star| < |\mathcal{C}|$, then their networks are disjoint. The proposition reflects a general preference of the partners to share as few consultants as possible: all else equal, fewer shared consultants means that a partner can employ more of the consultants in her network more often and so do better. The proof of the proposition makes use of this intuition.

We'll prove Proposition 2 using the following lemma.

Lemma 4. Covering.
Let Assumption 2 hold and let $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star, \ldots) \in \boldsymbol{E}$*, then* $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star \neq \emptyset$ *implies that* $\mathcal{C} \subset \mathcal{N}_A^\star \cup \mathcal{N}_B^\star$.

Proof. The proof is given in Section 4.4. □

The intuition is that if $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star \neq \emptyset$ and $\mathcal{C}$ is not in $\mathcal{N}_A^\star \cup \mathcal{N}_B^\star$, then there is at least one consultant who isn't in either partner's network. We show that one partner, say A, can always do strictly better by swapping this consultant for one who she shares with B.[49] Thus, $\boldsymbol{s}^\star$ cannot be an equilibrium, a contradiction. The lemma follows.

Proof of Proposition 2. There are two cases $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star = \emptyset$ and $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star \neq \emptyset$. If $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star = \emptyset$, then $|\mathcal{N}_A^\star \cap \mathcal{N}_B^\star| = 0$ and so it's impossible to find two subsets of $\mathcal{C}$ with a strictly smaller intersection. It follows that $\mathcal{N}_A^\star$ and $\mathcal{N}_B^\star$ are minimally overlapping.

If $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star \neq \emptyset$, we argue by contradiction. Suppose $\mathcal{N}_A^\star$ and $\mathcal{N}_B^\star$ aren't minimally overlapping, then there are $\mathcal{N}_A' \subset \mathcal{C}$ and $\mathcal{N}_B' \subset \mathcal{C}$ with $|\mathcal{N}_A^\star| = |\mathcal{N}_A'|$, $|\mathcal{N}_B^\star| = |\mathcal{N}_B'|$, and $|\mathcal{N}_A' \cap \mathcal{N}_B'| < |\mathcal{N}_A^\star \cap \mathcal{N}_B^\star|$. Since $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star \neq \emptyset$, Lemma 4 gives that $\mathcal{C} \subset \mathcal{N}_A^\star \cup \mathcal{N}_B^\star$, so the sets $\mathcal{N}_A^\star$ and $\mathcal{N}_B^\star \backslash \mathcal{N}_A^\star$ partition $\mathcal{C}$. It follows that $N = |\mathcal{C}| = |\mathcal{N}_A^\star| + |\mathcal{N}_B^\star| - |\mathcal{N}_A^\star \cap \mathcal{N}_B^\star|$, since $|\mathcal{N}_B^\star \backslash \mathcal{N}_A^\star| = |\mathcal{N}_B^\star| - |\mathcal{N}_A^\star \cap \mathcal{N}_B^\star|$. Thus, $N < |\mathcal{N}_A'| + |\mathcal{N}_B'| - |\mathcal{N}_A' \cap \mathcal{N}_B|$. Since $|\mathcal{N}_B' \backslash \mathcal{N}_A'| = |\mathcal{N}_B'| - |\mathcal{N}_A' \cap \mathcal{N}_B'|$, we've $N < |\mathcal{N}_A'| + |\mathcal{N}_B' \backslash \mathcal{N}_A'|$. Since $\mathcal{N}_A'$ and $\mathcal{N}_B' \backslash \mathcal{N}_A'$ are disjoint subsets of $\mathcal{C}$, we necessarily have that $|\mathcal{N}_A'| + |\mathcal{N}_B' \backslash \mathcal{N}_A'| \leq N$. Thus, $N < N$, a contradiction. □

Remark. In the Supplement we show that the conclusion of this proposition (appropriately generalized) holds when there are more than two partners.

[49]This insight depends critically on the heterogeneity assured by Assumption 2. In the Supplement, we give an example where the conclusion of Lemma 4 doesn't apply because heterogeneity fails.

5.3.3 Employment List and Employment Efficient Equilibria

In this subsection we introduce the concepts of employment lists and employment efficiency, we argue that these properties are intuitive refinements, and we show that every equilibrium with these properties is payoff equivalent to a simple equilibrium. We begin by formally defining these properties.

Definition. A $\boldsymbol{s} = (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}$ has an *employment list for partner i* if $\mathcal{N}_i \neq \emptyset$ implies that we can write $\mathcal{N}_i$ as an ordered list $\{j_1, j_2, \ldots, j_{|\mathcal{N}_i|}\}$ such that, for each $x \in X$ with $\sigma_{1i}(\mathcal{N}_i, x) \neq \emptyset$, we have $\sigma_{1i}(\mathcal{N}_i, x) = \{j_1, j_2, \ldots, j_{|\sigma_{1i}(\mathcal{N}_i,x)|}\}$. We say $\boldsymbol{s}$ has *employment lists* if it has employment lists for both A and B.

If a strategy profile has employment lists, then when i gets project x first, then she always employs the first $|\sigma_{1i}(\mathcal{N}_i, x)|$ consultants on her list when she employs anyone. For instance, in the equilibrium in Example 1, B's employment list is $\{2, 1\}$ as she employs consultant 2 to carry out project x_1 and consultants 1 and 2 to carry out project x_2.

Definition. A $\boldsymbol{s} = (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}$ is *employment efficient for partner i* if, for each $x \in X$, we have:
(i) $\sigma_{1i}(\mathcal{N}_i, x) \subset \mathcal{N}_i \backslash \mathcal{N}_{-i}$ when $|\sigma_{1i}(\mathcal{N}_i, x)| \leq |\mathcal{N}_i \backslash \mathcal{N}_{-i}|$, and
(ii) $\mathcal{N}_i \backslash \mathcal{N}_{-i} \subset \sigma_{1i}(\mathcal{N}_i, x)$ when $|\sigma_{1i}(N_i, x)| > |\mathcal{N}_i \backslash \mathcal{N}_{-i}|$.
We say $\boldsymbol{s}$ is *employment efficient* if it's employment efficient for both A and B.

When partner i moves first, she may employ either "exclusive" consultants in $\mathcal{N}_i \backslash \mathcal{N}_{-i}$ or shared consultants in $\mathcal{N}_i \cap \mathcal{N}_{-i}$. Since every shared consultant i employs is one less consultant that $-i$ may use, i reduces $-i$'s payoff by employing shared consultants. Thus, it's inefficient for i to employ shared consultants before she's employed every one of her exclusive consultants. Employment efficiency requires that this inefficiency not occur. Specifically, it requires (i) that if i wants to employ less than $|\mathcal{N}_i \backslash \mathcal{N}_{-i}|$ consultants, then she chooses behavioral strategies that employ only exclusive consultants, and (ii) that i wants to employ more than $|\mathcal{N}_i \backslash \mathcal{N}_{-i}|$ consultants, she chooses behavioral strategies that employ all of her exclusive consultants. An example of such a strategy is the equilibrium given in Example 1.

The next lemma comments on the relationship between simple strategies, employment list strategies, and employment efficient strategies.

Lemma 5. Simple Strategies Have Employment Lists and are Employment Efficient.
The vector of simple strategies $\tilde{\boldsymbol{s}}(n_A, n_B)$ *has employment lists and is employment efficient for all* $(n_A, n_B) \in \{0, \ldots, N\}^2$.

Proof. Obvious and omitted. □

We say that a $\boldsymbol{s}^\star \in \boldsymbol{E}$ is an **employment list and employment efficient equilibrium** (abbreviated ELEE equilibrium) if it has employment lists and is employment efficient. We use $\boldsymbol{E}_{LE}$ to denote the set of ELEE equilibria. In light of Lemma 5, we've $\boldsymbol{E}_S \subset \boldsymbol{E}_{LE}$. In fact, the set of ELEE equilibria is substantially larger than the set of simple equilibria. For instance, it contains all equilibria that differ from a simple equilibrium off of the equilibrium path, as well as all permutations of simple equilibria.

We focus on ELEE equilibria for two reasons. First, employment lists and employment efficiency are natural refinements. Employments lists are reasonable because they captures the intuitive idea that each partner uses an address book or other (mental) list to determine which consultants to employ. Employment efficiency is also natural because it's costless to implement (since both exclusive and shared consultants both cost a partner w) and it weakly increases efficiency. Second, ELEE equilibria are "robust." That is, there's an ELEE equilibrium for every parameterization of our game, whereas there are parameterizations for which there are no non-ELEE equilibria. We give an example of this in the Supplement.[50]

The next proposition characterizes the relationship between ELEE and simple equilibria.

Proposition 3. Payoff Equivalency of ELEE Equilibria and Simple Equilibria.
Let Assumption 2 hold. Then, each ELEE equilibrium is payoff equivalent to a simple equilibrium where both partners hold the same sized networks as they do in the ELEE equilibrium. That is, for each $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star, \ldots) \in \boldsymbol{E}_{LE}$*, we have that (i)* $\tilde{\boldsymbol{s}}(|\mathcal{N}_A^\star|, |\mathcal{N}_B^\star|) \in \boldsymbol{E}_S$ *and (ii) that* $U_i(\boldsymbol{s}^\star) = U_i(\tilde{\boldsymbol{s}}(|\mathcal{N}_A^\star|, |\mathcal{N}_B^\star|))$ *for each partner* i.

Proof. The proof is given in Section 4.4. □

The key insight behind the proposition is that whenever partner i moves first in the second stage and uses an employment list and employment efficient behavioral strategy, then her behavior in $\boldsymbol{s}^\star$ is a permutation of her behavior in the corresponding simple equilibrium. Thus, the second partner to move $-i$ has the same number of in-network consultants available in both equilibria. It follows that both partners get the same payoffs in these equilibria.

5.3.4 Welfare and Network Size, Opposition of Interests

Our goal in this subsection is to establish (i) that there are equilibria where A does best and B does worst and (ii) that, under a strict monotonicity condition, A does best whenever B does worst.

To understand the monotonicity condition, we first have to understand how rivalry affects the partners' payoffs when they play simple strategies. To these ends, let $b_A(n) =$

[50]Unfortunately, we cannot appeal to a Pareto dominance argument to justify the focus on ELEE equilibria. In the Supplement, we give an example of a non-ELEE equilibrium where A does strictly better than in any ELEE equilibria, while B does strictly worse than in any ELEE equilibria.

$\arg\max_{n' \in \{0,\dots,N\}} U_A(\tilde{\boldsymbol{s}}(n', n))$ and $b_B(n) = \arg\max_{n' \in \{0,\dots,N\}} U_B(\tilde{\boldsymbol{s}}(n, n'))$ denote A and B's best responses when playing simple strategies. Let $\overline{b}_i(n) = \max_{\leq} b_i(n)$ denote the maximal selection of partner i's best response. Occasionally, we'll write $\overline{b}_i(n, r_i, c_i)$ when we wish to emphasize the dependence of i's maximal selection on her reward and cost functions.

Lemma 6. Weak Monotonicity.
Each partner's payoff to a simple strategy is weakly decreasing in the size of the other partner's simple strategy when she responds optimally. That is, $U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(n), n))$ *and* $U_B(\tilde{\boldsymbol{s}}(n, \overline{b}_B(n))$ *are weakly decreasing in* n.

Proof. The proof is given in Section 4.4. □

The intuition for this result is that each partner exerts a weakly negative externality on the other by holding a larger network because the use of consultants is rivalrous. More precisely, as the size of B's simple strategy n_B grows, then $U_A(\tilde{\boldsymbol{s}}(n_A, n_B))$ decreases for each n_A since the probability A gets to use shared consultants decreases; the lemma follows.

Our monotonicity condition is that the partners' payoffs to simple strategies evaluated at best responses are *strictly* decreasing in the size of the other partner's network. Formally, we make the following assumption.

Assumption 3. Strict Monotonicity and Single-Valued Best Replies.
We have (i) $b_A(n)$ and $b_B(n)$ are single-valued for all $n \in \{0, \dots, N\}$ and (ii) $U_A(\tilde{\boldsymbol{s}}(b_A(n), n))$ and $U_B(\tilde{\boldsymbol{s}}(n, b_B(n))$ are strictly decreasing in n.

Part (i) is guaranteed if the partners' payoffs to (simple strategies) are strictly quasi-concave or if the partners commit to a selection of their best replies before the game begins. Part (ii) holds whenever both partners find it desirable to hold large networks (e.g., the networking cost is low and the returns to difficult projects are high) as then an increase in the size of her rival's network always increases the number of shared consultants. It's readily verified that the payoffs in Example 1 satisfy both parts of this assumption.

Now we're in the position to give the main result of this subsection. Let $\overline{\boldsymbol{W}}(i) = \{\boldsymbol{s}^\star \in \boldsymbol{E}_{LE} | U_i(\boldsymbol{s}^\star) \geq U_i(\boldsymbol{s}) \text{ for all } \boldsymbol{s} \in \boldsymbol{E}_{LE}\}$ be the set of ELEE equilibria where partner i does best. Also, let $\underline{\boldsymbol{W}}(i) = \{\boldsymbol{s}^\star \in \boldsymbol{E}_{LE} | U_i(\boldsymbol{s}^\star) \leq U_i(\boldsymbol{s}) \text{ for all } \boldsymbol{s} \in \boldsymbol{E}_{LE}\}$ be the set of ELEE equilibria where i does worst. We'll occasionally write $\overline{\boldsymbol{W}}(i, r_i, c_i)$ and $\underline{\boldsymbol{W}}(i, r_i, c_i)$ when we wish to emphasize the dependence of these sets on i's reward and cost functions.

Proposition 4. Opposing Interests.
Let Assumption 2 hold, then there's an ELEE equilibrium where A does best and B does worst and vice-versa, i.e., $\overline{\boldsymbol{W}}(A) \cap \underline{\boldsymbol{W}}(B)$ *and* $\overline{\boldsymbol{W}}(B) \cap \underline{\boldsymbol{W}}(A)$ *are non-empty. When Assumption 3 also holds, then in A does best in any ELEE equilibrium where B does worst and vice versa, i.e.,* $\overline{\boldsymbol{W}}(A) = \underline{\boldsymbol{W}}(B)$ *and* $\overline{\boldsymbol{W}}(B) = \underline{\boldsymbol{W}}(A)$.

The proposition gives that there are equilibria where A and B's **interests are opposed**, i.e., where A does best and B does worst and vice versa. It also tells us that A does best precisely when B does worst when our monotonicity condition holds.

We'll prove the proposition by going to the Auxiliary Game. Since this game is supermodular, it has a largest equilibrium, which A weakly most-prefers and B weakly least-prefers (per Lemma 6), and a smallest equilibrium, which B weakly most-prefers and A weakly least-prefers (per Lemma 6). When Assumption 3 holds, the partners' preferences over equilibria become strict. Thus, A does best (worst) in only the largest (smallest) equilibrium and B does worst (worst) only in the largest (smallest) equilibrium. We establish these facts in Lemma 7 below. We then deduce Proposition 4 from this lemma and Proposition 3.

Recall that the set of equilibria of the Auxiliary Game $\boldsymbol{F}$ is a complete lattice. Let $(\overline{z}_A, \overline{z}_B)$ be the maximal element of $\boldsymbol{F}$ and let $(\underline{z}_A, \underline{z}_B)$ be its minimal element. Let $\overline{\boldsymbol{W}}_{\boldsymbol{F}}(i) = \{(z_A^\star, z_B^\star) \in \boldsymbol{F} | U_i(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_i(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ for all $(z_A, z_B) \in \boldsymbol{F}\}$ denote the set of equilibria in the Auxiliary Game where partner i does best. Let $\underline{\boldsymbol{W}}_{\boldsymbol{F}}(i) = \{(z_A^\star, z_B^\star) \in \boldsymbol{F} | U_i(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \leq U_i(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ for all $(z_A, z_B) \in \boldsymbol{F}\}$ denote the set of all equilibria in the Auxiliary Game where i does worst.

Two preliminary facts will be useful. First, $U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B)) = U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - z_B), N - z_B)$ for all $(z_A, z_B) \in \boldsymbol{F}$ as both z_A and $\overline{b}_A(N - z_B)$ were picked to maximize $U_A(\tilde{\boldsymbol{s}}(n, N - z_B))$ by choice of $n \in \{0, \ldots, N\}$, the former by definition of equilibrium and the latter by construction. Second, $U_B(\tilde{\boldsymbol{s}}(z_A, N - z_B)) = U_B(\tilde{\boldsymbol{s}}(z_A, \overline{b}_B(z_A))$ for all $(z_A, z_B) \in \boldsymbol{F}$ as z_B was picked to maximize $U_B(\tilde{\boldsymbol{s}}(z_A, N - n))$ by choice of $n \in \{0, \ldots, N\}$ and $\overline{b}_B(z_A)$ was picked to maximize $U_B(\tilde{\boldsymbol{s}}(z_A, n))$ by choice of $n \in \{0, \ldots, N\}$.

Lemma 7. Equilibria with Opposing Interests in the Auxiliary Game.
Let $(\overline{z}_A, \overline{z}_B)$ be the maximal element of $\boldsymbol{F}$ and let $(\underline{z}_A, \underline{z}_B)$ be its minimal element. We have (i) $(\overline{z}_A, \overline{z}_B) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$ and (ii) $(\underline{z}_A, \underline{z}_B) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(B) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$. If Assumption 3 also holds, then we have (iii) $\overline{\boldsymbol{W}}_{\boldsymbol{F}}(A) = \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B) = \{(\overline{z}_A, \overline{z}_B)\}$ and (iv) $\overline{\boldsymbol{W}}_{\boldsymbol{F}}(B) = \underline{\boldsymbol{W}}_{\boldsymbol{F}}(A) = \{(\underline{z}_A, \underline{z}_B)\}$.

Proof. In light of Lemma 6, parts (i) and (ii) are almost obvious. But, we prove part (i) for completeness; the argument for part (ii) is analogous. Subsequently, we prove part (iii) since a similar argument gives part (iv).

We begin by establishing part (i). First, we show that $U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) \geq U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ for all $(z_A, z_B) \in \boldsymbol{F}$. Write

$$U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) = U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}_B), N - \overline{z}_B))$$
$$\geq U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - z_B), N - z_B)) = U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B)).$$

The equalities are due to our first fact. The inequality is due to Lemma 6 since $N - \overline{z}_B \leq N - z_B$ by the maximality of $(\overline{z}_A, \overline{z}_B)$. Next we show that $U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) \leq U_A(\tilde{\boldsymbol{s}}(z_A, N - z))$ for all $(z_A, z_B) \in \boldsymbol{F}$. Write

$$U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) = U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, \overline{b}_B(\overline{z}_A))) \leq U_B(\tilde{\boldsymbol{s}}(z_A, \overline{b}_B(z_A))) = U_B(\tilde{\boldsymbol{s}}(z_A, N - z_B)),$$

where the equalities are due to our second fact and the inequality is due to Lemma 6 as $z_A \leq \overline{z}_A$ by maximality.

Now we establish part (iii) by showing that $(\overline{z}_A, \overline{z}_B)$ is the unique element of $\overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$ and that it's also the unique element of $\underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$. To begin, let $(z_A^\star, z_B^\star) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$. Then, for every $(z_A, z_B) \in \boldsymbol{F}$,

$$\begin{aligned} U_A(\tilde{\boldsymbol{s}}(b_A(N - z_B^\star), N - z_B^\star) &= U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) \\ &\geq U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B)) = U_A(\tilde{\boldsymbol{s}}(b_A(N - z_B), N - z_B), \end{aligned}$$

where the equalities are due to our first fact and the inequality is due to $(z_A^\star, z_B^\star) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$. Since $U_A(\tilde{\boldsymbol{s}}(b_A(n), n)$ is strictly decreasing by Assumption 3, this implies $z_B \leq z_B^\star$ for all z_B played in some equilibrium. Thus, $z_B^\star = \overline{z}_B$. Since A's best response is single valued by Assumption 3, we have $z_A^\star = b_A(N - z_B^\star) = b_A(N - \overline{z}_B) = \overline{z}_A$. Hence, $(z_A^\star, z_B^\star) = (\overline{z}_A, \overline{z}_B)$, implying that $\overline{\boldsymbol{W}}_{\boldsymbol{F}}(A) = \{(\overline{z}_A, \overline{z}_B)\}$.

Now consider $\underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$. Let $(z_A^\star, z_B^\star) \in \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$. Then, for every $(z_A, z_B) \in \boldsymbol{F}$,

$$U_B(\tilde{\boldsymbol{s}}(z_A^\star, b_B(z_A^\star)) = U_B(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) \leq U_B(\tilde{\boldsymbol{s}}(z_A, N - z_B)) = U_B(\tilde{\boldsymbol{s}}(z_A, b_B(z_A)).$$

Since $U_A(\tilde{\boldsymbol{s}}(n, b_B(n))$ is strictly decreasing, we have $z_A^\star = \overline{z}_A$. Thus, $z_B^\star = \overline{z}_B$ since B's best reply is unique, implying $(z_A^\star, z_B^\star) = (\overline{z}_A, \overline{z}_B)$. Hence, $\underline{\boldsymbol{W}}_{\boldsymbol{F}}(B) = \{(\overline{z}_A, \overline{z}_B)\}$. □

Two facts will be useful in the proof of Proposition 4. First, if $\boldsymbol{s} = (\mathcal{N}_A, \ldots, \mathcal{N}_B, \ldots) \in \overline{\boldsymbol{W}}(A)$, then $(|\mathcal{N}_A|, N - |\mathcal{N}_B|) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$.[51] Second, if $\boldsymbol{s}^\star = (\mathcal{N}_A, \ldots, \mathcal{N}_B, \ldots) \in \underline{\boldsymbol{W}}(B)$, then $(|\mathcal{N}_A|, N - |\mathcal{N}_B|) \in \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$.[52]

Proof of Proposition 4. We begin by showing that $\overline{\boldsymbol{W}}(A) \cap \underline{\boldsymbol{W}}(B)$ is non-empty. Subsequently, we'll show that $\overline{\boldsymbol{W}}(A) = \underline{\boldsymbol{W}}(B)$ when Assumption 3 holds. The arguments that (i) $\overline{\boldsymbol{W}}(B) \cap \underline{\boldsymbol{W}}(A) \neq \emptyset$ and (ii) that $\overline{\boldsymbol{W}}(B) = \underline{\boldsymbol{W}}(A)$ when Assumption 3 holds are analogous.

[51]To see this, suppose that $\boldsymbol{s} = (\mathcal{N}_A, \ldots, \mathcal{N}_B, \ldots) \in \overline{\boldsymbol{W}}(A)$ and that $(|\mathcal{N}_A|, N - |\mathcal{N}_B|) \notin \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$. Then, $U_A(\boldsymbol{s}^\star) = U_A(\tilde{\boldsymbol{s}}(|\mathcal{N}_A|, |\mathcal{N}_B|)) < U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B))$ by Lemma 7 and Proposition 3, where $(\overline{z}_A, \overline{z}_B)$ is the maximal element of $\boldsymbol{F}$. Since $\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B) \in \boldsymbol{E}_S \subset \boldsymbol{E}_{LE}$ by Lemmas 3 and 5, $U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) \leq U_A(\boldsymbol{s})$. Thus, $U_A(\boldsymbol{s}) < U_A(\boldsymbol{s})$, an impossibility.

[52]The argument for this fact is analogous to the argument for the first fact.

We first establish that $\overline{\boldsymbol{W}}(A) \cap \underline{\boldsymbol{W}}(B)$ is non-empty. Consider $(\overline{z}_A, \overline{z}_B)$. Lemma 7 and the fact that $\boldsymbol{s}' = (\mathcal{N}'_A, \ldots, \mathcal{N}'_B, \ldots) \in \boldsymbol{E}_S$ if and only if $(|\mathcal{N}'_A|, N - |\mathcal{N}'_B|) \in \boldsymbol{F}$ (per the construction of simple equilibria) imply

$$U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) \geq U_A(\boldsymbol{s}') \text{ and } U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) \leq U_B(\boldsymbol{s}') \text{ for all } \boldsymbol{s}' \in \boldsymbol{E}_S.$$

Thus, Proposition 3 implies that $\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B) \in \overline{\boldsymbol{W}}(A) \cap \underline{\boldsymbol{W}}(B)$. (To see this, let $\boldsymbol{s}^\star \in \boldsymbol{E}_{LE}$, then Proposition 3 gives that $U_A(\boldsymbol{s}^\star) = U_A(\boldsymbol{s}')$ and $U_B(\boldsymbol{s}^\star) = U_B(\boldsymbol{s}')$ for an $\boldsymbol{s}' \in \boldsymbol{E}_S$. Thus, $U_A(\boldsymbol{s}^\star) \leq U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B))$ and $U_B(\boldsymbol{s}^\star) \geq U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B))$. Since this holds for all elements of $\boldsymbol{E}_{LE}$, we've $\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B) \in \overline{\boldsymbol{W}}(A)$ and $\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B) \in \underline{\boldsymbol{W}}(B)$.)

Now we establish that $\overline{\boldsymbol{W}}(A) = \underline{\boldsymbol{W}}(B)$ when Assumption 3 holds. First, we show that $\overline{\boldsymbol{W}}(A) \subset \underline{\boldsymbol{W}}(B)$. Let $\boldsymbol{s}^\star = (\mathcal{N}^\star_A, \ldots, \mathcal{N}^\star_B, \ldots) \in \overline{\boldsymbol{W}}(A)$, the our first fact gives $(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$. Thus, Lemma 7 gives that $(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|) \in \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$. The construction of simple equilibria then implies that $U_B(\tilde{\boldsymbol{s}}(|\mathcal{N}^\star_A|, |\mathcal{N}^\star_B|)) \leq U_B(\boldsymbol{s}')$ for all $\boldsymbol{s}' \in \boldsymbol{E}_S$. Thus, Proposition 3 gives that $\boldsymbol{s}^\star \in \underline{\boldsymbol{W}}(B)$ since every other ELEE equilibrium maps to a simple equilibrium with a (weakly) higher payoff. Next, we establish that $\underline{\boldsymbol{W}}(B) \subset \overline{\boldsymbol{W}}(A)$. Let $\boldsymbol{s}^\star = (\mathcal{N}^\star_A, \ldots, \mathcal{N}^\star_B, \ldots) \in \underline{\boldsymbol{W}}(B)$. Our second fact gives $(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|) \in \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$. Lemma 7 then implies that $(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$. Then the construction of simple equilibria implies $U_A(\tilde{\boldsymbol{s}}(|\mathcal{N}^\star_A|, |\mathcal{N}^\star_B|)) \geq U_A(\boldsymbol{s}')$ for all $\boldsymbol{s}' \in \boldsymbol{E}_S$. Proposition 3 then gives that $\boldsymbol{s}^\star \in \overline{\boldsymbol{W}}(A)$. It follows that that $\overline{\boldsymbol{W}}(A) = \underline{\boldsymbol{W}}(B)$. □

5.3.5 Welfare and Network Size, Comparative Statics

In this subsection, we characterize how the partners' payoffs and network sizes shift in ELEE equilibria where their interests are opposed as their reward and cost functions shift. We consider what happens in the event of the following shift.

Assumption 4. Changes in Rewards and Costs.
The cost and reward functions change as follows:
(i) A's reward function r_A increases to r'_A, i.e., $r_A(x) \leq r'_A(x)$ for each project x.
(ii) A's cost function c_A decreases to c'_A, i.e., $c'_A(n) \leq c_A(n)$ for all n.
(iii) B's cost and reward functions don't change.
Also, $c_A(n) - c'_A(n)$ is weakly increasing in n.

The last part of the assumption is a technical requirement. It's satisfied, for instance, when A has constant marginal costs of networking.[53]

Proposition 5. Comparative Statics in Rewards and Costs.

[53]If A's cost decreases from κ_A to κ'_A, then $c_A(n) - c'_A(n) = (\kappa_A - \kappa'_A)n$, which is trivially increasing in n.

Let Assumptions 2 and 4 hold. As A's reward and cost functions shift, then A's payoff increases and B's payoff decreases in the ELEE equilibria that are best for A and worst for B. That is, for $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star) \in \overline{\boldsymbol{W}}(A, r_A, c_A) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$ *and* $\boldsymbol{s}' = (\mathcal{N}_A', \ldots, \mathcal{N}_B') \in \overline{\boldsymbol{W}}(A, r_A', c_A') \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$*, we have that*

$$U_A(\boldsymbol{s}', r_A', c_A') \geq U_A(\boldsymbol{s}^\star, r_A, c_A) \text{ and } U_B(\boldsymbol{s}', r_B, c_B) \leq U_B(\boldsymbol{s}^\star, r_B, c_B).$$

If Assumption 3 also holds both before and after the shifts in rewards and costs, then the size of A's network increases and the size of B's network decreases, i.e., $|\mathcal{N}_A'| \geq |\mathcal{N}_A^\star|$ *and* $|\mathcal{N}_B'| \leq |\mathcal{N}_B^\star|$*. An analogous result holds for the ELEE equilibria that are best for B and worst for A.*

We'll prove this proposition by going to the Auxiliary Game and showing that as A's reward and cost functions shift, her best responses increase – see Lemma 8 below. This increases the size of A's network and decreases the size of B's network in the maximal and minimal equilibria – see Lemma 9 below. Then, Lemma 6 and the shift in rewards and costs imply that A's payoffs in these equilibria increases, while B's payoffs in decreases, while Lemma 7 implies that the size of A's network increases and the size of B's network decreases – see Lemma 10 below. We'll use this last result and Proposition 3 to deduce Proposition 5.

Let $\mathcal{S}$ and $\mathcal{S}'$ be subsets of $\{0, \ldots, N\}^n$, with $n \geq 1$. Recall that $\mathcal{S}$ is less than $\mathcal{S}'$ in the **strong set order**, denoted $\preceq$, if $(\min(n_1, n_1'), \ldots, \min(n_n, n_n')) \in \mathcal{S}$ and $(\max(n_1, n_1'), \ldots, \max(n_n, n_n')) \in \mathcal{S}'$ for all $(n_1, \ldots, n_n) \in \mathcal{S}$ and $(n_1', \ldots, n_n') \in \mathcal{S}'$. For each $z \in \{0, \ldots, N\}$, let $\phi_A(z) = \arg\max_{z' \in \{0,\ldots,N\}} U_A(\tilde{\boldsymbol{s}}(z', N - z))$ and $\phi_B(z) = \arg\max_{z' \in \{0,\ldots,N\}} U_A(\tilde{\boldsymbol{s}}(z, N - z'))$ denote A and B's best responses in the Auxiliary Game.[54] We'll occasionally write $\phi_A(z, r_A, c_A)$ and $\phi_B(z, r_B, c_B)$ when we wish to emphasize the dependence of these best replies on the partners' reward and cost functions.

Lemma 8. Shifts in Best Responses.
Let Assumption 4 hold, then for each $z \in \{0, \ldots, N\}$*, we have* $\phi_A(z, r_A, c_A) \preceq \phi_A(z, r_A', c_A')$*.*

Proof. The proof is given in Section 4.4. □

The key insight of this proof is that when r_A increases and c_A decreases, then A never does worse by expanding the size of her network. (While the intuition is simple, our formal argument needs the technical part of Assumption 4 to deliver the result.) Thus, her best response shifts out. Consequently, A winds up holding a larger network in equilibrium, while B winds up holding a smaller network.

Lemma 9. Comparisons of Extremal Elements.

[54]Notice that $\phi_A(z) = b_A(N - z)$ and that $\phi_B(z) = \{z' \in \{0, \ldots, N\} | N - z' \in b_B(z)\}$.

Let Assumption 4 hold. Let $\boldsymbol{F}$ denote the set of equilibria in the Auxiliary Game before the parameter shift and let $\boldsymbol{F}'$ denote the set of equilibria after the parameter shift. Let $(\overline{z}_A, \overline{z}_B)$ and $(\underline{z}_A, \underline{z}_B)$ be the maximal and minimal elements of $\boldsymbol{F}$ and let $(\overline{z}'_A, \overline{z}'_B)$ and $(\underline{z}'_A, \underline{z}'_B)$ be the maximal and minimal elements of $\boldsymbol{F}'$. Then $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ and $(\underline{z}_A, \underline{z}_B) \leq (\underline{z}'_A, \underline{z}'_B)$.

Proof. The proof is given in Section 4.4. □

Lemma 10. Comparative Statics in the Auxiliary Game.
Let Assumption 4 hold. For all $(z^\star_A, z^\star_B) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A, r_A, c_A) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B, r_B, c_B)$ and all $(z'_A, z'_B) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A, r'_A, c'_A) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B, r_B, c_B)$, we have that

$$U_A(\tilde{\boldsymbol{s}}(z'_A, N - z'_B), r'_A, c'_A) \geq U_A(\tilde{\boldsymbol{s}}(z^\star_A, N - z^\star_B), r_A, c_A) \text{ and} \tag{5.1}$$

$$U_B(\tilde{\boldsymbol{s}}(z'_A, N - z'_B), r_B, c_B) \leq U_B(\tilde{\boldsymbol{s}}(z^\star_A, N - z^\star_B), r_B, c_B). \tag{5.2}$$

If Assumption 3 also holds both before and after the shifts in rewards and costs, then $(z^\star_A, z^\star_B) \leq (z'_A, z'_B)$. Analogous result hold for equilibria were B does best and A does worst.

Proof. We'll only establish the lemma for the equilibria of the Auxiliary Game where A does best and B does worst. The argument for all equilibria where B does best and A does worst is analogous. We begin by proving that equation (5.1) is true. Subsequently, we'll prove the result for network sizes is true.

Let $(\overline{z}_A, \overline{z}_B)$ and $(\overline{z}'_A, \overline{z}'_B)$ be as in the statement of Lemma 9. To simplify notation, for all $(n_A, n_B) \in \{0, \ldots, N\}^2$, let $U_i(n_A, n_B)$ denote $U_i(\tilde{\boldsymbol{s}}(n_A, N - n_B), r_i, c_i)$ for each partner i and let $U'_A(n_A, n_B)$ denote $U_A(\tilde{\boldsymbol{s}}(n_A, N - n_B), r'_A, c'_A)$. Lemma 7 implies that $U_A(z^\star_A, z^\star_B) = U_A(\overline{z}_A, \overline{z}_B)$, that $U_B(z^\star_A, z^\star_B) = U_B(\overline{z}_A, \overline{z}_B)$, that $U'_A(z'_A, z'_B) = U'_A(\overline{z}'_A, \overline{z}'_B)$, that $U_B(z'_A, z'_B) = U_B(\overline{z}'_A, \overline{z}'_B)$. Thus, we only need to show that

$$U'_A(\overline{z}'_A \overline{z}'_B) \geq U_A(\overline{z}_A, \overline{z}_B) \text{ and } U_B(\overline{z}'_A \overline{z}'_B) \leq U_B(\overline{z}_A, \overline{z}_B) \tag{5.3}$$

to establish equations (5.1) and (5.2).

Let's prove (5.3) for A. Let $\overline{b}_A(n)$ denote $\overline{b}_A(n, r_A, c_A)$ and let $\overline{b}'_A(n)$ denote $\overline{b}_A(n, r'_A, c'_A)$. Write

$$\begin{aligned} U_A(\overline{z}_A, \overline{z}_B) &= U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}_B), N - \overline{z}_B), r_A, c_A) \\ &\leq U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}'_B), N - \overline{z}'_B), r_A, c_A) \\ &\leq U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}'_B), N - \overline{z}'_B), r'_A, c'_A) \\ &\leq U_A(\tilde{\boldsymbol{s}}(\overline{b}'_A(N - \overline{z}'_B), N - \overline{z}'_B), r'_A, c'_A) \\ &= U'_A(\overline{z}'_A, \overline{z}'_B) \end{aligned}$$

The first line is standard – see the facts before Lemma 7. Since $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ by Lemma 9, the second line follows from Lemma 6 as $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ implies that $N - \overline{z}'_B \leq N - \overline{z}_B$. The third line follows from the fact A's costs fall and her rewards increase. The fourth line follows from the optimality of $\overline{b}'_A(\cdot)$. The fifth line is standard. Since the argument for B is analogous, equation (5.3) holds.

It remains to show that the size of A's network increases and that the size of B's network decreases. Since Assumption 3 holds, Lemma 7 implies $(z^\star_A, z^\star_B) = (\overline{z}_A, \overline{z}_B)$ and that $(z'_A, z'_B) = (\overline{z}'_A, \overline{z}'_B)$. Thus, the desired result follows directly from Lemma 9. □

Proof of Proposition 5. We'll only establish the result for ELEE equilibria where A does best and B does worst. The argument for ELEE equilibria where B does best and A does worst is analogous.

Let $\boldsymbol{s}^\star = (\mathcal{N}^\star_A, \ldots, \mathcal{N}^\star_B, \ldots) \in \overline{\boldsymbol{W}}(A, r_A, c_A) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$ and let $\boldsymbol{s}' = (\mathcal{N}'_A, \ldots, \mathcal{N}'_B, \ldots) \in \overline{\boldsymbol{W}}(A, r'_A, c'_A) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$. The two facts before the Proof of Proposition 4 imply that $(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A, r_A, c_A) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B, r_B, c_B)$ and that $(|\mathcal{N}'_A|, N - |\mathcal{N}'_B|) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A, r'_A, c'_A) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B, r_B, c_B)$. Thus, Lemma 10 gives

$$\begin{aligned} U_A(\tilde{\boldsymbol{s}}(|\mathcal{N}'_A|, N - |\mathcal{N}'_B|), r'_A, c'_A) &\geq U_A(\tilde{\boldsymbol{s}}(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|), r_A, c_A) \text{ and} \\ U_B(\tilde{\boldsymbol{s}}(|\mathcal{N}'_A|, N - |\mathcal{N}'_B|), r_B, c_B) &\leq U_B(\tilde{\boldsymbol{s}}(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|), r_B, c_B). \end{aligned}$$

Since $U_i(\boldsymbol{s}^\star) = U_i(\tilde{\boldsymbol{s}}(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|))$ and $U_i(\boldsymbol{s}') = U_i(\tilde{\boldsymbol{s}}(|\mathcal{N}'_A|, N - |\mathcal{N}'_B|))$ for each partner i by Proposition 3, $U_A(\boldsymbol{s}', r'_A, c'_A) \geq U_A(\boldsymbol{s}^\star, r_A, c_A)$ and $U_B(\boldsymbol{s}', r_B, c_B) \leq U_B(\boldsymbol{s}^\star, r_B, c_B)$.

If Assumption 3 also holds, then we have $(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|) \leq (|\mathcal{N}'_A|, N - |\mathcal{N}'_B|)$ by Lemma 11. Thus, $|\mathcal{N}^\star_A| \leq |\mathcal{N}'_A|$ and $|\mathcal{N}^\star_B| \geq |\mathcal{N}'_B|$. □

5.3.6 Welfare and Network Size, Comparison of Payoffs and Network Sizes

In this subsection, we show that A has a larger network and payoff than B in any ELEE equilibrium that's best for her and worst for B whenever she has a reward or cost advantage, i.e., when the next assumption holds.

Assumption 5. Partners' Costs and Rewards.
For each $x \in X$, we have that $P_A(x) = P_B(x)$ and $r_A(x) \geq r_B(x)$. For all n, we also have (i) $c_A(n) \leq c_B(n)$ and (ii) $c_B(n) - c_A(n)$ is weakly increasing.

Proposition 6. Comparison of Payoffs and Network Sizes.
Let Assumptions 2 and 5 hold, then A earns more than B in any ELEE equilibrium where she does best and B does worst, i.e., $U_A(\boldsymbol{s}^\star) \geq U_B(\boldsymbol{s}^\star)$ for all $\boldsymbol{s}^\star = (\mathcal{N}^\star_A, \ldots, \mathcal{N}^\star_B, \ldots) \in$

$\overline{\boldsymbol{W}}(A) \cap \underline{\boldsymbol{W}}(B)$. *If Assumption 3 also holds, then A has a larger network than B, i.e.,* $|\mathcal{N}_A^\star| \geq |\mathcal{N}_B^\star|$.

Outside of these equilibria, however, A needn't have a higher payoff or a larger network than B – we give an example of this in the Supplement. We'll establish the proposition by first showing that when A has B's reward and cost functions, then A does weakly better than B and holds a weakly larger network than B in the maximal equilibrium of the Auxiliary Game – see the next lemma. Then we'll prove Proposition 6 by applying Proposition 5.

Lemma 11. An Intermediate Result.
Let Assumptions 2 and 5 hold, then, $U_A(\boldsymbol{s}^\star, r_B, r_B) \geq U_B(\boldsymbol{s}^\star, r_B, r_B)$ *for all* $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star, \ldots) \in \overline{\boldsymbol{W}}(A, r_B, c_B) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$. *If Assumption 3 also holds, then* $|\mathcal{N}_A^\star| \geq |\mathcal{N}_B^\star|$.

The intuition is that, under the hypothesis of the lemma, A holds a larger network than B in the largest equilibrium of the Auxiliary Game because of supermodularity/rivalry. Thus, Lemma 6 implies that A makes more than B. The lemma then follows from Proposition 3 when coupled with Lemma 7.

Proof. We'll first establish that A does better than B. Subsequently, we'll establish that A's network is weakly larger than B's network. Let $\boldsymbol{F}$ denote the equilibrium set of the Auxiliary Game when A has B's reward and cost function and let $(\overline{z}_A, \overline{z}_B)$ be it's maximal element.

We need a preliminary fact, that $N - \overline{z}_B \leq \overline{z}_A$. To see this, recall that $\phi_A(z) = b_A(N - z)$ and that $\phi_B(z) = \{z' \in \{0, \ldots, N\} | N - z' \in b_B(z)\}$. Since $\overline{z}_A \in \phi_A(\overline{z}_B)$ and $\overline{z}_B \in \phi_B(\overline{z}_A)$, we have $\overline{z}_A \in b_A(N - \overline{z}_B)$ and $N - \overline{z}_B \in b_B(\overline{z}_A)$. Since $b_A = b_B$, it follows that $N - \overline{z}_B \in b_A(\overline{z}_A)$ and $\overline{z}_A \in b_B(N - \overline{z}_B)$. Hence, $N - \overline{z}_B \in \phi_A(N - \overline{z}_A)$ and that $N - \overline{z}_A \in \phi_B(N - \overline{z}_B)$. So $(N - \overline{z}_B, N - \overline{z}_A) \in \boldsymbol{F}$, implying $(N - \overline{z}_B, N - \overline{z}_A) \leq (\overline{z}_A, \overline{z}_B)$, which gives $N - \overline{z}_B \leq \overline{z}_A$.

