Essays in Monetary and Fiscal Policy

by

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Abstract

This thesis consists of three chapters on monetary and fiscal policy. The first chapter explores the importance of redistribution in explaining why monetary policy has aggregate effects on household consumption. I argue that traditional representative agent models focusing on substitution effects ignore a key component of the monetary policy transmission mechanism, which exists because those who gain from accommodative monetary policy have higher marginal propensities to consume (MPCs) than those who lose. I use a sufficient statistic approach to show that, provided households' elasticities of intertemporal substitution are reasonably small, redistributive effects can be as important as substitution effects in explaining the response of aggregate consumption to real interest rate changes in the U.S. My calibrated general equilibrium model predicts that, if U.S. mortgages all had adjustable rates, the effect of interest-rate changes on consumer spending would more than double and would be asymmetric, with rate increases reducing spending by more than cuts would increase it.

The second chapter, joint with Matthew Rognlie, explains why a monetary union between countries (such as the Eurozone today) may lead to a stronger fiscal union. Since exchange rates can no longer adjust to offset shocks, the presence of nominal rigidities implies that fiscal risk-sharing becomes more valuable in a monetary union. As a result, countries in such a union are capable of overcoming their lack of commitment to fiscal transfers. However, inefficient equilibria without fiscal transfers remain possible. We derive implications for the optimal policy of the central bank when the fiscal union is under stress.

The third chapter, also joint with Matthew Rognlie, studies the possibility that feedbacks between sovereign bond spreads and governments' desire to default may lead to multiple equilibria in sovereign debt markets. We show that such multiplicity does not exist in the infinite-horizon model of Eaton and Gersovitz (1981), a widely adopted benchmark for quantitative analyses of these markets. Our proof may be important to understand Euro government bond markets, and calls for renewed attention on the theoretical conditions that are needed for sovereign debt models to generate multiple equilibria.

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Contents

Chapter 1

Monetary Policy and the Redistribution Channel

The role that monetary policy plays in redistributing income and wealth has been at the center stage of discussions between economic commentators and policymakers in the past few years.[1] There is a clear sense that households are not all equally affected by low interest rates, but no consensus on who gains and who loses. Consider the many ways in which accommodative monetary policy can affect an asset holder: he may lose from low returns, or benefit due to capital gains; he may lose from inflation eroding his savings, or benefit if a recession is avoided.

A conventional view is that redistribution is a side effect of monetary policy changes, separate from the issue of aggregate stabilization which these changes aim to achieve. This view is implicit in most models of the monetary policy transmission mechanism, which feature a representative agent. By contrast, in this paper I argue that redistribution is a *channel* through which monetary policy affects macroeconomic aggregates. Specifically, I contend that the redistributive effects of *accommodative* monetary policy contribute to *increasing* aggregate consumption demand, because those who gain have higher marginal propensities to consume (MPCs) than those who lose—and inversely for contractionary monetary policy. The simple argument goes back to Tobin (1982):

> Aggregation would not matter if we could be sure that the marginal propensities to spend from wealth

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[1]For recent examples see Martin Wolf's contention that "the Federal Reserve's policies have benefited the relatively well off; it is trying to raise the prices of assets which are overwhelmingly owned by the rich." ("Why inequality is such a drag on economies", Financial Times, September 30, 2014) or William D. Cohan, "How Quantitative Easing Contributed to the Nation's Inequality Problem", New York Times Dealbook, October 24, 2014. Among central bankers' speeches on the topic, see Benoît Coeuré, "Savers Aren't Losing Out", November 11, 2013; James Bullard, "Income Inequality and Monetary Policy: A Framework with Answers to Three Questions", June 26, 2014 or Yves Mersch, "Monetary Policy and Economic Inequality", October 17, 2014.

> were the same for creditors and for debtors. But [...] the population is not distributed between debtors and creditors randomly. Debtors have borrowed for good reasons, most of which indicate a high marginal propensity to spend from wealth or from current income.

However, I find that Tobin's distinction between debtors and creditors is not precise enough to evaluate the aggregate effect of redistribution via monetary policy on household spending. Because accommodative monetary policy tends to raise inflation *and* to lower real interest rates, there are at least two dimensions of heterogeneity—and hence two channels of redistribution—to consider. Let me discuss these in turn.

First, current and future inflation revalue nominal balance sheets—the first by redenominating assets and liabilities, and the second by increasing nominal interest rates and hence the rates at which future flows are discounted. Nominal creditors lose and nominal debtors gain: this is the *Fisher channel*, which has a long history in the literature since Fisher (1933). This channel has been explored by Doepke and Schneider (2006a), who measure the balance sheet exposures of various sectors and groups of households in the United States to different inflation scenarios. Net nominal positions (NNPs) quantify these exposures for the case of unexpected increases in the price level.

Real interest rate changes create a second, more subtle form of redistribution. Households' balance sheets do not only consist of their financial assets and liabilities: they also include their future incomes and consumption plans. Hence, to determine if someone benefits from falls in real interest rates, one should not look at the increase in the prices of the financial assets that this person holds. Instead, one should consider whether his total assets have longer durations than his total liabilities. *Unhedged interest rate exposures* (UREs)—the difference between maturing assets and liabilities at a point in time—are the correct measure of balance-sheet exposures to real interest rate changes, just like net nominal positions are for price level changes. For example, agents whose financial wealth is primarily invested in short-term certificates of deposit tend to have positive UREs, while those with large investments in long-term bonds or adjustable-rate mortgage holders tend to have negative UREs. Real interest rate falls redistribute away from the first group towards the second group: this is what I call the *interest rate exposure channel.*

In this paper I focus my attention on this second channel for two reasons. First, equilibrium real interest rates fluctuate over the business cycle, making the redistribution they induce important at all times. By contrast, the Fisher channel is likely to be more muted in countries with low and stable inflation—though it is very important during large regime shifts in monetary policy such as the ones that motivate Doepke and Schneider (2006a), or during debt deflation episodes (Bernanke, 1983). Second, while Tobin (1982) and many others have long argued that nominal debtors have higher MPCs than nominal creditors, the literature has not asked whether *agents with unhedged borrowing requirements* (negative URE) have higher MPCs than *agents with unhedged savings needs* (positive URE). In this paper I find that they do, using two complementary approaches: one reduced-form and one structural. I first develop a theory to identify a sufficient statistic, the *cross-sectional covariance between MPCs and UREs*, that quantifies the effect of redistribution via real interest rates on aggregate demand. I measure this statistic in the data and find that it is plausibly large. I then calibrate a dynamic general equilibrium model that confirms this finding. I use the model to assess the performance of my sufficient statistic in predicting the aggregate effect of changes in monetary policy, and to run various counterfactual experiments.

In first part of the paper, I develop my sufficient statistic approach. In partial equilibrium, I consider an optimizing agent with a given initial balance sheet, who values nondurable consumption and leisure, and is subject to an arbitrary change in his income path and in the whole term structure of nominal and real interest rates. I show that, irrespective of the form of his utility function, the wealth effect component of his consumption response is given by the product of his marginal propensity to consume—the partial derivative of his consumption with respect to a current increase in his income—with a balance-sheet revaluation term in which NNPs and UREs appear. Importantly, I show that, for the case of transitory shocks, this result is robust to the presence of incomplete markets, idiosyncratic risk, and (certain kinds of) borrowing constraints. In other words, the MPC out of a windfall income transfer is relevant to determine the response of optimizing consumers to *any other* change in their balance sheet—in particular those induced by inflation or real interest rate changes. To the best of my knowledge, this result is new to the incomplete-markets consumption literature.[2]

I then sum across the individual-level predictions and exploit the fact that financial assets and liabilities net out in general equilibrium to obtain the response of aggregate consumption to simultaneous transitory shocks to the real interest rate, output, and the price level. Independently of agents' utility functions, the specification of production, and the asset market structure, the cross-sectional covariances between MPCs and exposures to the aggregate shocks are sufficient statistics for the part of this response that is due to redistribution.[3] I denote by $\mathcal{E}_r$ the covariance between MPCs and UREs, and call it the *redistribution elasticity of aggregate demand* with respect to real interest rates.

My theory shows that it is possible to decompose the change in aggregate consumption that results from a change in real interest rates into the sum of a redistribution component (which depends on $\mathcal{E}_r$) and an intertemporal substitution component (which depends on consumers' Elasticities of Intertemporal Substitution, or EIS). There is a large literature estimating "the" EIS—a key parameter in dynamic macroeconomic models—using aggregate or household-level data. While there is no agreement on its exact value, most studies point to a number between 0 and 2, with some consensus in macroeconomics for a value below 1 (see for example Hall, 2009 or Havránek, 2013). To assess how large the redistribution channel is without taking a stand on the value of the EIS, I derive a measure σ^* that corresponds to the average level of this elasticity that makes the interest rate exposure channel and the substitution channel equal in magnitude, for the case of a purely transitory change in the real interest rate and exogenous labor supply. σ^* is positive when

[2]A precursor to this finding is Kimball (1990), who provides equations that characterize MPC, analyzes how it depends on income uncertainty and market structure, and notes in passing that "the marginal propensity to consume out of wealth figures into the interest elasticity of consumption, as a factor in the wealth effect term".

[3]In addition to the Fisher and the interest rate exposure channels, my analysis highlights the presence of an *earnings heterogeneity channel* which arises when monetary policy does not change all household incomes proportionally. In section 1.5.6 I discuss the role of this channel in determining the aggregate effect of monetary policy in the context of my general equilibrium model.

$\mathcal{E}_r$ is negative; that is, when negative-URE households have higher marginal propensities to consume than positive-URE ones. I define a method for measuring UREs, in an exercise similar to Doepke and Schneider's work on exposures to price changes.

Turning to the data, I estimate $\sigma^* = 0.12$ in Italy using a survey containing a self-reported measure of MPC (Jappelli and Pistaferri, 2014), and $\sigma^* = 0.3$ in the United States using a procedure that exploits the randomized timing of tax rebates as a source of identification for MPC (Johnson, Parker and Souleles, 2006, hereafter JPS). These numbers confirm that there exists a redistribution channel of monetary policy, acting in the same direction as the substitution channel.[4] They also show that this channel is quantitatively significant, especially when the EIS is reasonably small. Therefore, representative-agent analyses that abstract from redistribution can be significantly off.

My reduced-form approach has the virtue of being very robust, but it requires precise measures of MPC and URE, which are challenging to obtain jointly. Moreover, it cannot directly handle persistent changes in real interest rates, nor can it be used to perform policy experiments that affect MPCs or UREs themselves. A calibrated general equilibrium model addresses these shortcomings by imposing more structure. I construct a Bewley-Huggett-Aiyagari incomplete markets model with nominal, long-term, circulating private IOUs (as in Huggett, 1993) and endogenous labor supply. The model features rich heterogeneity in MPCs and UREs, and perfect aggregation of labor supply owing to GHH preferences (Greenwood, Hercowitz and Huffman, 1988).

I calibrate the model to the U.S. economy and quantitatively evaluate, in its steady-state, the size of all my sufficient statistics. I find that the interest rate exposure channel has the same sign, and comparable magnitude, as in my reduced-form analysis. I also find that the Fisher channel is consistent with a substantial increase in demand from an unexpected increase in the price level. Hence, in the model, *both* channels should contribute to the expansionary effects of accommodative monetary policy. I confirm this by studying the economy's transitional dynamics after unanticipated shocks.

In the version of the model with flexible prices, redistributive shocks have no output effects due to the absence of wealth effects on labor supply. Thus, I study the effects of these shocks on equilibrium interest rates, and find that these effects depend on the economy's maturity structure. When financial assets are long term,[5] the interest rate exposure channel is smaller, which lowers the total elasticity of aggregate demand to real interest rates, and increases the fluctuations in the natural rate of interest in response to exogenous shocks. Intuitively, under a long maturity structure, debtors—the high-MPC agents in the economy—roll over a smaller fraction of their liabilities each period, and their consumption plans are therefore less sensitive to changes in real interest rates.

Under sticky prices, the transmission mechanism of monetary policy works as follows. A surprise fall

[4] This finding is also consistent with recent survey evidence from the United Kingdom reported in Bank of England (2014). Faced with a hypothetical two-percentage-point increase in interest rates, overwhelmingly more mortgagors reported that they would cut spending, and that they would do so by larger amounts, than savers reported that they would increase spending.

[5] In my model, all financial assets have an offsetting liability within the household sector, so that experiments shortening the maturity of one necessarily shorten the maturity of the other.

in the nominal interest rate creates a drop in the real interest rate, which boosts demand—both through the substitution channel and through the interest rate exposure channel. The latter effect is stronger when the economy's maturity structure is shorter. This increase in demand then gets amplified through four channels. First, the average income increase translates into a spending increase though the economy's average MPC. Second, consumers' incomes are impacted in heterogeneous ways: in my model, the increase in output disproportionately benefits the high earners, who have lower MPCs; hence the earnings heterogeneity channel acts as a mitigating factor. Third, higher hours worked increase spending on their own, due to the complementarity between consumption and labor supply inherent in GHH preferences. Finally, as in standard New Keynesian models, firms' marginal costs are raised, which can create inflation and boost demand further via the Fisher channel.[6] In my calibration, the fixed point of this process of income-spending increases reflects substantial amplification of monetary policy shocks.

I use the model to ask the extent to which monetary policy transmission would differ if the all assets and liabilities in the U.S. economy—calibrated with an EIS of 0.5—were short term. I find that monetary policy shocks would have more than double their current effect on household nondurable consumption. This is consistent with the cross-country structural vector autoregression study of Calza, Monacelli and Stracca (2013), which finds that consumption reacts much more strongly to identified monetary policy shocks in countries where mortgages predominantly have adjustable rates. It also confirms a widely held-view in policy circles that mortgage structure plays a role in the monetary policy transmission mechanism (Cecchetti, 1999; Miles, 2004). One interpretation is that the substitution effect is stronger in these countries, since agents effectively participate more in financial markets. My paper offers an alternative interpretation that does not rely on limited participation: in adjustable-rate mortgage (ARM) countries, monetary policy affects household spending predominantly by redistributing wealth.[7]

Finally, I derive in the context of the model a first-order approximation for the impulse response to a one-time monetary policy shock—it involves MPC-based sufficient statistics—and I compare this prediction to the full nonlinear impulse response to a shock. While the approximation is excellent for any small shock, as well as for larger increases in interest rates, in my ARM calibration I find that it overpredicts the increase in output that results from a moderate fall in the policy rate. This asymmetry in the effects of monetary policy comes from the differential response of borrowers at their credit limit to rises and falls in income: while these borrowers save an important fraction of the gains they get from low interest rates—which reduce the payments they have to make on the credit limit—they adjust spending one for one with every dollar increase in the payments they have to make when interest rates rise. The prediction that interest rate hikes lower output more than falls increase it has received support in the empirical literature (Cover, 1992; de Long

[6]In the current version of the model, I shut down the Fisher channel by assuming that prices are fully sticky. In general, the Fisher channel will increase in strength with the degree of price flexibility.

[7]Aside from limited participation, I am leaving a number of other redistributive channels out of my analysis. First, since I abstract away from aggregate risk, in my framework monetary policy cannot change risk premia. Second, since I assume that all assets are remunerated at the risk-free rate, my analysis does not address the unequal incidence of inflation due to larger cash holdings by the poor (Erosa and Ventura, 2002; Albanesi, 2007). Hence my analysis can best be seen to apply to conventional monetary policy actions in modern developed countries with low and stable inflation targets.

and Summers, 1988; Tenreyro and Thwaites, 2013). An influential interpretation of this fact, which dates back to Keynes, relies on the presence of downward nominal wage rigidities. My explanation is that MPC differences are smaller for falls than for rises in interest rates, so that the redistribution channel is smaller for the former than for the latter.

Literature review. This paper contributes to several strands of the literature.

First, an extensive empirical literature has documented that marginal propensities to consume are large and heterogenous in the population (Parker, 1999; Johnson et al., 2006; Parker, Souleles, Johnson and McClelland, 2013; Broda and Parker, 2014; see Jappelli and Pistaferri, 2010 for a survey). In particular, the literature has found a dependence of MPCs on household balance sheet positions, which motivates my analysis (Mian, Rao and Sufi, 2013; Mian and Sufi, 2014; Baker, 2014; Jappelli and Pistaferri, 2014). Recently, di Maggio, Kermani and Ramcharan (2014) and Keys, Piskorski, Seru and Yao (2014) have measured the consumption response of households to changes in the interest rates they pay on their mortgages. My theory shows that these papers quantify an important leg of the redistribution channel of monetary policy. In appendix 1.7.2.2, I illustrate how such cross-sectional studies of the link between household consumption and balance sheet structure can be mapped into my theoretical framework.

Two empirical papers have explored two of the channels of monetary policy that I highlight in isolation, focusing on the redistribution itself rather than its aggregate demand effect. As described above, Doepke and Schneider (2006a) quantify the redistribution that the Fisher channel could create under different scenarios of surprise inflation. Coibion, Gorodnichenko, Kueng and Silvia (2012) use the Consumer Expenditure Survey and find that identified monetary policy accommodations lower income inequality. To the extent that lower income agents have higher MPCs, this suggests that the *earnings heterogeneity channel* (a term I borrow from their paper) may increase aggregate demand following falls in interest rates.[8]

On the theoretical front, my work belongs to a long tradition in macroeconomics that uses micro-founded general equilibrium models to explain why unanticipated increases in nominal interest rates tend to lower household spending. The vast majority of the New Keynesian literature—with which I share my focus on household consumption, and my emphasis on sticky prices in the last section—analyzes this question in a framework where households do not have net financial positions (see the textbook expositions of Woodford, 2003 and Galí, 2008). In this context, intertemporal substitution is the dominant reason why consumers respond to changes in real interest rates.[9] My analysis brings wealth effects to the forefront of the analysis of monetary policy shocks, by emphasizing that, when MPCs and balance sheets are heterogenous, there exists a direct or "first-round" effect of redistribution that transmits real interest rate shocks to aggregate

[8] My consumer theoretic analysis may help to refine the theoretical framework of Coibion et al. (2012). In particular, their *savings redistribution channel* ("an unexpected increase in interest rates or decrease in inflation [which] benefits savers and hurts borrowers") can usefully be decomposed into my *Fisher channel* and my *interest rate exposure channel*: the two are very different. Typical borrowers with fixed-rate mortgages, for example, have much to lose from decreases in inflation but close to nothing from rises in real interest rates.

[9] Wealth effects do matter in New Keynesian representative-agent models, but only through "second-round", general equilibrium effects (higher interest rates lower aggregate spending which in turn lowers household incomes).

consumption demand.

My analysis is also related to a literature that emphasizes the role of firms' or banks' net worth in amplifying the effects of monetary policy on investment (for example, Bernanke and Gertler, 1995; Bernanke, Gertler and Gilchrist, 1999; Adrian and Shin, 2010; Brunnermeier and Sannikov, 2014). While this literature mainly stresses the Fisher channel, both asset-liability duration mismatches and changes in borrowers' interest expenses are understood to create a link between monetary policy and net worth. In this paper, I show that the concept of unhedged interest rate exposures makes these insights applicable to the study of consumption, the largest component of output.

A number of models use redistribution as a mechanism that can generate macroeconomic effects of monetary policy even without nominal rigidities. Among these are models that emphasize limited participation (Grossman and Weiss, 1983; Rotemberg, 1984; Alvarez, Atkeson and Edmond, 2009) or wealth effects on labor supply (Sterk and Tenreyro, 2014 combine them with substitution into durable goods purchases). By contrast, I consider agents who participate in financial markets at all times, and I generally abstract away from wealth effects on labor supply, so I introduce nominal rigidities to translate the link between redistribution and demand that I highlight into an effect on final output.

Since the pioneering work of Harberger (1964), sufficient statistics have been used in public finance to evaluate the welfare effect of hypothetical policy changes in a way that is robust to the specifics of the underlying structural model (see Chetty, 2009 for a survey). My sufficient statistics are useful to evaluate the impact on *aggregate demand* of hypothetical changes in macroeconomic aggregates in a similarly robust way. All that is required is information on household balance sheets, income and consumption levels, and their MPCs. Farhi and Werning (2013b) bridge the public finance approach and my positive approach: MPCs enter as sufficient statistics for their optimal macro-prudential interventions under nominal rigidities.

The importance of MPC differences in the determination of aggregate demand is well understood by the theoretical literature on fiscal transfers (for example Galí, López-Salido and Vallés, 2007; Oh and Reis, 2012; Farhi and Werning, 2013a; McKay and Reis, 2013). MPC differences between borrowers and savers, in particular, have been explored a source of aggregate effects from shocks to asset prices (King, 1994), or to borrowing constraints (Eggertsson and Krugman, 2012; Guerrieri and Lorenzoni, 2015; Korinek and Simsek, 2014). Guerrieri and Lorenzoni (2015) notably study, as I do, a Bewley-Huggett-Aiyagari model with rich heterogeneity in asset positions and MPCs (together with heterogeneity in marginal propensities to work). None of these studies, however, focus on the role of MPC differences in generating aggregate effects of monetary policy.

My dynamic general equilibrium model belongs to a recent literature studying monetary policy in New Keynesian environments with incomplete markets. Sheedy (2014) focuses on a normative question: when the only available assets are nominal and risk-free, the central bank can exploit its influence on the price level to ameliorate market incompleteness through the Fisher channel. He explores the quantitative importance of this objective, relative to the traditional aggregate stabilization role of monetary policy. Two papers close to my work are Gornemann, Kuester and Nakajima (2012) and McKay, Nakamura and Steinsson (2014). Gornemann et al. (2012) examine quantitatively the population distribution of consumption and welfare gains after monetary policy shocks, in a calibrated model that matches the U.S. business cycle moments, as well as its wealth distribution in steady state. They find that a small fraction of wealthy agents gain, and others lose, from contractionary monetary policy shocks. My framework suggests that this can arise from rich households both earning more in profits *and* gaining from positive unhedged interest rate exposures. It also suggests a simple way to calculate, in their model, the part of the aggregate output effect that is due to redistribution, a question which motivates them but which they do not address. McKay et al. (2014) compare, as I do, the aggregate effects of monetary policy shocks when markets are incomplete relative to the representative-agent case, but they focus on shocks announced well in advance. They show that incomplete markets dampen the effect of this forward guidance. This contrasts to the response to contemporaneous monetary policy shocks that I highlight, which tends to be higher under incomplete markets—especially when debt is short-term debt—because of the negative correlation between marginal propensities to consume and unhedged interest rate exposures that the model generates.

Finally, a few other dynamic general equilibrium models examine the impact of mortgage structure on the monetary transmission mechanism. As in my paper, Calza et al. (2013) and Rubio (2011) find, in the calibrations of their models, significantly larger effects from monetary policy shocks under variable- than under fixed-rate mortgages (FRMs). I highlight the role of unhedged interest rate exposures in accounting for part of these results. Garriga, Kydland and Sustek (2013) study a flexible-price, limited participation model and also find much larger output effects under ARMs than FRMs, mainly acting through investment.

Layout. The remainder of this paper is structured as follows. Section 1.1 presents a partial equilibrium decomposition of consumption responses to shocks into substitution and wealth effects, starting from a single-agent model under perfect foresight and showing that the result survives under incomplete markets. Section 1.2 provides my aggregation result and some simple general equilibrium applications. Section 1.3 assesses the quantitative magnitude of the redistribution channel by measuring σ^* in survey data. Finally, section 1.4 builds and calibrates a Huggett model, which section 1.5 uses to assess quantitatively the role of the economy's maturity structure in shaping the cyclical properties of the natural rate of interest and the ability of monetary policy to increase household spending. Section 1.5 also investigates the asymmetric effects of increases and cuts in interest rates. Section 1.6 concludes.

1.1 Household balance sheets and wealth effects

In this section, I consider the role of households' balance sheets in determining their consumption and labor supply adjustments to a macroeconomic shock. I first highlight the forces at play in a life-cycle labor supply model (Modigliani and Brumberg, 1954; Heckman, 1974) featuring perfect foresight and balance sheets with an arbitrary maturity structure. Following an unexpected shock, balance sheet revaluations—as well as marginal propensities to consume and work—play a crucial role in determining both the welfare response and the wealth effects on consumption and labor supply (theorem 2). I isolate in particular the role of unhedged interest rate exposures. Under certain conditions, the result in theorem 2 survives the addition of idiosyncratic income uncertainty (theorem 9) and therefore applies to a large class of microfounded models of consumption behavior.

1.1.1 Perfect-foresight model

Consider a household with arbitrary non-satiable preferences over nondurable consumption $\{c_t\}$ and hours of work $\{n_t\}$.[10] I assume no uncertainty for simplicity: the same insights obtain when markets are complete. The household is endowed with a stream of real unearned income $\{y_t\}$. He has perfect foresight over the general level of prices $\{P_t\}$ and the path of his nominal wages $\{W_t\}$, and holds long-term nominal and real contracts. Time is discrete, but the horizon may be finite or infinite, so I do not specify it in the summations. The agent solves the following utility maximization problem:

$$\begin{aligned} \max \quad & U(\{c_t, n_t\}) \\ \text{s.t.} \quad & P_t c_t = P_t y_t + W_t n_t + ({}_{t-1}B_t) + \sum_{s\geq 1} ({}_tQ_{t+s})\left({}_{t-1}B_{t+s} - {}_tB_{t+s}\right) \\ & \quad + P_t\left({}_{t-1}b_t\right) + \sum_{s\geq 1} ({}_tq_{t+s})\, P_{t+s}\left({}_{t-1}b_{t+s} - {}_tb_{t+s}\right) \end{aligned} \tag{1.1.1}$$

In the flow budget constraint (1.1.1), ${}_tB_{t+s}$ denotes a nominal payment the household arranges in period t to be paid out to him in period $t+s$, whereas ${}_tb_{t+s}$ denotes a payment in real terms. Correspondingly, ${}_tQ_{t+s}$ is the time-t price of a nominal zero-coupon bond paying at $t+s$, and ${}_tq_{t+s}$ the price of a real zero-coupon bond. This asset structure is the most general one that can be written for this dynamic environment with no uncertainty. Examples of nominal assets include deposits, long-term bonds or most typical mortgages. Examples of real assets include stocks (which here pay a riskless real dividend stream and therefore are priced according to the risk-free discounted value of this stream), inflation-indexed government bonds, or price-level adjusted mortgages.

The only restriction on the environment is an assumption of no arbitrage, which results in a Fisher equation for the nominal term structure:

$${}_tQ_{t+s} = ({}_tq_{t+s})\frac{P_t}{P_{t+s}} \quad \forall t, s$$

I begin the analysis of the consumer problem at $t=0$. The environment allows for a very rich description of the household's initial holdings of financial assets, denoted by the consolidated claims, nominal $\{{}_{-1}B_t\}_{t\geq 0}$ and real $\{{}_{-1}b_t\}_{t\geq 0}$, due in each period. I write the initial real term structure as $q_t \equiv ({}_0q_t)$ and real wages at t as $w_t \equiv \frac{W_t}{P_t}$.

Using either a terminal condition if the economy has finite horizon, or a transversality condition if the

[10] The analysis extends in a straightforward way to include additional endowment goods—such as housing—in preferences, and this is omitted for brevity.

economy has infinite horizon, the flow budget constraints consolidate as expected into an intertemporal budget constraint:

$$\sum_{t\geq 0} q_t c_t = \sum_{t\geq 0} q_t \left(y_t + w_t n_t + ({}_{-1}b_t) + \left(\frac{{}_{-1}B_t}{P_t} \right) \right) \tag{1.1.2}$$

Just as in any consumer theory problem, this one features a degree of indeterminacy in prices. The budget constraint (1.1.2) is unchanged when all discount rates q_t are multiplied by a constant.[11] I choose the *present value* normalization, $q_0 = 1$, which discounts future cash flows to date-0 terms.[12] I define present value wealth, W, as the value of the right-hand side of (1.1.2) when $q_0 = 1$. W can in turn be decomposed as the sum of human wealth (the present value of all future income) and financial wealth W^F:

$$W^F = \sum_{t\geq 0} q_t \left(({}_{-1}b_t) + \left(\frac{{}_{-1}B_t}{P_t} \right) \right)$$

From (1.1.2) it is clear that all financial assets with the same present value—that deliver the same level of financial wealth W^F—are equivalent from the point of view of the solution. This observation is summarized in the following proposition:

Proposition 1. *Financial assets with the same initial present value W^F deliver the same solution to the consumer problem.*

This framework predicts that a given household, holding a mortgage with outstanding nominal principal L (normalizing the price level at $P_0 = 1$), formulates the same plan $\{c_t, n_t\}_{t\geq 0}$ for consumption and labor supply irrespective of whether this liability is in the form of:

a) an adjustable-rate mortgage (ARM): ${}_{-1}B_0 = -L$ (the household has sold a short-term bond which is rolled over at the going nominal market interest rate every period)

b) a fixed-rate mortgage (FRM), where nominal payments are fixed and contracted in advance for T periods: ${}_{-1}B_t = -M$ for $t = 0 \ldots T$ (the outstanding principal is calculated as the outstanding present value of mortgage payments M: $L = \sum_{t=0}^{T} Q_t M$)

c) a price-level adjusted mortgage (PLAM), where the payments are pre-specified in real terms and get revalued with the price level: ${}_{-1}b_t = -m$ for $t = 0 \ldots T$ (the outstanding principal discounts these payments using the *real* term structure: $L = \sum_{t=0}^{T} q_t m$)

1.1.2 Adjustment after a shock

I now conduct an exercise where, *keeping balance sheets fixed* at $\{{}_{-1}B_t\}_{t\geq 0}$ and $\{{}_{-1}b_t\}_{t\geq 0}$, all of the variables relevant to the consumer choice problem are altered at once:

a) the price level $\{P_0, P_1 \ldots\}$

[11] Note that there is no *nominal* indeterminacy in the usual sense, since the nominal denomination of assets means that the problem is generally different when the path for the price level is altered.

[12] For some purposes, in particular when thinking through the impact of changes of the real interest rate at date 0 only, it will be convenient to choose a *future value* normalization, with $q_T = 1$ at some arbitrary date T in the future.

b) the real term structure[13] $\{q_0 = 1, q_1, q_2 \ldots\}$

c) the agent's unearned income sequence $\{y_0, y_1 \ldots\}$

d) the stream of real wages $\{w_0, w_1 \ldots\}$

I consider the *first-order* change in consumption dc_0, labor supply dn_0, and welfare dU that results from this change in the environment. It is simplest to interpret this exercise as an *unanticipated shock*, to which dc_0, dn_0 and dU are the approximate impulse responses on impact. However, the analysis in this section is entirely partial equilibrium. In general equilibrium, the changes in a)-d) are linked and the income terms may include insurance payments; I postpone discussing these issues to section 1.2.

Several important quantities from consumer theory are defined along the initial path and can be evaluated at the initial sequence of prices, wealth and utility level: those include the marginal propensity to consume $MPC = \frac{\partial c_0}{\partial y_0}$, the marginal propensity to supply labor $MPN = \frac{\partial n_0}{\partial y_0}$, and the Hicksian (or compensated) demand elasticities $\epsilon^h_{x_0,y_t} = \frac{\partial x^h_0}{\partial y_t}\frac{y_t}{x_0}$ for $x \in \{c, n\}$ and $y \in \{q, w\}$.[14] Slutsky's equations can then be adapted to this dynamic context (see appendix 1.7.3.1) to yield:

Theorem 2 (Generalized impulse response.). *To first order, the date-0 responses of consumption, labor supply and welfare to the considered change are given by*

$$dc_0 \simeq MPCd\Omega + c_0\left(\sum_{t\geq 0}\epsilon^h_{c_0,q_t}\frac{dq_t}{q_t} + \sum_{t\geq 0}\epsilon^h_{c_0,w_t}\frac{dw_t}{w_t}\right)$$

$$dn_0 \simeq MPNd\Omega + n_0\left(\sum_{t\geq 0}\epsilon^h_{n_0,q_t}\frac{dq_t}{q_t} + \sum_{t\geq 0}\epsilon^h_{n_0,w_t}\frac{dw_t}{w_t}\right)$$

$$dU \simeq U_{c_0}d\Omega$$

where $d\Omega = dW - \sum_{t\geq 0} c_t dq_t$, the net-of-consumption wealth change, is given by

$$\begin{aligned} d\Omega = & \underbrace{\sum_{t\geq 0}(q_t y_t)\frac{dy_t}{y_t}}_{\text{Real unearned income change}} + \underbrace{\sum_{t\geq 0}(q_t w_t n_t)\frac{dw_t}{w_t}}_{\text{Real earned income change}} \\ & + \underbrace{\sum_{t\geq 0} q_t\left(y_t + w_t n_t + \left(\frac{{}_{-1}B_t}{P_t}\right) + ({}_{-1}b_t) - c_t\right)\frac{dq_t}{q_t}}_{\text{Revaluation of net savings flows}} - \underbrace{\sum_{t\geq 0} Q_t\left(\frac{{}_{-1}B_t}{P_0}\right)\frac{dP_t}{P_t}}_{\text{Revaluation of net nominal position}} \end{aligned} \tag{1.1.3}$$

These equations separate the substitution and the wealth effects that result from the shock. In general, consumers substitute intertemporally with respect to current and all future interest rate and wage changes. All wealth effects get aggregated into a net term, $d\Omega$, which affects consumption and labor supply after multiplication by the marginal propensity to consume and work, respectively. Most parametrizations of utility specify consumption and leisure to be *normal*, that is, to increase with exogenous increases in income ($MPC > 0$ and $MPN \leq 0$).

[13] To prevent arbitrage opportunities after the shock, I assume that the nominal term structure adjusts instantaneously to make the Fisher equation hold at the post-shock sequences of interest rates and prices.

[14] Appendix 1.7.3.2 specifies these elasticities for several standard parametrizations of utility.

Note that theorem 2 makes no assumption on horizon or the form of the utility function. This is in line with Campbell (2006)'s recommendation for "normative household finance [to] emphasize results that are robust to alternative specifications of household utility". The only assumption is that of a linear budget constraint with fixed, known prices, which Deaton and Muellbauer (1980) call the *neoclassical model.*

Determinants of the net wealth change. The net wealth change $d\Omega$ in (1.1.3) is the key expression determining both welfare and wealth effects in theorem 2. Observe first the structure of this term: it is a sum of products of *balance-sheet exposures* by *changes in aggregates.* For example, the exposure to a price increase at date t is the net nominal payment stream due to be received at that date: $Q_t\left(\frac{{}_{-1}B_t}{P_0}\right)$. An unexpected increase dP_t in the price level at t lowers the present value of this stream and creates a balance-sheet loss for a nominal asset holder which is valued at $-Q_t\left(\frac{{}_{-1}B_t}{P_0}\right)\frac{dP_t}{P_t}$.

Just as an increase in the price level at date t "acts" upon the net nominal payment stream at that date, equation (1.1.3) shows that changes in real discount rates act upon the present value of what I call *unhedged interest rate exposures* (UREs)

Definition 3. A household's date-t *unhedged interest rate exposure* measured at date -1 is the difference between *all* his maturing assets (including his income) and liabilities (including his planned consumption) at time t:

$$ {}_{-1}URE_t \equiv y_t + w_t n_t + \left(\frac{{}_{-1}B_t}{P_t}\right) + ({}_{-1}b_t) - c_t $$

Hence ${}_{-1}URE_t$ represents the *net saving requirement* of the household at time t, from the point of view of date -1. Because it includes the *stocks* of financial assets that mature at date t rather than interest *flows*, it might significantly diverge from traditional measures of savings at date t, in particular if investment plans have very short durations. The following examples and observations help clarify the role of UREs.

Example 4 (No wealth effect)**.** Consider a household whose initial financial assets are entirely indexed to inflation, and are arranged so that dividend payments match the difference between his planned consumption path and his other sources of income:

$$ {}_{-1}B_t = 0; \quad {}_{-1}b_t = c_t - (y_t + w_t n_t) \quad \forall t \tag{1.1.4} $$

This household has no exposure to price changes or real interest rate changes at any date (${}_{-1}URE_t = 0$ $\forall t$). Following a shock that does not change his income ($dy_t = dw_t = 0$ $\forall t$), his consumption and labor supply responses are purely driven by substitution effects, and his welfare is unaffected to first order. (The second-order term is positive, reflecting the gain from the ability to reoptimize at the new prices.)

Observation 1. The composition of a household's balance sheet is important to understand his consumption, labor supply, and welfare response to changes in interest rates.

The financial balance sheet described by equations (1.1.4) is sometimes called the *Arrow trading plan,* where all trades are pre-arranged. In practice, a household investing all wealth in real annuities might achieve a plan close to this one. According to proposition 1, any investment plan for a given level of financial wealth leads to the same life-cycle path for consumption and labor supply before the shock. But theorem 2 makes clear that these plans have different consumption and welfare implications after the shock. With short durations (for example, ${}_{-1}b_t = ({}_{-1}B_t) = 0$ for $t \geq 1$), date-0 unhedged interest rate exposure and net asset position are closely aligned.

Observation 2. Asset value changes give incomplete information to understand the effects of monetary policy on household welfare.

In the model just presented and in its extensions in the rest of the paper, monetary policy influences asset values through two channels: risk-free real discount rate effects and expected inflation effects. But these asset value changes do not enter $d\Omega$ directly, so they are not relevant *on their own* to understand who gains and who loses from monetary policy, contrary to what popular discussions sometimes imply. For example, it is sometimes argued that accommodative monetary policy benefits bondholders by increasing bond prices. Theorem 2 shows that such a conclusion cannot be reached without knowledge of the consumption plan a given bondholder is trying to finance. Monetary policy has no effect on bondholders whose dividend streams initially match the difference between their target consumption and other sources of income. Accommodative monetary policy benefits households who hold long-term bonds to finance short-term consumption, through the capital gains it generates. It hurts households who finance a long consumption stream with short-term bonds, by lowering the rates at which they reinvest their wealth.[15] Unhedged interest rate exposures constitute the welfare-relevant metric for the impact of real interest rate changes on households.

Example 5 (Purely transitory change). Suppose one-off unexpected inflation revises all prices by $\frac{dP_t}{P_t} = \frac{dP}{P}$ for $t \geq 0$, the real interest rate changes for one period only ($\frac{dq_t}{q_t} = -dr$ for $t \geq 1$), and income and wages change at $t = 0$ only by $dy_0 = dy$, $dw_0 = dw$. Dropping all date-0 subscripts for ease of notation, I obtain

$$\begin{aligned} d\Omega &= dy + ndw + \sum_{t\geq 1} q_t \left({}_{-1}URE_t\right)(-dr) - \sum_{t\geq 0} Q_t \left(\frac{{}_{-1}B_t}{P_0}\right)\frac{dP}{P} \\ &= dy + ndw + URE \cdot dr - NNP \cdot \frac{dP}{P} \end{aligned} \tag{1.1.5}$$

In (1.1.5), the household's net nominal position is defined as the market value of his nominal liabilities: $NNP \equiv \sum_{t\geq 0} Q_t \left(\frac{{}_{-1}B_t}{P_0}\right)$. The term that appears as the relevant balance-sheet exposure to a transitory dr change is the unhedged interest rate exposure for date zero,

$$URE \equiv {}_{-1}URE_0$$

[15]This remark echoes the prescriptions of the literature on long-term portfolio choice, which, following the pioneering work of Merton (1971), argues that long-horizon investors should invest in long-term inflation indexed bonds (Campbell and Viceira, 2002). My analysis focuses on ex-post wealth effects rather than ex-ante portfolio choice.

as can be seen by using the intertemporal budget constraint[16]

$$URE + \sum_{t \geq 1} q_t \left({}_{-1}URE_t \right) = 0 \tag{1.1.6}$$

A temporary rise in the real interest rate benefits a household with a short-term unhedged saving need—such as a holder of short-term financial assets. The exact present-value balance-sheet gain is given by $URE \cdot dr$.

Example 6 (Permanent change). Consider an unexpected increase in the inflation rate, $\frac{dP_t}{P_t} = td\pi$, and a permanent change in the real interest rate, $\frac{dq_t}{q_t} = -tdr^\ell$, for $t \geq 1$. I obtain:

$$d\Omega = \underbrace{\sum_{t \geq 1} tq_t \left({}_{-1}URE_t \right) \left(-dr^\ell \right)}_{\equiv -URE^\ell} - \underbrace{\sum_{t \geq 0} tQ_t \left(\frac{{}_{-1}B_t}{P_0} \right)}_{\equiv NNP^\ell} d\pi \tag{1.1.7}$$

It can be useful to write the exposure of a balance sheet to a permanent change in inflation as $NNP^\ell = NNP \cdot D^N$, where $D^N \equiv \frac{NNP^\ell}{NNP}$ is the *duration* of the position. On the other hand, the present value of the time path of unhedged interest rate exposures is zero by (1.1.6). Appendix 1.7.2.1 presents an example where URE^ℓ, as defined in (1.1.7), has the same sign as URE and is much larger in magnitude, though this need not be true in general.

From the welfare change to the behavioral response. Doepke and Schneider (2006a) calculate NNP and NNP^ℓ for various groups of U.S. households and show that these numbers are large and heterogenous in the population: they are very positive for rich, old households and negative for the young middle class with fixed-rate mortgage debt. Theorem 2 shows that these numbers are not only relevant for welfare, but also *for the predicted behavioral response* to these inflation scenarios. For example, in partial equilibrium, before real interest rates adjust and induce intertemporal substitution, an agent's consumption response to an unexpected increase in the price level is given, to first order, by $MPC \cdot NNP \cdot \frac{dP}{P}$.

Example 7 (Purely transitory change, separable utility). Assume that utility is separable over time and that labor is supplied inelastically, $U(\{c_t\}) = \sum_{t \geq 0} \beta^t u(c_t)$. Then Appendix 1.7.3.3 shows that the consumption response at date 0, $dc \equiv dc_0$, is given, to first order, by

$$dc \simeq \underbrace{MPC \cdot \left(dy + ndw + URE \cdot dr - NNP \frac{dP}{P} \right)}_{\text{Wealth effect}} - \underbrace{\sigma c (1 - MPC)\, dr}_{\text{Substitution effect}} \tag{1.1.8}$$

where σ is the local elasticity of intertemporal substitution:

$$\sigma \equiv -\frac{u'(c)}{cu''(c)} \tag{1.1.9}$$

A temporary increase in the real interest rate lowers consumption demand via intertemporal substitution, and increases it via a wealth effect for an agent with positive URE (respectively, decreases it for an agent with

[16] An alternative way to see the role of the date-0 unhedged interest rate exposure is to use future value normalization for discount rates: $dq_0 = q_0 dr$. After rediscounting to express the wealth change in present value, the relevant term in $d\Omega$ is $\frac{1}{q_0}\left[q_0 \left({}_{-1}URE_0 \right) dr \right] = URE \cdot dr$

negative URE). Note that the Hicksian elasticity is scaled down from the Frisch elasticity σ by a $1 - MPC$ term (this is a general result for separable preferences; see Houthakker, 1960).

Example 8 (Mortgage type and consumption wealth effects)**.** Theorem 2 predicts that adjustable-rate mortgage holders lower their consumption by more than fixed-rate mortgage holders in response to increases in interest rates because of a wealth effect. Appendix 1.7.2 develops this argument in detail using a numerical example, shows the accuracy of the approximation implied by theorem 2 for different interest-rate scenarios, and proposes a structural interpretation to regressions that compare households with exogenously different mortgage types.

Even though theorem 2 assumes no uncertainty and perfect foresight, it applies directly to environments with uncertainty but where markets are complete, *except* for the shock that is unexpected (all summations are then over states as well as dates). An important feature of all these environments is that the marginal propensity to consume, MPC, is the same out of all forms of wealth ($\frac{\partial c_0}{\partial y_0} = \frac{\partial c_0}{\partial W}$): a dollar received today has the same impact on consumption as a hypothetical amount received in the future, provided that amount is worth a dollar in present value terms. However, a key result of the next section is that theorem 2—and in particular equation (1.1.8)—extends to environments with incomplete markets and idiosyncratic income uncertainty.

1.1.3 The consumption response to shocks under incomplete markets

A large empirical literature, cited in the introduction, measures the marginal propensity to consume out of transitory income shocks. In this section, I show that the theoretical MPC out of current income ($MPC = \frac{\partial c}{\partial y}$) remains a key sufficient statistic for predicting behavior with respect to other changes in consumer balance sheets. I first present the general framework under which my results can be derived, and then present theorem 9 under the case of inelastic labor supply. I develop extensions in appendix 1.7.3.3.

I consider a dynamic, incomplete-market partial equilibrium consumer choice model. The consumer faces an idiosyncratic process for real wages $\{w_t\}$ and unearned income $\{y_t\}$, and has utility function over the sequence $\{c_t, n_t\}$, which has an expected utility form and is separable over time:

$$\mathbb{E}\left[\sum_t \beta^t U(c_t, n_t)\right] \tag{1.1.10}$$

The horizon is still not specified in the summation. As we will see, it will only influence behavior through its impact on the MPC. To model market incompleteness in a general form, I assume that the consumer can trade in N stocks and as well as in a nominal long-term bond. In period t, stocks pay real dividends $\mathbf{d}_t = (d_{1t} \dots d_{Nt})$ and can be purchased at real prices $\mathbf{S}_t = (S_{1t} \dots S_{Nt})$; the consumer's portfolio of shares is denoted by θ_t. The long-term bond is modeled as in Hatchondo and Martinez (2009): it can be bought at time t at price Q_t and is a promise to pay a geometrically declining nominal coupon with pattern $(1, \delta_N, \delta_N^2, \dots)$

starting at date $t+1$ and until the horizon end-point. The household's budget constraint at date t is therefore

$$P_t c_t + Q_t\left(\Lambda_{t+1} - \delta_N \Lambda_t\right) + \theta_{t+1} \cdot P_t \mathbf{S}_t = P_t y_t + P_t w_t n_t + \Lambda_t + \theta_t \cdot \left(P_t \mathbf{S}_t + P_t \mathbf{d}_t\right)$$

Each period, a borrowing limit may restrict trading. I write this constraint with the generic form

$$\mathbb{S}\left(\Lambda_{t+1}, \theta_{t+1}; P_t, Q_t, \mathbf{S}_t\right) \geq 0 \tag{1.1.11}$$

and pay special attention to borrowing limits that have the form

$$\mathbb{S}^*\left(\Lambda_{t+1}, \theta_{t+1}; P_t, Q_t, \mathbf{S}_t\right) \equiv Q_t \Lambda_{t+1} + \theta_{t+1} \cdot P_t \mathbf{S}_t + \frac{\overline{D} P_t}{R_t} \geq 0 \tag{1.1.12}$$

for some $\overline{D}$, where R_t is the real interest rate at t. $\mathbb{S}^*$ is a natural specification in that it restricts the real market value of all claims viewed from date $t+1$ to be bounded below by a constant $-\overline{D}$.