We first establish that A does better than B. Since both partners' have the same costs and rewards, $U_A(\tilde{\boldsymbol{s}}(n_A, n_B), r_B, c_B) = U_B(\tilde{\boldsymbol{s}}(n_B, n_A), r_B, c_B)$, i.e., the game is symmetric; implying that both partners have the same best response, i.e., that $b_A(n) = b_B(n)$. Consequently,

$$\begin{aligned} U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B), r_B, c_B) &= U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}_B), N - \overline{z}_B), r_B, c_B) \\ &\geq U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(\overline{z}_A), \overline{z}_A), r_B, c_B) \\ &= U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, \overline{b}_B(\overline{z}_A)), r_B, c_B) \\ &= U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B), r_B, c_B). \end{aligned}$$

The first line is standard – see the facts before Lemma 7. The second line is due to Lemma 6 and the fact that $N - \overline{z}_B \leq \overline{z}_A$. The third line is due to the symmetry of the game. The last line is standard. Since $\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B) \in \overline{\boldsymbol{W}}(A, r_B, c_B) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$ by Proposition 3 (the argument is analogous to the Proof of Proposition 4), we have the desired result.

It remains to show that $|\mathcal{N}_A^\star| \geq |\mathcal{N}_B^\star|$. By the preliminary fact of the Proof of Proposition 6, $(|\mathcal{N}_A^\star|, N - |\mathcal{N}_B^\star|) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A, r_B, c_B) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B, r_B, c_B)$. Thus, Lemma 7 gives $(|\mathcal{N}_A^\star|, N - |\mathcal{N}_B^\star|) = (\overline{z}_A, \overline{z}_B)$ as Assumption 3 holds. The desired result now follows from the preliminary fact. □

Proof of Proposition 6. This is almost obvious. Since $r_A \geq r_B$, $c_A \leq c_B$, and $c_B - c_A$ is weakly increasing, Proposition 5 gives that $U_A(\boldsymbol{s}', r_A, c_A) \geq U_A(\boldsymbol{s}^\star, r_B, c_B)$ and $U_B(\boldsymbol{s}', r_B, c_B) \leq U_B(\boldsymbol{s}^\star, r_B, c_B)$ and that $|\mathcal{N}_A'| \geq |\mathcal{N}_A^\star|$ and $|\mathcal{N}_B'| \leq |\mathcal{N}_B^\star|$ for all $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star, \ldots) \in \overline{\boldsymbol{W}}(A, r_B, c_B) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$ and $\boldsymbol{s}' = (\mathcal{N}_A', \ldots, \mathcal{N}_B', \ldots) \in \overline{\boldsymbol{W}}(A, r_A, c_A) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$.

We've been a bit loose in the use of Assumption 3; strictly speaking, we haven't verified that it holds in the counterfactual where A has B's reward and cost function. To see that it does hold, recall that $U_B(\tilde{\boldsymbol{s}}(b_B(n, r_B, c_B), n), r_B, c_B)$ is strictly decreasing in n by hypothesis. Since

$$U_A(\tilde{\boldsymbol{s}}(b_A(n, r_B, c_B), n), r_B, c_B) = U_B(\tilde{\boldsymbol{s}}(b_B(n, r_B, c_B), n), r_B, c_B),$$

we have that A's optimal payoff is also strictly decreasing in n when she has B's reward and cost functions. □

5.3.7 Efficiency of Equilibria

In this section we show, via an example, that equilibria may be ex-ante inefficient. We also discuss why this is so and prove that over-investment is the "usual culprit." The next definition is useful.

Definition. We say that an $\boldsymbol{s} \in \boldsymbol{S}$ is *efficient* if it maximizes ex-ante surplus, i.e., if $\boldsymbol{s}$ solves

$$\max_{\boldsymbol{s}' \in \boldsymbol{S}} U_A(\boldsymbol{s}') + U_B(\boldsymbol{s}').$$

Since $\boldsymbol{S}$ is finite, there's always an efficient strategy. However, this efficient strategy need not be an equilibrium. The next example illustrates.

Example 2. Inefficient Equilibria.

Let $\mathcal{C} = \{1, 2\}$ be the set of consultants and let $X = \{x_1\}$ be the set of projects. Let $d(x_1) = 2$ be the difficulty of the project, let $r_A(x_1) = 25$, and let $r_B(x_1) = 5$. Let $w = 1$ and suppose both partners have a constant marginal networking cost of $1/2$. Also, let $P_i(x_1) = 1$ for each partner i.

In the unique equilibrium path outcome, A and B to hold network $\{1, 2\}$, employ both consultants to complete the project when they move first in the second stage, and employ no consultants when they move second. Thus, ex-post, the second partner wastes resources networking with consultants she can't employ. This is undesirable from a social standpoint. Instead, because of A's high reward function, it's uniquely efficient to have A hold the network $\{1, 2\}$ and employ both consultants whenever she moves, and to have B hold the empty network. It follows that no equilibrium is efficient. $\triangle$

To see why, exactly, an efficient strategy can't be an equilibrium in the example, suppose both players initially follow an efficient strategy. Then A earns 22 and B earns 0. Suppose B adds consultants 1 and 2 to her network. She incurs a cost of 1, but now can complete project x_1 whenever she moves before A, so her expected payoff is $3/2 - 1 = 1/2$. Thus, B has incentive to defect from $\boldsymbol{s}'$. Specifically, she has an incentive to **over-invest** in her network, i.e., form a larger network that is socially efficient. In over-investing, B increases the total cost of networking from 1 to 2 and so diminishes social welfare. Additionally, B also exerts a negative externality on A: she reduces the probability that A completes her high value project from 1 to $1/2$. This causes A's contribution to welfare to fall to $\frac{23}{2} - 1 = 10.5$, which further reduces welfare.

In light of this, it's natural to wonder if over-investment is the reason efficient strategies fail to be equilibria. The next result establishes that, at least for simple strategies, it is. Let $\boldsymbol{S}_S$ be the set of simple strategies, i.e.,

$$\boldsymbol{S}_S = \{\boldsymbol{s} \in \boldsymbol{S} | \boldsymbol{s} = \tilde{\boldsymbol{s}}(n_A, n_B) \text{ for some } (n_A, n_B) \in \{0, \ldots, N\}^2\}.$$

Proposition 7. Efficient Simple Strategies and Over-Investment.
Let $\boldsymbol{s} = (\boldsymbol{s}_A, \boldsymbol{s}_B) = (\mathcal{N}_A, \ldots, \mathcal{N}_B, \ldots) \in \boldsymbol{S}_S$ be efficient. If $\boldsymbol{s}$ is not an equilibrium, then one of the two partners strictly benefits from over-investing in her network. That is, $\boldsymbol{s} \notin \boldsymbol{E}$ implies that either (i) a $n \in \{|\mathcal{N}_A| + 1, \ldots, N\}$ such that $U_A(\tilde{\boldsymbol{s}}_A(n), \boldsymbol{s}_B) > U_A(\boldsymbol{s})$ or (ii) a $n \in \{|\mathcal{N}_B| + 1, \ldots, N\}$ such that $U_B(\boldsymbol{s}_A, \tilde{\boldsymbol{s}}_B(n)) > U_B(\boldsymbol{s})$.

Proof. The proof is given in Section 4.8 as a corollary to a more general result which we omit for simplicity since it's statement requires several intermediate concepts that are introduced in Section 4.8. $\square$

The key insights of the proof are (i) that simple strategies are best replies to simple strategies and (ii) that, when both partners play simple strategies, then one partner does better when the other decreases the size of her network (per Lemma 6). Thus, if one partner, say A, defects from an efficient simple strategy, she does best by defecting to another simple strategy. If A plays a simple strategy where her network is smaller than in the efficient simple strategy, then B does better and social welfare improves. Since this is impossible as the original strategy is efficient, we necessarily have that A defects to a simple strategy where she plays a larger network.

5.3.8 Extension, Asymmetric Probabilities of Moving First

So far, we've assumed that both partners have an equal chance of moving first in the second stage. However, it's sometimes the case that one partner is a better salesperson than the other and, thus, is capable of obtaining projects more quickly. Naturally, this partner naturally has a greater chance of moving first in the second stage. In this section, we model this by allowing one partner, say A, to move first more frequently than B, and we characterize how changes in this probability effect the partners' payoffs and network sizes.[55]

Environment

Let $\alpha \in [0, 1]$ be the probability that A gets her project first in the second stage, so $1-\alpha$ is the probability B gets her project first. Thus, for each strategy vector $\boldsymbol{s}$, A's and B's ex-ante payoffs are

$$U_A(\boldsymbol{s}, \alpha) = \sum_{(x_A, x_B)\in X^2} (\alpha\, u_{1A}(\boldsymbol{s}, x_A, x_B) + (1-\alpha)u_{2A}(\boldsymbol{s}, x_A, x_B))P_A(x_A)P_B(x_B)$$

$$U_B(\boldsymbol{s}, \alpha) = \sum_{(x_A, x_B)\in X^2} ((1-\alpha)u_{1B}(\boldsymbol{s}, x_B, x_A) + \alpha\, u_{2B}(\boldsymbol{s}, x_B, x_A))P_A(x_A)P_B(x_B).$$

It's readily verified that, for each value of α, our all of our existing results hold – an equilibrium exists, all equilibria are minimally overlapping, A's and B' equilibrium interests are opposed, and so on.

Results

We write $\overline{\boldsymbol{W}}(i, \alpha)$ and $\underline{\boldsymbol{W}}(i, \alpha)$ to emphasize the dependence of the sets of best and worst equilibria for partner i on the probability that A moves first in the second stage.

Proposition 8. Comparative Statics in α.
Let Assumption 2 hold. As the probability that A moves first increases from α to α', A's payoff increases and B payoff decreases in the ELEE equilibria that are best for A and worst for B. That is, for $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star) \in \overline{\boldsymbol{W}}(A, \alpha) \cap \underline{\boldsymbol{W}}(B, \alpha)$ and $\boldsymbol{s}' = (\mathcal{N}_A', \ldots, \mathcal{N}_B') \in \overline{\boldsymbol{W}}(A, \alpha') \cap \underline{\boldsymbol{W}}(B, \alpha')$, we have

$$U_A(\boldsymbol{s}', \alpha') \geq U_A(\boldsymbol{s}^\star, \alpha) \text{ and } U_B(\boldsymbol{s}', \alpha') \leq U_B(\boldsymbol{s}^\star, \alpha).$$

If Assumption 3 also holds both before and after the shifts in α, then the size of A's network increases and the size of B's network decreases, i.e., $|\mathcal{N}_A'| \geq |\mathcal{N}_A^\star|$ and $|\mathcal{N}_B'| \leq |\mathcal{N}_B^\star|$. An analogous result holds for the ELEE equilibria that are best for B and worst for A.

[55] As previously mentioned, we consider three additional extensions in the Supplement that examine multiple partners, a production technology with residual value, and heterogeneous labor costs.

The intuition for this result is that it's better to move first in the second stage because then one is not subject to less rivalry. Thus, an increase in α increases A and B's best responses in the Auxiliary Game – see Lemma 12 below. This, in turn, increases the size of A's network and decreases the size of B's network in the maximal and minimal equilibria – see Lemma 13 below. Then, Lemma 6 implies that A's payoffs in these equilibria increases and that B's payoffs in decreases, while Lemma 7 implies that the size of A's network increases and the size of B's network decreases – see Lemma 14 below. This last result along with Proposition 3 imply Proposition 8. We defer the proof to give the following corollary.

Corollary 1. Comparison of Payoffs and Network Sizes.
Let Assumptions 2 and 5 hold, and let $\alpha' \geq 1/2$, then A earns more than B in any ELEE equilibrium where she does best and B does worst, i.e., $U_A(\boldsymbol{s}^\star, \alpha') \geq U_B(\boldsymbol{s}^\star, \alpha')$ for all $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star) \in \overline{\boldsymbol{W}}(A, \alpha') \cap \underline{\boldsymbol{W}}(B, \alpha')$. If Assumption 3 also holds when A moves first with probability $1/2$ and with probability α', then A holds a larger network than B, i.e., $|\mathcal{N}_A^\star| \geq |\mathcal{N}_B^\star|$.

Proof. Set $\alpha = 1/2$, then Proposition 6 gives that $U_A(\boldsymbol{s}^\star, 1/2) \geq U_B(\boldsymbol{s}^\star, 1/2)$ for all $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star) \in \overline{\boldsymbol{W}}(A, 1/2) \cap \underline{\boldsymbol{W}}(B, 1/2)$ and that $|\mathcal{N}_A^\star| \geq |\mathcal{N}_B^\star|$. Applying Proposition 8 gives that $U_A(\boldsymbol{s}', \alpha') \geq U_B(\boldsymbol{s}', \alpha')$ and that $|\mathcal{N}_A'| \geq |\mathcal{N}_B'|$ for all $\boldsymbol{s}' = (\mathcal{N}_A', \ldots, \mathcal{N}_B') \in \overline{\boldsymbol{W}}(A, \alpha') \cap \underline{\boldsymbol{W}}(B, \alpha')$. $\square$

When $\alpha' < 1/2$, however, this result need not be true. For instance, if $\alpha = 0$, then A may optimally choose the empty network and earn zero, while B earns a strictly positive amount, because the parameters are such that B optimally employs all of the consultants when she moves first.

We'll prove Proposition 8 by formalizing the intuition we sketched above. We write $\phi_A(z, \alpha)$ and $\phi_B(z, \alpha)$ to emphasize the dependence of A's and B's best replies in the Auxiliary Game on the probability A moves first.

Lemma 12. Shifts in Best Responses.
Suppose α increase to α', then for each $z \in \{0, \ldots, N\}$, we've $\phi_A(z, \alpha) \preceq \phi_A(z, \alpha')$ and $\phi_B(z, \alpha) \preceq \phi_B(z, \alpha')$.

Proof. The proof is given in Section 4.4. $\square$

Lemma 13. Comparisons of Extremal Elements.
Let $\boldsymbol{F}$ denote the set of equilibria in the Auxiliary Game at α and let $\boldsymbol{F}'$ denote the set of equilibria at α', where $\alpha \leq \alpha'$. Let $(\overline{z}_A, \overline{z}_B)$ and $(\underline{z}_A, \underline{z}_B)$ be the maximal and minimal elements of $\boldsymbol{F}$ and let $(\overline{z}'_A, \overline{z}'_B)$ and $(\underline{z}'_A, \underline{z}'_B)$ be the maximal and minimal elements of $\boldsymbol{F}'$. Then $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ and $(\underline{z}_A, \underline{z}_B) \leq (\underline{z}'_A, \underline{z}'_B)$.

Proof. Analogous to the Proof of Lemma 9 and omitted. □

Lemma 14. Welfare Comparative Statics in the Auxiliary Game.
Let $\alpha \leq \alpha'$. For all $(z_A^\star, z_B^\star) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A,\alpha) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B,\alpha)$ and all $(z_A', z_B') \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A,\alpha') \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B,\alpha')$, we have that

$$U_A(\tilde{\boldsymbol{s}}(z_A', N - z_B'), \alpha') \geq U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star), \alpha) \text{ and}$$
$$U_B(\tilde{\boldsymbol{s}}(z_A', N - z_B'), \alpha') \leq U_B(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star), \alpha).$$

If Assumption 3 also holds both before and after the shift in α, then $(z_A^\star, z_B^\star) \leq (z_A', z_B')$. Analogous result hold for equilibria were B does best and A does worst.

Proof. The proof is given in Section 4.4. □

Proof of Proposition 8. In light of Lemma 14, the proof is analogous to the Proof of Proposition 5 and, thus, is omitted. □

5.4 Omitted Proofs

In this section we give the proofs that were omitted from the main text.

Proof of Lemma 1

To establish Lemma 1, we need the following result.

Lemma A1. Payoffs when Partners Play Simple Strategies.
Let $i \in \{A, B\}$ and let $(n_i, n_{-i}) \in \{0, \dots, N\}^2$. Then,

$$U_i(\tilde{\boldsymbol{s}}(n_i, n_{-i})) = \sum_{\{x|d(x)\leq n_i\}} (r_i(x) - w\, d(x)) \frac{P_i(x)}{2} + \sum_{l=1}^{n_i} \sum_{\{x|d(x)=l\}} (r_i(x) - w\, d(x)) g_i(l, n_{-i}) \frac{P_i(x)}{2} - c_i(n_i)$$

where, for $l \in \{1, \dots, N\}$,

$$g_i(l, n_{-i}) = 1 - \sum_{\{x|N+1-l\leq d(x)\leq n_{-i}\}} P_{-i}(x).$$

Proof. This result is purely computational. We perform the computation for A as the it's analogous for B. The claim is true if $n_A = 0$. Thus, we take $n_A \geq 1$.

We begin by computing A's payoff when she gets her project first. It's useful to think about the difficulty of A's project x_A. If $d(x_A) \leq n_A$, then A employs $\tilde{\sigma}_{1A} = \{1, \dots, d(x_A)\}$, completes her project, and earns $r_A(x_A) - w\, d(x_A)$ (before networking costs). If $d(x_A) > n_A$, then A employs $\tilde{\sigma}_{1A} = \emptyset$, does not complete her project, and earns 0. Thus, before accounting for network costs, A's expected payoff from getting her project first are

$$\sum_{\{x|d(x)\leq n_A\}} (r_A(x) - w\, d(x))P_A(x).$$

Next, we compute A's payoff when she gets her project second. Let x_A and x_B be A and B's projects. We need to consider the possibility that A can't complete her project. Let $l = d(x_A)$. Since B follows her simple strategy there are four sub-cases: (i) $d(x_B) > n_B$ and $l > n_A$, (ii) $d(x_B) \leq n_B$ and $l > n_A$, (iii) $d(x_B) > n_B$ and $l \leq n_A$, and (iv) $d(x_B) \leq n_B$ and $l \leq n_A$. If cases (i) or (ii), then $\tilde{\sigma}_{2A} = \emptyset$ and so A earns nothing.

If case (iii), then $\tilde{\sigma}_{2A} = \{1, \ldots, l\}$, A completes her project, and earns $r_A(x_A) - w\, d(x_A)$ (before networking costs). Since A follows her simple strategy, she completes her project if and only if B doesn't employ a consultant with an index at or below l.[56] Since B employs no consultants, the desired result follows.

If case (iv), then A might not complete her project because B might employ consultants with indices at or below l. Since B follows her simple strategy, she'll employ a consultant with an index at or below l if an only if she gets a project x_B with $d(x_B) \geq N + 1 - l$.[57] Thus, if $d(x_B) \geq N + 1 - l$, A does not complete her project and earns 0. However, if $d(x_B) < N + 1 - l$ or $d(x_B) > n_B$, A completes her project and earns $r_A(x_A) - w\, d(x_A)$.

It follows that, A's expected payoff from getting project x_A second are:
(1) 0 if $d(x_A) > n_A$, and
(2) $(r_A(x_A) - w\, d(x_A))(1 - \sum_{\{x|N+1-l\leq d(x)\leq n_B\}} P_B(x))$ if $d(x_A) \leq n_A$.
The first follows directly from sub-cases (i) and (ii). The second follows from sub-cases (iii) and (iv), where A gets $r_A(x_A) - w\, d(x_A)$ if either $d(x_B) < N+1-l$ or $d(x_B) > n_B$, an event of probability $1 - \sum_{\{x|N+1-l\leq d(x)\leq n_B\}} P_B(x) = g_A(l, n_B)$. Thus, before networking costs, A's expected payoff from getting her project second is

$$\sum_{\{x|d(x)\leq n_A\}} (r_A(x) - w\, d(x))g_A(d(x), n_B)P_A(x).$$

Since A has a 1/2 chance of getting her project first and a 1/2 chance of getting her project

[56] To see this, suppose B employs a set of consultants $\mathcal{T}$ and recall that A's network $\tilde{\mathcal{N}}_A = \{1, \ldots, n_A\}$ under $\tilde{\mathbf{s}}(n_A, n_B)$. If $\mathcal{T} \not\subset \{l+1, \ldots, N\}$, then $\tilde{\sigma}_{2A} = \emptyset$ (by construction) and A fails to complete her project. If, however, $\mathcal{T} \subset \{l+1, \ldots, N\}$, then $|\tilde{\mathcal{N}}_A \backslash \mathcal{T}| \geq l$, and so $\tilde{\sigma}_{2A} = \{1, \ldots, l\}$, implying A completes her project.

[57] Recall that B employs $\{N+1-d(x_B), \ldots, N\}$. If $d(x_B) < N+1-l$, then B doesn't employ a with an index at or below l as $d(x_B) < N+1-l \implies N+1-d(x_B) > l$. If, however, $d(x_B) \geq N+1-l$, then B employs a consultant with an index at or below l as $d(x_B) \geq N+1-l \implies N+1-d(x_B) \leq l$.

second, her expected payoff inclusive of networking costs are

$$\sum_{\{x|d(x)\le n_A\}} (r_A(x) - w\,d(x))\frac{P_A(x)}{2} + \sum_{\{x|d(x)\le n_A\}} (r_A(x) - w\,d(x))g_A(d(x_A), n_B)\frac{P_A(x)}{2} - c_A(n_A).$$

Reordering the second sum shows that this expression is equivalent to the one given in the statement of the lemma. □

Proof of Lemma 1. This result is largely computational. We prove it for A as the argument for B is analogous. Let n_A, n_B, and n'_B be in $\{0, \ldots, N\}$ with $n'_B \ge n_B$. We first establish that the difference $U_A(\tilde{\boldsymbol{s}}(n_A, n_B)) - U_A(\tilde{\boldsymbol{s}}(n_A, n'_B))$ is weakly increasing in n_A, for all $n_A \in \{0, \ldots, N\}$. Given this, for $n'_A \ge n_A$ with $n'_A \in \{0, \ldots, N\}$,

$$U_A(\tilde{\boldsymbol{s}}(n'_A, n_B)) - U_A(\tilde{\boldsymbol{s}}(n'_A, n'_B)) \ge U_A(\tilde{\boldsymbol{s}}(n_A, n_B)) - U_A(\tilde{\boldsymbol{s}}(n_A, n'_B))$$
$$U_A(\tilde{\boldsymbol{s}}(n'_A, n_B)) + U_A(\tilde{\boldsymbol{s}}(n_A, n'_B)) \ge U_A(\tilde{\boldsymbol{s}}(n'_A, n'_B)) + U_A(\tilde{\boldsymbol{s}}(n_A, n_B)).$$

That is, $U_A(\tilde{\boldsymbol{s}}(n_A, n_B))$ is submodular in (n_A, n_B). It follows that $U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ is supermodular in $(z_A, z_B) \in \{0, \ldots, N\}^2$.

It remains to establish that $U_A(\tilde{\boldsymbol{s}}(n_A, n_B)) - U_A(\tilde{\boldsymbol{s}}(n_A, n'_B))$ is weakly increasing in n_A. To these ends, use Lemma A1 to write

$$U_A(\tilde{\boldsymbol{s}}(n_A, n_B)) - U_A(\tilde{\boldsymbol{s}}(n_A, n'_B)) = \sum_{l=1}^{n_A} \sum_{\{x|d(x)=l\}} (r_A(x) - w\,d(x))(g_A(l, n_B) - g_A(l, n'_B))\frac{P_A(x)}{2}.$$

Since $g_A(l, n)$ is weakly decreasing in n and Assumption 1 holds, the sum on the right hand side is weakly increasing in n_A. □

Proof of Lemma 3

To prove Lemma 3, we need to show that a partner always does best by playing her simple strategy whenever the other plays a simple strategy. We'll make this argument via three lemmas. The first lemma describes an optimal way for a partner to employ her consultants, given her network and given that the other partner plays a simple strategy. The second lemma calculates a partner's payoff to a network under these optimal employment strategies. The third lemma uses the first two lemmas to show that a simple strategy is a best response to a simple strategy. This lets prove Lemma 3.

Remark A1. Optimal Behavior in the Second Stage.

It's useful to characterize an optimal behavior for partner i in the second stage. Suppose i gets project x and has non-empty network $\mathcal{N}$. (If $\mathcal{N}$ is empty, i cannot employ any consultants and so gets 0.) If i gets her project first, then it's optimal for her to employ exactly $d(x)$ consultants when $d(x) \leq |\mathcal{N}|$ and to employ zero consultants otherwise. If i gets her project second, then it's optimal for her to employ exactly $d(x)$ consultants when $d(x) \leq |\mathcal{N} \backslash \mathcal{T}|$ and to employ zero consultants otherwise, where $\mathcal{T}$ is the set of consultants initially employed by $-i$.

Simply, i may employ up to (i) $|\mathcal{N}|$ consultants when she gets her project first or (ii) $|\mathcal{N}_i \backslash \mathcal{T}|$ consultants when she gets her project second. Let n be the number of consultants i employs. If $n < d(x)$, then i fails to complete her project and earns $-w\,n$ (before networking costs). If, however, $n \geq d(x)$, then i earns $r_i(x) - w\,n$ (before networking costs). Since $w \geq 0$ and $r_i(x) - w\,d(x) \geq 0$, it follows that it's best for i to set $n = d(x)$ if she has at that many consultants available and to set $n = 0$ otherwise.

Next we develop pair of second stage behavioral strategies that implements a partner's optimal behavior, given her network and given the other partner plays her simple strategy. To these ends, we introduce the following notation.

Let $\mathcal{N}$ be a non-empty network of consultants, and let $n = |\mathcal{N}|$. The **standard form of $\boldsymbol{\mathcal{N}}$ for $\boldsymbol{A}$** is a secondary labeling of the consultants in $\mathcal{N}$ such that the *lowest* indexed consultant receives the label m_1, the second lowest indexed consultant receives the label m_2, and so on until the highest indexed consultant gets the label m_n. For instance, if $\mathcal{N} = \{1, 5, 2, 4\}$, then 1 is labeled m_1, 2 is labeled m_2, 4 is labeled m_3, and 5 is labeled m_4, so we write $\mathcal{N} = \{m_1, m_2, m_3, m_4\}$. The **standard form of $\boldsymbol{\mathcal{N}}$ for $\boldsymbol{B}$** is a secondary labeling of the consultants in $\mathcal{N}$ such that the *highest* indexed consultant receives the label m_1, the second highest indexed consultant gets the label m_2, and so on until the lowest indexed consultant gets the label m_n.

We now develop a candidate pair of second stage behavioral strategies for A. Let $\mathcal{N}_A$ be a network for A. If $\mathcal{N}_A$ is empty, let

$$\hat{\sigma}_{1A}^{\mathcal{N}_A}(\mathcal{N}, x) = \hat{\sigma}_{2A}^{\mathcal{N}_A}(\mathcal{N}, \mathcal{N}', x) = \emptyset \text{ for all } (\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X.$$

If $\mathcal{N}_A$ is non-empty, let $\{m_1, \ldots, m_n\}$ be the standard form of $\mathcal{N}_A$ for A, where $n = |\mathcal{N}_A|$. For each $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$, let

$$\hat{\sigma}_{1A}^{\mathcal{N}_A}(\mathcal{N}, x) = \begin{cases} \{m_1, m_2 \ldots, m_{d(x)}\} & \text{if } d(x) \leq |\mathcal{N}| \text{ and } \mathcal{N} = \mathcal{N}_A \\ \emptyset & \text{else,} \end{cases}$$

$$\hat{\sigma}_{2A}^{\mathcal{N}_A}(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{m_1, m_2, \ldots, m_{d(x)}\} & \begin{array}{c} \text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|, \mathcal{N}' \subset \{m_{d(x)} + 1, \ldots, N\}, \\ \text{and } \mathcal{N} = \mathcal{N}_A \end{array} \\ \emptyset & \text{else.} \end{cases}$$

We refer to $\hat{\sigma}_{1A}^{\mathcal{N}_A}$ and $\hat{\sigma}_{2A}^{\mathcal{N}_A}$ as ***A*'s hat strategies given $\boldsymbol{\mathcal{N}_A}$**.[58] Under these strategies, when A gets project x, she employs the $d(x)$ consultants in her network with the *lowest* indices, when these consultants are available and there are $d(x)$ consultants in her network. Observe that, when $\mathcal{N}_A = \{1, \ldots, n\}$ for some $n \in \{0, \ldots, N\}$, then $m_1 = 1$, $m_2 = 2$, …, and $m_n = n$, so the hat strategies are equivalent to A's second stage strategies under her simple strategy.

Consider B. Let $\mathcal{N}_B$ be a network for B. If $\mathcal{N}_B$ is empty, let

$$\hat{\sigma}_{1B}^{\mathcal{N}_B}(\mathcal{N}, x) = \hat{\sigma}_{2A}^{\mathcal{N}_B}(\mathcal{N}, \mathcal{N}', x) = \emptyset \text{ for all } (\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X.$$

If $\mathcal{N}_B$ is non-empty, let $\{m_1, \ldots, m_n\}$ be the standard form of $\mathcal{N}_B$ for B, where $n = |\mathcal{N}_B|$. For each $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$, let

$$\hat{\sigma}_{1B}^{\mathcal{N}_B}(\mathcal{N}, x) = \begin{cases} \{m_1, m_2 \ldots, m_{d(x)}\} & \text{if } d(x) \leq |\mathcal{N}| \text{ and } \mathcal{N} = \mathcal{N}_B \\ \emptyset & \text{else,} \end{cases}$$

$$\hat{\sigma}_{2B}^{\mathcal{N}_B}(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{m_1, m_2, \ldots, m_{d(x)}\} & \begin{array}{r} \text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|, \ \mathcal{N}' \subset \{1, \ldots, m_{d(x)} - 1\}, \\ \text{and } \mathcal{N} = \mathcal{N}_B \end{array} \\ \emptyset & \text{else.} \end{cases}$$

We refer to $\hat{\sigma}_{1B}^{\mathcal{N}_B}$ and $\hat{\sigma}_{2B}^{\mathcal{N}_B}$ as ***B*'s hat strategies given $\boldsymbol{\mathcal{N}_B}$**. Under these strategies, when B gets project x, she employs the $d(x)$ consultants in her network with the *highest* indices, when these consultants are available there are $d(x)$ consultants in her network. Observe that, when $\mathcal{N}_B = \{N+1-n, \ldots, N\}$ for some $n \in \{0, \ldots, N\}$, then $m_1 = N$, $m_2 = N-1$, …, and $m_n = N+1-n$, so the hat strategies are equivalent to A's second stage strategies under her simple strategy.

The next lemma shows that the hat strategies function as desired, i.e., given partner i's network, they implement i's optimal behavior when $-i$ plays a simple strategy.

Lemma A2. Optimality of Hat Strategies.
LA2.1 : Let $\mathcal{N}_A \subset \mathcal{C}$, then, for all $(\sigma_{1A}, \sigma_{2A})$ such that $(\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}) \in \boldsymbol{S}_A$ and all $n_B \in \{0, \ldots, N\}$,

$$U_A(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{s}_B(n_B)) \geq U_A(\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \tilde{s}_B(n_B)).$$

LA2.2 : Let $\mathcal{N}_B \subset \mathcal{C}$, then, for all $(\sigma_{1B}, \sigma_{2B})$ such that $(\mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}_B$ and all $n_A \in \{0, \ldots, N\}$,

$$U_B(\tilde{s}_A(n_A), \mathcal{N}_B, \hat{\sigma}_{1B}^{\mathcal{N}_B}, \hat{\sigma}_{2B}^{\mathcal{N}_B}) \geq U_B(\tilde{s}_A(n_A), \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}).$$

[58]It is readily verified that $(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}) \in \boldsymbol{S}_A$ for every $\mathcal{N}_A \subset \mathcal{C}$.

Proof. We'll prove $LA2.1$ as the argument for $LA2.2$ is analogous. Since A has expected utility, it suffices to show that

$$u_{1A}(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{s}_B(n_B), x_A, x_B) \geq u_{1A}(\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \tilde{s}_B(n_B), x_A, x_B) \tag{5.4}$$
$$u_{2A}(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{s}_B(n_B), x_A, x_B)) \geq u_{2A}(\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \tilde{s}_B(n_B), x_A, x_B)), \tag{5.5}$$

for each $(x_A, x_B) \in X^2$. Since this argument is trivial if $\mathcal{N}_A = \emptyset$ because A always gets 0 when she gets her project first or second, we take $\mathcal{N}_A$ to be non-empty. Let $\{m_1, \dots, m_{n_A}\}$ be the standard form of $\mathcal{N}_A$, where $n_A = |\mathcal{N}_A|$.

Let $(x_A, x_B) \in X^2$. Suppose A gets her project first. If $d(x_A) \leq |\mathcal{N}_A|$, then $\hat{\sigma}_{1A}^{\mathcal{N}_A} = \{m_1, \dots, m_{d(x)}\}$ and so A completes her project. If, however, $d(x_A) > |\mathcal{N}_A|$, then $\hat{\sigma}_{1A}^{\mathcal{N}_A} = \emptyset$ and A doesn't complete her project. This behavior is optimal per Remark A1, so we necessarily have that equation (5.4) holds.

Now, suppose A gets her project second. Let $l = d(x_A)$. Since B follows her simple strategy, there are four cases: (i) $d(x_B) > n_B$ and $l > n_A$, (ii) $d(x_B) > n_B$ and $l \leq n_A$, (iii) $d(x_B) \leq n_B$ and $l > n_A$, and (iv) $d(x_B) \leq n_B$ and $l \leq n_A$. If (i) or (iii), then $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \emptyset$ as $l > |\mathcal{N}_A|$ and A does not complete her project. This is optimal per Remark A1. Thus, we necessarily have that equation (5.5) holds.

If (ii), then B employs no consultants, i.e., $\tilde{\sigma}_{1B} = \emptyset$. Since $d(x_A) \leq n_A = |\mathcal{N}_A \backslash \tilde{\sigma}_{1B}|$, we've $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \{m_1, \dots, m_l\}$, so A completes her project. This is also optimal per Remark A1. Thus, we necessarily have that equation (5.5) holds.

If (iv), then A may or may not complete her project. Since A follows her hat strategy, she completes her project if and only if B doesn't employ a consultant with an index at or below m_l.[59] Since B follows her simple strategy, she'll employ a consultant with an index at or below m_l if and only if $d(x_B) \geq N + 1 - m_l$.[60]

If $d(x_B) \geq N + 1 - m_l$, then $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \emptyset$ and A does *not* complete her project. This is optimal by Remark A1. Specifically, B employs all consultants with indices of $N+1-d(x_B)$ and above, let $\mathcal{T}$ denote this set. Since $m_l \geq N + 1 - d(x_B)$, $\mathcal{N}_A$ less $\mathcal{T}$ is a *subset* of $\{m_1, \dots, m_{l-1}\}$. Thus, $l > |\mathcal{N}_A \backslash \mathcal{T}|$ and so Remark A1 gives that it's best for A not to complete her project. Thus, we necessarily have that equation (5.5) holds.

If $d(x_B) < N + 1 - m_l$, then $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \{m_1, \dots, m_l\}$ and A completes her project. This is

[59] To see this, suppose B employs a set of consultants $\mathcal{T}$. If $\mathcal{T} \not\subset \{m_l + 1, \dots, N\}$, then $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \emptyset$ (by construction) and A fails to complete her project. If, however, $\mathcal{T} \subset \{m_l + 1, \dots, N\}$, then $|\mathcal{N}_A \backslash \mathcal{T}| \geq l$ and so $\tilde{\sigma}_{2A}^{\mathcal{N}_A} = \{m_1, \dots, m_l\}$, implying A completes her project.

[60] Recall that B employs $\{N+1-d(x_B), \dots, N\}$. If $d(x_B) < N+1-m_l$, then B doesn't employ a consultant with an index at or below m_l as $d(x_B) < N+1-m_l \implies N+1-d(x_B) > m_l$. If, however, $d(x_B) \geq N+1-m_l$, then B employs a consultant with an index at or below m_l as $d(x_B) \geq N+1-m_l \implies N+1-d(x_B) \leq m_l$.

optimal by Remark A1. Simply, B employs all consultants with indices of $N+1-d(x_B)$ or above, let $\mathcal{T}$ denote this set. Since $N+1-d(x_B) > m_l$, $\mathcal{N}_A$ less $\mathcal{T}$ *contains* $\{m_1, \ldots, m_l\}$. It follows that $|\mathcal{N}_A \backslash \mathcal{T}| \geq l$, so Remark A1 gives that it's best for A to complete her project. Thus, equation (5.5) necessarily holds. $\square$

The next lemma calculates the partner's expected payoffs to different networks when they use their hat strategies in the second stage and the other partner plays a simple strategy.

Lemma A3. Payoffs to Different Networks.
Let $i \in \{A, B\}$, *let* $n_{-i} \in \{0, \ldots, N\}$, *and let* $\mathcal{N}_i \subset \mathcal{C}$.
LA3.1 : If $\mathcal{N}_i = \emptyset$, *then*

$$U_i(\mathcal{N}_i, \hat{\sigma}_{1i}^{\mathcal{N}_i}, \hat{\sigma}_{2i}^{\mathcal{N}_i}, \tilde{\boldsymbol{s}}_{-i}(n_{-i})) = 0.$$

LA3.2 : If $\mathcal{N}_i \neq \emptyset$, *let* $\{m_1, \ldots, m_n\}$ *be the standard form of* $\mathcal{N}_i$ *for partner* i, *where* $n = |\mathcal{N}_i|$. *Then,*

$$\begin{aligned} U_i(\mathcal{N}_i, \hat{\sigma}_{1i}^{\mathcal{N}_i}, \hat{\sigma}_{2i}^{\mathcal{N}_i}, \tilde{\boldsymbol{s}}_{-i}(n_{-i})) = &\sum_{\{x|d(x)\leq n\}} (r_i(x) - w\, d(x)) \frac{P_i(x)}{2}, \\ &+ \sum_{l=1}^{n} \sum_{\{x|d(x)=l\}} (r_i(x) - w\, d(x)) \frac{P_i(x)}{2} h_i^{\mathcal{N}_i}(l, n_{-i}) - c_i(n) \end{aligned}$$

where, for $l \in \{1, \ldots, N\}$,

$$h_i^{\mathcal{N}_i}(l, n_{-i}) = \begin{cases} 1 - \sum_{\{x|N+1-m_l \leq d(x) \leq n_{-i}\}} P_{-i}(x) & \text{if } i = A \\ 1 - \sum_{\{x|m_l \leq d(x) \leq n_{-i}\}} P_{-i}(x) & \text{if } i = B. \end{cases}$$

Proof. The argument is analogous to the proof of Lemma 1. The only difference is when a partner moves second. We document this difference and omit the balance of the argument. We focus on $i = A$ as the argument when $i = B$ is analogous. As usual we take $\mathcal{N}_A \neq \emptyset$ to avoid trivialities.

Let x_A and x_B be A and B's projects. Let $l = d(x_A)$ and let $n_A = |\mathcal{N}_A|$. There are four cases: (i) $d(x_B) > n_B$ and $l > n_A$, (ii) $d(x_B) \leq n_B$ and $l > n_A$, (iii) $d(x_B) > n_B$ and $l \leq n_A$, and (iv) $d(x_B) \leq n_B$ and $l \leq n_A$. If (i) or (ii), then $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \emptyset$ and A earns nothing, while if (iii) then B employs no consultants, so $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \{m_1, \ldots, m_l\}$ and A earns $r_A(x_A) - w\, l$.

If (iv), then A may or may not complete her project. Since A follows $\hat{\sigma}_{2A}^{\mathcal{N}_A}$, she'll complete her project if and only if B employs no consultant with an index at or below m_l. Since B follows her simple strategy, she'll employ a consultant with an index at or below m_l if and only if $d(x_B) \geq N+1-m_l$. Thus, A earns $r_A(x_A) - w\, l$ if $d(x_B) < N+1-m_l$ and earns 0 if $d(x_B) \geq N+1-m_l$.

It follows that, A's expected payoff from getting project x_A second are:
(i) 0 if $d(x_A) > n_A$, and
(ii) $(r_A(x_A) - w\, d(x_A))(1 - \sum_{\{x|N+1-m_l \leq d(x) \leq n_B\}} P_B(x))$ if $d(x_A) \leq n_A$.
The first is follows directly from cases (i) and (ii). The second follows from cases (iii) and (iv), where A gets $r_A(x_A) - w\, d(x_A)$ if $d(x_B) < N + 1 - m_l$ or $d(x_B) > n_B$, an event of probability $1 - \sum_{\{x|N+1-m_l \leq d(x) \leq n_B\}} P_B(x)$. Thus, before networking costs, A's expected payoff from getting her project second is

$$\sum_{\{x|d(x) \leq n_A\}} (r_A(x) - w\, d(x)) h_A^{\mathcal{N}_A}(d(x_A), n_B) P_A(x).$$

The lemma follows. □

The next lemma shows that a partner always does best by playing a simple strategy when the other partner plays a simple strategy.

Lemma A4. Optimality of Simple Strategies.
$LA4.1$: *Let* $\mathcal{N}_A \subset \mathcal{C}$*, let* $n_A = |\mathcal{N}_A|$*, and let* $n_B \in \{0, \ldots, N\}$*. Then,*

$$U_A(\tilde{\boldsymbol{s}}_A(n_A), \tilde{\boldsymbol{s}}_B(n_B)) \geq U_A(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{\boldsymbol{s}}_B(n_B)).$$

$LA4.2$: *Let* $\mathcal{N}_B \subset \mathcal{C}$*, let* $n_B = |\mathcal{N}_B|$*, and let* $n_A \in \{0, \ldots, N\}$*. Then,*

$$U_B(\tilde{\boldsymbol{s}}_A(n_A), \tilde{\boldsymbol{s}}_B(n_B)) \geq U_B(\tilde{\boldsymbol{s}}_A(n_A), \mathcal{N}_B, \hat{\sigma}_{1B}^{\mathcal{N}_B}, \hat{\sigma}_{2B}^{\mathcal{N}_B}).$$

The key insight of $LA4.1$ is that the (ex-ante) probability B employs a consultant is monotone increasing in the index of the consultant – B employs consultant N with the highest probability, consultant $N - 1$ with the second highest probability, and so on. Since A wants to network with consultants she'll be able to employ with high probability, she does best by networking with the lowest indexed consultants.

Proof. We prove $LA4.1$ as the argument for $LA4.2$ is analogous. If $\mathcal{N}_A$ is empty, the result is true as A makes zero. Thus, we take $\mathcal{N}_A$ to be non-empty.