After dividing through by the price level at time t, defining real bond positions as $\lambda_t \equiv \frac{\Lambda_t}{P_{t-1}}$ and writing $\Pi_t \equiv \frac{P_t}{P_{t-1}}$ for the inflation rate between $t-1$ and t, the budget constraint becomes

$$c_t + Q_t\left(\lambda_{t+1} - \delta_N \frac{\lambda_t}{\Pi_t}\right) + \left(\theta_{t+1} - \theta_t\right) \cdot \mathbf{S}_t = y_t + w_t n_t + \frac{\lambda_t}{\Pi_t} + \theta_t \cdot \mathbf{d}_t$$

Define the household's net nominal position at t as the real market value of his nominal assets:

$$NNP_t \equiv \left(1 + Q_t \delta_N\right) \frac{\lambda_t}{\Pi_t}$$

and the time-t unhedged interest rate exposure as:

$$URE_t \equiv y_t + w_t n_t + \frac{\lambda_t}{\Pi_t} + \theta_t \cdot \mathbf{d}_t - c_t = Q_t\left(\lambda_{t+1} - \delta_N \frac{\lambda_t}{\Pi_t}\right) + \left(\theta_{t+1} - \theta_t\right) \cdot \mathbf{S}_t$$

Note that, to the extent that the portfolio choice problem has a unique solution at date $t-1$, URE_t is uniquely defined in each state at time t. Suppose that at a given time 0, the consumer is making an unconstrained choice between consumption, hours worked and investment into all available assets, without hitting the constraint in (1.1.11). His optimization problem can be represented using the recursive formulation

$$\max_{c,n,\lambda',\theta'} U(c,n) + \underbrace{\beta \mathbb{E}\left[V\left(\lambda', \theta'; y', w', Q', \Pi', \mathbf{d}', \mathbf{S}'\right)\right]}_{\equiv W(\lambda', \theta')}$$

$$\text{s.t.} \quad c + Q\left(\lambda' - \delta_N \frac{\lambda}{\Pi}\right) + \left(\theta' - \theta\right)\mathbf{S} = y + wn + \frac{\lambda}{\Pi} + \theta \mathbf{d} \tag{1.1.13}$$

Note that the function V corresponds to the value from optimizing given a starting real level of bonds λ' and shares θ', and includes the possibility of hitting future borrowing constraints.

Consider the predicted effects on consumption resulting from a simultaneous unexpected change in unearned income dy, the real wage dw, the price level $\frac{dP}{P} = \frac{d\Pi}{\Pi}$ and the real interest rate dr. Since the changes are purely transitory, the only effect on asset prices from the real interest rate change comes from the change in discounting: $\frac{dQ}{Q} = \frac{dS_j}{S_j} = -dr$ for $j = 1 \ldots N$. Similarly, transitory changes do not alter the value from future optimization starting at (λ', θ')— that is, the function W is unchanged. One can now apply the implicit function theorem to the set of $N+2$ first-order conditions which, together with the budget constraint, characterize the solution to the problem in (1.1.13). Writing $MPC = \frac{\partial c}{\partial y}$, the partial derivatives of consumption with respect to all changes can then be expressed as a function of MPC and a limited set of constants and elasticities that depend on the utility function. One crucial such constant is the local elasticity of intertemporal substitution (1.1.9). In the simplest case where n hours are inelastically supplied

$$U(c,n) = u(c) \tag{1.1.14}$$

the result from Example 7 carries through:

Theorem 9. *Suppose utility is separable (1.1.10) and labor supply is inelastic (1.1.14). Assume that either a) the consumer is locally optimizing, or b) the consumer is at a binding borrowing constraint (MPC=1) and* $\mathbb{S} = \mathbb{S}^*$. *Then his total change in consumption dc resulting from a purely transitory, simultaneous small change in income dy, wages dw, the price level dP and the real interest rate dr is given by (1.1.8):*

$$dc \simeq MPC\left(dy + ndw + UREdr - NNP\frac{dP}{P}\right) - \sigma c\,(1 - MPC)\,dr \tag{1.1.8}$$

The full proof is given in appendix 1.7.3.3. The intuition is simple: when the consumer is locally optimizing, MPC summarizes the way in which he reacts to all balance-sheet revaluations, income being only one such revaluation. When the borrowing limit *is* binding, his consumption adjustment depends on the way the borrowing limit changes when the shock hits. Under a specification that maintains the real value of claims next period as a constant ($\mathbb{S} = \mathbb{S}^*$), the changes in dr and dP free up borrowing capacity exactly in the amount $UREdr - NNP\frac{dP}{P}$.

Appendix 1.7.3.4 shows that theorem 9 extends to situations with endogenous labor supply, and conjectures an extension to multi-period shocks. I now explore the implications of this result for the general equilibrium aggregation.

1.2 Aggregation and the redistribution channel

This section shows how the microeconomic demand responses derived in section 1.1 aggregate in general equilibrium to explain the economy-wide response to shocks in a large class of heterogenous-agent models (theorem 10). The key insight is that aggregate consumption responds to redistribution according to sufficient statistics—covariances between marginal propensities to consume and agents' balance-sheet exposures to macroeconomic aggregates—that are independent of the particular model generating these MPCs and exposures.

1.2.1 Fixed balance sheets in response to shocks

Taking theorem 2 as a starting point for general equilibrium aggregation following a shock requires assuming that fixed balance sheets are a useful first step for such an exercise. This assumption is restrictive in that it gives balance sheets an active role—contrary to a complete-markets world in which they are pure accounting devices—but it does not rule out the possibility of insurance. First, as noted in Example 4, balance sheets can be *arranged* so as not to generate wealth effects from price-level or real interest rate changes. Second, at the level of generality of the next section, it is possible to count ex-post insurance transfers between agents as part of the individual-level revisions in income after shocks. My assumption is that insurance beyond what is observable in balance sheets is limited enough that fixed balance sheets are indeed a more useful starting point than fixed Pareto weights in the analysis of redistributive effects of monetary policy.

In the later sections, I will progressively make the fixed-balance sheet assumption stronger to fully understand its implications. When I specify the production side of the model I will rule out insurance transfers; and in the dynamic general equilibrium model of sections 1.4 and 1.5, I will further assume that balance sheets are chosen without anticipating the shocks.[17]

1.2.2 Aggregation result

Consider a general-equilibrium closed economy populated by $i = 1 \ldots I$ heterogenous agents facing the same prices. There is no government, labor is the only factor of production, and there is no aggregate risk: a net supply of $j = 1 \ldots N$ trees pays dividends in terms of consumption goods which are fixed in the aggregate (though they can vary with consumers' idiosyncratic states). Absent a government, there is no net supply of nominal assets, and absent accumulable capital, agents cannot save or borrow in the aggregate. Goods market clearing then requires that aggregate consumption be equal to aggregate income from all sources, including dividends from trees. Writing $d_{t,j}$ for the aggregate dividend of tree j, this condition (omitting $t = 0$ subscripts) is

$$C \equiv \sum_{i=1}^{I} c_i = \sum_{i=1}^{I} y_i + \sum_{i=1}^{I} w_i n_i + \sum_{j=1}^{N} d_j \equiv Y$$

Equivalently, the aggregate date-0 unhedged interest exposure of the economy is zero:

$$\sum_{i=1}^{I} URE_i = 0 \tag{1.2.1}$$

while the absence of net supply of financial assets is a statement that the aggregate net nominal position of the economy is zero:

$$\sum_{i=1}^{I} NNP_i = 0 \tag{1.2.2}$$

Consider a change that upsets the equilibrium for one period only. Aggregation of consumer responses as described by theorem 9 shows that a change in monetary policy operates on aggregate consumption via five channels, as described by the following theorem.

Theorem 10 (Aggregate demand response to a one-time shock). *In a general equilibrium where heterogenous agents have inelastic labor supply and are all either locally optimizing, or subject to a borrowing constraint*

[17]The assumption of fixed balance sheets is the one adopted by the overwhelming majority of papers in the financial frictions literature (Kiyotaki and Moore, 1997; Bernanke et al., 1999). It is well known that complete markets without additional frictions reduce the strength of these effects very significantly (Krishnamurthy, 2003; di Tella, 2013; Carlstrom, Fuerst and Paustian, 2014). Opportunities to hedge against macroeconomic shocks are likely to be more limited for households than they are for firms.

of the form $\mathbb{S}^*$, *the changes* dC, dY_i, dP, dr *are linked to first order by*

$$dC \simeq \underbrace{\left(\sum_i \frac{Y_i}{Y} MPC_i\right) dY}_{\text{Aggregate income channel}} + \underbrace{\mathcal{C}\text{ov}_I \left(MPC_i, dY_i - Y_i \frac{dY}{Y}\right)}_{\text{Earnings heterogeneity channel}} - \underbrace{\mathcal{C}\text{ov}_I \left(MPC_i, NNP_i\right)}_{\text{Fisher channel}} \frac{dP}{P}$$

$$+ \left(\underbrace{\mathcal{C}\text{ov}_I \left(MPC_i, URE_i\right)}_{\text{Interest rate exposure channel}} - \underbrace{\sum_i \sigma_i \left(1 - MPC_i\right) c_i}_{\text{Substitution channel}} \right) dr \tag{1.2.3}$$

Theorem 10 shows that, in a large class of models with heterogenous agents, a small set of sufficient statistics is enough to understand and predict the first-order response of aggregate demand to a macroeconomic shock.

Suppose that[18]

$$\text{Cov}_I \left(MPC_i, URE_i\right) < 0 \quad \text{and} \quad \text{Cov}_I \left(MPC_i, NNP_i\right) < 0 \tag{1.2.4}$$

so that agents with unhedged borrowing requirements (respectively net nominal borrowers) have higher marginal propensities to consume than agents with unhedged savings needs (respectively net nominal asset holders). Suppose that, as a result of a transitory monetary accommodation, the real interest rate falls ($dr < 0$) and the price level unexpectedly increases ($\frac{dP}{P} > 0$).[19] Then both the interest-rate exposure channel and the Fisher channel contribute *independently* to increasing aggregate demand—the right-hand side of (1.2.3). If, as a result, output increases and this disproportionately benefits high-MPC agents, the earnings heterogeneity channel raises aggregate demand further. The covariance terms in equation (1.2.3) constitute the redistribution channel of monetary policy.

The rest of this paper explores the implications of Theorem 10. First, in section 1.2.3 I show it can be used to solve for endogenous variables as a function of exogenous ones under various specifications of the production side of the economy. Section 1.3 uses household-level cross-sectional data to measure $\text{Cov}_I \left(MPC_i, URE_i\right)$, confirming that it is negative and plausibly large. Section 1.4 builds a full general equilibrium model and generates the negative covariances in (1.2.4) endogenously.

1.2.3 Three general equilibrium applications

Theorem 10 holds irrespective of the underlying model generating MPCs and exposures at the micro level, as well as the relationship between dY, dP and dr at the macro level. Here I develop three examples of specifications of the production side of the economy. Under each, one endogenous aggregate can be solved for as a function of one exogenous one. These examples show that the interest-rate exposure channel is always a key component of the elasticity of aggregate demand to a change in the real interest rate, and preview a

[18] For clarity, I distinguish in my notation the aggregate $\mathcal{C}\text{ov}_I \left(a_i, b_i\right) \equiv \sum_i a_i b_i - \frac{1}{I} \left(\sum_i a_i\right) \left(\sum_i b_i\right)$ from the more conventional $\text{Cov}_I \left(a_i, b_i\right) \equiv \frac{1}{I} \mathcal{C}\text{ov}_I \left(a_i, b_i\right)$ which appear in the elasticity expressions below. I also define $\mathbb{E}_I \left[a_i\right] \equiv \frac{1}{I} \sum_i a_i$.

[19] In practice, monetary accommodations tend to affect the price level with a lag, but Theorem 10 applies irrespective of the time horizon, which can be chosen to be long enough for these effects to operate.

few of the results from my calibrated model in sections 1.4 and 1.5. I find it useful to introduce the following definitions.

Definition 11. The *redistribution elasticities of aggregate demand* to the real interest rate $\mathcal{E}_r$ and the price level $\mathcal{E}_P$ are defined, respectively, as

$$\mathcal{E}_r \equiv \frac{\mathrm{Cov}_I\left(MPC_i, URE_i\right)}{\sum_i c_i} = \mathrm{Cov}_I\left(MPC_i, \frac{URE_i}{\mathbb{E}_I\left[c_i\right]}\right)$$
$$\mathcal{E}_P \equiv -\mathrm{Cov}_I\left(MPC_i, \frac{NNP_i}{\mathbb{E}_I\left[c_i\right]}\right)$$

while the average elasticity of intertemporal substutition $\overline{\sigma}$ and the Hicksian scaling factor S are defined as

$$\overline{\sigma} \equiv \mathbb{E}_I\left[\sigma_i \frac{(1-MPC_i)\,c_i}{\mathbb{E}_I\left[(1-MPC_i)\,c_i\right]}\right]$$
$$S \equiv \mathbb{E}_I\left[(1-MPC_i)\frac{c_i}{\mathbb{E}_I\left[c_i\right]}\right]$$

Example 12 (Current account response upon opening up to international trade).

Consider an endowment economy with flexible prices, where individual i has a fixed endowment income Y_i. The economy is initially closed, with $C_t = Y_t$ and a current account $CA_t = Y_t - C_t = 0$ at all times. The sequence of real interest rates $\{r_0, r_1, r_2 \ldots\}$ clears the goods and asset markets. Suppose that at date 0 the economy unexpectedly opens up to international goods and asset trade, and that all agents can access international financial markets at the prevailing sequence of world interest rates $\{r_0^*, r_1, r_2, \ldots\}$. For instance, $r_0^* < r_0$ could capture a temporary savings glut in the rest of the world. Using (1.2.3), the resulting instantaneous change in the current account as a share of output is

$$\frac{dCA}{Y} = -\frac{dC}{C} \simeq -\left(\mathcal{E}_r - \overline{\sigma}S\right)dr \tag{1.2.5}$$

where $dr = \frac{r_0^* - r_0}{1+r_0}$. When $r_0^* < r_0$ and condition (1.2.4) holds so that $\mathcal{E}_r < 0$, opening up to trade creates a consumption boom and a deficit on the current account whose magnitude is given by (1.2.5) to first order.

Example 13 (Equilibrium real interest rate change in response to exogenous shocks).

Consider again the economy of example 12. Assume that $t = 1$ is the terminal date at which all financial assets must be repaid; initial financial assets are arbitrary. In order to understand how the redistribution channel alters the determination of equilibrium real interest rates, consider the following two unexpected shocks at date 0. The first shock is an endowment change that affects each agent proportionally, $\frac{dY_i}{Y_i} = \frac{dY}{Y}$; it can be thought of as a total factor productivity shock that "lifts all boats equally". In equilibrium, the goods market must clear: $dC = dY$. Using (1.2.3), this induces an equilibrium real interest rate change equal to

$$dr \simeq \frac{1 - \mathbb{E}_I\left[\frac{Y_i}{Y}MPC_i\right]}{\mathcal{E}_r - \overline{\sigma}S}\frac{dY}{Y} \tag{1.2.6}$$

An increase in the aggregate endowment raises desired savings, and (1.2.6) expresses the amount by which this in turn depresses the real interest rate.

The second experiment is an unexpected increase in the price level, $\frac{dP}{P}$, that does not change agents' endowments. The equilibrium interest rate change at date 0 is now

$$dr \simeq -\frac{\mathcal{E}_P}{\mathcal{E}_r - \overline{\sigma} S}\frac{dP}{P} \tag{1.2.7}$$

Here, assuming (1.2.4) still holds, the rise in the price level creates a positive demand pressure via the Fisher channel, which is counteracted by an increase in the equilibrium real rate of interest, mitigating this pressure via the substitution channel and the interest-rate exposure channel. These two examples are an illustration of the following proposition:

Proposition 14. *A larger interest-rate exposure channel (a more negative $\mathcal{E}_r$) dampens the fluctuations in the equilibrium real rate of interest in response to exogenous shocks.*

Example 15 (Output effects of monetary policy under nominal rigidities).

Consider an economy with perfectly sticky wages. Labor is the only source of income. Agent i initially supplies labor n_i; agents accommodate demand increases by working more at the going wage, with the burden of increased labor supply falling on all agents in proportion to their current level of work.[20] A perfectly competitive firm has production function $Y = \sum_{i=1}^{I} n_i$. Real wages are therefore $w_i = 1$, and individual incomes are $Y_i = n_i$.

Suppose the central bank has a steady-state policy of setting its nominal interest rate equal to the natural rate of interest and is targeting the natural level of output. Consider the consequence of a one-period deviation from this level by $d\iota$, followed by a return to the steady-state policy. Since wages are perfectly sticky and firms have constant returns to production, consumer price inflation is zero, and the nominal interest rate change creates a real interest rate change $dr = d\iota$. Since the central bank stabilizes future incomes, all resulting changes take place over one period, and Theorem 10 applies.

In this case, the Fisher channel is shut down because nominal prices do not change ($dP = 0$), and the earnings heterogeneity channel is shut down given the equiproportionate rule for demand accommodation. Hence (1.2.3) allows us to solve for the demand increase as

$$\frac{dC}{C} \simeq \frac{1}{1 - \mathbb{E}_I\left[\frac{Y_i}{Y} MPC_i\right]}\left(\mathcal{E}_r - \overline{\sigma} S\right) dr \tag{1.2.8}$$

A given fall in the real interest rate generates a larger increase in demand when $\mathcal{E}_r$ is more negative. This is summarized in the following proposition.

Proposition 16. *A larger interest-rate exposure channel (a more negative $\mathcal{E}_r$) amplifies the real effects of monetary policy shocks.*

In the special case of a representative agent ($I = 1$) with EIS σ, we have $\mathcal{E}_r = 0$, $\overline{\sigma} = \sigma$ and $S = 1 - MPC$;

[20] I consider this model of the labor market and rationing rule for simplicity and continuity of exposition with Theorem 10. The same results obtain in a more conventional New Keynesian model with flexible wages, but require the use of a modified version of Theorem 10 that allows for substitution effects on consumption as real wages change (see section 1.2.4.1).

and equation (1.2.8) yields the well-known New Keynesian impulse response:

$$\begin{aligned}\frac{dC}{C} &\simeq -\frac{1}{1-MPC}\sigma\left(1-MPC\right)dr \\ &= -\sigma dr\end{aligned}$$

One can interpret the cancellation of the MPC terms as the fixed point of an infinite round of income-spending increases, involving the traditional Keynesian multiplier, $\frac{1}{1-MPC}$.

1.2.4 Extensions

1.2.4.1 Elastic labor supply

While theorem 10 applies to cases in which labor supply is inelastic or, by extension, to those in which all agents are off their labor supply curves, it can readily be extended to include elastic labor supply. These extensions are useful, for example, when the model has a standard New Keynesian production side, with sticky prices but flexible wages. Changes in real wages induce substitution effects which are not accounted for in equation (1.2.3).

The extension to a case where preferences are separable between consumption and leisure is straightforward. Write ψ_i for agent i's Frisch elasticity of labor supply. Theorem 9′ in appendix 1.7.3.4 shows that expression (1.2.3) obtains with two simple adjustments: one for the fact that the income term now includes the substitution response of hours, $dY_i = dy_i + (1+\psi_i)\,n_i dw_i$, and another scaling up MPC in the intertemporal substitution term to take into account the fact that part of every increase in income is used on leisure: the new expression for the substitution channel is $\sum_i \sigma_i \left(1 - MPC_i\left(1+\frac{\psi_i}{\sigma_i}\frac{w_i n_i}{c_i}\right)\right) c_i$.

Under non-separable preferences, increases in hours worked change the marginal utility of consumption. When preferences have the particular form of complementarity embedded in $U(c,n) = u(c - v(n))$, appendix theorem 9″ shows that a new term appears reflecting the increase in agents' desired consumption following an increase in hours:

$$\begin{aligned}dC \simeq\; & \underbrace{\left(\sum_i \frac{Y_i}{Y} MPC_i\right) dY}_{\text{Aggregate income channel}} + \underbrace{\mathcal{C}\text{ov}_I\left(MPC_i, dY_i - Y_i\frac{dY}{Y}\right)}_{\text{Earnings heterogeneity channel}} - \underbrace{\mathcal{C}\text{ov}_I\left(MPC_i, NNP_i\right)}_{\text{Fisher channel}}\frac{dP}{P} \\ & + \left(\underbrace{\mathcal{C}\text{ov}_I\left(MPC_i, URE_i\right)}_{\text{Interest rate exposure channel}} - \underbrace{\sum_i \sigma_i \xi_i \left(1-MPC_i\right) c_i}_{\text{Substitution channel}}\right) dr + \underbrace{\sum_i \psi_i\left(1-MPC_i\right) n_i dw_i}_{\text{Complementarity channel}} \qquad (1.2.9)\end{aligned}$$

where $0 < \xi_i < 1$ is a constant. The complementarity channel that arises in (1.2.9) has been argued to be reasonable to explain, for example, the observed hump shapes in the life-cycle profile of earnings and consumption (Heckman, 1974; Aguiar and Hurst, 2005).

1.2.4.2 Outside assets and trading with other sectors

The market clearing equations (1.2.1) and (1.2.2), which can be rewritten in terms of cross-sectional means, $\mathbb{E}_I[URE_i] = \mathbb{E}_I[NNP_i] = 0$, respectively state that the unhedged interest rate exposure and net nominal positions of the household sector must be zero. Equivalently, unexpected real interest rate and price-level changes create *pure redistribution* within the household sector. There are reasons to expect these equalities to fail in the data. It is useful to reflect upon why this might be true and discuss how this may alter the aggregate predictions from the model.

Doepke and Schneider (2006a) find that the net nominal position of U.S. households is positive. This means households tend to lose in the aggregate, mostly to the benefit of the government sector, from an unexpected rise in inflation. Similarly, there are reasons to expect to find a positive aggregate URE in the data. The main one is that the household sector tends to be maturity mismatched, holding relatively short-term assets (deposits) and relatively long-term liabilities (fixed-rate mortgages); this is the natural counterpart to the reverse situation in the banking sector. In addition, in periods where the government is increasing its debt and has large flow borrowing requirements, these flows must be financed and households are natural counterparts for them.[21]

Since households are the ultimate claimants on the financial sector and the government, gains to these sectors that occur as a result of lower real interest rates or unexpected inflation must ultimately be rebated back to them. Consider the case of a purely Ricardian model, such as a simple Real Business Cycle model. There, we know that the trading plan between the household and the government is irrelevant. If the government happens to be a flow borrower (have negative URE) when a shock takes place which results in lower real interest rates, this creates a present-value gain to the government, and a lump-sum transfer to the representative household must take place to ensure that both agents' present-value budget constraints are still satisfied. The same holds true for the maturity-mismatched financial sector, which might rebate gains from lower interest rates through lower fees or higher dividends.[22]

When $\mathbb{E}_I[URE_i] > 0$ and strictly no transfer takes place, the redistribution elasticity $\mathcal{E}_r$ needs to be replaced by a term I call the "no rebate" elasticity,

$$\mathcal{E}_r^{NR} = \mathbb{E}_I[MPC_i]\frac{\mathbb{E}_I[URE_i]}{\mathbb{E}_I[c_i]} \tag{1.2.10}$$

However, given the logic that gains and losses must be rebated, this term is only a partial-equilibrium upper bound. An agnostic procedure is to assume a uniform rebating rule, in which case the covariance formula applies. Rebate rules might in practice target higher or lower MPC agents, so that the precise number may depart from the covariance expression in either direction. In the measurement part that follows, I use $\mathcal{E}_r$ as

[21] In the United States, households do not hold many government securities directly. The major counterparts to the recent \$8trn increase in federal debt have been the U.S. financial sector, the Federal Reserve system, and the Rest of the World. See Bank of England (2012) and McKinsey Global Institute (2013) for sectoral studies of winners and losers from the current monetary policy environment.

[22] In a world where financial frictions are important, the "stealth recapitalization" of the banking sector (Brunnermeier and Sannikov, 2012, 2014) from lower real interest rates may have large additional effects on aggregate demand, notably via investment, beyond the ones induced by a rebating of gains to the household sector.

my benchmark, and compute the no-rebate $\mathcal{E}_r^{NR}$ as a robustness check.

Open-economy considerations can strengthen this rebating logic further. In the international financial accounts of the United States, the Rest of the World has long liabilities (FDI) and shorter assets (Treasury securities)—that is, its aggregate URE is positive—and therefore it tends to lose when interest rates fall, creating additional gains that must ultimately accrue to households.

Although my results in the next sections suggest otherwise, it is interesting to note the theoretical possibility that the interest-rate exposure term—either $\mathcal{E}_r$ or $\mathcal{E}_r^{NR}$ if none of the gains to other sectors are rebated to households—may not only be positive, but larger than $\overline{\sigma}S$. In this case, real interest rate increases *raise* aggregate consumption demand, altering significantly the conventional understanding of how monetary policy operates.[23]

1.3 Measuring the redistribution elasticity of aggregate demand

A clear picture that emerges from section 1.2 is that the overall elasticity of aggregate demand to changes in real interest rates is, in general, the sum of a redistribution component and an intertemporal substitution component. In a representative-agent model, the latter is the only channel of transmission from real interest rates to consumption. The magnitude of this substitution channel depends crucially on the size of the elasticity of intertemporal substitution (EIS). In this section I treat the EIS as a parameter σ and ask how large the redistribution component (which I call the interest rate exposure channel) is, *relative* to the substitution channel at any given σ.

In order to do this, I make several simplifications. I maintain my focus on a purely transitory change in the real interest rate, for which theorem 10 established the existence of a sufficient statistic for the interest rate exposure channel. Longer-run changes in real interest rates tend to increase both the intertemporal substitution term and the redistribution term (see appendix 1.7.2 and section 1.5), so the relative magnitudes that I obtain here plausibly extrapolate to these changes as well. I assume that all agents have the same EIS σ and supply labor inelastically—cases with endogenous labor supply reduce the size of the substitution channel, so that this assumption provides a lower bound on the relative magnitude of the redistribution component. Manipulating (1.2.3), I find that the *partial elasticity* of demand to the real interest rate, $\frac{\partial C}{C}\frac{1}{\partial r}$, is given by[24]

$$\left(\underbrace{\mathrm{Cov}_I\left(MPC_i, \frac{URE_i}{\mathbb{E}_I[c_i]}\right)}_{\mathcal{E}_r} - \sigma \underbrace{\mathbb{E}_I\left[(1-MPC_i)\frac{c_i}{\mathbb{E}_I[c_i]}\right]}_{S}\right) \tag{1.3.1}$$

[23] This theoretical possibility is sometimes mentioned in economic discussions of monetary policy. See Raghuram Rajan ("Interestingly [...] low rates could even hurt overall spending"), "Money Magic", Project Syndicate, November 11, 2013; or Charles Schwab "Raise Interest Rates, Make Grandma Smile", Wall Street Journal, November 20, 2014

[24] This is only a partial elasticity since it holds other macroeconomic aggregates fixed. As illustrated in section 1.2.3, in general equilibrium, changes in real interest rates are accompanied by other adjustments in macroeconomic quantities which also have effects consumption. For example, in the case of monetary policy shocks in a New Keynesian model, the central-bank-induced change in the real interest rate creates multiplier effects, and the general equilibrium elasticity of demand to dr is larger than the partial elasticity (see Example 15).

A key finding of this section is that the redistribution elasticity of aggregate demand $\mathcal{E}_r$ is *negative*, so that the redistribution effect and the substitution effect act in the same direction. I further quantify the magnitude of (the absolute value of) $\mathcal{E}_r$ so that it can be compared to $\sigma S < \sigma$. I also measure the "no rebate" (NR) version of this elasticity, $\mathcal{E}_r^{NR}$, defined in (1.2.10), which assumes that gains and losses to other domestic agents are not rebated to households and is therefore a lower bound for the effects of redistribution on aggregate demand (see the discussion in section 1.2.4).

Since all quantities in (1.3.1) are measurable at the household or at the group level *except* for the EIS σ, a useful way of organizing the results is to determine the value of the EIS that would make the substitution and the redistribution effects equal in magnitude. I call this value σ^*; it is defined by

$$\sigma^* \equiv \frac{-\mathcal{E}_r}{S} = \frac{-\mathrm{Cov}_I\left(MPC_i, \frac{URE_i}{\mathbb{E}_I[c_i]}\right)}{\mathbb{E}_I\left[(1-MPC_i)\frac{c_i}{\mathbb{E}_I[c_i]}\right]}$$

Knowing σ^* allows us to say how much a representative-agent model should *add* to its assumed EIS of σ to correctly predict the magnitude of the economy's response to one-time shocks. Taking full account of the redistribution channel is more complex than assuming that the representative-agent is "more elastic" with respect to real interest rate changes—both because longer-term changes to interest rates do not simply scale the redistribution and the substitution component, and because the redistribution component is a function of other model primitives such as the market structure. Nevertheless, the value of σ^* is a useful starting point in evaluating the importance of redistribution in the transmission of shocks.

The literature has used different ways to measure the marginal propensity to consume out of transitory income shocks (see Jappelli and Pistaferri, 2010 for a survey). It is first important to determine what qualifies as a "transitory income shock" from the point of view of the theory as outlined above. A simple approach has been to ask households to self-report the part of any hypothetical windfall that they would immediately spend. This has the benefit of circumventing the general equilibrium issue of determining the source of this windfall. The Italian Survey of Household Income and Wealth (SHIW) contains such a question in 2010 (Jappelli and Pistaferri, 2014), and I use data from this survey as my first measure of MPC.

One concern with self-reported answers to hypothetical situations is that they are not informative about how households would actually behave in these situations. For this reason, the literature has looked at cleanly identified settings allowing estimation of MPC from actual behavior. Perhaps the best setting such settings are the large-scale 2001 and 2008 U.S. tax rebates, whose timing of receipt was randomized (Johnson et al., 2006; Parker et al., 2013). Since these studies exploit variation in timing for a policy announced ahead of time, they identify the MPC out of an *expected* increase in income. This is, in general, different from the theoretically-consistent MPC out of an unexpected increase. In a benchmark incomplete market model, unless borrowing constraints are binding and are not adjusting in response to the expected tax rebate, two consumers that differ only in their timing of receipt should adjust their consumption profile by similar amounts when they receive the news (reflecting the net gain from the present value of the transfer as well as Ricardian offsets) and not react differentially when they receive the transfer. However, to the extent that borrowing constraints are rigid and binding, or if households are surprised by the receipt despite its announcement, the estimation gets closer to the MPC that is important for the theory; and in general provides a lower bound for it.[25] I use the data from the 2001 rebates collected in the U.S. Consumer Expenditure Survey (CEX) and analyzed in Johnson et al. (2006) as my second measure of MPC.

Section 1.3.1 explains how one should conceptually measure a theory-consistent URE from household-level survey data. The data requirements are very stringent: one needs consumption, income and detailed information about assets and liabilities at the household level. Surveys such as the SHIW and the CEX do not measure assets and liabilities very precisely, so that there is likely to be some measurement error in URE. For example, more detailed measures of U.S. household wealth, such as those from the Federal Reserve's Survey of Consumer Finances, tend to show that asset positions (and hence UREs) are much more dispersed than what is reported in the CEX. This is likely to bias my estimate of σ^* downwards. In section 1.3.2 I present the two datasets, and in sections 1.3.3 and 1.3.4 I measure my key moments $\mathcal{E}_r^{NR}$, $\mathcal{E}_r$ and σ^* using the two procedures for calculating MPC just outlined.

1.3.1 Conceptual measurement issues

As defined in section 1.1.2, URE_i measures the total resource flow that a household i needs to invest over the first period of his consumption plan. From the surveys, I construct URE_i as

$$URE_i = Y_i - C_i + B_i - D_i \tag{1.3.2}$$

where Y_i is income from all sources including dividends and interest payments, C_i is consumption including expenditures on durable goods[26] as well as mortgage payments and installments on consumer credit, and B_i and D_i represent, respectively, asset and liability stocks that mature over the period.

Even though $\mathcal{E}_r$ is a unitless number, the choice of time units is important: MPC needs to be measured over a period consistent with the choice of time units for the numerator and denominator of $\frac{URE_i}{\mathbb{E}_I[c_i]}$. Ideally, all measurement would be done over a quarter, which is the frequency at which models analyzing monetary policy are calibrated. This is what I do with the CEX.[27]

Consumption, income and MPC in the SHIW are only available at annual frequency. Because it is not obvious how to translate an annual measure of MPC into a quarterly one, I measure $\mathcal{E}_r$ once at annual frequency, and once at quarterly frequency where I use $MPC^Q = \frac{MPC^A}{3}$ to reflect the tendency of households

[25] See Kaplan and Violante (2014) for another discussion of the interpretation of the coefficients estimated by JPS, and a model in which even households with positive total assets can be at a binding limit on their liquid transactions account, and modify their consumption upon receipt but not upon news of the transfer.

[26] The theory presented in section 1.1 can be extended to include durable goods. Wealth effects depend on a measure of unhedged interest rate exposures that subtracts durable *expenditures*. This is intuitive: a consumer who has a plan to borrow large amounts to finance a durable good is hurt by a rise in real interest rates that raises the financing cost. However, the presence of durable goods also creates additional substitution effects between nondurable and durable consumption, since a real interest rate increase raises the user cost of durables and makes consumers substitute away from them. This effect only mitigates the substitution effect on nondurable consumption further.

[27] Households are interviewed every quarter. Although they are asked to report a monthly break-down of expenditure by month, most researchers aggregate the data back to quarterly frequency to prevent recall-driven serial correlation in consumption.

with precautionary savings motives to spend more of their income in the first quarter of a one-off transfer receipt.[28]

Given the limited information regarding asset maturities in both the SHIW and the CEX, maturing asset stocks (B_i) are difficult to determine precisely. In my quarterly calibrations, I treat time and savings deposits as maturing in the quarter. In annual calibrations, I treat them as maturing within the year. These are likely to be good effective lower and upper bounds for deposit durations. Doepke and Schneider (2006a) calculate that, since the beginning of the 2000s, the average duration of U.S. financial assets has been around 4 years. Whenever I have good information on assets beyond deposits, I count one-fourth of the stock towards B_i in the annual calibration (respectively one-sixteenth in the quarterly calibration). I treat adjustable-rate mortgages, just as deposits, as maturing liability stocks within the quarter or within the year depending on the calibration. Fixed-rate mortgages are not counted additionally in B_i since mortgage payments—which include amortization—are already subtracted from URE as part of the consumption measure.

Finally, I assume away timing differences in the reporting of consumption and income in my calculation of URE. To the extent that these create noise in my URE estimate, they may tend to raise the observed cross-sectional dispersion in URE but will also likely reduce its covariance with *MPC,* so that it is unclear that it will impart a clear directional bias in my estimate of $\mathcal{E}_r$.

1.3.2 Data

Survey of Household Income and Wealth 2010 Italy's 2010 Survey of Household Income and Wealth (SHIW) contains a question which can be used as an empirical measure of MPC:

> "Imagine you unexpectedly receive a reimbursement equal to the amount your household earns in a month. How much of it would you save and how much would you spend? Please give the percentage you would save and the percentage you would spend."

Jappelli and Pistaferri (2014) present a detailed analysis of the data and of the empirical determinants of MPC.[29]

Consumer Expenditure Survey, 2001-2002 (JPS sample) My data for the Consumer Expenditure Survey comes from the Johnson et al. (2006) (JPS) dataset, which I merged with the main survey data to add information on total consumption expenditures, as well as assets and liabilities separately. The dataset covers households with interviews between February 2001 and March 2002. I restrict my sample to households who have income information. This leaves me with 9,443 interviews of 4,583 households.

[28] This theoretical pattern is also consistent with empirical behavior (see the dynamic specifications of Johnson et al., 2006; Parker et al., 2013 and Broda and Parker, 2014).

[29] Note that the time frame for MPC is not specified in the question, as issue that is left unresolved in Jappelli and Pistaferri (2014). A follow-up question in the 2012 SHIW separates durable and nondurable consumption, and specifies the time frame as a full year. The equivalent "MPC" out of both durable and nondurable consumption has close to the same distribution as that of MPC in the 2010 SHIW (respective means are 47 in 2010 and 45 in 2010) which suggests that households tended to assume that the question referred to the full year.

	SHIW		CEX	
Variable	mean	n.s.d.	mean	n.s.d.
Income from all sources (Y_i, per year)	36,114	0.90	45,617	1.01
Consumption incl. mortgage payments (C_i, per year)	27,976	0.61	36,253	0.79
Deposits and maturing assets (B_i)	14,200	1.45	7,147	0.77
ARM mortgage liabilities and consumer credit (D_i)	6,228	1.03	2,872	0.22
Unhedged interest rate exposure (URE_i, per year)	16,110	1.92	13,639	1.27
Unhedged interest rate exposure (URE_i, per Q)	10,007	7.07	6,616	3.39
Marginal Propensity to Spend (annual)	0.47	0.35		
Count	7,951		9,443	

"mean" is the sample mean computed using sample weights (in € for SHIW; current USD for CEX)
"n.s.d" is the normalized standard deviation, $sd_I\left(\frac{X_i}{\mathbb{E}_I[C_i]}\right)$ for $X_i = Y_i, C_i, B_i, URE_i$ and $sd_I(MPC_i)$ for MPC

Table 1.1: Main summary statistics from the two datasets

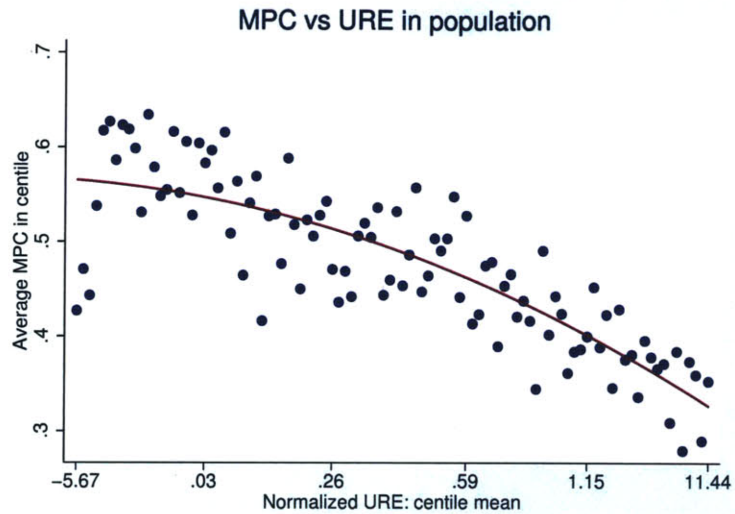
I define maturing assets are the sum of total assets in checking, brokerage and other accounts, savings, S&L, credit unions and other accounts as well as one fourth of the amount in U.S. savings bonds. Because the CEX has poor data on mortgages, I count all of "total amount owed to creditors" from the fifth interview towards D_i is order to calculate my unhedged interest rate exposure measure.

Summary statistics Table 1.1 reports the main relevant summary statistics from the two datasets, with appendix 1.7.1.2 providing further details. For each dataset, the first column reports sample means in current euro for the SHIW (respectively current dollars for the CEX). Consumption is always below income at the mean. This is in part because the coverage of the consumption data in the surveys is less than the whole of personal consumption expenditures from national accounts, in part because consumption tends to be underreported due to imperfect recall. The second column reports a normalized population standard deviation measure, where all variables except for MPC are normalized by the sample mean of consumption, which allows comparison of cross-sectional dispersions across surveys and is consistent with the normalization implicit in the definition of $\mathcal{E}_r$ in (1.3.1).

1.3.3 The redistribution elasticity in the SHIW

Figure 1.3.1 illustrates that the empirical correlation between MPC and URE is negative in the SHIW. This is reminiscent of the finding from Jappelli and Pistaferri (2014) that MPC covaries with net liquid assets in this survey. A direct implication is that $\mathcal{E}_r < 0$: falls in interest rates increase demand via the redistribution channel.

Table 1.2 computes the key moments $\mathcal{E}_r^{NR}$, $\mathcal{E}_r$ and σ^* using the household-level information. Sampling uncertainty is taken into account using the survey's sampling weights. The main quantitative result is that, depending on the frequency at which the estimation is done, σ^* is around 0.1. In other words, and using my preferred, annual-frequency estimate, I find that *in Italy, the redistribution channel explains as much of the demand response to changes in real interest rates as the substitution channel if the EIS is equal to 0.12.*



The figure presents the average reported MPC in each percentile of URE

Figure 1.3.1: Correlation between MPC and URE in the population

Time Horizon		Annual		Quarterly	
Parameter		Estimate	95% C.I.	Estimate	95% C.I.
No-rebate elasticity	$\widehat{\mathcal{E}_r^{NR}}$	0.21	[0.17; 0.23]	0.15	[0.12; 0.19]
Redistribution elasticity	$\widehat{\mathcal{E}_r}$	-0.06	[-0.09; -0.04]	-0.07	[-0.10; -0.03]
Scaling factor	$\widehat{S}$	0.55	[0.53; 0.57]	0.85	[0.83; 0.87]
Equivalent EIS	$\widehat{\sigma^*} = -\frac{\widehat{\mathcal{E}_r}}{\widehat{S}}$	0.12	[0.06; 0.17]	0.08	[0.03; 0.12]

All statistics computed using survey weights

Table 1.2: Measures of $\mathcal{E}_r^{NR}$, $\mathcal{E}_r$ and σ^* using the SHIW

This baseline number is regarded by some as a plausible value for the EIS (for example Hall, 1988, or the meta-analysis in Havránek, 2013). Hence this first measure already suggests that the redistribution channel has quantitative legs.

Combining information from tables 1.1 and 1.2, we can decompose the annual $\mathcal{E}_r$ measure as

$$\mathcal{E}_r = \underbrace{\text{Corr}_I\left(MPC_i, \frac{URE_i}{\mathbb{E}_I\left[c_i\right]}\right)}_{\simeq -0.09} \underbrace{\text{Sd}_I\left(MPC_i\right)}_{0.35} \underbrace{\text{Sd}_I\left(\frac{URE_i}{\mathbb{E}_I\left[c_i\right]}\right)}_{1.92}$$

The low absolute value for the correlation between MPC and URE suggests that $\mathcal{E}_r$ could plausibly be several times larger in a setting with less measurement error in MPCs and UREs. The next section takes a different approach to measuring MPCs and finds precisely this.[30]

[30] A feature of table 1.2 is that the no-rebate redistribution elasticity $\mathcal{E}_r^{NR}$ is positive, in other words, the negative correlation between MPC and URE is not enough, in this case, to overwhelm the positive average URE in the data. In addition to all the reasons given in section 1.2.4 for why $\mathcal{E}_r$ is a more reasonable calculation than $\mathcal{E}_r^{NR}$ in a world where measurement is perfect, the SHIW's total consumption data is sparse. Hence average consumption is underestimated and average URE is overestimated in the survey. This problem is not as severe in the CEX which has a more comprehensive measure of consumption than the SHIW, but it is still present. My benchmark estimate $\mathcal{E}_r$ removes this bias exactly if underreporting in consumption is uncorrelated with MPC.

1.3.4 The redistribution elasticity from the 2001 tax rebates in the CEX

In this section I take a different route towards calculating the value of my key redistribution elasticities. Instead of relying on a survey-based measure, I compute the MPC out of the 2001 tax rebate using the Johnson et al. (2006) (JPS) procedure, stratifying by URE. I then use the estimates by bin to form a covariance. To be specific, I split the sample into J groups ranked by their URE. I then run the main JPS estimating equation

$$C_{i,m,t+1} - C_{i,m,t} = \alpha_m + \beta X_{i,t} + \sum_{j=1}^{J} MPC_j R_{i,t+1} QURE_{i,j} + u_{i,t+1} \tag{1.3.3}$$

where $C_{i,m,t}$ is the level of household i's consumption expenditures in month m and at date t, α_m are month fixed effects absorbing seasonal variation in expenditures, $X_{i,t}$ are the controls used by JPS in their main specification (age and changes in family composition), $R_{i,t+1}$ is the dollar amount of the rebate at $t+1$, and $QURE_{i,j}$ is a dummy indicating that household i's URE is in group $j = 1 \ldots J$. This procedures exploits variation in timing of the rebate across households in the same exposure group to identify the propensity to consume out of the expected one-time transfer that the stimulus payment provides.

In each URE bin, I next calculate the average normalized URE, $NURE_j$, as the average over households in group j of $\frac{URE_i}{\bar{c}}$, where $\bar{c}$ is average consumption expenditure in the sample. I finally compute my estimators as

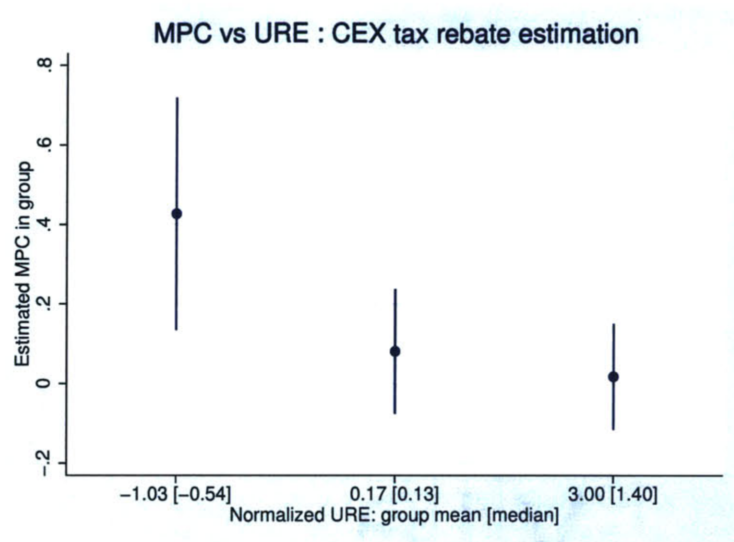
$$\begin{aligned}
\widehat{\mathcal{E}_r^{NR}} &= \frac{1}{J}\sum_{j=1}^{J} MPC_j NURE_j \\
\widehat{\mathcal{E}_r} &= \widehat{\mathcal{E}_r^{NR}} - \left(\frac{1}{J}\sum_{j=1}^{J} MPC_j\right)\left(\frac{1}{J}\sum_{j=1}^{J} NURE_j\right) \\
\widehat{S} &= 1 - \left(\frac{1}{J}\sum_{j=1}^{J} MPC_j\right)
\end{aligned}$$

where MPC_j is the point estimate in group j from (1.3.3).[31] In order to take into account sampling uncertainty, I compute the distribution of these estimators using a Monte-Carlo procedure, resampling the panel at the household level with replacement.

Figure 1.3.2 illustrates the procedure for $J = 3$, using expenditures on food as the headline consumption estimate. There is a clear gradient in MPC, with households with lower (and on average negative) URE displaying a much higher marginal propensity to consume, confirming my claim that $\mathcal{E}_r < 0$.

Table 1.3 repeats the exercise of table 1.2, where this time the moment estimation is done at the group and not the individual level. The quantitative results are large. Using food consumption as the source of MPC estimation in (1.3.3), σ^* is estimated to be 0.3, which is well within the range of typical values for the

[31] Note that I simply take $\widehat{S}$ to be the sample counterpart to $1 - \mathbb{E}_I[MPC]$. The procedure cannot simultaneously give an estimate of the covariance between MPC and consumption. In the SHIW data, the difference between average MPC and consumption-weighted MPC is small, so this is unlikely to significantly affect the value of σ^*.



The figure presents the estimated MPC, together with 95% confidence intervals, in each URE bin

Figure 1.3.2: MPC estimated in URE bins (JPS procedure, food consumption)

Consumption measure		Food		All nondurable	
Parameter		Estimate	95% C.I.	Estimate	95% C.I.
No-rebate elasticity	$\widehat{\mathcal{E}_r^{NR}}$	-0.12	[-0.27; 0.02]	-0.33	[-0.65; -0.02]
Redistribution elasticity	$\widehat{\mathcal{E}_r}$	-0.24	[-0.42; -0.07]	-0.64	[-0.97; -0.32]
Scaling factor	$\widehat{S}$	0.82	[0.69; 0.95]	0.56	[0.33; 0.78]
Equivalent EIS	$\widehat{\sigma^*} = -\frac{\widehat{\mathcal{E}_r}}{\widehat{S}}$	0.30	[0.05; 0.54]	1.15	[0.24; 2.07]

Confidence intervals are bootstrapped by resampling households 100 times with replacement

Table 1.3: Moments of the redistribution channel computed using the JPS procedure

EIS. Using all nondurable consumption instead, σ^* becomes as high as 1.15, suggesting that redistribution may even play a dominant role in the transmission of shocks to real interest rates. Even the no-rebate elasticity $\mathcal{E}_r^{NR}$ is negative, so that the negative correlation $\mathcal{E}_r$ is strong enough to overwhelm the effect of a positive aggregate URE. Appendix tables 1.8 and 1.9 also shows these results to be robust to using medians instead of means within URE bins, and to using any different number of bins J for 2 to 10. In every case, the correlation is very negative and the standard deviation of MPC is substantial. Even though the standard deviation of URE is only 1.27 in this dataset (table 1.1), these numbers combine to make σ^* large in magnitude and give the redistribution channel of monetary policy quantitative support.

1.3.5 Towards a calibrated general equilibrium model

In this section I proposed a quantification of the redistribution channel that operates through real interest rates. The main caveat is that this calculation is subject to a substantial degree of measurement error, as illustrated by the range of values for σ^* obtained across datasets and methods. Section 1.3.1 discussed why unhedged interest rate exposures are difficult to measure: in particular, it is known that consumption tends to be measured with error in surveys, and a precise attribution of financial stocks to flow interest rate exposures is difficult without many more details on the composition of wealth (in particular, asset duration) than is available in most surveys. Another issue, given the impossibility of observing individual households' actual plans for income and consumption over the long run, is that no empirical methodology can evaluate the redistribution channel induced by a *long-term* change in real interest rates, which requires knowledge of future URE terms.

Building a fully-specified general equilibrium model gives up the benefits of the sufficient statistic approach, but it addresses these shortcomings. It allows us to evaluate the interest-rate exposure channel based on a model-consistent measure of MPC, and to provide a cross-check on the reduced-form exercise of this section. In addition, it allows us to evaluate the other components of the redistribution channel identified in theorem 10, and it allows all these components to interact in determining macroeconomic responses to shocks. This is the goal of the next sections.

1.4 A Huggett model with long-term nominal assets

I now build a calibrated, general equilibrium model that features heterogeneity in unhedged interest rate exposures and marginal propensities to consume. Its preferences and market structure are of the Bewley-Huggett-Aiyagari class—a benchmark for both microeconomic and macroeconomic analyses of consumption behavior (see the surveys in Heathcote, Storesletten and Violante, 2009 and Attanasio and Weber, 2010), which can generate average marginal propensities to consume in line with the empirical evidence (Carroll, Slacalek and Tokuoka, 2014). I depart from the standard model by introducing assets that are nominal and have long maturities.

I choose a simple specification for the production side of the economy, just rich enough to illustrate how the forces highlighted in theorem 10 all interact in general equilibrium. Thus, I assume that labor is the only factor of production, with preferences such that there are no wealth effects on labor supply. This assumption has great modeling appeal, because it allows for a simple aggregation result for GDP. It may also be a reasonable description of preferences for the study of cyclical phenomena such as the ones I am interested in.[32] I also assume that all claims in the economy are pure circulating private IOUs, as in Huggett (1993). Two observations motivate this choice. First, in the Flow of Funds data of the United States, total household financial liabilities and interest-paying assets held directly are roughly balanced at any point in time (see Appendix 1.7.1.1). Second, the rates of return on these assets are directly influenced by changes

[32]While wealth effects on labor supply are important to understand long-run trends, and even though some papers do point towards diminished labor supply in response to receipts of inheritances (Holtz-Eakin, Joulfaian and Rosen, 1993) or lottery prizes (Imbens, Rubin and Sacerdote, 2001), I am not aware of evidence that these effects act over very short horizons. Most households are plausibly not on their short-run labor supply curve: one way to capture this would be to assume that rigidities prevent labor markets from clearing instantly. In a model where labor markets do clear, such as the one I build, it is then natural to specify the absence of wealth effects as stemming from preferences. This preference specification has received support in structurally estimated macroeconomic models (Schmitt-Grohé and Uribe, 2012). A model without income effects on labor supply in the presence of heterogeneity is also considered a good benchmark in several other contexts: analyses of the Mirrleesian model of income taxation (for example Diamond, 1998), and studies of capital taxation (Aiyagari, 1995) or redistribution with incomplete markets (Heathcote, 2005; Correia, 2010).

in monetary policy. In this model, therefore, changes in inflation or in real interest rates do not create an aggregate wealth effect on their own—only purely redistributive effects.

In order to focus on the role of individual-level heterogeneity in determining the aggregate response to shocks, I continue to consider equilibria that feature perfect foresight over macroeconomic aggregates and are perturbed by unexpected shocks. In this section I describe the model and properties of these equilibria for given initial distributions. I calibrate the model to the United States, and compute the moments of the redistribution channel identified in theorem 10. In section 1.5 I will consider transitional dynamics following shocks, and illustrate the quantitative importance of the redistribution channel for the determination of the natural rate of interest and for the aggregate effects of monetary policy shocks.

1.4.1 Environment

The model is an infinite-horizon economy with a continuum of ex-ante identical, but ex-post heterogenous households indexed by $i \in [0,1]$. Agents face idiosyncratic uncertainty with respect to their productivity $\{e_t^i\}$ and their discount factor $\{\beta_t^i\}$. The process for the idiosyncratic state $\mathbf{s}_t^i = (e_t^i, \beta_t^i)$ is uncorrelated across agents and follows a Markov chain $\Gamma(\mathbf{s}'|\mathbf{s})$ over time. This Markov chain is assumed to have a stationary distribution $\varphi(\mathbf{s})$, which is also the initial cross-sectional distribution of idiosyncratic states. There is no aggregate uncertainty: the path $\{\mathbf{S}_t\}$ for all macroeconomic variables is perfectly anticipated.

Household i has GHH preferences over the sequence $\{c_t^i, n_t^i\}$:

$$\mathbb{E}\left[\sum_{t=0}^{\infty} (\beta_t^i)^t u\left(c_t^i - v\left(n_t^i\right)\right)\right] \tag{1.4.1}$$

where the outside felicity function u is such that the elasticity of intertemporal substitution in net consumption is a constant σ, and the disutility function v over working hours has constant elasticity ψ,

$$u(g) = \frac{g^{1-\sigma^{-1}}}{1-\sigma^{-1}} \quad v(n) = b\frac{n^{1+\psi^{-1}}}{1+\psi^{-1}}$$

where b is a constant.

The final good that enters consumers' utility is produced with a technology

$$Y_t = \left[\int_0^1 \left(x_t^j\right)^{\frac{\epsilon-1}{\epsilon}} dj\right]^{\frac{\epsilon}{\epsilon-1}}$$

where x_t^j is the quantity of intermediate good $j \in [0,1]$ used as input and $\epsilon > 1$ is the constant elasticity of substitution across goods. All intermediate goods are produced with an identical linear technology

$$x_t^j = A_t l_t^j \tag{1.4.2}$$

where $l_t^j = \int_i e_t^i n_t^{i,j} di$ is the number of efficiency units of work entering the production of good j (with $n_t^i = \int_j n_t^{i,j} dj$) and A_t is aggregate productivity.

1.4.2 Markets and government

Households. Households only have access one type of nominal, risk-free, long-term bond with rate of decay δ_N (see section 1.1.3). They are subject to an affine tax schedule on labor income. Each period, a household with productivity e_t^i seeks to maximize (1.4.1) subject to the budget constraint

$$P_t c_t^i + Q_t \left(\Lambda_{t+1}^i - \delta_N \Lambda_t^i\right) = (1-\tau) W_t e_t^i n_t^i + P_t T_t + \Lambda_t^i$$

where P_t is the nominal price of the final good, Q_t the nominal price of a bond paying the sequence of coupons $\left(1, \delta_N, \delta_N^2, \ldots\right)$ starting in period $t+1$, Λ_t^i the nominal coupon payment due at time $t+1$, τ the marginal tax rate on labor income, W_t the nominal market wage per efficient unit of work, and T_t a real lump-sum transfer from the government, common across individuals.

A borrowing constraint further limits the size of bond issuances so that the market value of real end-of-period liabilities is bounded below by a limit $\overline{D}_t$ at time t:

$$Q_t \Lambda_{t+1}^i \geq -\overline{D}_t P_t \tag{1.4.3}$$

Maximization with respect to n_t^i yields a static first-order condition for hours supplied:

$$n_t^i = \left[\frac{1}{b} e_t^i (1-\tau) \frac{W_t}{P_t}\right]^{\psi} = \left[\frac{1}{b} e_t^i w_t\right]^{\psi} \tag{1.4.4}$$

where $w_t \equiv (1-\tau) \frac{W_t}{P_t}$ is the post-tax real market wage.

Define i's *net consumption* g_t^i and the net income function z as

$$g_t^i \equiv c_t^i - v\left(n_t^i\right) \qquad z(e, w) \equiv \max_{\hat{n}} \left\{w \cdot e \cdot \hat{n} - v(\hat{n})\right\}$$

The consumer's idiosyncratic state is summarized by his real bond position $\lambda_t^i \equiv \frac{\Lambda_t^i}{P_{t-1}}$. From his point of view, the relevant components of the aggregate state are $\left(w_t, T_t, Q_t, \Pi_t, \overline{D}_t\right) \subseteq \mathbf{S}_t$, where $\Pi_t = \frac{P_t}{P_{t-1}}$ denotes the inflation rate at t. His optimization problem is characterized by the Bellman equation:

$$\begin{aligned} V_t(\lambda, \mathbf{s}) = \quad & \max_{g,\lambda'} \quad u(g) + \beta(\mathbf{s}) \mathbb{E}\left[V_{t+1}(\lambda', \mathbf{s}') \,|\mathbf{s}\right] \\ & \text{s.t.} \quad g + Q_t \left(\lambda' - \delta_N \frac{\lambda}{\Pi_t}\right) = z(e(\mathbf{s}), w_t) + T_t + \frac{\lambda}{\Pi_t} \\ & \qquad\; Q_t \lambda' \geq -\overline{D}_t \end{aligned} \tag{1.4.5}$$

Proposition 17. *The consumption policy function $c_t(\lambda, \mathbf{s})$ is concave in bond holdings λ, and strictly concave for λ sufficiently high that the borrowing constraint does not bind.*

Proposition 17, which follows from a result of Carroll and Kimball (1996),[33] implies that the marginal propensity to consume is declining in bonds held, generating a natural link between MPCs and asset positions which is key to the redistribution channel.

Firms. The final good is produced by a perfectly competitive firm, which takes as given the prices $\left\{P_t^j\right\}$ of intermediate goods. Profit maximization leads to a final good price of P_t, zero profits (so that it is not

[33] See Appendix 1.7.4 for the proofs to all propositions of this section.

necessary to be specific about the firm's ownership), and isoelastic demand for intermediate goods:

$$x_t^j = \left(\frac{P_t^j}{P_t}\right)^{-\epsilon} Y_t \quad \text{where} \quad P_t \equiv \left[\int_0^1 \left(P_t^j\right)^{1-\epsilon} dj\right]^{\frac{1}{1-\epsilon}} \tag{1.4.6}$$

Labor markets are competitive and wages are fully flexible. Every intermediate goods firm $j \in [0,1]$ produces under monopolistic competition. The firm's nominal profits in period t, when its current price is p, are given by

$$F_t^j(p) = px_t^j(p) - W_t l_t^j(p) = \left(p - \frac{W_t}{A_t}\right)\left(\frac{p}{P_t}\right)^{-\epsilon} Y_t \tag{1.4.7}$$

I consider two assumptions on price setting. When prices are flexible, producers set them in each period and state to maximize (1.4.7). This results in an identical price across all firms in each period t, equal to

$$P_t = P_t^j = \frac{\epsilon}{\epsilon - 1}\frac{W_t}{A_t} \tag{1.4.8}$$

I also consider the opposite assumption of perfectly sticky prices. This is meant to capture the macroeconomic adjustments that take place under nominal rigidities in the simplest possible way; under the shocks I consider, incentives to change prices vanish in the long run. Under sticky prices, firm j has a price P_t^j at time t and cannot change it. It accommodates the demand x_t^j that is forthcoming at that price by hiring workers at the going real wage, and its profits are determined by $F_t^j(P_t^j)$.

Fiscal policy. To simplify the treatment of firm ownership given the double heterogeneity of consumers and firms, I assume that the government owns all the firms. Each period, it collects their nominal profits and runs the personal income tax system. Moreover, it maintains a strict balanced budget every period, and therefore sets the lump-sum transfer equal to total collections:

$$P_t T_t = \int_j F_t^j(P_t^j) + \tau \int_i W_t e_t^i n_t^i di \tag{1.4.9}$$

1.4.3 Aggregation and analysis

This section present several aggregation results that obtain when consumers choose hours worked optimally and markets for intermediate goods, final goods, and labor clear. The latter two conditions are expressed as

$$C_t \equiv \mathbb{E}_I\left[c_t^i\right] = Y_t \qquad \int_j l_t^j dj = \int_i e_t^i n_t^i di \tag{1.4.10}$$

Proposition 18. *When households optimally choose labor supply and intermediate-goods and labor markets clear, (per capita) GDP is equal to*

$$Y_t = \frac{1}{\triangle_t} A_t N(w_t) \tag{1.4.11}$$

where $N(w_t) \equiv \kappa w_t^{\psi}$ *is the total supply of effective hours,* $\kappa \equiv \frac{1}{b^{\psi}}\mathbb{E}\left[e^{1+\psi}\right]$ *is a constant, and* $\triangle_t \equiv \int_{j=0}^1 \left(\frac{P_t^j}{P_t}\right)^{-\epsilon} dj \geq 1$ *is a measure of price dispersion.*

Proposition (18) illustrates the simplicity of aggregation in this model, despite the heterogeneity of consumers and firms. GDP is only a function of three parameters: technology A_t, a summary measure of heterogeneity in production outcomes $\triangle_t$, and aggregate labor supply which depends only on the net-of-tax real wage w_t.

Define gross nonfinancial income Y_t^i as i's real earnings inclusive of the lump-sum tax, and net financial income by subtracting the disutility of labor supply from gross income:

$$Y_t^i \equiv w_t e_t^i n_t^i + T_t \qquad Z_t^i \equiv z\left(e_t^i, w_t\right) + T_t$$

Proposition 19. *When households optimally choose labor supply and intermediate-goods and labor markets clear, the tax intercept share of GDP is:*

$$\frac{T_t}{Y_t} = \widetilde{\tau}_t$$

where $\widetilde{\tau}_t$ is the labor wedge, *a summary measure of the economy's distortions:*

$$1 - \widetilde{\tau}_t = \frac{w_t}{A_t}\triangle_t = (1-\tau)\frac{W_t}{P_t A_t}\triangle_t \tag{1.4.12}$$

Moreover, households i's total nonfinancial gross and net incomes as a share of per capita GDP are:

$$\frac{Y_t^i}{Y_t} = (1-\widetilde{\tau}_t)\frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]} + \widetilde{\tau}_t \qquad \frac{Z_t^i}{Y_t} = \frac{1-\widetilde{\tau}_t}{1+\psi}\frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]} + \widetilde{\tau}_t \tag{1.4.13}$$

Note that $\mathbb{E}_I\left[Y_t^i\right] = Y_t$. The balanced-budget and full profit taxation assumptions therefore lead to a very simple link between the labor wedge and the progressivity of the tax system in this model. Fundamental inequality in productivity e_t^i translates into inequality in pre-tax earnings $\left(e_t^i\right)^{1+\psi}$. As $\widetilde{\tau}_t$ varies from zero to one, agents' relative incomes alternate between their fundamental level and the one arising under perfect equality. A higher $\widetilde{\tau}_t$, in turn, indicates a more distorted economy—which may result from high tax rates, high monopoly power or a negative output gap (a recession) under sticky prices.[34]

Definition 20. The *moments of the redistribution channel* in the model are defined as follows. Gross-of-tax income-weighted MPC $\mathcal{M}_t^g$ and (net-of tax) income-weighted MPC $\mathcal{M}_t$ are

$$\mathcal{M}_t^g \equiv \mathbb{E}_I\left[\frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]}MPC_t^i\right] \qquad \mathcal{M}_t \equiv \mathbb{E}_I\left[\frac{Y_t^i}{Y_t}MPC_t^i\right]$$

The redistribution elasticities with respect to the labor wedge $\widetilde{\tau}$, the price level P, and the real interest rate r are

$$\mathcal{E}_{\tau,t} \equiv -\mathrm{Cov}_I\left(MPC_t^i, \frac{Y_t^i}{Y_t}\right) \quad \mathcal{E}_{P,t} \equiv -\mathrm{Cov}_I\left(MPC_t^i, \frac{NNP_t^i}{\mathbb{E}_I\left[c_t^i\right]}\right) \qquad \mathcal{E}_{r,t} \equiv \mathrm{Cov}_I\left(MPC_t^i, \frac{URE_t^i}{\mathbb{E}_I\left[c_t^i\right]}\right)$$

Note that the average MPC is $\mathbb{E}_I\left[MPC_t^i\right] = \mathcal{M}_t + \mathcal{E}_{\tau,t}$. Finally, the share of net in gross consumption and the Hicksian scaling factor are

$$\xi_t^i \equiv 1 - \frac{v\left(n_t^i\right)}{c_t^i} \qquad S_t \equiv \mathbb{E}_I\left[\xi_t^i\left(1 - MPC_t^i\right)\frac{c_t^i}{\mathbb{E}_I\left[c_t^i\right]}\right]$$

All of these cross-sectional moments can be directly computed from the policy functions in an equilibrium.

[34] In a flexible-price steady-state, $\widetilde{\tau}_t = \tau^*$ is a constant and Y_t is declining τ^* (see proposition 23). This implies a simple tradeoff between efficiency and equity, which one can resolve by setting the steady-state tax rate τ.

In particular,

$$MPC_t^i = \frac{\Pi_t}{1+\delta_N Q_t}\frac{\partial c_t\left(\lambda, \mathbf{s}^i\right)}{\partial \lambda} \qquad URE_t^i = Y_t^i + \frac{\lambda_t^i}{\Pi_t} - c_t^i \qquad NNP_t^i = \left(1+Q_t\delta_N\right)\frac{\lambda_t^i}{\Pi_t}$$

Because of the concavity of the consumption function, we expect $\mathcal{E}_{P,t} > 0$ and $\mathcal{E}_{r,t} < 0$ provided that all assets and liabilities are short term ($\delta_N = 0$), since in this case URE_t^i is simply $Q_t\lambda_{t+1}^i$. In the steady-state calibration we will see that $\mathcal{E}_r < 0$ also holds for longer maturities—consistent with the reduced-form analysis presented in section 1.3. A key result, however, will be that the absolute value of $\mathcal{E}_r$ is declining in δ_N. The cross-sectional moments from definition 20 are useful to predict the response of aggregate consumption to macroeconomic shocks that last for one period, as the following proposition (a version of theorem 10 in the model) indicates.

Proposition 21 (Response to one-time shocks in the model). *Assume that final goods, intermediate goods, and labor markets initially clear. Consider a shock at t that changes dY_t, $d\widetilde{\tau}_t$, dR_t and dw_t for one period only, and revises all future prices by dP_t. Then aggregate consumption changes by approximately*

$$\frac{dC_t}{C_t} \simeq \mathcal{M}_t\frac{dY_t}{Y_t} + \mathcal{E}_{\tau,t}\frac{d\widetilde{\tau}_t}{1-\widetilde{\tau}_t} + \mathcal{E}_{P,t}\frac{dP_t}{P_t} + \left(\mathcal{E}_{r,t} - \sigma S_t\right)\frac{dR_t}{R_t} + \psi\left(1-\mathcal{M}_t^g\right)\left(1-\widetilde{\tau}_t\right)\frac{dw_t}{w_t} \tag{1.4.14}$$

Moreover, provided that the tax rate τ is kept constant:

$$\frac{d\widetilde{\tau}_t}{1-\widetilde{\tau}_t} = -\frac{dw_t}{w_t} + \frac{dA_t}{A_t} - \frac{d\triangle_t}{\triangle_t}$$

and provided that the current account remains closed:

$$\frac{dC_t}{C_t} = \frac{dY_t}{Y_t} = \frac{dA_t}{A_t} + \psi\frac{dw_t}{w_t} - \frac{d\triangle_t}{\triangle_t}$$

Each of the term in (1.4.14) corresponds to one of the channels highlighted in (1.2.9). In sections 1.5.4-1.5.6, I will apply the proposition to evaluate the accuracy of the theorem's prediction relative to the full nonlinear solution to the response of shocks in the calibrated model.

1.4.4 Equilibrium and steady-state with flexible prices

Define the real interest rate between t and $t+1$ as

$$R_t = \frac{1+\delta_N Q_{t+1}}{Q_t \Pi_{t+1}} \tag{1.4.15}$$

When prices are fully flexible, R_t is determined in equilibrium. Through its influence on the nominal bond price Q_t, the monetary authority controls the path of the inflation rate Π_t directly (for example, pinning it down through a Taylor rule). All price level changes are perfectly anticipated by households and firms, and have no effect on real variables. The following defines a flexible-price equilibrium:

Definition 22. Given an initial distribution $\Psi_0(\mathbf{s},\lambda)$ over idiosyncratic states and bond positions, an initial price level P_0, and paths for productivity $\{A_t\}$, inflation $\{\Pi_t\}$ and borrowing limits $\left\{\overline{D}_t\right\}$, a flexible-price equilibrium is a sequence of consumption rules $\{c_t(\mathbf{s},\lambda)\}$, next-period bond choices $\{\lambda_{t+1}(\mathbf{s},\lambda)\}$, distributions $\{\Psi_t(\mathbf{s},\lambda)\}$ and aggregate prices $\{R_t, Q_t, P_t, W_t, w_t, \triangle_t\}$ and quantities $\{N_t, C_t, Y_t, T_t\}$ such that:

consumers optimally choose hours worked according to (1.4.4) and make a dynamic consumption and bond purchase decision consistent with (1.4.5), final and intermediate-goods firms maximize profits, leading to (1.4.8), the price level evolves according to $P_t = \Pi_t P_{t-1}$ and bond prices according to (1.4.15), the government's budget constraint (1.4.9) is satisfied, markets for labor, intermediate goods, final output and bonds clear:

$$C_t \equiv \int c_t(\mathbf{s}, \lambda)\, d\Psi_t(\mathbf{s}, \lambda) = Y_t \quad Q_t \int \lambda_{t+1}(\mathbf{s}, \lambda)\, d\Psi_t(\mathbf{s}, \lambda) = 0 \tag{1.4.16}$$

and the evolution of the bond distribution is consistent with $\{\lambda_{t+1}(\mathbf{s}, \lambda)\}$.

In equilibrium, using the results in section 1.4.3, the following proposition holds:

Proposition 23. *When prices are flexible, the labor wedge is constant*

$$\widetilde{\tau}_t = \tau^* \equiv 1 - (1-\tau)\frac{\epsilon - 1}{\epsilon} \quad \Delta_t = 1 \quad \forall t$$

Moreover, output—and therefore aggregate consumption—is entirely determined by current productivity, the degree of monopoly power, and the tax system:

$$Y_t = \kappa (1-\tau^*)^{\psi} A_t^{1+\psi} \quad \forall t$$

A corollary of proposition 23 is that (unexpected) redistributive policies—such as targeted lump-sum transfers from one group of agents to another, or inflationary shocks that erode the real value of debts and assets—do not affect aggregate output. They do, however, change *relative* consumption and welfare levels, as well as the market-clearing real interest rate, in a way that depends on the strength of the redistribution channel as explored in section 1.5.4. This result can be viewed as a useful benchmark, highlighting the importance of general equilibrium when thinking through the aggregate effects of redistributive policy.

I define a steady-state as a flexible-price equilibrium with constant productivity A, debt limit $\overline{D}$ and inflation Π, attaining a constant real interest rate R^* and a stationary distribution for bonds $\Psi(\mathbf{s}, \lambda)$. The steady-state has the following important property:

Proposition 24 (Invariance of steady-state to the maturity structure). *Two economies that differ only in their maturity structure of financial assets and liabilities δ_N attain the same steady-state interest rate R^*, with the same joint distribution over bond market values $b = Q\lambda$ and idiosyncratic states* $\mathbf{s}$.

The logic behind this proposition is simple. When agents face a constant term structure of interest rates R^*, short and long-term assets span the same set of contingencies, and a constant borrowing limit specification (1.4.3) is is also neutral with respect to maturity. Crucially for my argument, unhedged interest rate exposures do vary with δ_N. Changing δ_N allows us to change asset durations $\frac{\Pi R}{\Pi R - \delta_N}$ and the strength of the interest rate exposure channel without changing any of the other steady-state properties of the model.

1.4.5 Equilibrium with fully sticky prices

When prices are fully sticky, the inflation rate is constant at $\Pi_t = 1$, and the monetary authority, by changing the nominal interest rate, controls the real interest rate R_t directly.

Definition 25. Given an initial distribution $\Psi_0(\mathbf{s}, \lambda)$ over idiosyncratic states and bond positions, an initial distribution for prices $\left\{P_0^j\right\}$, and paths for productivity $\{A_t\}$, the real interest rate $\{R_t\}$ and borrowing limits $\{\overline{D}_t\}$, a sticky-price equilibrium is a sequence of consumption rules $\{c_t(\mathbf{s}, \lambda)\}$, next-period bond choices $\{\lambda_{t+1}(\mathbf{s}, \lambda)\}$, distributions $\{\Psi_t(\mathbf{s}, \lambda)\}$ and aggregate variables $\{\Pi_t, Q_t, P_t, W_t, w_t, N_t, C_t, Y_t, T_t, \Delta_t\}$ such that: consumers optimally choose hours worked according to (1.4.4) and make a dynamic consumption and bond purchase decision consistent with (1.4.5), final goods firm maximize profits and intermediate-goods firms satisfy demand at prices $\left\{P_0^j\right\}$, the price level is a constant $P_t = P_0$, $\Pi_t = 1$, bond prices evolve according to (1.4.15), the government's budget constraint (1.4.9) is satisfied, markets for labor, consumption and bonds clear as in (1.4.16), and the evolution of the bond distribution is consistent with $\{\lambda_{t+1}(\mathbf{s}, \lambda)\}$.

In a fully sticky-price equilibrium, when the central bank temporarily and unexpectedly sets a real interest rate R_t below its "natural" level that prevails under flexible prices R_t^*, aggregate demand increases due to both a substitution and a redistribution effect, and firms accommodate by producing more. Price dispersion Δ_t is fixed at its initial level Δ_0. According to (1.4.11), an increase in production Y_t requires an increase in the real wage w_t. This leads to a fall in the labor wedge (1.4.12), and therefore to a fall in the progressivity of the tax system.[35] In section 1.5.5 I will discuss how all these macroeconomic changes interact with the heterogeneity in determining the aggregate effect of monetary policy shocks.

1.5 Model-based evaluation of the redistribution channel

1.5.1 Steady-state calibration and solution method

I perform my calibration at quarterly frequency. I target an annual equilibrium real interest rate of 3% and a household debt/PCE ratio 113%—the U.S. level for 2013, which in that year is virtually equal to the stock of interest-paying assets held by the household sector.[36] I also target an average asset duration of 4.5 years. This is an average of the durations of U.S. household assets and liabilities reported by Doepke and Schneider (2006a) at the end of their data sample (see their figure 3). I consider a flexible-price steady-state with no inflation: $\Pi = 1$. These considerations imply a choice of $\delta_N = 0.95$.[37]

In order to discipline the micro-level facts, I use the 2009 wave of the Panel Study of Income Dynamics (PSID). I aim to obtain, in equilibrium, asset dispersions that match that those observed in that survey, and

[35]Note that this result is not associated with a traditional automatic stabilizer logic. During booms, firm profits are low since their prices are too low; since the government collects these profits and runs a balanced budget, these low profits diminish lump-sum transfers to households. Hence current booms—low wedges —exacerbate relative income inequality and current recessions mitigate it.

[36]Sources: NIPA and U.S. Financial Accounts. See Appendix 1.7.1.1 for details.

[37]The duration of the nominal bond is $\frac{R\Pi}{R\Pi - \delta_N} = \mathcal{D}$, so $\frac{\delta_N}{\Pi} = R\left(1 - \frac{1}{\mathcal{D}}\right)$.

Parameter	Description	Value
β^I	Impatient discount factor	0.95
β^P	Patient discount factor	0.993
ψ	Elasticity of labor supply	1
σ	Elasticity of substitution in net consumption	0.5
δ_N	Asset/liability coupon decay rate	0.95
$\overline{D}$	Borrowing limit (% of annual PCE per capita)	185%

Table 1.4: Calibration parameters

to bring consumers' marginal propensities to consume in the model in line with the empirical evidence of papers such as Johnson et al. (2006), Parker et al. (2013) and Baker (2014). I target an average quarterly marginal propensity to consume of 0.25—which is close to a consensus number from the empirical literature. I achieve these joint objectives through a combination of relatively tight borrowing limits ($\overline{D} = 7.4$, or 185% of per capita annual consumption, when the natural borrowing limit is over 1300%) and a preference process where agents alternate between patience (discount factor β^P) and impatience (discount factor β^I). I specify that the stationary population distribution must contain patient and impatient agents in equal numbers, and that consumers stay in their patience state for 50 years on average. This process is meant to capture slow-moving preference heterogeneity. It is similar to the one which Krusell and Smith (1998) found useful to match the wealth distribution and which Carroll et al. (2014) used to generate high marginal propensities to consume on average in the population.

Table 1.4 summarizes my benchmark parameters. Since my calibration for the income process targets pre-tax earnings rather than productivity, the elasticity of labor supply ψ does not play a major role in the flexible price version of my model. However, it does play a role in determining the response of real wages to monetary policy shocks in the sticky price version. Since the structural vector autoregression evidence (for example Christiano, Eichenbaum and Evans (2005)) is for a muted response, I calibrate ψ on the high end of what standard estimates from analyses of panel data imply, and set $\psi = 1$. I set the elasticity of intertemporal substitution in net consumption to $\sigma = 0.5$, which is well within the range of typical calibrations. Since I am ultimately interested in comparing the substitution and the redistribution channel, this allows me a priori not to stack the cards against the substitution channel.[38]

The mix of labor income tax τ and the elasticity of substitution between goods ϵ is irrelevant, conditional on the labor wedge τ^*. I calibrate this wedge jointly with the earnings process. I then normalize GDP and average hours to 1 per quarter, which allows me to calibrate A and b given τ^*, ψ and the process for e.[39] Further details on the calibration and the numerical solution technique are provided in appendix 1.7.5.

		Steady-state value	
		$\delta_N = 0.95$	$\delta_N = 0$
Redistribution elasticity for r	$\mathcal{E}_r$	-0.09	-1.76
Hicksian scaling factor	S	0.57	
Equivalent EIS	$\sigma^* = -\frac{\mathcal{E}_r}{S}$	0.15	3
Total partial elasticity to r	$\mathcal{E}_r - \sigma S$	-0.38	-2.05
Average MPC	$\mathbb{E}_I\left[MPC^i\right]$	0.25	
Average MPC (net-income weighted)	$\mathcal{M}$	0.16	
Average MPC (gross-income weighted)	$\mathcal{M}^g$	0.09	
Redistribution elasticity for P	$\mathcal{E}_P$	1.77	
Redistribution semi-elasticity for $\widetilde{\tau}$	$\mathcal{E}_\tau/(1-\tau^*)$	0.16	

Table 1.5: Calibration outcomes for the steady-state moments of Proposition 21

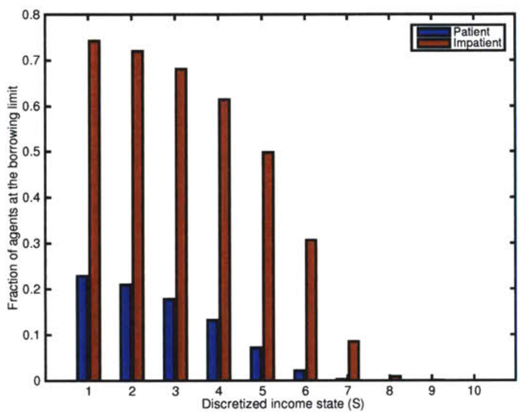


Figure 1.5.1: Share of constrained agents

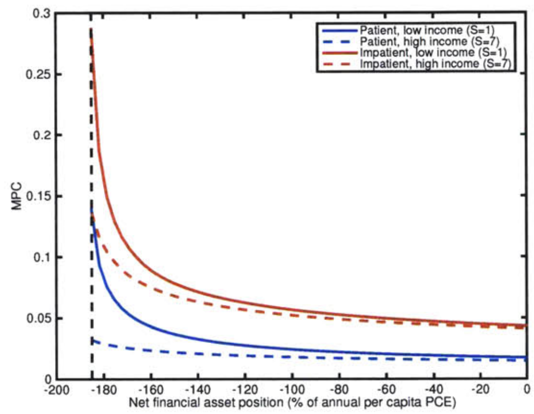


Figure 1.5.2: MPCs by asset and state

1.5.2 Calibration outcomes

Table 1.5 displays the model statistics from definition 20, which proposition 21 showed to be important to analyze the redistribution channel. The average MPC is 0.25 per quarter, which is in line with the empirical evidence. 22% of borrowers are at their borrowing limit, so that it is important to understand their behavior when interest rates change. The model generates a large dispersion in MPCs, with some agents with high cash-on-hand only slightly above the permanent-income level (below 0.01), and many constrained agents with much larger MPCs (figure 1.5.1).[40]

Given the long maturities, the standard deviation of unhedged interest rate exposures is moderate. The

[38]The size of the redistribution channel, however, depends endogenously on σ in this model since the EIS determines the sizes of asset positions that agents are ready to take; so in equilibrium higher σ is associated with both a higher substitution channel—σS in Definition 20—and a higher redistribution channel ($\mathcal{E}_r$)

[39]This involves backing A and b from $Y = (1-\tau^*)^\psi \mathbb{E}\left[e^{1+\psi}\right] \frac{A^{1+\psi}}{b^\psi} = 1$ and $\frac{\mathbb{E}[n]}{Y} = \frac{\mathbb{E}\left[e^\psi\right]}{\mathbb{E}\left[e^{1+\psi}\right]A} = 1$

[40]Note that while the MPC of agents exactly at the borrowing limit is equal to 1 by definition, the Lagrange multiplier on their borrowing constraint is small enough that even a small positive transfer (lower than the amount it takes to move them to the next point on the asset grid) leads them to start smoothing substantially, as their MPC falls discontinuously to a number below 0.5. In other words, the large persistence in earnings creates an incentive for households to smooth non-infinitesimal positive shocks to income. This has an important implication for the asymmetry of the reaction to positive and negative shocks to interest rates.

redistribution elasticity $\mathcal{E}_r = -0.09$ has a magnitude which is comparable to the one I obtained in the SHIW, and below that I obtained in the CEX. However, if we change maturities to be much shorter ($\delta_N = 0$, which is the short-term debt typically modeled with a duration of 1 quarter), this elasticity rises dramatically, much above the assumed level of the EIS. This important finding has implications for the determination of the natural rate of interest under flexible prices (section 1.5.4) and for the effects of monetary policy changes (section 1.5.5).

One unexpected feature of the model comes from the elasticity of aggregate demand with respect to rises in the price level, $\mathcal{E}_P = 1.77$. This number implies a very powerful redistribution through the Fisher channel. There is a strong intuition for this result. Inflation redistributes along the asset dimension, which in this class of models is highly correlated with MPC (see proposition 17). On the other hand, incomes and MPCs are much less correlated (their covariance in the model is -0.09): agents can smooth shocks to income quite well, and MPCs are driven by preference heterogeneity much more than by income heterogeneity. Figure 1.5.2 illustrates these statements in the context of the calibration. The gradient is clearly much higher with respect to asset level than it is with respect to income (moving from solid to dashed lines). As I discuss in section 1.5.4, this implies that targeted income-redistributive policies have less demand impact than price level changes in this world with nominal assets. First, I need to discuss how borrowing limits adjust away from steady-state.

1.5.3 Transition after shocks and borrowing constrained agents

In the following sections I will compute transitional dynamics starting from steady-state. Following the logic of section 1.1, I compute the impulse-response, perfect-foresight path for macroeconomic aggregates in deviations from steady-state, following an unexpected shock and holding asset positions fixed. The channels illustrated in theorem 10 all play a role. I illustrate the comparison between economies that vary in terms of the strength of their interest rate exposure channels by comparing the benchmark U.S. calibration to the calibration with short maturities only.

An important question in computing transitional dynamics is how borrowing limits adjust. A natural choice for borrowing limits is one that holds the *real coupon payment in the next period fixed*:

$$\overline{D_t} = Q_t \overline{d} \quad \forall t \tag{1.5.1}$$

The borrowing limit can then be written

$$\lambda_{t+1} \geq -\overline{d}$$

and is equivalent to a restriction on *flow payments* in the next period (which we can think of mortgage payments including amortization when durations are long), as opposed to the present value of liabilities, as would be implied if $\overline{D_t}$ was not adjusting following changes in Q_t.

In addition to being a natural one, the specification of the adjustment process in (1.5.1) implies that proposition 21 holds exactly, including for agents at a binding borrowing limit. It is crucial to understand how these agents are affected depending on the maturity of the debt in the economy, δ_N. For simplicity, assume that the inflation rate is constant at $\Pi_t = 1$, so that nominal and real interest rates are equal. Consider an agent with income y_t^i who maintains himself at the borrowing limit in an initial steady-state where the real interest rate is R and the bond price is constant at $Q = \frac{1}{R-\delta_N}$. His consumption is equal to his income, minus the interest payment on the value of the borrowing limit $\overline{D} = Q\overline{d}$.

$$c_t^i = Y_t^i - \overline{D}(R-1)$$

Across economies with different debt maturities δ_N, $\overline{D}$ is a constant, so that the steady-state payments are the same, but the exposure of these payments to real interest rate changes differ. Indeed we can decompose:

$$\overline{D}(R-1) = (R-\delta_N)\overline{D} - \overline{D}(1-\delta_N) = \overline{d} + \underline{URE}$$

where $\overline{d} = (R-\delta_N)\overline{D}$ is the part that is precontracted and $\underline{URE} = -\overline{D}(1-\delta_N)$ the part that is subject to interest changes. Hence, economies with different δ_N involve very different levels of unhedged interest rate exposures for borrowing-constrained agents, ranging from the full principal $-\overline{D}$ when $\delta_N = 0$ to none when $\delta_N = 1$. In the benchmark calibration with $\delta_N = 0.95$, the minimum income level of agents is $Y^i = 0.413$, $\overline{d} = 0.413$, and $\underline{URE} = -0.358$. In other words, these highly indebted agents use their full income for interest payments and amortization, and then borrow to maintain their consumption level, so that their effective interest payments are $(R-1)\overline{D} = 0.055$. On the other hand, in the ARM calibration, $\overline{d} = 7.455$ and $\underline{URE} = -7.4$.

Suppose that current and future real interest rates increase, so that Q_t falls, and suppose the change is small enough that it does not alter the agent's decision to stay at the borrowing limit. His new consumption level is then

$$\begin{aligned} c_t^{i*} &= Y_t^i - \left(\overline{d} + \frac{Q_t}{Q}\underline{URE}\right) \\ &= Y_t^i - \overline{D}(R-1) - \left(\frac{Q_t - Q}{Q}\right)\underline{URE} \end{aligned}$$

For short-term changes, $\frac{Q_t-Q}{Q} \simeq -\frac{dR}{R}$ and we obtain the prediction from theorem 21, $c_t^{i*} - c_t^i \simeq \underline{URE}\frac{dR}{R}$, which is largest in an economy with $\delta_N = 0$.[41]

1.5.4 The redistribution channel and the natural rate of interest

Proposition 23 made clear that under flexible prices, aggregate output in the model is only a function of current technological parameters and taxes. The real interest rate provides the adjustment mechanism that brings demand back in line with supply in response to shocks that redistribute income through taxation or real wealth through inflation. This section expands upon the examples provided in example 13 in the context of a full dynamic determination of the natural rate of interest, and explores the quantitative ability of the sufficient statistics provided given in Table 1.5 to predict the first-order response to small shocks.

[41] Note that for changes in real interest rates that last for more than a period, there is a larger change in Q_t in an economy with long-term debt, so less debt is rolled over but the price is more affected, offsetting part of the first effect.

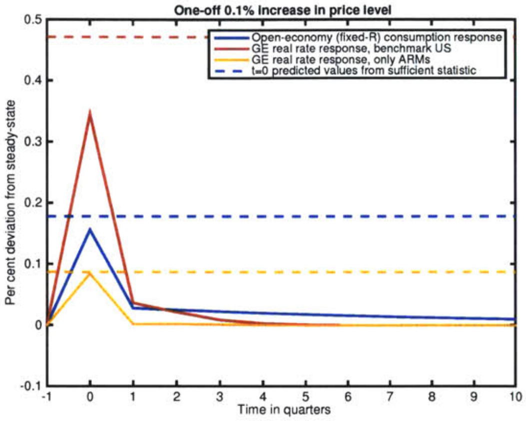


Figure 1.5.3: Price level shock $\frac{dP}{P} = 0.1\%$

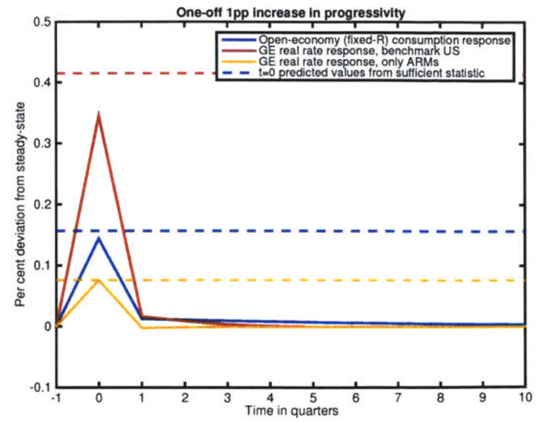


Figure 1.5.4: Tax progressivity $d\tau = 1\text{pp}$

I look at two classes of shocks that act purely through redistribution. The first one is a 0.1% one-off increase in the price level that redistributes via the Fisher channel.[42] The second is a 1 percentage point effective rise in the progressivity of the tax system, from $\tau^* = 40\%$ to 41%, which is entirely performed through lump-sum redistribution and therefore does not affect work incentives. The steady-state moments computed in section 1.5.2 predict that these two shocks should have approximately the same effect on the equilibrium real interest rate, because (to first order) they have the same aggregate demand effect.

Figures 1.5.3 and 1.5.4 plots the dynamic path of adjustment of the economy to both shocks. As a benchmark, the blue line plots the consumption response in a hypothetical open economy where the real interest rate did not change. Dotted lines correspond to the first-order approximations of these responses using the sufficient statistics approach. While both shocks are fairly small, they nevertheless translate into a fairly substantial 0.17% aggregate consumption effect in partial equilibrium. In general equilibrium, the real interest rate adjusts more in the benchmark calibration than in the US calibration, illustrating again the proposition presented in example 13. Moreover, the first-order approximation gets responses that are close to correct relative to the nonlinear solution, in particular for the demand response in partial equilibrium. Methodologically, these results therefore illustrate the usefulness of the sufficient statistic approach in predicting general equilibrium effects. Theoretically, they stress the importance of the redistribution channel, and of the maturity structure, in determining the natural rate of interest.

1.5.5 The redistribution channel and monetary policy shocks

I now consider the way in which the redistribution channel alters the effects of monetary policy shocks. Starting from the steady-state, I assume that the central bank unexpectedly lowers the real interest rate, and then lets it gradually return to its steady-state level with the following path:

[42] This shock is similar in spirit to that performed by Doepke and Schneider (2006b), but I consider a special class of models in which I know GDP stays constant. I focus attention on a small shock to evaluate the accuracy of my sufficient statistics; section 1.5.6 explores the case of large shocks.

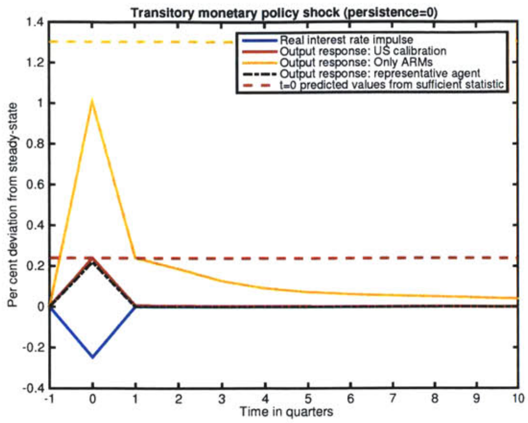


Figure 1.5.5: M.P. shock ($\rho_R = 0$)

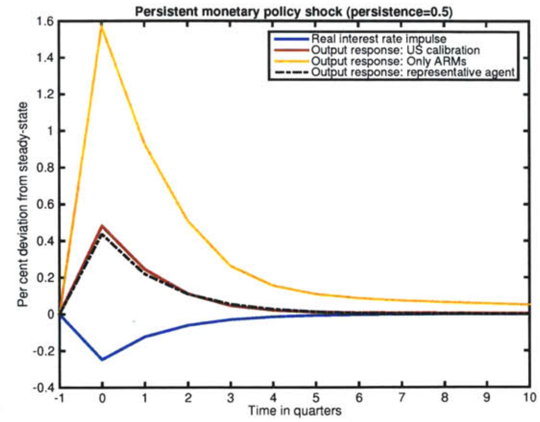


Figure 1.5.6: M.P. shock ($\rho_R = 0.5$)

$$R_t - R^* = \rho_R \left(R_{t-1} - R^*\right)$$

This corresponds to the typical *monetary policy shock* that is analyzed in the New Keynesian literature, for the special case in which prices are fully sticky.