Write $\tilde{\boldsymbol{s}}_A(n_A) = (\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1A}, \tilde{\sigma}_{2A})$ and recall that $\hat{\sigma}_{1A}^{\tilde{\mathcal{N}}_A} = \tilde{\sigma}_{1A}$ and $\hat{\sigma}_{2A}^{\tilde{\mathcal{N}}_A} = \tilde{\sigma}_{2A}$ since $\tilde{\mathcal{N}}_A = \{1, \ldots, n_A\}$, i.e., A's hat strategies are the same as her second stage strategies under her simple strategy of size n_A. Thus, $U_A(\tilde{\boldsymbol{s}}_A(n_A), \tilde{\boldsymbol{s}}_B(n_B)) = U_A(\tilde{\mathcal{N}}_A, \hat{\sigma}_{1A}^{\tilde{\mathcal{N}}_A}, \hat{\sigma}_{2A}^{\tilde{\mathcal{N}}_A}, \tilde{\boldsymbol{s}}_B(n_B))$. So we only need to show that

$$U_A(\tilde{\mathcal{N}}_A, \hat{\sigma}_{1A}^{\tilde{\mathcal{N}}_A}, \hat{\sigma}_{2A}^{\tilde{\mathcal{N}}_A}, \tilde{\boldsymbol{s}}_B(n_B)) - U_A(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{\boldsymbol{s}}_B(n_B)) \geq 0.$$

Since both of A's networks are of size n_A, Lemma A3 gives

$$U_A(\tilde{\mathcal{N}}_A, \hat{\sigma}_{1A}^{\tilde{\mathcal{N}}_A}, \hat{\sigma}_{2A}^{\tilde{\mathcal{N}}_A}, \tilde{\boldsymbol{s}}_B(n_B)) - U_A(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{\boldsymbol{s}}_B(n_B)) = \sum_{l=1}^{n_A} \sum_{\{x|d(x)=l\}} (r_A(x) - w\, d(x)) \frac{P_A(x)}{2} (h_A^{\tilde{\mathcal{N}}_A}(l, n_B) - h_A^{\mathcal{N}_A}(l, n_B)).$$

Since Assumption 1 holds and $P_A \geq 0$, we only need to show that $h_A^{\tilde{\mathcal{N}}_A}(l, n_B) - h_A^{\mathcal{N}_A}(l, n_B) \geq 0$.

Let $\{\tilde{m}_1, \ldots, \tilde{m}_{n_A}\}$ be the standard from of $\tilde{\mathcal{N}}_A$ for A and let $\{m_1, \ldots, m_{n_A}\}$ be the standard form of $\mathcal{N}_A$ for A. Thus,

$$h_A^{\tilde{\mathcal{N}}_A}(l, n_B) - h_A^{\mathcal{N}_A}(l, n_B) = \sum_{\{x|N+1-m_l \leq d(x) \leq n_B\}} P_B(x) - \sum_{\{x|N+1-\tilde{m}_l \leq d(x) \leq n_B\}} P_B(x).$$

Observe that, for every $k \in \{1, \ldots, n_A\}$, we've $\tilde{m}_k \leq m_k$.[61] Thus, $N + 1 - m_l \leq N + 1 - \tilde{m}_l$. It follows that

$$h_A^{\tilde{\mathcal{N}}_A}(l, n_B) - h_A^{\mathcal{N}_A}(l, n_B) = \sum_{\{x|N+1-m_l \leq d(x) < N+1-\tilde{m}_l\}} P_B(x) \geq 0,$$

where the inequality follows from the fact that $P_B \geq 0$. $\square$

Proof of Lemma 3. Let $(z_A^\star, z_B^\star)$ be an equilibrium of the Auxiliary Game. We'll prove that $\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) = (\tilde{\boldsymbol{s}}_A(z_A^\star), \tilde{\boldsymbol{s}}_B(N - z_B^\star))$ is an equilibrium by showing that

$$U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_A(\boldsymbol{s}_A, \tilde{\boldsymbol{s}}_B(N - z_B^\star)) \text{ for all } \boldsymbol{s}_A \in \boldsymbol{S}_A \text{ and}$$
$$U_B(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_B(\tilde{\boldsymbol{s}}_A(z_A^\star), \boldsymbol{s}_B) \text{ for all } \boldsymbol{s}_B \in \boldsymbol{S}_B.$$

We'll establish this for A since the argument for B is analogous.

Since $(z_A^\star, z_B^\star)$ is an equilibrium of the Auxiliary Game,

$$U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_A(\tilde{\boldsymbol{s}}(n_A, N - z_B^\star)) \text{ for all } n_A \in \{0, \ldots, N\}.$$

Thus, Lemma A4 implies that

$$U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_A(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{\boldsymbol{s}}_B(N - z_B^\star)) \text{ for all } \mathcal{N}_A \subset \mathcal{C}.$$

[61] To see this, note that for every $j \in \{1, \ldots, n_A\}$, we have $j \leq m_j$. Simply, $m_1 \geq 1$. Since $m_2 > m_1$, we necessarily have $m_2 \geq m_1 + 1 \geq 2$. Continuing establishes the desired result. Since $\tilde{m}_1 = 1$, $\tilde{m}_2 = 2$, $\ldots$, and $\tilde{m}_{n_A} = n_A$, we have that $\tilde{m}_k \leq m_k$ for every $k \in \{1, \ldots, n_A\}$.

Hence, Lemma A2 gives

$$U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_A(\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \tilde{\boldsymbol{s}}_B(N - z_B^\star)) \text{ for all } (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}) \in \boldsymbol{S}_A.$$

The desired result follows. □

PROOF OF LEMMA 4

To prove Lemma 4, we first need to refine our understanding of the partners' optimal behavior in the second stage. Once we do this, we'll prove a preliminary lemma: that partners always behave according to Remark A1 in any equilibrium when Assumption 2 holds. Subsequently, we'll prove Lemma 4.

Remark A2. Unique Optimal Behavior in the Second Stage.
Under Assumption A2, the optimal behavior described in Remark A1 is the *unique* optimal behavior in the second stage. This follows from the facts $w > 0$ and $r_i(x) - w\,d(x) > 0$ for each partner i and each $x \in X$. (Since $r_i(x) - w\,d(x) > 0$, it's always best for i to complete project x by employing at least $d(x)$ consultants. Since $w > 0$, it's best for i to never employ more than $d(x)$ consultants.) Consequently, if partner i behaves in any other manner, she can do *strictly* better by switching to the behavior described in Remark A1.

With this in mind, we introduce a notion inspired by sequential rationality. Let $\boldsymbol{s} = (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}$ and let $\mathcal{E}_i^{\boldsymbol{s}}$ denote the set of sets of consultants partner i employs under $\boldsymbol{s}$ when she gets her project first, i.e.,

$$\mathcal{E}_i^{\boldsymbol{s}} = \{\mathcal{T} \in \mathbb{P}(\mathcal{C}) | \mathcal{T} = \sigma_{1i}(\mathcal{N}_i, x) \text{ for some } x \in X\}.$$

Definition. We say that a $\boldsymbol{s} = (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}$ is *weakly rational for partner* i if she behaves optimally in the second stage when the other partner follows $\boldsymbol{s}$. That is, if, for each $(\mathcal{T}, x) \in \mathcal{E}_{-i}^{\boldsymbol{s}} \times X$, we have:
(i) $|\sigma_{1i}(\mathcal{N}_i, x)| = d(x)$ when $d(x) \leq |\mathcal{N}_i|$ and $|\sigma_{1i}(\mathcal{N}_i, x)| = 0$ when $d(x) > |\mathcal{N}_i|$; and
(ii) $|\sigma_{2i}(\mathcal{N}_i, \mathcal{T}, x)| = d(x)$ when $d(x) \leq |\mathcal{N}_i \backslash \mathcal{T}|$ and $|\sigma_{2i}(\mathcal{N}_i, \mathcal{T}, x)| = 0$ when $d(x) > |\mathcal{N}_i \backslash \mathcal{T}|$.
We say that $\boldsymbol{s}$ is *weakly rational* if it is weakly rational for both partners A and B.

Notice that simple strategies of any size are always weakly rational. The next lemma describes another type of weakly rational strategy vector.

Lemma A5. Every Equilibrium is Weak Rationality.
Let Assumption 2 hold, then each $\boldsymbol{s}^\star \in \boldsymbol{E}$ *is weakly rational.*

The lemma follows from the fact that every project occurs with positive probability. Thus, if a partner follows a strategy that isn't weakly rational, she can do strictly better in expectation by switching to a strategy that is weakly rational, violating the conjecture of equilibrium.

Proof. This is almost obvious, we only give the proof for completeness. We argue by contradiction. Let $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star, \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star) \in \boldsymbol{E}$. We suppose, without loss, that A's strategy $(\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star)$ isn't weakly rational for her; the argument is analogous for B. We'll establish that $(\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star)$ isn't a best response for A when B plays according to $\boldsymbol{s}^\star$, a contradiction.

There are three ways in which $(\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star)$ may not be weakly rational:

1. There is an $x' \in X$ such that $|\sigma_{1A}^\star(\mathcal{N}_A^\star, x')| \neq d(x')$ when $d(x) \leq |\mathcal{N}_A^\star|$ or $|\sigma_{1A}^\star(\mathcal{N}_A^\star, x)| \neq 0$ when $d(x) > |\mathcal{N}_A^\star|$.

   In this case, upon getting project x first, A does strictly better by employing exactly $d(x)$ consultants when $d(x) \leq |\mathcal{N}_A^\star|$ or employing no consultants when $d(x) > |\mathcal{N}_A^\star|$.

2. There is an $(\mathcal{T}', x') \in \mathcal{E}_B^{\boldsymbol{s}} \times X$, such that $|\sigma_{2A}^\star(\mathcal{N}_A^\star, \mathcal{T}', x')| \neq d(x')$ when $d(x') \leq |\mathcal{N}_A^\star \backslash \mathcal{T}'|$ or $|\sigma_{2A}^\star(\mathcal{N}_A^\star, \mathcal{T}', x')| \neq 0$ when $d(x') > |\mathcal{N}_A^\star \backslash \mathcal{T}'|$.

   In this case, upon getting project x' second and observing B employ consultants in $\mathcal{T}'$, A does strictly better by employing exactly $d(x')$ consultants when $d(x') \leq |\mathcal{N}_A^\star \backslash \mathcal{T}'|$ or employing no consultants when $d(x') > |\mathcal{N}_A^\star \backslash \mathcal{T}'|$.

3. Both 1 and 2.

We'll only consider the second case, since it's the hardest and the other two are analogous.

Suppose case 2 occurs. Let σ_{2A}' be a new strategy for A such that $\sigma_{2A}'(\mathcal{N}, \mathcal{N}', x) = \sigma_{2A}^\star(\mathcal{N}, \mathcal{N}', x)$ for all $(\mathcal{N}, \mathcal{N}', x) \in (\mathbb{P}(\mathcal{C})^2 \times X) \backslash \{(\mathcal{N}_A^\star, \mathcal{T}', x')\}$ and such that $\sigma_{2A}'(\mathcal{N}_A^\star, \mathcal{T}', x')$ selects $d(x')$ consultants from $\mathcal{N}_A^\star \backslash \mathcal{T}'$ when $d(x') \leq |\mathcal{N}_A^\star \backslash \mathcal{T}'|$ or select no consultants when $d(x') > |\mathcal{N}_A^\star \backslash \mathcal{T}'|$. Let $\boldsymbol{s}' = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}', \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star)$.

Consider the difference of A's payoff under $\boldsymbol{s}'$ and $\boldsymbol{s}^\star$. Since (i) A has the same network in both strategies, (ii) follows the same behavioral strategy when she gets her project first, and (iii) follows the same behavioral strategy when she gets her project second, unless B gets a project that causes her to employ $\mathcal{T}'$ and A gets project x',

$$U_A(\boldsymbol{s}') - U_A(\boldsymbol{s}^\star) = \sum_{\{x_B | \sigma_{1B}^\star(\mathcal{N}_B^\star, x_B) = \mathcal{T}'\}} (u_{2A}(\boldsymbol{s}', x', x_B) - u_{2A}(\boldsymbol{s}^\star, x', x_B)) \frac{P_A(x')P_B(x_B)}{2}.$$

Since $P_A > 0$ and $P_B > 0$, this difference is strictly positive if $u_{2A}(\boldsymbol{s}', x', x_B) > u_{2A}(\boldsymbol{s}^\star, x', x_B)$, which is exactly the case since A's behavior under $\boldsymbol{s}'$ is optimal and her behavior under $\boldsymbol{s}^\star$

is sub-optimal. Thus, $(\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star)$ is not a best response for A when B plays according to $\boldsymbol{s}^\star$. $\square$

Proof of Lemma 4. We argue by contradiction. Suppose that $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star \neq \emptyset$ and that $\mathcal{C} \not\subset \mathcal{N}_A^\star \cup \mathcal{N}_B^\star$. Then there is a consultant $j \in \mathcal{C}\backslash(\mathcal{N}_A^\star \cup \mathcal{N}_B^\star)$. We'll show that A can do strictly better by swapping a consultant $k \in \mathcal{N}_A^\star \cap \mathcal{N}_B^\star$ for consultant j, when B follows $\boldsymbol{s}^\star$. It follows that $\boldsymbol{s}^\star$ cannot be an equilibrium.

We proceed by constructing a "post-swap" strategy for A and showing that this strategy leads to a strictly higher payoff than A gets in equilibrium. Write $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star, \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star)$ for the equilibrium given in the statement of this lemma.

Let $X_A \subset X$ such that $x \in X_A$ implies $d(x) = |\mathcal{N}_A^\star \backslash \mathcal{N}_B^\star| + 1$. Also, let $X_B \subset X$ such that $x \in X_B$ implies $d(x) = |\mathcal{N}_B^\star|$.[62] By Lemma A5, $x \in X_B$ if and only if $\sigma_{1B}^\star(\mathcal{N}_B^\star, x_B) = \mathcal{N}_B^\star$.[63]

We now write down a post-swap strategy for A, which we denote $(\mathcal{N}_A', \sigma_{1A}', \sigma_{2A}')$. Since A swaps k for j, we have $\mathcal{N}_A' = \{j\} \cup \mathcal{N}_A^\star \backslash \{k\}$. We'll choose σ_{1A}' so that, when A observes her network is $\mathcal{N}_A'$ and observes her project is x, she employs exactly the same consultants as she would under $\sigma_{1A}^\star$, when she observes her network is $\mathcal{N}_A^\star$ and observes her project is x, save she swaps k for j. Formally, for every $x \in X$, let

$$\sigma_{1A}'(\mathcal{N}_A', x) = \begin{cases} \sigma_{1A}^\star(\mathcal{N}_A^\star, x) & \text{if } k \notin \sigma_{1A}^\star(\mathcal{N}_A^\star, x) \\ \{j\} \cup \sigma_{1A}^\star(\mathcal{N}_A^\star, x)\backslash\{k\} & \text{if } k \in \sigma_{1A}^\star(\mathcal{N}_A^\star, x). \end{cases}$$

And for every $(\mathcal{N}, x) \in (\mathbb{P}(\mathcal{C})\backslash \mathcal{N}_A') \times X$, let $\sigma_{1A}'(\mathcal{N}, x) = \emptyset$.

We'll choose σ_{2A}' so that, when A observes her network is $\mathcal{N}_A'$, observes B employ $\mathcal{T} \in \mathcal{E}_B^{\boldsymbol{s}^\star}$, and observes her project is x, she employs exactly the same consultants as she would under $\sigma_{2A}^\star$, when she observes her network is $\mathcal{N}_A^\star$, observes B employ $\mathcal{T}$, and observes her project is x, save she swaps k for j. Formally, for every $(\mathcal{T}, x) \in \mathcal{E}_B^{\boldsymbol{s}^\star} \times X$, let

$$\sigma_{2A}'(\mathcal{N}_A', \mathcal{T}, x) = \begin{cases} \sigma_{2A}^\star(\mathcal{N}_A^\star, \mathcal{T}, x) & \text{if } k \notin \sigma_{2A}^\star(\mathcal{N}_A^\star, \mathcal{T}, x) \\ \{j\} \cup \sigma_{2A}^\star(\mathcal{N}_A^\star, \mathcal{T}, x)\backslash\{k\} & \text{if } k \in \sigma_{2A}^\star(\mathcal{N}_A^\star, \mathcal{T}, x). \end{cases}$$

And for every $(\mathcal{N}, \mathcal{N}', x) \in (\mathbb{P}(\mathcal{C})^2 \times X) \setminus \big(\{\mathcal{N}_A'\} \times \mathcal{E}_B^{\boldsymbol{s}^\star} \times X\big)$, let $\sigma_{2A}'(\mathcal{N}, \mathcal{N}', x) = \emptyset$.

We make one modification to σ_{2A}' before proceeding: if A observes B employ $\mathcal{N}_B^\star$ and

[62] Observe that X_A and X_B are non-empty. Since j isn't in both partner's networks, we have $|\mathcal{N}_A| \leq N-1$ and $|\mathcal{N}_B| \leq N-1$. Thus, we have $0 \leq |\mathcal{N}_A^\star \backslash \mathcal{N}_B^\star| \leq N-1$. Hence, Assumption 2 gives that there's a x' and x'' in X such that $d(x') = |\mathcal{N}_A^\star \backslash \mathcal{N}_B^\star| + 1$ and $d(x'') = |\mathcal{N}_B^\star|$.

[63] To see this, let $x \in X_B$. Since every equilibrium is weakly rational, we must have $|\sigma_{1B}^\star(\mathcal{N}_B^\star, x)| = d(x) = |\mathcal{N}_B^\star|$, which implies $\sigma_{1B}^\star(\mathcal{N}_B^\star, x) = \mathcal{N}_B^\star$. Conversely, let $x \notin X_B$. Then either $d(x) < |\mathcal{N}_B^\star|$, implying $|\sigma_{1B}^\star(\mathcal{N}_B^\star, x)| < |\mathcal{N}_B^\star|$, or $d(x) > |\mathcal{N}_B^\star|$, implying $|\sigma_{1B}^\star(\mathcal{N}_B^\star, x')| = 0$. In both cases, $\sigma_{1B}^\star(\mathcal{N}_B^\star, x')$ is a strict subset of $\mathcal{N}_B^\star$.

then gets a project in X_A, she employs all of $\mathcal{N}'_A \backslash \mathcal{N}^\star_B$. Formally, for every $x \in X_A$, let $\sigma'_{2A}(\mathcal{N}'_A, \mathcal{N}^\star_B, x) = \mathcal{N}'_A \backslash \mathcal{N}^\star_B$. It is readily verified that $(\mathcal{N}'_A, \sigma'_{1A}, \sigma'_{2A}) \in \boldsymbol{S}_A$. Let $\boldsymbol{s}' = (\mathcal{N}'_A, \sigma'_{1A}, \sigma'_{2A}, \mathcal{N}^\star_B, \sigma^\star_{1B}, \sigma^\star_{2B})$.

Before we establish that A makes strictly under $\boldsymbol{s}'$ than $\boldsymbol{s}^\star$, we need two preliminary facts. First, $|\sigma'_{1A}(\mathcal{N}'_A, x)| = |\sigma^\star_{1A}(\mathcal{N}^\star_A, x)|$ for all $x \in X$, i.e., A uses the same number of consultants to complete project x under $\boldsymbol{s}'$ and $\boldsymbol{s}^\star$ when she moves first in the second stage. This is a direct consequence the construction of σ'_{1A}. Second, for all $(x_A, x_B) \in X^2 \backslash (X_A \times X_B)$,

$$|\sigma'_{2A}(\mathcal{N}'_A, \sigma^\star_{1B}(\mathcal{N}^\star_B, x_B), x_A)| = |\sigma^\star_{2A}(\mathcal{N}^\star_A, \sigma^\star_{1B}(\mathcal{N}^\star_B, x_B), x_A)|.$$

That is, given A and B's projects are in $X^2 \backslash (X_A \times X_B)$, then A uses the same number of consultants to complete x_A under $\boldsymbol{s}'$ and $\boldsymbol{s}^\star$ when she moves second. Let's establish this. Since $\sigma^\star_{1B}(\mathcal{N}^\star_B, x_B) = \mathcal{N}^\star_B$ if and only if $x_B \in X_B$, we have $(\sigma^\star_{1B}(\mathcal{N}^\star_B, x_B), x_A) \in \{\mathcal{N}^\star_B\} \times X_A$ if and only if $(x_A, x_B) \in X_A \times X_B$. Thus, $(\sigma^\star_{1B}(\mathcal{N}^\star_B, x_B), x_A) \in \big(\mathcal{E}^{\boldsymbol{s}^\star}_B \times X\big) \setminus (\{\mathcal{N}^\star_B\} \times X_A)$ if and only if $(x_A, x_B) \in X^2 \setminus (X_A \times X_B)$. Since $|\sigma'_{2A}(\mathcal{N}'_A, \mathcal{T}, x_A)| = |\sigma^\star_{2A}(\mathcal{N}^\star_A, \mathcal{T}, x_A)|$ for all $(\mathcal{T}, x_A) \in \big(\mathcal{E}^{\boldsymbol{s}^\star}_B \times X\big) \setminus (\{\mathcal{N}^\star_B\} \times X_A)$ by construction,[64] we have the secondary preliminary fact.

Recall that A's ex-post payoff depends only on the size of her network and the number of consultants she employs. Since A holds the same sized network under $\boldsymbol{s}'$ and $\boldsymbol{s}^\star$, the first preliminary fact implies $u_{1A}(\boldsymbol{s}', x_A, x_B) = u_{1A}(\boldsymbol{s}^\star, x_A, x_B)$ for all $(x_A, x_B) \in X^2$, while the second preliminary fact implies that $u_{2A}(\boldsymbol{s}', x_A, x_B) = u_{2A}(\boldsymbol{s}^\star, x_A, x_B)$ for all $(x_A, x_B) \in X^2 \backslash (X_A \times X_B)$. Thus,

$$U_A(\boldsymbol{s}') - U_A(\boldsymbol{s}) = \sum_{(x_A, x_B) \in X_A \times X_B} (u_{2A}(\boldsymbol{s}', x_A, x_B) - u_{2A}(\boldsymbol{s}^\star, x_A, x_B)) \frac{P_A(x_A) P_B(x_B)}{2}.$$

Under $\boldsymbol{s}^\star$, when A moves second, she employs no consultants when her project x_A is in X_A and B's project x_B is in X_B. Since B employs her entire network, A is left with $|\mathcal{N}^\star_A \backslash \mathcal{N}^\star_B|$ consultants to possibly employ. Since $x \in X_A$ implies $d(x) > |\mathcal{N}^\star_A \backslash \mathcal{N}^\star_B|$, it's optimal for A to employ no consultants per Remarks A1 and A2. Since $\boldsymbol{s}^\star$ is an equilibrium, Lemma A5 tells us that this is exactly what A does. Hence, $u_{2A}(\boldsymbol{s}^\star, x_A, x_B) = -c_A(|\mathcal{N}^\star_A|)$.

Under $\boldsymbol{s}'$, when A moves second, she employs $\mathcal{N}'_A \backslash \mathcal{N}^\star_B$ if her project x_A is in X_A and B's project x_B is in X_B by construction of σ'_{2A}. Since $|\mathcal{N}'_A \backslash \mathcal{N}^\star_B| = |\{j\} \cup \mathcal{N}^\star_A \backslash \mathcal{N}^\star_B| = 1 + |\mathcal{N}^\star_A \backslash \mathcal{N}^\star_B|$ (as $j \notin \mathcal{N}^\star_B$ and $k \in \mathcal{N}^\star_B$) and $d(x_A) = 1 + |\mathcal{N}^\star_A \backslash \mathcal{N}^\star_B|$, A completes her project and gets a payoff of $u_{2A}(\boldsymbol{s}^\star, x_A, x_B) = r_A(x_A) - w\, d(x_A) - c_A(\mathcal{N}^\star_A)$.

[64]This equality does not hold on $\{\mathcal{N}^\star_B\} \times X_A$ due to our modification.

It follows that

$$U_A(\boldsymbol{s}') - U_A(\boldsymbol{s}) = \sum_{(x_A,x_B)\in X_A\times X_B} (r_A(x_A) - w\, d(x_A))P_A(x_A)P_B(x_B) > 0,$$

where the strict inequality follows from Assumption 2. □

Proof of Proposition 3

To prove Proposition 3, we first establish a useful technical lemma concerning the payoffs of a partner in a certain type of strategy. We then leverage this lemma to prove the proposition. We say that a $\boldsymbol{s} = (\mathcal{N}_A, \ldots, \mathcal{N}_B, \ldots) \in \boldsymbol{S}$ has the **covering property** if $\mathcal{N}_A \cap \mathcal{N}_B \neq \emptyset$ implies $\mathcal{C} \subset \mathcal{N}_A \cup \mathcal{N}_B$. Lemma 4 gives that all equilibria have the covering property under Assumption 2.

Lemma A6. Payoff Equivalent Strategies.
*LA*6.1 : *Let* $\boldsymbol{s} = (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}$ *such that: (i)* $\boldsymbol{s}$ *has the covering property, (ii)* $\boldsymbol{s}$ *is employment efficient for B, (iii)* $\boldsymbol{s}$ *is weakly rational for A, and (iv)* σ_{1B} *fulfills part (i) of the definition of weak rationality for B. Then,* $U_A(\boldsymbol{s}) = U_A(\tilde{\boldsymbol{s}}(|\mathcal{N}_A|, |\mathcal{N}_B|))$.
*LA*6.2 : *Let* $\boldsymbol{s} = (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}$ *such that: (i)* $\boldsymbol{s}$ *has the covering property, (ii)* $\boldsymbol{s}$ *is employment efficient for A, (iii)* $\boldsymbol{s}$ *is weakly rational for B, and (iv)* σ_{1A} *fulfills part (i) of the definition of weak rationality for A. Then,* $U_B(\boldsymbol{s}) = U_B(\tilde{\boldsymbol{s}}(|\mathcal{N}_A|, |\mathcal{N}_B|))$.

The intuition behind this lemma is exactly the same as the intuition behind Proposition 3: whenever partner i moves first in the second stage, her behavior is a permutation of her behavior in the corresponding simple equilibrium.

Proof. We prove $LA6.1$ as the argument for $LA6.2$ is analogous. Let $\tilde{\boldsymbol{s}}$ denote $\tilde{\boldsymbol{s}}(|\mathcal{N}_A|, |\mathcal{N}_B|) = (\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1A}, \tilde{\sigma}_{2A}, \tilde{\mathcal{N}}_B, \tilde{\sigma}_{1B}, \tilde{\sigma}_{2B})$. We prove the lemma by establishing that (i) $u_{1A}(\boldsymbol{s}, x_A, x_B) = u_{1A}(\tilde{\boldsymbol{s}}, x_A, x_B)$ and (ii) $u_{2A}(\boldsymbol{s}, x_A, x_B) = u_{2A}(\tilde{\boldsymbol{s}}, x_A, x_B)$ for every $(x_A, x_B) \in X^2$. It follows that $U_A(\boldsymbol{s}) = U_A(\tilde{\boldsymbol{s}})$. Let $n_A = |\mathcal{N}_A|$ and let $n_B = |\mathcal{N}_A|$.

We need to establish a preliminary result: $|\mathcal{N}_A \cap \mathcal{N}_B| = |\tilde{\mathcal{N}}_A \cap \tilde{\mathcal{N}}_B|$. There are two cases, $\mathcal{N}_A \cap \mathcal{N}_B = \emptyset$ and $\mathcal{N}_A \cap \mathcal{N}_B \neq \emptyset$. If $N_A \cap N_B = \emptyset$, then $\mathcal{N}_A$, $\mathcal{N}_B$, and $\mathcal{S} = \mathcal{C} \backslash (\mathcal{N}_A \cup \mathcal{N}_B)$ partition $\mathcal{C}$. Thus, $N = n_A + n_B + |\mathcal{S}|$, implying $N \geq n_A + n_B$. Since

$$|\tilde{\mathcal{N}}_A \cap \tilde{\mathcal{N}}_B| = |\{N+1-n_B, \ldots, n_A\}| = \begin{cases} 0 & \text{if } n_A + n_B \leq N \\ n_A + n_B - N & \text{if } n_A + n_B > N, \end{cases}$$

(by definition of simple strategies) we have $|\tilde{\mathcal{N}}_A \cap \tilde{\mathcal{N}}_B| = 0$. Thus, $|N_A \cap N_B| = |\tilde{N}_A \cap \tilde{N}_B|$.

If $\mathcal{N}_A \cap \mathcal{N}_B \neq \emptyset$, then the covering property implies $\mathcal{C} \subset \mathcal{N}_A \cup \mathcal{N}_B$. Thus, we've that $\mathcal{N}_A \backslash \mathcal{N}_B$, $\mathcal{N}_B \backslash \mathcal{N}_A$, and $\mathcal{N}_A \cap \mathcal{N}_B$ partition $\mathcal{C}$, so $N = |\mathcal{N}_A \backslash \mathcal{N}_B| + |\mathcal{N}_B \backslash \mathcal{N}_A| + |\mathcal{N}_A \cap \mathcal{N}_B|$.

Since $|\mathcal{N}_i \backslash \mathcal{N}_{-i}| = n_i - |\mathcal{N}_i \cap \mathcal{N}_i|$ for $i \in \{A, B\}$, we have $|\mathcal{N}_A \cap \mathcal{N}_B| = n_A + n_B - N$. Since $\mathcal{N}_A \cap \mathcal{N}_B \neq \emptyset$, we have $|\mathcal{N}_A \cap \mathcal{N}_B| > 0$, implying $n_A + n_B - N > 0$. Thus, $|\tilde{\mathcal{N}}_A \cap \tilde{\mathcal{N}}_B| = n_A + n_B - N$ and so $|\mathcal{N}_A \cap \mathcal{N}_B| = |\tilde{\mathcal{N}}_A \cap \tilde{\mathcal{N}}_B|$.

Suppose A gets project first. Let x_A and x_B be A and B's projects. Since $\boldsymbol{s}$ is weakly rational for A, $|\sigma_{1A}(\mathcal{N}_A, x_A)| = d(x_A)$ when $d(x_A) \leq n_A$ and $|\sigma_{1A}(\mathcal{N}_A, x_A)| = 0$ when $d(x_A) > n_A$. Likewise, $|\tilde{\sigma}_{1A}(\mathcal{N}_A, x_A)| = d(x_A)$ when $d(x_A) \leq n_A$ and $|\tilde{\sigma}_{1A}(\mathcal{N}_A, x_A)| = 0$ when $d(x) > n_A$ by definition of a simple strategy. Thus, A employs the same number of consultants under both strategies, i.e., $|\sigma_{1A}(\mathcal{N}_A, x_A)| = |\tilde{\sigma}_{1A}(\mathcal{N}_A, x_A)|$. Since A holds the same sized network under $\boldsymbol{s}$ and $\tilde{\boldsymbol{s}}$ and since A's ex-post earnings are determined by the number of consultants she employs and the size of her network, we have $u_{1A}(\boldsymbol{s}, x_A, x_B) = u_{1A}(\tilde{\boldsymbol{s}}, x_A, x_B)$. Since (x_A, x_B) were arbitrary, the desired result follows.

Suppose A gets her project second. Let x_A and x_B be A and B's projects. There are four cases: (i) $d(x_B) > n_B$ and $d(x_A) > n_A$, (ii) $d(x_B) > n_B$ and $d(x_A) \leq n_A$, (iii) $d(x_B) \leq n_B$ and $d(x_A) > n_A$, and (iv) $d(x_B) \leq n_B$ and $d(x_A) \leq n_A$. In each case, we establish that A's behavioral strategies σ_{2A} and $\tilde{\sigma}_{2A}$ select the same number of consultants when B follows σ_{1B} and $\tilde{\sigma}_{1B}$ respectively, i.e., that $|\sigma_{2A}(\mathcal{N}_A, \sigma_{1B}(\mathcal{N}_B, x_B), x_A)| = |\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)|$. Since A's ex-post payoff is determined entirely by the size of her network, which is the same under both strategies, and the number of consultants that she employs, we have $u_{2A}(\boldsymbol{s}, x_A, x_B) = u_{2A}(\tilde{\boldsymbol{s}}, x_A, x_B)$. Since (x_A, x_B) were arbitrary, the desired result follows.

If (i), then $|\sigma_{1B}(\mathcal{N}_B, x_B)| = 0$ by the weak rationality of σ_{1B} and $|\sigma_{2A}(\mathcal{N}_A, \emptyset, x_A)| = 0$ by the weak rationality of $\boldsymbol{s}$ for A. Likewise, $|\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)| = 0$ and $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \emptyset, x_A)| = 0$ by the construction of the simple strategies. Thus, $|\sigma_{2A}(\mathcal{N}_A, \emptyset, x_A)| = |\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \emptyset, x_A)|$.

If case (ii), then $|\sigma_{1B}(\mathcal{N}_B, x_B)| = 0$ by the weak rationality of σ_{1B} and $|\sigma_{2A}(\mathcal{N}_A, \emptyset, x_A)| = d(x_A)$ by the weak rationality of $\boldsymbol{s}$ for A. Likewise, $|\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)| = 0$ and $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \emptyset, x_A)| = d(x_A)$ by the construction of simple strategies. Thus, $|\sigma_{2A}(\mathcal{N}_A, \emptyset, x_A)| = |\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \emptyset, x_A)|$.

If case (iii), then $|\sigma_{1B}(\mathcal{N}_B, x_B)| = d(x_B)$ by the weak rationality of σ_{1B} and $|\sigma_{2A}(\mathcal{N}_A, \sigma_{1B}(\mathcal{N}_B, x_B), x_A)| = 0$ by the weak rationality of $\boldsymbol{s}$ for A. Likewise, $|\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)| = d(x_B)$ and $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)| = 0$ by the construction of simple strategies. Thus, $|\sigma_{2A}(\mathcal{N}_A, \sigma_{1B}(\mathcal{N}_B, x_B), x_A)| = |\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)|$.

If case (iv), then $|\sigma_{1B}(\mathcal{N}_B, x_B)| = d(x_B)$ by the weak rationality of σ_{1B}. It's useful to think about the number of consultants left for A after B moves. Since $\boldsymbol{s}$ is employment efficient for B, $\sigma_{1B}(\mathcal{N}_B, x_B) \subset \mathcal{N}_B \backslash \mathcal{N}_A$ when $d(x_B) \leq |\mathcal{N}_B \backslash \mathcal{N}_A|$, leaving A with n_A consultants in $\mathcal{N}_A$. If $d(x_B) > |\mathcal{N}_B \backslash \mathcal{N}_A|$, then employment efficiency implies B employs all consultants in $\mathcal{N}_B \backslash \mathcal{N}_A$ and so employs $d(x_B) - |\mathcal{N}_B \backslash \mathcal{N}_A|$ consultants from $\mathcal{N}_A \cap \mathcal{N}_B$. This leaves A with $|\mathcal{N}_A \cap \mathcal{N}_B| - (d(x_B) - |\mathcal{N}_B \backslash \mathcal{N}_A|)$ consultants in $\mathcal{N}_A \cap \mathcal{N}_B$ and with $|\mathcal{N}_A \backslash \mathcal{N}_B|$ consultants in

$\mathcal{N}_A \backslash \mathcal{N}_B$. Thus, there are

$$|\mathcal{N}_A \backslash \sigma_{1B}(\mathcal{N}_B, x_B)| = \begin{cases} n_A & \text{if } d(x_B) \leq |\mathcal{N}_B \backslash \mathcal{N}_A| \\ N - d(x_B) & \text{if } d(x_B) > |\mathcal{N}_B \backslash \mathcal{N}_A| \end{cases}$$

consultants left for A after B moves under $\boldsymbol{s}$. Since $\tilde{\boldsymbol{s}}$ is also employment efficient and $\tilde{\sigma}_{1B}$ satisfies part (i) of the definition of weak rationality for B, an analogous argument gives that there are

$$|\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)| = \begin{cases} n_A & \text{if } d(x_B) \leq |\tilde{\mathcal{N}}_B \backslash \tilde{\mathcal{N}}_A| \\ N - d(x_B) & \text{if } d(x_B) > |\tilde{\mathcal{N}}_B \backslash \tilde{\mathcal{N}}_A| \end{cases}$$

consultants left for A after B moves. Since (i) $|\mathcal{N}_B \backslash \mathcal{N}_A| = n_B - |\mathcal{N}_A \cap \mathcal{N}_B|$, (ii) $|\tilde{\mathcal{N}}_B \backslash \tilde{\mathcal{N}}_A| = n_B - |\tilde{\mathcal{N}}_B \cap \tilde{\mathcal{N}}_A|$, and (iii) $|\mathcal{N}_A \cap \mathcal{N}_B| = |\tilde{\mathcal{N}}_A \cap \tilde{\mathcal{N}}_B|$, we have

$$|\mathcal{N}_A \backslash \sigma_{1B}(\mathcal{N}_B, x_B)| = |\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)|. \tag{5.6}$$

That is, A has the same number of consultants left after B moves in both $\boldsymbol{s}$ and $\tilde{\boldsymbol{s}}$.

Since $\boldsymbol{s}$ is weakly rational for A, we've $|\sigma_{2A}(\mathcal{N}_A, \sigma_{1B}(\mathcal{N}_B, x_B), x_A)| = d(x_A)$ when $d(x_A) \leq |\mathcal{N}_A \backslash \sigma_{1B}(\mathcal{N}_B, x_B)|$ and $|\sigma_{2A}(\mathcal{N}_A, \sigma_{1B}(\mathcal{N}_B, x_B), x_A)| = 0$ when $d(x_A) > |\mathcal{N}_A \backslash \sigma_{1B}(\mathcal{N}_B, x_B)|$. Likewise, $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)| = d(x_A)$ when $d(x_A) \leq |\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)|$ and $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)| = 0$ when $d(x_A) > |\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)|$ by construction of simple strategies.[65] Hence, equation (5.6) gives

$$|\sigma_{2A}(\mathcal{N}_A, \sigma_{1B}(\mathcal{N}_B, x_B), x_A)| = |\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)|.$$

The desired result follows. □

Corollary A1. Employment Efficient Equilibria and Simple Strategies.
Let Assumption 2 hold and let $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star, \ldots) \in \boldsymbol{E}$ *be employment efficient, then* $U_i(\boldsymbol{s}^\star) = U_i(\tilde{\boldsymbol{s}}(|\mathcal{N}_A^\star|, |\mathcal{N}_B^\star|))$ *for each partner* i.

Proof. Since every equilibrium has the covering property by Lemma 4 and is weakly rational by Lemma A5, the antecedents of Lemma 6 are satisfied. The corollary follows. □

[65] Let's establish these facts. Suppose that $d(x_A) > |\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)|$, then the the definition of $\tilde{\sigma}_{2A}$ gives that $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)| = 0$. Now suppose that $d(x_A) \leq |\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)|$, then the definition of $\tilde{\sigma}_{2A}$ gives that $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)| = d(x_A)$ if $\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B) \subset \{d(x_A)+1, \ldots, N\}$. We'll establish that $\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B) \subset \{d(x_A)+1, \ldots, N\}$. Recall that $\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B) = \{N+1-d(x_B), \ldots, N\}$ and $\tilde{\mathcal{N}}_A = \{1, \ldots, n_A\}$. Thus, $\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B) = \{1, \ldots, \min\{n_A, N - d(x_B)\}\}$. Hence, $d(x_A) \leq |\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)|$ implies that $d(x_A) \leq N - d(x_B)$. It follows that $N + 1 - d(x_B) \geq d(x_A) + 1$ and so $\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B) \subset \{d(x_A)+1, \ldots, N\}$.

This corollary, by itself, does not imply Proposition 3 because *both* partners' strategies are different in $\boldsymbol{s}^\star$ and $\tilde{\boldsymbol{s}}(|\mathcal{N}_A^\star|, |\mathcal{N}_B^\star|)$, even when $\boldsymbol{s}^\star$ is an equilibrium. Thus, one partner may have a profitable defection available under the latter strategy. As we show in the next proof, employment lists mitigate this concern.

Proof of Proposition 3. Let $\boldsymbol{s}^\star = (\boldsymbol{s}_A^\star, \boldsymbol{s}_B^\star) = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star, \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star) \in \boldsymbol{E}_{LE}$, let $n_A^\star = |\mathcal{N}_A^\star|$ and $n_B^\star = |\mathcal{N}_B^\star|$. Let $\tilde{\boldsymbol{s}}$ denote $\tilde{\boldsymbol{s}}(n_A^\star, n_B^\star)$. We'll prove the proposition by showing that $(n_A^\star, N - n_B^\star) \in \boldsymbol{F}$, i.e., is an equilibrium of the Auxiliary Game, as then Lemma 3 implies $\tilde{\boldsymbol{s}}(n_A^\star, n_B^\star) \in \boldsymbol{E}_S$.

We argue that $(n_A^\star, N - n_B^\star) \in \boldsymbol{F}$ by contradiction. Suppose that $(n_A^\star, N - n_B^\star) \notin \boldsymbol{F}$, then at least one partner, say A, does strictly better in the Auxiliary Game by picking a new network size n', with $n' \in \{0, \ldots, N\}$ and $n' \neq n_A^\star$, i.e.,

$$U_A(\tilde{\boldsymbol{s}}(n', n_B^\star))) > U_A(\tilde{\boldsymbol{s}}(n_A^\star, n_B^\star)). \tag{5.7}$$

The argument is analogous if B is the partner who does better.

Construct a network $\mathcal{N}_A'$ of size n' for A as follows. If $n' \leq |\mathcal{C} \backslash \mathcal{N}_B^\star|$, let $\mathcal{N}_A'$ consist of n' elements of $\mathcal{C} \backslash \mathcal{N}_B^\star$. If $n' > |\mathcal{C} \backslash \mathcal{N}_B^\star|$, then $\mathcal{N}_B^\star \neq 0$ as $n' \leq N$ and so B has an employment list $\{j_1, \ldots, j_{n_B^\star}\}$ for $\boldsymbol{s}^\star$. Let $\mathcal{N}_A'$ consist of $\mathcal{C} \backslash \mathcal{N}_B^\star$ and the last $n' - |\mathcal{C} \backslash \mathcal{N}_B^\star|$ elements of B's employment list. Thus,

$$\mathcal{N}_A' = \mathcal{C} \backslash \mathcal{N}_B^\star \cup \{j_{n_B^\star}, j_{n_B^\star - 1}, \ldots, j_{\psi+1}, j_\psi\},$$

where $\psi = n_B^\star + 1 - (n' - |\mathcal{C} \backslash \mathcal{N}_B^\star|)$. Also, construct weakly rational strategies σ_{1A}' and σ_{2A}' for A when her network is $\mathcal{N}_A'$ and B follows $\boldsymbol{s}^\star$. For all $x \in X$, let $\sigma_{1A}'(\mathcal{N}_A', x)$ select $d(x)$ consultants from $\mathcal{N}_A'$ when $d(x) \leq n'$ and let $\sigma_{1A}'(\mathcal{N}_A', x)$ be empty when $d(x) > n'$. For all other $(\mathcal{N}, x) \in \mathbb{P}(\mathcal{C}) \times X$, let $\sigma_{1A}'(\mathcal{N}, x) = \emptyset$. For all $(\mathcal{T}, x) \in \mathcal{E}_B^{\boldsymbol{s}^\star} \times X$, let $\sigma_{2A}'(\mathcal{N}_A', \mathcal{T}, x)$ select $d(x)$ consultants from $\mathcal{N}_A' \backslash \mathcal{T}$ when $d(x) \leq |\mathcal{N}_A' \backslash \mathcal{T}|$ and select no consultants when $d(x) > |\mathcal{N}_A' \backslash \mathcal{T}|$. For all other $(\mathcal{N}, \mathcal{T}, x) \in \mathcal{E}_B^{\boldsymbol{s}^\star} \times X$, let $\sigma_{2A}'(\mathcal{N}, \mathcal{T}, x) = \emptyset$.