Figures 1.5.5 and 1.5.6 show the effect on aggregate GDP of a transitory and a persistent monetary policy shock. The initial impulse is a 100 annualized basis points fall in the nominal interest rate, which translates into a fall in the real interest rate (the light blue line) since prices are fully sticky. In the benchmark U.S. calibration (red line), the response of the economy to a monetary policy shock is slightly above that predicted by representative-agent version of the model, consistent with a redistribution channel amplifying the effects of real interest rates changes. However, while $\sigma^* = 0.15$ and $\sigma = 0.5$ may suggest that the total response with heterogeneity should be 30% above that without, the actual magnitude is lower because earnings heterogeneity channel lowers the multiplier from income increases, as explained below. For an equivalent economy with short maturities, the response is more than twice as large as both the benchmark U.S. response and the representative-agent response. There are two equivalent interpretations for this result.[43] The first interpretation is that unhedged interest rate exposures are smaller for all agents in the economy, and in particular for the high-MPC borrowers, so that their consumption is less affected by a change in the real interest rate. The second interpretation is that when assets have long maturities, as they do in the benchmark, expansionary monetary policy creates capital gains for asset holders and upward revaluation of liabilities for borrowers. These redistribute against the MPC gradient and make monetary policy less potent in affecting output. With short maturities, on the other hand, real interest rate changes create redistribution that is more aligned with the MPC gradient. This explains the larger effects of monetary policy shocks on output in this case. This prediction of the model is consistent with the cross-country structural VAR evidence

[43]The figures also illustrate that the redistribution channel is not simply making the representative-agent "more elastic". The relative magnitudes of the redistribution and the substitution channel is above 4 in the case of a short-lived shock.

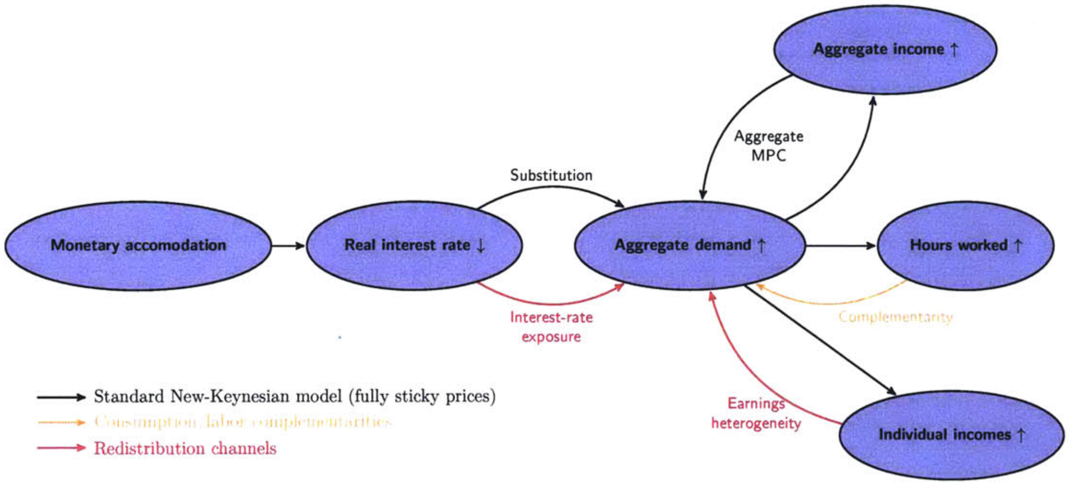


Figure 1.5.7: Monetary policy transmission mechanism in the model

presented in Calza et al. (2013). It suggests that wealth redistribution is the primary reason why monetary policy affects consumption in a country like the United Kingdom where mortgages have adjustable rates.

Figure 1.5.5 also compares the output response in the full solution of the model to the predictions from the first-order approximation, discussed in more detail below. The approximation is excellent, especially for the benchmark calibration, illustrating again the fruitfulness of the sufficient statistic approach in quantifying the aggregate effects of monetary policy. Section 1.5.6 will explain why the response in the ARM-only calibration is below that from the first-order approximation.

Understanding the role of redistribution Consider the case $\rho_R = 0$: the shock lasts for only one period. With fully rigid prices the Fisher channel is shut down: $dP = d\triangle = 0$. Applying proposition 21 at the steady-state, and dropping time subscripts for ease of notation, the following identities hold:

$$\frac{dC}{C} = \frac{dY}{Y} \qquad -\frac{d\widetilde{\tau}}{1-\widetilde{\tau}} = \frac{dw}{w} = \frac{1}{\psi}\frac{dY}{Y}$$

and the predicted response of consumption and GDP to a monetary policy shock is

$$\frac{dC}{C} = \frac{(\mathcal{E}_r - \sigma S)}{\tau^* (1 - \mathcal{M}^g) + \mathcal{M}^g - \mathcal{M} + \frac{\mathcal{E}_\tau}{\psi}} \frac{dR}{R} = -\mu^{het} S (\sigma^* + \sigma) \frac{dR}{R} \tag{1.5.2}$$

Figure 1.5.7 helps interpret equation (1.5.2). Consider first the representative agent version of this model. For such a case we have, from definition 20, $S = \xi (1 - MPC)$, $\mathcal{E}_r = \mathcal{E}_\tau = 0$, and $\mathcal{M}^g = \mathcal{M} = MPC$. Hence (1.5.2) yields[44]

$$\frac{dC}{C} = \frac{-\sigma \xi (1 - MPC)}{\tau^* (1 - MPC)} \frac{dR}{R} = -\frac{\xi}{\tau^*} \sigma \frac{dR}{R} \tag{1.5.3}$$

In the standard New Keynesian model, a monetary accommodation lowers the real interest rate, which raises aggregate demand through the substitution channel, and gets further amplified through the aggregate income channel. The MPC cancels out of this calculation, leaving the denominator of (1.5.3). Relative

[44] Appendix 1.7.4.7 verifies that this expression also obtains from writing the loglinearized equations characterizing the model.

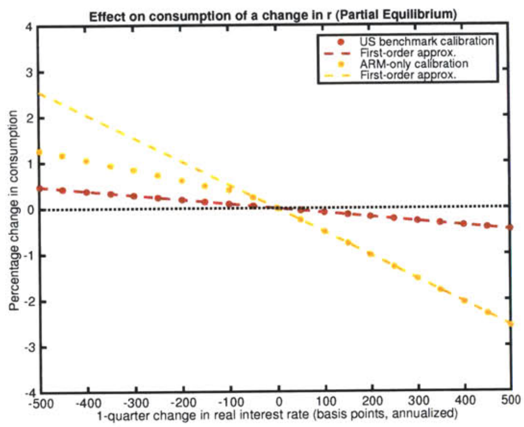


Figure 1.5.8: Change in r: partial equilibrium

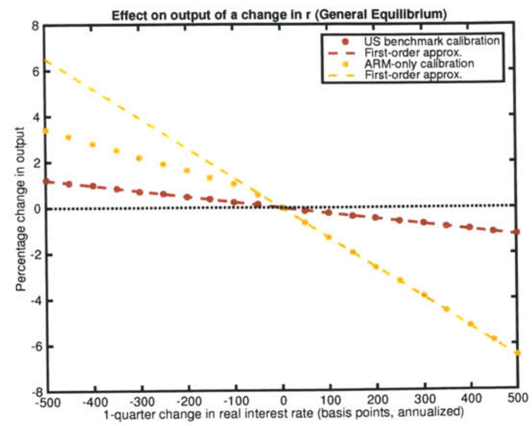


Figure 1.5.9: Change in r: general equilibrium

to the standard model, this model features consumption/labor complementarities from GHH preferences. These provide strong amplification of increases in hours worked on consumption. Combining the effect of this multiplier from the scaling of net consumption to gross consumption, we obtain a factor $\xi/\tau^* = 1.75$ multiplying $\sigma \frac{dR}{R}$ in (1.5.3).

In the presence of heterogeneity, the redistribution channel enters at two levels. First, lower real interest rates boost aggregate demand though the interest rate exposure channel as well as the substitution channel ($\mathcal{E}_r < 0$), which adds a negative term to the numerator of (1.5.3). Second, monetary policy raises incomes heterogeneously: wages rise and profits fall, resulting in a lower tax intercept and less effective redistribution during the boom. Because high income agents have lower MPC, this acts to lower demand. Given the calibrated $\psi = 1$, this results in a factor of $\mu^{het} S = 1.47$ multiplying $(\sigma^* + \sigma) \frac{dR}{R}$ in (1.5.2). Hence, in this particular calibration, the "impulse" reponse to a monetary policy shock is larger (from the interest-rate exposure channel), but the multiplier is lower (from the earnings heterogeneity channel) than what the representative-agent model predicts; with the first effect dominating slightly.[45]

Adding price adjustment to the picture would only enhance the departures of the impulse response to monetary policy from the representative-agent benchmark. As is clear from the magnitude of $\mathcal{E}_P$ and from section 1.5.4, the Fisher channel is strong in the calibrated model, since MPCs and nominal asset positions are highly correlated. This suggests that inflation can be a powerful amplification mechanism of monetary policy acting through redistribution.

1.5.6 Asymmetric effects of increases and cuts in interest rates

This section explores in more detail the quality of the first-order approximation (1.5.2), relative to the full nonlinear solution of the model, according to the size of the one-off real interest rate change of $dr = \frac{dR}{R}$.

[45] This multiplier effect is only illustrative of the role of the earnings heterogeneity channel in determining the full response to monetary policy shocks. A model with wage rigidities would alter the sign of this channel (which would be more consistent with the empirical evidence of Coibion et al. (2012)) and increase the multiplier relative to the case without heterogeneity.

In figure 1.5.8 I show the aggregate consumption effects that result from this change in partial equilibrium, that is, before any general equilibrium effects on income—as in the current account experiment of Example 12. The corresponding first-order approximation—the numerator of (1.5.2)—is very good, including for relatively large increases in the real interest rate, as well as for small decreases. However, in the ARM calibration, it overpredicts the consumption response for falls in the real interest rate of 100bp or more. This asymmetric effect can be traced back to the asymmetric behavior of the 22% of agents who are at their borrowing limit in this economy in response to increases and falls in income. While they have to cut consumption one for one in response to falls in income, their MPC out of moderate increases in income is below 0.3, as figure 1.5.2 illustrated. Because their debt is short term, falls in interest rates have large effects on their interest payments. This reduces effective MPC differences, and therefore the size of the redistribution channel, that results from falls in interest rates. Increases in interest rates do not have the same feature, since the MPC of borrowers out of increases in income payments is exactly one, as captured by the sufficient statistics.

Figure 1.5.9 shows that this asymmetric effect remains in general equilibrium. Is is slightly less strong than in partial equilibrium because the incomes of poor agents—which make up a disproportionate fraction of agents at the borrowing limit—do not rise as fast as those of the rest of the population as the tax system becomes less progressive in the boom.

This type of asymmetric effects of monetary policy changes receives support from the empirical evidence (see for example Cover, 1992; de Long and Summers, 1988 and recently Tenreyro and Thwaites, 2013). My explanation, which has to do with asymmetric MPC differences in response to policy rate changes, provides an alternative to the traditional Keynesian interpretation of this fact, which relies on downward nominal wage rigidities.[46]

1.6 Conclusion

This paper contributes to our understanding of the role of heterogeneity in the transmission mechanism of monetary policy. I established the precise sense in which a systematic covariance between agents' marginal propensities to consume and exposures to macroeconomic shocks generates a redistribution channel. I showed that the covariance between marginal propensities to consume and unhedged interest rate exposures is a sufficient statistic for the importance of the redistribution channel through real interest rates, and proposed a measure of this statistic in survey data. My results suggest that, if the Elasticity of Intertemporal Substitution is about 0.3—regarded by many as a plausible value—changes in real interest rates could be affecting consumption demand via redistribution as much as they do via the standard substitution channel present in representative-agent models. This is a quantitatively large effect which suggests that, while

[46]While my U.S. benchmark calibration does not feature asymmetric effects of interest rates, in practice, the refinancing option embedded in fixed rate mortgages in the United States is likely to create an asymmetric effect in the opposite direction from the one I stress in this section.

heterogeneity and inequality are important issues in their own right, they are also key to understanding aggregate consumption dynamics.

An important finding of my paper is that the monetary policy transmission mechanism operates differently in economies with short and long asset maturities, due to the different nature of the redistribution caused by changes in interest rates. This finding expands upon the popular view that lower interest rates benefit holders of long-term assets and are prejudicial to holders of short-term assets. When assets have relatively long maturities, lower real interest rates indeed tend to benefit asset holders and therefore to redistribute against the gradient of MPC in the economy. This makes interest rate cuts less effective at increasing aggregate demand. My calibrated model suggests that these effects—which one can interpret as a quantification of the macroeconomic effects of market incompleteness—are also large. It also suggests that monetary policy and mortgage design policies are intertwined, confirming a widely-held view in policy circles.

My results capture some of the general equilibrium, macroeconomic consequences of the presence of large and heterogeneous marginal propensities to consume, which are a robust feature of household micro data. Beyond the role of wealth redistribution for the macroeconomic effects of fiscal or monetary policy, this raises other questions—such as the role of inequality in the distribution of income in determining aggregate demand—which I leave for future research.

1.7 Appendix

1.7.1 Data

1.7.1.1 Aggregate data

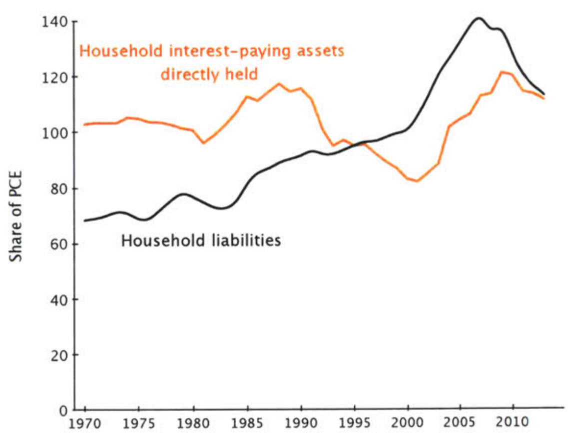


Figure 1.7.1: Monetary assets and liabilities

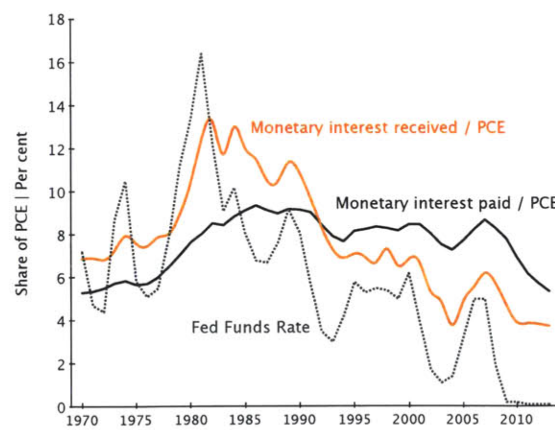


Figure 1.7.2: Monetary interest flows

Data sources: monetary interest paid and received by households is from NIPA table 7.11 (lines 15 and 32 respectively). Personal Consumption Expenditure (PCE) is from NIPA table 2.3.5, line 1. Household liabilities and interest-paying assets are from the U.S. Financial Accounts (Board of Governors Z.1 release).

Interest-paying liabilities is the sum of mortgages (table L.217, line 7) and consumer credit (table L.222, line 1). Interest-paying assets includes time and savings deposits (L.205[13]) and credit market instruments (the sum of L.208[18], L.209[6], L.210[6], L.211[8], L.212[13], L.216[36], L.217[14] and L.222[3])

1.7.1.2 Micro data

Tables 1.6 and 1.7 present summary statistics from the surveys used in section 1.3 with more information on the distributions than those available in table 1.1.

Survey of Household Income and Wealth 2010

A striking feature of the Italian data is that fewer than 10% of households report to own mortgages. The share of adjustable mortgages is around 50%, which is consistent with official sources (Eurosystem, 2009).

Variable	count	mean	p5	p25	p50	p75	p95
Income from all sources (Y_i, €/year)	7,951	36,114	9,565	19,857	30,719	45,340	81,320
Consumption incl. mortgage payments (C_i, €/year)	7,951	27,976	10,700	17,060	24,000	33,600	57,600
Deposits and maturing assets (B_i, €)	7,951	14,200	0	1,000	5,156	15,054	50,000
ARM mortgage liabilities and consumer credit (D_i, €)	7,951	6,228	0	0	0	0	26,800
Unhedged interest rate exposure (€/yr)	**7,951**	**16,110**	**-21,862**	**1,093**	**10,974**	**26,646**	**71,610**
Unhedged interest rate exposure (€/Q)	**7,951**	**10,007**	**-17,328**	**594**	**6,407**	**16,871**	**52,054**
Total fixed-income financial assets (€)	7,951	15,133	0	0	0	4,285	99,000
Total financial liabilities (€)	7,951	27,481	0	1,359	7,000	24,064	91,104
Marginal Propensity to Spend	7,951	47	0	20	50	80	100

All statistics are computed using survey weights

Table 1.6: Summary statistics from the Italian SHIW 2010

Consumer Expenditure Survey

Variable	count	mean	p5	p25	p50	p75	p95
Income from all sources ($/year)	9,443	45,617	6,700	18,612	36,000	62,828	115,000
Consumption incl. mortgage payments ($/year)	9,443	36,253	9,544	18,724	28,464	44,114	90,296
Deposits and maturing assets ($/year)	9,443	7,147	0	0	0	2,100	30,100
Consumer credit ($)	9,443	2,872	0	0	0	2,000	14,600
Unhedged interest rate exposure ($/yr)	9,443	13,639	-41,948	-6,000	6,192	27,616	84,126
Unhedged interest rate exposure ($/Q)	9,443	6,616	-16,514	-2,288	1,377	7,813	35,520
Net liquid assets ($)	7,202	7,340	0	5	1,329	6,400	38,000

All statistics are computed using survey weights

Table 1.7: Summary statistics from the JPS CEX sample

1.7.1.3 Additional data tables

Consumption measure	Food		All nondurable	
	Estimate	95% C.I	Estimate	95% C.I
$\widehat{\mathcal{E}_r^{PE}}$	-0.065	[-0.14; 0.01]	-0.17	[-0.33 -0.02]
$\widehat{\mathcal{E}_r}$	-0.123	[-0.21; -0.04]	-0.32	[-0.48; -0.16]
$\widehat{S}$	0.82	[0.69; 0.95]	0.56	[0.33; 0.78]
$\widehat{\sigma^*} = -\frac{\widehat{\mathcal{E}_r}}{\widehat{S}}$	0.149	[0.03; 0.27]	0.58	[0.12; 1.04]

Confidence intervals are bootstrapped by resampling households 100 times with replacement

Table 1.8: Estimates from table 1.3 using median instead of mean URE per bin

J	2	3	4	5	7	10
$\widehat{\sigma^*}$	0.25	0.30	0.29	0.28	0.35	0.31
95% C.I.	[0.03; 0.47]	[0.05; 0.55]	[0.05; 0.52]	[0.04; 0.53]	[0.08; 0.62]	[0.00; 0.60]

Confidence intervals are bootstrapped by resampling households 100 times with replacement

Table 1.9: Estimated $\widehat{\sigma^*}$, using J bins of URE (consumption measure: food expenditures)

1.7.2 Mortgage type and unhedged interest-rate exposures

1.7.2.1 An example

Consider a consumer who, midway through his life-cycle (at a date labelled $t = 0$), has an amount L in debt which must be repaid before his death at date T. The general level of prices is constant and normalized to $P = 1$. I abstract away from intertemporal substitution in labor supply and assume that this consumer faces a certain stream of unearned income $y_t = y$, and has expected utility with constant elasticity of substitution σ over consumption alone:

$$U(\{c_t, n_t\}) = \sum_{t=0}^{T-1} \beta^t \left(\frac{c_t^{1+\frac{1}{\sigma}}}{1+\frac{1}{\sigma}} \right) \tag{1.7.1}$$

The rate of interest is initially equal to the rate of time preference: $q_t = \beta^t$. It is clear that the solution to this optimization problem is

$$c_t = c = y - R_T L$$

where R_T is the annuitization factor for T periods, which is also the household's marginal propensity to consume, $MPC = \sum_{t=0}^{T-1} q_t = R_T = \frac{1-\beta}{1-\beta^T}$. The market value of financial wealth is projected to evolve as $W_t^F = -\frac{1-\beta^{T-t}}{1-\beta^T} L$, so that the debt is repaid in full by year T.

Consider the following three financial arrangements, each having the same present value $-L$, and so, as per proposition 1, leading to the same lifetime consumption plan under the initial path for real interest rates:

a) an adjustable-rate mortgage: ${}_{-1}B_{ARM,0} = -L$, ${}_{-1}B_{ARM,t} = 0$ for $t > 0$

b) a fixed-rate mortgage: ${}_{-1}B_{FRM,t} = -R_T L$ for $t = 0 \ldots T-1$

c) a bullet loan, due at the end of life, ${}_{-1}B_{bullet,T-1} = -\frac{1}{\beta^{T-1}} L$, ${}_{-1}B_{bullet,T-1} = 0$ for $t < T-1$

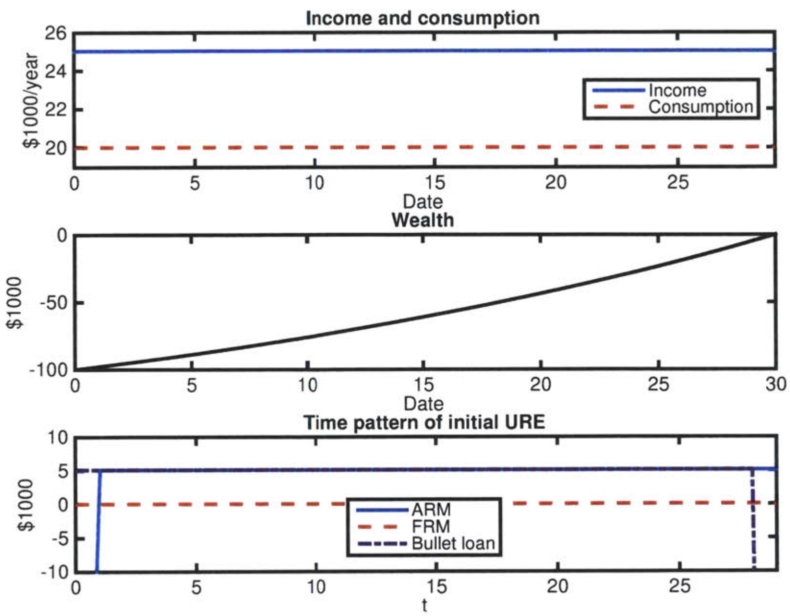


Figure 1.7.3: Solution to the life-cycle model and time path of URE under different mortgages

Figure 1.7.4 illustrates the solution under a simple calibration in which income is constant at $y = \$25\,000$ per year, initial debt is $L = \$100\,000$, the discount factor is $\beta = 0.97$, the elasticity of intertemporal substitution is $\sigma = 0.25$ and the horizon is $T = 30$ years. Here the annual real interest rate is $R = 3.1\%$ and the annual marginal propensity to consume is $MPC = R_T = 5\%$, so that consumption is equal to $c = \$20\,000$ per year. The remaining \$5000 can be interpreted as payment of interest and principal under the ARM and FRM plans, or as a savings build-up towards repayment of the bullet loan. The bottom panel of the figure shows the time path of unhedged interest rate exposures for each of the three mortgage types under consideration. Under an ARM, URE is very negative at date 0[47] and then becomes positive and equal to $R_T L$ throughout the consumer's remaining life. A bullet loan has a symmetric pattern of UREs, positive everywhere and very negative in the last period. A Fixed-Rate Mortgage, in this case, achieves the Arrow trading plan with URE exactly equal to zero throughout the life-cycle.

Suppose real interest rates unexpectedly change at date 0, and consider the impact on the consumer's welfare and consumption level in the first year following the change. Appendix 1.7.3.2 shows that if real

[47] The figure is truncated at URE=-\$10,000 for readability. Under an ARM, date-0 URE is $-(1 - R_T)\,L \simeq -\$95\,000$. Under a bullet loan, (non-discounted) URE in the last period is $-\$236\,000$.

present-value discount factors change by $\{dq_s\}$, under the utility function (1.1.5), theorem 2 specializes to

$$dc_0 = MPCd\Omega + \sigma c\left[MPC\sum_{s=1}^{T-1} dq_s\right]$$

$$dU = u'(c)\,d\Omega$$

$$d\Omega = \sum_{s=1}^{T-1} q_s\left(_{-1}URE_s\right)\frac{dq_s}{q_s}$$

For example, a *one-time change* dr in the real interest rate between periods 0 and 1 changes discount factors by $dq_s = -q_s dr$ for $s \geq 1$. We saw in (1.1.8) that the resulting consumption change is

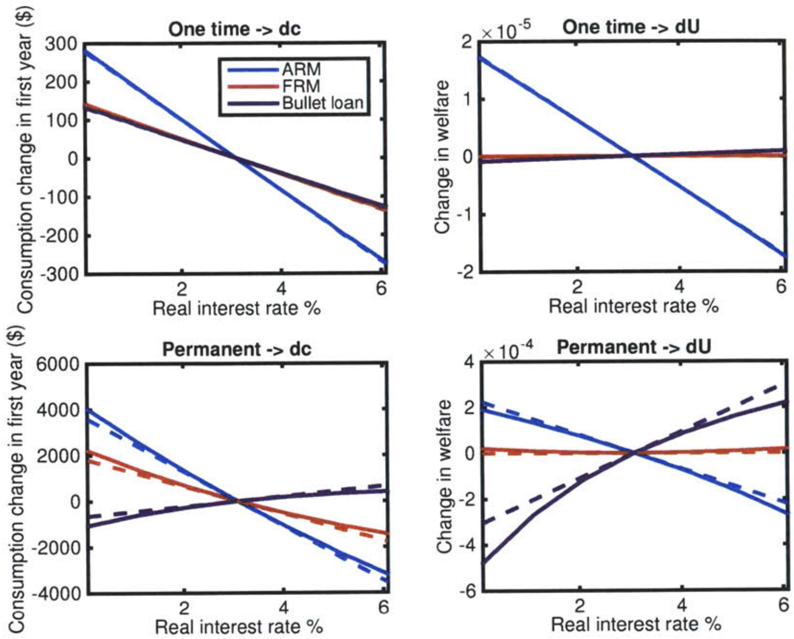
$$dc_0 \quad = \quad \left[MPC \cdot URE - \sigma c\left(1 - MPC\right)\right] dr \tag{1.7.2}$$

The top panels of figure 1.7.4 display the first-year consumption change dc_0 and change in welfare dU under various levels for the real interest rate in the first year. A rise in the real interest above its initial level of $R = 3.1\%$ always decreases consumption from intertemporal substitution, but by much more moderate amounts when the consumer holds an FRM or a bullet loan than when he holds an ARM. The consumption change of an FRM holder is entirely due to intertemporal substitution, since he has no interest rate exposure and so never experiences wealth effects. Quantitatively, this substitution effect amounts to $-\sigma c\left(1 - MPC\right)$, which is a fall of about \$50 in annual consumption per percentage point rise in the real interest rate. Since the date-0 URE term is positive but small for the bullet loan holder, his wealth effect from the rise in interest rates translates into a small increase in welfare and a level of consumption slightly above that of the FRM holder. On the other hand, ARM holders experience a very large wealth loss from this temporary rise in interest rates since their initial unhedged interest rate exposure is almost as large as their loan balance. Precisely, $URE_{ARM} = -\left(1 - R_T\right) L$, implying a \$950 present-value loss—translating into an additional fall in consumption of \$50—per percentage point fall in the short-term real interest rate.

This simple calibration—where under ARMs, income and substitution effects are of comparable magnitudes—illustrates the quantitative results in the paper. In the population, marginal propensities to consume tend to be much larger than the 5% annual number in this simple example, and balance sheet positions can be much larger than the 5 times annual consumption assumed here. To the extent that these positions are relatively short term, the income effect can therefore overwhelm the substitution effect for many individuals. In the aggregate, when MPCs and UREs are aligned, this can imply that real interest rate changes affect consumption through redistribution as much as through substitution. Section 1.3 makes this observation more precise.

The bottom panel of figure 1.7.4 considers the effect of a *permanent* change in the real interest rate throughout the consumer's remaining life. Such a change dr modifies present-value discount factors by $dq_s = -q_s s dr$ for $s \geq 1$. From (1.1.7),

$$d\Omega^{\ell} = \left[\sum_{s=1}^{T-1} q_s s\left(_{-1}URE_s\right)\right]\left(-dr\right) \equiv URE^{\ell} \cdot dr$$



The figure shows the actual consumption change (solid lines) and the predictions from the approximation of theorem 2 (dashed lines) under various types of mortgages. Differences between colored lines are due to differential wealth effects from the rate change.

Figure 1.7.4: Effect of a change in real interest rates on dc and dU with different mortgages

Define the duration of a "perpetuity" bond paying 1 in each year between date 1 and date $T-1$ as

$$D^{\ell} = \frac{\sum_{s=1}^{T-1} s q_s}{\sum_{s=1}^{T-1} q_s}$$

Hence theorem 2 implies that the first-order response of consumption in response to this shift in the real yield curve is

$$dc_0^{\ell} = \left[MPC \cdot URE^{\ell} - \sigma c \left(1 - MPC\right) D^{\ell}\right] dr$$

The bottom panel of figure 1.7.4 shows that the approximation is excellent for small changes in the real interest rate. Long-term changes create larger substitution effects than short-term changes (in the calibration, the long-term substitution effect is $D^{\ell} = 12.8$ years times the short-term one). This is visible by comparing the response of fixed-rate mortgage holders for whom wealth effects are zero under all scenarios. But long-term real interest rate changes also create much larger wealth effects. In the calibration, for an ARM holder $URE^{\ell} = -\$1.2\text{m}$, which is a present-value loss of \$12 000 per percentage point rise in the real interest rate, translating into a \$600 of additional cut in annual consumption. For a bullet loan holder $URE^{\ell} = \$1.7\text{m}$, enough for the wealth effect of a prolonged change in real interest rates to overwhelm its substitution effect, as is visible in the bottom left panel of figure 1.7.4.

Comparing across lines on the left panels of figure 1.7.4 at a given interest rate level, we difference out substitution effects on consumption and obtain the difference in wealth effects across mortgage types m and m', $\sum_{t\geq 0} q_t \left({}_{-1}URE_{m,t}\right) - \sum_{t\geq 0} q_t \left({}_{-1}URE_{m't}\right)$. Another way of looking at this difference in difference is to notice that it subtracts out the valuation of future consumption and income, with the remainder being a difference in financial wealth alone, $\sum_{t\geq 0} q_t \left({}_{-1}b_{m,t}\right) - \sum_{t\geq 0} q_t \left({}_{-1}b_{m',t}\right)$. When interest rates rise, we might observe households with ARMs lowering their consumption more than households with FRMs: the theory predicts that this is due to a wealth effect, which can be quantified based on observable balance sheet information; and not—at least to the extent that borrowing constraints do not strictly bind—a "disposable income" effect. The next section builds on this observation and provides a structural interpretation to studies that regress consumption changes on mortgage type.

1.7.2.2 Interpretation of consumption/balance-sheet regressions

Consider a regression that compares households with a standard fixed-rate mortgage to those with an adjustable-rate mortgage around a monetary policy change. For a cross-section of households i, we have data on their change in consumption $\triangle c_i$ around the event, and an indicator of whether they held an adjustable-rate mortgage ARM_i in the pre-period. Suppose that we run

$$\triangle c_i = \alpha + \beta ARM_i + \epsilon_i \tag{1.7.3}$$

and that we have an ideal instrument for ARM_i, so that the assignment to balance-sheets can be taken to be as good as random. As an example, suppose we use as an instrument a dummy for whether the mortgage is above or below the conforming loan limit, CLL_i. Vickery (2007) and Moench, Vickery and Aragon (2010) show that this is a powerful predictor of mortgage choice, with loans above the conforming loan limit having a much higher chance of being adjustable rates; suppose further that the exclusion restriction is valid so that the only influence of conforming loan limits on consumption changes is through mortgage type. Write $\triangle c_{i,ARM}$ for the change in consumption consumer i if he holds an adjustable-rate mortgage and $\triangle c_{i,FRM}$ for the change in consumption that would result if he held a fixed-rate mortgage with the same principal outstanding. The instrumented regression of $\triangle c_i$ on ARM_i then produces a treatment effect[48] of

$$\begin{aligned} \beta &= \mathbb{E}\left[\triangle c_{i,ARM} - \triangle c_{i,FRM}\right] \\ &= \mathbb{E}\left[c_{i,ARM} - c_{i,FRM}\right] \end{aligned} \tag{1.7.4}$$

Write $\overline{c_i}$ for i's consumption plan absent any change in monetary policy. Then according to theorem 2,

$$\begin{aligned} c_{i,ARM} - c_{i,FRM} &= \left(c_{i,ARM} - \overline{c_i}\right) - \left(c_{i,FRM} - \overline{c_i}\right) \\ &\simeq dc_{i,ARM} - dc_{i,FRM} \\ &= MPC_i \left(dW^F_{i,ARM} - dW^F_{i,FRM}\right) \end{aligned}$$

In other words, the regression differences out all of the influences on consumption that are induced by the monetary policy change (in particular, intertemporal substitution and general equilibrium effects on income)

[48]For example, the case of the CLL instrument, this result requires that potential outcomes be independent of the assignment to the CLL cutoff, that the exclusion restriction be valid, and that the first stage be positive. With heterogenous treatment effects, (1.7.4) is technically a local average treatment effect, where the average is taken over "compliers" whose decision to purchase an adjustable rate mortgage is changed by the conforming loan limit. See Imbens and Angrist (1994).

and captures purely the differential balance-sheet revaluation experienced by the household under the two potential assignments to mortgage structure.[49]

Both the ARM and the FRM are nominal contracts. Since an ARM contract is entirely short term, its present value is isolated from movements in the nominal term structure $\frac{dQ_t}{Q_t} = \frac{dq_t}{q_t} - \frac{dP_t}{P_t}$, so:

$$dW^F_{i,ARM} = 0$$

A consumer with a fixed-rate mortgage that specifies nominal payments M_i, on the other hand, experiences a financial wealth revaluation from a change in the nominal term structure $\{dQ_t\}$ of

$$dW^F_{i,FRM} = -\sum_{t=1}^{T_i} Q_t \left(\frac{M_i}{P_0}\right) \frac{dQ_t}{Q_t}$$

Hence, a fixed-rate borrower experiences a present-value gain from a rise in interest rates that lowers discount rates dQ_t, since the monetary policy change lowers the present value of his liabilities.[50]

For example, a parallel rise in the term structure due to a combination of a permanent rise in inflation and/or a permanent rise in the real interest rate, where all nominal interest rates are altered by $d\iota$, leads to

$$dW_{i,FRM} = -D_i \cdot L_i \cdot d\iota$$

where L_i is the consumer's mortgage principal outstanding, and D_i its duration. In that case

$$\beta = -\mathbb{E}\left[MPC_i \cdot D_i \cdot L_i\right] d\iota \qquad (1.7.5)$$

that is, the regression coefficient has a structural interpretation as an average product of MPCs by individual characteristics (mortgage durations and balances) that may be part of the dataset.[51] In this case we could use $\frac{\triangle c_i}{D_i L_i}$ as a left-hand side variable instead of $\triangle c_i$ in (1.7.3), and infer an average MPC.

This result is a useful benchmark, which runs counter to the intuition that consumption should fall because the rise in interest rates is akin to a "negative income shock" for the adjustable-rate mortgage holder: it shows that it is more accurate to think of it as a negative *wealth* shock, whose relevant denominator is the change in wealth and not the change in mortgage payments. However, it is important to highlight that it derived under the condition that borrowing constraints do not bind. If borrowing constraints bind for many consumers—as they do in the model of section 1.4—the result may down, depending on how borrowing limits adjust, and the income shock interpretation may be closer to accurate for some households. Another caveat is that in the case of *falls* in interest rates, in countries like the United States, the difference in financial wealth effect between FRM and ARM holders is lower than suggested by equation (1.7.5) due to the possibility of

[49] As I show in section 1.1.3, this result is true even if households face incomplete markets, providing borrowing constraints are not binding and the change is transitory. The result appears extends to any change in the term structure, providing borrowing constraints do not bind during the time over which the financial wealth effects operate (see Conjecture 27, work in progress).

[50] In section 1.7.2.1 we saw that the ARM borrower experienced a negative wealth effect, while the FRM borrower experienced none, following a rise in interest rates. The two results are not contradictory: this section focuses on financial wealth revaluations alone, while the previous section included human wealth in the calculation. Of course both ways of looking at this differential prediction yield the same conclusion.

[51] The bottom left panel of 1.7.4 provides a quantitative example: if interest rates rise permanently by 1% per year, the average marginal propensity to consume is 5%, the average mortgage liability is \$100 000, and average mortgage duration is 12 years, then we may expect $\beta = -(.05)\,(12)\,(100\,000)\,(.01) = -\$600$, that is, adjustable-rate-mortgage borrowers consume \$600 less per year than fixed-rate mortgage borrowers as a result of the interest rate change.

refinancing FRMs at a low cost.

The general idea behind equation (1.7.5) is that a perfect instrument differences out general equilibrium effects of monetary policy changes and singles out the out the first-round source of differential consumption adjustments across consumers—which here is a relative wealth change. In two recent papers, di Maggio et al. (2014) and Keys et al. (2014) investigate the response of car expenditures across counties with different shares of adjustable vs fixed rate mortgages after the recent fall in interest rates.[52] Like other county-level regressions such as those of Mian et al. (2013), such a regression may capture general equilibrium effects on local income of wage changes, in addition to the first-round effect of monetary policy on wealth. The models in my paper are a step towards understanding these general equilibrium effects in a world where aggregate demand is affected by the wealth distribution.

1.7.3 Proofs for section 1.1

1.7.3.1 Proof of theorem 2

Proof. It is convenient to define the expenditure function over the sequences $\{q_t\}$ and $\{w_t\}$:

$$e\left(\{q_t\},\{w_t\},U\right)=\min\left\{\sum_t q_t\left(c_t-w_tn_t\right)\quad \text{s.t.}\quad U\left(\{c_t,n_t,h_t\}\right)\geq U\right\}$$

and let c_t^h, n_t^h be the resulting Hicksian demands. The envelope theorem implies a version of Shephard's lemma:

$$\begin{aligned} e_{q_t} &= c_t-w_tn_t \quad \forall t \\ e_{w_t} &= -q_tn_t \quad \forall t \end{aligned}$$

Define the indirect utility function to attain unearned wealth $\widetilde{W}=\sum_{t\geq 0}q_t\left(y_t+\left({}_{-1}b_t\right)+\left(\frac{{}_{-1}B_t}{P_t}\right)\right)$ (wealth exclusive of earned income whose price is changing) as

$$V\left(\{q_t\},\{w_t\},\widetilde{W}\right)=\max\left\{U\left(\{c_t,n_t\}\right)\quad \text{s.t.}\quad \sum_t q_t\left(c_t-w_tn_t\right)=\widetilde{W}\right\}$$

and let c_t, n_t be the resulting Marshallian demands. Differentiating along the identities

$$\begin{aligned} c_0^h\left(\{q_t\},\{w_t\},U\right) &= c_0\left(\{q_t\},\{w_t\},e\left(\{q_t\},\{w_t\},U\right)\right) \\ n_0^h\left(\{q_t\},\{w_t\},U\right) &= n_0\left(\{q_t\},\{w_t\},e\left(\{q_t\},\{w_t\},U\right)\right) \end{aligned}$$

we find that Marshallian derivatives are

$$\begin{aligned} \frac{\partial c_0^h}{\partial q_t} &= \frac{\partial c_0}{\partial q_t}+\frac{\partial c_0}{\partial \widetilde{W}}e_{q_t} & \frac{\partial c_0^h}{\partial w_t} &= \frac{\partial c_0}{\partial w_t}+\frac{\partial c_0}{\partial \widetilde{W}}e_{w_t} \\ \frac{\partial n_0^h}{\partial q_t} &= \frac{\partial n_0}{\partial q_t}+\frac{\partial n_0}{\partial \widetilde{W}}e_{q_t} & \frac{\partial n_0^h}{\partial w_t} &= \frac{\partial n_0}{\partial w_t}+\frac{\partial n_0}{\partial \widetilde{W}}e_{w_t} \end{aligned}$$

[52] The assignment of counties to mortgage structure is not quasi-random—in part because higher house prices increase the fraction of loans above the conforming loan limit—but both papers have a way of controlling for this.

denoting $MPC \equiv \frac{\partial c_0}{\partial \widetilde{W}} = \frac{\partial c_0}{\partial W}$ and $MPN \equiv \frac{\partial n_0}{\partial \widetilde{W}} = \frac{\partial n_0}{\partial W}$,[53] and using Shephard's lemma we obtain

$$\frac{\partial c_0^h}{\partial q_t} = \frac{\partial c_0}{\partial q_t} + MPC \cdot (c_t - w_t n_t) \qquad \frac{\partial c_0^h}{\partial w_t} = \frac{\partial c_0}{\partial w_t} - MPC \cdot q_t n_t$$
$$\frac{\partial n_0^h}{\partial q_t} = \frac{\partial n_0}{\partial q_t} + MPN \cdot (c_t - w_t n_t) \qquad \frac{\partial n_0^h}{\partial w_t} = \frac{\partial n_0}{\partial w_t} - MPN \cdot q_t n_t$$

Applying a Taylor expansion to the consumption function $c_0(\{q_t\}, \{w_t\}, \{P_t\}, \{y_t\})$ and using the above values for derivatives evaluated at the initial sequence $\{q_t\}, \{w_t\}, \{P_t\}, \{y_t\}$, we have, for sufficiently small changes in aggregates:

$$\begin{aligned}
dc_0 &\simeq \sum_{t\geq 0} \frac{\partial c_0}{\partial q_t} dq_t + \sum_{t\geq 0} \frac{\partial c_0}{\partial w_t} dw_t + \frac{\partial c_0}{\partial \widetilde{W}} d\widetilde{W} \\
&\simeq \sum_{t\geq 0} \left(\frac{\partial c_0^h}{\partial q_t} - MPC \cdot (c_t - w_t n_t) \right) dq_t + \sum_{t\geq 0} \left(\frac{\partial c_0^h}{\partial w_t} + MPC \cdot q_t n_t \right) dw_t \\
&\quad + MPC \left(\sum_{t\geq 0} \left(y_t + ({}_{-1}b_t) + \left(\frac{{}_{-1}B_t}{P_t} \right) \right) dq_t - \sum_{t\geq 0} q_t \left(\frac{{}_{-1}B_t}{P_t} \right) \frac{dP_t}{P_t} + \sum_{t\geq 0} q_t dy_t \right) \\
&\simeq c_0 \sum_{t\geq 0} \underbrace{\frac{q_t}{c_0} \frac{\partial c_0^h}{\partial q_t}}_{\epsilon^h_{c_0,q_t}} \frac{dq_t}{q_t} + c_0 \sum_{t\geq 0} \underbrace{\frac{w_t}{c_0} \frac{\partial c_0^h}{\partial w_t}}_{\epsilon^h_{c_0,w_t}} \frac{dw_t}{w_t} + MPC d\Omega
\end{aligned}$$

where

$$\begin{aligned}
d\Omega &= \sum_{t\geq 0} q_t dy_t + \sum_{t\geq 0} q_t n_t dw_t \\
&\quad + \sum_{t\geq 0} \left(y_t + w_t n_t + ({}_{-1}b_t) + \left(\frac{{}_{-1}B_t}{P_t} \right) - c_t \right) dq_t - \sum_{t\geq 0} q_t \left(\frac{{}_{-1}B_t}{P_t} \right) \frac{dP_t}{P_t} \\
&= \sum_{t\geq 0} q_t y_t \frac{dy_t}{y_t} + \sum_{t\geq 0} q_t w_t n_t \frac{dw_t}{w_t} \\
&\quad + \sum_{t\geq 0} q_t \left(y_t + w_t n_t + ({}_{-1}b_t) + \left(\frac{{}_{-1}B_t}{P_t} \right) - c_t \right) \frac{dq_t}{q_t} - \sum_{t\geq 0} Q_t \left(\frac{{}_{-1}B_t}{P_0} \right) \frac{dP_t}{P_t}
\end{aligned}$$

where the last line uses the Fisher equation, $\frac{q_t}{P_t} = \frac{Q_t}{P_0}$ to rewrite future real wealth in date-0 terms. Using the same calculation for n_0, we obtain the labor supply response formula in theorem 2. The welfare response follows from application of the envelope theorem to the indirect utility function:

$$\begin{aligned}
\frac{\partial V}{\partial q_t} &= -U_{c_0}(\{c_t, n_t, h_t\}) \cdot (c_t - w_t n_t) \\
\frac{\partial V}{\partial w_t} &= U_{c_0}(\{c_t, n_t, h_t\}) \cdot (q_t n_t) \\
\frac{\partial V}{\partial \widetilde{W}} &= U_{c_0}(\{c_t, n_t, h_t\})
\end{aligned}$$

[53] Under a present-value normalization with $q_0 = 1$, $MPC = \frac{\partial c_0}{\partial y_0}$ and $MPN = \frac{\partial n_0}{\partial y_0}$

therefore a Taylor expansion yields

$$\begin{aligned}
dU &\simeq \sum_{t\geq 0}\frac{\partial V}{\partial q_t}dq_t + \sum_{t\geq 0}\frac{\partial V}{\partial w_t}dw_t + \frac{\partial V}{\partial \widetilde{W}}d\widetilde{W} \\
&\simeq U_{c_0}\left(\{c_t, n_t\}\right)\cdot\left(\sum_{t\geq 0}\left(w_t n_t - c_t\right)dq_t + \sum_{t\geq 0} q_t n_t dw_t + d\widetilde{W}\right) \\
&\simeq U_{c_0}\left(\{c_t, n_t\}\right)\cdot d\Omega
\end{aligned}$$

as was to be shown.