Let $\boldsymbol{s}_A' = (N_A', \sigma_{1A}', \sigma_{2A}')$. Because of the construction, we have that $\boldsymbol{s}_A' \in \boldsymbol{S}_A$ and that $(\boldsymbol{s}_A', \boldsymbol{s}_B^\star)$ is weakly rational for A.

We'll prove that $U_A(\boldsymbol{s}_A', \boldsymbol{s}_B^\star) = U_A(\tilde{\boldsymbol{s}}(n', n_B^\star))$. Given this, we have

$$U_A(\boldsymbol{s}_A^\star, \boldsymbol{s}_B^\star) \geq U_A(\boldsymbol{s}_A', \boldsymbol{s}_B^\star) = U_A(\tilde{\boldsymbol{s}}(n', n_B^\star)),$$

where the weak inequality follows from the fact $(\boldsymbol{s}_A^\star, \boldsymbol{s}_B^\star)$ is an equilibrium. Corollary A1 gives that $U_A(\boldsymbol{s}_A^\star, \boldsymbol{s}_B^\star) = U_A(\tilde{\boldsymbol{s}}(n_A^\star, n_B^\star))$, since $\boldsymbol{s}^\star$ is employment efficient. Thus, we have

$$U_A(\tilde{\boldsymbol{s}}(n_A^\star, n_B^\star)) \geq U_A(\tilde{\boldsymbol{s}}(n', n_B^\star)),$$

a contradiction of our initial supposition (5.7).

It remains to establish that $U_A(\boldsymbol{s}'_A, \boldsymbol{s}_B^\star) = U_A(\tilde{\boldsymbol{s}}(n', n_B^\star))$. This follows from Lemma A6, we just need to verify that the antecedents hold. To these ends, recall that $(\boldsymbol{s}'_A, \boldsymbol{s}_B^\star)$ is weakly rational for A by construction. Since $\boldsymbol{s}^\star$ is an equilibrium, Lemma A5 gives that $\sigma_{1B}^\star$ satisfies part (i) of the definition of weak rationality for B. Additionally, $(\boldsymbol{s}'_A, \boldsymbol{s}_B^\star)$ has the covering property: if $\mathcal{N}'_A \cap \mathcal{N}_B^\star \neq \emptyset$, then $\mathcal{C} \backslash \mathcal{N}_B^\star \subset \mathcal{N}'_A$ by construction and so $\mathcal{C}$ is in $\mathcal{N}'_A \cup \mathcal{N}_B^\star$. Finally, $(\boldsymbol{s}'_A, \boldsymbol{s}_B^\star)$ is employment efficient for B. If $\mathcal{N}'_A \cap \mathcal{N}_B^\star = \emptyset$, this is trivially the case. If $\mathcal{N}'_A \cap \mathcal{N}_B^\star \neq \emptyset$, observe that $\mathcal{N}_B^\star \backslash \mathcal{N}'_A = \{j_1, \ldots, j_{\psi-1}\}$ since we constructed $\mathcal{N}'_A$ to contain the last $n' - |\mathcal{C} \backslash \mathcal{N}_B^\star|$ elements of B's employment list. Since B sticks to her employment list when she gets a project x, we've $\sigma_{1B}^\star(\mathcal{N}_B^\star, x) = \{j_1, \ldots, j_{d(x)}\}$ when B employs consultants. Thus, if $d(x) \leq |\mathcal{N}_B^\star \backslash \mathcal{N}'_A| = \psi - 1$, we have $\sigma_{1B}^\star \subset \mathcal{N}_B^\star \backslash \mathcal{N}'_A$. If, however, $d(x) > \psi - 1$, then $\mathcal{N}_B^\star \backslash \mathcal{N}'_A \subset \sigma_{1B}^\star$. Thus, Lemma A6 applies. □

Proof of Lemma 6

Proof of Lemma 6. We'll prove the lemma for A as the argument for B is analogous. Let $n < n'$. Since Assumption 1 holds, Lemma A1 gives that $U_A(\tilde{\boldsymbol{s}}(n_A, n_B))$ is weakly decreasing in n_B for all n_A. Thus,

$$U_A(\tilde{\boldsymbol{s}}(\bar{b}_A(n), n)) \geq U_A(\tilde{\boldsymbol{s}}(\bar{b}_A(n'), n)) \geq U_A(\tilde{\boldsymbol{s}}(\bar{b}_A(n'), n')).$$

The first inequality is due to the optimality of $\bar{b}_A(n)$ and the second inequality is due to the fact $U_A(\cdot)$ is weakly decreasing in n. The lemma follows. □

Proof of Lemma 8

We prove Lemma 8 by applying Topkis' Monotonicity Theorem.

Proof of Lemma 8. We'll establish that $\phi_A(z, r_A, c_A) \preceq \phi_A(z, r'_A, c'_A)$. Let $\theta \in \{0, 1\}$ and let

$$f(z_A, z_B, \theta) = \begin{cases} U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B), r_A, c_A) & \text{if } \theta = 0 \\ U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B), r'_A, c'_A) & \text{if } \theta = 1. \end{cases}$$

We'll show that $f(z_A, z_B, \theta)$ is supermodular in (z_A, θ) for each $z_B \in \{0, \ldots, N\}$. Given this, Theorem 2.8.1 in Topkis [68] implies that $\rho(z_B, \theta) = \arg\max_{z_A \in \{0, \ldots, N\}} f(z_A, z_B, \theta)$ is weakly increasing in θ, i.e., that $\rho(z_B, 0) \preceq \rho(z_B, 1)$ for each $z_B \in \{0, \ldots, N\}$. Since $\phi_A(z, r_A, c_A) = \rho(z, 0)$ and $\phi_A(z, r'_A, c'_A) = \rho(z, 1)$, it follows that $\phi_A(z, r_A, c_A) \preceq \phi_A(z, r'_A, c'_A)$ for all $z \in$

$\{0, \ldots N\}$.

To show that $f(z_A, z_B, \theta)$ is supermodular in (z_A, θ), let $\theta' = 1$ and $\theta = 0$. Lemma A1 gives

$$f(z_A, z_B, \theta') - f(z_A, z_B, \theta) = U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B), r'_A, c'_A) - U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B), r_A, c_A)$$

$$\begin{aligned} f(z_A, z_B, \theta') - f(z_A, z_B, \theta) = &\sum_{\{x|d(x) \leq z_A\}} (r'_A(x) - r_A(x)) \frac{P_A(x)}{2} \\ &+ \sum_{l=1}^{z_A} \sum_{\{x|d(x)=l\}} (r'_A(x) - r_A(x)) g_A(l, N - z_B) \frac{P_A(x)}{2} + (c_A(z_A) - c'_A(z_A)). \end{aligned}$$

Since $r'_A \geq r_A$ and $c_A \geq c'_A$ (and $g_A \geq 0$ and $P_A \geq 0$), this sum is positive. Additionally, the sum is increasing in z_A – the first two terms are trivially increasing in z_A and the last term is increasing in z_A by Assumption 4. Thus, for $z'_A \geq z_A$,

$$\begin{aligned} f(z'_A, z_B, \theta') - f(z'_A, z_B, \theta) &\geq f(z_A, z_B, \theta') - f(z_A, z_B, \theta) \\ f(z'_A, z_B, \theta') + f(z_A, z_B, \theta) &\geq f(z_A, z_B, \theta') + f(z'_A, z_B, \theta), \end{aligned}$$

that is, f is supermodular in (z_A, θ). □

Proof of Lemma 9

Proof of Lemma 9. This is almost obvious. We'll establish that $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ as the case for the minimal elements is analogous. We'll work with the maximal selections of ϕ_A and ϕ_B for simplicity, which we denote $\overline{\phi}_A$ and $\overline{\phi}_B$. Let $\overline{\Phi} = (\overline{\phi}_A(z_B, r_A, c_A), \overline{\phi}_B(z_A, r_B, c_B))$ and let $\overline{\Phi}' = (\overline{\phi}_A(z_B, r'_A, c'_A), \overline{\phi}_B(z_A, r_B, c_B))$. Since $\overline{\phi}_A$ and $\overline{\phi}_B$ are increasing functions by Lemma 1 and Theorem 2.8.1 of Topkis [68], $\overline{\Phi}$ and $\overline{\Phi}'$ are increasing functions that map $\{0, \ldots, N\}^2$ into itself. Thus, Tarski's Fixed Point Theorem gives that $\overline{\boldsymbol{F}}$ the set of fixed points of $\overline{\Phi}$ and $\overline{\boldsymbol{F}}'$ the set of fixed points of $\overline{\Phi}'$ are non-empty complete lattices. It follows that $(\overline{z}_A, \overline{z}_B)$ is the maximal element of $\overline{\boldsymbol{F}}$ and $(\overline{z}'_A, \overline{z}'_B)$ is the maximal element of $\overline{\boldsymbol{F}}'$.

Lemma 8 implies that $\overline{\phi}_A(z_B, r_A, c_A) \leq \overline{\phi}_A(z_B, r'_A, c'_A)$. Thus, $\overline{\Phi} \leq \overline{\Phi}'$. We can now establish the desired result. Let $D = \{\overline{z}_A, \ldots, N\} \times \{\overline{z}_B, \ldots, N\}$. Since $\overline{\Phi}'(\overline{z}_A, \overline{z}_B) \geq \overline{\Phi}(\overline{z}_A, \overline{z}_B) = (\overline{z}_A, \overline{z}_B)$ and $\overline{\Phi}'$ is increasing, we've that $\overline{\Phi}'$ takes D into itself. Hence, Tarski's Fixed Point Theorem gives that there's a $(z'_A, z'_B) \in D \cap \overline{\boldsymbol{F}}'$. It follows that $(\overline{z}_A, \overline{z}_B) \leq (z'_A, z'_B) \leq (\overline{z}'_A, \overline{z}'_B)$. □

Proof of Proposition 7

We prove Proposition 7 by proving a more general result and then deriving the proposition as a corollary. Let $\boldsymbol{S}_E$ be the set of employment efficient strategies, let $\boldsymbol{S}_R$ be the set of weakly rational strategies, and let $\boldsymbol{S}_C$ be the set of strategies with the covering property. Let $\boldsymbol{S}_{CER} = \boldsymbol{S}_E \cap \boldsymbol{S}_C \cap \boldsymbol{S}_R$.

Proposition A1. Efficiency and Over-Investment.
Let Assumption 2 hold and let $\boldsymbol{s} = (\boldsymbol{s}_A, \boldsymbol{s}_B) = (\mathcal{N}_A, \ldots, \mathcal{N}_B, \ldots) \in \boldsymbol{S}_{CER}$ *be efficient. If* $\boldsymbol{s}$ *is not an equilibrium, then one of the two partners strictly benefits from over-investing in her network. That is,* $\boldsymbol{s} \notin \boldsymbol{E}$ *implies that either (i) there is a* $\boldsymbol{s}'_A = (\mathcal{N}'_A, \sigma'_{1A}, \sigma'_{2A})$ *such that* $U_A(\boldsymbol{s}'_A, \boldsymbol{s}_B) > U_A(\boldsymbol{s})$ *and* $|\mathcal{N}'_A| > |\mathcal{N}_A|$*, or (ii) there is a* $\boldsymbol{s}'_B = (\mathcal{N}'_B, \sigma'_{1B}, \sigma'_{2B})$ *such that* $U_B(\boldsymbol{s}_A, \boldsymbol{s}'_B) > U_B(\boldsymbol{s})$ *and* $|\mathcal{N}'_B| > |\mathcal{N}_B|$.

Proof. Since $\boldsymbol{s} \notin \boldsymbol{E}$, at least one partner benefits by unilaterally defecting. Without loss, suppose this partner is A and let $\boldsymbol{s}'_A$ be A's new strategy, so $U_A(\boldsymbol{s}'_A, \boldsymbol{s}_B) > U_A(\boldsymbol{s})$.

We first establish that there is a strategy $\boldsymbol{s}''_A$ such that (i) $(\boldsymbol{s}''_A, \boldsymbol{s}_B) \in \boldsymbol{S}_{CER}$ and (ii) $U_A(\boldsymbol{s}''_A, \boldsymbol{s}_B) \geq U_A(\boldsymbol{s}'_A, \boldsymbol{s}_B)$. We establish this fact in three steps. First, observe that we may construct an (interim) strategy $\boldsymbol{s}^R_A$ such that $(\boldsymbol{s}^R_A, \boldsymbol{s}_B) \in \boldsymbol{S}_R$ and $U_A(\boldsymbol{s}^R_A, \boldsymbol{s}_B) \geq U_A(\boldsymbol{s}'_A, \boldsymbol{s}_B)$. To do this, let $\boldsymbol{s}^R_A$ specify the same network as $\boldsymbol{s}'_A$, while specifying weakly rational behavioral strategies for A under the supposition that B follows $\boldsymbol{s}_B$. (We can construct such behavioral strategies because our game is finite.) Since Assumption 2 holds, Remarks A1 and A2 give that A gets a higher payoff under $\boldsymbol{s}^R_A$ than under $\boldsymbol{s}'_A$. (The weak inequality follows from the fact $\boldsymbol{s}'_A$ may be in $\boldsymbol{S}_R$.)

Second, observe that we may construct a strategy $\boldsymbol{s}^{RC}_A$, such that (i) $(\boldsymbol{s}^{RC}_A, \boldsymbol{s}_B) \in \boldsymbol{S}_{CR}$ and (ii) $U_A(\boldsymbol{s}^{RC}_A, \boldsymbol{s}_B) \geq U_A(\boldsymbol{s}^R_A, \boldsymbol{s}_B)$. We do this in the same manner as in the Proof of Lemma 4: swap each of A's shared consultants for a consultant who aren't in either partners' network (until all consultants are in a partner's network) and, given A's post swap network, choose weakly rational strategies for A under the hypothesis B follows $\boldsymbol{s}_B$. Since Assumption 2 holds and $\boldsymbol{s}_B$ satisfies part (i) of the definition of weak rationality for B (as $(\boldsymbol{s}^R_A, \boldsymbol{s}_B)$ is weakly rational), an argument analogous to the Proof of Lemma 4 gives that A does weakly better under $\boldsymbol{s}^{RC}_A$ than under $\boldsymbol{s}^R_A$, given B plays $\boldsymbol{s}_B$. (The weak inequality follows from the fact $\boldsymbol{s}^R_A$ may be in $\boldsymbol{S}_C$.)

Third, observe that we may construct a strategy $\boldsymbol{s}^{ARC}_A$ such that $(\boldsymbol{s}^{ARC}_A, \boldsymbol{s}_B) \in \boldsymbol{S}_{CER}$ and (ii) $U_A(\boldsymbol{s}^{ARC}_A, \boldsymbol{s}_B) = U_A(\boldsymbol{s}^{RC}_A, \boldsymbol{s}_B)$. We do this by simply re-ordering A's behavioral strategy to be employment efficient, i.e., to recommend that A employ exclusive consultants before shared consultants. Such a re-ordering doesn't change A's payoff as she employs exactly the same number of workers before and after the re-ordering. We take $\boldsymbol{s}''_A = \boldsymbol{s}^{ACR}_A$ to complete the argument.

Since $(\boldsymbol{s}_A, \boldsymbol{s}_B) \in \boldsymbol{S}_{CER}$ and $(\boldsymbol{s}''_A, \boldsymbol{s}_B) \in \boldsymbol{S}_{CER}$, Lemma A6 gives that $U_i(\boldsymbol{s}) = U_i(\tilde{\boldsymbol{s}}(n_A, n_B))$

and $U_i(\boldsymbol{s}''_A, \boldsymbol{s}_B) = U_i(\tilde{\boldsymbol{s}}(n''_A, n_B))$ for each partner i, where n_i is the size of i's network under $\boldsymbol{s}$ and n''_A is the size of i's network in $\boldsymbol{s}''_A$. By hypothesis,

$$U_A(\boldsymbol{s}''_A, \boldsymbol{s}_B) = U_A(\tilde{\boldsymbol{s}}(n''_A, n_B)) > U_A(\boldsymbol{s}). \tag{5.8}$$

Since $\boldsymbol{s}$ is efficient on $\boldsymbol{S}$,

$$\begin{aligned} U_A(\tilde{\boldsymbol{s}}(n''_A, n_B)) &\leq U_A(\boldsymbol{s}) + U_B(\boldsymbol{s}) - U_B(\tilde{\boldsymbol{s}}(n''_A, n_B)) \\ &= U_A(\boldsymbol{s}) + U_B(\tilde{\boldsymbol{s}}(n_A, n_B)) - U_B(\tilde{\boldsymbol{s}}(n''_A, n_B)). \end{aligned}$$

Since Lemma A1 gives

$$U_B(\tilde{\boldsymbol{s}}(n_A, n_B)) - U_B(\tilde{\boldsymbol{s}}(n''_A, n_B)) = \sum_{l=1}^{n_B} \sum_{\{x|d(x)=l\}} (r_B(x) - w\, d(x)) \frac{P_B(x)}{2} (g_B(l, n_A) - g_B(l, n''_A)),$$

we have

$$U_A(\tilde{\boldsymbol{s}}(n''_A, n_B)) \leq U_A(\boldsymbol{s}) + \sum_{l=1}^{n_B} \sum_{\{x|d(x)=l\}} (r_B(x) - w\, d(x)) \frac{P_B(x)}{2} (g_B(l, n_A) - g_B(l, n''_A)). \tag{5.9}$$

To complete the proof, we argue by contradiction. Suppose $n''_A \leq n_A$, i.e., A holds a smaller network after defecting. Then, $g_B(l, n_A) - g_B(l, n''_A) \leq 0$ as $g_B(l, n)$ is weakly decreasing in n. Thus, equation (5.9) implies that $U_A(\boldsymbol{s}) \geq U_A(\tilde{\boldsymbol{s}}(n''_A, n_B)) = U_A(\boldsymbol{s}''_A, \boldsymbol{s}_B)$. This is a contradiction of equation (5.8). It follows that $n''_A > n_A$. □

Remark. We cannot extend Proposition A1 to all efficient strategy vectors. The reason is that efficiency may require behavior that isn't weakly rational. The intuition is that it may be in society's interest for the first partner to pass on a difficult and low value project so as to allow the second partner a better chance of completing a complete a difficult project of very high value.

Proof of Proposition 7. Since $\boldsymbol{S}_S \subset \boldsymbol{S}_{CER}$ by Lemma 5 and the fact every simple strategy is weakly rational, Proposition 7 is an immediate corollary of Proposition A1. □

Proof of Lemma 12

Proof of Lemma 12. The proof is analogous to the proof of Lemma 8. We'll prove that $\phi_A(z, \alpha) \preceq \phi_A(z, \alpha')$ by showing that A's payoff in the Auxiliary Game is supermodular in (z, α). The argument that $\phi_B(z, \alpha) \preceq \phi_B(z, \alpha')$ is analogous.

Let $\alpha' \geq \alpha$. Lemma A1 (appropriately modified), gives that

$$U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha') - U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha) = \sum_{l=1}^{z_A} \sum_{\{x|d(x)=l\}} (r_A(x) - w\, l) P_A(x)(1 - g_A(l, z_B))\, (\alpha' - \alpha).$$

Since $g_A \in [0, 1]$, we have that $(1 - g_A(l, z_B)) \geq 0$. Thus, the summand on the right-hand-side is positive as Assumption 1 and $P_A \geq 0$. Hence, the sum is increasing in z_A. It follows for $z'_A \geq z_A$ that

$$\begin{aligned} U_A(\tilde{\boldsymbol{s}}(z'_A, z_B), \alpha') - U_A(\tilde{\boldsymbol{s}}(z'_A, z_B), \alpha) \geq U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha') - U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha) \\ U_A(\tilde{\boldsymbol{s}}(z'_A, z_B), \alpha') + U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha) \geq U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha') + U_A(\tilde{\boldsymbol{s}}(z'_A, z_B), \alpha). \end{aligned}$$

That is, $U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha)$ is supermodular in (z_A, α) for each $z_B \in \{0, \ldots, N\}$. It follows from Topkis' Monotonicity Theorem (Theorem 2.8.1 [68]) that $\phi_A(z, \alpha) \preceq \phi_A(z, \alpha')$ for all $z \in \{0, \ldots N\}$. □

Proof of Lemma 14

Proof of Lemma 14. Analogous to the proof of Lemma 10. We'll establish the result for the equilibria that are best for A and worst for B. An analogous argument applies at the equilibria that are best for B and worst for A. Let $(\overline{z}_A, \overline{z}_B)$ and $(\overline{z}'_A, \overline{z}'_B)$ be as in the statement of Lemma 13. To simplify notation, for all $(n_A, n_B) \in \{0, \ldots, N\}^2$, let $U_i(n_A, n_B)$ denote $U_i(\tilde{\boldsymbol{s}}(n_A, N - n_B), \alpha)$ and let $U'_i(n_A, n_B)$ denote $U_i(\tilde{\boldsymbol{s}}(n_A, N - n_B), \alpha')$ for each partner i. Lemma 7 implies that $U_A(z^\star_A, z^\star_B) = U_A(\overline{z}_A, \overline{z}_B)$, that $U_B(z^\star_A, z^\star_B) = U_B(\overline{z}_A, \overline{z}_B)$, that $U'_A(z'_A, z'_B) = U'_A(\overline{z}'_A, \overline{z}'_B)$, that $U'_B(z'_A, z'_B) = U_B(\overline{z}'_A, \overline{z}'_B)$. Thus, we only need to show that

$$U'_A(\overline{z}'_A \overline{z}'_B) \geq U_A(\overline{z}_A, \overline{z}_B) \text{ and } U'_B(\overline{z}'_A \overline{z}'_B) \leq U_A(\overline{z}_A, \overline{z}_B) \tag{5.10}$$

to establish the display equation of the lemma. Let's prove (5.10) for A. Let $\overline{b}_A(n)$ denote $\overline{b}_A(n, \alpha)$ and let $\overline{b}'_A(n)$ denote $\overline{b}_A(n, \alpha')$. Write

$$\begin{aligned} U_A(\overline{z}_A, \overline{z}_B) &= U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}_B), N - \overline{z}_B), \alpha) \\ &\leq U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}'_B), N - \overline{z}'_B), \alpha) \\ &\leq U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}'_B), N - \overline{z}'_B), \alpha') \\ &\leq U_A(\tilde{\boldsymbol{s}}(\overline{b}'_A(N - \overline{z}'_B), N - \overline{z}'_B), \alpha') \\ &= U'_A(\overline{z}'_A, \overline{z}'_B) \end{aligned}$$

The and fifth lines are standard. Since $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ by Lemma 13, the second line follows from Lemma 6 as $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ implies that $N - \overline{z}'_B \leq N - \overline{z}_B$. The third line follows from the Proof of Lemma 12, where we showed $U_A(\tilde{\boldsymbol{s}}(n_A, n_B), \alpha') - U_A(\tilde{\boldsymbol{s}}(n_A, n_B), \alpha) \geq 0$ for all (n_A, n_B). The fourth line follows from optimality. Since the argument for B is analogous, we've (5.10).

It remains to show that the size of A's network increases and that the size of B's network decreases. Since Assumption 3 holds, Lemma 7 implies $(z^\star_A, z^\star_B) = (\overline{z}_A, \overline{z}_B)$ and that $(z'_A, z'_B) = (\overline{z}'_A, \overline{z}'_B)$. Thus, the desired result follows directly from Lemma 13. □

6 Appendix: Supplement to Chapter 2

This chapter collects several supplemental results to the games introduced in Chapter 2. Please note that all references (e.g., Proposition X, Lemma Z, etc.) in this chapter refer to the items in Chapter 2.

- Section 5.1 shows that the General Game and the Effort Game don't preserve the classic lattice or law of demand properties of the Marriage and Assignment Games.

- Section 5.2 proves Lemma 1.

- Section 5.3 gives a generalization of Proposition 1 that allows for certain discontinuous payoffs – it requires the payoff functions be increasing transformations of upper-semicontinuous functions. It also gives several examples that illustrate why existence fails absent sufficient continuity or compactness. It closes by discussing how to generalize the agreement space to allow players to agree to gambles with infinite outcomes, like investment strategies.

- Section 5.4 gives a few additional results on the structure of matchings in the General Game and the Effort Game. For the General Game, it gives a "Rural Hospitals Theorem," i.e., a test to determine whether a player is partnered in every stable allocation. For the Effort Game, it (i) shows that the set of stable allocations partitions the type space into the set of players who are always, sometimes, and never partnered, and (ii) gives strong conditions that ensure higher types produce strictly higher benefits than lower types. A consequence of (ii) is that every interior stable allocation exhibits assortative matching in types.

- Section 5.5 gives two additional comparatives statics results for the Effort Game. First, it presents an example that shows the conclusions of Proposition 9 don't always hold. Second, it shows a novel and somewhat strong comparative statics result: when there are an equal number of men and women who find each other individually rational, then a decrease in fixed cost of effort increases the set of stable allocations (in the set of inclusion) and so optimal social welfare increases.

- Section 5.6 discusses welfare maximization in the Effort Game and gives an example of an Effort Game where *no* stable allocation maximizes welfare. Since other examples can be readily constructed, we conclude that welfare maximizing stable allocations do not usually exist.

6.1 Loss of Lattice Structure

This section shows that the General Game and the Effort Game don't preserve the classical lattice properties of (i) the Marriage Game in allocation space or (ii) the Assignment Game in payoff space.[66] In the Marriage Game, this failure is due to the fact a certain intermediate binary relation isn't anti-symmetric. In the Assignment Game, this failure occurs because the stable set may not to contain the meet or join of its elements.

We begin by considering the lattice property in allocation space. Consider a binary relation $\preccurlyeq_{\mathcal{M}}$ on $\mathcal{A}$ such that, for any two allocations $(\phi, \bar{\boldsymbol{x}}) \in \mathcal{A}$ and $(\phi', \bar{\boldsymbol{x}}') \in \mathcal{A}$, $(\phi, \bar{\boldsymbol{x}}) \preccurlyeq_{\mathcal{M}} (\phi', \bar{\boldsymbol{x}}')$ if every man does *weakly* better in $(\phi', \bar{\boldsymbol{x}}')$ than in $(\phi, \bar{\boldsymbol{x}})$.[67] The relation $\preccurlyeq_{\mathcal{M}}$ is reflexive and transitive on both $\mathcal{A}$ and $\mathcal{S}$. In the Marriage Game, Roth and Sotomayor (Theorem 2.16, [60]) establish that $\preccurlyeq_{\mathcal{M}}$ is antisymmetric on $\mathcal{S}$ by showing that $(\mathcal{S}, \preccurlyeq_{\mathcal{M}})$ is a lattice when preferences (over matches and agreements) are strict. However, $(\mathcal{S}, \preccurlyeq_{\mathcal{M}})$ is generally not a lattice when players are indifferent because $\preccurlyeq_{\mathcal{M}}$ is not antisymmetric. The next example illustrates.

Example S1. Lack of Antisymmetry.

Let $\mathcal{M} = \{1, 2\}$, $\mathcal{W} = \{3\}$, and let $X = \{0, 1\}$. When a man m and woman w are matched, their payoffs from agreement/action $x \in \mathbb{R}$ are $u_m(w, x) = x$ and $u_w(m, x) = 1$. If a player is single, he or she gets zero.

There are four stable allocations:

1. Man 1 is matched to woman 3 with agreement 0 while man 2 is single with an arbitrary agreement.

2. Man 1 is matched to woman 3 with agreement 1 while man 2 is single with an arbitrary agreement.

3. Man 2 is matched to woman 3 with agreement 0 while man 1 is single with an arbitrary agreement.

4. Man 2 is matched to woman 3 with agreement 1 while man 1 is single with an arbitrary agreement.

Let (u_1, u_2) record the payoffs of each man. Then we have that the men's earnings are $(0, 0)$ in (1), $(1, 0)$ in (2), $(0, 0)$ in (3), and $(0, 1)$ in (4).

[66] Other classical properties of the Marriage and Assignment Games also fail to hold in our games – e.g., the "law of demand" and the "Rural Hospitals Theorem." We omit a discussion of the former and discuss the latter briefly in Section 4.

[67] The subsequent discussion is unchanged if we define this relation in terms of woman's payoffs.

It's easily seen that $\preccurlyeq_{\mathcal{M}}$ isn't antisymmetric. Consider stable allocations (1) and (3). In both of these allocations, the men earn zero, so (1) $\preccurlyeq_{\mathcal{M}}$ (3) and (3) $\preccurlyeq_{\mathcal{M}}$ (1). Yet, we know that (1) $\neq$ (3), it follows that $\preccurlyeq_{\mathcal{M}}$ isn't antisymmetric. Thus, $(\mathcal{S}, \preccurlyeq_{\mathcal{M}})$ isn't a lattice.

It follows, for instance, that $\mathcal{S}$ needn't contain a "man-preferred" stable allocation, i.e., an allocation that all men prefer to any other stable allocation. Indeed, man 1 likes (2) most and man 2 likes it least, while man 2 likes (4) most and man 2 likes it least. △

As the next example shows, even the additional structure of the Effort Game is of no help in restoring the antisymmetry of $\preccurlyeq_{\mathcal{M}}$.

Example S2. Lack of Antisymmetry in the Effort Game.

Let $\mathcal{M} = \{1\}$, $\mathcal{W} = \{2, 3\}$, and let $X = [0, 1]^2$. Let θ_1 be arbitrary but set $\theta_2 = \theta_3$, so women 2 and 3 have the same benefit production function and the same cost of effort function and assume that Assumption 4 holds and that agreeability is strict (i.e., there is an $\boldsymbol{x} \in X$ such that (i) $u_1(2, \boldsymbol{x}) > 0$ and $u_2(1, \boldsymbol{x}) > 0$).

By the remark after the Proof of Lemma 6, there's a stable allocation that maximizes man 1's payoff. Let $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{3^\star})$ denote this allocation. For the sake of argument, assume that $\phi^\star(1) = 2$. We can then create a new stable allocation $(\phi', \bar{\boldsymbol{x}}')$ by having 1 and 3 match with agreement $\boldsymbol{x}^{1^\star}$ and having 2 be single. Since 2 and 3 are clones, we have $u_1(\phi', \bar{\boldsymbol{x}}') = u_1(\phi^\star, \bar{\boldsymbol{x}}^\star)$, $u_3(\phi', \bar{\boldsymbol{x}}') = u_2(\phi^\star, \bar{\boldsymbol{x}}^\star)$, and $u_2(\phi', \bar{\boldsymbol{x}}') = u_3(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Thus, $(\phi', \bar{\boldsymbol{x}}')$ is stable and also maximizes 1's payoff. Hence, $(\phi^\star, \bar{\boldsymbol{x}}^\star) \preccurlyeq_{\mathcal{M}} (\phi', \bar{\boldsymbol{x}}')$ and $(\phi', \bar{\boldsymbol{x}}') \preccurlyeq_{\mathcal{M}} (\phi^\star, \bar{\boldsymbol{x}}^\star)$, yet $(\phi', \bar{\boldsymbol{x}}') \neq (\phi', \bar{\boldsymbol{x}}')$, implying $\preccurlyeq_{\mathcal{M}}$ is not antisymmetric on $\mathcal{S}$ and that $(\mathcal{S}, \preccurlyeq_{\mathcal{M}})$ isn't a lattice. (If $\phi^\star(1) = 3$, reverse the roles of the women in the preceding argument.) △

Remark. There is an active literature that seeks to weaken the conditions under which $\preccurlyeq_{\mathcal{M}}$ is antisymmetric on $\mathcal{S}$. See, for instance, Eriksson and Karlander [26] and Sotomayor [66].

We next consider the lattice property in payoff space. Let $\mathcal{U} \subset \mathbb{R}^M$ denote the set of payoffs obtained by men in $\mathcal{S}$, i.e.,

$$\mathcal{U} = \{(v_1, \ldots, v_M) \in \mathbb{R}^M | \text{there is a } (\phi^\star, \bar{\boldsymbol{x}}^\star) \in \mathcal{S} \text{ with } v_i = u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) \, \forall \, i \in \mathcal{M}\}.$$

The analogue of $\preccurlyeq_{\mathcal{M}}$ on $\mathcal{U}$ is the standard order $\leq$. In the Assignment Game, Demange and Gale (Property 2, [24]) establish that $\mathcal{U}$ is a lattice under $\leq$. However, $(\mathcal{U}, \leq)$ may not be a lattice in the General Game because $\mathcal{U}$ may fail to contain the meet/join of its elements. Again, consider Example 1. Herein, $\mathcal{U} = \{(0,0), (1,0), (0,1)\}$. Since $\sup_{\leq}\{(1,0), (0,1)\} = (1,1)$ is not in $\mathcal{U}$, $(\mathcal{U}, \leq)$ isn't a lattice. It's unclear whether $\mathcal{U}$ must be a lattice in the Effort Game.

6.2 Proof of Lemma 1

In this section, we prove Lemma 1. We do this by adapting Roth's [59] generalization of the Deferred Acceptance algorithm.[68] We make and maintain the following assumption throughout this section.

Assumption. The set of feasible agreements X is finite.

It is convenient to refer to a match between man m and woman w with agreement $\boldsymbol{x}$ as a triple $(m, w, \boldsymbol{x})$. Likewise, we write $(i, i, \boldsymbol{x})$ for the match between player i and himself/herself with agreement $\boldsymbol{x}$. Let O be the finite set representing all possible matches and agreements. For each man m and for each $o' = (m', w', \boldsymbol{x}') \in O$ with $m' = m$, we write $\tilde{u}_m(o') \equiv u_m(w', \boldsymbol{x}')$ for the payoff of m to o'. Likewise, for each woman w and each $o' = (m', w', \boldsymbol{x}') \in O$ with $w' = w$, we write $\tilde{u}_w(o') \equiv u_w(m', \boldsymbol{x}')$ the payoff of w to o'.

For each man m, let $O_m = \{m\} \times \{\mathcal{W} \cup \{m\}\} \times X$ be m's finite set of (possible) matches and agreements.

Algorithm. Modified Deferred Acceptance (Men Propose).

Step 0 Match every man and woman to himself or herself and assign each an arbitrary feasible agreement. For each woman w, let $(w, w, \boldsymbol{x}^w)$ denote w's current match and agreement.

Step 1 For each man m:

Let m select his most-preferred (possible) match and agreement, i.e., his most-preferred element in O_m according to $\tilde{u}_m$, breaking ties in an arbitrary manner. Label his selection o_m^1. We interpret o_m^1 as m's new offer.

For each woman w:
Let R_w^1 be the collection of offers she receives and $(w, w, \boldsymbol{x}^w)$, i.e.,

$$R_w^1 = \{(w, w, \boldsymbol{x}^w)\} \cup \{(m', w', \boldsymbol{x}') \in \cup_{m \in \mathcal{M}} \{o_m^1\} | w' = w\}.$$

From this set, w selects her most-preferred element according to $\tilde{u}_w(\cdot)$, breaking ties in an arbitrary manner. Label her selection o_w^1. We interpret o_w^1 as the offer w tentatively accepts, and we interpret every element in $R_w^1 \backslash \{o_w^1\}$ as a rejected offer. (We allow w to accept remaining single including $(w, w, \boldsymbol{x}^w)$.)

[68] Another way to prove Lemma 1 would be as follows. First, force players to rank the matches/agreements to which they are indifferent – this is possible since the set of feasible agreements is finite. This ranking indices a strict preference relationship for each player. Second, feed the strict induced preferences into Hatfield and Milgrom's [34] algorithm to obtain an allocation that is stable under the induced preferences. Third, observe that since the induced preferences respect the strict part of the original preferences, the allocation found in the second step is a stable under the original preferences.

$\vdots$ $\qquad\qquad$ $\vdots$

Step l $\quad$ For each man m:

1. If m's offer $o_m^{l-1} = (m, w, \boldsymbol{x})$ was accepted by woman w in step $l-1$, i.e., if $o_w^{l-1} = o_m^{l-1}$, then he sets $o_m^l = o_m^{l-1}$, i.e., his offer remains in-force.
2. If m's offer o_m^{l-1} was not accepted in step $l-1$, either:
   (a) He made an offer to himself, i.e., $o_m^{l-1} = (m, m, \boldsymbol{x})$.
   If so, then he sets $o_m^l = o_m^{l-1}$, i.e., he makes the same offer to himself.
   (b) His offer was rejected, i.e., $o_m^{l-1} = (m, w, \boldsymbol{x}) \neq o_w^{l-1}$.
   If so, then he select the most-preferred element in O_m that was not rejected in an earlier step, i.e., select the most-preferred element of $O_m \backslash \cup_{j=1}^{l-1} \{o_m^j\}$, breaking ties in an arbitrary manner. Label his selection o_m^l, we interpret this is m's new offer.

For each woman w, let

$$R_w^l = \{(w, w, \boldsymbol{x}^w)\} \cup \{(m', w', \boldsymbol{x}') \in \cup_{m \in \mathcal{M}} \{o_m^l\} | w' = w\}.$$

From this set, she selects her most-preferred offer, breaking ties in an arbitrary manner. Label her selection o_w^l. We interpret o_w^l as the offer w tentatively accepts, and we interpret every element in $R_w^l \backslash \{o_w^l\}$ as a rejected offer.

Stopping $\quad$ The algorithm stops in step T when, for each man m, either (i) m makes himself an offer, i.e., $o_m^T = (m, m, \boldsymbol{x})$, or (ii) the offer m makes to woman w is accepted by w, i.e., $o_m^T = (m, w, \boldsymbol{x})$ implies that $o_w^T = o_m^T$. ○

Two observations are useful.

Observation S1. *Each man m makes (at most) a finite number of new offers.*

Since O_m is finite, there are only a finite number of offers m (weakly) prefers to being single. Since m never remakes offers that have been rejected (i.e., if m's offer o_m^l is rejected in step l, he chooses a new offer from $O_m \backslash \cup_{j=1}^{l} \{o_m^j\}$ in step $l+1$), once he accumulates a sufficiently large number of rejections, he's exhausted every offer that he prefers to bachelorhood and, thus, he remains single by selecting an offer of the form $(m, m, \boldsymbol{x})$. Hence, he stops making new offers.

Observation S2. *A woman w accepts better offers as the algorithm progresses, i.e., for two steps l and l' with $l \leq l'$, we have $u_w(o_w^l) \leq u_w(o_w^{l'})$.*

This follows from the proposal structure: If w accepts an offer at step l, the man who made it makes it again in step $l+1$. (If this offer is her outside option, it's present at $l+1$ by default.) Thus, w cannot do no worse at $l+1$ than she did at l. The observation follows from induction.

Lemma S1. Stopping.
The algorithm stops after a finite number of steps, i.e., $T < \infty$.

Proof. We establish the result by contradiction. Suppose $T = \infty$, i.e., the algorithm does not stop. Then, in each step, some man makes an offer to a woman that she rejects. In the next step, this man necessarily makes a new offer. Thus, in each step (after step zero), at least one man makes a new offer. Since there are a finite number of men, we have that at least one man makes an infinite number of new offers. This contradicts Observation S1. □

When the algorithm stops, there is a o_m^T for each man m and o_w^T for each woman w. We use these to construct an matching and vector of agreement $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$ as follows:

1. For each $m \in \mathcal{M}$:
   (a) If $o_m^T = (m, m, \boldsymbol{x})$ for some $\boldsymbol{x} \in X$, set $\phi^\star(m) = m$ and set $\boldsymbol{x}^{m^\star} = \boldsymbol{x}$.
   (b) If $o_m^T = (m, w, \boldsymbol{x})$ for some $(w, \boldsymbol{x}) \in \mathcal{W} \times X$, set $\phi^\star(m) = w$ and set $\boldsymbol{x}^{m^\star} = \boldsymbol{x}$.
2. For each $w \in \mathcal{W}$:
   (a) If $o_w^T = (w, w, \boldsymbol{x})$ for some $\boldsymbol{x} \in X$, set $\phi^\star(w) = w$ and set $\boldsymbol{x}^{w^\star} = \boldsymbol{x}$.
   (b) If $o_w^T = (m, w, \boldsymbol{x})$ for some $(m, \boldsymbol{x}) \in \mathcal{M} \times X$, set $\phi^\star(w) = m$ and set $\boldsymbol{x}^{w^\star} = \boldsymbol{x}$.

By the stopping condition and the method of construction, we have that $\phi^\star$ is a matching and that $\boldsymbol{x}^{i^\star} = \boldsymbol{x}^{\phi^\star(i)^\star}$ for all $i \in \mathcal{N}$.[69] Thus, $(\phi^\star, \bar{x}^\star)$ is an allocation. Observe that

$$u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) = \tilde{u}_i(o_i^T)$$

for all $i \in \mathcal{N}$.

Lemma S2. Stability.
The allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable.

Proof. The allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is automatically feasible and individually rational since (i) all offers specify agreements in X and (ii) players don't make or accept offers that make

[69]Simply, if man m has offer $o_m^T = (m, w, \boldsymbol{x})$ then woman w has accepted offer o_m^T. Thus, steps (1.a) and (2.b) give $\phi^\star(m) = w$ and $\phi^\star(w) = m$, while steps (1.b) and (2.b) give $\boldsymbol{x}^{m^\star} = \boldsymbol{x} = \boldsymbol{x}^{w^\star}$. The argument is analogous if a man or a woman made an offer to himself or herself.

them worse off than if they're single. Thus, we only need to establish stability. To these ends, we'll establish, for each man m and each woman w, that $u_m(w, \boldsymbol{x}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for an $\boldsymbol{x} \in X$ implies that $u_w(m, \boldsymbol{x}) \leq u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Given this, m and w can't block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, stability follows.