□

1.7.3.2 Value of elasticities for common utility functions

Inelastic labor supply Suppose that labor supply is inelastic (so that we can consider all income, $Y_t = y_t + nw_t$, as unearned) and utility is time-separable, but not necessarily homothetic

$$U\left(\{c_t, n\}\right) = \sum_t \beta^t u\left(c_t\right) \tag{1.7.6}$$

The budget constraint is

$$\sum_{t\geq 0}\frac{q_t}{q_0}c_t = W$$

Online appendix OA.1.1 derives the following facts: the marginal propensity to consume is

$$MPC = \left(1 + \sum_{t\geq 1}\frac{q_t}{q_0}\frac{\sigma\left(c_t\right)c_t}{\sigma\left(c_0\right)c_0}\right)^{-1} \tag{1.7.7}$$

and the Hicksian elasticities are:

$$\begin{aligned}
\epsilon^h_{c_0,q_t} &= MPC\frac{q_t}{q_0}\sigma\left(c_t\right)\frac{c_t}{c_0} \quad t\geq 1 \\
\epsilon^h_{c_0,q_0} &= -\sigma\left(c_0\right)\left(1 - MPC\right)
\end{aligned}$$

Hence, under inelastic labor supply, theorem 2 predicts that

$$dc_0 = MPCd\Omega - c_0\sigma\left(c_0\right)\left(1 - MPC\right)\frac{dq_0}{q_0} + MPC\sum_{t\geq 1}\frac{q_s}{q_0}c_s\sigma\left(c_s\right)\frac{dq_s}{q_s}$$

$$dU = u'\left(c_0\right)d\Omega$$

$$d\Omega = \sum_{t\geq 0} q_t dY_t + \sum_{t\geq 0} q_t\left(Y_t + \left(\frac{{}_{-1}B_t}{P_t}\right) + \left({}_{-1}b_t\right) - c_t\right)\frac{dq_t}{q_t} - \sum_{t\geq 0} Q_t\left(\frac{{}_{-1}B_t}{P_0}\right)\frac{dP_t}{P_t}$$

Application: one-time change For the case of a one-time change, $\frac{dP_t}{P_t} = \frac{dP}{P}$ for $t \geq 0$, and $\frac{dq_t}{q_t} = -dr$ for $t \geq 1$, we have

$$\begin{aligned} MPC\left(\sum_{t\geq 1}\frac{q_s}{q_0}c_s\sigma\left(c_s\right)\right)(-dr) &= MPC\left(\sigma\left(c_0\right)c_0\right)\left(MPC^{-1}-1\right) \\ &= -\sigma\left(c_0\right)c_0\left(1-MPC\right)dr \end{aligned}$$

Combining with $d\Omega$ from (1.1.5) we obtain

$$dc_0 = MPC\cdot\left(dy + ndw + URE\cdot dr - NNP\cdot\frac{dP}{P}\right) - c_0\sigma\left(c_0\right)\left(1-MPC\right)dr$$

Separable preferences over consumption and labor Consider the standard macroeconomic specification of preferences

$$U\left(\{c_t,n_t\}\right) = \sum_t \beta^t\left(u\left(c_t\right)-v\left(n_t\right)\right) \quad u\left(c\right) = \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \quad v\left(n\right) = b\frac{n^{1+\frac{1}{\psi}}}{1+\frac{1}{\psi}} \tag{1.7.8}$$

Online appendix OA.1.2 derives the values of Hicksian elasticities ϵ^h and marginal propensities in the general case. Here I just report their value for an infinite horizon model around a steady-state with no growth, that is, where $\frac{q_s}{q_0} = \beta^s$ and $w_s = w^*$, $\forall s$. These are elasticities relevant to determine the impulse responses in many RBC models, for example. Writing $\vartheta \equiv \frac{w^*n^*}{c^*}$ for the share of earned income in consumption and $\kappa \equiv \frac{\frac{\psi}{\sigma}\vartheta}{1+\frac{\psi}{\sigma}\vartheta} \in (0,1)$ we have:

ϵ^h	q_0	$q_s, s \geq 1$	w_0	$w_s, s \geq 1$	Marg. propensity	
c_0	$-\sigma\beta$	$\sigma(1-\beta)\beta^s$	$\sigma\kappa(1-\beta)$	$\sigma\kappa(1-\beta)\beta^s$	MPC	$(1-\kappa)(1-\beta)$
n_0	$\psi\beta$	$-\psi(1-\beta)\beta^s$	$\psi(1-\kappa(1-\beta))$	$-\psi\kappa(1-\beta)\beta^s$	MPN	$-\frac{1}{w^*}\kappa(1-\beta)$

Table 1.10: Steady-state moments, separable preferences

GHH preferences Consider now a GHH preference specification

$$U\left(\{c_t,n_t\}\right) = \sum_t \beta^t u\left(c_t - v\left(n_t\right)\right) \quad u\left(c\right) = \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \quad v\left(n\right) = b\frac{n^{1+\frac{1}{\psi}}}{1+\frac{1}{\psi}}$$

Define $\xi = \left(1-\frac{v(n^*)}{c^*}\right) = \left(1-\frac{w^*n^*}{c^*}\left(\frac{\psi}{1+\psi}\right)\right) \in (0,1)$, online appendix OA.1.3 shows that the relevant elasticities, in a steady-state with no growth, are given by the following table:

ϵ^h	q_0	$q_s, s \geq 1$	w_0	$w_s, s \geq 1$	Marg. propensity	
c_0	$-\sigma\xi\beta$	$\sigma\xi(1-\beta)\beta^s$	$(1+\psi)(1-\xi)$	0	MPC	$1-\beta$
n_0	0	0	ψ	0	MPN	0

Table 1.11: Steady-state moments, GHH preferences

1.7.3.3 Proof of theorem 9

This section proves a main lemma and derives theorem 9 in the case where $N = 0$. Online appendix OA.2 shows that the result extends to the case with an arbitrary number of stocks.

Lemma 26. *Let $c(z,b,q)$ and $q(z,b,q)$ be the solution to the following consumer choice problem under concave preferences over current consumption $u(c)$ and assets $V(a)$*

$$\begin{aligned} \max \quad & u(c) + V(a) \\ s.t. \quad & c + q(a-b) = z \end{aligned}$$

then to first order

$$dc \simeq MPC\left(dz - (a-b)\,dq + qdb\right) - \sigma(c)\,c\,(1-MPC)\,\frac{dq}{q}$$

where $\sigma(c) \equiv -\frac{u'(c)}{cu''(c)}$ is the local elasticity of intertemporal substitution and $MPC = \frac{\partial c}{\partial z}$

Proof. The following first-order conditions are necessary and sufficient for optimality:

$$u'(c) = \frac{1}{q}V'(a) \tag{1.7.9}$$

I first obtain the expression for MPC by considering an increase in income dz alone. Consider how that increase is divided between consumption and savings. (1.7.9) implies

$$u''(c)\,dc = \frac{1}{q}V''(a)\,da \tag{1.7.10}$$

where the changes dc and da are related to dz through the budget constraint

$$dc + qda = dz \tag{1.7.11}$$

Define $MPC = \frac{\partial c}{\partial z}$ and $MPA = \frac{\partial a}{\partial z}$. Then (1.7.10) implies

$$\frac{MPA}{MPC} = \frac{qu''(c)}{V''(a)}$$

Combining with (1.7.11), the marginal propensity to consume satisfies

$$MPC = \frac{\partial c}{\partial z} = 1 - qMPA = 1 - \frac{q^2u''(c)}{V''(a)}MPC \tag{1.7.12}$$

and is equal to

$$MPC = \frac{1}{1 + q^2\frac{u''(c)}{V''(a)}} = \frac{V''(a)}{V''(a) + q^2u''(c)}$$

Consider now the overall effect on c and a of a change in z, b, and q. Applying the implicit function theorem to the system of equations

$$\begin{cases} qu'(c) - V'(a) = 0 \\ c + q(a-b) - z = 0 \end{cases}$$

results in the following expression for partial derivatives:

$$\begin{bmatrix} \frac{\partial c}{\partial z} & \frac{\partial c}{\partial b} & \frac{\partial c}{\partial q} \\ \frac{\partial a}{\partial z} & \frac{\partial a}{\partial b} & \frac{\partial a}{\partial q} \end{bmatrix} = -\underbrace{\begin{bmatrix} qu''(c) & -V''(a) \\ 1 & q \end{bmatrix}}_{\equiv A}^{-1} \begin{bmatrix} 0 & 0 & u'(c) \\ -1 & -q & (a-b) \end{bmatrix}$$

now

$$\det(A) = q^2 u''(c) + V''(a) = \frac{V''(a)}{MPC}$$

and so

$$A^{-1} = \frac{MPC}{V''(a)} \begin{bmatrix} q & V''(a) \\ -1 & qu''(c) \end{bmatrix}$$

therefore

$$\begin{bmatrix} \frac{\partial c}{\partial z} & \frac{\partial c}{\partial b} & \frac{\partial c}{\partial q} \end{bmatrix} = MPC \begin{bmatrix} \frac{-q}{V''(a)} & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & u'(c) \\ -1 & -q & (a-b) \end{bmatrix}$$

This results directly in

$$\begin{aligned} \frac{\partial c}{\partial z} &= MPC \\ \frac{\partial c}{\partial b} &= qMPC \end{aligned}$$

Moreover, using (1.7.12) together with $u'(c) \equiv -c\sigma(c)\, u''(c)$ we find

$$-q\frac{u'(c)}{V''(a)}MPC = \frac{\sigma(c)\,c}{q}q^2\frac{u''(c)}{V_{aa}(a,\tilde{q})}MPC = \frac{\sigma(c)\,c}{q}(1-MPC)$$

so that

$$\frac{\partial c}{\partial q} = \frac{\sigma(c)\,c}{q}(1-MPC) - (a-b)\,MPC$$

Using a first-order Taylor expansion, the total differential is then approximately

$$\begin{aligned} dc &\simeq \frac{\partial c}{\partial z}dz + \frac{\partial c}{\partial b}db + \frac{\partial c}{\partial q}dq \\ &= MPC\,(dz + qdb - (a-b)\,dq) + \sigma(c)\,c\,(1-MPC)\,\frac{dq}{q} \end{aligned} \tag{1.7.13}$$

as claimed. □

Proof of theorem 9. **Case a)**: For simplicity I restrict to the special case where $N = 0$, so that the the long-term bond constitutes the only means of transferring wealth through time. Online appendix OA.2 shows that the result extends to cases with $N > 0$. The notation of theorem 9 can be cast using that of lemma 26 by using the mapping

$$q \equiv Q \quad z \equiv y + wn + \frac{\lambda}{\Pi} \quad a \equiv \lambda' \quad b \equiv \delta_N \frac{\lambda}{\Pi}$$

with $\frac{dP}{P} = \frac{d\Pi}{\Pi}$ and $\frac{dQ}{Q} = -dr$. Hence $dz = dy + ndw - \frac{\lambda}{\Pi}\frac{dP}{P}$, $db = -\delta_N \frac{\lambda}{\Pi}\frac{dP}{P}$ and $\frac{dq}{q} = -dr$; so

$$dz + qdb - (a-b)\,dq = dy + ndw - \underbrace{(1+Q\delta_N)\frac{\lambda}{\Pi}}_{\text{NNP}}\frac{dP}{P} + \underbrace{\left(\lambda' - \delta_N\frac{\lambda}{\Pi}\right)Q\,dr}_{\text{URE}}$$

Inserting this equation into (1.7.13) yields the desired result.

Case b): The consumption of an agent at the borrowing limit when $\mathbb{S} = \mathbb{S}^*$ is given by

$$c = y + wn + (1 + Q\delta_N)\frac{\lambda}{\Pi} + \theta \cdot (\mathbf{d} + \mathbf{S}) + \frac{\overline{D}}{R}$$

under the considered change,

$$dc \simeq dy + ndw - \underbrace{(1 + Q\delta_N)\frac{\lambda}{\Pi}}_{\text{NNP}}\frac{dP}{P} + \underbrace{\left(Q\delta_N\frac{\lambda}{\Pi} + \theta \cdot \mathbf{S} + \frac{\overline{D}}{R}\right)}_{-\text{URE}}(-dr)$$

Since this agent has $MPC = 1$, and hence satisfies $\sigma c (1 - MPC) = 0$, the result follows. □

1.7.3.4 Extensions

Separable preferences Suppose that preferences are separable between consumption and hours worked:

$$U(c, n) = u(c) - v(n) \tag{1.7.14}$$

Define the elasticity of intertemporal substitution as in (1.1.9) and the Frisch elasticity of labor supply (local to the point of initial choice) as

$$\psi \equiv \frac{v'(n)}{nv''(n)}$$

Also define the total marginal propensity to spend in the present by

$$\widetilde{MPC} = MPC\left(1 + \frac{\psi}{\sigma}\frac{wn}{c}\right)$$

when income is increased by 1, consumption rises by MPC and hours worked fall by $MPN = -\frac{\psi}{\sigma}\frac{n}{c}MPC$, lowering earned income by $\frac{\psi}{\sigma}\frac{wn}{c}MPC$.

Theorem 9′. *When preferences are separable as given by (1.7.14), the change in consumption dc resulting from a purely transitory, simultaneous change in dy, dw, dP and dr is given by*

$$dc \simeq MPC\left(dy + (1 + \psi)\,ndw + UREdr - NNP\frac{dP}{P}\right) - \sigma c\left(1 - \widetilde{MPC}\right)dr \tag{1.7.15}$$

The proof is given in online appendix OA.3, and is similar in spirit to that of theorem 9. The separability of preferences delivers a clean result regarding the way in which increases in wages translate into current consumption. From the point of view of the consumption decision itself, it is sufficient to know how much earnings would change if the marginal utility of consumption was held constant, that is, $(1 + \psi)\,ndw$.

Non-separable preferences Suppose now that preferences are not separable. Instead, assume that consumption and work effort complement each other in utility, to the point that wealth effects on labor supply are inexistent: preferences take the form

$$U(c, n) = u(c - v(n)) \tag{1.7.16}$$

Also denote by ξ the share of net consumption in gross consumption:

$$\xi = 1 - \frac{v(n)}{c} = 1 - \left(1 + \frac{1}{\psi}\right)\frac{wn}{c}$$

Theorem 9″. *When preferences are non-separable as given by (1.7.16), the change in consumption dc resulting from a purely transitory, simultaneous change in dy, dw, dP and dr is given by*

$$dc \simeq MPC\left(dy + n(1+\psi)\,dw + UREdr - NNP\frac{dP}{P}\right) + (1 - MPC)\,\psi n dw - \sigma c\xi\,(1 - MPC)\,dr$$

The proof is given in online appendix OA.4. Instead of contributing equally to the increase in consumption dc as in theorem 9′, the two components of the increase in earnings induced by higher wages no longer have a symmetric role. The non-behavioral part is treated by the consumer as unearned income and contributes to an increase in consumption through the MPC. The behavioral part ($wdn = \psi ndw$), however, now has a one-for-one effect on consumption, due to the particular form of complementarity between hours and consumption assumed here.

Long-term change Consider preferences that are separable over consumption only, and a finite horizon T

$$\mathbb{E}\left[\sum_{t=1}^{T}\beta^t u(c_t)\right]$$

In section 1.7.3.2 I derived the implications of theorem 2 under separable preferences of this kind. The following is a conjecture that the results carry through under incomplete markets, provided that borrowing constraints never bind, as is the case in a Bewley model under the natural borrowing limit. Uncertain future terms need to be appropriately evaluated given the information available at the initial date, which involves the use of a modified probability measure.

Conjecture 27. *When markets are incomplete with respect to idiosyncratic shocks and borrowing constraints never bind, the consumption change to an unexpected shock to the paths for $\{q_t\}$ and $\{P_t\}$ under perfect foresight is given by*

$$dc_0 = MPCd\Omega - c_0\sigma(c_0)(1 - MPC)\frac{dq_0}{q_0} + MPC\sum_{t\geq 1}\frac{q_t}{q_0}\mathbb{E}_0^{\mathbb{Q}}\left[c_t\sigma(c_t)\right]\frac{dq_t}{q_t}$$

$$d\Omega = \sum_{t\geq 0} q_t\mathbb{E}_0^{\mathbb{Q}}\left[URE_t\right]\frac{dq_t}{q_t} - \sum_{t\geq 0} Q_t\left(\frac{{}_{-1}B_t}{P_0}\right)\frac{dP_t}{P_t}$$

where $\mathbb{Q}$ is a measure with density

$$\xi_T = \frac{u''(c_T)\,MPC_T}{\mathbb{E}\left[u''(c_T)\,MPC_T\right]} \qquad \left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]_t = \mathbb{E}_t\left[\xi_T\right]$$

1.7.4 Proofs for section 1.4

1.7.4.1 Proof of proposition 17

When the borrowing constraint binds, the consumption function is linear in λ. Otherwise, an Euler equation characterizes consumers' optimal consumption plans. The envelope theorem applied to z, $\frac{dz(e,w)}{de} = w \cdot$

n, shows that these consumers are exposed to residual idiosyncratic risk provided that $w(\mathbf{S}_t) > 0$. An adaptation of the Carroll and Kimball (1996) results then shows that the net consumption function g is strictly concave in λ. The result follows because $c = g + v(n)$ where n is independent of λ.

1.7.4.2 Proof of proposition 18

Integrating (1.4.10) using households' first-order condition for labor supply (1.4.4) as well as the existence of a stationary distribution $\varphi(\mathbf{s}_t)$ for idiosyncratic states, I obtain

$$\int_i e_t^i n_t^i di = \kappa w_t^{\psi} \equiv N(w_t) \tag{1.7.17}$$

where $\kappa = \frac{1}{b^{\psi}} \int [e(\mathbf{s})]^{1+\psi} \varphi(\mathbf{s})\, d\mathbf{s} = \frac{1}{b^{\psi}} \mathbb{E}\left[e^{1+\psi}\right]$ is a time-invariant cross-sectional moment of idiosyncratic productivity. Hence the supply curve of total effective hours worked, $N(\cdot)$, has the same constant elasticity ψ with respect to the economy's post-tax real wage $w_t \equiv (1-\tau)\frac{W_t}{P_t}$ as individual labor supply (1.4.4).

Next, integrating (1.4.6) and (1.4.2) across firms, and using intermediate-good market clearing, GDP is equal to

$$\begin{aligned} Y_t &= \frac{1}{\triangle_t} \int_{j=0}^{1} x_t^j dj \\ &= \frac{1}{\triangle_t} A_t \int_{j=0}^{1} l_t^j dj \end{aligned}$$

where

$$\triangle_t = \int_{j=0}^{1} \left(\frac{P_t^j}{P_t}\right)^{-\epsilon} dj$$

Using labor market clearing (1.4.10), together with (1.7.17), we have

$$Y_t = \frac{1}{\triangle_t} A_t \int_i e_t^i n_t^i di = \frac{1}{\triangle_t} A_t N(w_t) \tag{1.4.11}$$

1.7.4.3 Proof of proposition 19

Using the government budget constraint constraint (1.4.9), the definition of firm profits in (1.4.7), and labor market clearing (1.4.10)

$$\begin{aligned} P_t T_t &= \int_j F_t^j (P_t^j) dj + \tau \int_i W_t e_t^i n_t^i di \\ &= \int_j P_t^j x_t^j dj - W_t \int_j l_t^j dj + \tau \int_i W_t e_t^i n_t^i di \\ &= P_t Y_t - (1-\tau) W_t N(w_t) \end{aligned}$$

Exploiting the relationship between aggregate hours and output in (1.4.11)

$$\begin{aligned} T_t &= Y_t - (1-\tau) \frac{W_t}{P_t} \frac{\triangle_t}{A_t} Y_t \\ &= Y_t \underbrace{\left(1 - (1-\tau) \frac{W_t}{P_t} \frac{\triangle_t}{A_t}\right)}_{\equiv \widetilde{\tau}_t} \end{aligned}$$

Moreover, noting from (1.7.17) that

$$e_t^i n_t^i = \frac{1}{b^\psi}\left(e_t^i\right)^{1+\psi} w_t^\psi = \frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]} N\left(w_t\right) \tag{1.7.18}$$

we find that real non-financial income for an individual with productivity e_t^i is

$$\begin{aligned} Y_t^i &\equiv w_t e_t^i n_t^i + T_t \\ &= w_t N\left(w_t\right) \frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]} + \widetilde{\tau}_t Y_t \end{aligned}$$

using (1.4.11) again,

$$\begin{aligned} Y_t^i &= w_t \frac{\triangle_t}{A_t} \frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]} Y_t + \widetilde{\tau}_t Y_t \\ &= Y_t \left(\left(1-\widetilde{\tau}_t\right) \frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]} + \widetilde{\tau}_t \right) \end{aligned}$$

Finally, note that from the definition of net income z,

$$\begin{aligned} z\left(e,w\right) &= we\left(\frac{1}{b}we\right)^{\psi} - \frac{b}{1+\psi^{-1}}\left(\frac{1}{b}we\right)^{1+\psi} \\ &= \frac{1}{b^\psi}\left(we\right)^{1+\psi}\left(1-\frac{1}{1+\psi^{-1}}\right) \\ &= \frac{1}{1+\psi}\frac{1}{b^\psi}\left(we\right)^{1+\psi} \end{aligned}$$

Hence

$$\begin{aligned} Z_t^i &= z\left(e_t^i, w_t\right) + T_t \\ &= \frac{1}{1+\psi}\frac{1}{b^\psi}\left(w_t\right)^{\psi+1}\left(e_t^i\right)^{1+\psi} + T_t \end{aligned}$$

abd using (1.7.18) we finally obtain

$$\begin{aligned} Z_t^i &= \frac{1}{1+\psi} w_t N\left(w_t\right) \frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]} + T_t \\ &= Y_t \left(\frac{1-\widetilde{\tau}_t}{1+\psi} \frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]} + \widetilde{\tau}_t \right) \end{aligned}$$

1.7.4.4 Proof of proposition 21

First rewrite the extension of theorem 10 for GHH preferences in (1.2.9) in terms of elasticities, using the fact that all individuals have a common net EIS and Frisch elasticity ($\sigma^i = \sigma$, $\psi^i = \psi$), together with market clearing:

$$C_t = E_I\left[c_t^i\right] = Y_t$$

This yields

$$\frac{dC_t}{C_t} \simeq \underbrace{\mathbb{E}_I\left[\frac{Y_t^i}{Y_t}MPC_t^i\right]}_{\mathcal{M}_t}\frac{dY_t}{Y_t}+\frac{1}{Y_t}\text{Cov}_I\left(MPC_t^i, dY_t^i - Y_t^i\frac{dY_t}{Y_t}\right) - \underbrace{\text{Cov}_I\left(MPC_t^i, \frac{NNP_t^i}{\mathbb{E}_I\left[c_t^i\right]}\right)}_{\mathcal{E}_{P,t}}\frac{dP_t}{P_t}$$

$$+\left(\underbrace{\text{Cov}_I\left(MPC_t^i, \frac{URE_t^i}{\mathbb{E}_I\left[c_t^i\right]}\right)}_{\mathcal{E}_{r,t}} - \sigma\underbrace{\mathbb{E}_I\left[\xi_t^i\left(1-MPC_t^i\right)\frac{c_t^i}{\mathbb{E}_I\left[c_t^i\right]}\right]}_{\mathcal{S}_t}\right)\frac{dR_t}{R_t}+\frac{\psi}{Y_t}\mathbb{E}_I\left[\left(1-MPC_t^i\right)n_t^i e_t^i\right]dw_t$$

Next, using (1.4.13),

$$Y_t^i = Y_t\left(\frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]}+\widetilde{\tau}_t\left(1-\frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]}\right)\right)$$

so

$$\begin{aligned} dY_t^i - \frac{Y_t^i}{Y_t}dY_t &= Y_t\left(1-\frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]}\right)d\widetilde{\tau}_t \\ &= -\frac{Y_t^i - Y_t}{1-\widetilde{\tau}_t}d\widetilde{\tau}_t \end{aligned}$$

hence

$$\frac{1}{Y_t}\text{Cov}_I\left(MPC_t^i, dY_t^i - Y_t^i\frac{dY_t}{Y_t}\right) = \underbrace{-\text{Cov}_I\left(MPC_t^i, \frac{Y_t^i}{Y_t}\right)}_{\mathcal{E}_{\tau,t}}\frac{d\widetilde{\tau}_t}{1-\widetilde{\tau}_t}$$

Finally, from (1.7.18),

$$\frac{e_t^i n_t^i}{Y_t} = \frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]}\frac{N\left(w_t\right)}{Y_t} = \frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]}\frac{\Delta_t w_t}{A_t}\frac{1}{w_t} = \frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]}\frac{1-\widetilde{\tau}_t}{w_t}$$

so that

$$\begin{aligned} \frac{\psi}{Y_t}\mathbb{E}_I\left[\left(1-MPC_t^i\right)n_t^i e_t^i\right]dw_t &= \psi\left(1-\widetilde{\tau}_t\right)\mathbb{E}_I\left[\left(1-MPC_t^i\right)\frac{\left(e_t^i\right)^{1+\psi}}{\mathbb{E}\left[e^{1+\psi}\right]}\right]\frac{dw_t}{w_t} \\ &= \psi\left(1-\widetilde{\tau}_t\right)\left(1-\mathcal{M}_t^g\right)\frac{dw_t}{w_t} \end{aligned}$$

Combining all results yields the formula in (1.4.14). The other results in proposition 21 obtain by differentiating $1-\widetilde{\tau}_t = \frac{w_t}{A_t}\Delta_t$, from which we find

$$-\frac{d\widetilde{\tau}_t}{1-\widetilde{\tau}_t} = \frac{dw_t}{w_t} - \frac{dA_t}{A_t} + \frac{d\Delta_t}{\Delta_t}$$

as well as by differentiating $Y_t = \frac{\kappa}{\Delta_t}A_t n_t^{\psi}$, which yields

$$\frac{dY_t}{Y_t} = -\frac{d\Delta_t}{\Delta_t} + \frac{dA_t}{A_t} + \psi\frac{dw_t}{w_t}\ .$$

1.7.4.5 Proof of proposition 23

When prices are flexible, intermediate good price setting results in (1.4.8). Hence all firms set the same nominal price at all times ($\Delta_t = 1$), and the post-tax real wage is

$$w_t = (1-\tau)\frac{W_t}{P_t} = (1-\tau)\frac{\epsilon-1}{\epsilon}A_t = (1-\tau^*)A_t$$

where τ^* is the constant defined as

$$\tau^* \equiv 1 - (1-\tau)\frac{\epsilon-1}{\epsilon}$$

then, from the definition of the labor wedge in (1.4.12), it follows that

$$\widetilde{\tau}_t = \tau^* \quad \forall t$$

and it follows from proposition 18 that

$$Y_t = A_t N(w_t) = \kappa(1-\tau^*)^{\psi} A_t^{1+\psi}$$

1.7.4.6 Proof of proposition 24

In flexible price equilibrium, real wages are the constant $(1-\tau^*)A$, the tax intercept is the constant $T = \tau^* Y$, and the bond price is the constant $Q = \frac{1}{\Pi R - \delta_N}$. Hence, the budget constraint and borrowing constraint in (1.4.5) rewrite

$$\begin{aligned} g + \frac{1}{\Pi R - \delta_N}\lambda' &= z(e(\mathbf{s}), (1-\tau^*)A) + T + \frac{R}{\Pi R - \delta_N}\lambda' \\ \frac{1}{\Pi R - \delta_N}\lambda' &\geq -\overline{D} \end{aligned}$$

Define gross assets as $a = \frac{1}{\Pi R - \delta_N}\lambda + \overline{D}$ and cash-on-hand as

$$\chi = z(e(\mathbf{s}), (1-\tau^*)A) + T + \frac{R}{\Pi R - \delta_N}\lambda + \overline{D}$$

The consumer's net consumption policy is then the solution to the stationary Bellman equation

$$V(\chi; \mathbf{s}) = \max_{\hat{a} \geq 0} u(\chi - \hat{a}) + \beta \mathbb{E}\left[V\left(Z(\mathbf{s}) + R\hat{a} - \overline{D}(R-1); \mathbf{s}'\right)|\mathbf{s}\right] \tag{1.7.19}$$

The maturity structure parameter δ_N does not enter equation (1.7.19). Hence in steady state, aggregate consumption demand is neutral with respect to this parameter. Since aggregate labor supply is also unaffected by it, this proves the proposition.

1.7.4.7 Monetary policy shock in the representative-agent model

Following the tradition in the New Keynesian literature, I present the equations in log-linearized form. Letting $r_t = \log \frac{R_t}{R^*}$, the Euler equation is:

$$\mathbb{E}_t[\hat{c}_{t+1}] - \hat{c}_t = \sigma \xi r_t + (1-\tau^*)(\mathbb{E}_t[\hat{n}_{t+1}] - \hat{n}_t) \tag{1.7.20}$$

where $\xi = 1 - \frac{v(N)}{C} = 1 - (1-\tau^*)\frac{\psi}{1+\psi}$. Equation (1.7.20) illustrates the amplification mechanism inherent in the complementarities between hours and consumption. Since productivity is unchanged following a shock to the real interest rate, a change in consumption must be matched by a change in hours worked, $\widehat{c_t} = \widehat{n_t}$. This in turn raises the marginal utility of consumption. In this way, a one percentage point increase in consumption around the steady-state raises net consumption by τ^* per cent. Since consumers substitute intertemporally with respect to *net* consumption, a multiplier of $\frac{1}{\tau^*}$ applies from net to gross consumption. In equilibrium, $\hat{c}_t$ follows the following dynamic equation in response to interest rate shocks:

$$\hat{c}_t = \mathbb{E}_t\left[\hat{c}_{t+1}\right] - \frac{\sigma\xi}{\tau^*}r_t$$

which can be solved for the path of $\widehat{c_t}$ given the path for r_t, assuming that the central bank's commitment to future stabilization ensures a return to the initial steady-state in the long run.

1.7.5 Calibration of the earnings process and solution technique

The steady-state wedge τ^* plays a crucial role in the analysis. As is clear from proposition 19, it determines the degree of inequality in earnings, and hence the strength of the precautionary savings motive, given a process for idiosyncratic uncertainty.[54] I therefore calibrate it jointly with the productivity process as follows.

Since Lillard and Weiss (1979) and MaCurdy (1982), a large literature has fitted earnings processes to panel data on labor earnings, in particular to PSID data on male earnings. A consensus from the literature is that the earnings process features an important degree of persistence: the data on annual, log pre-tax labor earnings is reasonably described by an $AR(1)$ process with a large autoregressive root, possibly a unit root.

Since my model with infinitely-lived agents exploits the existence of a stationary distribution to define steady-state aggregates, I postulate that individual-level productivity follows an $AR(1)$ process in logs at quarterly frequency

$$\log e_t^i = \rho \log e_{t-1}^i + \sigma_e\sqrt{1-\rho^2}\epsilon_t^i \quad \epsilon_t^i \sim \mathcal{N}(0,1) \tag{1.7.21}$$

with $\rho < 1$. (1.7.21) admits the stationary distribution $\log e^{SS} \sim \mathcal{N}\left(0, \sigma_e^2\right)$. In the steady-state of my model, pre-tax earnings $e_t^i n_t^i$ are proportional to $\left(e_t^i\right)^{1+\psi}$. This implies that the steady-state variance of log earnings is $(1+\psi)^2\sigma_e^2$. Moreover the process, sampled at annual frequency, is an $AR(1)$ with root ρ^4.

In the PSID, the cross-sectional standard deviation of log head pre-tax earnings in 2009 is 1.04. This number been relatively stable over time since 1968. I therefore set $\sigma_e = \frac{1.04}{1+\psi}$. I then discretize the idiosyncratic productivity process by using a 10-point Markov Chain using the procedure described in Tauchen (1986). Figure 1.7.5 plots the PSID Lorenz curve for post-tax earnings against that obtained in the model, illustrating that a lognormal distribution fits the majority of the earnings distribution very well.

Typical calibrations of the earnings process in this class of models (for example, Aiyagari, 1995; Floden and Lindé, 2001 or Guerrieri and Lorenzoni, 2015) assume smaller standard deviations for log earnings than

[54] In section 1.5.5 we saw that it is also crucially related to the output multiplier when prices are sticky.

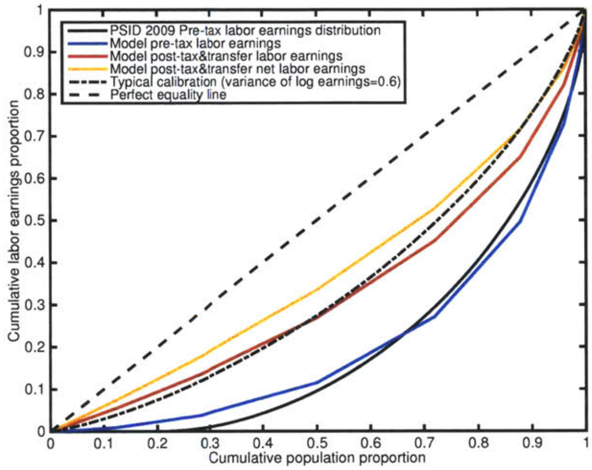


Figure 1.7.5: Calibration of pre- and post-tax earnings process

1.04, because they calibrate residual earnings uncertainty and therefore do not seek to match the earnings distribution as I do here. In my model, the driver of consumption-smoothing behavior is *post*-tax-and-transfer earnings, whose standard deviation is controlled by τ^*. I therefore set $\tau^* = 0.4$ to match the post-tax labor earnings distribution that these studies typically take (with a standard deviation of logs of 0.6).

The value of ρ^4 is more controversial. Two papers that estimate the process in the PSID and use it to calibrate an incomplete-market model are Heaton and Lucas (1996) and Floden and Lindé (2001). The former use a value of 0.53, while the latter use 0.91. I settle for $\rho^4 = 0.8$, implying a quarterly degree of persistence $\rho = 0.9457$.

Numerical solution technique To compute policy functions for steady states and transitional dynamics, I use the method of endogenous gridpoints (Carroll, 2006; Guerrieri and Lorenzoni, 2015). The algorithm for finding the steady-state is standard. For transitional dynamics, I use the following procedure. Starting from a date T at which the economy is assumed to have returned to steady-state, I compute the policy functions backwards using the path for known macroeconomic variables. I then compute the bond distribution forward from the initial distribution Ψ_0. For each date at which there is an excess demand or supply in the goods market, I adjust the real interest rate (for the flexible-price economy) or the real wage (for the sticky-price economy) in the direction of market clearing. Using this new path, I start the procedure again and iterate until convergence. Details are provided in the online appendix, section OB.

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Chapter 2

Monetary Union Begets Fiscal Union

> ***Transfer criterion***
> *Countries that agree to compensate each other for adverse shocks form an optimum currency area.*
>
> *Baldwin and Wyplosz (2012), Chapter 15*

A simplified narrative of the recent crisis in the Eurozone can be given as follows. Following the adoption of the single currency, a number of "periphery" countries progressively lost competitiveness as their real unit labor costs grew faster than the union average. When the global financial crisis hit in 2008, the accumulated internal imbalances came into sharp focus. Given their fixed nominal exchange rate vis-à-vis other Eurozone members, the only way periphery countries could achieve the real depreciation needed to regain competitiveness was through a painful process of economic contraction bringing about falls in domestic prices. Indeed, this adjustment process was so damaging to periphery economies that there was mounting speculation that some of them might leave the Euro. Monetary union was under stress. But it ultimately did not break: Eurozone countries were strongly bound to their single currency despite the large stabilization costs that it induced.

Meanwhile, periphery countries' large external and fiscal deficits became increasingly difficult to finance. Some of the financing was bridged through bailout packages, but their size was limited by reluctance from core country taxpayers who were supposed to fund them. Eurozone fiscal union was only implicit, and core countries hit their participation constraint.

Our paper studies fiscal unions subject to such participation constraints, contrasting their role inside and outside a monetary union. To capture features of fiscal union that resemble those of the Eurozone today, we assume that countries have limited ability to commit to risk-sharing. Specifically, we require that cross-country transfers always be part of a subgame-perfect equilibrium of a repeated game: in our model, countries only make transfers if these transfers are backed by credible promises of future reciprocity. This emphasis on reciprocity is supported by a recent study from the IMF (Allard, Brooks, Bluedorn, Bornhorst,

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Christopherson, Ohnsorge, Poghosyan and IMF Staff Team (2013)), which finds that "with a risk-sharing mechanism in place over a sufficiently long period, all current euro area members would have benefited from transfers at some point in time". Meanwhile, we capture the costs of monetary union by introducing nominal rigidities. When countries are part of a monetary union, the central bank can stabilize the union as a whole, but not each individual country—leading to overheating in some countries and to recessions in others. These stabilization costs are absent under independent monetary policy, where each country's central bank can stabilize its own economy separately.

Our primary result is that monetary union enhances fiscal union. The stabilization costs induced by monetary union *also* make countries more willing to share risks. This is due to an interaction between the degree of risk-sharing and the ability of the union-wide central bank to stabilize its members: when countries share risks better, there is less divergence in the stance of monetary policy that is appropriate for each country individually. In our benchmark model, this idea is illustrated starkly by the *risk-sharing miracle*: when its member countries share risks perfectly, the union-wide central bank is able to stabilize all of them simultaneously.

Our paper proposes a mechanism through which monetary union between countries leads to a stronger fiscal union. By doing so, it contributes to *sequencing theory*, a field of international relations that studies how one type of economic cooperation can lay the foundation for the next. This literature often takes as a starting point an interpretation of Balassa (1962), according to which regional integration takes place by progressively climbing the steps of the "integration staircase" depicted in figure 2.0.1 (Gustavsson (1999), Estevadeordal and Suominen (2008), Baldwin (2012)). Central to sequencing theory is the existence of spillovers that make integration gather momentum by begetting further cooperation in other areas. For example, Haas (1958), in his famous study of the European Coal and Steal Community in the early 1950s, emphasizes the ability of the newly-founded institution to support special interest groups that pushed for broader economic integration, eventually leading to the European Economic Community. Our paper microfounds the spillovers that enable countries to climb the last step of the integration staircase: in a monetary union, the absence of fiscal union becomes more costly, and countries internalize this when deciding on fiscal cooperation.

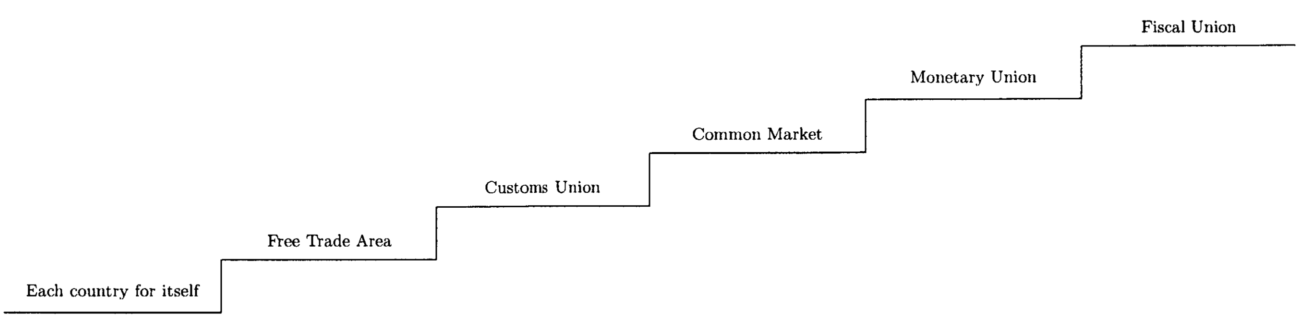

Figure 2.0.1: The integration staircase

Our paper also adds to the theory of Optimum Currency Areas (Mundell (1961), McKinnon (1963), Kenen (1969)) by proposing a new tradeoff, between stabilization costs and *risk-sharing benefits* of monetary union. While the cost side of the ledger—the difficulty of monetary union to cope with asymmetric shocks—has been well understood since at least Friedman (1953), the benefit side has lacked comparatively compelling microfoundations. The literature emphasizes such diverse advantages as the elimination of transactions costs or the gains from reduced uncertainty, which we find less tangible than the risk-sharing benefits stressed in this paper. Another recent paper revisiting the benefits side of the monetary union ledger is Chari, Dovis and Kehoe (2013). Their emphasis is on the benefits to central bank credibility, pointing out that a union may reduce overall inflationary bias to the extent that the shocks that create a desire to inflate are not perfectly correlated across countries. This channel is absent in our model, in which we shut off all forms of inflationary bias.

One way to read our paper is thus, in the light of the OCA literature, as a direct argument for why countries might form monetary unions despite their apparent costs. Our argument is that these costs may in fact be the seed of the benefits: once the monetary union is joined, the possibility of high stabilization costs in the absence of fiscal cooperation enforces the cooperation itself, and limits the incurrence of the costs.

Having established our main result, we go further and ask what the union-wide central bank can do, given additional commitment power, to proactively encourage the fiscal union. We find that it can help, by departing from its traditional role of aggregate stabilization and committing to accomodative monetary policy in volatile times. Accomodative monetary policy helps because it creates an overall boom in the union, which in turn effectively relaxes countries' participation constraints and facilitates transfers. This incentive effect is new to the literature on optimal monetary policy in currency unions.

Our model puts together two distinct strands of the literature. The first is the literature on limited commitment (Kehoe and Levine (1993), Coate and Ravallion (1993), Kocherlakota (1996), Alvarez and Jermann (2000), Ligon, Thomas and Worrall (2002)). This literature derives endogenous constraints on risk-sharing by giving agents the option to leave transfer arrangements at any point in time, and it focuses on the best outcomes that are sustainable given these constraints. When countries run an independent monetary policy, our setup reduces to the one described by this literature, and the same forces—the degree of patience, risk-aversion, and the persistence of the idiosyncratic endowment processes—drive the feasible amount of risk-sharing.

We combine the constraints on risk-sharing featured by the limited commitment literature with the constraints on monetary policy emphasized by the modern international economics literature on currency unions in the presence of nominal rigidities (Benigno (2004), Galí and Monacelli (2005), Galí and Monacelli (2008)). This literature provides a microfoundation for the stabilization costs that arise in currency unions, as the central bank is generally unable to fully stabilize each member country and must balance out the cross-country distortions it creates by setting its policy instrument at an intermediate level.

Our paper is close in spirit and in form to Farhi and Werning (2013), who also study the benefits of a fiscal union of the kind we describe — cross-country insurance arrangements — within a monetary union. While they focus on the constrained inefficiency of private arrangements and the need for government interventions to reach a constrained-optimal outcome, we shut off private international financial markets and study a constraint faced by the *governments* in their implementation of the optimal outcome. In doing so, we extend their framework to allow for a full game-theoretic analysis of monetary and fiscal policy. In most of our paper, the risk-sharing miracle holds and the constrained-optimal outcome is also first-best, a case which is of limited interest in Farhi and Werning (2013), but which we regard as capturing in an elegant way the widespread view that alignment of fiscal policy limits the costs of monetary union. Under the risk-sharing miracle, nominal rigidities only create a cost to the extent that a limited commitment friction binds and prevents countries from reaching a full risk-sharing outcome. As we briefly discuss, this would no longer be true if we allowed for shocks to preferences or nontradables productivity.

Another paper that discusses monetary union in the presence of a limited commitment friction is Fuchs and Lippi (2006). Their focus is on the short-term commitment benefits brought about by monetary union, in a situation where independent central banks might otherwise be tempted to follow beggar-thy-neighbor policies. They use a reduced form to specify country preferences over the level of the monetary policy instrument. In contrast, we abstract away from the interesting possibility that monetary union might break up, but we fully endogenize fiscal and monetary policy, assuming both policies maximize country welfare subject to the constraints of the environment (limited commitment and nominal rigidities). This allows us to study the rich ways in which these two frictions—and these two types of policies—interact.

Many recent policy discussions have been focused on the need to establish fiscal federalism in the Euro Area. Our model recognizes the importance of these efforts. Except in the special case where our mechanism is so powerful as to endogenously lead countries to share risks perfectly, the limited commitment friction does create costs, so it is valuable to try and mitigate it. In fact, the risk-sharing miracle implies that if countries could overcome the commitment friction altogether, they would be able to attain the first best. In practice, it has been difficult to get countries to sign formal agreements regarding contingent future fiscal transfers. We have two insights to add here. First, we emphasize that because it is in the private interest of countries to internalize the macroeconomic externalities associated with monetary union (Farhi and Werning (2013)), a stronger fiscal union might emerge on its own: the set of possible equilibria is enlarged, though countries may take time to move to the more cooperative one. Second, our normative analysis suggests that proactive monetary policy might be used as an imperfect substitute to fiscal union, nudging countries into sharing risks better.

The rest of our paper is organized as follows. Section 2.1 introduces our main framework, laying out our model's game-theoretic foundations and defining our equilibrium concept. It then proves a number of properties of equilibria which simplify the analysis in the rest of the paper. Among these are the risk-sharing miracle (Theorem 36) and the ability to sustain any subgame perfect equilibrium using strategies that revert to autarky following any deviation (Theorem 41). Section 2.2 considers a case where countries' endowments satisfy a simple symmetry condition, and delivers two main results that substantiate our claim that monetary union enhances fiscal union. Theorem 46 shows any risk-sharing arrangement that is sustainable under independent monetary policy is also sustainable under monetary union. It is a sharp illustration of the sense in which the monetary union improves risk-sharing. Theorem 47 shows that, under certain cases, the monetary-union-induced improvement in risk sharing is so powerful that it can take countries all the way from autarky to first best. Section 2.3 proposes a normative analysis of monetary policy when fiscal union is subject to a limited commitment friction. Theorem 50 shows that it is valuable to provide aggregate stimulus in times of high volatility in order to create a macroeconomic environment favorable to transfers between countries. Section 2.4 concludes. All proofs are in Appendix 2.5.

2.1 Main framework

2.1.1 Fundamentals

Two countries $i = 1, 2$ live forever and have identical preferences. Each values the stream of tradables consumption $\{C^i_{T,t}\}$, nontradables consumption $\{C^i_{NT,t}\}$ and labor $\{N^i_t\}$ according to the utility function

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u\left(C^i_{T,t}, C^i_{NT,t}, N^i_t\right)\right]$$

where we specify the felicity function to be log in tradables and nontradables, and isoelastic in labor:

$$u\left(C_T, C_{NT}, N\right) = \log\left(C_T\right) + \alpha\left(\log\left(C_{NT}\right) - \frac{N^{1+\phi}}{1+\phi}\right) \tag{2.1.1}$$

All goods are perishable, tradables goods are nonproduced, nontradable goods have to be consumed where they are produced, and labor is immobile across countries.

There exists another country in the world that is also endowed with tradables. The only purpose of this country in the model is to provide the reference unit of account, as Section 2.1.1.3 will discuss.

2.1.1.1 Tradable goods

Aggregate uncertainty is decribed by a finite-state Markov process $\{s_t\}$ with elements $s_t \in \mathbf{S}$ and transition matrix Π. A history of length t is denoted by $s^t = (s_0, \ldots, s_t)$. We write $s^\tau \succeq s^t$ to indicate that s^τ is a successor node of s^t.

Each country has a risky endowment $E^i_T(s_t)$ of an identical, freely tradable good. The aggregate state s_t thus determines the distribution of tradable endowments across the two countries. For now, tradable endowment shocks are the only source of uncertainty in the model.

Assumption 28 (External balance). *The union achieves external balance in each history s^t:*

$$C_T^1\left(s^t\right)+C_T^2\left(s^t\right)=E_T^1\left(s_t\right)+E_T^2\left(s_t\right)\equiv E_T\left(s_t\right)\quad\forall s^t=\left(s_0,\ldots,s_t\right)$$

Assumption 29 (Strict benefits from tradables risk-sharing). *For all $s\in\mathbf{S}$, there exists $s'\in\mathbf{S}$ such that*

$$\frac{E_T^1\left(s\right)}{E_T^2\left(s\right)}\neq\frac{E_T^1\left(s'\right)}{E_T^2\left(s'\right)}$$

Since countries have concave expected utility over tradable consumption, assumptions 28 and 29 together imply that there exist ex-ante utility gains from sharing risks by arranging state-contingent transfers of tradables. Like Farhi and Werning (2013), we call such an arrangement a "fiscal union". In our model, the extent of risk-sharing is limited by a commitment friction which we will soon describe.

2.1.1.2 Nontradable goods

Nontradable goods are produced from labor by a continuum of firms. We abstract from uncertainty regarding nontradable production, and discuss the consequences of relaxing this assumption in Section 2.2.3. In each country i, there is a continuum of firms $j\in[0,1]$ that each operate the simple technology

$$Y_{NT}^{i,j}=N^{i,j}$$

in each period[1]. The consumer's utility value from the consumption of each variety is given by the CES aggregator

$$C_{NT}^i=\left(\int_{j=0}^1\left(C_{NT}^{i,j}\right)^{\frac{\epsilon-1}{\epsilon}}dj\right)^{\frac{\epsilon}{\epsilon-1}}\qquad\epsilon>1\tag{2.1.2}$$

Consumption of each variety must equal production, $C_{NT}^{i,j}=Y_{NT}^{i,j}$, and labor market clearing requires $N^i=\int_j N^{i,j}dj$.

Our assumed preferences and production structure are intended to make the first-best level of nontradables a simple reference point. Since all firms have the same technology, efficiency requires them to produce equally, in which case $C_{NT}^{i,j}=N^{i,j}=C_{NT}^i=N^i$ for all j. Optimizing utility (2.1.1) subject to this constraint gives $C_{NT}^i=N^i=1$.

Lemma 30. *An efficient allocation of production requires $C_{NT}^{i,j}=N^{i,j}=1$, $\forall i,j$*

In equilibrium, production may depart from this efficient level as a result of monopoly power and nominal rigidities.

[1]To reduce notation, we suppress dependence on the time and state whenever this is unambiguous.

2.1.1.3 Rest of the world and units of account

In order to discuss exchange rate regimes, we need to be specific about units of account. We assume that the homogenous tradable good is traded as part of a world market, and that its foreign-currency price is normalized to $P_T^*(h^t) = 1$ in all histories h^t (h^t consists of the exogenous state s^t as well as the history of previous actions by all agents, as described below). The foreign currency, which we call the dollar, then provides a natural reference unit of account, and we assume that transfers between countries are specified in that unit of account. With this interpretation, assumption 28 amounts to imposing that the two countries have a closed capital account vis-à-vis the rest of the world.

We think of monetary policy as fixing the nominal exchange rate $\mathcal{E}^i(h^t)$ in each history—that is, the number of units of domestic currency it stands ready to buy or sell per dollar. By the law of one price for tradables, the exchange-rate policy of the central bank effectively fixes the domestic currency price of the tradable good at $P_T^i(h^t) = \mathcal{E}^i(h^t)$ in every history h^t. The key difference between flexible exchange rates and a monetary union is that, in the latter, the union-wide central bank has to set a common exchange rate $\mathcal{E}^1(h^t) = \mathcal{E}^2(h^t)$ in each history.

2.1.2 Timing and equilibrium

As ensured by Assumption 29, there are gains from risk-sharing in tradable goods. We study an environment where transfers between countries emerge as part of a subgame perfect equilibrium of a repeated game. Three distinct types of actors participate in this game: monopolistically competitive firms setting prices for nontradable goods, fiscal authorities for each country, and—depending on the monetary regime—either a common central bank for both countries or two independent central banks.

The timing within each period is given in Figure 2.1.1. We start by outlining the sequence of events informally, before describing each part in detail in Sections 2.1.2.1-2.1.2.4.

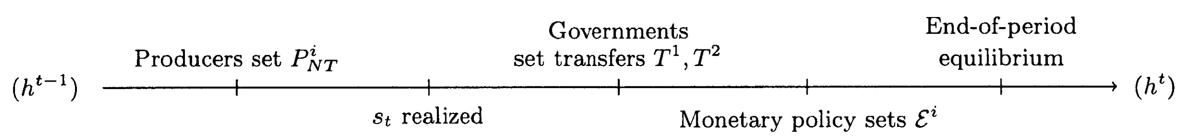

Figure 2.1.1: Timing

At the beginning of period t, h^{t-1} includes the history of all previous actions and states. Each actor in period t has a pure strategy conditional on both the incoming history h^{t-1} and any preceding actions within period t. First, before the state is realized, firms set nontradable goods prices (in the domestic unit of account) based on h^{t-1}, producing a price distribution φ_t^i in each country. Once the state s_t is realized, the fiscal authority in each country makes a transfer T^i based on h^{t-1}, $\{\varphi_t^i\}$, and s_t. As discussed in Section 2.1.1.3, this transfer is specified in the international unit of account. Next, exchange rates $\mathcal{E}_t^i$ are chosen either separately in each country or—in the case of a monetary union—commonly for both, based on h^{t-1},

$\{\varphi_t^i\}$, s_t, and $\{T_t^i\}$. Finally, based on the state and all actions thus far in the period, the end-of-period market determines production and consumption according to household demand.

In the following sections, we proceed by backward induction, describing how the outcome at each step within the period is determined, taking subsequent strategies as given. As depicted in Figure 2.1.1, there are four steps of interest: end-of-period equilibrium, monetary policy, fiscal policy, and pricesetting. These are the subjects of Sections 2.1.2.1-2.1.2.4, respectively.

2.1.2.1 End-of-period equilibrium

Households. At the end of the period, households in country i are faced with prices $P_{NT}^{i,j}$, P_T^i, and W^i, as well as profits $\psi^{i,j}$ earned from each firm j's production and a lump sum tax t^i from the domestic government. The nontradable goods prices $P_{NT}^{i,j}$ have already been set, while the prices P_T^i and W^i are determined on a Walrasian market. We assume that households do not have access to financial markets. We could allow them to access domestic financial markets without loss of generality: since the government has access to a lump-sum tax, Ricardian equivalence would hold.

Households optimally choose consumption

$$\left\{C_T^i, C_{NT}^{i,j}, N^i\right\} \in \arg\max_{\{\hat{C}_T^i, \hat{C}_{NT}^{ij}, \hat{N}^i\}} \left(\log\left(\hat{C}_T^i\right) + \alpha\left(\log\left(\hat{C}_{NT}^i\right) - \frac{\left(\hat{N}^i\right)^{1+\phi}}{1+\phi}\right)\right)$$

$$\text{s.t.} \quad P_T^i \hat{C}_T^i + \int_{j=0}^{1} P_{NT}^{i,j}\hat{C}_{NT}^{i,j}\mathrm{d}j \le P_T^i E_T^i + W^i\hat{N}^i + \int_{j=0}^{1}\psi^{i,j}dj - t^i \tag{2.1.3}$$

where $\hat{C}_{NT}^i$ is the aggregator in (2.1.2). The following lemma is derived from the first-order conditions of the problem.

Lemma 31. *At an optimum of the consumer problem, tradable and nontradable consumption are proportional:*

$$C_{NT}^i = \alpha\frac{P_T^i}{P_{NT}^i}C_T^i \tag{2.1.4}$$

good-specific non-tradable demand is

$$C_{NT}^{i,j} = \left(\frac{P_{NT}^{i,j}}{P_{NT}^i}\right)^{-\epsilon} C_{NT}^i \tag{2.1.5}$$

and labor supply is

$$N^i = \left(\frac{W^i}{P_{NT}^i}\frac{1}{C_{NT}^i}\right)^{\frac{1}{\phi}} \tag{2.1.6}$$

where $P_{NT}^i = \left(\int_j \left(P_{NT}^{i,j}\right)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}}$ *is the price index associated with* $\left\{P_{NT}^{i,j}\right\}$.

Profits of nontradable goods producers. We assume that governments subsidize the labor cost of firms at a rate $\tau_L^i = -\frac{1}{\epsilon}$. Given its price $P_{NT}^{i,j}$, producer j honors demand $C_{NT}^{i,j}$ by hiring $N^{i,j}$ workers and generates profits

$$\psi^{i,j} = \left(P_{NT}^{i,j} - \left(1+\tau_L^i\right) W^i\right) N^{i,j} \tag{2.1.7}$$

which are remitted to households as part of their budget (2.1.3).

Government. As determined earlier in the period, the government of country i sends an international transfer of T^i and receives one of T^{-i}, both denominated in dollars. It operates the labor tax, and rebates all profits to households. Its budget constraint, expressed in domestic currency units, is then

$$t^i = \mathcal{E}^i \left(T^{-i} - T^i\right) + \tau_L^i W^i \int_j N^{i,j} dj \tag{2.1.8}$$

Law of one price for tradable goods. As per its decision earlier in the period, the central bank sets its nominal exchange rate $\mathcal{E}^i$ against the dollar. Given this exchange rate, domestic residents can purchase tradable goods from the rest of the world at price $\mathcal{E}^i$ or from the domestic market at price P_T^i. Equilibrium requires that the two be equated, to prevent pure arbitrage profits:

$$P_T^i = \mathcal{E}^i \tag{2.1.9}$$

Market clearing for labor. Each firm hires to meet demand based on the price it posted. A firm with posted price $P_{NT}^{i,j}$ must hire

$$N^{i,j} = Y_{NT}^{i,j} = \left(\frac{P_{NT}^{i,j}}{P_{NT}^i}\right)^{-\epsilon} C_{NT}^i \tag{2.1.10}$$

where C_{NT}^i is country i's aggregate nontradable demand. Labor market clearing requires that

$$N^i = \int_j N^{i,j} dj = \Delta_{NT}^i C_{NT}^i \tag{2.1.11}$$

were $\Delta_{NT} \equiv \int_j \left(\frac{P_{NT}^{i,j}}{P_{NT}^i}\right)^{-\epsilon} dj \geq 1$ is a measure of price dispersion.

Indirect utility function. Conditional on the state and realized actions earlier in the period, end-of-period equilibrium is a nominal wage W^i, a tradables price P_T^i, household quantities $\left\{C_T^i, C_{NT}^{i,j}, N^i\right\}$, firm quantities $\left\{Y_{NT}^{i,j}, \psi^{i,j}, N^{i,j}\right\}$ and a domestic government transfer t^i, such that household optimization conditions (2.1.4)-(2.1.6) are satisfied, household budgets are balanced (2.1.3), the law of one price (2.1.9) holds, firms produce to meet customer demand according to (2.1.10) and generate profits (2.1.7), the government balances its budget (2.1.8), and the labor market clears (2.1.11).

Lemma 32. *Given $\{\varphi^i\}, s, \{T^1, T^2\}, \mathcal{E}^i$, end of period equilibrium is unique. The nontradable price index* *and price dispersion are given by*

$$P_{NT}^{i}=\left(\int p^{1-\epsilon}d\varphi^{i}(p)\right)^{\frac{1}{1-\epsilon}};\quad \Delta_{NT}=\int\left(\frac{p}{P_{NT}^{i}}\right)^{-\epsilon}d\varphi^{i}(p)$$

Country i consumes tradables

$$C_{T}^{i}=E_{T}^{i}(s)+T^{-i}-T^{i}$$

while on the nontradable side its production, consumption, and labor are given by

$$Y_{NT}^{i}=C_{NT}^{i}=\alpha\frac{\mathcal{E}^{i}}{P_{NT}^{i}}C_{T}^{i};\quad N_{T}^{i}=\alpha\frac{\mathcal{E}^{i}}{P_{NT}^{i}}\Delta_{NT}^{i}C_{T}^{i}$$

The country attains indirect utility

$$v^{i}(\{\varphi^{i}\},s,\{T^{1},T^{2}\},\mathcal{E}^{i})=\log\left(C_{T}^{i}\right)+\alpha\left(\log\left(\alpha\frac{\mathcal{E}^{i}}{P_{NT}^{i}}C_{T}^{i}\right)-\frac{\left(\alpha\frac{\mathcal{E}^{i}}{P_{NT}^{i}}\Delta_{NT}^{i}C_{T}^{i}\right)^{1+\phi}}{1+\phi}\right)\tag{2.1.12}$$

If we define $V^{i}(h^{t-1})$ to be the expected utility of a country starting at history h^{t-1}, the following recursion holds, leaving dependence of equilibrium objects on history implicit for simplicity of notation:

$$V^{i}(h^{t-1})=\sum_{s_t}\pi(s_t|s_{t-1})\left(v^{i}(\{\varphi_{t}^{i}\},s_t,\{T_{t}^{1},T_{t}^{2}\},\mathcal{E}_{t}^{i})+\beta V^{i}(h^{t})\right)\tag{2.1.13}$$

2.1.2.2 Central bank

Monetary authority's objective. In each period the central bank acts after observing the price distribution for nontradables $\{\varphi^{i}\}$, the state s, and the government transfers of $\{T^{1},T^{2}\}$, by setting the nominal exchange rate. We consider two monetary regimes. Under independent monetary policy, country i's central bank sets the exchange rate $\mathcal{E}^{i}$ to maximize agent welfare in the end-of-period equilibrium:

$$\mathcal{E}^{i}=\arg\max_{\hat{\mathcal{E}}^{i}}v^{i}(\{\varphi^{i}\},s,\{T^{i}\},\hat{\mathcal{E}}^{i})\tag{2.1.14}$$

Under monetary union, we assume that a unified central bank sets the common exchange rate $\mathcal{E}\equiv\mathcal{E}^{1}=\mathcal{E}^{2}$ to maximize an equally weighted average of country welfare:

$$\mathcal{E}=\arg\max_{\hat{\mathcal{E}}}\frac{1}{2}v^{1}(\{\varphi^{1}\},s,\{T^{1},T^{2}\},\hat{\mathcal{E}})+\frac{1}{2}v^{2}(\{\varphi^{2}\},s,\{T^{1},T^{2}\},\hat{\mathcal{E}})\tag{2.1.15}$$

Beyond the natural choice of a weighted average of country welfare as an objective for the central bank in the union, the objective objective function (2.1.15) embodies two assumptions. The first one is an assumption of equal weights: this is natural given that countries have identical preferences and hence an equally-sized efficient nontradable sector (Lemma 30). The second is the assumption of a static objective. We make this assumption to prevent the central bank from becoming involved as a intertemporal player in the repeated game. It is equivalent to restricting the set of subgame perfect equilibria to exclude fiscal strategies that depend on past monetary policy. Among other things, this eliminates equilibria where the central bank uses monetary policy to punish current deviators from the fiscal union, and is itself incentivized to enforce punishments because future fiscal cooperation depends on its actions.

Ruling out equilibria where the central bank can act as a strategic enforcer of fiscal union allows us to focus on the more direct channels through which monetary and fiscal union are related. Since a primary message of this paper is that monetary union encourages fiscal risk-sharing, we view this as a conservative choice: these more elaborate equilibria only strengthen the monetary authority's role in fiscal union. Later, in Section 2.3, we will explore an alternative timing that allows the central bank to behave more strategically.

Stabilization. Let $\tau^i(s)$ denote the labor wedge in country i in end of period equilibrium, defined such that $1-\tau^i(s)$ is the ratio of the marginal rate of substitution to the marginal rate of transformation between labor and aggregate nontradables:

$$1-\tau^i(s) \equiv C^i_{NT}(s)\,\Delta^i_{NT}(N^i(s))^{\phi} \tag{2.1.16}$$

Lemma 33. *An independent central bank in country i, maximizing (2.1.14), sets*

$$\tau^i(s) = 0 \tag{2.1.17}$$

and therefore achieves the first-best in equilibrium ($C^{i,j}_{NT} = N^{i,j} = 1$, $\forall i, j$). The central bank in a monetary union, maximizing (2.1.15), sets

$$\frac{1}{2}\tau^1(s) + \frac{1}{2}\tau^2(s) = 0 \tag{2.1.18}$$

An independent central bank simply sets the labor wedge in its own country to 0. Since we will show in Section 2.1.2.4 that nontradable pricesetting results in no price dispersion ($\Delta^i_{NT} = 1$), a labor wedge of 0 is equivalent to the efficient level of nontradable consumption and production given in Lemma 30. Monetary policy then replicates the outcome that would prevail under flexible prices. The central bank in a monetary union sets an average of labor wedges to 0.

In this light, we can view objectives (2.1.14) and (2.1.15) as rules for stabilizing aggregate demand, a traditional role of monetary policy. These optimality conditions are featured by the literature on optimal monetary policy in currency unions (Benigno (2004), Galí and Monacelli (2008), Farhi and Werning (2013)).

2.1.2.3 Transfer policy

The government of each country i has a pure transfer strategy $T^i(h^{t-1}, (\varphi^1_t, \varphi^2_t), s_t)$, which specifies a transfer in period t conditional on the full history h^{t-1} from earlier periods, as well as the nontradable price distributions $(\varphi_t^1, \varphi_t^2)$ and exogenous state s_t realized thus far in period t. We restrict these transfers to lie in the interval $\left[0, E_T^i(s_t) - \epsilon\right]$ for some $\epsilon > 0$. This ensures compactness of the strategy set and thus that all values are finite.

In subgame perfect equilibrium, the transfer T^i in period t is set so that

$$T^i = \arg\max_{\hat{T}^i} v^i(\{\varphi_t^i\}, s_t, \{\hat{T}^i, T^{-i}\}, \mathcal{E}_t^i) + \beta V^i(h^t) \tag{2.1.19}$$

where $\{\varphi_t^i\}$ and s_t are already known, T^{-i} is given by the equilibrium strategy of the other country, $\mathcal{E}_t^i$ is given by the central bank's optimal response characterized in Lemma 33, and $V^i(h^t)$ is defined in (2.1.13). $V^i(h^t)$ implicitly incorporates the reaction of future transfers to the current decision.

Equation (2.1.19) shows that when choosing its transfer policy, the government internalizes the effect this transfer has on current indirect utility, taking into account the direct effect of the transfer on tradables consumption, as well as the indirect effect of the transfer on the nontradable side of the economy and the reaction of the central bank to the transfer. But the main tradeoff embedded in (2.1.19) is between present and future: by choosing a lower transfer $\hat{T}^i$, country i can usually improve its current utility v^i, but this may be at the expense of future utility V^i. Positive transfers on the equilibrium path are sustained by strategies that, off the equilibrium path, punish deviating countries with lower future transfers.

2.1.2.4 Nontradable pricesetting

Nontradable pricesetters in country i maximize expected profits (2.1.7) in end-of-period equilibrium, weighted by the stochastic discount factor of the country i household.

Lemma 34. *In each country i, in equilibrium all nontradable pricesetters j set the same price $P_{NT}^{i,j} = P_{NT}^i$, so there is no price dispersion ($\triangle_{NT}^i = 1$) and the price distribution φ^i is degenerate. The labor wedge is then*

$$\tau^i(s) = 1 - C_{NT}^i(s)^{1+\phi}$$

and P_{NT}^i is such that the expected labor wedge (2.1.16) in country i across all states is 0:

$$\sum_s \pi(s|s_{-1})\tau^i(s) = 0 \tag{2.1.20}$$

Note that (2.1.20), which sets the expected labor wedge for a country equal to 0, is consistent with the characterization of monetary policy in both (2.1.17)—which sets the ex-post labor wedge in the country to 0—and (2.1.18)—which sets the ex-post average of labor wedges across both countries to 0. This is necessary for equilibrium to exist, and it is due to the labor subsidy $\tau_L^i = -\frac{1}{\epsilon}$, which ensures the constrained efficiency of the monopolist's pricesetting problem. As explained in more detail in the proof of Lemma 34 (appendix 2.5), without this subsidy, pricesetters and the central bank target inconsistent conditions, as the central bank tries to inflate away the effects of the monopolistic markup; anticipating this, pricesetters set even higher prices. Here, contrary to other models of the inflationary bias (Barro and Gordon (1983), Clarida, Galí and Gertler (1999)), there is no cost on either side from setting higher prices and this process has no fixed point unless $\tau_L^i = -\frac{1}{\epsilon}$.

2.1.3 Characterizing outcomes on the equilibrium path

Since the full set of subgame perfect equilibria is difficult to characterize, we first examine behavior on the equilibrium path. In Section 2.1.3.1, we show that given the on-path net transfers, it is possible to derive all other on-path quantities and relative prices. We follow up in Section 2.1.3.2 by demonstrating what we call the *risk-sharing miracle*: any transfer rule that achieves perfect risk sharing in tradable goods also achieves the first best on the nontradable side. Finally, in Section 2.1.3.3, we show that any on-path transfer rule attainable in subgame perfect equilibrium can be attained in an SPE of a much more specific form. This will vastly simplify the study of attainable on-path outcomes in the rest of the paper.

2.1.3.1 The sufficiency of net transfers

Consider any subgame perfect equilibrium. Following any exogenous history s^t, on the equilibrium path there are transfers $T^1(s^t)$ and $T^2(s^t)$. Let $T(s^t) \equiv T^1(s^t) - T^2(s^t)$ be the *net transfer* from country 1 to country 2.

Lemma 35. *Given $T(s^t)$, all quantities and relative prices on the equilibrium path are uniquely determined.*

Proof. We know from Lemma 34 that there is no price dispersion: $\Delta_{NT}^i = 1$. The characterization of end-of-period equilibrium in Lemma 32 then shows that $C_T^i(s^t)$, $C_{NT}^i(s^t)$, and $N^i(s^t)$ are uniquely determined by $T(s^t)$ and the relative prices $\mathcal{E}^i(s^t)/P_{NT}^i(s^{t-1})$, as given by the following equations:

$$C_T^i = E_T^i(s) + (-1)^i T \qquad C_{NT}^i = N_T^i = \alpha \frac{\mathcal{E}^i}{P_{NT}^i} C_T^i$$

Thus if we can show that the relative prices $\mathcal{E}^i(s^t)/P_{NT}^i(s^{t-1})$ are uniquely determined by $T(s^t)$, our result will follow.

In equilibrium, the labor wedge τ^i (2.1.16) is given as a function of $\mathcal{E}^i/P_{NT}^i$ and C_T^i by

$$\tau^i = 1 - \left(\alpha \frac{\mathcal{E}^i}{P_{NT}^i} C_T^i\right)^{1+\phi} \tag{2.1.21}$$

In the case of independent monetary policy, equation (2.1.17) shows that each country's central bank always sets the labor wedge equal to zero in every state. Given this, price-setters' optimality conditions (2.1.20) are automatically satisfied. We can then invert (2.1.21) to obtain all relative prices

$$\frac{\mathcal{E}^i(s^t)}{P_{NT}^i(s^{t-1})} = \frac{1}{\alpha\left(E_T^i(s^t) + (-1)^i T(s^t)\right)}$$

In the case of a monetary union, perfect stabilization may no longer be possible. Conditional on s^{t-1}, (2.1.18) and (2.1.20) give a set of $S+2$ equations for the labor wedges $\tau^i\left(s^{t-1}, s_t\right)$, one of which is redundant. There are $S+1$ unknown relative prices $\frac{\mathcal{E}^i\left(s^{t-1}, s_t\right)}{P^1_{NT}\left(s^{t-1}\right)}$ and $\frac{P^2_{NT}\left(s^{t-1}\right)}{P^1_{NT}\left(s^{t-1}\right)}$, matching the number of nonredundant equations. The proof in appendix 2.5 shows there always exists a unique solution for these relative prices.

Note that always is some nominal indeterminacy. In the independent monetary policy case, each country can have its own price level in every period; in the monetary union the overall price level is undetermined in every period. Such indeterminacy is the result of our assumption that prices are reset in every period, and has no allocative consequences.

□

2.1.3.2 Risk-sharing miracle

Even though perfect stabilization is generally not feasible in monetary union, there is an important special case in which it is.

Theorem 36 (Risk-sharing miracle). *If in period t, the net transfers $T\left(s^t\right)$ achieve first-best risk sharing across all states s_t, the first best is also achieved for the nontradable side, even in monetary union.*

Proof. Under independent monetary policy, the first best is always achieved (Lemma 33). Under monetary union, the first best in country i requires that

$$\tau^i\left(s^t\right) = 1 - \frac{\mathcal{E}\left(s^t\right)}{P^i_{NT}\left(s^{t-1}\right)} C^i_T\left(s^t\right) = 0 \tag{2.1.22}$$

First-best tradable risk sharing achieves $\frac{C^2_T\left(s^t\right)}{C^1_T\left(s^t\right)} = \lambda$ for some constant λ. Relative prices

$$\frac{P^2_{NT}\left(s^{t-1}\right)}{P^1_{NT}\left(s^{t-1}\right)} = \lambda \quad \text{and} \quad \frac{\mathcal{E}\left(s^t\right)}{P^1_{NT}\left(s^{t-1}\right)} = \frac{1}{C^1_T\left(s^t\right)}$$

then imply

$$\frac{\mathcal{E}\left(s^t\right)}{P^2_{NT}\left(s^{t-1}\right)} = \frac{1}{C^1_T\left(s^t\right)}\frac{1}{\lambda} = \frac{1}{C^2_T(s^t)}$$

At those prices, (2.1.22) is satisfied simultaneously in both countries. With the labor wedge equal to zero in both countries and in all states, the equilibrium conditions for monetary policy (2.1.18) and price-setting (2.1.20) are then trivially satisfied. □

The intuition behind the risk-sharing miracle is that when countries share risks appropriately through fiscal policy, they make the appropriate stance of monetary policy identical across countries. The central bank, by setting its policy instrument as an average of the desirable level for each country, is therefore able to stabilize them both simultaneously. In this way, the risk-sharing miracle is a sharp illustration of the longstanding view that closer fiscal union reduces the difficulties created by a common currency.

2.1.3.3 Implementation using grim-trigger equilibria

Since we are interested in attainable on-path outcomes for quantities and relative prices, our analysis will be facilitated by the result in this Section, which shows that such outcomes can be implemented by a grim-trigger strategy.

Definition 37. A grim-trigger strategy that sustains a given net transfer rule $T(s^t)$ specifies that if net transfers $T(s^\tau)$ have been observed for all $s^\tau \prec s^t$, countries make transfers

$$T^1(h^t) = \max\{T(s^t), 0\} \quad T^2(h^t) = \max\{-T(s^t), 0\}$$

and otherwise, they each make transfer $T^i(h^t) = 0$.

For the off-path permanent choice of $T^i(h^t) = 0$ to be part of a subgame perfect equilibrium, the choice of $T^i = 0$ must constitute a Nash equilibrium of the stage game. Although this will generally be the case, in our environment it is in principle possible for countries to be in such extreme booms that they find unilateral transfers preferable to autarky, because these transfers lead to decreased demand for nontradable goods and relieve the pressure on the domestic economy. We rule this possibility out with the following assumption:

Assumption 38 (No voluntary unilateral transfer in autarky). *Parameters are such that, for countries living in autarky within a monetary union, it is never desirable to make unilateral transfers.*

Appendix 2.5 provides the assumption on primitives to which assumption 38 is equivalent. It also provides a simpler and stronger sufficient condition: when the countries are relatively open, in the sense that

$$\alpha < \frac{8}{1 + \max_s\left\{\frac{E_T^1(s)}{E_T^2(s)}; \frac{E_T^2(s)}{E_T^1(s)}\right\}} \tag{2.1.23}$$

neither country ever wants to make unilateral transfers in autarky, irrespective of the way nontradables prices were set.

Assumption 39 (Ex-ante option to withdraw). *At the beginning of each period, countries can commit not to make any outgoing transfer and to refuse any incoming transfer.*

Although countries cannot commit to making any particular level of transfer, assumption 39 imparts them with a small level of commitment, intended to rule out the possibility of an expectations trap where self-sustaining transfers arise only because price-setters expect them to happen, delivering lower utility to countries than what they could get if they lived in autarky forever. The following result follows from assumptions 38 and 39:

Lemma 40. *In monetary union, the autarky allocation is subgame perfect and provides the lowest utility level to both agents of any subgame-perfect equilibrium.*

Proof. Since transfers are restricted to lie in $\left[0, E_T^i(s) - \epsilon\right]$, the set of values achievable in SPE is bounded, so it has minimum M. Consider any subgame perfect equilibrium that attains the ex-ante value V. By assumption 39, a permanent deviation that attains the flow value of autarky in each period is always available, delivering V^{aut}, so $V \geq V^{aut}$. As a consequence of assumption 38, autarky is a static Nash equilibrium, so its infinite repetition is subgame perfect, showing $V^{aut} \geq M$. Hence $V \geq V^{aut} \geq M$ for any value V attained by a subgame-perfect equilibrium, and in particular for the minimal value M, showing that $V^{aut} = M$. □

We conclude with this Section's main theorem.

Theorem 41. *Any net transfer rule $T(s^t)$ attainable in subgame perfect equilibrium is also attainable in an SPE where countries follow grim-trigger strategies.*

Proof. Consider a subgame perfect equilibrium. By definition of subgame perfection, the value $V^i(h^t)$ attained on path for country i after history h^t is higher than the value of any possible deviation: $V^i(h^t) \geq V^{dev,i}(h^t)$. Since any deviation is itself subgame perfect, from Lemma 40, in turn we have $V^{dev,i}(h^t) \geq V^{aut,i}(s^t)$ after every history. As in Section 2.1.3.1, denote by $T(s^t)$ the net transfer from country 1 to country 2 that arises on the equilibrium path. Consider then replacing the subgame perfect equilibrium with a grim trigger strategy sustaining the transfer rule $T(s^t)$. By Lemma 35, this strategy delivers the same on-path equilibrium outcomes, and so delivers the same value $V^i(s^t)$ on the equilibrium path. By the above argument, we therefore have

$$V^i\left(s^t\right) \geq V^{aut,i}\left(s^t\right) \quad \forall i,\ \forall s^t \tag{2.1.24}$$

Since the considered grim-trigger strategy delivers $V^{aut,i}(s^t)$ after any deviation, (2.1.24) shows that this strategy constitutes a subgame-perfect equilibrium that delivers the same net transfer rule $T(s^t)$ as the initial SPE. □

Theorem 41 follows the traditional approach to repeated games, where sustainable outcomes are characterized using the worst off-path punishment (Abreu (1988)). It shows that in our complicated game, the worst punishment is still autarky, just as in the traditional limited commitment literature (see for example Kocherlakota (1996)).

2.2 Risk-sharing benefits of monetary union

In this section, we develop our main results using a symmetric structure for endowments and transfer strategies.

Assumption 42. *Countries' endowment processes are ex-ante symmetric, as follows. There exists a finite-state Markov process z_t with elements $z_t \in \mathbf{Z}$ and transition matrix Π^z, and a pair of endowment levels*

$E^H(z) \geq E^L(z)$ *for each* $z \in \mathbf{Z}$. *Countries' endowment processes* E^i *are such that*

$$\Pr\left(E^i = E^H(z_t)\,|z^t\right) = \Pr\left(E^i = E^L(z_t)\,|z^t\right) = \frac{1}{2} \quad i \in \{1,2\}$$

This process allows an arbitrary degree of persistence in both the level and the volatility of union-wide tradable output, but imposes that countries' relative fortunes have an equal chance of being reversed in every period.

Definition 43. A Markov transfer rule is a function $T(z|z_{-1})$ that specifies the transfer from the country with endowment $E^H(z)$ to the country with endowment $E^L(z)$ at any $z \in \mathbf{Z}$ following $z_{-1} \in \mathbf{Z}$.

When restricting endowments to be ex-ante symmetric and countries' fiscal arrangements to Markov transfer rules, the analysis of ex-ante price setting is simplified dramatically, as the following Lemma illustrates.

Lemma 44. *When countries have endowment processes governed by assumption 42 and when they follow Markov transfer rules, their relative nontradable prices are always equal under monetary union:*

$$\frac{P^1_{NT}\left(s^{t-1}\right)}{P^2_{NT}\left(s^{t-1}\right)} = 1 \quad \forall s^{t-1}$$

In particular, we can normalize both these prices to 1. This consequence of symmetry allows us to cut through the complexity imposed by the relative price-setting decisions of producers in each country. It guarantees that the real exchange rate, $\frac{P_T(s^t)}{P_{NT}(s^{t-1})} = \frac{\mathcal{E}(s^t)}{P_{NT}(s^{t-1})}$ is the *same* in both countries of the monetary union at any point in time. This allows us to provide sharp comparisons of the feasible degree of risk-sharing under independent monetary policy and monetary union, respectively.

2.2.1 Improved risk-sharing under monetary union

Definition 45. A Markov transfer rule with some risk sharing is a Markov transfer rule $T(z|z_{-1})$ such that $T(z|z_{-1}) \in \left[0, \frac{E^H(z)+E^L(z)}{2}\right]$ for all $z, z_{-1} \in \mathbf{Z}$.

Under Markov transfer rules with some risk sharing, endowments and tradable consumption levels are ordered as

$$E^L(z) \leq C^L_T(z) \leq C^H_T(z) \leq E^H(z) \tag{2.2.1}$$

Theorem 46. *Any Markov transfer rule with some risk sharing that is achievable in SPE under independent monetary policy is also achievable in SPE in a monetary union.*

Proof. Under independent monetary policy, countries' nontradable sides are always at their efficient level in every period (Lemma 33). Consider a Markov transfer rule with some risk sharing achievable under this monetary regime. By Theorem 41, the same transfer rule is achievable under an SPE that reverts to autarky following any deviation. By definition of subgame perfection, the H country does not want to refrain from the transfer at any node. Using the Markov structure, there is such a participation constraint for every $z \in \mathbf{Z}$, expressed as

$$\underbrace{\log\left(E_T^H(z)\right) - \log\left(C_T^H(z)\right)}_{\text{One-shot gain from defaulting}}$$
$$\underbrace{\leq \beta \sum_z \frac{\tilde{\pi}(z'|z)}{2}\left[\left(\log\left(C_T^L(z')\right) - \log\left(E^L(z')\right)\right) - \left(\log\left(E^H(z')\right) - \log\left(C_T^H(z')\right)\right)\right]}_{\text{Expected loss from lack of future risk-sharing}} \tag{2.2.2}$$

where the probabilities $\tilde{\pi}(z'|z)$, given in the appendix, take into account the relevant mix of future probabilities and discounting. Given concavity of the log function and (2.2.1), the loss of future risk sharing that comes from reversion to autarky entails costs, and these costs overwhelm the one-off gains from leaving the union when (2.2.2) is satisfied.

Consider sustaining the same SPE under monetary union using the same on-path and off-path actions. Lemma 44 implies that both countries have the same real exchange rate at every node, so they evaluate tradable consumption levels using the same indirect utility function

$$\tilde{v}_z(C_T) = \log(C_T) + \alpha\left(\log(\alpha\epsilon_z(C_T)C_T) - \frac{1}{1+\phi}(\alpha\epsilon_z(C_T)C_T)^{1+\phi}\right)$$

where $\epsilon_z(C_T)$ is the equilibrium reaction of the central bank to the level of tradable consumption C_T when the state is z, given in appendix 2.5. If we can check that a version of (2.2.2) holds with log replaced by $\tilde{v}_z$, this guarantees that the participation constraint for the H country is met in every state z under monetary union. Since the L participation constraint is trivially satisfied at every node given that $E^L(z) \leq C_T^L(z)$, $T(z|z_{-1})$ is indeed sustainable in a monetary union SPE and the theorem follows.

Appendix 2.5 gives a formal argument, but here we illustrate why (2.2.2) does hold with the indirect utility function $\tilde{v}_z$. Figure 2.2.1 illustrates that the costs from less future risk-sharing are greater: by the risk-sharing miracle, a rule that delivers perfect risk-sharing $T(z') = \frac{E^H(z')+E^L(z')}{2}$ at every future node attains the same utility for the country as it does under independent monetary policy, but deviations are now more costly because of the macroeconomic externalities associated with the central bank's inability to perfectly stabilize. Figure 2.2.1 also illustrates that the benefits of leaving are smaller: a country tempted to leave the fiscal union is already overheated, and leaving the union exacerbates this boom.

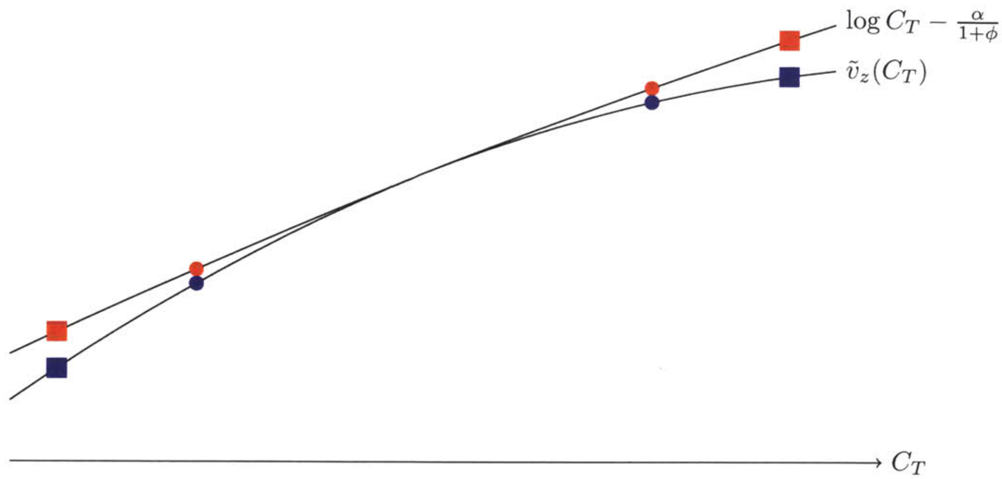

Figure 2.2.1: Costs and benefits of participating in fiscal union under alternative monetary regimes

□

2.2.2 An example of powerful improvement

We now show that the risk-sharing incentives can improved so much under monetary union as to transport countries from autarky to first-best.

Theorem 47. *There exists a combination of parameters and endowment processes such that autarky is the only feasible risk-sharing outcome when countries run an independent monetary policy, but a SPE with full risk-sharing is possible under monetary union.*

Proof. We consider the simplest possible example of our framework: the symmetric iid case with union-wide tradable output equal to 1. In the notation above, the Markov chain is reduced to a point $z = 1$ in every period, $E^H = e$ and $E^L = 1 - e$, with $e > \frac{1}{2}$. In this context, a Markov transfer is a value T, such that $\left(C_T^L, C_T^H\right) = (1 - e + T, e - T)$. We consider transfers that improve risk-sharing, in other words $T \in \left[0, \frac{1}{2} - e\right]$.

Appendix 2.5 demonstrates formally the following propositions. Under independent monetary policy, our setup collapses to a simple limited commitment model. Some risk sharing ($T > 0$) is feasible if the country currently in the high state is patient enough to value the benefits from future reciprocity: its discount factor must be above a lower bound, $\underline{\beta}^{indep} = 2(1 - e)$. Conversely, a deviation from first-best risk sharing, where $T = \frac{1}{2} - e$, is valuable if its discount factor is below an upper bound $\overline{\beta}^{indep}$, derived from the participation constraint of the country with endowment e.

Consider now the case of a monetary union, and assume countries are achieving perfect risk-sharing $T = \frac{1}{2} - e$. Due to the risk-sharing miracle (theorem 36), their nontradables side is perfectly stabilized, so their values from the transfer arrangement are identical to those under independent monetary policy.

However, a deviation now entails an additional cost $c(\alpha, \phi, e)$ coming from the fact that the defaulting country will experience a macroeconomic boom. Going forward, the central bank loses its ability to stabilize both countries simultaneously, which creates additional utility costs $c(\alpha, \phi, e)$ and $c(\alpha, \phi, 1-e)$ in each state. This implies that the discount factor threshold above which full risk-sharing is feasible is now $\overline{\beta}^{union}(\alpha, \phi, e)$, a function of parameters governing the nontradable side.

Figure 2.2.2 illustrates these thresholds, for an illustrative calibration with $e = 0.7$ and $\phi = 1$. When $\alpha = 0$, the nontradable side is inexistent and the discount factor threshold for first-best under monetary union and independent monetary policy coincide: $\overline{\beta}^{indep} = \overline{\beta}^{union}(0, \phi, e)$. Countries with discount factors below $\underline{\beta}^{indep} = 0.6$ cannot sustain any risk-sharing under independent monetary policy, countries with discount factors above $\underline{\beta}^{indep}$ can sustain first-best, and countries with intermediate discount factors can sustain some, but not full risk-sharing. As α grows and countries nontradables side becomes more important, it becomes easier to sustain risk-sharing. For α around 2.5, countries with discount factors around 0.5 cannot not sustain *any* risk-sharing under independent monetary policy but can sustain full risk-sharing under monetary union. If α becomes too large ($\alpha \geq \overline{\alpha}(e, \phi)$), the autarky punishment is no longer subgame perfect: the country high state is in such a boom under autarky that it prefers to make a unilateral transfer to cool its tradable side. □

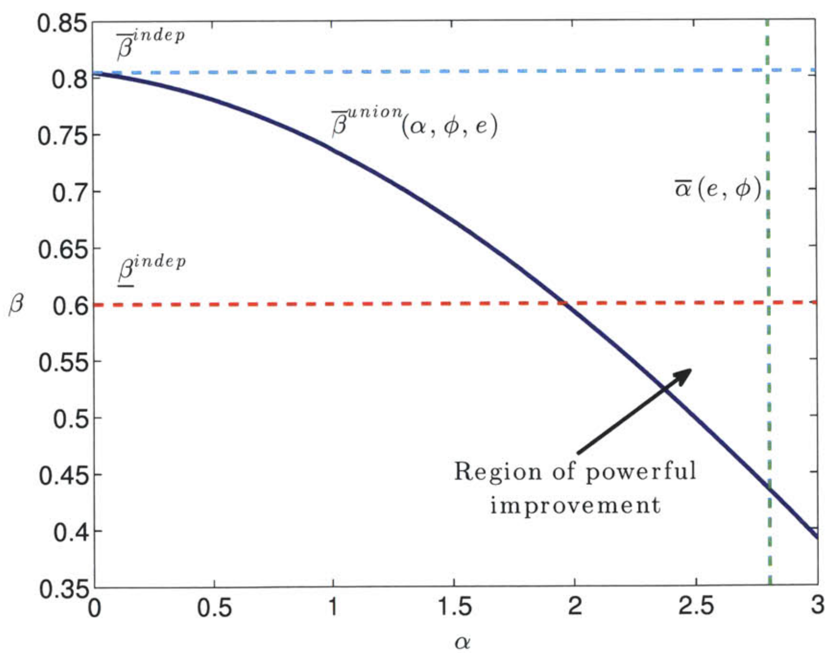

Figure 2.2.2: Illustration of thresholds for theorem 47

2.2.3 Discussion

The theorems in this section are two facets of our claim that monetary union begets fiscal union. By specializing the framework of Section 2.1 to a case where endowments and transfer rules have limited history dependence, we are able to prove in theorem 46 that fiscal union can improve risk sharing in a particularly clear sense: any transfer rule that was feasible under independent monetary policy is still feasible under monetary union. And theorem 47 shows that it is possible to find powerful improvements in this class of equilibria. We now discuss the generality of these results, by considering what would happen if we relaxed some of the assumptions imposed.

Consider relaxing the assumptions on symmetry of endowments and transfer rules. The modern literature on limited commitment (Kocherlakota (1996), Alvarez and Jermann (2000), Ligon et al. (2002)) emphasizes that it is in general possible to sustain subgame-perfect outcomes that improve upon Markov transfer rules. In the equilibria characterized by this literature, the amount a country owes depends not only on its current and previous state, but on the full history of past shocks, which an endogenous state variable (promised utility) keeps track of. For this more general class of equilibria, theorem 46 is in general no longer true. In particular, the country hitting its participation constraint is no longer unambiguously the country which would experience a boom if if left the union: a country with a history of very bad shocks may be held at its participation constraint as it is called upon to pay back in a mild state, even if its endowment is still relatively low. However, even under this class of equilibria, there is still a sense in which risk-sharing is ameliorated under monetary union: the discount factor thresholds to attain first-best are ordered $\overline{\beta}^{union} \leq \overline{\beta}^{indep}$. In fact, it is generally possible to find powerful improvements as in theorem 47 for these more general endowment structures. Because of the risk-sharing miracle, the optimal policies in the fiscal union involve first-best risk sharing, which is simple to characterize.

Another way to relax assumptions is to add shocks to the nontradable side of the economy. Such shocks can be modelled in our framework by assuming that preferences for nontradables are dependent on the exogenous state: $\alpha^i(s)$. In this case, the risk-sharing miracle is in general no longer true, as can be seen by the following argument. Assume that countries share risks to tradables perfectly, so that their relative tradables consumption is constant across all states. Since under monetary union they share the same nominal exchange rate, their relative nontradable consumption in a state s is then, from households' first-order condition,

$$\frac{C^1_{NT}(s)}{C^2_{NT}(s)} = \lambda \frac{\alpha^1(s)}{\alpha^2(s)} \tag{2.2.3}$$

where λ is a constant reflecting the risk-sharing rule and nontradable prices that are constant across all states. Unless α^1 and α^2 vary proportionally across states, (2.2.3) is incompatible with efficient consumption of nontradables, which still requires that $C^1_{NT}(s) = C^2_{NT}(s) = 1$ (Lemma 44). The constrained-efficient outcome that takes into account nominal rigidities, which fiscal union would reach absent the limited commitment constraint, does not feature perfect stabilization in each country (Farhi and Werning (2013)). This means that joining a monetary union necessarily entails some welfare losses from imperfect stabilization, but even in this case, there is still a force pushing for welfare gains from improved incentives to share risks, so the overall welfare benefit from transiting into monetary union might be positive. In this sense the overall message of the model — the risk-sharing benefits of monetary union have to be balanced against the stabilization costs — is unchanged by the presence of shocks to nontradables.