Let m be a man and suppose that $u_m(w, \boldsymbol{x}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for some woman w and $\boldsymbol{x} \in X$, i.e., tuple $(m, w, \boldsymbol{x})$ gives m a higher payoff than he gets from his T-step offer o_m^T. Since m makes better offers before worse ones, we have that m offered $(m, w, \boldsymbol{x})$ before he offered o_m^T. Since m only makes new offers if his current offer is rejected, we also have w rejected offer $(m, w, \boldsymbol{x})$. Let l be the step in which this rejection occurred and let o_w^l be the offer w accepted in step l. Since w accepts one of the best offers she receives, we have $\tilde{u}_w(o_w^l) \geq u_w(m, \boldsymbol{x})$. Thus, $u_w(m, \boldsymbol{x}) \leq u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$ by Observation S2. □

Proof of Lemma 1. Run the Modified Deferred Acceptance algorithm. By Lemma S1, the algorithm stops produces an allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ that is stable by Lemma S2. □

6.3 Discontinuous Payoffs and Existence

In this section, we give three results. The first result generalizes Proposition 1 to allow for upper semi-continuous functions. The second result further generalizes Proposition 1 to allow the payoffs to be increasing transformations of upper semi-continuous functions. We then present two examples; the first illustrates the utility of these generalizations, while the second demonstrates their inessentiality. Motivated by the second example, we take a first step towards developing necessary conditions for existence by providing sufficient conditions for non-existence. We close by discussing a generalization of the agreement space that's useful for modeling agreements about gambles and by providing an illustrative example.

6.3.1 Existence for Upper-Semicontinuous Payoffs

We begin by replacing Assumption 1 with the following weaker assumption.

Assumption S1. Compactness and Upper-Semicontinuity.
The set of feasible agreements X is compact and, for each man m and each woman w, $u_m(w', \boldsymbol{x})$ and $u_w(m', \boldsymbol{x})$ are upper-semicontinuous in $\boldsymbol{x}$ for all women w' and for all men m' respectively.

Proposition S1. Generalization of Proposition 1
Let Assumption S1 hold, then a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exists.

Proof. When payoffs are upper-semicontinuous, the proof of Proposition 1 is unchanged since Lemmas 3 and 4 remain true. Simply, $u_i(\phi, \bar{\boldsymbol{x}})$ is upper-semicontinuous. Thus, for each player i, the set $\{\bar{\boldsymbol{x}}|u_i(\phi,\bar{\boldsymbol{x}}) \geq 0\}$ is closed and, for each $\alpha \in \mathbb{R}$, the set $\{\bar{\boldsymbol{x}}|u_i(\phi,\bar{\boldsymbol{x}}) < \alpha\}$ (Efe [50], Proposition 4). It follows that (i) $F(\phi)$ is the intersection of closed sets and is closed and (ii) $D_\phi(c)$ is the finite intersection of opens sets and so is open. Hence, Lemmas 3 and 4 hold. □

We cannot extend this result to all non-upper-semicontinuous functions. Simply, if the functions are not upper semicontinuous, then the $D_\phi(c)$ sets need not be open. If this happens, then we cannot argue for the existence of a stable allocation using the logic of Proposition 1. The next example illustrates.

Example S3. Non-Upper-Semicontinuity.

Let $\mathcal{M} = \{1\}$ and $\mathcal{W} = \{2\}$. Let $X = [0,1]$ and suppose that when players match, their payoffs from agreement $x \in \mathbb{R}$ are

$$u_1(2,x) = \begin{cases} x & \text{if } x < 1/2 \\ 0 & \text{if } x \geq 1/2 \end{cases} \text{ and } u_2(1,x) = \begin{cases} x & \text{if } x < 1/2 \\ 0 & \text{if } x \geq 1/2. \end{cases}$$

Observe that these payoffs are not upper-semicontinuous.

There are two possible matchings in Φ, ϕ_s and ϕ_p. Under ϕ_s, both players are single, so $\phi_s(1) = 1$ and $\phi_s(2) = 2$. Under ϕ_p, both players are partnered, i.e., $\phi_p(1) = 2$.

Consider the allocations that man 1 and woman 2 can block, i.e., consider the $D_\phi(m,w,\boldsymbol{x})$ sets. We have

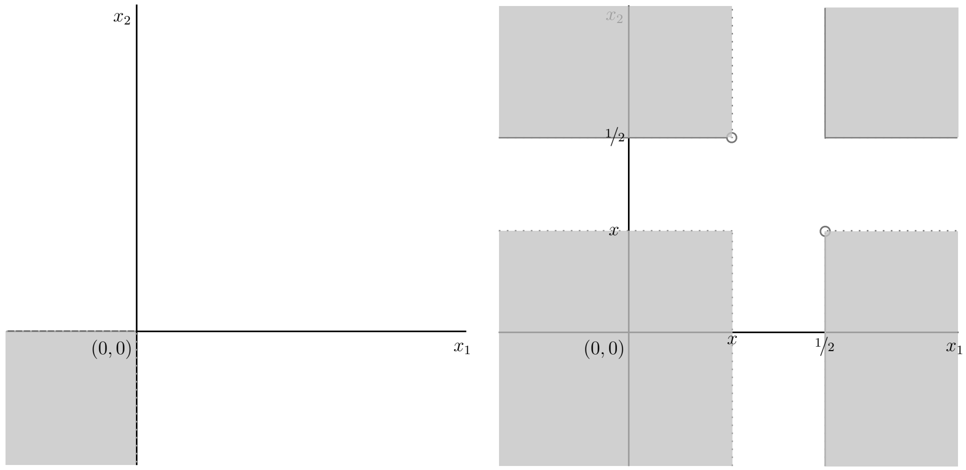
$$\begin{aligned} D_{\phi_s}(1,2,x) &= \{\bar{\boldsymbol{x}} \in \mathbb{R}^2 |\, u_1(2,x) > u_1(\phi_s,\bar{\boldsymbol{x}}) \text{ and } u_2(1,x) > u_2(\phi_s,\bar{\boldsymbol{x}})\} \\ &= \begin{cases} \mathbb{R}^2 & \text{if } \boldsymbol{x} \in (0, 1/2) \\ \emptyset & \text{else,} \end{cases} \end{aligned}$$

since (i) $u_1(2,x)$ and $u_2(1,x)$ are strictly positive if and only if $x \in (0, 1/2)$ and (ii) single players get zero.

We also have that

$$\begin{aligned} D_{\phi_p}(1,2,x) &= \{\bar{\boldsymbol{x}} \in \mathbb{R}^2 |\, u_1(2,x) > u_1(\phi_p,\bar{\boldsymbol{x}}) \text{ and } u_2(1,x) > u_2(\phi_p,\bar{\boldsymbol{x}})\} \\ &= \begin{cases} (-\infty,0)^2 & \text{if } x \in \{0\} \cup [1/2, 1] \\ (-\infty,x)^2 \cup [1/2,\infty)^2 \cup (-\infty,x) \times [1/2,\infty) \\ \qquad \cup [1/2,\infty) \times (-\infty,x) & \text{if } x \in (0, 1/2). \end{cases} \end{aligned}$$

Figure 3.1 illustrates both cases. Let's compute the second line. If $x \in \{0\} \cup [1/2, 1]$, then $u_1(2,x) = 0$ and $u_2(1,x) = 0$. Thus, the player can only block $(\phi_p,\bar{\boldsymbol{x}})$ with $\bar{\boldsymbol{x}} \leq (0,0)$. If



(a) When $x \in \{0\} \cup [1/2, 1]$ (b) When $x \in (0, 1/2)$

The gray regions and the solid lines are $D_{\phi_p}(\cdot)$, the dashed lines are the boundaries.

Figure 6.1: Illustration of $D_{\phi_p}(1, 2, x)$

$x \in (0, 1/2)$, then $u_1(2, x) = x$ and $u_2(1, \boldsymbol{x}) = x$. Thus, the man and the woman block any $(\phi_p, \bar{\boldsymbol{x}}) = (\phi_p, x^1, x^2)$ with (i) $x_1 < x$ and $x_2 < x$, (ii) $x_1 < x$ and $x_2 \geq 1/2$, (iii) $x_1 \geq 1/2$ and $x_2 < x$, or (iv) $x_1 \geq 1/2$ and $x_2 \geq 1/2$.

While $D_{\phi_s}(1, 2, x)$ is open for each $x \in X$, $D_{\phi_p}(1, 2, x)$ is not open. Thus, we cannot apply the logic of Proposition 1 to conclude that this game has a stable allocation.

In fact, we can prove that no stable allocation exists by arguing from primitives. Since man 1 and woman 2 get zero from (i) being single or (ii) pairing and taking an action in $\{0\} \cup [1/2, 1]$, it is best to have them match and take an action $x \in (0, 1/2)$. Thus, every candidate stable allocation $(\phi, \bar{\boldsymbol{x}})$ has $\phi(1) = 2$ and $\bar{\boldsymbol{x}} = (x, x)$ with $x \in (0, 1/2)$. Since $(0, 1/2)$ is open, there is a $y \in (0, 1/2)$ that both players prefer to x, i.e., $u_1(2, y) = y > u_1(\phi, \bar{\boldsymbol{x}}) = x$ and $u_2(1, y) = y > u_2(\phi, \bar{\boldsymbol{x}}) = x$. Thus, the players block the candidate. Since this holds for every $x \in (0, 1/2)$, there is no stable allocation. $\triangle$

6.3.2 Extending Existence

Fortunately, there is a way we can extend Proposition S1 to certain types of functions that are not upper-semicontinuous. The next lemma shows that stable allocations are preserved by weakly increasing transformations of the payoffs. We use $\{u_i\}_{i \in \mathcal{N}}$ to denote the payoffs of a game. Recall that if $f(x) : \mathbb{R} \to \mathbb{R}$ is a weakly increasing function, then $f(x) > f(y)$ implies that $x > y$.

Lemma S3. Preservation under Increasing Transformation.

Let $(\phi^\star, \bar{x}^\star)$ be a stable allocation when the payoffs are $\{u_i\}_{i\in\mathcal{N}}$. For each $i \in \mathcal{N}$, let $f_i : \mathbb{R} \to \mathbb{R}$ be a weakly increasing function such that $f_i(0) = 0$.[70] *Then, $(\phi^\star, \bar{x}^\star)$ is a stable allocation when the payoffs are $\{f_i(u_i)\}_{i\in\mathcal{N}}$. Additionally, if f_i is strictly increasing for each $i \in \mathcal{N}$ and $(\phi^\star, \bar{x}^\star)$ is also Pareto optimal when the payoffs are $\{u_i\}_{i\in\mathcal{N}}$, then $(\phi^\star, \bar{x}^\star)$ is Pareto optimal when the payoffs are $\{f_i(u_i)\}_{i\in\mathcal{N}}$.*

Proof. The first part of the proof is almost obvious since monotone transforms don't affect the individual rationality or the blocking relationship. Let $\{u_i\}_{i\in\mathcal{N}}$ denote the payoffs and let $\{f_i(u_i)\}_{i\in\mathcal{N}}$ denote the payoffs after the transformation. Let $\mathcal{S}$ denote the set of stable allocations when payoffs are $\{u_i\}_{i\in\mathcal{N}}$ and let $\mathcal{S}'$ denote the set of stable allocations after the transformation. Since Assumption 1 holds, $\mathcal{S}$ is non-empty. Our goal is to show that $\mathcal{S} \subset \mathcal{S}'$.

We prove $\mathcal{S} \subset \mathcal{S}'$ via contradiction. Let $(\phi^\star, \bar{x}^\star) \in \mathcal{S}$ and suppose $(\phi^\star, \bar{\boldsymbol{x}}^\star) \notin \mathcal{S}'$. There are two possibilities (i) $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is not individually rational or (ii) $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is blocked. If (i), then there is a player i with $f_i(u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)) < 0$. This implies that $u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) < 0$, a contradiction. If (ii), then there's a man m, a woman w, and an agreement $\boldsymbol{x} \in X$ such that

$$f_m(u_m(w, \boldsymbol{x})) > f_m(u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)) \text{ and } f_w(u_w(m, \boldsymbol{x})) > f_w(u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)).$$

Since the functions f_m and f_w are increasing, we have

$$u_m(w, \boldsymbol{x}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \text{ and } u_w(m, \boldsymbol{x}) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star).$$

That is, m and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction.

To complete the proof, we only need show the preservation of Pareto optimality. We do this via contradiction. Let $(\phi^\star, \bar{x}^\star)$ be a Pareto stable allocation when the payoffs are $\{u_i\}_{i\in\mathcal{N}}$ and let f_i be strictly increasing for each $i \in \mathcal{N}$. Suppose that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is not Pareto optimal when payoffs are $\{f_i(u_i)\}_{i\in\mathcal{N}}$. Then, there is a $(\phi', \bar{\boldsymbol{x}}')$ with (i) $f_i(u_i(\phi', \bar{\boldsymbol{x}}')) \geq f_i(u_i(\phi^\star, \bar{\boldsymbol{x}}^\star))$ for all $i \in \mathcal{N}$ and (ii) $f_i(u_i(\phi', \bar{\boldsymbol{x}}')) > f_i(u_i(\phi^\star, \bar{\boldsymbol{x}}^\star))$ for some $i \in \mathcal{N}$. Since f_i is strictly increasing, we have (i) $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$ and (ii) $u_i(\phi', \bar{\boldsymbol{x}}') > u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for some $i \in \mathcal{N}$. Thus, $(\phi^\star, \bar{x}^\star)$ is not Pareto, a contradiction. □

The lemma implies that interior (Pareto) stable allocations are also preserved under weakly increasing transformations. Notice that the transformation function does not depend on the identity of player i's match. Indeed this is necessary: if f_i depended on the identity of i's match then the transformation would change i's strict ranking of matches and agreements, causing the lemma to fail. The next result is our general existence result.

Assumption S2. Generalization of Assumption 1.

[70] We require $f_i(0) = 0$ so that i's value of being single remains zero after we transform payoffs.

The set of feasible agreements X is compact and players' payoffs $\{u_i\}_{i\in\mathcal{N}}$ are such that:
(i) $u_i = f_i(\tilde{u}_i)$ for each player i, where $f_i : \mathbb{R} \to \mathbb{R}$ is a weakly increasing function for each player i such that $f_i(0) = 0$.
(ii) $\tilde{u}_i(i, \boldsymbol{x}) = 0$ for each player i.
(ii) for each man m and each woman w, the payoffs $\tilde{u}_m(w', \boldsymbol{x})$ and $\tilde{u}_w(m', \boldsymbol{x})$ are upper-semicontinuous in $\boldsymbol{x}$ for all women w' and for all men m' respectively.

Proposition S2. General Existence.
Let Assumption S2 hold, then there is a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

Proof. By Proposition S1, there is a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ in the game with payoffs $\{\tilde{u}_i\}_{i\in N}$. By Lemma S3, $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ remains stable when the payoffs are $\{u_i\}_{i\in\mathcal{N}}$. □

Proposition S2 is useful because it ensures existence for certain games without upper-semicontinuous payoffs. The next example illustrates.

Example S4. Illustration of Proposition S2.

Let there be an equal number of men and women, i.e., $N = 2M$, and let $X = [0, 2]^2$. When a man m and woman w are matched, their payoffs from agreement $(x_1, x_2) \in \mathbb{R}^2$ are

$$u_m(w, x_1, x_2) = \begin{cases} 1 & \text{if } x_1 + x_2 \leq 2 \\ 2 & \text{if } x_1 + x_2 > 2 \end{cases} \quad \text{and } u_w(m, x_1, x_2) = \begin{cases} 2 & \text{if } x_1 + x_2 \leq 2 \\ 1 & \text{if } x_1 + x_2 > 2. \end{cases}$$

As usual, single players get zero.

It is readily verified that $u_m(\cdot)$ is lower semicontinuous and that $u_w(\cdot)$ is upper semicontinuous. Thus, to apply Proposition S2, we need to show that $u_m(\cdot)$ is an increasing transformation of an upper semicontinuous function for each man m. Consider the increasing function

$$f_m(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } 0 < t \leq 2 \\ 2 & \text{if } t > 2, \end{cases}$$

and the payoff $\tilde{u}_m(w, x_1, x_2) = x_1 + x_2$ when $w \in \mathcal{W}$ and $\tilde{u}_m(m, x_1, x_2) = 0$. The function $f_m(t)$ is clearly increasing and $\tilde{u}_m$ is clearly continuous in (x_1, x_2). Since (i) $u_m(w, x_1, x_2) = f_m(\tilde{u}_m(w, x_1, x_2))$ for each woman w and since $u_m(m, x_1, x_2) = f_m(\tilde{u}_m(m, x_1, x_2))$, the antecedents of Proposition S2 are satisfied. Hence, there is a stable allocation.

One class of stable allocations is described as follows. Let $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$ be such that (i) each man m is matched to a woman, i.e., $\phi^\star(m) \in \mathcal{W}$, and (ii) all players have the same agreement with their match, i.e., $\boldsymbol{x}^{i^\star} = (x_1^\star, x_2^\star) \in X$ for all $i \in \mathcal{N}$. Let's establish the stability of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. For concreteness, take $x_1^\star + x_2^\star \le 2$. Then $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) = 1$ for each man m and $u_w(\phi^\star, \bar{\boldsymbol{x}}^\star) = 2$ for each woman w. For some man m' and woman w' to block, we need an $(x_1, x_2) \in X$ such that $u_{m'}(w', x_1, x_2) > 1$ and $u_{w'}(m', x_1, x_2) > 2$. Inspection shows that such an (x_1, x_2) does not exist.

Observe that the set of achievable payoffs for a man m and a woman w is

$$\begin{aligned} V_{mw} &= \{(v_m, v_w) \in \mathbb{R}^2 | \text{ there's a } \boldsymbol{x} \in X \text{ with } u_m(w, \boldsymbol{x}) = v_m \text{ and } u_w(m, \boldsymbol{x}) = v_w\} \\ &= \{(1,2), (2,1)\}. \end{aligned}$$

Thus, we have another example that satisfies none of the antecedents of the classical existence results of [4, 34, 41, 59]. △

Remark. Throughout the main paper and this section, we have assumed that payoffs are (i) defined on all of $\mathbb{R}^k$ and (ii) are real-valued. Both assumptions may be weakened – we may define payoffs on a subset of $\mathbb{R}^k$ containing X and let payoffs take values in the extended reals.[71] We omit a formal treatment since it adds no insight. We do, however, discuss the usefulness of payoffs defined over function spaces at the end of this section.

So far, we have taken X to be compact subset of $\mathbb{R}^k$. Unfortunately, there is very little that we can say when X is open or unbounded. Basically, if X is open, then the $F(\phi)$ sets may fail to be compact, even when payoffs are upper semicontinuous. This prevents us from using the arguments of Proposition 1 to prove existence. The next example illustrates.

Example S5. Non-Compactness.

Let $\mathcal{M} = \{1\}$ and $\mathcal{W} = \{2\}$. Let $X = (0, 1)$ and suppose that the man and woman match players match, their payoffs from agreement $x \in \mathbb{R}$ are

$$u_1(2, x) = x \text{ and } u_2(1, x) = x.$$

There are two possible matchings in Φ, ϕ_s and ϕ_p. Under ϕ_s, both players are single, i.e., $\phi_s(1) = 1$ and $\phi_s(2) = 2$. Under ϕ_p, both players are partnered, i.e., $\phi_p(1) = 2$.

Consider the $F(\phi)$ sets. We have

$$\begin{aligned} F(\phi_s) &= \{\bar{\boldsymbol{x}} \in \mathbb{R}^2 | \, \bar{\boldsymbol{x}} \in X^2 \cap A(\phi_s) \text{ and } u_i(\phi_s, \bar{\boldsymbol{x}}) \ge 0 \text{ for all } i \in \mathcal{N}\} \\ &= \{\bar{\boldsymbol{x}} \in \mathbb{R}^2 | \, \bar{\boldsymbol{x}} \in X^2\}, \end{aligned}$$

[71] The chief difficulty with relaxing (i) is ensuring that $D_\phi(c)$, or an analogous object, is open. A solution is to define payoffs on a set X' which contains an open set O such that $X \subset O \subset X'$ and then "restrict" $D_\phi(c)$ to O.

where the second line is due to the facts (i) $u_i(\phi_s, \bar{\boldsymbol{x}}) = 0$ and (ii) $A(\phi_s) = \mathbb{R}^2$. We also have

$$\begin{aligned} F(\phi_p) &= \{\bar{\boldsymbol{x}} \in \mathbb{R}^2 |\, \bar{\boldsymbol{x}} \in X^2 \cap A(\phi_p) \text{ and } u_i(\phi_p, \bar{\boldsymbol{x}}) \geq 0 \text{ for all } i \in \mathcal{N}\} \\ &= \{(x^1, x^2) \in X^2 | x^1 = x^2\} = \{(x^1, x^2) \in (0,1)^2 | x^1 = x^2\}, \end{aligned}$$

where the second line is due to the facts that (i) $A(\phi_p) = \{(x^1, x^2) \in \mathbb{R}^2 | x^1 = x^2\}$ and (ii) $u_i(\phi, \bar{\boldsymbol{x}}) \geq 0$ when $\bar{\boldsymbol{x}} \geq 0$. Since $X = (0,1)$, we have that $F(\phi_s)$ is the open unit square and we have that $F(\phi_p)$ is the 45-degree line intersected with the open unit square. Thus, we cannot apply the logic of Proposition 1 to conclude that this game has a stable allocation.

In fact, we can prove that this game has no stable allocation; we do this by arguing from primitives. Since man 1 and woman 2 get zero from being single, it is best to have them match and take an action $x \in (0,1)$. Thus, every candidate stable allocation $(\phi, \bar{\boldsymbol{x}})$ has $\phi(1) = 2$ and $\bar{\boldsymbol{x}} = (x, x)$ with $x \in (0,1)$. Since X is open, there is a $y \in X$ that both players prefer to x, i.e., $u_1(2, y) = y > u_1(\phi, \bar{\boldsymbol{x}}) = x$ and $u_2(1, y) = y > u_2(\phi, \bar{\boldsymbol{x}}) = x$. Thus, the players block the candidate. Since this holds for every $x \in (0,1)$, there is no stable allocation. △

Remark. If X is open, payoffs are upper-semicontinuous, and intolerability holds, then one may demonstrate that there is a stable allocation: consider the closure of X and apply Proposition S2 to conclude the existence of a stable allocation, then notice that this allocation is interior and so remains stable when we return to the original game.

6.3.3 Inessential

The next example shows that Assumption S2 is inessential for the existence of a stable allocation.

Example S6. Stable Allocations and Violations of Assumption S2.

Let $\mathcal{M} = \{1\}$ and $\mathcal{W} = \{2\}$. Let $X = (0,1)$ and suppose that when the man and woman match, their payoffs from agreement $x \in \mathbb{R}$ are

$$u_1(2, x) = \begin{cases} x & \text{if } x < 1/2 \\ 0 & \text{if } x \geq 1/2 \end{cases} \text{ and } u_2(1, x) = \begin{cases} x & \text{if } x < 2/3 \\ 0 & \text{if } x \geq 2/3. \end{cases}$$

As usual, both players get zero from being single.

We need to establish that these payoffs aren't increasing transformations of upper-semicontinuous functions (in the sense of Assumption S2). The easiest way to do this is to construct two auxiliary games A_1 and A_2. In game A_1, both players have the man's payoff, i.e.,

$$u_1^{A_1}(2,x) = u_2^{A_1}(1,x) = \begin{cases} x & \text{if } x < 1/2 \\ 0 & \text{if } x \geq 1/2, \end{cases}$$

and the set of feasible agreements is $[0,1]$. In game A_2, both players have the woman's payoff, i.e.,

$$u_1^{A_2}(2,x) = u_2^{A_2}(1,x) = \begin{cases} x & \text{if } x < 2/3 \\ 0 & \text{if } x \geq 2/3, \end{cases}$$

and the set of feasible agreements is $[0,1]$. In both games, the payoff to being single is zero.

Consider the first auxiliary game, which is exactly the same game as in Example S3. If $u_1^{A_1}$ and $u_2^{A_1}$ were increasing transformations of upper-semicontinuous functions (in the sense of Assumption S2), then Proposition S2 gives that there is a stable allocation. Yet, we know from Example S3 that there is no stable allocation in this game. It follows that the man's payoff is not an increasing transformation of an upper-semicontinuous function.

We use the second auxiliary game to establish that the woman's payoff is not an increasing transformation of an upper-semicontinuous function. To do this, first need to establish that there is no stable allocation game A_2. Given this, it again follows that the woman's payoff is not an increasing transformation of an upper-semicontinuous function.

It remains to establish that there is no stable allocation in game A_2. Since man 1 and woman 2 get zero from (i) being single or (ii) pairing and taking an action in $\{0\} \cup [2/3, 1]$, it is best to have them match and take an action $x \in (0, 2/3)$. Thus, every candidate stable allocation $(\phi, \bar{\boldsymbol{x}})$ has $\phi(1) = 2$ and $\bar{\boldsymbol{x}} = (x,x)$ with $x \in (0, 2/3)$. Since $(0, 2/3)$ is open, there is a $y \in (0, 2/3)$ that both players prefer to x, i.e., $u_1^{A_2}(2,y) = y > u_1^{A_2}(\phi, \bar{\boldsymbol{x}}) = x$ and $u_2^{A_2}(1,y) = y > u_2^{A_2}(\phi, \bar{\boldsymbol{x}}) = x$. Thus, the players block the candidate. Since this holds for every $x \in (0, 2/3)$, there is no stable allocation in game A_2.

In the original game, there are a continuum of stable allocations that are described by $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, where $\phi^\star(1) = 2$ and $\bar{\boldsymbol{x}}^\star = (x,x)$ with $x \in [1/2, 2/3)$. That is, man 1 and woman 2 match and take an action $x \in [1/2, 2/3)$. Such an allocation is trivially feasible and individually rational. Let's establish that it's stable. If man 1 and woman 2 were to take an action y less than x, then woman 2 would be strictly worse off. While if man 1 and woman 2 were to take an action y greater than x, then man 1 would not be strictly better off. It follows that man 1 and woman 2 cannot block, implying stability. $\triangle$

6.3.4 Necessary Conditions?

Since Assumption S2 is inessential, it's natural to wonder what conditions are necessary. The next proposition is a first step towards answering this question since it gives strong sufficient conditions for nonexistence.

Let X be the (possibly compact) set of feasible agreements and let m be an arbitrary man. We say that m's payoff function $u_m(w, \boldsymbol{x})$ is **unsatisfied**, if for each woman w and each $\boldsymbol{x} \in X$, we have

$$u_m(w, \boldsymbol{x}) < \sup_{\boldsymbol{x}' \in X} u_m(w, \boldsymbol{x}').$$

That is, $u_m(w, \boldsymbol{x})$ is unsatisfied if there is no best agreement for m in X. A woman's payoff function is unsatisfied under the analogous condition. For instance, the payoff functions in Examples S3 and S6 are unsatisfied, while upper-semicontinuous functions are not unsatisfied when X is compact because they obtain their maxima.

If $u_m(w, \boldsymbol{x})$ is unsatisfied, then for each woman w, there is an "improving sequence" $\{\boldsymbol{x}_w(j)\}_{j=1}^{\infty}$, with $\boldsymbol{x}_w(j) \in X$ for all j, such that

$$u_m(w, \boldsymbol{x}_w(j)) \to \sup_{\boldsymbol{x}' \in X} u_m(w, \boldsymbol{x}')$$

as $j \to \infty$. Indeed, such a sequence must exist as for any point $\boldsymbol{x} \in X$, we can find another point $\boldsymbol{x}'$ such that $u_m(w, \boldsymbol{x}') > u_m(w, \boldsymbol{x})$. (If not, then $\boldsymbol{x}'$ is the largest element of $u_m(w, X)$, a contradiction of the fact $u_m(w, \boldsymbol{x}) < \sup_{\boldsymbol{x}' \in X} u_m(w, \boldsymbol{x}')$.)

Proposition S3. Nonexistence of Stable Allocations.
Suppose that, for each man m and each woman w, we have (i) $u_m(w, \boldsymbol{x}) > 0$ and $u_w(m, \boldsymbol{x}) > 0$ for some $\boldsymbol{x} \in X$, (ii) $u_m(w, \boldsymbol{x})$ and $u_w(m, \boldsymbol{x})$ are unsatisfied, and (iii) m and w share an improving sequence, then there is no stable allocation.

Proof. We'll show that every feasible and individually rational allocation is blocked; thus, there is no stable allocation. Let $(\phi, \bar{\boldsymbol{x}})$ be a feasible and individually rational allocation. There are two cases: (a) some man and woman are matched by ϕ or (b) no man and woman are matched by ϕ.

Consider case (a). Let m and w be a man and woman such that $\phi(m) = w$. We have that

$$u_m(\phi, \bar{\boldsymbol{x}}) < \alpha_m = \sup_{\boldsymbol{x}' \in X} u_m(w, \boldsymbol{x}') \text{ and } u_w(\phi, \bar{\boldsymbol{x}}) < \alpha_w = \sup_{\boldsymbol{x}' \in X} u_w(m, \boldsymbol{x}'),$$

where the strict inequalities are due to unsatisfiability. By (iii), man m and woman w share an improving sequence $\{\boldsymbol{x}(k)\}_{k=1}^{\infty}$, along which $u_m(w, \boldsymbol{x}(k)) \to \alpha_m$ and $u_w(m, \boldsymbol{x}(k)) \to \alpha_w$. It follows that there is a finite K such that $k \geq K$ implies (i) $|\alpha_m - u_m(w, \boldsymbol{x}(k))| <$

$|\alpha_m - u_m(\phi, \bar{\boldsymbol{x}})|$ and (ii) $|\alpha_w - u_w(m, \boldsymbol{x}(k))| < |\alpha_w - u_w(\phi, \bar{\boldsymbol{x}})|$. Thus, at $\boldsymbol{x}(K)$, we have $u_m(w, \boldsymbol{x}(K)) > u_m(\phi, \bar{\boldsymbol{x}})$ and $u_w(m, \boldsymbol{x}(K)) > u_m(\phi, \bar{\boldsymbol{x}})$. That is, man m and woman w block $(\phi, \bar{\boldsymbol{x}})$.

Consider case (b). Let m and w be an arbitrary man and woman. Since all players are single, $u_m(\phi, \bar{\boldsymbol{x}}) = u_w(\phi, \bar{\boldsymbol{x}}) = 0$. Thus, by (i), man m and woman w block $(\phi, \bar{\boldsymbol{x}})$. □

The most significant drawback of the result is that a shared improving sequence is a strong form of preference alignment. Yet, without this alignment we know that there may be a stable allocation. For instance, in Example S6 there is no shared improving sequence because an improving sequence for man 1 is any sequence that converges to $1/2$ from below, while an improving sequence for woman 2 is any sequence that converges to $2/3$ from below.

Interestingly, there may be stable allocations when the payoffs are unsatisfied and there is a shared improving sequence. Thus, while Assumption S2 rules out payoffs being unsatisfied; unsatisfaction itself is consistent with the existence of a stable allocation.[72]

6.3.5 Generalizing the Agreement Space and Gambles

In this subsection, we extend the agreement space to allow for agreements over functions (e.g., investment strategies), argue existence, and illustrate our approaches.

Let $\mathcal{X}$ be a metric space (e.g., the space of all continuous functions from the unit interval into itself) which denotes the set of all conceivable agreement and let $X \subset \mathcal{X}$ denote the set of feasible agreements. As usual, each player has a payoff over matches and agreements, which we denote as $u_m : (\mathcal{W} \cup \{m\}) \times \mathcal{X} \to \mathbb{R}$ for a man m and $u_w : (\mathcal{M} \cup \{w\}) \times \mathcal{X} \to \mathbb{R}$ for a woman w.

Broadly, there are two approaches for establishing existence when men and women agree to elements of $\mathcal{X}$: (i) parameterization or (ii) direct choice. By parameterization, we mean that there's a function $I : \mathbb{R}^k \to \mathcal{X}$ such that $I(\cdot)$ is one-to-one between $Y = I^{-1}(X)$ and X. (For instance, if $\mathcal{X}$ is the set of all normal density functions, then $I(y_1, y_2) = N(y_1, y_2)$.) Thus, when a man and a woman match and select $x \in X$, they're effectively selecting point $\boldsymbol{y} = I^{-1}(x)$ in $\mathbb{R}^k$. Hence, it's without loss to suppose that they chooses a point in Y instead

[72] For example, suppose that $X = [0, 1]$, that $\mathcal{M} = \{1\}$ and $\mathcal{W} = \{2\}$, and that

$$u_1(2, x) = u_2(1, x) = \begin{cases} x - 3 & \text{if } x < 1/2 \\ -3 & \text{if } x \geq 1/2. \end{cases}$$

Then, when man 1 and woman 2 are matched, the supremum of the payoffs is $-5/2$ and an improving sequence for both players is any sequence that eventually converges to $1/2$ from below. However, since both players can get zero by remaining single, a stable allocation is for them to both be single and have arbitrary agreements.

of in X. Consequently, a stable allocation exists by Proposition 1, subject to the restrictions that (i) payoffs are continuous in x, (ii) $I(\cdot)$ is continuous, and (iii) Y is compact.

Unfortunately, an index function does not always exist, so players may need to directly choose their agreements. When this is the case, a stable allocation exists when X is compact and payoffs are continuous in x on $\mathcal{X}$. The formal argument is analogous to the proof of Proposition 1. While this approach is limited (because it's hard to verify that a given X is compact), it can be useful as this next result illustrates.

Example S7. Reproving Theorem 1 of Chiappori and Reny [15].

In Chiappori and Reny [15] men and women with heterogeneous risk preferences match to share risk. Specifically, each player i has a concave utility of wealth function $u_i : \mathbb{R} \to \mathbb{R}$ and a random income y_i with absolutely continuous cumulative distribution F_{y_i} and support $[0, 1/2]$. When a man m and woman w match, they commit to an income sharing rule $x(y)$ that specifies how much of their joint random income $y_{mw} = y_m + y_w$ the man gets ex-post, leaving $y - x(y)$ for the woman. Thus, m and w have payoffs

$$u_m(w, x) = \mathbb{E}(u_m(x(y_{mw}))) \text{ and } u_w(m, x) = \mathbb{E}(u_w(y_{mw} - x(y_{mw}))),$$

where $\mathbb{E}$ denotes the standard expectations operator. As usual, we assume single players get zero for simplicity.[73]

Clearly, payoffs are defined for all sharing rules so $\mathcal{X} = \{f | f : [0, 1] \to [0, 1]\}$. Let X be the set of equicontinuous sharing rules in $\mathcal{X}$. Since $u_m(w, x)$ and $u_w(m, x)$ are continuous in x and X is compact by the Arzela-Ascoli Theorem, we may conclude that a stable allocation exists. That is, we obtain Chiappori and Reny's Theorem 1. $\triangle$

6.4 Structure of Matchings

This section gives three results on the structure of matchings in the General Game and the Effort Game. The first result concerns the General Game; it establishes that a player who makes a strictly positive payoff in some stable allocation is partnered in every stable allocation. The rest of the results concern the Effort Game. The second result shows that the set of interior stable allocations partitions the type space in an intuitive way. The third result shows that, under strong conditions, higher types produce strictly higher benefits than lower types. A consequence is that every interior stable allocation exhibits assortative matching in types.

[73] Chiappori and Reny actually let their single players consume their incomes, so each single player i gets $\mathbb{E}(u_i(y_i))$ instead of zero. Our argument readily generalizes to their case.

6.4.1 General Game and a Rural Hospitals Theorem

The goal of this subsection is to establish the following result, which gives test for whether a player is matched in every stable allocation.

Proposition S4. A Rural Hospitals Theorem.
Let Assumption 3 hold. If a player i is partnered in a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) > 0$, then i is partnered in every stable allocation.

This result, also known as "Rural Hospitals Theorem," is well known in the Marriage Game (with strict preferences) and the Assignment Game, e.g., Roth and Sotomayor's [60] Theorem 1.22 and Demange and Gale's [24] Property 1. Our contribution is two-fold: we extend it to a more general environment and we highlight it's dependency on Assumption 3.

As mentioned, the proposition fails when Assumption 3 does not hold. For instance, if X has empty interior then the proposition fails, as is the case in Example S1. Likewise, if local nonsatiation fails then the proposition fails, as is the case in a variant of Example 2 where there's only one woman. Additionally, if intolerability fails then the proposition fails. For example, consider a game where $\mathcal{M} = \{1, 2\}$, $\mathcal{W} = \{3\}$, $X = [0, 1]$, and all players receive a payoff of x when matched and zero when single. Then there are two stable allocations: (i) 1 and 3 match with agreement 1, while 2 is single, and (ii) 2 and 3 match with agreement 1, while 1 is single. Player 1 earns a strictly positive amount in (i) and yet is single in (ii).

The next lemma is our first step in establishing the proposition.

Lemma S4. Improvement and Harm.
Let Assumption 3 hold and let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $(\phi', \bar{\boldsymbol{x}}')$ be two stable allocations. If $u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) > u_i(\phi', \bar{\boldsymbol{x}}')$, then $u_{\phi^\star(i)}(\phi^\star, \bar{\boldsymbol{x}}^\star) < u_{\phi^\star(i)}(\phi', \bar{\boldsymbol{x}}')$.

The lemma shows that when one player does strictly better in stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ than in stable allocation $(\phi', \bar{\boldsymbol{x}}')$, then this player's match in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ does strictly better in $(\phi', \bar{\boldsymbol{x}}')$ than in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

Proof. Without loss, let i be a man. Since $u_i(\phi', \bar{\boldsymbol{x}}') \geq 0$ per individual rationality, we have $u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) > 0$, which implies that i has a partner. Let $w = \phi^\star(i)$, and let $\boldsymbol{x}^{i^\star}$ denote i's agreement in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Since $u_i(w, \boldsymbol{x}^{i^\star}) = u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) > u_i(\phi', \bar{\boldsymbol{x}}')$, the stability of $(\phi', \bar{\boldsymbol{x}}')$ gives that

$$u_w(m, \boldsymbol{x}^{i^\star}) \leq u_w(\phi', \bar{\boldsymbol{x}}').$$

Our goal is to show that the inequality in the display equation is strict.

Observe that $\boldsymbol{x}^{i^\star}$ is interior. If not, then $\boldsymbol{x}^{i^\star} \in \partial X \cap X \subset \partial X$, so either $u_i(w, \boldsymbol{x}^{i^\star}) = u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) < 0$ or $u_w(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_w(m, \boldsymbol{x}^{i^\star}) < 0$ by intolerability. So $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ cannot be stable, a contradiction.

Since $\boldsymbol{x}^{i^\star}$ is interior, there is an $r > 0$ such that $B_r(\boldsymbol{x}^{i^\star}) \subset X$. Continuity of i's payoff gives that we can pick r small enough such that $u_i(w, \boldsymbol{x}) > u_i(\phi', \bar{\boldsymbol{x}}')$ for all $\boldsymbol{x} \in B_r(\boldsymbol{x}^{i^\star})$. Local nonsatiation of w's payoff gives that there is an $\boldsymbol{x}' \in B_r(\boldsymbol{x}^{i^\star})$ such that $u_w(m, \boldsymbol{x}') > u_w(m, \boldsymbol{x}^{i^\star})$. Now, if $u_w(m, \boldsymbol{x}^{i^\star}) = u_w(\phi', \bar{\boldsymbol{x}}')$, then i and w block $(\phi', \bar{\boldsymbol{x}}')$ via agreement $\boldsymbol{x}'$, a violation of stability. Thus, we must have $u_w(m, \boldsymbol{x}^{i^\star}) < u_w(\phi', \bar{\boldsymbol{x}}')$. □

The next corollary is originally due to Demange and Gale [24].

Corollary S1. Decomposition.
Let Assumption 3 hold and let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $(\phi', \bar{\boldsymbol{x}}')$ be two stable allocations. Let $A = \{m \in \mathcal{M} | u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) > u_m(\phi', \bar{\boldsymbol{x}}')\}$ and let $B = \{w \in \mathcal{W} | u_w(\phi^\star, \bar{\boldsymbol{x}}^\star) < u_w(\phi', \bar{\boldsymbol{x}}')\}$. Then, $\phi^\star(A) = B$ and $\phi'(B) = A$.

Proof. First, recall that everyone in A is partnered under $\phi^\star$ and everyone in B is partnered under ϕ' as they all earn strictly positive payoffs by hypothesis. By Lemma S4, we have $\phi^\star(A) \subset B$ and $\phi'(B) \subset A$. Suppose that $\phi^\star(A) \neq B$. Then, there is a woman $w \in B$ for whom $\phi^\star(w) \notin A$, a contradiction of Lemma S4. Thus, $\phi^\star(A) = B$. The analogous argument gives $\phi'(B) = A$. □

Proof of Proposition S4. We argue by contradiction. Without loss, suppose man m is matched in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and has $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) > 0$, yet is single in $(\phi', \bar{\boldsymbol{x}}')$. Then $m \in A$. So by Corollary S1, $m \in \phi'(B)$ implying m is matched in $(\phi', \bar{\boldsymbol{x}}')$, a contradiction. □

6.4.2 Effort Game And Partitions of the Endowed Types

In this subsection, we show that men and women with sufficiently high types are "always" partnered, while those with sufficiently low types are always single. We make the following assumption.

Assumption S3. Strict Types and Strict Agreeability.
The endowment of types $\{\theta_i\}_{i \in \mathcal{N}}$, the benefit function $b(y)$, and the cost function $c(\theta, y)$ are such that,
(i) No two men have the same type and no two women have the same type.
(ii) Agreeability is strict for at least one man and woman, i.e., there is a man m, a woman w, and an $\boldsymbol{x} \in X$ such that $u_m(w, \boldsymbol{x}) > 0$ and $u_w(m, \boldsymbol{x}) > 0$.
(iii) Intolerability holds.

Given a matching $\phi \in \Phi$, there is a partnered man with lowest type since the set of players is finite. Let $\theta^{\mathbb{M}}(\phi)$ record his type, where $\theta^{\mathbb{M}} : \Phi \to \Theta$. Analogously, there is a partnered woman with lowest type. Let $\theta^{\mathbb{W}}(\phi)$ record her type, where $\theta^{\mathbb{W}} : \Phi \to \Theta$. Thus, every partnered man has a type greater than or equal to $\theta^{\mathbb{M}}(\cdot)$, while every partnered woman has a type greater than or equal to $\theta^{\mathbb{W}}(\cdot)$. Recall $\Phi_{\mathcal{S}}$ is the set of matchings that are part of stable allocations, i.e., is the set of matching such that, for each $\phi \in \Phi_{\mathcal{S}}$, there is an $\bar{\boldsymbol{x}} \in X^N$ so that $(\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}$.