2.3 Optimal joint monetary and fiscal policy in the union

2.3.1 Alternative timing and the role of monetary policy

In Sections 2.1 and 2.2, we assumed that the central bank sets the exchange rate after countries announced their transfers. With static welfare maximization as its objective, the central bank was limited to stabilizing the aggregate economy ex post, without any commitment power or ability to internalize the sustainability of fiscal union. This assumption allowed us to evaluate the direct effects of monetary union, without considering the central bank as a strategic actor in its own right—which is inevitably a more speculative exercise.

In this section, we broaden the role of monetary policy, allowing the central bank in a monetary union to commit to an exchange rate policy at the beginning of each period, while retaining its objective of within-period welfare maximization. We now replace the timing from Figure 2.1.1 with that depicted in Figure 2.3.1:

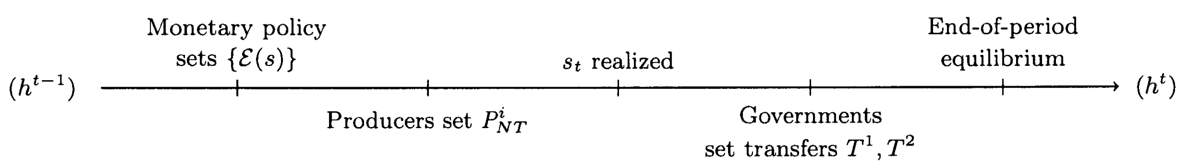

Figure 2.3.1: Alternative Timing

The choice of a state-contingent $\{\mathcal{E}(s)\}$ at the beginning of the period is driven by expected welfare maximization

$$\mathcal{E}(s) = \arg\max_{\{\hat{\mathcal{E}}(s)\}} \sum_s \pi(s) \left(\frac{1}{2} v^1(\{\varphi^1\}, s, \{T^1(s), T^2(s)\}, \hat{\mathcal{E}}(s)) + \frac{1}{2} v^2(\{\varphi^2\}, s, \{T^1(s), T^2(s)\}, \hat{\mathcal{E}}(s)) \right) \quad (2.3.1)$$

where the dependence of price distributions $\{\varphi^i\}$ and transfers $T^i(s)$ on monetary policy (through the reaction functions of nontradable pricesetters and governments) in (2.3.1) is left implicit.

Since the central bank moves first, it can internalize the effect of its decision on governments' incentives to make transfers, and it will not necessarily find aggregate stabilization optimal—thus overturning the result from Lemma 33. It may instead devise policy that actively encourages sustained fiscal union, expanding upon the complementarity between monetary and fiscal union derived in Section 2.2.

2.3.2 Expansionary monetary policy and aggregate dispersion

To illustrate the role of monetary policy in this new environment, we specialize Assumption 42 to a simpler case where the stochastic process for endowments is iid across periods, and symmetric within each period.

Assumption 48. *There exist finitely many $z \in \mathbf{Z}$, each of which is associated with a probability $\pi(z)$ and a pair of endowment levels $E^H(z) \geq E^L(z)$. Endowments are iid across periods, and in each period are drawn such that for each z,*

$$\Pr\left(E^1 = E^H(z) \text{ and } E^2 = E^L(z)\right) = \Pr\left(E^1 = E^L(z) \text{ and } E^2 = E^H(z)\right) = \frac{1}{2}\pi(z)$$

We will characterize the optimal relationship between the stance of monetary policy and the distribution of endowments across states. As in Section 2.2, we consider equilibria with symmetric strategies that depend only on the current state (and whether there has yet been a deviation), rather than depending on the full history of past actions. We also repeat Assumption 38 by ruling out extreme cases where the boom in a country is so great that a unilateral transfer is worthwhile.

Assumption 49. *Consider equilibria where the government receiving endowment H makes transfer $T(z; d)$, where $z \in \mathbf{Z}$ is the aggregate state and $d \in \{0, 1\}$ is an indicator specifying whether play is on or off the equilibrium path. Also restrict attention to equilibria where unilateral transfers are never worthwhile.*

Observe that the restriction on strategies in Assumption 49, along with the symmetry of the endowment process, ensures, just as in Lemma 44, that price-setters in both countries set the same nontradable price. We can again normalize this price to 1: $P^1_{NT} = P^2_{NT} \equiv 1$.

Now suppose that, out of all the equilibria consistent with Assumption 49, we aim to characterize the equilibrium with the highest expected welfare. Quantities in this equilibrium must solve the following planning problem:

$$\max \quad \sum_z \pi(z) \left(w(E^H_T(z) - T(z), p(z)) + w(E^L_T(z) + T(z), p(z))\right) \tag{2.3.2}$$

$$\text{s.t.} \quad w(E^H_T(z), p(z)) - w(E^H_T(z) - T(z), p(z)) \leq V \quad \forall z \tag{2.3.3}$$

$$\sum_z \pi(z) \cdot \frac{\tau^H(z) + \tau^L(z)}{2} = 0 \tag{2.3.4}$$

where $p(z) \equiv P_T(z)/P_{NT} = 1$ is the relative price of tradables and nontradables in state z, $w(C_T, p) \equiv \log C_T + \alpha\left(\log(\alpha p C_T) - \frac{(\alpha p C_T)^{1+\phi}}{1+\phi}\right)$ is the indirect utility function corresponding to C_T, and V is the difference between expected future welfare along the equilibrium path and expected future welfare following a deviation. (2.3.3) is simply the participation constraint, which is necessary to ensure that the government with the high endowment makes transfer $T(z)$ rather than deviating and hoarding its entire endowment; while (2.3.4) is imposed by nontradable pricesetting, following (2.1.20) in Lemma 34.

In the previous environment, Lemma 33 showed that the central bank stabilizes the aggregate economy of the monetary union, setting the average labor wedge across both countries to zero. More generally, the average labor wedge summarizes the nontradable side of the union economy: a negative labor wedge corresponds to an aggregate boom, while a positive labor wedge corresponds to an aggregate bust. In an optimum equilibrium that solves the planning problem (2.3.2)-(2.3.4), the central bank no longer seeks stabilization in every state. Instead, there is a remarkably simple relationship between macroeconomic conditions and dispersion $E^H(z)/E^L(z)$ of the endowments, captured in the following theorem.

Theorem 50. *Consider the endowment process given by Assumption 48 and equilibria as described by Assumption 49. In the subgame perfect equilibrium with maximal expected welfare, $(\tau^H(z)+\tau^L(z))/2$ is weakly decreasing in the dispersion $E^H(z)/E^L(z)$ between endowments. It is strictly decreasing in $E^H(z)/E^L(z)$ for any z such that risk-sharing is neither perfect ($C_T^H(z) = C_T^L(z)$) nor absent ($C_T^H(z) = E^H(z)$).*

In other words, in the optimal equilibrium, the central bank *creates booms when dispersion is high.* The intuition for this result is simple: when endowments are more dispersed, risk sharing is more important, and the country with a high endowment will be more willing to make a transfer when it is experiencing more of a boom. The central bank is willing to accept the cost of a partly overheated union-wide economy in order to create this boom and facilitate risk sharing in the fiscal union.

More concretely, consider the participation constraint (2.3.3), with the indirect utility function w expanded:

$$(1+\alpha)\left(\log(E_T^H(z)) - \log(E_T^H(z) - T(z))\right) \\ - \frac{\alpha}{1+\phi}\alpha^{1+\phi}p(z)^{1+\phi}\left(E_T^H(z)^{1+\phi} - (E_T^H(z) - T(z))^{1+\phi}\right) \leq V \quad (2.3.5)$$

Given the stipulation in Assumption 49 that unilateral transfers are not optimal, the left side of (2.3.5) must be increasing in $T(z)$; when the transfer is larger, making a transfer is less desirable relative to autarky. Since higher $p(z)$ decreases the total value on the left and relaxes the constraint, it enables higher transfers. Hence when transfers are particularly valuable—as in cases of high dispersion—it is worthwhile to raise $p(z)$ to the point where there is an aggregate boom, trading off the (initially) second-order costs of aggregate overexpansion against the first-order benefits of better risk sharing.

This is depicted graphically in Figure 2.3.2, which plots the welfare $w(C_T, p)$ of a country with consumption C_T and relative price p for two different values of p, the price $p = p'$ that achieves aggregate stabilization and the higher price $p = p''$ that creates an aggregate boom. Squares depict endowments, while circles depict consumption after transfers. When the central bank raises the relative price from p' to p'', at high levels of consumption there is even more of a boom, attenuating the within-period welfare loss from making a transfer and leaving the high-endowment country's participation constraint easier to satisfy. The change in monetary policy causes a first-order decrease in welfare for the booming high-endowment country and a first-order increase in welfare for the depressed low-endowment country, netting out to only a second-order loss for the union as a whole when p'' is close to the stabilization level p'. At the margin, this loss is offset by the benefits from easing the participation constraint.

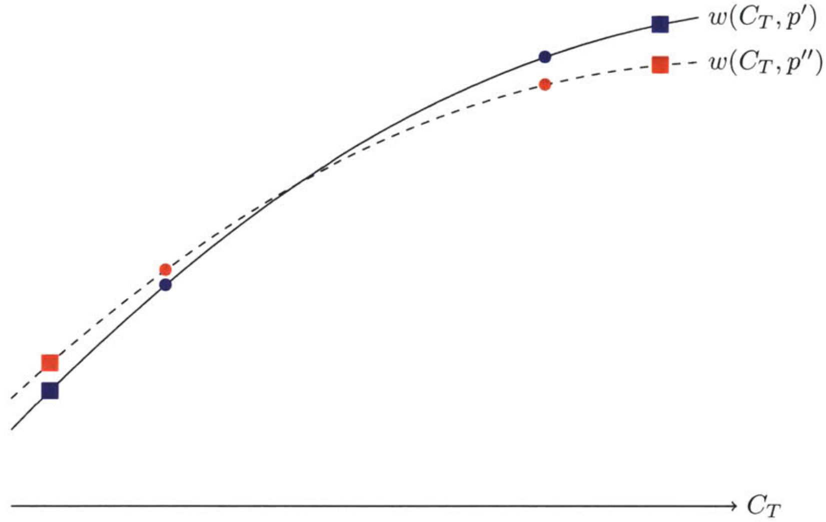

Figure 2.3.2: Welfare and transfer incentives under different monetary policies

The logic of Figure 2.3.2 suggests that the central bank should create a boom whenever the participation constraint is binding—which will generally be the case when perfect risk sharing is not achieved. Of course, it is not possible for the central bank to create a boom in every state: the nontradable sector sets prices that achieve aggregate stabilization in expectation, as indicated by (2.3.4). The central bank can only shape the *relative* pattern of boom and bust across states, not create booms across the board—and, as Theorem 50 finds, in the best equilibrium it chooses to create booms in states with more dispersed endowments, when it is particularly important to loosen participation constraints and encourage transfers.

2.4 Conclusion

In this paper, we have examined how monetary union facilitates the formation of a stronger fiscal union. We have seen, in an important special case, how any risk-sharing arrangement that can be sustained under independent monetary policy can also be sustained under monetary union. In fact, dramatic improvement is possible: even when no risk-sharing is possible in equilibrium with independent policy, monetary union can sometimes bring governments to a first-best outcome.

The key force is the complementarity of monetary and fiscal union: when countries more effectively share risks, their outcomes are more closely aligned, and a central bank that stabilizes the union-wide economy can come closer to stabilizing each individual economy as well. This not only makes fiscal union a desirable counterpart to monetary union—as is widely understood—but also provides a channel through which monetary integration can enable otherwise unsustainable risk-sharing. Without independent monetary policy as a fallback, governments have a greater stake in preserving joint fiscal arrangements. This bears out the progression from monetary to fiscal union envisaged in the literature on sequencing theory; and it suggests a possible upside to a common currency, when most models featuring nominal rigidities offer only drawbacks.

Further exploring the role of monetary policy in a fiscal union, we found that when a union-wide central bank behaves strategically—taking the sustainability of the fiscal union into account—the optimal rule is expansionary when dispersion within the monetary union is high. Booms make governments more willing to provide transfers, and these transfers are most important when outcomes vary greatly within the union. A singleminded emphasis on stabilization is not optimal.

To what extent are the forces in this paper visible in practice? Certainly the Euro Area today is far from perfect risk sharing—the willingness of the core to subsidize the periphery has clear limits. At the same time, the level of cross-country support under monetary union, although often frustratingly limited, has greatly exceeded what came before. There have been multiple bailouts and transfer schemes—both explicit and implicit, often taking the form of below-market lending—made with the explicit intent of preserving the common currency's viability.

Our model offers other reasons for guarded optimism. Although we show in Theorem 46 that monetary union expands the set of attainable risk sharing equilibria, we cannot be sure that governments will immediately take advantage of this feature by coordinating on the better equilibria. (After all, autarky is always an equilibrium as well.) But as participants become aware of the heightened importance of fiscal union when exchange rates are no longer free to adjust, they may learn to play the equilibrium with improved risk sharing.

Monetary policy also has an important role. In recent years, dispersion in Euro Area outcomes has coincided with the deepest aggregate slump in decades. This is exactly the opposite of the optimal arrangement in Theorem 50, which prescribes monetary accommodation that creates a boom whenever members' fates diverge and the fiscal union is under stress. To some extent, this inconsistency may be due to limitations on monetary policy that this model leaves out—in particular, the zero lower bound. But the insufficiently expansionary policy in the Euro Area is also partly by choice: the ECB raised rates in 2011, just as the fiscal prospects of peripheral countries were rapidly deteriorating, and it has since been hesitant to employ unconventional expansionary policy. Our model implies that for the full promise of fiscal union to be realized, very different choices are needed from monetary policymakers.

Monetary union begets fiscal union—but not necessarily overnight.

2.5 Appendix: proofs

Proof of Lemma 30. Substituting production into preferences

$$\log\left(C_T\right)+\alpha\left(\frac{\epsilon}{\epsilon-1}\log\left(\int_j\left(C_{NT}^j\right)^{\frac{\epsilon-1}{\epsilon}}dj\right)-\frac{\left(\int_j C_{NT}^j dj\right)^{1+\phi}}{1+\phi}\right)$$

and taking a first-order condition with respect to C_{NT}^k shows that

$$\frac{\left(C_{NT}^k\right)^{-\frac{1}{\epsilon}}}{\int_j\left(C_{NT}^j\right)^{\frac{\epsilon-1}{\epsilon}}dj}=\left(\int_j C_{NT}^j dj\right)^{\phi}\quad\forall k$$

This shows that the efficient allocation is constant consumption across goods, $C_{NT}^k=C_{NT}^*\ \forall k$, and further that C_{NT} satisfies

$$\frac{1}{C_{NT}}=C_{NT}^{\phi}\quad\Rightarrow\quad C_{NT}^*=1$$

□

Proof of Lemma 33. Define the function $C_{NT}^i(e)=\alpha\frac{e}{P_{NT}^i}C_T^i$, and note that $\frac{dC_{NT}^i(e)}{de}=\alpha\frac{C_T^i}{P_{NT}^i}=\frac{C_{NT}^i(e)}{e}$. Under independent monetary policy, the central bank chooses the exchange rate e to maximize

$$v^i(e)=\log\left(C_T^i\right)+\alpha\left(\log\left(C_{NT}^i(e)\right)-\frac{\left(\Delta_{NT}^i C_{NT}^i(e)\right)^{1+\phi}}{1+\phi}\right)$$

Its first order condition is

$$\alpha\frac{C_{NT}^i(e)}{e}\left(\frac{1}{C_{NT}^i(e)}-\left(\Delta_{NT}^i\right)^{1+\phi}\left(C_{NT}^i(e)\right)^{\phi}\right)=0$$

resulting in

$$1-\left(\Delta_{NT}^i C_{NT}^i\right)^{1+\phi}=0$$

combining the definition of the labor wedge in (2.1.16) with labor market clearing (2.1.11), we obtain

$$\tau^i=1-C_{NT}^i\Delta_{NT}^i\left(N^i\right)^{\phi}=1-\left(C_{NT}^i\Delta_{NT}^i\right)^{1+\phi}=0$$

The central bank's exchange rate choice $\mathcal{E}^i\equiv e$ is given by

$$\mathcal{E}^i=\frac{1}{\alpha}\frac{P_{NT}^i}{\Delta_{NT}^i C_T^i}\tag{2.5.1}$$

Under joint monetary policy, the central bank chooses e to maximize

$$\frac{1}{2}v^1(e) + \frac{1}{2}v^2(e)$$

resulting in the first-order condition

$$\begin{aligned}
\frac{1}{2}\left(\frac{C_{NT}^1}{e}\right)\left(\frac{1}{C_{NT}^1} - \left(\Delta_{NT}^1\right)^{1+\phi}\left(C_{NT}^1\right)^{\phi}\right) + \frac{1}{2}\left(\frac{C_{NT}^1}{e}\right)\left(\frac{1}{C_{NT}^2} - \left(\Delta_{NT}^2\right)^{1+\phi}\left(C_{NT}^2\right)^{\phi}\right) &= 0 \\
\frac{1}{2}\left(1 - \left(\Delta_{NT}^1 C_{NT}^1\right)^{1+\phi}\right) + \frac{1}{2}\left(1 - \left(\Delta_{NT}^2 C_{NT}^2\right)^{1+\phi}\right) &= 0 \\
\frac{1}{2}\tau^1 + \frac{1}{2}\tau^2 &= 0
\end{aligned}$$

which is equation (2.1.18). That is, the central bank's exchange rate choice $\mathcal{E}$ solves

$$\frac{1}{2}\left(\Delta_{NT}^1 \alpha \frac{\mathcal{E}}{P_{NT}^1} C_T^1\right)^{1+\phi} + \frac{1}{2}\left(\Delta_{NT}^2 \alpha \frac{\mathcal{E}}{P_{NT}^2} C_T^2\right)^{1+\phi} = 1 \tag{2.5.2}$$

resulting in

$$\mathcal{E} = \frac{1}{\alpha}\left(\frac{1}{2}\left(\frac{P_{NT}^1}{\Delta_{NT}^1 C_T^1}\right)^{-(1+\phi)} + \frac{1}{2}\left(\frac{P_{NT}^2}{\Delta_{NT}^2 C_T^2}\right)^{-(1+\phi)}\right)^{-\frac{1}{1+\phi}} \tag{2.5.3}$$

□

Proof of Lemma 34. Using the relationship between consumer nontradable and tradable demand (2.1.4), we can rewrite firm profits (2.1.7) given price p, when the nominal wage is $W^i(s)$, the nominal exchange rate is $\mathcal{E}^i(s)$, and the price index for nontradables is $P_{NT}^i(s)$, as

$$\psi^{i,j}(p) = \left(p - \left(1+\tau_L^i\right) W^i(s)\right)\left(\frac{p}{P_{NT}^i(s)}\right)^{-\epsilon} \alpha \frac{\mathcal{E}^i(s)}{P_{NT}^i(s)} C_T^i(s) \tag{2.5.4}$$

Firms maximize the expected level of (2.5.4), valuing nominal profits in each state using the nominal stochastic discount factor $\frac{1}{\mathcal{E}^i(s) C_T^i(s)}$:

$$\begin{aligned}
\Pi(p) &= \sum_s \pi(s|s_{-1}) \frac{1}{\mathcal{E}^i(s) C_T^i(s)} \left(p - \left(1+\tau_L^i\right) W^i(s)\right)\left(\frac{p}{P_{NT}^i(s)}\right)^{-\epsilon} \alpha \frac{\mathcal{E}^i(s)}{P_{NT}^i(s)} C_T^i(s) \\
&= \alpha \sum_s \pi(s|s_{-1}) \left(p - \left(1+\tau_L^i\right) W^i(s)\right)\left(\frac{p}{P_{NT}^i(s)}\right)^{-\epsilon} \frac{1}{P_{NT}^i(s)}
\end{aligned}$$

The first-order condition yields

$$P_{NT}^{ij} = \left(1+\tau_L^i\right) \frac{\epsilon}{\epsilon - 1} \frac{\sum_s \pi(s|s_{-1}) \left(\frac{1}{P_{NT}^i(s)}\right)^{1-\epsilon} W^i(s)}{\sum_s \pi(s|s_{-1}) \left(\frac{1}{P_{NT}^i(s)}\right)^{1-\epsilon}} \tag{2.5.5}$$

Hence, all firms set the same price $P_{NT}^{i,j} = P_{NT}^{i} = P_{NT}^{i}(s)\ \forall s$, and (2.5.5) simplifies to

$$P_{NT}^{i} = \left(1+\tau_{L}^{i}\right)\frac{\epsilon}{\epsilon-1}\sum_{s}\pi\left(s|s_{-1}\right)W^{i}\left(s\right) \tag{2.5.6}$$

From the household labor supply condition (2.1.6) we know that

$$\frac{W^{i}\left(s\right)}{P_{NT}^{i}} = C_{NT}^{i}\left(s\right)\left(N^{i}\left(s\right)\right)^{\phi} = 1-\tau^{i}\left(s\right)$$

where the latter equality holds because price dispersion is $\Delta_{NT}^{i}\left(s\right)=1$ in all states. We obtain

$$\sum_{s}\pi\left(s|s_{-1}\right)\tau^{i}\left(s\right) = 1-\frac{1-\frac{1}{\epsilon}}{1+\tau_{L}^{i}}$$

Since in each country the labor subsidy is set at the level $\tau_{L}^{i} = -\frac{1}{\epsilon}$, equation (2.1.20) obtains. Note that if the subsidy is set at any level $\tau_{L}^{i} > -\frac{1}{\epsilon}$, the optimal price-setting problem is to equalize the expected labor wedge to a strictly positive number, a condition inconsistent with the central bank's exchange-rate setting policy.

To understand the logic behind the equilibrium determination of relative prices, we further characterize the price level consistent with a given expectation of the central banks' exchange rate policy $\left\{\mathcal{E}^{i}\left(s\right)\right\}$. Using market clearing (2.1.11) with $\Delta_{NT}^{i}\left(s\right)=1$, we obtain $\frac{W^{i}(s)}{P_{NT}^{i}} = \left(C_{NT}^{i}\left(s\right)\right)^{1+\phi}$. Using (2.1.4) and (2.1.9) into equation (2.5.6), we then find

$$\left(1+\tau_{L}\right)\frac{\epsilon}{\epsilon-1}\left(\sum_{s}\pi\left(s|s_{-1}\right)C_{NT}^{i}\left(s\right)^{1+\phi}\right) = \left(1+\tau_{L}\right)\frac{\epsilon}{\epsilon-1}\left(\sum_{s}\pi\left(s|s_{-1}\right)\left(\frac{\alpha\mathcal{E}^{i}\left(s\right)C_{T}^{i}\left(s\right)}{P_{NT}^{i}}\right)^{1+\phi}\right) = 1$$

and results in an equilibrium price level of

$$P_{NT}^{i} \quad = \quad \left(\left(1+\tau_{L}\right)\frac{\epsilon}{\epsilon-1}\right)^{\frac{1}{1+\phi}}\alpha\left(\sum_{s\in\mathbf{S}}\pi\left(s|s_{-1}\right)\left(\mathcal{E}^{i}\left(s\right)C_{T}^{i}\left(s\right)\right)^{1+\phi}\right)^{\frac{1}{1+\phi}} \tag{2.5.7}$$

When $\tau_{L}^{i} > -\frac{1}{\epsilon}$, monopolists collectively target a price that is higher than an average of $\mathcal{E}^{i}\left(s\right)C_{T}^{i}\left(s\right)$. This is inconsistent with central bank optimality. For example, under independent monetary policy, the central bank's exchange rate decision (2.5.1) leads to $\mathcal{E}^{i}\left(s\right)C_{T}^{i}\left(s\right) = \frac{1}{\alpha}P_{NT}^{i}$ in every state, and it is easy to see that these equations do not have a fixed point with positive prices. □

Proof of Lemma 35. Suppressing the dependence on s^{t-1}, the central bank's exchange rate choice (2.5.3) in the absence of price dispersion is

$$\mathcal{E}(s) = \frac{1}{\alpha}\left(\frac{1}{2}\left(\frac{P_{NT}^1}{C_T^1(s)}\right)^{-(1+\phi)} + \frac{1}{2}\left(\frac{P_{NT}^2}{C_T^2(s)}\right)^{-(1+\phi)}\right)^{-\frac{1}{1+\phi}} \tag{2.5.8}$$

while the monopolists' price setting conditions under the labor subsidy $\tau_L^1 = \tau_L^2 = -\frac{1}{\epsilon}$ lead to nontradable prices given by (2.5.7)

$$P_{NT}^1 = \alpha\left(\sum_{s\in\mathbf{S}} \pi(s|s_{-1})\left(\mathcal{E}(s)\,C_T^1(s)\right)^{1+\phi}\right)^{\frac{1}{1+\phi}} \tag{2.5.9}$$

$$P_{NT}^2 = \alpha\left(\sum_{s\in\mathbf{S}} \pi(s|s_{-1})\left(\mathcal{E}(s)\,C_T^2(s)\right)^{1+\phi}\right)^{\frac{1}{1+\phi}} \tag{2.5.10}$$

Define $\rho \equiv \frac{P_{NT}^2}{P_{NT}^1}$ as the relative price of nontradables in the two countries. Exploiting the homogeneity of equations, using (2.5.8) we obtain that

$$\frac{\mathcal{E}(s)}{P_{NT}^1} = \frac{1}{\alpha}\left(\frac{1}{2}\left(\frac{1}{C_T^1(s)}\right)^{-(1+\phi)} + \frac{1}{2}\left(\frac{\rho}{C_T^2(s)}\right)^{-(1+\phi)}\right)^{-\frac{1}{1+\phi}} \tag{2.5.11}$$

and this allows to rewrite (2.5.9) as

$$\begin{aligned} 1 &= \alpha\left(\sum_{s\in\mathbf{S}} \pi(s|s_{-1})\left(\frac{\mathcal{E}(s)}{P_{NT}^1}C_T^1(s)\right)^{1+\phi}\right)^{\frac{1}{1+\phi}} \\ &= \left(\sum_{s\in\mathbf{S}} \pi(s|s_{-1})\left(\frac{1}{2} + \frac{1}{2}\left(\frac{\rho C_T^1(s)}{C_T^2(s)}\right)^{-(1+\phi)}\right)^{-1}\right)^{\frac{1}{1+\phi}} \end{aligned}$$

The relative price ρ is therefore a solution to

$$\sum_{s\in\mathbf{S}} \pi(s|s_{-1})\left(\frac{1}{2} + \frac{1}{2}\left(\frac{C_T^2(s)}{\rho C_T^1(s)}\right)^{(1+\phi)}\right)^{-1} = 1 \tag{2.5.12}$$

The function $\rho \to \sum_{s\in\mathbf{S}} \pi(s|s_{-1})\left(\frac{1}{2} + \frac{1}{2}\left(\frac{C_T^2(s)}{\rho C_T^1(s)}\right)^{(1+\phi)}\right)^{-1}$ is strictly increasing in ρ, with limits $\frac{1}{2}$ as $\rho \to 0$ and 2 as $\rho \to \infty$, so (2.5.12) has a unique solution. Equation (2.5.12) then delivers a unique solution for all relative prices $\frac{\mathcal{E}(s)}{P_{NT}^1}$, $s \in \mathbf{S}$, and therefore for $\frac{\mathcal{E}(s)}{P_{NT}^2} = \frac{\mathcal{E}(s)}{P_{NT}^1}\frac{1}{\rho}$, $s \in \mathbf{S}$ also. These relative prices are consistent with (2.5.10) because satisfaction of (2.5.12) ensures that

$$\begin{aligned} 0 &= \sum_{s \in \mathbf{S}} \pi(s|s_{-1}) \frac{\frac{1}{2} - \frac{1}{2}\left(\frac{C_T^2(s)}{\rho C_T^1(s)}\right)^{(1+\phi)}}{\frac{1}{2} + \frac{1}{2}\left(\frac{C_T^2(s)}{\rho C_T^1(s)}\right)^{(1+\phi)}} \\ &= \sum_{s \in \mathbf{S}} \pi(s|s_{-1}) \frac{\frac{1}{2}\left(\frac{\rho C_T^1(s)}{C_T^2(s)}\right)^{(1+\phi)} - \frac{1}{2}}{\frac{1}{2}\left(\frac{\rho C_T^1(s)}{C_T^2(s)}\right)^{(1+\phi)} + \frac{1}{2}} \\ &= \sum_{s \in \mathbf{S}} \pi(s|s_{-1}) \left(1 - \frac{1}{\frac{1}{2}\left(\frac{\rho C_T^1(s)}{C_T^2(s)}\right)^{(1+\phi)} + \frac{1}{2}}\right) \end{aligned}$$

so we also have

$$\sum_{s \in \mathbf{S}} \pi(s|s_{-1}) \left(\frac{1}{2} + \frac{1}{2}\left(\frac{\rho C_T^1(s)}{C_T^2(s)}\right)^{(1+\phi)}\right)^{-1} = 1$$

This result is another illustration of the redundancy of equations: using $\tau^i(s)$ to denote the labor wedge in country i and state s, price-setting in country 1 implies $\sum \pi(s|s_{-1})\tau^1(s) = 0$, and monetary policy implies $\tau^1(s) + \tau^2(s) = 0$ for all s, from we obtain $\sum \pi(s|s_{-1})\tau^2(s) = 0$. □

Another proof of theorem 36. Following the proof of Lemma 35, consider the special case where transfers ensure full risk-sharing ($\frac{C_T^2(s)}{C_T^1(s)} = \lambda$ for all states s). The unique solution to (2.5.12) is then clearly $\rho = \lambda$. In this case $\frac{\mathcal{E}(s)}{P_{NT}^1} = \frac{1}{\alpha C_T^1(s)}$, $\frac{\mathcal{E}(s)}{P_{NT}^2} = \frac{1}{\alpha C_T^2(s)}$, and nontradables are at their efficient level in both countries and in all states. This is the risk-sharing miracle. □

Sufficient conditions for assumption 38. Consider the period indirect utility function attained by country i when choosing to make transfer T, taking as given the transfer T^{-i} made by the other country as well as the equilibrium reaction of the central bank to $\{T, T^{-i}\}$, which we denote $\mathcal{E}(T)$ to make the dependence of the exchange rate on the transfer explicit. Denote $C_T^i(T) = E_T^i + T^{-i} - T$ and $C_{NT}^i(T) = \alpha \frac{\mathcal{E}(T)}{P_{NT}^i} C_T^i(T)$. Note that

$$\begin{aligned} \frac{dC_{NT}^i(T)}{dT} &= \alpha \frac{1}{P_{NT}^i}\left(\mathcal{E}'(T)C_T^i(T) - \mathcal{E}(T)\right) \\ &= -\alpha \frac{\mathcal{E}(T)}{P_{NT}^i}\left(1 - \frac{\mathcal{E}'(T)}{\mathcal{E}(T)}C_T^i(T)\right) \\ &= -\frac{C_{NT}^i(T)}{C_T^i(T)}\left(1 - \frac{\mathcal{E}'(T)}{\mathcal{E}(T)}C_T^i(T)\right) \end{aligned}$$

In a Nash equilibrium where current fiscal policy only has impact on current utility, T is chosen to maximize

$$v^i(T) = \log\left(C_T^i(T)\right) + \alpha\left(\log\left(C_{NT}^i(T)\right) - \frac{\left(C_{NT}^i(T)\right)^{1+\phi}}{1+\phi}\right)$$

The first-order condition of this problem is

$$\begin{aligned}
\frac{dv^i(T)}{dT} &= -\frac{1}{C_T^i(T)} - \alpha\frac{C_{NT}^i(T)}{C_T^i(T)}\left(1-\frac{\mathcal{E}'(T)}{\mathcal{E}(T)}C_T^i(T)\right)\left(\frac{1}{C_{NT}^i(T)} - C_{NT}^i(T)^{\phi}\right) \\
&= -\frac{1}{C_T^i(T)}\left[1+\alpha\left(1-\frac{\mathcal{E}'(T)}{\mathcal{E}(T)}C_T^i(T)\right)\underbrace{\left(1-C_{NT}^i(T)^{1+\phi}\right)}_{\tau^i(T)}\right]
\end{aligned}$$

where $\tau^i(T)$ is country i's labor wedge, a determined by the central bank's reaction to $\left(T, T^{-i}\right)$ and to the prices in place. From (2.5.2), this is done to enforce

$$\frac{1}{2}\left(\alpha\frac{\mathcal{E}(T)}{P_{NT}^1}C_T^1(T)\right)^{1+\phi} + \frac{1}{2}\left(\alpha\frac{\mathcal{E}(T)}{P_{NT}^2}C_T^2(T)\right)^{1+\phi} = 1$$

differentiating we find

$$\frac{1}{2}\left(\alpha\frac{\mathcal{E}(T)}{P_{NT}^1}C_T^1(T)\right)^{1+\phi}\left[\frac{\mathcal{E}'(T)}{\mathcal{E}(T)} + \frac{C_T^{1'}(T)}{C_T^1(T)}\right] + \frac{1}{2}\left(\alpha\frac{\mathcal{E}(T)}{P_{NT}^2}C_T^2(T)\right)^{1+\phi}\left[\frac{\mathcal{E}'(T)}{\mathcal{E}(T)} + \frac{C_T^{2'}(T)}{C_T^2(T)}\right] = 0$$

which leads to the central bank adjustment rule

$$\frac{\mathcal{E}'(T)}{\mathcal{E}(T)} = -\frac{1}{2}\left(1-\tau^1(T)\right)\frac{C_T^{1'}(T)}{C_T^1(T)} - \frac{1}{2}\left(1-\tau^2(T)\right)\frac{C_T^{2'}(T)}{C_T^2(T)}$$

For country i, an increase in own transfer reduces home country tradable consumption and increases foreign: $C_T^{i'}(T) = -1$ and $C_T^{-i'}(T) = 1$, so

$$\begin{aligned}
\frac{\mathcal{E}'(T)C_T^i(T)}{\mathcal{E}(T)} &= \frac{1}{2}\left(1-\tau^i(T)\right) - \frac{1}{2}\left(1-\tau^{-i}(T)\right)\frac{C_T^i(T)}{C_T^{-i}(T)} \\
&= \frac{1}{2}\left(1-\tau^i(T)\right) - \frac{1}{2}\left(1+\tau^i(T)\right)\frac{C_T^i(T)}{C_T^{-i}(T)} \\
&= \frac{1}{2}\left(1-\frac{C_T^i(T)}{C_T^{-i}(T)}\right) - \frac{1}{2}\tau^i(T)\left(1+\frac{C_T^i(T)}{C_T^{-i}(T)}\right)
\end{aligned}$$

and finally

$$1-\frac{\mathcal{E}'(T)C_T^i(T)}{\mathcal{E}(T)} = \frac{1}{2}\left(1+\frac{C_T^i(T)}{C_T^{-i}(T)}\right)\left(1+\tau^i(T)\right)$$

an expression which is always strictly positive, given that $\tau^i(T) > -1$. We therefore have

$$\frac{dv^i(T)}{dT} = -\frac{1}{C_T^i(T)}\left[1+\alpha\frac{1}{2}\left(1+\frac{C_T^i(T)}{C_T^{-i}(T)}\right)\left(1+\tau^i(T)\right)\tau^i(T)\right]$$

The function $\tau \mapsto 1+\alpha\frac{1}{2}\left(1+\frac{C_T^i}{C_T^{-i}}\right)(1+\tau)\tau$ has value 1 at both $\tau = 0$ and $\tau = -1$. Its minimum

$1-\alpha\frac{1}{8}\left(1+\frac{C_T^i(T)}{C_T^{-i}(T)}\right)$ is attained at $\tau=-\frac{1}{2}$. If

$$\alpha<\frac{8}{1+\frac{C_T^i(T)}{C_T^{-i}(T)}}$$

then $\frac{v^i(T)}{dT}<0$ and country i prefers not to make a transfer.

A sufficient condition for countries not to want to make transfers in autarky is then

$$\alpha<\frac{8}{1+\max_s\left\{\frac{E_T^1(s)}{E_T^2(s)};\frac{E_T^2(s)}{E_T^1(s)}\right\}}$$

more generally, the condition is that $1+\alpha\frac{1}{2}\left(1+\frac{E_T^i(s)}{E_T^j(s)}\right)\left(1+\tau^i(s)\right)\tau^i(s)>0$ for the labor wedge in country i and state s that results from central bank optimization given an autarkic fiscal policy, that is

$$\tau^i(s)=1-\frac{1}{\frac{1}{2}+\frac{1}{2}\left(\frac{\rho E_T^i(s)}{E_T^j(s)}\right)^{-(1+\phi)}}$$

where ρ is the solution to (2.5.12). □

Proof of Lemma 44. Suppose that countries have endowment processes governed by assumption 42 and follow Markov transfer rules. From (2.5.12), given any z_{-1}, the relative nontradables price $\rho=\frac{P_{NT}^2(z_{-1})}{P_{NT}^1(z_{-1})}$ solves

$$\sum_{s\in S}\pi(z|z_{-1})\left\{\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\left(\frac{E^H(z)-T(z,z_{-1})}{\rho\left(E^L(z)+T(z,z_{-1})\right)}\right)^{(1+\phi)}\right)^{-1}+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\left(\frac{E^L(z)+T(z,z_{-1})}{\rho\left(E^H(z)-T(z,z_{-1})\right)}\right)^{(1+\phi)}\right)^{-1}\right\}=1 \tag{2.5.13}$$

making use of the equality

$$\frac{1}{2}\frac{1}{\frac{1}{2}+\frac{1}{2}x}+\frac{1}{2}\frac{1}{\frac{1}{2}+\frac{1}{2}\frac{1}{x}}=1\quad\forall x$$

applied separately to each pair $x=\left(\frac{E^H(z)-T(z,z_{-1})}{E^L(z)+T(z,z_{-1})}\right)^{(1+\phi)}$, we see that $\rho=1$ is a solution to (2.5.13), hence its unique solution by Lemma 35. □

Proof of theorem 46. By Lemma 44, the relative nontradables price in monetary union is $\rho=1$. This delivers the indirect utility function

$$\tilde{v}(C)=\log(C)+\alpha\left(\log\left(\alpha\epsilon_z(C)C\right)-\frac{\left(\alpha\epsilon_z(C)C\right)^{1+\phi}}{1+\phi}\right)$$

where the common real exchange rate for both countries is given by

$$\epsilon_z(C)=\frac{1}{\alpha}\left(\frac{1}{2}\left(\frac{1}{C}\right)^{-(1+\phi)}+\frac{1}{2}\left(\frac{1}{E(z)-C}\right)^{-(1+\phi)}\right)^{-\frac{1}{1+\phi}} \tag{2.5.14}$$

with $E(z) = E^H(z) + E^L(z)$. This implies that

$$g_z(C) \equiv \alpha \epsilon_z(C) C = \left(\frac{1}{2} + \frac{1}{2}\left(\frac{C}{E(z) - C}\right)^{-(1+\phi)}\right)^{-\frac{1}{1+\phi}} \tag{2.5.15}$$

is a strictly increasing function of C. $g_z(C)$ attains 1 at $C = \frac{E(z)}{2}$.

The proof of theorem (46) now follows. By definition of subgame perfection, the H country does not want to refrain from the transfer at any node z^t:

$$\begin{aligned} &\log\left(C_T^H(z^t)\right) + \beta \sum_{z^h \succeq z^t} \frac{\pi(z^h|z^t)}{2}\left(\log\left(C_T^H(z^t)\right) + \log\left(C_T^L(z^t)\right)\right) \\ &\geq \log\left(E_T^H(z^t)\right) + \beta \sum_{z^h \succeq z^t} \frac{\pi(z^h|z^t)}{2}\left(\log\left(E_T^H(z^t)\right) + \log\left(E_T^L(z^t)\right)\right) \end{aligned}$$

Using the Markov structure, the participation constraints for all $z \in \mathbf{Z}$ can be written

$$\underbrace{\log\left(E_T^H(z)\right) - \log\left(C_T^H(z)\right)}_{\text{One-shot gain from defaulting}} \leq \underbrace{\beta \sum_{z} \frac{\tilde{\pi}(z'|z)}{2}\left[\left(\log\left(C_T^L(z')\right) - \log\left(E^L(z')\right)\right) - \left(\log\left(E^H(z')\right) - \log\left(C_T^H(z')\right)\right)\right]}_{\text{Expected loss from lack of future risk-sharing}}$$

where $\tilde{\pi}(z'|z)$ are the elements of the matrix $\tilde{\Pi} = \Pi^z(I - \beta\Pi^z)^{-1}$, which take into account the relevant mix of future probabilities and discounting. Due to the ordering (2.2.1) and the concavity of log, we have

$$\left(\log\left(C_T^L(z)\right) - \log\left(E^L(z)\right)\right) - \left(\log\left(E^H(z)\right) - \log\left(C_T^H(z)\right)\right) \geq 0$$

Consider sustaining the same SPE under monetary union using the same on-path and off-path actions. Lemma 44 implies that both countries have the same real exchange rate at every node, so they evaluate tradable consumption levels using the same indirect utility function

$$\tilde{v}_z(C) = \log(C) + \alpha\left(\log(\alpha\epsilon_z(C)C) - \frac{1}{1+\phi}(\alpha\epsilon_z(C)C)^{1+\phi}\right)$$

where $\epsilon_z(C)$ is given in (2.5.14). Recall from (2.5.15) that $g_z(C) \equiv \alpha\epsilon_z(C)C$ is strictly monotone and attains the value 1 (the efficient nontradables consumption level) at $C = \frac{E^H(z)+E^L(z)}{2}$. This is a consequence of the risk-sharing miracle. Since

$$f(x) \equiv \alpha\left(\log(x) - \frac{1}{1+\phi}x^{1+\phi}\right)$$

is a concave function with a maximum at $x = 1$, the function $\tilde{v}_z(C) - \log(C) = f(g_z(C))$ is single-peaked with a maximum at $\frac{E^L(z)+E^H(z)}{2}$. In particular

$$\begin{aligned} f\left(g_z\left(E^H(z)\right)\right) &\leq f\left(g_z\left(C_T^H(z)\right)\right) \quad \forall z \\ f\left(g_z\left(E^L(z)\right)\right) &\leq f\left(g_z\left(C_T^L(z)\right)\right) \quad \forall z \end{aligned}$$

from which it follows that

$$\tilde{v}_z\left(E^H(z)\right) - \tilde{v}_z\left(C_T^H(z)\right) \leq \log\left(E^H(z)\right) - \log\left(C_T^H(z)\right) \quad \forall z$$

and that

$$\log\left(C_T^L(z)\right) - \log\left(E^L(z)\right) \leq \tilde{v}_z\left(C_T^L(z)\right) - \tilde{v}_z\left(E^L(z)\right) \quad \forall z$$

Combining these inequalities, we obtain

$$\begin{aligned} \tilde{v}_z\left(E_T^H(z)\right) - \tilde{v}_z\left(C_T^H(z)\right) &\leq \log\left(E_T^H(z)\right) - \log\left(C_T^H(z)\right) \\ &\leq \beta \sum_z \frac{\tilde{\pi}(z'|z)}{2}\left[\left(\log\left(C_T^L(z)\right) - \log\left(E^L(z)\right)\right) - \left(\log\left(E^H(z)\right) - \log\left(C_T^H(z)\right)\right)\right] \\ &\leq \beta \sum_z \frac{\tilde{\pi}(z'|z)}{2}\left[\left(\tilde{v}_z\left(C_T^L(z)\right) - \tilde{v}_z\left(E^L(z)\right)\right) - \left(\tilde{v}_z\left(E^H(z)\right) - \tilde{v}_z\left(C_T^H(z)\right)\right)\right] \end{aligned}$$

which guarantees that the participation constraint for the H country is met in every state z under monetary union, as claimed. □

Proof of theorem 47. We consider the simplest possible example of our framework: the symmetric iid case with union-wide tradable output equal to 1. In the notation above, the Markov chain is reduced to a point $z = 1$ in every period, $E^H = e$ and $E^L = 1 - e$, with $e > \frac{1}{2}$. In this context, a Markov transfer is a value T, such that $\left(C_T^L, C_T^H\right) = (1 - e + T, e - T)$. We consider transfers that improve risk-sharing, in other words $T \in \left[0, \frac{1}{2} - e\right]$.