Proposition S5. Endowed Type Thresholds.
Suppose the benefit function $b(y)$, the cost function $c(\theta, y)$, and the type endowment $\{\theta_i\}_{i\in\mathcal{N}}$ are such that Assumptions 5, 6, and S3 hold. Then, there are two thresholds $\underline{\theta} = \min\{\theta^{\mathbb{M}}(\Phi_{\mathcal{S}})\}$ and $\overline{\theta} = \max\{\theta^{\mathbb{M}}(\Phi_{\mathcal{S}})\}$, such that for any man m with type θ_m,
(i) $\theta_m < \underline{\theta}$ implies m is single in every interior stable allocation,
(ii) when $\underline{\theta} < \overline{\theta}$, we have $\underline{\theta} \le \theta_m < \overline{\theta}$ implies m is partnered in some interior stable allocations and single in others, and
(iii) $\theta_m \ge \overline{\theta}$ implies m is partnered in every interior stable allocation.
The analogous result holds for women, though the thresholds may differ by gender.

That is, the set of stable matchings partitions the set of endowed types into three regions. Men with a sufficiently high type are "always" matched. Men with a sufficiently low type are always single. And men with intermediate types may be single in some interior stable allocations and partnered in others because they earn zero in the interior stable allocations in which they're partnered (see Proposition S4).[74]

Remark. These thresholds depend on the type endowment. Thus, for fixed $b(y)$ and $c(\theta, y)$, the thresholds will change as the type endowment changes.

Proof. Observe that $\theta^{\mathbb{M}}(\Phi_{\mathcal{S}})$ is finite. This follows from the fact that $\Phi_{\mathcal{S}}$ is nonempty by Assumption 4 and is finite since Φ is finite. Thus, $\theta^{\mathbb{M}}(\Phi_{\mathcal{S}})$ has a maximal element $\overline{\theta}$ and minimal element $\underline{\theta}$ because. By Assumption S3, every man has a different type. Thus, in each matching $\phi \in \Phi$ there is exactly *one* partnered man with lowest type $\theta^{\mathbb{M}}(\phi)$. Let $m(\phi)$ give his index.

We prove part (iii) first. Consider an arbitrary stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Since $m(\phi^\star)$'s type is less than or equal to $\overline{\theta}$, $\theta_m \ge \overline{\theta}$ implies either (i) that m has a type strictly higher than $m(\phi^\star)$, i.e., $\theta_m > \theta^{\mathbb{M}}(\phi^\star)$, or (ii) that m is the lowest type partnered man, i.e., $\theta_m = \theta^{\mathbb{M}}(\phi^\star)$. Thus, m is either partnered by Lemma 9 or by definition of $\theta^{\mathbb{M}}(\cdot)$. Since the stable allocation was arbitrary, the part (iii) follows.

Part (i) is almost trivial. When $\theta_m < \underline{\theta}$, then in every stable allocation the type of m is below $\theta^{\mathbb{M}}(\cdot)$, implying m is single. Part (ii) is straightforward. We take $\underline{\theta} < \overline{\theta}$ to avoid trivialities. Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be a stable allocation with $\theta^{\mathbb{M}}(\phi^\star) = \underline{\theta}$ and let $(\phi', \bar{\boldsymbol{x}}')$ be a stable

[74] Observe that the set of men with types equal to or greater than $\overline{\theta}$ is non-empty. Simply, in the matching $\phi^\star \in \Phi_{\mathcal{S}}$ where $\theta^{\mathbb{M}}(\cdot)$ attains it's maximum $\overline{\theta}$, we have that the man whose type is $\theta^{\mathbb{M}}(\phi^\star)$ is an element of $\{m|\theta_m \ge \overline{\theta}\}$. In contrast, the set of men with types strictly less than $\underline{\theta}$ may be empty, e.g., Example 1.

allocation with $\theta^{\mathbb{M}}(\phi') = \overline{\theta}$. When $\underline{\theta} \leq \theta_m < \overline{\theta}$, we have (i) m is single in $(\phi', \bar{\boldsymbol{x}}')$ because $\theta_m < \theta^{\mathbb{M}}(\phi')$ and (ii) m is partnered in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ because $\theta_m \geq \theta^{\mathbb{M}}(\phi^\star)$. The former is obvious, the latter follows from an argument analogous to the above paragraph. □

Remark. Since there are a finite number of type endowments because Θ is finite, there are only a finite number of endowments satisfying Assumption S3 for a fixed benefit function $b(y)$ and a fixed cost function $c(\theta, y)$. Thus, an immediate implication of Proposition S5 is that there are two thresholds

$$\overline{\theta}^\dagger = \max\{\overline{\theta}_{\{\theta_i\}_{i\in\mathcal{N}}} | \{\theta_i\}_{i\in\mathcal{N}} \text{ satisifies Assumption S3}\}$$
$$\underline{\theta}^\dagger = \min\{\underline{\theta}_{\{\theta_i\}_{i\in\mathcal{N}}} | \{\theta_i\}_{i\in\mathcal{N}} \text{ satisifies Assumption S3}\}$$

such that:
(i) If a man's type is above $\overline{\theta}^\dagger$, he's partnered in every interior stable allocation induced by every type endowment meeting Assumption S3, given $b(y)$ and $c(\theta, y)$
(ii) If his type is below $\underline{\theta}^\dagger$, he's single in every interior stable allocation induced by every type endowment meeting Assumption S3, given $b(y)$ and $c(\theta, y)$.

6.4.3 Effort Game, A Strict Version of Proposition 6, and Assortative Matching

Proposition 6 of the main text established that higher types produce weakly higher benefits than lower types. The purpose of this section is to give a stronger result: that higher types produce *strictly* higher benefits than lower types under the next assumption. A corollary is that every interior stable allocation exhibits assortative matching in types.

Assumption S4. Continuous Differentiability and Strict Monotonicity.
The benefit function $b(y)$ is continuously differentiable and has $\frac{\partial b(y)}{\partial y} > 0$ on $[0, \beta]$. The cost function $c(\theta, y)$ is also continuously differentiable and has $\frac{\partial c(\theta,y)}{\partial y} > 0$ for all $(\theta, y) \in \Theta \times [0, \beta]$.

Not only is differentiability restrictive, but this assumption also implies that a player's effort cannot provide a direct benefit to himself or herself since cost is strictly increasing.

Proposition S6. The Strong Version of Proposition 6.
Let Assumptions 4, 5, and S4 hold, let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, and let $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$ and $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{W}}$ be the associated benefit groups. Then, higher type men produce strictly higher benefits, i.e., if two men m and m' are partnered, then $\theta_m < \theta_{m'}$ implies $m \in G_l^{\mathbb{M}}$ and $m' \in G_j^{\mathbb{M}}$ with $j < l$. The analogous result holds for women.

The proof (given at the end of this sub-section) proceeds in two steps. First, we'll show that differentiability and the Pareto optimality of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ imply that a matched man and woman must have an agreement that sets their marginal rates of substitution (MRS) to be equal. Second, we'll establish that if a benefit group contains more than one type of player, then the equal MRS condition is violated for some man and woman because lower types have strictly higher marginal costs per Assumptions 5 and S4. Hence, each benefit group only contains one type of player. Proposition S6 then follows from Proposition 6.

This result is sensitive to the continuity of effort. For example, when $\mathcal{M} = \{1, 2\}$, $\mathcal{W} = \{3, 4\}$, $b(y) = y$, $c(\theta, y) = y/\theta$, $\theta_1 = \theta_3 = 1$ and $\theta_2 = \theta_4 = 2$, and the set of feasible agreements is $\{0, 1\}^2$, then it's stable for man 1 and woman 4 to match with agreement $(1, 1)$ and for man 2 and woman 3 to match with agreement $(1, 1)$. That is, different type men are in the same benefit group and the matching is negative assortative.

An immediate corollary of Proposition S6 is the following.

Corollary S2. All Interior Stable Allocations Exhibit Assortative Matching in Types. *Let Assumptions 4, 5, and S4 hold, then every interior stable allocation* $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *exhibits assortative matching in types.*

Proof. Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation and let $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$ and $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{W}}$ be the associated benefit groups. Let m and m' be two partnered men with $\theta_{m'} > \theta_m$. Then $m \in G_j^{\mathbb{M}}$ and $m' \in G_l^{\mathbb{M}}$ with $l < j$ by Proposition S6. Thus, $\phi^\star(m) \in G_j^{\mathbb{W}}$ and $\phi^\star(m') \in G_l^{\mathbb{W}}$ by Proposition 5, which implies $\theta_{\phi^\star(w)} \leq \theta_{\phi^\star(w')}$ per the contraposition of Proposition S6. Hence, $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exhibits assortative matching in types. □

We now prove Proposition S6 by formalizing the argument we sketched above.

Lemma S5. Equal MRS Condition. *Let Assumptions 4 and S4 hold, let* $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *be an interior stable allocation, and let* $\{z_i^\star\}_{i \in \mathcal{N}}$ *be the associated efforts. If man* m *is matched to woman* w*, then*

$$MRS_m(z_m^\star, z_w^\star) = \frac{\dfrac{\partial u_m(w, z_m^\star, z_w^\star)}{\partial x_1}}{\dfrac{\partial u_m(w, z_m^\star, z_w^\star)}{\partial x_2}} = \frac{\dfrac{\partial u_w(m, z_m^\star, z_w^\star)}{\partial x_1}}{\dfrac{\partial u_w(m, z_m^\star, z_w^\star)}{\partial x_2}} = MRS_w(z_m^\star, z_w^\star).$$

Proof. Since $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$ is Pareto optimal by Corollary 1, we have that $(z_m^\star, z_w^\star)$ solves

$$\max_{(x_1, x_2) \in X} u_m(w, x_1, x_2) \text{ s.t. } u_w(m, x_1, x_2) \geq u_w(\phi^\star, \bar{\boldsymbol{x}}^\star).$$

(If not then there is an (x_1', x_2') which serves as a Pareto improvement for m and w and so $(\phi', \bar{\boldsymbol{x}}')$ with $\phi' = \phi^\star$ and $\bar{\boldsymbol{x}}' = (\boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{m-1^\star}, x_1', x_2', \boldsymbol{x}^{m+1^\star}, \ldots, \boldsymbol{x}^{w-1^\star}, x_1', x_2', \boldsymbol{x}^{w+1^\star}, \ldots, \boldsymbol{x}^{N^\star})$ is a Pareto improvement for all players, a contradiction.) Since (i) $u_m(w, x_1, x_2)$ and

$u_w(m, x_1, x_2)$ are continuously differentiable, (ii) $u_w(m, x_1, x_2) = u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is the only binding constraint at $(z^\star_m, z^\star_w)$ per the interiority of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and the strict monotonicity of the payoffs in efforts, and (iii) $\frac{\partial u_w(m,z^\star_m,z^\star_w)}{\partial x_1} > 0$ and $\frac{\partial u_w(m,z^\star_m,z^\star_w)}{\partial x_2} > 0$, Theorem 1.18 of de la Fuente (p. 293 [23]) gives there is a $\lambda \in \mathbb{R}_+$ such that

$$\underbrace{\begin{bmatrix} \frac{\partial u_m(w,z^\star_m,z^\star_w)}{\partial x_1} \\ \frac{\partial u_m(w,z^\star_m,z^\star_w)}{\partial x_2} \end{bmatrix}}_{A} + \lambda \underbrace{\begin{bmatrix} \frac{\partial u_w(m,z^\star_m,z^\star_w)}{\partial x_1} \\ \frac{\partial u_w(m,z^\star_m,z^\star_w)}{\partial x_2} \end{bmatrix}}_{B} = 0.$$

Since $A \neq 0$ and $B \neq 0$, we have that $\lambda \neq 0$. Solving this equation for λ gives

$$-\lambda = \frac{\frac{\partial u_m(w,z^\star_m,z^\star_w)}{\partial x_1}}{\frac{\partial u_w(m,z^\star_m,z^\star_w)}{\partial x_1}} = \frac{\frac{\partial u_m(w,z^\star_m,z^\star_w)}{\partial x_2}}{\frac{\partial u_w(m,z^\star_m,z^\star_w)}{\partial x_2}}.$$

A bit of algebra gives the lemma. □

Lemma S6. Homogenous Benefit Groups.
Let Assumptions 4, 5, and S4 hold, let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, and let $G^{\mathbb{M}}_1, \ldots, G^{\mathbb{M}}_{J_{\mathbb{M}}}$ and $G^{\mathbb{W}}_1, \ldots, G^{\mathbb{W}}_{J_{\mathbb{M}}}$ be the associated benefit groups. Then, it's the case that every benefit group only contains one type of player, i.e., for all $l \in \{1, \ldots, J_{\mathbb{M}}\}$, if men m and m' are in $G^{\mathbb{M}}_l$, then $\theta_m = \theta_{m'}$. The analogous result holds for women.

Proof. We argue by contradiction. Let m and m' be in $G^{\mathbb{M}}_l$ and suppose that $\theta_m < \theta_{m'}$. Let $w = \phi^\star(m)$ and $w' = \phi^\star(m')$. Also, let $z^\star_m$, $z^\star_w$, $z^\star_{m'}$, and $z^\star_{w'}$ denote these players' efforts. Since m and m' are in the same benefit group, w and w' are in $G^{\mathbb{W}}_l$ by Proposition 5. Thus, $z^\star_m = z^\star_{m'}$ and $z^\star_w = z^\star_{w'}$ as $b(y)$ is strictly increasing. Lemma S5 gives that

$$\frac{c_y(\theta_m, z^\star_m)}{b'(z^\star_w)} = \frac{b'(z^\star_m)}{c_y(\theta_w, z^\star_w)} \text{ and } \frac{c_y(\theta_{m'}, z^\star_{m'})}{b'(z^\star_{w'})} = \frac{b'(z^\star_{m'})}{c_y(\theta_{w'}, z^\star_{w'})},$$

where $b'(\cdot)$ denotes the first derivative of $b(\cdot)$ and $c_y(\cdot)$ denotes the first derivative of $c(\cdot)$ with respect to effort y. Which implies that

$$c_y(\theta_m, z^\star_m)\, c_y(\theta_w, z^\star_w) = c_y(\theta_{m'}, z^\star_m)\, c_y(\theta_{w'}, z^\star_w). \tag{6.1}$$

Since $\theta_m < \theta_{m'}$, we have $c_y(\theta_{m'}, z^\star_m) < c_y(\theta_m, z^\star_m)$ by Assumption 5. There are two cases: (i) $\theta_{w'} \geq \theta_w$ or $\theta_{w'} < \theta_w$. If $\theta_{w'} \geq \theta_w$, we have an immediate contradiction: $\theta_{w'} \geq \theta_w$ implies $c_y(\theta_{w'}, z^\star_w) \leq c_y(\theta_w, z^\star_w)$ by Assumption 5, so the right-hand-side of equation (6.1) is strictly smaller than the left-hand-side (as all marginal costs are strictly positive), implying the equation cannot hold. Thus, we proceed under (ii), so we have $\theta_m < \theta_{m'}$ and $\theta_{w'} < \theta_w$.

Construct a new allocation from $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ by re-matching (i) m with w' (ii) m' with w without changing their efforts, while leaving the matches and agreements of all other player alone. Since m, m', w, and w' are all in the same index benefit groups, this re-matching doesn't change their payoffs. (The formal argument is akin to the Proof of Proposition 8.) Thus, the new allocation is stable by Lemma 7. A second application of Lemma S5 now implies

$$c_y(\theta_m, z_m^\star)\, c_y(\theta_{w'}, z_w^\star) = c_y(\theta_{m'}, z_m^\star)\, c_y(\theta_w, z_w^\star).$$

The contradiction is immediate: since $\theta_m < \theta_{m'}$ implies $c_y(\theta_{m'}, z_m^\star) < c_y(\theta_m, z_m^\star)$ and since $\theta_{w'} < \theta_w$ implies $c_y(\theta_{w'}, z_w^\star) > c_y(\theta_w, z_w^\star)$, the right-hand-side is strictly smaller than the left-hand-side, an impossibility. □

Proof of Proposition S6. In light of Lemma S6, Proposition S6 follows directly from Proposition 6. □

Remark. Since our arguments depend on the classic equal MRS condition of Pareto optimality, we cannot dispense with differentiability or strict monotonicity of either the benefit or cost functions.

Remark. Under the additional assumptions that every man has a unique type, every woman has a unique type, and strict agreeability holds for all men and women, Corollary S2 allows us to conclude that the every interior stable allocation has the same matching: the highest type man and woman are matched, the second highest type man and woman are matched, and so on until all women have been matched (assuming there are fewer women then men), while all of the remaining men are single.

6.5 Additional Comparative Statics for the Effort Game

In this section, we give two additional comparative statics results for the Effort Game. We first give an example that shows the conclusions of Proposition 9 cannot be expected to always hold. Subsequently, we give a novel comparative statics result: that a decrease in fixed-cost and/or an increase in fixed-benefit increases the set of stable allocations (in the sense of inclusion) and increases optimal total welfare and increases the maximal effort that players exert.

6.5.1 Higher Types and Worse Outcomes

In this subsection, we give an example of a type increase that leaves at least one player strictly worse off for every pair of pre-increase and post-increase stable allocations we might compare. This example indicates that increases in the endowment of types don't always increase all players' *levels* of well-being.

Example S8. Increasing Types and Worsening Outcomes.

Let $\mathcal{M} = \{1,2\}$, $\mathcal{W} = \{3,4,5\}$, $X = [0,1]$, and let $b(y)$, $c(\theta,y)$, and $\{\theta_i\}_{i\in\mathcal{N}}$ satisfy Assumptions 4, 5, 6, and S4, as well as the following: for each man m and woman w, $u_m(w,\boldsymbol{x}) > 0$ and $u_w(m,\boldsymbol{x}) > 0$ for some $\boldsymbol{x} \in X$. Consider two endowments of types, $\{\theta_i\}_{i\in\mathcal{N}}$ and $\{\theta'_i\}_{i\in\mathcal{N}}$, where

(i) $\theta_1 > \theta_2$ and $\theta_3 > \theta_4 > \theta_5$, and

(ii) $\theta'_1 > \theta'_2$ and $\theta'_4 > \theta'_5 > \theta'_3$, where $\theta'_i \geq \theta_i$ for each player i.

Thus, woman 3 goes from the highest type woman in $\{\theta_i\}_{i\in\mathcal{N}}$ to the lowest type woman in $\{\theta'_i\}_{i\in\mathcal{N}}$, while all other players' rank-order types are unchanged.

Let $\mathcal{S}$ be the set of stable allocations when types are $\{\theta_i\}_{i\in\mathcal{N}}$ and let $\mathcal{S}'$ be the set of stable allocations when types are $\{\theta'_i\}_{i\in\mathcal{N}}$.

Claim. Payoff of Woman 3.

For each $(\phi,\bar{\boldsymbol{x}}) \in \mathcal{S}$, we have that $u_3(\phi,\bar{\boldsymbol{x}},\theta_3) \geq \bar{\epsilon} > 0$. For each $(\phi,\bar{\boldsymbol{x}}) \in \mathcal{S}'$, we have that $u_3(\phi,\bar{\boldsymbol{x}},\theta'_3) = 0$.

Proof. Let $(\phi,\bar{\boldsymbol{x}}) \in \mathcal{S}$. We begin by arguing that $u_3(\phi,\bar{\boldsymbol{x}},\theta_3) > 0$ via contradiction. If $u_3(\phi,\bar{\boldsymbol{x}},\theta_3) = 0$, then an argument analogous to the Proof of Lemma 9 shows that woman 3 can form a blocking pair with one of the men. Since this contradicts stability, we must have $u_3(\phi,\bar{\boldsymbol{x}},\theta_3) > 0$. Then, since $\mathcal{S}$ is "compact" by Lemma 5, we have that $\min_{(\phi,\bar{\boldsymbol{x}})\in\mathcal{S}} u_3(\phi,\bar{\boldsymbol{x}},\theta_3) = \bar{\epsilon} > 0$ by an argument analogous to the Proof of Lemma 6. The first part of the claim follows.

The second part of the claim is straightforward. Corollary 1 gives that $\mathcal{S}'$ is composed of (i) interior stable allocations and (ii) allocations where everyone's single. If (ii), then $u_3(\phi,\bar{\boldsymbol{x}},\theta'_3) = 0$. If (i), then Corollary S2 gives that $(\phi,\bar{\boldsymbol{x}})$ exhibits assortative matching in types, so we necessarily have that the men are matched to the high type women per our final assumption, implying woman 3 is single and earns nothing. □

It follows from the claim that woman 3's payoff collapses as a result of the increase in the types because she drops from the highest type woman to the lowest type woman. △

6.5.2 Set-based Comparative Statics of Fixed Cost

In this subsection, we give strong sufficient conditions which ensure that the stable set increases (in the sense of inclusion) when players fixed costs fall. (Note that Proposition 9 doesn't address changes in fixed costs because Assumption 7 ensures that *all* types have the *same* fixed costs. In addition, Lemma S3 doesn't address changes in fixed costs because such changes cannot be written in terms of increasing functions that are equal to zero at the origin.)

It's useful to write the cost function $c(\theta, y)$ as $c(\theta, y) = c_f(\theta) + c_v(\theta, y)$, where $c_f(\theta) = c(\theta, 0)$ is the fixed cost of effort and $c_v(\theta, y) = c(\theta, y) - c(\theta, 0)$ is the variable cost of effort. It's also useful to write the benefit function $b(\theta)$ as $b(y) = b_f + b_v(y)$ where $b_f = b(0)$ is fixed benefit and $b_v(y) = b(y) - b(0)$ is the variable benefit. Let $\mathcal{S}_I$ denote the set of interior stable allocations. The total effort exerted in an $(\phi, \bar{\boldsymbol{x}}) = (\phi, x_1^1, x_2^1, \ldots, x_1^N, x_2^N) \in \Phi \times \mathbb{R}^{kN}$ is $T_E(\phi, \bar{\boldsymbol{x}}) = \sum_{\{m|\phi(m)\neq m\}} x_1^m + x_2^m$.

Proposition S7. Changing the Set of Stable Allocations.
Let Assumption 7 hold. If fixed-cost $c_f(\theta)$ decreases or fixed-benefit b_f increases, then the set of interior stable allocations $\mathcal{S}_I$ increases, in the sense of inclusion, implying: (i) maximal social welfare in interior stable allocations $\max_{(\phi,\bar{\boldsymbol{x}})\in\mathcal{S}_I} T(\phi, \bar{\boldsymbol{x}})$ increases and (ii) the maximal effort players exert in interior stable allocations $\max_{(\phi,\bar{\boldsymbol{x}})\in\mathcal{S}_I} T_E(\phi, \bar{\boldsymbol{x}})$ increases.

That is, a decrease in fixed-cost or an increase in fixed benefits has the potential to increase social welfare and increase the amount of effort that players exert. To see the intuition, consider man m. The decrease in $c_f(\theta)$ increases m's benefit to being matched to woman w from $u_m(w, \boldsymbol{x})$ to $u_m(w, \boldsymbol{x}) + \kappa_m$ for each $\boldsymbol{x} \in X$, where $\kappa_m \geq 0$. Thus, (i) m finds more women and agreements to be individually rational and (ii) m's strict preference over women and agreements is unchanged. It follows from (ii) that if m is partnered in some stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, then he cannot be part of a blocking pair after the cost decrease. Since Assumption 7 ensures every player is partnered in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ (see Lemma S7 below), we have that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ remains stable after the change in fixed costs, so the stable set expands.[75] The comparative statics for social welfare and total effort follow. The proof of Proposition S7 formalizes this intuition.

We'll prove Proposition S7 via two lemmas. The first is that every player is partnered when Assumption 7 holds and so implies that every stable allocation is an interior stable allocation. (Recall that Assumption 7 implies Assumption 4, which implies Corollary 1 holds.) The second uses a contradiction argument to show that the set of stable allocations increases.

Lemma S7. All Players are Partnered.
Let Assumption 7 hold, then (i) every player is partnered in any stable allocation and (ii) every stable allocation is an interior stable allocation, i.e., $\mathcal{S} = \mathcal{S}_I$.

Proof. We establish (i) by contradiction. Suppose there is a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ where

[75] A decrease in the value of being single would also induce the same expansion of the stable set.

some man m is single. Then, because there are a equal number of men and women, some woman w is also single. Since m and w both earn zero in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and since strict agreeability holds, m and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction. The balance of the statement follows from Proposition 3. □

Lemma S8. An Increasing Stable Set.
Let Assumption 7 hold, and suppose that for each man m and each woman w, (i) $u_m(w, \boldsymbol{x})$ increases to $u_m(w, \boldsymbol{x}) + \kappa_m$ for each $(w, \boldsymbol{x}) \in \mathcal{W} \times \mathbb{R}^k$ and (ii) $u_w(m, \boldsymbol{x})$ increases to $u_w(m, \boldsymbol{x}) + \kappa_w$ for each $(m, \boldsymbol{x}) \in \mathcal{M} \times \mathbb{R}^k$, where $\kappa_m \geq 0$ and $\kappa_w \geq 0$ are weakly positive constants. Then $\mathcal{S}$ increases in the sense of inclusion.

Proof. Let $\{u_i\}_{i \in \mathcal{N}}$ denote the original payoffs and let $\{u_i'\}_{i \in \mathcal{N}}$ denote the payoffs after the increase. Let $\mathcal{S}$ denote the set of stable allocations when payoffs are $\{u_i\}_{i \in \mathcal{N}}$ and let $\mathcal{S}'$ denote the set of stable allocations when the payoffs are $\{u_i'\}_{i \in \mathcal{N}}$. Since Assumption 1 holds, $\mathcal{S}$ is nonempty. Our goal is to show that $\mathcal{S} \subset \mathcal{S}'$; we'll prove this via contradiction.

Let $(\phi^\star, \bar{x}^\star) \in \mathcal{S}$ and suppose $(\phi^\star, \bar{\boldsymbol{x}}^\star) \notin \mathcal{S}'$. There the only possibility is that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is blocked after the increase. (It cannot be that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ fails to be feasible, as X is unchanged, or that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ fails to be individually rational, as everyone's payoff to being matched has weakly increased.) Hence, then there is a man m, a woman w, and an agreement $\boldsymbol{x} \in X$ such that $u_m'(w, \boldsymbol{x}) > u_m'(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_w'(m, \boldsymbol{x}) > u_w'(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Since every player is partnered in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ per Lemma S7, we have

$$u_m(w, \boldsymbol{x}) + \kappa_m > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) + \kappa_m \text{ and } u_w(m, \boldsymbol{x}) + \kappa_w > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star) + \kappa_m.$$

This implies that $(\phi^\star, \bar{\boldsymbol{x}}^\star) \notin \mathcal{S}$, a contradiction. □

Proof of Proposition S7. Let $\{u_i\}_{i \in \mathcal{N}}$ denote players' payoffs. When fixed-cost decline from $c_f(\theta)$ to $c_f'(\theta)$, where $c_f'(\theta) \leq c_f(\theta)$ for all $\theta \in \Theta$, or fixed-benefit increases from b_f to b_f', where $b_f \leq b_f'$, then each man m and each woman w payoffs' increase from $u_m(w, x_1, x_2)$ to $u_m(w, x_1, x_2) + \kappa_m$ for all $(w, \boldsymbol{x}) \in \mathcal{W} \times \mathbb{R}^2$ and from $u_w(m, x_1, x_2)$ to $u_w(m, x_1, x_2) + \kappa_m$ for all $(m, \boldsymbol{x}) \in \mathcal{M} \times \mathbb{R}^2$ respectively, where $\kappa_m = c_f(\theta_m) - c_f'(\theta_m) + b_f' - b_f \geq 0$ and $\kappa_w = c_f(\theta_w) - c_f'(\theta_w) + b_f' - b_f \geq 0$. Thus, Lemma S8 give that the set of stable allocations $\mathcal{S}$ increases, equivalently, that the set of interior stable allocations $\mathcal{S}_I$ increases (per Lemma S7).

It remains to show that $\max_{(\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}} T(\phi, \bar{\boldsymbol{x}})$ and $\max_{(\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}} T_E(\phi, \bar{\boldsymbol{x}})$ both exist and increase. Existence follows either from Lemma 6 or an argument analogous to the Proof of Lemma 6. That $\max_{(\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}} T_E(\phi, \bar{\boldsymbol{x}})$ increases follows directly from the fact $\mathcal{S}$ increases. That $\max_{(\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}} T(\phi, \bar{\boldsymbol{x}})$ increases follows from the facts that $\mathcal{S}$ increases and that players' utilities are shifted upwards. □

6.6 Welfare Maximization in the Effort Game

Lemma 10 shows that symmetric stable allocations maximize total welfare. In light of this, one may ask if there is a welfare maximizing stable allocation in the Effort Game. The following example shows that the answer is no.

Example S9. Lack of a Welfare Maximizing Stable Allocation.
Suppose $X = [0,2]^2$. Let $\mathcal{M} = \{1,2\}$ and let $\mathcal{W} = \{3,4\}$. Let $b(y) = y$ and let $c(\theta, y) = \frac{1}{\theta}y^2 + \frac{1}{8}$. Let $\theta_1 = \theta_3 = 2$, let $\theta_2 = 1$, and let $\theta_4 = 3/2$. It's readily verified that Assumptions 4, 5, and 6 hold, so all of our results apply, save Proposition 9.

Let's turn to welfare maximization. Since (i) the cost functions are strictly convex, (ii) agreeability holds for all players, and (iii) the payoffs are additive in benefit and cost, in every welfare maximizing allocation: man 1 exerts effort 1, man 2 exerts effort $1/2$, woman 3 exerts effort 1, and woman 4 exerts effort $3/4$. There are two ways to implement this:

1. Man 1 and woman 3 match with agreement $(1,1)$, while 2 and 4 match with agreement $(1/2, 3/4)$. Herein, 1 earns $3/8$, 2 earns $3/8$, 3 earns $3/8$, and 4 earns 0.

2. Man 1 and woman 4 match with agreement $(1, 3/4)$, while 2 and 3 match with agreement $(1/2, 1)$. Herein, 1 earns $1/8$, 2 earns $5/8$, 3 earns $-1/8$, and 4 earns $4/8$.

Since the second allocation isn't individually rational for woman 3, only the first allocation can be stable. We claim that the first allocation is blocked by 1 and 4, with agreement $(9/10, 1)$. At this agreement, 1 earns $47/100$, which exceeds $3/8$, and 4 earns $11/40$, which exceeds zero. It follows that no stable allocation maximizes welfare.

The reason that stable allocations don't maximize total welfare is that 4's incentives are not aligned with what's socially optimal. At the welfare maximizing allocation, she has an incentive to over-invest in effort to secure a more lucrative relationship with 1. The reason 4 faces this incentive is because the type endowment is asymmetric. (If we were to restore symmetry by changing man 2's type to $\theta_2 = 3/2$, then we'd remove 4's desire to match with man 1 and so we'd recover the existence of a welfare maximizing stable allocation per Lemma 10.) Since this misalignment is generic (as examples like this one can be readily constructed), we conclude that welfare maximizing stable allocations generally do not exist. $\triangle$

7 Appendix: Supplement to Chapter 3

This chapter collects several supplemental results to the game introduced in Chapter 3. Please note that all references (e.g., Proposition X, Lemma Z, etc.) in this chapter refer to the items in Chapter 3.

- Section 6.1 discusses asymmetric type endowments and establishes conditions for existence with such endowments.
- Section 6.2 shows how to generalize all of the results of Chapter 3 to $K \geq 2$ sided games.

7.1 Asymmetric Type Endowments

In this subsection, we discuss asymmetric type endowments. We begin by giving an example with asymmetric types. We use this example to (i) illustrate that symmetric allocations may be unstable when types are asymmetric and (ii) illustrate that stable allocations can exist when types are asymmetric. Subsequently, we discuss how one may partially extend our existence results to games with asymmetric types.

Example A1. A Simple Effort Game.

Suppose $\mathcal{T} = \{1, 2\}$, $\mathcal{A} = \{3, 4\}$, and $\mathcal{C} = \{5, 6\}$. Let $X = [0, 4]^3$, $b(\theta, y) = y$, and $d(\theta, y) = y^2/\theta$. In addition, let $\theta_1 = 1$, $\theta_2 = 2$, and $\theta_3 = \theta_4 = \theta_5 = \theta_6 = 1$.

Symmetric Allocation. One symmetric allocation is $(\phi', \bar{\boldsymbol{x}}')$ where (i) town 1, architect 3, and construction company 5 are matched with agreement $(1, 1, 1)$ and (ii) town 2, architect 4, and construction company 6 are matched with agreement $(2, 2, 2)$. This allocation isn't stable. To see this, consider town 1, architect 4, and construction company 6. They block if there's a $(x_t, x_a, x_c) \in [0, 4]^3$ such that

$$x_a + x_c - (x_t)^2 > 1,\ x_t + x_c - (x_a)^2 > 0, \text{ and } x_t + x_a - (x_c)^2 > 0.$$

It's easily checked that $(1, 1, 4/3)$ is such a point. Since every other symmetric allocation involves the same efforts as $(\phi', \bar{\boldsymbol{x}}')$ but a different matching of architects and construction companies to towns, it follows that no symmetric allocation is stable.

Stable Allocation. It's readily verified that one stable allocation is $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ where (i) 1, 3, and 5 are matched with agreement $(0.5567, 1.3982, 1.3982)$ and (ii) 2, 4, and 6 are matched with agreement $(0.9232, 1.5832, 1.5832)$. In this allocation, town 1 earns 2.48, town 2 earns 2.74, and the remaining players all earn zero. This allocation was chosen to maximize the town's payoffs subject to their partners' individual rationality constraints. Thus, there are no blocking pairs since all of architects and construction companies are the same type. △

While the example shows that a stable allocation may exist when the type endowment is asymmetric, we cannot argue a general existence result using the logic of Proposition 1: this logic is inexorably tied to the symmetry of the type endowment and doesn't extend. That said, we can give two partial results.

Lemma A1. Existence with Asymmetric Type Endowments.
There's a stable allocation when there's only one town and there are multiple architects and construction companies. (Likewise, a stable allocation always exists if there's only one architect or only one construction company.)

Proof. Let t be the unique town. We assume there's an a architect, a c construction company, and agreement $\boldsymbol{x} \in X$ such that t, a, and c earn strictly positive payoffs at $\boldsymbol{x}$. This is without loss because, if this weren't the case, then it would be stable for all players to be single.

For each pair $(a, c) \in \mathcal{A} \times \mathcal{C}$, let

$$V(a, c) = \max_{\boldsymbol{x} \in X} u_t(a, c, \boldsymbol{x}) \text{ s.t. } u_a(t, c, \boldsymbol{x}) \geq 0 \text{ and } u_c(t, a, \boldsymbol{x}) \geq 0,$$

and let $M(a, c)$ be the set of maximizers. (Take $V(a, c) = 0$ if there's no maximum for a given (a, c) because there are no individually rational agreements for all three players.) Let $(\tilde{a}, \tilde{c})$ solve

$$\max_{(a,c) \in \mathcal{A} \times \mathcal{C}} V(a, c),$$

and let $\tilde{\boldsymbol{x}} \in M(\tilde{a}, \tilde{c})$. (This solution exists by our initial assumption.)

A stable allocation is for t to match with architect $\tilde{a}$ and construction company $\tilde{c}$ with agreement $\tilde{\boldsymbol{x}}$, while all other architects and construction companies are single. This allocation is feasible and individually rational by construction. It also cannot be blocked because (i) no other pair of architect or construction company could offer t a strictly higher payoff and (ii) there are no other towns to offer $\tilde{a}$ and $\tilde{c}$ a better deal. □

To give our second result, we simplify by assuming all players have the *same* benefit function. We say that $d(\theta, y)$ is an **interval step-function in $\boldsymbol{\theta}$** if there is a finite collection of points $\{\tilde{\theta}_k\}_{k=1}^K$ and a finite collection of real-valued functions $\{d_k(y)\}_{k=1}^{K+1}$ such that

$$d(\theta, y) = \begin{cases} d_1(y) & \text{if } \theta \leq \tilde{\theta}_1 \\ d_2(y) & \text{if } \tilde{\theta}_1 < \theta \leq \tilde{\theta}_2 \\ \vdots & \vdots \\ d_K(y) & \text{if } \tilde{\theta}_{K-1} < \theta \leq \tilde{\theta}_K \\ d_{K+1}(y) & \text{if } \tilde{\theta}_K < \theta. \end{cases}$$

We say that the type endowment $\{\theta_i\}_{i\in\mathcal{N}}$ is **effectively symmetric** if there are an equal number of towns, architects, and construction companies with types "in-between" each step of $d(\theta, y)$, i.e., if

$$\begin{aligned} &\text{(i) } |\{t \in \mathcal{T}|\theta_t < \tilde{\theta}_1\}| = |\{a \in \mathcal{A}|\theta_a < \tilde{\theta}_1\}| = |\{c \in \mathcal{C}|\theta_c < \tilde{\theta}_1\}|, \\ &\text{(ii) } |\{t \in \mathcal{T}|\tilde{\theta}_{k-1} < \theta_t \leq \tilde{\theta}_k\}| = |\{a \in \mathcal{A}|\tilde{\theta}_{k-1} < \theta_a \leq \tilde{\theta}_k\}| \\ &\qquad = |\{c \in \mathcal{C}|\tilde{\theta}_{k-1} < \theta_c \leq \tilde{\theta}_k\}| \text{ for all } k \in \{2, \ldots, K\}, \text{ and} \\ &\text{(iii) } |\{t \in \mathcal{T}|\theta_t > \tilde{\theta}_K\}| = |\{a \in \mathcal{A}|\theta_a > \tilde{\theta}_K\}| = |\{c \in \mathcal{C}|\theta_c < \tilde{\theta}_K\}|. \end{aligned}$$

Notice that there may be no players with types in-between some steps.

Lemma A2. Existence with Asymmetric Type Endowments.
Let Assumption 1 hold and let all players have the same benefit function. Then, there's a stable allocation when $d(\theta, y)$ is a step function in θ and the type endowment is effectively symmetric.

Proof. Since $d(\theta, y)$ is constant in θ between $\tilde{\theta}_{k-1}$ and $\tilde{\theta}_k$, all players with types in $(\tilde{\theta}_{k-1}, \tilde{\theta}_k]$ have the same cost function. Thus, we can assign these players type $\tilde{\theta}_k$ without changing their payoffs since all players have the same benefit function. When we do this for each k, then there are an equal number of towns, architects, and construction companies with type $\tilde{\theta}_k$. This environment is now isomorphic to the environment we considered when we proved Proposition 1. The lemma follows. □

7.2 Games with $K \geq 2$ Sides

In this section, we consider an environment with $K \geq 2$ groups of players and we argue that our results generalize. We'll only sketch selected proofs because they're straightforward extensions of the proofs given in the main text.

There are $K \geq 2$ finite, equally sized groups of players G_1, G_2, …, G_K. (Let $\mathcal{N} = \cup_{i=1}^K G_i$

and $N = |\mathcal{N}|$.) Each player i may either be single or may be matched to one member of every other group. (For instance, if i is in group G_1, then she's either single or matched to a member of G_2, a member of G_3, ..., and a member of G_K.) A matching ϕ is a function that records each player's matches. When players $i_1 \in G_1$, $i_2 \in G_2$, ..., and $i_K \in G_K$ match, they select an agreement $\boldsymbol{x} = (x_1, \ldots, x_K) \in \mathbb{R}^K$ which gives their joint efforts – x_j is the effort of player i_j. As usual, players have a benefit function $b(\cdot)$ and a cost function $d(\cdot)$, and the value of being single is zero. The payoff of player i_j to being matched with players $i_1 \in G_1$, $i_2 \in G_2$, ..., $i_{j-1} \in G_{j-1}$, $i_{j+1} \in G_{j+1}$, ..., and $i_K \in G_K$ at agreement $\boldsymbol{x} = (x_1, \ldots, x_K)$ is

$$u_{i_j}(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_K, \boldsymbol{x}) = (\sum_{l \neq j} b(\theta_l, x_l)) - d(\theta_{i_j}, x_j).$$

An allocation $(\phi, \bar{\boldsymbol{x}})$ is **stable** if (i) it's feasible, i.e., $\bar{\boldsymbol{x}} \in X^N = [0, \beta]^{KN}$, (ii) it's individually rational, i.e., $u_i(\phi, \bar{\boldsymbol{x}}) \geq 0$ for each player i, and (iii) it's unblocked, i.e., there's no $(i_1, \ldots, i_K) \in \Pi_{l=1}^K G_l$ and $\boldsymbol{x} \in X$ such that $u_j(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_K, \boldsymbol{x}) > u_j(\phi, \bar{\boldsymbol{x}})$ for all $j \in \{i_1, \ldots, i_K\}$.

We extend universal agreeability (Assumption 1) and symmetry to this environment as follows. **Universal agreeability** now requires that for all $(\theta_1, \ldots, \theta_K) \in \Theta^K$, there is an $\boldsymbol{x} = (x_1, \ldots, x_K) \in X$ such that $\sum_{l \neq j} b(\theta_l, x_l) - c(\theta_j, x_j) > 0$ for each $j \in \{1, \ldots, K\}$. The type endowment $\{\theta_i\}_{i \in \mathcal{N}}$ is **symmetric** if (i) all groups are endowed with the same types, i.e., $\cup_{i \in G_j} \{\theta_i\} = \cup_{i \in G_k} \{\theta_i\}$ for all j and k in $\{1, \ldots, K\}$, and if (ii) there are an equal number of players in each group with the same type, i.e., for each $\theta \in \Theta$, we have $|\{i \in G_j | \theta_i = \theta\}| = |\{i \in G_k | \theta_i = \theta\}|$ for all j and k in $\{1, \ldots, K\}$.

The following is our generalization of Proposition 1.

Proposition B1. Generalization of Proposition 1.
Let universal agreeability hold and let the endowment of types $\{\theta_i\}_{i \in \mathcal{N}}$ be symmetric, then a welfare maximizing stable allocation exists.

The argument is analogous to the proof of Proposition 1: we first construct the symmetric allocation, then we establish that it has the requisite properties. Existence then follows from the fact a stable allocation always exists.

Construction. Symmetric Allocation.
Let universal agreeability hold and let type endowment $\{\theta_i\}_{i \in \mathcal{N}}$ be symmetric. We begin by ordering the players. For each group, we list players in descending order of their types, breaking ties arbitrarily. Label the first player in group G_k's list p_1^k, the second player p_2^k, and so on down to the last player p_T^k.

We construct a **symmetric allocation** $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = (\phi^\star, \boldsymbol{x}^{1^\dagger}, \ldots, \boldsymbol{x}^{N^\dagger})$ as follows. Set $\phi^\dagger$

such that

$$\phi^{\dagger}(p_l^k) = (p_l^1, \ldots, p_l^{k-1}, p_l^{k+1}, \ldots, p_l^K),$$

for each $l \in \{1, \ldots, T\}$ and each group $k \in \{1, \ldots, K\}$. Set $\bar{\boldsymbol{x}}^{\dagger}$ such that $\boldsymbol{x}^{p_l^1} = \boldsymbol{x}^{p_l^2} = \cdots = \boldsymbol{x}^{p_l^K} = (x_l^{\dagger}, \ldots, x_l^{\dagger})$ where

$$x_l^{\dagger} = \max\{\arg\max_{y \in [0,\beta]} (K-1)b(\theta_{p_l^1}, y) - d(\theta_{p_l^1}, y)\}, \tag{7.1}$$

for each $l \in \{1, \ldots, T\}$. ○

Observe that a symmetric allocation always exists per our standard continuity assumptions on the benefit and cost functions.

Lemma B1. Analogue of Lemma 1.
Let universal agreeability hold and let the endowment of types $\{\theta_i\}_{i \in \mathcal{N}}$ be symmetric. Then a symmetric allocation $(\phi^{\dagger}, \bar{\boldsymbol{x}}^{\dagger})$ is stable and welfare maximizing. In addition, for each player i, we have $\theta_i = \theta_j = \theta_k$ for each $(j, k) = \phi^{\dagger}(i)$ and

$$u_i(\phi^{\dagger}, \bar{\boldsymbol{x}}^{\dagger}) = \max_{y \in [0,\beta]} (K-1)b(\theta_i, y) - d(\theta_i, y). \tag{7.2}$$

Proof of Proposition B1. Obvious in light of Lemma B1 and omitted. □

Sketch of Proof of Lemma B1. The argument is virtually the same as the one employed in the main text. We begin by observing that all matched players have the same types, i.e., that $\theta_{p_l^1} = \theta_{p_l^2} = \cdots = \theta_{p_l^K}$ for all l. From this, it follows that equation (7.2) holds. Individual rationality now follows from universal agreeability, and feasibility is a non-issue. It only remains to check no blocking. As in the main text we argue by contradiction: if a group of players block, then it must be that their combined payoff exceeds the maximum of the sum of their payoffs per equation (7.2), which is impossible. The argument for welfare maximization is similar to the one given in the main text and is omitted. □

Propositions 2 and 3 readily generalize to K-sided games because they only depend on (i) the supermodularity and monotonicity of the benefit function and (ii) the submodularity of the cost function, which are both independent of the number of sides (per the additive nature of payoffs). We omit a detailed discussion because it adds no economic insight.

8 Appendix: Supplement to Chapter 4

This chapter collects several supplemental to the game presented in Chapter 4. Please note that all references (e.g., Proposition X, Lemma Z, etc.) in this chapter refer to the items in Chapter 4.

- Section 7.1 shows how to construct an equilibrium where the partners employ their consultants optimally both on and off of the equilibrium path in the second stage. Since partners maximize their expected payoffs in the first stage, it follows that when they play this equilibrium they behave optimally whenever they move. We also use this equilibrium to construct a Perfect Bayesian Equilibrium.

- Section 7.2 discusses non-ELEE equilibria via three examples. The first example demonstrates that there may be non-ELEE equilibria. The second example shows that non-ELEE equilibria don't exist under certain parameterizations, indicating that non-ELEE equilibria aren't "robust." The third example shows that ELEE equilibria do not Pareto dominate non-ELEE equilibria and vice versa.

- Section 7.3 provides two counter examples to results in the main text. The first example shows that minimal overlap may fail absent Assumption 2. The second example shows that the conclusion of Proposition 7 is sensitive to the equilibrium selected.

- Section 7.4 considers three different extensions to our game. The first extension allows for multiple partners. While this extension kills our welfare and existence arguments, we find that our minimal overlap result remains true. The second extension considers a production technology with residual value. We establish that an equilibrium exists, that our welfare results about *simple equilibria* remain true, and that our results about minimal overlap hold in this extension. The third extension examines heterogeneous labor costs. We argue that our existing results hold in this extension and we develop comparative statics for welfare (and network size) as labor costs change.

8.1 Optimal Second Stage Behavior

In this section, we show that there's an equilibrium where the partners behave "optimally" in the second stage. Since the partners always behave optimally in the first stage, it follows that they behave optimally, in this equilibrium, whenever they move. We give this result to allay concerns about non-credible, off-path threats. We also sketch the construction of a Perfect Bayesian Equilibrium from this equilibrium.

For simplicity, we assume that Assumption 2 holds throughout this section. This allows us to simplify the presentation and the construction. Recall from Remark A2 that partner i does best in the second stage, given her project x, by employing exactly $d(x)$ consultants, provided she has at least $d(x)$ consultants available, and by passing on her project otherwise. In light of this, we say that $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star, \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star) \in \boldsymbol{S}$ is a **equilibrium with optimal second stage behavior** if $\boldsymbol{s}^\star$ is an equilibrium and, for each partner i,

1. For each $(\mathcal{N}, x) \in \mathbb{P}(\mathcal{C}) \times X$, the strategy $\sigma_{1i}^\star(\mathcal{N}, x)$ selects $d(x)$ consultants in $\mathcal{N}$ when $d(x) \leq |\mathcal{N}|$ and selects 0 consultants when $d(x) > |\mathcal{N}|$.
2. For each $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$, the strategy $\sigma_{2i}^\star(\mathcal{N}, \mathcal{N}', x)$ selects $d(x)$ consultants in $\mathcal{N} \backslash \mathcal{N}'$ when $d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|$ and selects 0 consultants when $d(x) > |\mathcal{N} \backslash \mathcal{N}'|$.

We'll construct extensions of simple strategies that induce an equilibrium with optimal second stage behavior.

An **extended simple strategy for $\boldsymbol{A}$** of size $n \in \{0, \ldots, N\}$ is a tuple $\tilde{\boldsymbol{s}}_A^e(n) = (\tilde{\mathcal{N}}_A^e, \tilde{\sigma}_{1A}^e, \tilde{\sigma}_{2A}^e)$ such that:

1. We have
$$\tilde{\mathcal{N}}_A^e = \{1, \ldots, n\}.$$
2. For all $(\mathcal{N}, x) \in \mathbb{P}(\mathcal{C}) \times X$,
$$\tilde{\sigma}_{1A}^e(\mathcal{N}, x) = \begin{cases} \{1, \ldots, d(x)\} & \text{if } d(x) \leq |\mathcal{N}| \text{ and } \mathcal{N} = \tilde{\mathcal{N}}_A^e \end{cases}$$
otherwise, $\tilde{\sigma}_{1A}^e$ (arbitrarily) selects $d(x)$ consultants in $\mathcal{N}$ when $d(x) \leq |\mathcal{N}|$ and selects the empty set when $d(x) > |\mathcal{N}|$.
3. For all $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$,
$$\tilde{\sigma}_{2A}^e(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{1, \ldots, d(x)\} & \begin{array}{r} \text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|, \mathcal{N}' \subset \{d(x)+1, \ldots, N\}, \\ \text{and } \mathcal{N} = \tilde{\mathcal{N}}_A^e, \end{array} \end{cases}$$
otherwise, $\tilde{\sigma}_{2A}^e$ (arbitrarily) selects $d(x)$ consultants in $\mathcal{N} \backslash \mathcal{N}'$ when $d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|$ and selects the empty set when $d(x) > |\mathcal{N} \backslash \mathcal{N}'|$.

This strategy implements A's simple strategy on the equilibrium path. Off path, it tells A to behave optimally. (We've omitted the specific construction of the off-path branches of the strategy for simplicity.)

An **extended simple strategy for $\boldsymbol{B}$** of size $n \in \{0, \ldots, N\}$ is a tuple $\tilde{\boldsymbol{s}}_B^e(n) = (\tilde{\mathcal{N}}_B^e, \tilde{\sigma}_{1B}^e, \tilde{\sigma}_{2B}^e)$ such that:

1. We have

$$\tilde{\mathcal{N}}^e_B = \{N+1-n, \dots, N\}.$$

2. For all $(\mathcal{N}, x) \in \mathbb{P}(\mathcal{C}) \times X$,

$$\tilde{\sigma}^e_{1B}(\mathcal{N}, x) = \begin{cases} \{N+1-d(x), \dots, N\} & \text{if } d(x) \leq |\mathcal{N}| \text{ and } \mathcal{N} = \tilde{\mathcal{N}}^e_B \end{cases}$$

otherwise, $\tilde{\sigma}^e_{1B}$ (arbitrarily) selects $d(x)$ consultants in $\mathcal{N}$ when $d(x) \leq |\mathcal{N}|$ and selects the empty set when $d(x) > |\mathcal{N}|$.

3. For all $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$,

$$\tilde{\sigma}^e_{2B}(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{1, \dots, d(x)\} & \text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|,\ \mathcal{N}' \subset \{1, \dots, N-d(x)\}, \\ & \text{and } \mathcal{N} = \tilde{\mathcal{N}}^e_B, \end{cases}$$

otherwise, $\tilde{\sigma}^e_{2B}$ (arbitrarily) selects $d(x)$ consultants in $\mathcal{N} \backslash \mathcal{N}'$ when $d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|$ and selects the empty set when $d(x) > |\mathcal{N} \backslash \mathcal{N}'|$.

This concludes the description of the strategies.

The following proposition is our main result. For each $(n_A, n_B) \in \{0, \dots, N\}$, let $\tilde{\boldsymbol{s}}^e(n_A, n_B) = (\tilde{\boldsymbol{s}}^e_A(n_A), \tilde{\boldsymbol{s}}^e_B(n_B))$ denote A and B's joint vector of extended simple strategies.

Proposition S1. An Equilibrium with Optimal Second Stage Behavior.
Let Assumption 2 hold. There is a $(n^\star_A, n^\star_B) \in \{0, \dots, N\}^2$ such that $\tilde{\boldsymbol{s}}^e(n^\star_A, n^\star_B)$ is an equilibrium with optimal second stage behavior.

Since $\tilde{\boldsymbol{s}}^e(n^\star_A, n^\star_B)$ is an equilibrium, both partners pick their networks to maximize their expected payoffs given their behavioral employment strategies. Thus, the partners behave optimally in $\tilde{\boldsymbol{s}}^e(n^\star_A, n^\star_B)$ whenever they may be called upon to move. It follows that there are no non-credible threats in $\tilde{\boldsymbol{s}}^e(n^\star_A, n^\star_B)$. We'll prove the proposition after the next remark.

Remark. Since our game is one of imperfect and incomplete information because the partners don't observe each others' networks or projects, it's arguable that the most appropriate solution concept for our game is Perfect Bayesian Equilibrium (PBE). It's straightforward to construct a PBE from $\tilde{\boldsymbol{s}}^e(n^\star_A, n^\star_B)$ because the partner's beliefs about each other's networks are payoff irrelevant and the partners behave optimally under $\tilde{\boldsymbol{s}}^e(n^\star_A, n^\star_B)$.

Let $(\phi_A^\star, \phi_B^\star, \tilde{\boldsymbol{s}}^e(n_A^\star, n_B^\star))$ be a candidate PBE, where

$$\phi_A^\star(\mathcal{N}, x) = \begin{cases} P_B(x) & \text{if } \mathcal{N} = \{N+1-n_B^\star, \ldots, N\} \\ 0 & \text{else.} \end{cases}$$

$$\phi_B^\star(\mathcal{N}, x) = \begin{cases} P_A(x) & \text{if } \mathcal{N} = \{1, \ldots, n_A^\star\} \\ 0 & \text{else.} \end{cases}$$

It's easily verified that the beliefs $(\phi_A^\star, \phi_B^\star)$ are consistent with the joint strategy $\tilde{\boldsymbol{s}}^e(n_A^\star, n_B^\star)$ and Bayes' rule.[76] Thus, we only need to show that partners' expected payoffs are maximized whenever they move. But this follows directly from Proposition S1 and the prior discussion since partners don't care about each other's networks in the second stage.

We'll prove Proposition 1 via three intermediate lemmas. First, we'll establish that the extended simple strategies always fulfill parts (1) and (2) of the definition of an equilibrium with optimal second stage behavior. Second, we'll establish that the partners (ex-ante) payoffs are the same under their extended simple strategies and their simple strategies. Third, we'll use this result to establish that there's an equilibrium in extended simple strategies. The proposition then follows.

Lemma S1. Number of Consultants Employed.
For all $(n_A, n_B) \in \{0, \ldots, N\}^2$, we have $\tilde{\boldsymbol{s}}^e(n_A, n_B) = (\tilde{\mathcal{N}}_A^e, \tilde{\sigma}_{1A}^e, \tilde{\sigma}_{2A}^e, \tilde{\mathcal{N}}_B^e, \tilde{\sigma}_{1B}^e, \tilde{\sigma}_{2B}^e)$ is such that, for all $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$,

$$|\tilde{\sigma}_{1i}(\mathcal{N}, x)| = \begin{cases} d(x) & \text{if } d(x) \leq |\mathcal{N}| \\ 0 & \text{if } d(x) > |\mathcal{N}| \end{cases} \text{ and } |\tilde{\sigma}_{2i}(\mathcal{N}, \mathcal{N}', x)| = \begin{cases} d(x) & \text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'| \\ 0 & \text{if } d(x) > |\mathcal{N} \backslash \mathcal{N}'|, \end{cases}$$

for $i \in \{A, B\}$.

Proof. The is a direct consequence of the construction of the extended simple strategies. □

Lemma S2. Payoff Equivalency.

[76] Strictly speaking, $\phi_A^\star$ and $\phi_B^\star$ are not beliefs for every information set at which either partner may move. When partner i moves in either stage one or moves first in the second stage, she observes nothing about the other partner, so her belief is $\phi_i^\star$. However, when partner i moves second in the second stage, she observes that the other partner chose $\mathcal{T}$ and so has belief

$$\phi_i^\star(\mathcal{N}, x|\mathcal{T}) = \phi_i^\star(\mathcal{N}, x) \Big/ \sum_{\{(\mathcal{N}', x') | \tilde{\sigma}_{1-i}^e(\mathcal{N}', x') = \mathcal{T}\}} \phi_i^\star(\mathcal{N}', x')$$

about the other partner's type on the equilibrium path. Off the equilibrium path, we allow $\phi_i^\star(\mathcal{N}, x|\mathcal{T})$ to be arbitrary. Thus, $\phi_A^\star$ and $\phi_B^\star$ induce the requisite beliefs for each information set.

For all $i \in \{A, B\}$ *and all* $(n_A, n_B) \in \{0, \ldots, N\}^2$, *we have that* $U_i(\tilde{\boldsymbol{s}}(n_A, n_B)) = U_i(\tilde{\boldsymbol{s}}^e(n_A, n_B))$.

Proof. We'll prove this for A since the argument for B is analogous. Since A holds the same network under $\tilde{\boldsymbol{s}}$ and $\tilde{\boldsymbol{s}}^e$, we only need to show that A employs the same sets of consultants under both strategies whenever she moves. The desired result then follows because A's ex-post earnings depend only on the size of her network and the number of consultants she employs whenever she moves. Let $\tilde{\boldsymbol{s}}_A(n_A) = (\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1A}, \tilde{\sigma}_{2A})$ and let $\tilde{\boldsymbol{s}}^e_A(n_A) = (\tilde{\mathcal{N}}^e_A, \tilde{\sigma}^e_{1A}, \tilde{\sigma}^e_{2A})$. We'll use $\tilde{\boldsymbol{s}}$ to denote $\tilde{\boldsymbol{s}}(n_A, n_B)$ and $\tilde{\boldsymbol{s}}^e$ to denote $\tilde{\boldsymbol{s}}^e(n_A, n_B)$.

First, we establish that A employs the same set of consultants when she moves first in the second stage, i.e., that $\tilde{\sigma}_{1A}(\tilde{\mathcal{N}}_A, x) = \tilde{\sigma}^e_{1A}(\tilde{\mathcal{N}}^e_A, x)$ for all $x \in X$. Let x be A's project. Since A chooses the same network under $\tilde{\boldsymbol{s}}$ and $\tilde{\boldsymbol{s}}^e$, she observes $(\tilde{\mathcal{N}}_A, x) = (\tilde{\mathcal{N}}^e_A, x)$ when she moves. Thus, $\tilde{\sigma}_{1A} = \tilde{\sigma}^e_{1A} = \{1, \ldots, d(x)\}$ when $d(x) \le n_A$ and $\tilde{\sigma}_{1A} = \tilde{\sigma}^e_{1A} = \emptyset$ else. The desired result follows.

It follows that $\mathcal{E}^{\tilde{\boldsymbol{s}}}_A = \mathcal{E}^{\tilde{\boldsymbol{s}}^e}_A$, i.e., that A chooses the same subsets of consultants under both $\tilde{\boldsymbol{s}}$ and $\tilde{\boldsymbol{s}}^e$ when she moves first. Importantly, an analogous argument gives that

$$\mathcal{E}^{\tilde{\boldsymbol{s}}}_B = \mathcal{E}^{\tilde{\boldsymbol{s}}^e}_B,$$

i.e., B chooses the same subsets of consultants under both $\tilde{\boldsymbol{s}}$ and $\tilde{\boldsymbol{s}}^e$ when she moves first.

Second, we establish that A employs the same set of consultants when she moves second in the second stage, i.e., that $\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \mathcal{T}, x) = \tilde{\sigma}^e_{2A}(\tilde{\mathcal{N}}^e_A, \mathcal{T}, x)$ for all $(\mathcal{T}, x) \in \mathcal{E}^{\tilde{\boldsymbol{s}}}_B \times X$. Let $(\mathcal{T}, x) \in \mathcal{E}^{\tilde{\boldsymbol{s}}}_B \times X$. First, observe that if $d(x) \le |\tilde{\mathcal{N}}_A \backslash \mathcal{T}|$, then $\mathcal{T} \subset \{d(x)+1, \ldots, N\}$.[77] Thus, $\tilde{\sigma}_{2A} = \tilde{\sigma}^e_{2A} = \{1, \ldots, d(x)\}$ when $d(x) \le |\tilde{\mathcal{N}}_A \backslash \mathcal{T}|$ and $\tilde{\sigma}_{2A} = \tilde{\sigma}^e_{2A} = \emptyset$ when $d(x) > |\tilde{\mathcal{N}}_A \backslash \mathcal{T}|$. The desired result follows. □

Lemma S3. Equilibrium.
There is a $(n^\star_A, n^\star_B) \in \{0, \ldots, N\}^2$ *such that* $\tilde{\boldsymbol{s}}^e(n^\star_A, n^\star_B)$ *is an equilibrium.*

Proof. By Lemmas 1, 2, and 3, there is a $(n'_A, n'_B) \in \{0, \ldots, N\}^2$ such that $\tilde{\boldsymbol{s}}(n'_A, n'_B)$ is an equilibrium. We argue that $\tilde{\boldsymbol{s}}^e(n'_A, n'_B)$ is an equilibrium by contradiction. Suppose not, then one partner, say A, does strictly better by playing $(\mathcal{N}_A, \sigma_{1A}, \sigma_{2A})$ instead of playing $\tilde{\boldsymbol{s}}^e_A(n'_A)$ when B plays $\tilde{\boldsymbol{s}}^e_B(n'_B)$. Thus,

$$U_A(\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \tilde{\boldsymbol{s}}^e_B(n'_B)) > U_A(\tilde{\boldsymbol{s}}^e_A(n'_A, n'_B)) = U_A(\tilde{\boldsymbol{s}}_A(n'_A, n'_B)),$$

where the equality follows from Lemma S2. It follows that $\tilde{\boldsymbol{s}}(n'_A, n'_B)$ is not an equilibrium, a contradiction. □

[77] Since B follows her (extended) simple strategy, $\mathcal{T} = \{k, \ldots, N\}$ for some $k \in \{1, \ldots, N+1\}$. Thus, $\tilde{\mathcal{N}}_{\mathcal{A}} \backslash \mathcal{T} = \{1, \ldots, \min\{n_A, k-1\}\}$, so $d(x) \le |\tilde{\mathcal{N}}_{\mathcal{A}} \backslash \mathcal{T}|$ implies $d(x) \le k-1$ which implies $\mathcal{T} \subset \{d(x)+1, \ldots N\}$.

Proof of Proposition S1. The proposition is a consequence of Lemmas S1 and S3. □

8.2 Non-ELEE Equilibria

In this section, we provide three examples with non-ELEE equilibria. The first example shows that there are non-ELEE equilibria. The second example shows that non-ELEE equilibria do not exist under some parameterizations. This indicates that they aren't "robust." The third example shows that non-ELEE equilibria do not Pareto dominate ELEE equilibria and vice versa.

Example S1. A Non-ELEE Equilibrium.

Let $\mathcal{C} = \{1, 2, 3, 4\}$ be the set of consultants and let $X = \{x_1, x_2, x_3, x_3, x_5, x_6\}$ be the set of projects. Let $d(x_1) = d(x_2) = d(x_3) = d(x_4) = 1$, $d(x_5) = 4$, and $d(x_6) = 3$. Let

$$P_A(x) = \begin{cases} \frac{1}{40} & \text{if } x \in \{x_1, x_2, x_3, x_4\} \\ \frac{9}{10} & \text{if } x = x_5 \\ 0 & \text{else} \end{cases} \quad \text{and } P_B(x) = \begin{cases} 1 & \text{if } x = x_6 \\ 0 & \text{else,} \end{cases}$$

so A may get projects x_1, ..., x_5, while B always gets project x_6. Let A and B have constant marginal costs of networking of $\kappa_A = 1$ and let $\kappa_B = 3/4$ respectively. Let $r_A = r_B = r$ and let $\pi(x) = r(x) - w \cdot d(x)$ denote the payoff to completing project x with $d(x)$ consultants for both principles. Let $\pi(x_1) = \pi(x_2) = \pi(x_3) = \pi(x_4) = 1$, $\pi(x_5) = 100$, and $\pi(x_6) = 5$.

A non-ELEE equilibrium is $(\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star, \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star)$, where $\mathcal{N}_A^\star = \{1, 2, 3, 4\}$, $\mathcal{N}_B^\star = \{2, 3, 4\}$,

$$\sigma_{1A}^\star(\mathcal{N}, x) = \begin{cases} \{1\} & \text{if } x = x_1 \text{ and } \mathcal{N} = \mathcal{N}_A^\star \\ \{2\} & \text{if } x = x_2 \text{ and } \mathcal{N} = \mathcal{N}_A^\star \\ \{3\} & \text{if } x = x_3 \text{ and } \mathcal{N} = \mathcal{N}_A^\star \\ \{4\} & \text{if } x = x_4 \text{ and } \mathcal{N} = \mathcal{N}_A^\star \\ \{1, 2, 3, 4\} & \text{if } x = x_5 \text{ and } \mathcal{N} = \mathcal{N}_A^\star \\ \emptyset & \text{else,} \end{cases}$$

$$\sigma^\star_{1B}(\mathcal{N}, x) = \begin{cases} \{2,3,4\} & \text{if } x = x_6 \text{ and } \mathcal{N} = \mathcal{N}^\star_B \\ \emptyset & \text{else,} \end{cases}$$

$$\sigma^\star_{2A}(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{1\} & \text{if } x \in \{x_1, x_2, x_3, x_4\},\ \mathcal{N} = \mathcal{N}^\star_A, \text{ and } \mathcal{N}' = \{2,3,4\} \\ \emptyset & \text{else,} \end{cases}$$

$$\sigma^\star_{2B}(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{2,3,4\} & \text{if } x = x_6,\ \mathcal{N} = \mathcal{N}^\star_B, \text{ and } \mathcal{N}' = \{1\} \\ \emptyset & \text{else,} \end{cases}$$

where $\mathcal{N}$ and $\mathcal{N}'$ are subsets of $\mathcal{C}$.

It is fairly easy to show that $(\mathcal{N}^\star_A, \sigma^\star_{1A}, \sigma^\star_{2A})$ and $(\mathcal{N}^\star_B, \sigma^\star_{1B}, \sigma^\star_{2B})$ are mutual best responses. Let's verify that, for B, $(\mathcal{N}^\star_B, \sigma^\star_{1B}, \sigma^\star_{2B})$ is a best response to $(\mathcal{N}^\star_A, \sigma^\star_{1A}, \sigma^\star_{2A})$. As in Example 1, we'll do this by considering different networks for B. Since B only gets project x_6, it's never optimal for her to hold a network of size 1 or size 2. Thus, B's network consists of only consists of 0, 3, or 4 consultants. Let's examine what happens when B holds a network of each of these sizes.

- Suppose B chooses $\mathcal{N}_B = \emptyset$, then she earns 0.
- Suppose B chooses $\mathcal{N}_B = \{s_1, s_2, s_3\} \subset \mathcal{C}$. Let $\{s_4\} = \mathcal{C} \backslash \mathcal{N}_B$. Notice that when B moves second, she has three consultants available if and only if A gets project x_{s_4}. Thus, B's optimal behavior is to employ all three of her consultants if she gets her project first or if A gets project x_{s_4}, otherwise it's best for B to employ no consultants. Thus, B's expected payoff from $\mathcal{N}_B$ and behaving optimally is

$$\frac{1}{2}5 + \frac{1}{2}5\frac{1}{40} - 3\frac{3}{4} = \frac{5}{16}.$$

- Suppose chooses $\mathcal{N}_B = \{1,2,3,4\}$. Notice that when B moves second, she has three consultants available if and only if A gets a project in $\{x_1, \ldots, x_4\}$. Thus, B's optimal behavior is to employ three consultants when she gets her project first or when A gets a project in $\{x_1, \ldots, x_4\}$, otherwise it's best for B to employ no consultants. Thus, B's expected payoff from $\mathcal{N}_B$ and behaving optimally is

$$\frac{1}{2}5 + \frac{1}{2}5\frac{4}{40} - 4\frac{3}{4} = -1/4.$$

It follows that it's best for B to choose a three-consultant network, so $\mathcal{N}^\star_B = \{2,3,4\}$ is optimal for B. Given this, it's readily verified that $\sigma^\star_{1B}$ and $\sigma^\star_{2B}$ implement B's optimal employment behavior.

The argument that $(\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star)$ is a best response for A to $(\mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star)$ is a bit simpler. Since the expected reward for completing x_5 is so high, it's best for A to always hold a network of all four consultants. Thus, when she gets her project first, it's optimal that she employ all four of them if she gets project x_5 and that she employ one of them if she gets a project in $\{x_1, \ldots, x_4\}$. When A gets her project second, B employs $\{2, 3, 4\}$, leaving consultant 1 for A. Thus, if A gets projects x_1, x_2, x_3, or x_4 it's optimal for her to employ consultant 1, otherwise it's best for A to employ no consultants. It's easily seen that $\sigma_{1A}^\star$ and $\sigma_{2A}^\star$ implement A's optimal employment behavior given B follows $(\mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star)$.

This equilibrium is *not* employment efficient because whenever A gets projects x_2, x_3, or x_4 first, she does not employ consultant $\{1\} = \mathcal{N}_A^\star \backslash \mathcal{N}_B^\star$ despite the fact $|\sigma_{1A}^\star| = 1$. In fact, by rotating the consultant she uses in the way she does, A doesn't even use an employment list. By employing consultant 2, 3, and 4 for project of unit difficulty, A prevents B from completing x_6 when B moves second and so harms B. However, if A employed consultant 1 when she gets project x_2, x_3, or x_4 first, then (i) A's expected payoff is unchanged (since she continues to complete her projects) and (ii) B would be able to complete x_6 more often and so would have a larger expected payoff. It's readily seen that this change induces an ELEE equilibrium in which A gets the same payoff and B gets a strictly higher payoff. $\triangle$

Before we give the second example, we give the following lemma on the nature of equilibria when there is only one consultant.

Lemma. Equilibria with One Consultant.
Let $\mathcal{C} = \{1\}$, then there are only ELEE equilibria, i.e., $\boldsymbol{E}_{AL} = \boldsymbol{E}$.

Proof. Let $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star, \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star) \in \boldsymbol{E}$. Since each partner can only network with and employ one consultant, $\sigma_{1A}^\star$ and $\sigma_{1B}^\star$ are trivially employment efficient and have employment lists. $\square$

More generally, one can construct a (trivial) example where there are only ELEE equilibria for any N. (Simply set $X = \{x_1\}$ with $d(x_1) = N$ and let the reward for completing x_1 be very high. Then, in every equilibrium, every partner will always hold the complete network, will carry out the project whenever she moves first, and will pass on the project when she moves second. Thus, all equilibria are ELEE equilibria.) The next non-trivial example is one where there are only ELEE equilibria.

Example S2. Only ELEE Equilibria.

Let $\mathcal{C} = \{1, 2\}$ be the set of consultants and let $X = \{x_1, x_2\}$ be the set of projects. Let $d(x_1) = 1$ and $d(x_2) = 2$. Let $P_i(x) = 1/2$ for $x \in X$ each partner i. Let each partner i have a constant marginal cost of networking of 1, let $r_i(x_1) = 11$ and $r_i(x_2) = 12$, and let $w = 1$.

Claim. *There are only ELEE equilibria in this example.*

Strategy	Project x_1	Project x_2
1	$\emptyset$	$\emptyset$
2	$\emptyset$	$\{1\}$
3	$\{1\}$	$\emptyset$
4	$\{1\}$	$\{1\}$

Table 8.1: Possible $\sigma_{1B}(\mathcal{N}_B, x)$

Strategy	Project x_1	Project x_2	Strategy	Project x_1	Project x_2
1	$\emptyset$	$\emptyset$	9	$\{2\}$	$\emptyset$
2	$\emptyset$	$\{1\}$	10	$\{2\}$	$\{1\}$
3	$\emptyset$	$\{2\}$	11	$\{2\}$	$\{2\}$
4	$\emptyset$	$\{1,2\}$	12	$\{2\}$	$\{1,2\}$
5	$\{1\}$	$\emptyset$	13	$\{1,2\}$	$\emptyset$
6	$\{1\}$	$\{1\}$	14	$\{1,2\}$	$\{1\}$
7	$\{1\}$	$\{2\}$	15	$\{1,2\}$	$\{2\}$
8	$\{1\}$	$\{1,2\}$	16	$\{1,2\}$	$\{1,2\}$

Table 8.2: Possible $\sigma_{1B}(\mathcal{N}_B, x)$

We first establish that both partners do best by holding the complete network and employing consultants optimally, i.e., in the manner of Remark A1. (Since Assumption 2 holds, Remark A1 describes the unique optimal second stage behavior.) We argue this for A as the argument for B is analogous. Let $(\mathcal{N}_B, \sigma_{1B}, \sigma_{2B})$ be a strategy for B. There are four cases: (i) $\mathcal{N}_B = \emptyset$, (ii) $\mathcal{N}_B = \{1\}$, (iii) $\mathcal{N}_B = \{2\}$, and (iv) $\mathcal{N}_B = \{1,2\}$. Suppose case (i). Since B employs no consultants, a quick computation establishes that it's best for A to hold the complete network (and employ her consultants optimally).

Suppose case (ii). Then B may employ consultant 1 in four possible ways when she moves first. These are given by Table 1. In each case, A's best response selects the complete network. We illustrate this for the case of strategy 4. If A holds the empty network, then she gets 0 (when she employs her consultants optimally). If she holds network $\{1\}$, a quick computation shows that she earns $^6/_4$. If she holds network $\{2\}$, then she earns 4. If she holds network $\{1,2\}$, then she earns $^{22}/_4$. Thus, A's best response selects the complete network. An analogous argument gives that A optimally the complete network if case (iii).

Suppose case (iv). Then B may employ the consultants in 16 possible ways. These are enumerated in Table 2. In each case, it's readily verified that A's best response always selects the complete network. We illustrate for the case of strategy 8. If A holds the empty network, she gets 0. If she holds network $\{1\}$, she gets $^{12}/_8$ (when she employs consultants optimally). If she holds network $\{2\}$, then she gets $^{22}/_8$. If she holds network $\{1,2\}$, then she gets $^{34}/_8$. Thus, A's best response selects the complete network.

Since both partners always do best by holding the complete network, when they move first, they must employ one consultant when they get project x_1 and both consultants when they get project x_2. That is, A and B's best response always specifies a behavioral strategy of the form

$$\sigma_{1i}(\{1,2\},x) = \begin{cases} \{1\} & \text{if } x = x_1 \\ \{1,2\} & \text{if } x = x_2 \end{cases} \text{ or } \sigma_{1i}(\{1,2\},x) = \begin{cases} \{2\} & \text{if } x = x_1 \\ \{1,2\} & \text{if } x = x_2, \end{cases}$$

for $i \in \{A, B\}$. Both of these strategies are trivially employment efficient and have employment lists. The claim follows. □ △

Our next example illustrates that ELEE equilibria do not Pareto dominate non-ELEE equilibria and vice versa.

Example S3. Non-ELEE Welfare.

Let $\mathcal{C} = \{1,2\}$ be the set of consultants and let $X = \{x_1, x_2, x_3\}$ be the set of projects. Let $d(x_1) = d(x_2) = 1$ and let $d(x_3) = 2$. Let $r_i(x) = 6$ for all $i \in \{A, B\}$ and $x \in X$. Let $w = 1$, and let A and B have constant marginal networking costs of $\kappa_A = 0$ and $\kappa_B = {}^{41}\!/_{18}$ respectively. Let all projects be equally likely, i.e., $P_i(x) = {}^{1}\!/_{3}$ for all $x \in X$ and $i \in \{A, B\}$. We begin by characterizing the set of ELEE equilibria. Subsequently, we'll give our non-ELEE equilibria where A does better than in any ELEE equilibrium.

ELEE Equilibria. Since $\kappa_A = 0$, it's best for A to always play a simple strategy of size 2. Thus, B's payoff to a simple strategy of size n_B is,

$$\begin{cases} 0 & \text{if } n_B = 0 \\ \frac{1}{2}(\frac{2}{3}\,5) + \frac{1}{2}(\frac{2}{3}\,\frac{2}{3}\,5) - \frac{41}{18} = \frac{1}{2} & \text{if } n_B = 1 \\ \frac{1}{2}(5) + \frac{1}{2}(\frac{2}{3}\,\frac{2}{3}\,5) - \frac{82}{18} = -\frac{17}{18} & \text{if } n_B = 2. \end{cases}$$

Thus, B holds a network of size 2. It follows that the unique simple equilibrium is $\tilde{\boldsymbol{s}}(2,1)$. Thus, B earns ${}^{1}\!/_{2}$ in every ELEE equilibrium and A earns $\frac{1}{2}(5) + \frac{1}{2}(\frac{2}{3}\,5 + \frac{1}{3}\,\frac{1}{3}\,5) = {}^{85}\!/_{18}$ in every ELEE equilibrium – this follows from Proposition 3 and the uniqueness of the simple equilibrium. (Even though there's a unique simple equilibrium, there are multiple ELEE equilibria because we may alway permute the consultants' identities.)

A Non-ELEE Equilibrium. Consider $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star, \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star)$ such that $\mathcal{N}_A^\star =$

$\{1,2\}$, $\mathcal{N}_B^\star = \emptyset$,

$$\sigma_{1A}^\star(\mathcal{N}, x) = \begin{cases} \{1\} & \text{if } \mathcal{N} = \{1,2\} \text{ and } x = x_1 \\ \{2\} & \text{if } \mathcal{N} = \{1,2\} \text{ and } x = x_2 \\ \{1,2\} & \text{if } \mathcal{N} = \{1,2\} \text{ and } x = x_3 \\ \emptyset & \text{else,} \end{cases}$$

and

$$\sigma_{2A}^\star(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{1\} & \text{if } \mathcal{N} = \{1,2\},\ \mathcal{N}' = \emptyset, \text{ and } x \in \{x_1, x_2\} \\ \{1,2\} & \text{if } \mathcal{N} = \{1,2\},\ \mathcal{N}' = \emptyset, \text{ and } x = x_3 \\ \emptyset & \text{else.} \end{cases}$$

We claim this is an equilibrium. It's readily verified that if B holds the empty network, then A does best by holding network $\{1,2\}$ and employing one consultant when she gets project x_1 and two consultants when she gets project x_2 when she moves in the second stage. Since $\boldsymbol{s}^\star$ recommends exactly this behavior to A, we've that A does best by following $\boldsymbol{s}^\star$ when B does. The payoff of A in $\boldsymbol{s}^\star$ is $5 = {}^{90}\!/_{18}$.

Let's focus on B. We'll prove that $(\mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star)$ is a best response for B to $(\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star)$ by considering the different network's that B may hold. If $\mathcal{N}_B = \{1\}$, then B earns $\frac{1}{2}(\frac{2}{3}\,5) + \frac{1}{2}(\frac{2}{3}\frac{1}{3}\,5) - \frac{41}{18} = -{}^{1}\!/_{18}$ when she behaves optimally in the second stage, i.e., if she completes projects x_1 and x_2 whenever her consultant is free and passes otherwise. If $\mathcal{N}_B = \{2\}$, then B also earns $\frac{1}{2}(\frac{2}{3}\,5) + \frac{1}{2}(\frac{2}{3}\frac{1}{3}\,5) - \frac{41}{18} = -{}^{1}\!/_{18}$ when she behaves optimally in the second stage, i.e., if she completes projects x_1 and x_2 whenever her consultant is free and passes otherwise. If $\mathcal{N}_B = \{1,2\}$, then B earns $\frac{1}{2}(5) + \frac{1}{2}(\frac{2}{3}\frac{2}{3}\,5) - \frac{82}{18} = -{}^{17}\!/_{18}$ when she behaves optimally in the second stage, i.e., if she completes projects x_1, x_2, and x_3 whenever she has enough available consultants and passes otherwise. Since B earns 0 in $\boldsymbol{s}^\star$, she also does best by behaving according $\boldsymbol{s}^\star$ when A does.

Clearly, A does better in $\boldsymbol{s}^\star$ then she does in any ELEE equilibrium and B does worse than in any ELEE equilibrium. The reason A is able to do better than B is because of the fact she chooses a different consultant to perform each difficulty one project, i.e., she "randomizes." It's this randomization and B's high networking cost that makes it unprofitable for B to network with either consultant. (Partner A is willing to randomize because she finds it best to acquire the complete network and so is indifferent to the identities of the consultants she uses on each difficulty one project.) $\triangle$

8.3 Counter Examples

In this section we give two counter examples to results in the main text. The first counter example shows that minimal overlap may fail absent Assumption 2. The second example shows that the conclusions of Proposition 7 are sensitive to the equilibrium selected.

Example S4. A Failure of Minimal Overlap.

Let $\mathcal{C} = \{1, \ldots, 10\}$, let $X = \{x_1, x_2\}$, and let $d(x_1) = 8$ and $d(x_2) = 9$. Let $P_A(x_1) = 1$ and $P_B(x_2) = 1$, so A always gets project x_1 and B always gets project x_2. Let $r_i(x) = 100$ for $i \in \{A, B\}$ and let each partner have a constant marginal cost of networking of $1/2$. Also, let $w = 1$.

An equilibrium is $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star, \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star)$, where $\mathcal{N}_A^\star = \{1, \ldots, 8\}$, $\mathcal{N}_B^\star = \{1, \ldots, 9\}$,

$$\sigma_{1A}^\star(\mathcal{N}, x) = \begin{cases} \{1, \ldots, 8\}\} & \text{if } \mathcal{N} = \mathcal{N}_A^\star \text{ and } x = x_1 \\ \emptyset & \text{else} \end{cases}$$

$$\sigma_{1B}^\star(\mathcal{N}, x) = \begin{cases} \{1, \ldots, 9\} & \text{if } \mathcal{N} = \mathcal{N}_B^\star \text{ and } x = x_2 \\ \emptyset & \text{else,} \end{cases}$$

$$\sigma_{2A}^\star(\mathcal{N}, \mathcal{N}', x) = \sigma_{2B}^\star(\mathcal{N}, \mathcal{N}', x) = \emptyset,$$

where $\mathcal{N}$ and $\mathcal{N}'$ are subsets of $\mathcal{C}$ and x is a project.

We'll verify that this is an equilibrium for A as the argument for B is analogous. Since A's reward is so high, it's always best for her to employ eight consultants. Thus, A's network consists of 8, 9, or 10 consultants. As in Example 1, we'll show that A's equilibrium strategy is a best reply by considering different networks for her.

- Suppose A has network $\{1, \ldots, 10\}$. When she moves first, she optimally employs 8 of these 10 consultants and completes her project. When she moves second, she optimally employs none of these consultants: B employs 9 consultants, leaving A with 1, so A is unable to complete her project. Thus, A's expected payoff is $\frac{1}{2}92 - 5 = 41$.

- Suppose A has a network of nine consultants. Specifically, she has network $\{j_1, \ldots, j_9\}$, where j_k denotes an arbitrary consultant. When A moves first, she optimally employs 8 of these consultants. When she moves second, she employs none: no matter what network she chooses, she's left with at most 1 consultant after B moves and, thus, cannot complete her project. Thus, A's expected payoff is $\frac{1}{2}92 - \frac{9}{2} = 41.5$.

- Suppose A has a network of eight consultants. Specifically, she has network $\{j_1, \ldots, j_8\}$.

When she moves first, she employs her entire network. When she moves second, she employs no-one as no matter what network she's chosen, she doesn't have enough consultants to complete her project. Thus, A's expected payoff is $\frac{1}{2}92 - 4 = 42$.

It follows that A does best by following her equilibrium strategy.

This equilibrium lacks the minimal overlap property: this property would require either A or B to replace a shared consultant with consultant 10. The reason minimal overlap fails is that there's insufficient heterogeneity in project difficulty (in the sense of Assumption 2). If there were sufficient heterogeneity in project difficulty, then A would get a difficulty one project with positive probability and so would strictly benefit from swapping, say, consultant 1 for consultant 10 (since this swap would allow her to earn a strictly positive reward when she moves second and gets a difficulty one project). △

Example S5. A Cost Advantage and an Inferior Outcome.

Let $\mathcal{C} = \{1, 2\}$ be the set of consultants and let $X = \{x_1, x_2\}$ be the set of projects. Let $d(x_1) = 1$ and $d(x_2) = 2$. Let

$$P_A(x) = P_B(x) = \begin{cases} \frac{2}{3} & \text{if } x = x_1 \\ \frac{1}{3} & \text{if } x = x_2. \end{cases}$$

Let the partners have constant marginal costs of networking of $\kappa_A = 3/4$ and let $\kappa_B = 5/6$, so A has a lower cost of including a consultant in her network than B. Let $r_A = r_B = r$, and let r and w be such that $r(x_1) - w = 1$ and $r(x_2) - 2w = 5$.

An equilibrium in the Auxiliary Game is $(0, 0)$, i.e., A holds the empty network and B holds a network linking her to both consultants. To see this, first consider B. Since A employs no agents in the proposed equilibrium, B earns

$$\begin{cases} 0 & \text{if } z_B = 2 \\ 1\frac{2}{3} - \frac{5}{6} = -\frac{1}{6} & \text{if } z_B = 1 \\ 1\frac{2}{3} + 5\frac{1}{3} - \frac{10}{6} = \frac{4}{6} & \text{if } z_B = 0. \end{cases}$$

Thus, it's a best response for B to choose an action of 0, i.e., a network of size 2. Now consider A. Since B employs two agents, A earns

$$\begin{cases} 0 & \text{if } z_A = 0 \\ 1\frac{1}{2}\frac{2}{3} + 1\frac{1}{2}\frac{2}{3}\frac{2}{3} - \frac{3}{4} = \frac{-7}{36} & \text{if } z_A = 1 \\ \frac{1}{2}(1\frac{2}{3} + 5\frac{1}{3}) + \frac{1}{2}(1\frac{2}{3}\frac{2}{3}) - \frac{6}{4} = -\frac{1}{9} & \text{if } z_A = 2. \end{cases}$$

Thus, it's a best response for A to choose a network of size zero.

Lemma 3 gives that $\tilde{\boldsymbol{s}}(0, 2-0)$ is an equilibrium. In this equilibrium, the two display equations show that, $U_A(\tilde{\boldsymbol{s}}(0,2)) = 0$ and $U_B(\tilde{\boldsymbol{s}}(0,2)) = 2/3$. Thus, A's cost advantage doesn't help her obtain a higher payoff than B. The reason for this counterintuitive outcome is that $(0,0)$ is the *minimal* equilibrium of the Auxiliary Game, not the maximal equilibrium that's best for A. Since $(0,0)$ is the minimal equilibrium, Proposition 7 has no bite, implying A need not do better than B. This illustrates that the conclusion of Proposition 7 are sensitive to the equilibrium selected. △

8.4 Additional Extensions

In this section, we develop three additional extensions of our model.[78] The first extension allows for multiple partners. While this extension kills our welfare and existence arguments, we find that our minimal overlap result remains true. The second extension considers a production technology with residual value. We establish that an equilibrium exists, that our welfare results about *simple equilibria* remain true, and that our results about minimal overlap hold in this extension. The third extension examines heterogeneous labor costs. We argue that our existing results hold in this extension and we develop comparative statics for welfare as labor costs change.

8.4.1 Multiple Partners

In this subsection, we extend the model by allowing for multiple partners. While this extension kills our existence and welfare arguments, we show that our covering lemma continues to hold, implying that the partners' networks are minimally overlapping.

Environment. We suppose there are $k > 2$ partners labeled A, B, ..., K.[79] The first stage is exactly the same as in the main text, the partners select subsets of $\mathcal{C}$. In the second stage, each partner sequentially gets a project. As in the main text, we assume that all partners have an equal chance of getting their project first, second, ..., or last. The first partner to get a project may employ consultants from her network with impunity. The second partner may employ any consultants from her network who were not employed by the first. And, in general, the i-th partner to get a project may employ consultants in her network who weren't employed by any previous partners. The payoffs and production technology are as in the main text.

[78] A fourth extension that allows for fractional usage is available upon request.

[79] To clarify, we allow for more than 11 partners. In fact, we allow there to be an arbitrary, though finite number of partners between A and K.

Existence and Welfare. Unfortunately, the arguments behind our existence result (Proposition 1) and our welfare results (Proposition 4 to 7) fail when there are multiple partners. The problem is that any analogue of the Auxiliary Game is inherently submodular due to rivalry. Since there are more than two players, we cannot exploit the order inversion trick we used in the main text to make the Auxiliary Game a supermodular game. Thus, we cannot use Zhou's Fixed Point Theorem to show that the equilibrium set of the Auxiliary Game is a non-empty lattice. The possible emptiness of this set kills our equilibrium existence argument and the lack of lattice structure kills our welfare arguments. That said, an equilibrium may still exist, as the next examples illustrate. However, a general proof of existence and general proofs of the welfare results will require the development of new methods that are beyond the scope of this paper.

Example S6. An Equilibrium.

Suppose there are three partners A, B, and C. Let $X = \{x_1, x_2\}$ be the set of projects, with $d(x_1) = 1$ and $d(x_2) = 5$. Let $r_i(x) = 10$ for all $x \in X$, $w = 1/5$, and let all partners have a constant marginal cost of networking of $\kappa_i = 1/5$. Also, let $P_i(x) = 1/2$ for all $x \in X$.

Claim. *An equilibrium exists when* $N \geq 3 \max_{x \in X} d(x) = 15$.

Proof. Let $\mathcal{C}_A = \{1, \ldots, 5\}$, $\mathcal{C}_B = \{6, \ldots, 10\}$, and $\mathcal{C}_C = \{11, \ldots, 15\}$. We'll sketch the construction of an equilibrium by restricting A, B, and C to select consultants from $\mathcal{C}_A$, $\mathcal{C}_B$, and $\mathcal{C}_C$ respectively. Since it's costly to network with consultants, and no more than 5 are ever needed, the partners are happy to select networks in these sets. Thus, in the second stage, they never employ from each others' networks and so always complete their projects.

Consider A. Since $d(X) = \{1, 5\}$, she either networks with 1 or 5 consultants. If she networks with 1 and behaves optimally in the second stage (i.e., completes project x_1 and passes on project x_2), then she earns $5 - 3/10$. However, if she networks with 5 consultants and behaves optimally in the second stage (i.e., completes both projects x_1 and x_2 with exactly $d(x)$ consultants), then she earns 8. Thus, A does best by holding network $\mathcal{C}_A$. Likewise, B and C do best by holding networks $\mathcal{C}_B$ and $\mathcal{C}_C$.

It follows that it's an equilibrium for A to hold network $\mathcal{C}_A$, B to hold network $\mathcal{C}_B$, C to hold network $\mathcal{C}_C$, and for all three to behave optimally in the second stage. □ △

It's straight forward to generalize the logic used above to prove the following result.

Proposition S2. Existence with Multiple Partners.
Suppose there are $k > 2$ partners, then an equilibrium exists whenever $N \geq k \max_{x \in X} d(x)$.

The next example shows that equilibria may exist in non-trivial environments.

Example S7. Another Equilibrium.

Suppose there are three partners A, B, and C. Let $X = \{x_1, x_2\}$ be the set of projects, with $d(x_1) = 1$ and $d(x_2) = 2$. Let $\mathcal{C} = \{1, 2\}$. Let $r_i(x) = 10$ for all $x \in X$, $w = {}^1\!/\!_2$. Let

$$P_A(x) = P_B(x) = \begin{cases} 1 & \text{if } x = x_1 \\ 0 & \text{if } x = x_1 \end{cases} \text{ and } P_C(x) = \begin{cases} 0 & \text{if } x = x_1 \\ 1 & \text{if } x = x_1. \end{cases}$$

Also, let

$$c_A(n) = c_B(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 500 & \text{if } n = 2 \end{cases} \text{ and } c_C(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2. \end{cases}$$

We begin by pinning down the sizes of the networks A, B, and C hold in any equilibrium. Consider A. Since A faces a very high cost of holding a size two network, she only holds networks of size 0 or 1 in equilibrium. Suppose she holds a network of size 1, then she earns at least ${}^1\!/\!_3(10-{}^1\!/\!_2)-1 = 3.75$ in expectation because she always completes her project when she moves first in an equilibrium. Since she earns nothing from networking with 0 consultants, she always network with 1 consultant in an equilibrium. An analogous argument establishes that B always networks with 1 consultant in an equilibrium.

Now consider C. Since C always gets a difficulty 2 project and networking with a single consultant is costly, she holds a network of size 0 or 2 in equilibrium. Suppose C holds a network of size 2, then she earns at least ${}^1\!/\!_3(10-1)-1 = 2.75$ in expectation because she always completes her project when she moves first in an equilibrium. Since she earns nothing from networking with 0 consultants, she always network with 2 consultant in an equilibrium.

Thus, there are four possible equilibrium networks for A, B, and C:

1. $(N_A^\star, N_B^\star, N_C^\star) = (\{1\}, \{1\}, \{1, 2\})$
2. $(N_A^\star, N_B^\star, N_C^\star) = (\{1\}, \{2\}, \{1, 2\})$
3. $(N_A^\star, N_B^\star, N_C^\star) = (\{2\}, \{1\}, \{1, 2\})$
4. $(N_A^\star, N_B^\star, N_C^\star) = (\{2\}, \{2\}, \{1, 2\})$.

It's easily seen that (1) isn't part an equilibrium. If (1) and A behaves optimally in the second stage (i.e., completes project x_1 whenever her consultant is free and passes otherwise), then she earns ${}^1\!/\!_3(10-{}^1\!/\!_2)-1$. While if she were to switch to network $\{2\}$, then she'd earn ${}^1\!/\!_3(10-{}^1\!/\!_2)+{}^1\!/\!_3{}^1\!/\!_2(10-{}^1\!/\!_2)-1$. Thus, A "defects" from (1). Symmetry gives that A also defects from (4).

It follows that only (2) and (3) are possibly played in equilibrium. Consider (2). If A behaves optimally in the second stage, she earns $^1/_3(10 - ^1/_2) + ^1/_3{}^1/_2(10 - ^1/_2) - 1$, while if she defects to $\{1\}$, then she only earns $^1/_3(10 - ^1/_2) - 1$. Thus, A keeps her equilibrium network. Likewise, B has no incentive to defect, and C has no incentive to defect. It follows that it's an equilibrium for A, B, and C to hold the networks given by (2) and behave optimally in the second stage. (Optimal behavior for C is to complete x_2 if both of her consultants are available and pass otherwise.) By symmetry, (3) also constitutes an equilibrium collection of networks. $\triangle$

Covering. Now we turn to our positive result: the covering lemma remains true when there are multiple partners.

Proposition S3. Covering with Multiple Partners.
Suppose there are $k > 2$ partners and Assumption 2 holds. Let $\mathcal{N}_A^\star$, ..., $\mathcal{N}_K^\star$ be the partners' networks in an equilibrium, then $\mathcal{N}_i^\star \cap \mathcal{N}_j^\star \neq \emptyset$ for some partners i and j implies that $\mathcal{C} \subset \mathcal{N}_A^\star \cup \cdots \cup \mathcal{N}_K^\star$.

Proof. We only sketch the argument as it's analogous to the Proof of Lemma 4. We argue by contradiction. Suppose that $\mathcal{N}_i^\star \cap \mathcal{N}_j^\star \neq \emptyset$ and $\mathcal{C} \not\subset \mathcal{N}_A^\star \cup \cdots \cup \mathcal{N}_K^\star$, then there's a consultant c who is not in any partner's network. Consider i. When j moves first and i moves second, then i has access to $|\mathcal{N}_i \backslash \mathcal{N}_j|$ consultants when j gets a project of difficulty $|\mathcal{N}_j|$. Now, if i were to swap a consultant in $\mathcal{N}_i^\star \cap \mathcal{N}_j^\star$ for c, then she'd have $|\mathcal{N}_i \backslash \mathcal{N}_j| + 1$ consultants to work with when j goes first and gets a project of difficulty $|\mathcal{N}_j|$. Thus, she has a strictly higher chance of completing the project she gets, implying her expected earnings go up (since Assumption 2 ensures all projects generate strictly positive payoffs and occur with strictly positive probability). This implies that i would defect by choosing a different network, a contradiction of the supposition of equilibrium. $\square$

This implies that in any equilibrium the partners' share as few consultants "as possible." That is, if i and j's networks overlap, then there's no to way to reduce the number of consultants they share while (i) keeping the size of all partner's networks constant and (ii) not growing the "overlap" of the other partner's networks. Let's formalize this.

Definition. Let $\mathcal{N}_A$, $\mathcal{N}_B$, ..., $\mathcal{N}_K$ be networks for the partners. We say that $\mathcal{N}_A$, ..., $\mathcal{N}_K$ are *minimally overlapping* if there does not exists a family of $\{\mathcal{N}_i'\}$ with $\mathcal{N}_i \subset \mathcal{C}$ such that (i) $|\mathcal{N}_i| = |\mathcal{N}_i'|$ for each partner i and (ii) $|\mathcal{N}_i' \backslash \cup_{j \in I} \mathcal{N}_j'| > |\mathcal{N}_i \backslash \cup_{j \in I} \mathcal{N}_j|$ for each partner i and every collection of partners I.

Thus, if $\{\mathcal{N}_i\}$ is minimally overlapping, there's no way to increase the number of consultants each partner has exclusive access to (part (ii)) while maintaining the sizes of their networks (part (i)). (This definition generalizes the definition of minimal overlap given in the main text because $|\mathcal{N}'_A \cap \mathcal{N}'_B| < |\mathcal{N}_A \cap \mathcal{N}_B|$ if and only if $|\mathcal{N}'_A \backslash \mathcal{N}'_B| > |\mathcal{N}_A \backslash \mathcal{N}_B|$ and $|\mathcal{N}'_B \backslash \mathcal{N}'_A| > |\mathcal{N}_B \backslash \mathcal{N}_A|$.)

Proposition S4. Minimal Overlap with Multiple Partners.
Suppose there are $k > 2$ partners and Assumption 2 holds. Then every equilibrium exhibits minimal overlap.

Proof. We argue by contradiction. Suppose the partners have networks $\{\mathcal{N}_i^\star\}$ in an equilibrium and $\{\mathcal{N}_i^\star\}$ isn't minimally overlapping, then there's a collection of networks $\{\mathcal{N}'_i\}$ where (i) and (ii) of the definition of minimal overlap hold. Without loss, assume $\mathcal{N}_i^\star \cap \mathcal{N}_j^\star \neq \emptyset$ for two partners i and j, then Proposition S3 gives $\mathcal{C} \subset \cup \mathcal{N}_i^\star$, implying

$$N = |\mathcal{N}_A^\star| + |\mathcal{N}_B^\star \backslash \mathcal{N}_A^\star| + |\mathcal{N}_C^\star \backslash \mathcal{N}_A^\star \cup \mathcal{N}_B^\star| + \cdots + |\mathcal{N}_K^\star \backslash \cup_{j=A}^J \mathcal{N}_j^\star|.$$

By (i) and (ii) we have

$$N < |\mathcal{N}'_A| + |\mathcal{N}'_B \backslash \mathcal{N}'_A| + \cdots + |\mathcal{N}'_K \backslash \cup_{j=A}^J \mathcal{N}'_j|.$$

But, each set on the right-hand-side is a disjoint subset of $\mathcal{C}$, implying

$$|\mathcal{N}'_A| + |\mathcal{N}'_B \backslash \mathcal{N}'_A| + \cdots + |\mathcal{N}'_K \backslash \cup_{j=A}^J \mathcal{N}'_j| \leq N.$$

Thus, $N < N$, a contradiction. □

8.4.2 Residual Value

In this subsection, we extend the model by considering a production technology with residual value. We show that an equilibrium exists and that all of our results statics *for simple equilibria* remain true. That said, this technology (i) doesn't nest the technology used in the main text and (ii) is unusual in that clients typically don't pay for half-finished work.[80]

Technology. For each partner i, let $\pi_i : X \times \mathbb{N} \to \mathbb{R}$ such that $\pi_i(x, 0) = 0$ for each $x \in X$. For each project x, we assume that $\pi_i(x, \cdot)$ is strictly increasing in l up to some finite, positive integer and is decreasing thereafter. For simplicity, we assume that both $\pi_A(x, l)$ and $\pi_B(x, l)$ are increasing up to the same integer. For each x, let $d(x)$ give the

[80] There are, of course, exceptions. For instance, the US government will frequently pay defense contractors to adapt/develop new technologies for new weapons systems. If the system proves ineffective and is cancelled, the contractor is still paid for their time.

integer after which $\pi_i(x, \cdot)$ starts to decrease, i.e.,

$$d(x) = \min_{\leq}\{l \in \mathbb{N}_+ | \pi_i(x, l) \leq \pi_i(x, l+1)\}.$$

Notice that $\pi_i(x, d(x)) \geq 0$ for each project x and each partner i, i.e., completing every project by using $d(x)$ consultants is a good. (This obviates the need for us to make an assumption like Assumption 1.)

The function $\pi_i(x, l)$ gives partner i's second-stage payoff when she gets project x and employs l consultants. From this payoff, i pays her networking costs. Thus, when i moves first in the second stage and has network $\mathcal{N}_i$, project x, and employs $\mathcal{S} \subset \mathcal{N}_i$, her ex-post payoff is

$$\pi_i(x, |\mathcal{S}|) - c_i(|\mathcal{N}_i|). \tag{8.1}$$

Analogously, when i moves second in the second stage, the other partner employs $\mathcal{T} \subset \mathcal{C}$, and i has network $\mathcal{N}_i$, project x, and employs $\mathcal{S} \subset \mathcal{N}_i \backslash \mathcal{T}$, her ex-post payoff is

$$\pi_i(x, |\mathcal{S}|) - c_i(|\mathcal{N}_i|). \tag{8.2}$$

While this technology allows a partner to capture some residual value from employing less than the ideal number of consultants, it doesn't nest the technology used in the main paper. The reason is that, in the main paper, we don't require that a partner's second-stage payoff be increasing in the number of consultants she employs up to the ideal point. For instance, in Example 1, if a partner gets project x_2, then her payoff before networking costs is zero if she employs no consultants, strictly negative if she employs one consultant, and strictly positive if she employs two consultants.

Optimal Behavior. In the second stage, partner i's optimal behavior depends on whether she moves first or second. When i moves first, and sees her network is $\mathcal{N}$ and her project is x, then it's best for her to employ $d(x)$ consultants if $d(x) \leq |\mathcal{N}|$ and to employ $|\mathcal{N}|$ consultants when $d(x) > |\mathcal{N}|$. Likewise, when i moves second, sees her network is $\mathcal{N}$, her project is x, and the other partner employed $\mathcal{T}$, then it's best for her to employ $d(x)$ consultants if $d(x) \leq |\mathcal{N} \backslash \mathcal{T}|$ and employ $|\mathcal{N} \backslash \mathcal{T}|$ consultants otherwise. This behavior differs from that described in Remark A1 in that, when i has less than $d(x)$ in-network consultants available, she employs all of these consultants instead of none of them.

Existence. We make the following assumption to ensure existence.

Assumption S1. A Property of the Production Technology.

For each $i \in \{A, B\}$ and $x \in X$,

$$\pi_i(x, a) + \pi_i(x, d) \geq \pi_i(x, b) + \pi_i(x, c),$$

where a, b, c, and d are points in $\{0, \ldots, d(x)\}$ such that $a \geq \max\{b, c, d\}$, $d \leq \min\{b, c\}$, and $a + d \geq c + b$.

The assumption holds, for instance, if $\pi_i(x, l)$ is sufficiently convex in l on $\{0, \ldots, d(x)\}$ or is "affine" in l, i.e., has form $\pi_i(x, l) = \alpha_x + \beta_x \, l$ on $\{0, \ldots, d(x)\}$, where $\beta_x > 0$.[81] This assumption ensures that, come the analogue of the Auxiliary Game, a type of supermodularity is preserved by the π_i functions.

To argue existence, consider the following variant of simple strategies. A **variant simple strategy for $\boldsymbol{A}$** of size $n \in \{0, \ldots, N\}$ is a tuple $\tilde{\boldsymbol{s}}_A^v(n) = (\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1A}, \tilde{\sigma}_{2A})$ such that, for every $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$,

$$\begin{aligned}
\tilde{\mathcal{N}}_A^v &= \{1, \ldots, n\} \\
\tilde{\sigma}_{1A}^v(\mathcal{N}, x) &= \begin{cases} \{1, \ldots, d(x)\} & \text{if } d(x) \leq |\mathcal{N}| \text{ and } \mathcal{N} = \tilde{\mathcal{N}}_A^v \\ \mathcal{N} & \text{else} \end{cases} \\
\tilde{\sigma}_{2A}^v(\mathcal{N}, \mathcal{N}', x) &= \begin{cases} \{1, \ldots, d(x)\} & \begin{array}{l} \text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|, \, \mathcal{N}' \subset \{d(x)+1, \ldots, N\}, \\ \qquad \text{and } \mathcal{N} = \tilde{\mathcal{N}}_A^v \end{array} \\ \mathcal{N} \backslash \mathcal{N}' & \text{else.} \end{cases}
\end{aligned}$$

Likewise, a **variant simple strategy for $\boldsymbol{B}$** of size $n \in \{0, \ldots, N\}$ is a tuple $\tilde{\boldsymbol{s}}_B^v(n) =$

[81] For instance, we might have

$$\pi_i(x, l) = \begin{cases} \alpha_x + \beta_x \, l & \text{if } l \leq \gamma(x) \\ \alpha_x + \beta_x \gamma(x) - k(l - d(x)) & \text{if } l > \gamma(x), \end{cases}$$

where $k(l - d(x))$ reflects an inefficiency from having too many consultants in the same place at the same time, and $k \geq 0$ and $\gamma : X \to \mathbb{N}_+$ is an exogenous function that gives the number of consultants needed to reach the idea point.

$(\tilde{\mathcal{N}}_B, \tilde{\sigma}_{1B}, \tilde{\sigma}_{2B})$ such that, for every $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$,

$$\tilde{\mathcal{N}}_B^v = \{N+1-n, \ldots, N\}$$

$$\tilde{\sigma}_{1B}^v(\mathcal{N}, x) = \begin{cases} \{N+1-d(x), \ldots, N\} & \text{if } d(x) \leq |\mathcal{N}| \text{ and } \mathcal{N} = \tilde{\mathcal{N}}_B^v \\ \mathcal{N} & \text{else} \end{cases}$$

$$\tilde{\sigma}_{2B}^v(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{N+1-d(x), \ldots, N\} & \begin{array}{c}\text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|, \mathcal{N}' \subset \{1, \ldots, N-d(x)\}, \\ \text{and } \mathcal{N} = \tilde{\mathcal{N}}_B^v\end{array} \\ \mathcal{N} \backslash \mathcal{N}' & \text{else.} \end{cases}$$

In these strategies, A and B employ $d(x)$ consultants when at least that many in-network consultants are available and employ every available, in-network consultant otherwise. Thus, these strategies trivially implement A's and B's optimal second stage behavior.

The next proposition is our main result.

Proposition S5. Existence of an Equilibrium.
Let Assumption S1 hold, then there is a $(n_A^\star, n_B^\star) \in \{0, \ldots, N\}^2$ such that $(\tilde{\boldsymbol{s}}_A^v(n_A^\star), \tilde{\boldsymbol{s}}_B^v(n_B^\star))$ is an equilibrium when the partners' ex-post payoffs are given by equations (8.1) and (8.2).

Sketch of Proof of Proposition S5. The argument mirrors the Proof of Proposition 1. First, we establish that the analog of the Auxiliary Game under the variant simple strategies is a two-player supermodular game. Second, we use an equilibrium of this game to induce an equilibrium of the full game. Since the methods are standard, we'll only sketch the argument.

First, notice that if A and B play variant simple strategies of sizes (n_A, n_B), then A earns

$$U_A((\tilde{\boldsymbol{s}}_A^v(n_A), \tilde{\boldsymbol{s}}_B^v(n_B))) = \frac{1}{2}\left(\sum_{\{x | d(x) \leq n_A\}} \pi_A(x, d(x)) P_A(x) + \sum_{\{x | d(x) > n_A\}} \pi_A(x, n_A) P_A(x) \right)$$
$$+ \frac{1}{2} \sum_{(x_A, x_B) \in X^2} \pi_A(x_A, \alpha(x_A, x_B, n_A, n_B)) P_A(x_A) P_B(x_B) - c_A(n_A), \tag{8.3}$$

where

$$\alpha(x_A, x_B, n_A, n_B) = \min\{d(x_A), n_A, N - d(x_B), N - n_B\}.$$

The calculation of the first term is straightforward. To see where the second term comes from, suppose A gets project x_A and B gets project x_B. Then B employs $\{\max\{N+1-n_B, N+1-d(x_B), \ldots, N\}$, so A's set of available in-network consultants is $\{1, \ldots, \min\{n_A, N-n_B, N-d(x_B)\}\}$. Thus, A employs $d(x)$ consultants if $d(x) \leq \min\{n_A, N-n_B, N-d(x_B)\}$

and $\min\{n_A, N - n_B, N - d(x_B)\}$ else. That is, A employs $\alpha(x_A, x_B, n_A, n_B)$ consultants. Partner B's payoff is analogous.

Second, we need to establish that the partners' payoffs to variant simple strategies are submodular. Without loss, we do this for A by establishing that $\pi_A(x, \alpha(x_A, x_B, n_A, n_B))$ is submodular in (n_A, n_B). Let $n'_A \geq n_A$ and $n'_B \geq n_B$. Then, $N - n_B \geq N - n'_B$, so the supermodularity of the $\min\{\cdot\}$ function (see Topkis [68] Example 2.6.2) gives

$$\alpha(x_A, x_B, n'_A, n_B) + \alpha(x_A, x_B, n_A, n'_B) \geq \alpha(x_A, x_B, n'_A, n'_B) + \alpha(x_A, x_B, n_A, n_B).$$

Let $a = \alpha(x_A, x_B, n'_A, n_B)$, $b = \alpha(x_A, x_B, n'_A, n'_B)$, $c = \alpha(x_A, x_B, n_A, n_B)$, and $d = \alpha(x_A, x_B, n_A, n'_B)$. Notice that a, b, c, and d are all less than or equal to $d(x)$. In addition, $a + d \geq b + c$ from the above display equation, as well as $a \geq b, c, d$ and $d \leq b, c$. Thus, Assumption S1 gives $\pi_A(x, a) + \pi_A(x, d) \geq \pi_A(x, b) + \pi_A(x, c)$, i.e., $\pi_A(x, \alpha(x_A, x_B, n_A, n_B))$ is submodular in (n_A, n_B). It follows that

$$\frac{1}{2} \sum_{(x_A, x_B) \in X^2} \pi_i(x_A, \alpha(x_A, x_B, n_A, n_B)) P_A(x_A) P_B(x_B)$$

is submodular in (n_A, n_B). Since the other two terms of (8.3) are trivially submodular in (n_A, n_B), the desired result follows. It follows that there's an equilibrium in the analogue of the Auxiliary Game.

Third, we need to show that such an equilibrium induces an equilibrium of the game as a whole. The key to doing this is establishing that a variant simple strategy is a best reply to a variant simple strategy. To do this, we introduce variant hat strategies. Let $\mathcal{N}_A$ be a network for A. If $\mathcal{N}_A$ is empty, let

$$\breve{\sigma}_{1A}^{\mathcal{N}_A}(\mathcal{N}, x) = \breve{\sigma}_{2A}^{\mathcal{N}_A}(\mathcal{N}, \mathcal{N}', x) = \emptyset \text{ for all } (\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X.$$

If $\mathcal{N}_A$ is non-empty, let $\{m_1, \ldots, m_n\}$ be the standard form of $\mathcal{N}_A$ for A, where $n = |\mathcal{N}_A|$.

For each $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$, let

$$\breve{\sigma}_{1A}^{\mathcal{N}_A}(\mathcal{N}, x) = \begin{cases} \{m_1, m_2 \dots, m_{d(x)}\} & \text{if } d(x) \leq |\mathcal{N}| \text{ and } \mathcal{N} = \mathcal{N}_A \\ \mathcal{N} & \text{if } d(x) > |\mathcal{N}| \text{ and } \mathcal{N} = \mathcal{N}_A \\ \emptyset & \text{else,} \end{cases}$$

$$\breve{\sigma}_{2A}^{\mathcal{N}_A}(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{m_1, m_2, \dots, m_{d(x)}\} & \text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|, \mathcal{N}' \subset \{m_{d(x)} + 1, \dots, N\}, \\ & \text{and } \mathcal{N} = \mathcal{N}_A \\ \mathcal{N} \backslash \mathcal{N}' & \text{if } d(x) > |\mathcal{N} \backslash \mathcal{N}'|, \mathcal{N}' \subset \{m_{d(x)} + 1, \dots, N\}, \\ & \text{and } \mathcal{N} = \mathcal{N}_A \\ \emptyset & \text{else.} \end{cases}$$

We refer to $\breve{\sigma}_{1A}^{\mathcal{N}_A}$ and $\breve{\sigma}_{2A}^{\mathcal{N}_A}$ as **A's variant hat strategies given $\mathcal{N}_A$**. Likewise, we define variant hat strategies for B. Let $\mathcal{N}_B$ be a network for B. If $\mathcal{N}_B$ is empty, let

$$\breve{\sigma}_{1B}^{\mathcal{N}_B}(\mathcal{N}, x) = \breve{\sigma}_{2B}^{\mathcal{N}_B}(\mathcal{N}, \mathcal{N}', x) = \emptyset \text{ for all } (\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X.$$

If $\mathcal{N}_B$ is non-empty, let $\{m_1, \dots, m_n\}$ be the standard form of $\mathcal{N}_B$ for B, where $n = |\mathcal{N}_B|$. For each $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$, let

$$\breve{\sigma}_{1B}^{\mathcal{N}_B}(\mathcal{N}, x) = \begin{cases} \{m_1, m_2 \dots, m_{d(x)}\} & \text{if } d(x) \leq |\mathcal{N}| \text{ and } \mathcal{N} = \mathcal{N}_B \\ \mathcal{N} & \text{if } d(x) > |\mathcal{N}| \text{ and } \mathcal{N} = \mathcal{N}_B \\ \emptyset & \text{else,} \end{cases}$$

$$\breve{\sigma}_{2B}^{\mathcal{N}_B}(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{m_1, m_2, \dots, m_{d(x)}\} & \text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|, \mathcal{N}' \subset \{1, \dots, m_{d(x)} - 1\}, \\ & \text{and } \mathcal{N} = \mathcal{N}_B \\ \mathcal{N} \backslash \mathcal{N}' & \text{if } d(x) > |\mathcal{N} \backslash \mathcal{N}'|, \mathcal{N}' \subset \{1, \dots, m_{d(x)} - 1\}, \\ & \text{and } \mathcal{N} = \mathcal{N}_B \\ \emptyset & \text{else.} \end{cases}$$

We refer to $\breve{\sigma}_{1B}^{\mathcal{N}_B}$ and $\breve{\sigma}_{2B}^{\mathcal{N}_B}$ as **B's variant hat strategies given $\mathcal{N}_B$**. Since A and B's variant hat strategies implement their optimal second behavior when (i) their own network is held constant and (ii) the other partner plays a simple strategy, they weakly payoff any other behavioral strategies. Consequently, we only need to show that A and B prefer playing their variant simple strategies to playing their variant hat strategies when their rival plays a variant simple strategy.

To these ends, suppose that A has strategy $(\mathcal{N}_A, \breve{\sigma}_{1A}^{\mathcal{N}_A}, \breve{\sigma}_{2A}^{\mathcal{N}_A})$ and B follows $\tilde{\boldsymbol{s}}_B^v(n)$. Let $\{m_1, \ldots, m_{|\mathcal{N}_A|}\}$ be the standard form of A's network. Then A earns

$$U_A(((\mathcal{N}_A, \breve{\sigma}_{1A}^{\mathcal{N}_A}, \breve{\sigma}_{2A}^{\mathcal{N}_A}), \tilde{\boldsymbol{s}}_B^v(n_B))) = \frac{1}{2}\left(\sum_{\{x|d(x)\le|\mathcal{N}_A|\}} \pi_A(x, d(x))P_A(x) + \sum_{\{x|d(x)>|\mathcal{N}_A|\}} \pi_A(x, n_A)P_A(x)\right) + \frac{1}{2}\sum_{(x_A,x_B)\in X^2} \pi_A(x_A, \beta(x_A, x_B, n_A, n_B))P_A(x_A)P_B(x_B) - c_A(n_A),$$

where

$$\beta(x_A, x_B, n_A, n_B) = \max_{m_l\in\{m_1,\ldots,m_{|\mathcal{N}_A|}\}} |\{m_1, \ldots, m_l\}| \text{ such that}$$
$$\text{(i) } |\{m_1, \ldots, m_l\}| \le d(x) \text{ and (ii) } m_l \le \max\{N - d(x_B), N - n_B\}.$$

Notice that the constraint $m_l \in \{m_1, \ldots, m_{|\mathcal{N}_A|}\}$ implies $\beta(\cdot) \le |\mathcal{N}_A|$. (To see this representation, notice that when B moves first, she either employs consultants $\{N+1-d(x_B), \ldots, N\}$ if $d(x_B) \le n_B$ or $\{N+1-n_B, \ldots, N\}$ if $d(x_B) > n_B$. Thus, consultants $\mathcal{F} = \{1, \ldots, \max\{N-n_B, N-d(x_B)\}\}$ are unused by B and possibly available to A. Now, A's variant hat strategy tells her to (i) employ the $d(x_A)$ lowest indexed consultants who are in $\mathcal{N}_A$ and in $\{1, \ldots, \max\{N-n_B, N-d(x_B)\}\}$ when $d(x_A)$ is smaller than $|\mathcal{N}_A \cap \mathcal{F}|$ or (ii) employ everyone in $|\mathcal{N}_A \cap \mathcal{F}|$ when $d(x_A) > \mathcal{N}_A \cap \mathcal{F}$. If (i), then A employs consultants $\{m_1, \ldots, m_l\}$ where $l = d(x_A)$, and we trivially have $m_l \in \mathcal{F} \iff m_l \le \max\{N-n_B, N-d(x_B)\}$. If (ii), then A employs consultants $\{m_1, \ldots, m_l\}$ where $l = \max\{N-n_B, N-d(x_B)\}$.)

We now establish that A (or B) does best by playing their simple network and variant hat strategies when B (A) plays a simple strategy. That is, we need to establish that

$$U_A((\boldsymbol{s}_A^v(|\mathcal{N}_A|), \boldsymbol{s}_B^v(n_B))) - U_A(((\mathcal{N}_A, \breve{\sigma}_{1A}^{\mathcal{N}_A}, \breve{\sigma}_{2A}^{\mathcal{N}_A}), \boldsymbol{s}_B^v(n_B))) \ge 0.$$

To these ends, write

$$U_A((\tilde{\boldsymbol{s}}_A^v(|\mathcal{N}_A|), \tilde{\boldsymbol{s}}_B^v(n_B))) - U_A(((\mathcal{N}_A, \breve{\sigma}_{1A}^{\mathcal{N}_A}, \breve{\sigma}_{2A}^{\mathcal{N}_A}), \tilde{\boldsymbol{s}}_B^v(n_B))) = \frac{1}{2}\sum_{(x_A,x_B)\in X^2} (\pi_A(x, \alpha(x_A, x_B, n_A, n_B)) - \pi_A(x, \beta(x_A, x_B, n_A, n_B)))P_A(x_A)P_B(x_B).$$

It suffices to establish that $\pi_A(x, \alpha(x_A, x_B, n_A, n_B)) - \pi_A(x, \alpha(x_A, x_B, n_A, n_B))$. Since $\alpha(\cdot) \le d(x_A)$ and $\beta(\cdot) \le d(x_A)$, it is enough to establish that $\alpha(\cdot) \ge \beta(\cdot)$ as $\pi_A(x_A, l)$ is strictly increasing on $\{0, \ldots, d(x_A)\}$. Let's do this via a contradiction argument. Suppose that

$\beta(\cdot) > \alpha(\cdot)$. Since $\beta(\cdot) \leq d(x)$, we have $\alpha < d(x)$. Thus, $\alpha = |\mathcal{N}_A|$ by definition of $\alpha(\cdot)$. It follows that $\beta(\cdot) > |\mathcal{N}_A|$, i.e., that A uses more consultants than she has in her network. This violates (ii) of the definition of $\beta(\cdot)$ and gives the needed contradiction.

It follows that A does best by playing a variant simple strategy when B does best by playing a variant simple strategy. As a result, an argument analogous to the Proof of Lemma 3 gives that the variant simple strategies implement an equilibrium when evaluated at an equilibrium of the supermodular analogue of the Auxiliary Game. □

Minimal Overlap and Welfare. We close this subsection by noting that many of the results of the main text are robust to this change in the production technology, provided we appropriately modify our assumptions.[82] For instance, every equilibrium exhibits minimal overlap because partners still have an incentive to swap a shared consultant for a free consultant. Likewise, all the welfare properties of *simple* equilibria remain true because the partners payoffs to variant simple strategies are weakly decreasing in the size of each other's networks – e.g., equation (8.3) is easily seen to be decreasing in n_B. (It's unclear, however, whether an analogue of Proposition 3 remains true.)

8.4.3 Heterogenous Labor Costs

In this subsection, we allow A and B to pay $w_A > 0$ and $w_B > 0$ for each consultant they employ. It's readily verified that all of the results of the main text continue to hold, provided we appropriately modify our assumptions.[83] Thus, our goal here is to show how cost heterogeneity shapes equilibrium welfare.

Equilibrium Welfare and Network Size. The next proposition is the main result of this subsection; it shows how changes in labor costs affect the partner's outcomes. We write $\overline{\boldsymbol{W}}(i, w_i)$ and $\underline{\boldsymbol{W}}(i, w_i)$ when we wish to emphasize the dependence of these sets on i's cost of labor. The next proposition is our main result.

Proposition S6. Comparative Statics of Labor Costs.
Let $P_i(x)(r_i(x) - w_i\, d(x)) > 0$ for each project x and each partner i. As w_A decreases to w'_A and w_B remains fixed, then A's payoff increases and B's payoff decreases in the ELEE equilibria that are best for A and worst for B. That is, for all $\boldsymbol{s}^\star = (\mathcal{N}^\star_A, \ldots, \mathcal{N}^\star_B) \in \overline{\boldsymbol{W}}(A, w_A) \cap \underline{\boldsymbol{W}}(B, w_B)$ and all $\boldsymbol{s}' = (\mathcal{N}'_A, \ldots, \mathcal{N}'_B) \in \overline{\boldsymbol{W}}(A, w'_A) \cap \underline{\boldsymbol{W}}(B, w_B)$,

$$U_A(\boldsymbol{s}', w'_A) \geq U_A(\boldsymbol{s}^\star, w_A) \text{ and } U_B(\boldsymbol{s}', w_B) \leq U_B(\boldsymbol{s}^\star, w_B).$$

[82] For instance, Assumption 2 becomes "$P_i(x)\pi_i(x, d(x)) > 0$ for each project x and each partner i." The other assumptions are modified in a similar manner.

[83] For instance, Assumption 1 becomes "$r_i(x) - w_i\, d(x) \geq 0$ for each project x and each partner i." The other assumptions are modified in a similar manner.

If Assumption 3 also holds both before and after the shift in wages, then the size of A's network increases and the size of B's network decreases, i.e., $|\mathcal{N}'_A| \geq |\mathcal{N}^\star_A|$ *and* $|\mathcal{N}'_B| \leq |\mathcal{N}^\star_B|$*. An analogous result holds for ELEE equilibria that are best for B and worst for A.*

The proof is very similar to the proof of Proposition 6. We'll show that decrease in w_A increases A's best responses in the Auxiliary Game. This, in turn, increases the size of A's network and decreases the size of B's network in the maximal and minimal equilibria. The desired comparative statics then follow from monotonicity/Lemma 6, the shift in costs, and Proposition 3 when coupled with Lemma 7. Since we've already seen this method of proof in the main text, we only establish that the decrease in w_A increases A's best reply in the Auxiliary Game.

We write $\phi_A(z, w_A)$ and $\phi_B(z, w_B)$ to emphasize the dependence of A's and B's best replies in the Auxiliary Game on their respective costs of labor.

Lemma S4. Shifts in Best Responses.
Suppose w_A *decreases to* w'_A*, then for each* $z \in \{0, \ldots, N\}$*, we have* $\phi_A(z, w_A) \preceq \phi_A(z, w'_A)$*.*

Proof. Let $\theta \in \{0, 1\}$ and let

$$f(z_A, z_B, \theta) = \begin{cases} U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B), w_A) & \text{if } \theta = 0 \\ U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B), w'_A) & \text{if } \theta = 1. \end{cases}$$

We'll show that $f(z_A, z_B, \theta)$ is supermodular in (z_A, θ) for each $z_B \in \{0, \ldots, N\}$. Given this, Theorem 2.8.1 in Topkis [68] implies that $\rho(z_B, \theta) = \arg\max_{z_A \in \{0,\ldots,N\}} f(z_A, z_B, \theta)$ is weakly increasing in θ, i.e., that $\rho(z_B, 0) \preceq \rho(z_B, 1)$ for each $z_B \in \{0, \ldots, N\}$. Since $\phi_A(z, w_A) = \rho(z, 0)$ and $\phi_A(z, w'_A) = \rho(z, 1)$, it follows that $\phi_A(z, w_A) \preceq \phi_A(z, w'_A)$ for all $z \in \{0, \ldots N\}$.

To show that $f(z_A, z_B, \theta)$ is supermodular in (z_A, θ), let $\theta' = 1$ and $\theta = 0$. Lemma A1 gives

$$\begin{aligned} f(z_A, z_B, \theta') - f(z_A, z_B, \theta) &= U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B), w'_A) - U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B), w_A) \\ &= \sum_{\{x | d(x) \leq z_A\}} (w_A - w'_A) d(x) \frac{P_A(x)}{2} \\ &\quad + \sum_{l=1}^{z_A} \sum_{\{x | d(x) = l\}} (w_A - w'_A) g_A(l, N - z_B) \frac{P_A(x)}{2}. \end{aligned}$$

Since $w_A \geq w'_A$, this sum is positive. Additionally, the sum is trivially increasing in z_A.

Thus, for $z'_A \geq z_A$,

$$f(z'_A, z_B, \theta') - f(z'_A, z_B, \theta) \geq f(z_A, z_B, \theta') - f(z_A, z_B, \theta)$$
$$f(z'_A, z_B, \theta') + f(z_A, z_B, \theta) \geq f(z_A, z_B, \theta') + f(z'_A, z_B, \theta),$$

that is, f is supermodular in (z_A, θ). □

Proof of Proposition S6. Analogous to the Proofs of Propositions 6 and 9 in light of Lemma S4 and so omitted. □

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