Under independent monetary policy, given the flow value from the nontradables side is always constant at $f^* = f(1) \equiv -\frac{\alpha}{1+\phi}$, the value of being in the high state under the contract is

$$V^H(T) = \log(e - T) + \frac{\beta}{1-\beta}\left(\frac{1}{2}\log(e - T) + \frac{1}{2}\log(1 - e + T)\right) + \frac{f^*}{1-\beta}$$

The participation constraint states that $V^H(T) \geq V^H(0)$. Since V^H is concave in T, there exists a $T > 0$ such that this constraint is satisfied if, and only if

$$\left.\frac{dV^H}{dT}\right|_{T=0} = -\frac{1}{e} + \frac{\beta}{1-\beta}\frac{1}{2}\left(-\frac{1}{e} + \frac{1}{1-e}\right) \geq 0$$

that is, if and only if

$$\beta \geq \underline{\beta}^{indep} = 2(1-e)$$

Suppose countries now are sharing risks perfectly $T = \frac{1}{2} - e$, therefore obtaining $\left(C_T^L, C_T^H\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and value

$$V^{FB} = \frac{1}{1-\beta}\left(\log\left(\frac{1}{2}\right) + f^*\right)$$

Consider the deviation of the country in the high state, which is punished by the grim trigger strategy. This deviation is valuable if

$$\underbrace{\log(e) - \log\left(\frac{1}{2}\right)}_{\text{One-shot gain from defaulting}} \geq \frac{\beta}{1-\beta}\frac{1}{2}\underbrace{\left(2\log\frac{1}{2} - \log(e) - \log(1-e)\right)}_{\text{Expected loss from lack of future risk-sharing}}$$

Which is equivalent to the condition

$$\beta \leq \overline{\beta}^{indep} = 2\frac{\log(e) - \log\left(\frac{1}{2}\right)}{\log(e) - \log(1-e)} \leq 1$$

Consider now the case of a monetary union, and assume countries are achieving perfect risk-sharing $T = \frac{1}{2} - e$. Due to the risk-sharing miracle (theorem 36), their nontradables side is perfectly stabilized and they obtain V^{FB}. However, a deviation now entails the additional cost

$$c(e) = f^* - f(g(e)) = f(1) - f(g(e)) = \alpha\left(\frac{g(e)^{1+\phi} - 1}{1+\phi} - \log(g(e))\right) \geq 0$$

where nontradable consumption, considering the central bank reaction, is

$$g(e) = \left(\frac{1}{2} + \frac{1}{2}\left(\frac{e}{1-e}\right)^{-(1+\phi)}\right)^{-\frac{1}{1+\phi}} > 1$$

for the country in the H state and and $g(1-e) < 1$ for the country in the L state. The condition for a profitable deviation is now

$$\log(e) - \log\left(\frac{1}{2}\right) - c(e) \geq \frac{\beta}{1-\beta}\frac{1}{2}\left(2\log\frac{1}{2} - \log(e) - \log(1-e) + c(e) + c(1-e)\right)$$

which shows clearly that monetary union both lowers the benefit and raises the costs of defaulting. A country therefore finds it profitable to deviate from first-best risk-sharing under monetary union when

$$\beta \leq \overline{\beta}^{union}(\alpha, \phi, e) = 2\frac{\log(e) - \log\left(\frac{1}{2}\right) - c(e)}{\log(e) - \log(1-e) + c(1-e) - c(e)}$$

It is simple to show that $\overline{\beta}^{union}(\alpha, \phi, e)$ is strictly decreasing in α with $\lim_{\alpha\to\infty}\overline{\beta}^{union}(\alpha, \phi, e) < 0$. This has two consequences. First,

$$\overline{\beta}^{union}(\alpha, \phi, e) \leq \overline{\beta}^{union}(0, \phi, e) = \overline{\beta}^{indep}$$

formalizing our claim that first-best risk-sharing is easier to sustain under monetary union than under independent monetary policy. Second, there always exists values of (α, ϕ, e) such that $\overline{\beta}^{union}(\alpha, \phi, e) \leq \underline{\beta}^{indep}$. To complete the claim that countries with discount factors $\overline{\beta}^{union}(\alpha, \phi, e) \leq \beta \leq \underline{\beta}^{indep}$ can only sustain autarky under independent monetary policy, but can sustain full risk-sharing under monetary union, one also needs to check that autarky under monetary union is indeed subgame perfect at those parameters. This is ensured by the condition

$$\frac{1}{e} + \frac{\alpha}{g(e)} \left(1 - (g(e))^{1+\phi}\right) g'(e) \geq 0$$

or equivalently

$$\alpha \leq \overline{\alpha}(e, \phi) = \frac{1}{\frac{\left(\frac{e}{1-e}\right)^{-(1+\phi)}}{1+\left(\frac{e}{1-e}\right)^{-(1+\phi)}} \frac{(g(e))^{1+\phi}-1}{1-e}}$$

□

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Chapter 3

Unique equilibrium in the Eaton-Gersovitz model of sovereign debt

A common view of sovereign debt markets is that they are prone to multiple equilibria: a market panic may inflate bond yields, deteriorate the sustainability of government debt and precipitate a default event, justifying investor fears. Indeed, Mario Draghi's speech in July 2012, announcing that the ECB was "ready to do whatever it takes" to preserve the single currency, and the subsequent creation of the Outright Monetary Transactions (OMT) program, are widely seen as having moved Eurozone sovereign debt markets out of an adverse equilibrium: since then, bond spreads have experienced dramatic falls as fears of default have receded.

At the same time, in the last decade, a booming quantitative literature in the line of Eaton and Gersovitz (1981)—initiated by Arellano (2008) and Aguiar and Gopinath (2006), and summarized in Aguiar and Amador (2015)—has studied sovereign debt markets using an infinite-horizon incomplete markets model for which no result on equilibrium multiplicity was known. An example of how the literature viewed this issue is in Hatchondo, Martinez and Sapriza (2009):

> Krusell and Smith (2003) show that, typically, there is a problem of indeterminacy of Markov-perfect equilibria in an infinite-horizon economy. In order to avoid this problem, we analyze the equilibrium that arises as the limit of the finite-horizon economy equilibrium.

In this paper, we show that equilibrium is *unique* in the benchmark infinite-horizon model with a Markov process for the exogenous driving state and exogenous value from default. Although we emphasize Markov perfect equilibrium—the usual equilibrium concept in the literature, and one for which our argument is especially direct—we prove that our core uniqueness result extends to subgame perfect equilibria more generally. We also extend our proof to several modifications of the benchmark model, as described below.

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The key to all our proofs is to rule out the possibility of a self-sustaining deterioration in the value of borrowing arising from a deterioration in the terms of borrowing. The intuition is simplest in our first modification of the benchmark model: one in which debt is restricted to be risk-free, as in Zhang (1997). In that case, each equilibrium features an endogenous debt limit, which is the most that can be incurred without the possibility that the government will want to default in the next period. If there are two equilibria with distinct limits, we consider two governments that are each at the limit in their respective equilibria. We argue that the government with less debt must have a strictly higher value: starting from that point, it can follow a strategy that parallels the strategy of the higher-debt government, maintaining its liabilities at a uniform distance and achieving higher consumption at every point by economizing on interest payments. But this contradicts the assumption that both governments start at their debt limits, where each must obtain the (constant) value of default. In short, once both governments have exhausted their debt capacity, the one with a strictly lower level of debt is strictly better off—meaning that this government should be able to borrow slightly more without running the risk of default, and cannot have exhausted its capacity after all.

Interestingly, this proof strategy by replication has echoes of that used by Bulow and Rogoff (1989) to rule out reputational equilibria in a similar class of models where sovereign governments retain the ability to save after defaulting. The original Bulow-Rogoff result is cast in a complete markets setting. In a second modification of the benchmark model, we specify the only punishment from default as the loss of ability to borrow. As an immediate corollary to our Eaton-Gersovitz uniqueness result, we then obtain the *incomplete markets Bulow-Rogoff result*: under this specification of default costs, the no-borrowing equilibrium is the unique equilibrium. Hence our general uniqueness result nests a key impossibility result for the sovereign debt literature.

We next explore the importance of the model's assumptions by relaxing each of them in turn, and then considering the robustness of our uniqueness result. We first consider a case where savings are exogenously bounded. We prove that uniqueness holds provided that the bound on savings is strictly positive. This provides continuity with the result of Passadore and Xandri (2014), who have found multiplicity when the bound is zero. Next, we consider a case where the value of default is no longer exogenous. When this endogeneity comes from the assumption that governments in default have a stochastic option to reenter markets (a typical assumption in the quantitative literature), we rule out multiplicity of the most widely suspected form—where bond prices in a favorable equilibrium dominate those in a self-fulfilling adverse one—and obtain complete uniqueness when shocks are independent and identically distributed. Finally, we explore the role of our timing and commitment assumptions by discussing the way in which they differ from those in other models from the literature that do feature multiple equilibria (Calvo, 1988, Cole and Kehoe, 2000 and Lorenzoni and Werning, 2014).

Our results are important because they show that the multiplicity intuition is *not* valid in a benchmark model that is accepted as a good description—both qualitative and quantitative—of sovereign debt markets.[1]

[1] In particular, under our assumptions, sunspots cannot influence equilibrium outcomes. Recently, Stangebye (2015) has explored the role of sunspots in two versions of the Eaton-Gersovitz model where our results do not apply—first, for short-term

They provide an additional analytical result for a model about which few such results exist, making use of a powerful new proof technique along the way. And they show that alternative strategies to compute Markov perfect equilibria should all converge to the same solution. Our results are not directly applicable to all the extensions of the Eaton-Gersovitz model that the quantitative sovereign debt literature has considered, but they do suggest that multiplicity is unlikely in many of these cases as well, and therefore that the literature's quantitative findings are probably not driven by a hidden equilibrium selection.[2]

Our objective is not to deny that sovereign debt markets can be prone to self-fulfilling crises, or that OMT may have ruled out a bad equilibrium. Instead, we hope that our results may help sharpen the literature's understanding of the assumptions that are needed for such multiple equilibria to exist. Our replication-based proof strategy may also be of independent interest, as a general technique for proving uniqueness of equilibrium in infinite-horizon games.

Layout. The rest of the paper is organized as follows. Section 3.1 lays out the benchmark Eaton-Gersovitz model, and establishes uniqueness of Markov perfect equilibrium and uniqueness of subgame perfect equilibrium. Section 3.2 adapts our main proof to two related models. Section 3.2.1 proves uniqueness in the Zhang (1997) model, where debt is restricted to be risk-free, and section 3.2.2 derives the incomplete markets version of the Bulow and Rogoff (1989) result as a corollary of our main uniqueness result. Section 3.3 considers the robustness of our results as we relax various assumptions. Section 3.3.1 considers exogenous restrictions on savings. Section 3.3.2 considers the case where reentry is allowed after default. Section 3.3.3 discusses our timing and commitment assumptions, contrasting them with those in the related literature. Section 3.4 concludes.

3.1 Equilibrium uniqueness in the benchmark model

3.1.1 Model description

We now describe what we call the *benchmark* infinite-horizon model with Markov income (see Aguiar and Amador, 2015). We focus first on Markov perfect equilibria, in which the current states b and s encode all the relevant history. In section 3.1.3 we will show that this is without loss of generality, since one can specify this model as a game whose only subgame perfect equilibria are Markov perfect equilibria.

An exogenous state s follows a discrete Markov chain with elements in $\mathcal{S}$, $|\mathcal{S}| = S \in \mathbb{N}$ and transition matrix $\pi(s'|s)$. Output $y(s)$ is a function of this underlying state.

At the beginning of each period, the government starts with some level of debt b. After observing the realization of s, it decides whether to repay b or default. If it does not repay, it receives an exogenous value

debt, when the domain of debt is exogenously restricted beyond what we consider in section 3.3.1, and second, for long-term debt.

[2]While our focus is on sovereign debt, the benchmark model we study also constitutes the core of a literature that analyzes unsecured consumer credit (Chatterjee, Corbae, Nakajima and Ríos-Rull, 2007), and we conjecture that equilibrium is also unique in many of the models used in that literature.

$V^d(s)$, which encodes all the consequences of default. For example, if default is punished by permanent autarky, with output also reduced by an exogenous cost $\tau(s) \in [0, y(s)]$, then V^d is defined recursively by $V^d(s) = u(y(s) - \tau(s), s) + \beta \mathbb{E}_{s'|s}[V^d(s')]$. It is important for the current result that $V^d(s)$ is exogenous and independent of b. We will allow for some endogeneity of $V^d(s)$ when we consider stochastic reentry options in section 3.3.2.

If the government does not default, it receives $y(s)$ as endowment, pays b, and issues new bonds b' that will be due next period, raising revenue $Q(b', s)$. Its (possibly state-dependent) flow utility from consumption is $u(c, s)$, so that the value V from repayment is given by

$$\begin{aligned} V(b, s) &= \max_{b'} u(c, s) + \beta \mathbb{E}_{s'|s}[V^o(b', s')] \\ &\quad \text{s.t.} \quad c + b = y(s) + Q(b', s) \end{aligned} \tag{3.1.1}$$

and the value V^o including the option to default at the beginning of a period is given by

$$V^o(b, s) = \max_{p \in \{0,1\}} pV(b, s) + (1 - p)V^d(s) \tag{3.1.2}$$

where $p = 1$ denotes the decision to repay and $p = 0$ denotes the decision to default.

Debt is purchased by risk-neutral international investors that demand an expected return of R. For convenience, we assume that when a government is indifferent between repayment and default, it chooses to repay: $p(b, s) = 1$ if and only if $V(b, s) \geq V^d(s)$. Since investors receive expected repayment $\mathbb{E}_{s'|s}[p(b', s')]$, if follows that the bond revenue schedule Q is

$$Q(b', s) = \frac{b'}{R} \mathbb{E}_{s'|s}[p(b', s')] = \frac{b'}{R} \mathbb{P}_{s'|s}[V(b', s') \geq V^d(s')] \tag{3.1.3}$$

We are now ready to define Markov perfect equilibrium, which is the typical focus in the literature.

Definition 1. A Markov perfect equilibrium is a set of policy functions $p(b, s)$, $c(b, s)$, $b'(b, s)$ for repayment, consumption and next period borrowing, value functions $V(b, s)$ and $V^o(b, s)$, and a bond revenue schedule $Q(b', s)$ such that (3.1.1)-(3.1.3) are satisfied.

We first prove an existence result. For this we need a number of technical assumptions. We highlight the following, because of the crucial role it will play for our main uniqueness proof in section 3.1.2.

Assumption 1. For each $s \in \mathcal{S}$, $u(c, s)$ is strictly increasing in c.

Proposition 51. *Under assumption 1 and additional standard assumptions, a Markov perfect equilibrium exists. In any equilibrium, $V(b, s)$ is strictly decreasing in b for each $s \in \mathcal{S}$, and there exists a set of default thresholds $\{b^*(s)\}_{s \in \mathcal{S}}$ such that the government repays in state s if and only if $b \leq b^*(s)$. Both V and Q are uniquely determined by the thresholds $\{b^*(s)\}_{s \in \mathcal{S}}$.*

The proof, developed in appendix 3.5.1, is constructive and relies on a fixed-point procedure similar to the one used by the quantitative literature to search for an equilibrium. As highlighted by Aguiar and Amador (2015), this procedure involves iterating on a monotone and bounded operator in the space of default thresholds. These iterations converge to a fixed point, and our proof verifies that this fixed point defines an equilibrium. Our additional assumptions (including the continuity of u, a no-Ponzi restriction, and an upper bound on R) ensure that value functions exist, are continuous and finite-valued, and that default thresholds $b^*(s)$ are uniquely defined by the equalities

$$V(b^*(s), s) = V^d(s) \tag{3.1.4}$$

The set $\{b^*(s)\}_{s\in\mathcal{S}}$ then characterizes the bond revenue schedule Q: following (3.1.3),

$$Q(b', s) = \frac{b'}{R}\mathbb{P}_{s'|s}\left[b' \leq b^*(s')\right] = \frac{b'}{R}\sum_{\{s':b'\leq b^*(s')\}}\pi(s'|s) \tag{3.1.5}$$

In the special case where $u(c, s) = u(c)$, income is i.i.d, and V^d is the expected value of autarky, it is possible to show that $b^*(s)$ is increasing in $y(s)$ (see Arellano, 2008), but such monotonicity is not needed for our proof.

3.1.2 Uniqueness of Markov perfect equilibrium

Suppose that we have two distinct revenue schedules Q and $\widetilde{Q}$, each derived via (3.1.5) from anticipated default thresholds $\{b^*(s)\}_{s\in\mathcal{S}}$ and $\left\{\widetilde{b}^*(s)\right\}_{s\in\mathcal{S}}$. Let V and $\widetilde{V}$ be the value functions for a government facing these schedules. To prove uniqueness of equilibrium, we need to show that at most one of these value functions can be consistent with the default thresholds that generate it—in other words, that we cannot have both $V(b^*(s), s) = V^d(s)$ and $\widetilde{V}(\widetilde{b}^*(s), s) = V^d(s)$ for all s.

The key observation of this paper is that we can derive a simple inequality for the two value functions V and $\widetilde{V}$, related to the maximum difference between the default thresholds. This inequality requires assumption 1 together with

Assumption 2. $R > 1$.

The basis of our inequality is a simple replication strategy we call *mimicking at a distance.* Suppose that $b^*(s)$ exceeds $\widetilde{b}^*(s)$ by at most $M > 0$. Then we show that it is always weakly better to start with debt of $b - M$ when facing prices $\widetilde{Q}$ than with debt of b when facing prices Q, and indeed strictly better whenever $V(b, s) \geq V^d(s)$. This observation, formalized in lemma 52, will ultimately be the basis of the proof that distinct equilibria are impossible in proposition 53.

The argument is as follows. The government with debt $b - M$ facing prices $\widetilde{Q}$ has the option to *mimic* the policy of the government with debt b facing prices Q—always defaulting at the same points, and otherwise choosing the same level of debt for the next period minus M. Before it defaults, this government is better off because it pays less to service debt, allowing it to consume more.

Debt service, in turn, costs less for two reasons. First, the mimicking government is less likely to be above the default thresholds assumed by its revenue schedule. This is due to the choice of M: since M is the maximum amount by which the default thresholds $b^*(s)$ exceed the thresholds $\widetilde{b}^*(s)$, as long as the government facing Q chooses debt of M less than the government it is mimicking, it is weakly less likely to exceed $\widetilde{b}^*(s)$ than the other government is to exceed $b^*(s)$. Second, the mimicking government has strictly less debt, meaning that the cost of providing an expected return of $R > 1$ on this debt is lower.

Following this policy, the mimicking government always consumes strictly more until default, implying strictly higher utility due to assumption 1. It thus obtains a weakly higher value, which is strictly higher as long as it does not default right away.

Lemma 52 (Mimicking at a distance.). *Let Q and $\widetilde{Q}$ be two distinct revenue schedules, with Q reflecting expected default thresholds $\{b^*(s)\}_{s\in S}$ and $\widetilde{Q}$ reflecting expected default thresholds $\{\widetilde{b}^*(s)\}_{s\in S}$. Let V and $\widetilde{V}$ be the respective value functions for governments facing these revenue schedules. Define*

$$M = \max_s b^*(s) - \widetilde{b}^*(s) \tag{3.1.6}$$

and assume without loss of generality that $M > 0$. Then, for any s and b,

$$\widetilde{V}(b - M, s) \geq V(b, s) \tag{3.1.7}$$

with strict inequality whenever $V(b,s) \geq V^d(s)$.

Proof. First, note that for any b' and s, applying (3.1.5) we have

$$\begin{aligned}
\widetilde{Q}(b' - M, s) &= \frac{(b' - M)}{R} \sum_{\{s' : b' - M \leq \widetilde{b}^*(s')\}} \pi(s'|s) \geq \frac{(b' - M)}{R} \sum_{\{s' : b' \leq b^*(s')\}} \pi(s'|s) \\
&> \left(\frac{b'}{R} \sum_{\{s' : b' \leq b^*(s')\}} \pi(s'|s) \right) - M = Q(b', s) - M
\end{aligned} \tag{3.1.8}$$

Thus the amount that a government with schedule $\widetilde{Q}$ can raise by issuing $b' - M$ of debt is always strictly larger than the amount that a government with schedule Q can raise by issuing b' of debt, minus M. The two intermediate inequalities in (3.1.8) reflect the two sources of this advantage. First, there are weakly more cases in which $b' - M \leq \widetilde{b}^*(s')$ than in which $b' \leq b^*(s')$, and this higher chance of repayment makes it possible to raise more. Second, since assumption 2 requires $R > 1$, issuing M less debt costs strictly less than M in foregone revenue in the current period.

Now we can formally define the *mimicking at a distance* policy. For any states and debt levels s and b, let the history s^0 be such that the state and debt owed at $t = 0$ are respectively s and b. The optimal strategy for a government facing schedule Q induces an allocation $\{c(s^t), b(s^{t-1}), p(s^t)\}_{s^t \succeq s^0}$ at all histories following

s^0.[3] We construct a policy for the government facing schedule $\widetilde{Q}$ in state s and debt level $b-M$ as follows. For every history $s^t \succeq s^0$, let

$$\widetilde{p}\left(s^t\right) = p\left(s^t\right)$$

and provided that $p\left(s^t\right) = 1$, choose consumption and next-period debt as

$$\begin{aligned} \widetilde{b}\left(s^t\right) &= b\left(s^t\right) - M \\ \widetilde{c}\left(s^t\right) &= c\left(s^t\right) + \widetilde{Q}\left(b\left(s^t\right) - M, s_t\right) - \left(Q\left(b\left(s^t\right), s_t\right) - M\right) \end{aligned} \tag{3.1.9}$$

This ensures that the budget constraint is satisfied at all histories s^t where repayment takes place:

$$\begin{aligned} \widetilde{c}\left(s^t\right) + \widetilde{b}\left(s^{t-1}\right) - \widetilde{Q}\left(\widetilde{b}\left(s^t\right), s_t\right) &= \widetilde{c}\left(s^t\right) + b\left(s^{t-1}\right) - M - \widetilde{Q}\left(b\left(s^t\right) - M, s_t\right) \\ &= c\left(s^t\right) + b\left(s^{t-1}\right) - Q\left(b\left(s^t\right), s_t\right) \\ &= y\left(s_t\right) \end{aligned}$$

Furthermore, using (3.1.8) we see that $\widetilde{c}(s^t) > c(s^t)$: when there is repayment, the mimicking policy (3.1.9) sets consumption $\widetilde{c}(s^t)$ equal to consumption $c(s^t)$ in the other equilibrium, plus a bonus $\widetilde{Q}\left(b\left(s^t\right) - M, s_t\right) - \left(Q\left(b\left(s^t\right), s_t\right) - M\right) > 0$ from lower debt costs.

The mimicking policy, of course, need not be optimal; but since it is feasible, it serves as a lower bound for $\widetilde{V}(b-M, s)$:

$$\begin{aligned} \widetilde{V}(b-M, s) &\geq \sum_{\widetilde{p}(s^t)=1} \beta^t \Pi\left(s^t\right) u\left(\widetilde{c}\left(s^t\right), s_t\right) + \sum_{\widetilde{p}(s^t)=0, \widetilde{p}(s^{t-1})=1} \beta^t \Pi\left(s^t\right) V^d(s_t) \\ &\geq \sum_{p(s^t)=1} \beta^t \Pi\left(s^t\right) u\left(c\left(s^t\right), s_t\right) + \sum_{p(s^t)=0, p(s^{t-1})=1} \beta^t \Pi\left(s^t\right) V^d(s_t) = V(b, s) \end{aligned}$$

with strict inequality whenever $p(s^0) = 1$ (or equivalently $b \leq b^*(s)$), since this implies $\widetilde{c}(s^0) > c(s^0)$ and $u(c, s_0)$ is strictly increasing in c thanks to assumption 1. □

An illustration of the mimicking policy used in lemma 52 is given in figures 3.1.1 and 3.1.2, which depict time paths in a hypothetical two-state case. In this case, debt starts relatively high and the high-income state $y(s_H)$ keeps recurring, leading the government to deleverage in anticipation of lower incomes in the future. Figure 3.1.1 shows the paths of b (filled circles) and the mimicking policy $\widetilde{b} = b - M$ (hollow circles), while figure 3.1.2 shows the paths of c (filled circles) and the consumption $\widetilde{c} = c + \widetilde{Q}(b-M, s) - (Q(b, s) - M)$ induced by the mimicking policy (hollow circles).

Although $\widetilde{c}$ is always greater than c in figure 3.1.2, the gap $\widetilde{c} - c$ differs across periods. This reflects fluctuations in the two sources of $\widetilde{c} - c$: differences in default premia, and the lower cost of servicing $\widetilde{b} = b - M$ rather than b. First, since both debt levels at $t = 2$ are above the respective default thresholds

[3] $b(s^t)$ is defined to be the amount of debt chosen at history s^t to be repaid in period $t+1$.

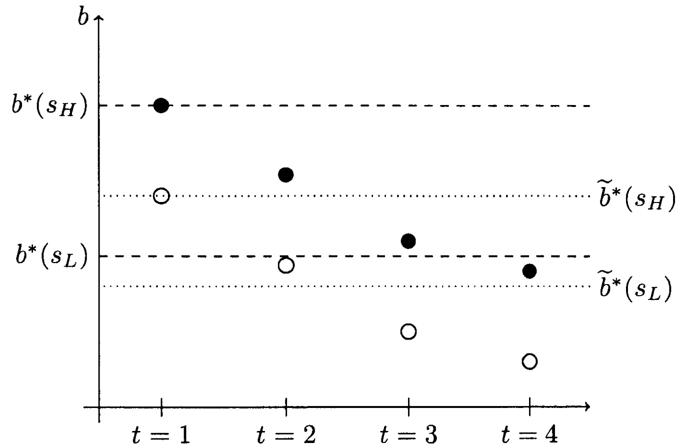

Figure 3.1.1: Example paths for b and $\widetilde{b}$.

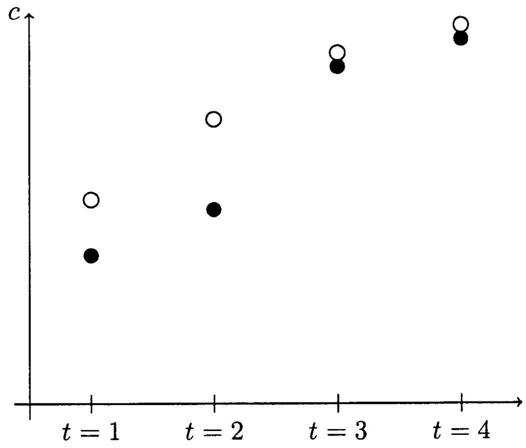

Figure 3.1.2: Example paths for c and $\widetilde{c}$.

for $y(s_L)$, there is no difference at $t = 1$ in the two default premia. At $t = 3$, however, the mimicking policy achieves a debt level below $\widetilde{b}^*(s_L)$, while the other policy has debt that remains above $b^*(s_L)$. Thus the default premium disappears at $t = 2$ for the mimicking policy while still being paid for the other policy, leading to an expansion in the gap $\widetilde{c} - c$. From $t = 4$ onward both policies achieve debt levels below their s_L default thresholds, leading to the disappearance of all default premia. This causes the gap $\widetilde{c} - c$ to compress starting at $t = 3$.

The central observation is that if it starts with debt $M = \max_s b^*(s) - \widetilde{b}^*(s)$ below the other government, the mimicking government can keep itself at the fixed distance M, achieving higher consumption along the way.

We now turn to the main result, which uses lemma 52 to rule out multiple equilibria (V, Q) and $(\widetilde{V}, \widetilde{Q})$ altogether.

Proposition 53. *In the benchmark model, Markov perfect equilibrium has a unique value function $V(b, s)$ and debt price schedule $Q(b, s)$.*

Proof. Suppose to the contrary that there are distinct equilibria (V, Q) and $\left(\widetilde{V}, \widetilde{Q}\right)$. Proposition 51 shows that these are characterized by their default thresholds $\{b^*(s)\}_{s \in \mathcal{S}}$ and $\left\{\widetilde{b}^*(s)\right\}_{s \in \mathcal{S}}$. Therefore, it suffices for us to show that the thresholds are unique.

Without loss of generality, assume that the maximal difference between b^* and $\widetilde{b}^*$ is positive and is attained in a state $\overline{s} \in \mathcal{S}$:

$$\max_s b^*(s) - \widetilde{b}^*(s) = b^*(\overline{s}) - \widetilde{b}^*(\overline{s}) = M > 0$$

Applying lemma 52 for $s = \bar{s}$ and $b = b^*(\overline{s}) = \widetilde{b}^*(\overline{s}) + M$, we know that

$$\widetilde{V}(\widetilde{b}^*(\overline{s}), \overline{s}) > V(b^*(\overline{s}), \overline{s})$$

But this contradicts the fact that $b^*(\bar{s})$ and $\widetilde{b}^*(\bar{s})$ are default thresholds, which requires $\widetilde{V}(\widetilde{b}^*(\bar{s}), \bar{s}) = V(b^*(\bar{s}), \bar{s}) = V^d(\bar{s})$. Thus our premise of distinct equilibria cannot stand. □

The intuitive thrust of lemma 52 and proposition 53 is that distinct debt revenue schedules cannot both be self-sustaining. No two schedules Q and $\widetilde{Q}$ can simultaneously rationalize their corresponding default thresholds $b^*(\bar{s})$ and $\widetilde{b}^*(\bar{s})$ in the state $\bar{s}$ where these thresholds differ most. Instead, the argument of lemma 52 shows that it is better for a government to start at the lower threshold $b^*(\bar{s})$ given schedule Q than to start at the higher threshold $\widetilde{b}^*(\bar{s})$ given schedule $\widetilde{Q}$; at this point, any advantages of $\widetilde{Q}$ over Q are outweighed by the heavier debt burden, and the former government can use a simple mimicking strategy to guarantee itself strictly higher consumption than the latter. It follows that these cannot both be default thresholds, which by definition must be equally desirable, with common value equal to the default value $V^d(\bar{s})$.

3.1.3 Uniqueness of subgame perfect equilibrium

The arguments used to prove proposition 53 can be extended to show that this model admits a unique subgame perfect equilibrium. While the Markov perfect concept exogenously restricts equilibrium to depend on a limited set of states, subgame perfect equilibria allow an arbitrary dependence of strategies at time t on the history h^{t-1} of past states and actions. The following result shows that the current states s and b summarize this dependence, demonstrating that the Markov concept—which has been the focus of much of the quantitative literature—is not restrictive. Proving this formally requires defining the game played by the government and international investors more precisely. Crucially, in this game, the value from government default is still exogenous—endogenizing the default option as part of the game is outside of the scope of this paper (see Kletzer and Wright, 2000, for such an exercise). Here we summarize our result, and relegate the description of the game and the proof to appendix 3.5.2. Let $V\left(h^{t-1}, s\right)$ be the value achieved by a government after history h^{t-1}, when the current exogenous state is s. Then the following result holds.

Proposition 54. *Consider two subgame perfect equilibria A and B. For any (b, s), and any histories (h_A, h_B) such that $b(h_A) = b(h_B) = b$, we have $V_A(h_A, s) = V_B(h_B, s)$.*

The key to the proof of proposition 54 is to show that, conditional on the exogenous state s, a government with higher debt must have lower value, independently of the equilibrium that is played *or* the history of past actions. This in turn relies on another mimicking argument, whereby a government with lower debt can always choose a strategy that ensures it higher consumption and higher future value than its higher-debt counterpart.

3.2 Application to other models

The argument used to prove uniqueness of equilibrium in section 3.1 is very general and can be used in other contexts, as the following applications illustrate.

3.2.1 Bewley models with endogenous debt limits

Consider a modification of the environment of section 3.1, in which lenders are restricted to offer a price of $\frac{1}{R}$ for every unit of debt that they buy. Borrowing must therefore be risk-free: this is the equilibrium defined in Zhang (1997). This restriction can be captured within the framework of the previous section by specifying that the price of non-riskless debt is zero. Instead of (3.1.3), the bond revenue schedule becomes

$$Q^z(b', s) = \frac{b'}{R}\mathbf{1}\left[V^z(b', s') \geq V^d(s') \quad \forall s'|s\right] \tag{3.2.1}$$

Define $\phi(s)$ as the value that satisfies $V^z(\phi(s), s) = V^d(s)$, and assume that for all s and s', $\pi(s'|s) > 0$. Then, writing $\varphi \equiv \min_s \{\phi(s)\}$, (3.2.1) becomes

$$Q^z(b', s) = \frac{b'}{R}\mathbf{1}_{\{b' \leq \varphi\}} \tag{3.2.2}$$

In other words, the model is a standard incomplete markets model in the tradition of Bewley (1977), with a debt limit φ determined endogenously by the requirement that the government should never prefer default.

We can immediately prove analogs of lemma 52 and proposition 53 in this new environment.

Lemma 55. *Consider two distinct equilibria with value functions V and $\widetilde{V}$ and debt limits $\widetilde{\varphi} < \varphi$. Then, letting $M = \varphi - \widetilde{\varphi}$, for any b and s we have*

$$\widetilde{V}(b - M, s) \geq V(b, s) \tag{3.2.3}$$

with strict inequality whenever $b \leq \varphi$.

Proof. Same as the proof of lemma 52, with (3.1.3) replaced by (3.2.2) and inequality (3.1.8) becoming

$$\widetilde{Q}^z(b' - M, s) = \frac{b' - M}{R}\mathbf{1}_{\{b' - M \leq \widetilde{\varphi}\}} = \frac{b' - M}{R}\mathbf{1}_{\{b' \leq \varphi\}} > \frac{b'}{R}\mathbf{1}_{\{b' \leq \varphi\}} - M = Q^z(b', s) - M \tag{3.2.4}$$

□

The intuition behind (3.2.3) and (3.2.4) is well known in this class of environment: an increase in the debt limit is equivalent to a translation of the value function, accompanied by a translation of the income process that reflects the interest costs of debt.[4] Our earlier inequality (3.1.8) can be interpreted as a generalization of this result.

Proposition 56. *In the model with riskless debt, Markov perfect equilibrium has a unique value function $V(b, s)$ and debt limit φ.*

Proof. Same as the proof of proposition 53, but using lemma 55 rather than lemma 52 . □

[4] See, for example, Ljungqvist and Sargent (2012).

As proposition 56 demonstrates, our approach for proving uniqueness is not limited to the standard Eaton-Gersovitz framework. The underlying replication argument can be adapted, with some modifications, to other environments common in the literature. As highlighted in the introduction, this particular application also illustrates the key intuition behind our main uniqueness result in section 3.1: a deterioration in the terms of borrowing cannot be self-sustaining in this class of models since, once governments have exhausted their debt capacity, those with less debt are always better off.

3.2.2 Bulow and Rogoff

Our proof is also related to that used by Bulow and Rogoff (1989) to rule out reputational equilibria in sovereign debt models where saving is allowed after default. As originally written, the Bulow-Rogoff result only applies directly to environments with complete markets, but a similar result also holds in the incomplete markets framework we study: if a government can save at a strictly positive net risk-free rate after defaulting, and there are no other exogenous penalties for default, then no debt can be sustained. Though this result has not—to our knowledge—been written formally until now, it has informally motivated the ingredients of modern variations on the Eaton-Gersovitz model, which all specify some exclusion from international markets after default, together with additional costs of default such as output losses.

Define $V^{nb}(b,s)$ to be the value function for a government that can save at the risk-free rate but not borrow.

$$\begin{aligned} V^{nb}(b,s) &= \max_{b'} u(c,s) + \beta \mathbb{E}_{s'|s}\left[V^{nb}(b',s')\right] \\ &\quad \text{s.t.} \quad c+b = y(s) + \frac{b'}{R},\ b' \leq 0 \end{aligned} \tag{3.2.5}$$

Now, specify $V^d(s) \equiv V^{nb}(0,s)$, such that when the government defaults, its debt is reset at 0 and it can subsequently save but not borrow. We can now prove the incomplete markets analog of Bulow and Rogoff (1989), as a special case of proposition 53.

Proposition 57 (Incomplete markets Bulow-Rogoff). *In the model with $V^d(s) = V^{nb}(0,s)$ (i.e. savings after default), no debt can be sustained: in the unique Markov perfect equilibrium, the default thresholds $b^*(s)$ equal 0 for all s, and $Q(b',s) = 0$ for all $b' \geq 0$. Hence $V(b,s) = V^{nb}(b,s)$.*

Proof. We first verify that when $V^d(s) = V^{nb}(0,s)$, there exists an equilibrium where the government will default for any positive amount of debt $b > 0$. This equilibrium is (V^{nb}, Q^{nb}), where V^{nb} is given in (3.2.5) and the government faces

$$Q^{nb}(b',s) = \begin{cases} \frac{b'}{R} & b' \leq 0 \\ 0 & b' \geq 0 \end{cases} \tag{3.2.6}$$

which is the revenue schedule induced by default thresholds identically equal to zero.

First, Q^{nb} generates V^{nb}. The budget constraint in (3.2.5) is effectively the same as the constraint in (3.1.1) given prices (3.2.6); although (3.2.5) does not allow $b' > 0$ while (3.1.1) does, positive borrowing $b' > 0$ will never be optimal given prices (3.2.6) because it raises no revenue. Moreover, proposition 51 shows that the value function generated by the prices in (3.2.6) is decreasing in b; hence whenever $b \leq 0$, we have $V^{nb}(b,s) \geq V^{nb}(0,s) = V^d(s)$ for all s, so that default is never optimal.

Second, the default thresholds corresponding to V^{nb} are identically equal to zero, thereby generating Q^{nb}. This also follows from the monotonicity of V^{nb} in b (proposition 51): since

$$V^{nb}(b,s) \geq V^d(s) = V^{nb}(0,s) \Longleftrightarrow b \leq 0 \quad \forall s$$

we have $b^*(s) = 0$ for all s.

Proposition 53 then implies that (V^{nb}, Q^{nb}) must be the unique Markov perfect equilibrium, and hence that there is no distinct equilibrium in which debt can be sustained. In particular, there is no equilibrium where the expectation of being able to borrow in the future is enough to discourage default and sustain some positive debt. This is the incomplete markets version of the Bulow and Rogoff (1989) result. □

Going back to the proof of proposition 53, the intuition behind this result is that once a government has already borrowed the maximum amount that can obtain a nonzero price, access to debt markets offers no benefits beyond access to a market for savings. It is impossible to borrow more until some debt is repaid—and rather than repay and reborrow, it is cheaper to default and then run savings up and down in a parallel way, achieving higher consumption by avoiding the costs of debt service. No amount of debt is sustainable: whenever a government has borrowed the maximum, it will default with certainty, and in anticipation creditors will never allow any debt.

This resembles the logic behind the original Bulow and Rogoff (1989) result, which observed that for a reputational debt contract in complete markets, there must always be some state of nature in which a government can default and use the amount demanded for repayment as collateral for a sequence of state-contingent "cash in advance" contracts that deliver strictly higher consumption in every future date and state. The main idea behind their proof carries over to our incomplete markets environment, once the cash in advance contracts are replaced with a simple, parallel savings strategy. Our contribution here is to show that this result is a special case of a much broader equilibrium uniqueness result.

3.3 Extensions of the benchmark model

The benchmark Eaton-Gersovitz model covered in section 3.1 is often modified to achieve greater realism or more tractable computation. Recently, Passadore and Xandri (2014) have studied multiplicity that arises when the domain of debt values is bounded from below, restricting savings. Although this assumption has not been microfounded in the literature, it can be justified by an appeal to the institutional difficulties some governments face in maintaining net savings; it can also make computation more tractable, as argued by Chatterjee and Eyigungor (2012), who exclude savings on their grid of debt values.[5] Meanwhile, both Aguiar and Gopinath (2006) and Arellano (2008) allow some probability of reentry each period for countries excluded from asset markets after default.

For bounded savings, we show in subsection 3.3.1 that a modified replication argument—using the concavity of u in addition to its monotonicity—continues to deliver uniqueness, except in a limit case where Passadore and Xandri (2014) have demonstrated multiplicity. This robustness is perhaps surprising, since the mimicking strategy behind lemma 52 and proposition 53 can rely on the government's ability to smooth consumption via savings rather than debt. For reentry, we show in subsection 3.3.2 that although our argument can no longer establish uniqueness, it still can rule out the most commonly hypothesized form of multiplicity, in which one equilibrium delivers uniformly better bond prices than another. It also implies uniqueness in the special case where states are independently and identically distributed. Finally, in subsection 3.3.3 we discuss other, more drastic modifications to the environment, in which our results no longer apply.

In sections 3.3.1 and 3.3.2, it will be necessary to impose more structure on the payout $V^d(s)$ from default. In particular, we assume that there is an exogenous flow utility $v^d(s)$ from each period spent in market exclusion following default, and that it is weakly less desirable than autarky: $v^d(s) \leq u(y(s), s)$.[6] This is common in the literature, which often specifies it in terms of an output cost $\tau(s)$ such that $v^d(s) = u(y(s)-\tau(s), s)$. When there is no reentry, V^d becomes the expected utility from receiving v^d in all subsequent periods:

$$V^d(s_0) = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t v^d(s_t) \middle| s_0\right] \tag{3.3.1}$$

3.3.1 Bound on savings

Here we consider the case of an exogenous bound on savings, which in our notation becomes a lower bound $\underline{b} \leq 0$ on debt. Section 3.1 already allowed for any upper bound $\bar{b}$ on debt.

The proof of lemma 52 no longer directly applies when savings are bounded, because the mimicking strategy (3.1.9) may no longer be feasible. Indeed, lemma 52 is no longer always true: a government with b debt, facing debt prices given by default thresholds $\{b^*(s)\}$, may be better off than a government with $b - M$ debt, facing debt prices given by default thresholds $\{b^*(s) - M\}$. When allowable savings are limited, the improved debt prices implied by higher default thresholds give the former government more ability to smooth consumption, and this benefit may outweigh the costs of a higher debt load—possibly justifying the higher default thresholds themselves. In this light, multiplicity seems plausible.

Nevertheless, it remains possible to demonstrate equilibrium uniqueness in nearly all cases. Whereas the approach in section 3.1 used only the strict monotonicity of u, here we need an additional assumption on u:

[5] In their numerical simulations, Chatterjee and Eyigungor (2012) find that this restriction does not bind.

[6] Imposing this structure involves some loss of generality, since we can no longer make the value of defaulting depend on state s without also affecting the value of being excluded from markets in state s after originally defaulting in state $s' \neq s$.

Assumption 3. For each $s \in \mathcal{S}$, $u(c, s)$ is concave in c.

With concave utility and bounded savings, higher debt prices are desirable because they make it cheaper for the government to smooth consumption using debt—but concavity also implies that the marginal gains from consumption smoothing decrease as the government does more of it. Initially, if the government faces very low debt prices, an improvement in these prices may dramatically increase the benefits of market access, so that the default probability falls by more than enough to sustain the price improvement. But eventually, as diminishing returns to consumption smoothing set in, price improvements are no longer self-sustaining. At some point, we reach an equilibrium—where for given bond prices, the benefits of market access produce a pattern of default exactly consistent with those prices. Lemma 58 and proposition 59, which are partial analogues of lemma 52 and proposition 53, demonstrate that this equilibrium is unique.

Formal argument. In many respects the formal argument echoes the replication argument from section 3.1, but there are also substantial differences. Rather than *mimicking at a distance*, which is no longer feasible, we use *compressed mimicking.* Given distinct revenue schedules Q and $\widetilde{Q}$ derived via (3.1.5) from default thresholds $\{b^*(s)\}$ and $\{\widetilde{b}^*(s)\}$, lemma 58 defines λ to be the minimum ratio between $\widetilde{b}^*(s) - \underline{b}$ and $b^*(s) - \underline{b}$. For any s and $\widetilde{b} - \underline{b} = \lambda(b - \underline{b})$, a government starting at $(\widetilde{b}, s)$ can *compress* by λ the optimal strategy for a government (which we call the *target*) starting at (b, s), choosing $\widetilde{b}(s^t) - \underline{b} = \lambda(b(s^t) - \underline{b})$ whenever the target government repays and defaulting whenever the target government defaults.

As in section 3.1, the mimicking government by construction obtains weakly better prices than the target government for its debt. Unlike in section 3.1, the mimicking government need not achieve higher consumption than the target government. Instead, because it is compressing the target's debt issuance plan by λ, in each period it obtains consumption $\widetilde{c}$ that is weakly higher than the convex combination $\lambda c + (1 - \lambda)y(s)$ of the target's consumption c and state-s autarky income $y(s)$. This inequality is strict when $\underline{b} < 0$, where the mimicking government can consume extra due to forgone financing costs. Concavity of u then implies that $u(\widetilde{c}, s)$ is strictly greater than $\lambda u(c, s) + (1 - \lambda)u(y(s), s)$. Summing the expected value across all periods, we obtain (3.3.3), the analog of (3.1.7); when $\underline{b} = 0$, the strict inequality can also follow from $v^d(s) < u(y(s), s)$.

Effectively, lemma 58 bounds the extent to which higher expected default thresholds, which induce higher debt prices and allow the government to smooth more easily, increase the value function relative to the value from default. If, relative to the minimum feasible debt $\underline{b}$, we scale up the government's level of debt by λ^{-1}, and we also scale up default thresholds by at most λ^{-1}, then (3.3.4) states that we cannot increase the value over default by more than λ^{-1}.

Lemma 58. *Let Q and $\widetilde{Q}$ be two distinct revenue schedules, with Q reflecting expected default thresholds $\{b^*(s)\}_{s \in \mathcal{S}}$ and $\widetilde{Q}$ reflecting expected default thresholds $\{\widetilde{b}^*(s)\}_{s \in \mathcal{S}}$. Let V and $\widetilde{V}$ be the respective value* *functions for governments facing these revenue schedules. Define*

$$\lambda \equiv \min_s \frac{\widetilde{b}^*(s) - \underline{b}}{b^*(s) - \underline{b}} \tag{3.3.2}$$

and assume, without loss of generality, that $0 \leq \lambda < 1$. *Assume also that either* $\underline{b} < 0$, *or* $v^d(s) < u(y(s), s)$ *for all* s. *Then for any* s *and* b *such that* $V(b,s) \geq V^d(s)$, *we have*

$$\widetilde{V}(\widetilde{b}, s) > (1-\lambda)V^d(s) + \lambda V(b,s) \tag{3.3.3}$$

where $\widetilde{b} - \underline{b} \equiv \lambda(b - \underline{b})$. *This can equivalently be written as*

$$\widetilde{V}(\widetilde{b}, s) - V^d(s) > \lambda\left(V(b,s) - V^d(s)\right) \tag{3.3.4}$$

Proof. In appendix 3.5.3. □

In contrast to (3.1.7), inequality (3.3.3) in lemma 58 does not show that $\widetilde{V}(\widetilde{b}, s)$ is higher than $V(b,s)$. Fortunately, this is not needed to establish uniqueness in proposition 59. Instead, (3.3.4) suffices to obtain a contradiction. Inequality (3.3.4) shows that if a government facing Q weakly prefers not to default at (b,s) (so that $V(b,s) - V^d(s) \geq 0$), then a government facing $\widetilde{Q}$ must *strictly* prefer not to default at $(\widetilde{b}, s)$ (so that $\widetilde{V}(\widetilde{b}, s) - V^d(s) > 0$). It is therefore impossible for both b and $\widetilde{b}$ to be default thresholds for their respective value functions.

Proposition 59. *If either* $\underline{b} < 0$ *or* $v^d(s) < u(y(s), s)$ *for all* s, *Markov perfect equilibrium has a unique value function* $V(b,s)$ *and debt price schedule* $Q(b,s)$.

Proof. If, to the contrary, we have distinct equilibria (V, Q) and $(\widetilde{V}, \widetilde{Q})$ with default thresholds $\{b^*(s)\}$ and $\{\widetilde{b}^*(s)\}$, define λ as in (3.3.2) and assume without loss of generality that $0 \leq \lambda < 1$.

Let $\overline{s}$ be the state where the minimum in (3.3.2) is obtained. Evaluating (3.3.4) at $\widetilde{b} = \widetilde{b}^*(\overline{s})$, $b = b^*(\overline{s})$, and $s = \overline{s}$, we obtain

$$0 = \widetilde{V}(\widetilde{b}^*(\overline{s}), \overline{s}) - V^d(\overline{s}) > \lambda\left(V(b^*(\overline{s}), \overline{s}) - V^d(\overline{s})\right) = 0$$

which is a contradiction. □

Proposition 59 clarifies the role of unlimited savings in the uniqueness result of proposition 53. Even if a government cannot save at all, equilibrium will be unique as long as there is some exogenous penalty for default, such that $v^d(s)$ is less than the flow utility $u(y(s), s)$ from consuming the endowment. This assumption is made in most of the quantitative literature (for example Arellano, 2008). Alternatively, if $v^d(s) = u(y(s), s)$, then even an infinitesimal capacity to save $\underline{b} = -\epsilon < 0$ is sufficient for uniqueness.

Combined with results elsewhere, proposition 59 offers a very sharp characterization of the conditions governing uniqueness. In the limit case excluded by the proposition,[7] with both no savings ($\underline{b} = 0$) and no default penalty aside from autarky ($v^d(s) = u(y(s))$), Passadore and Xandri (2014) demonstrate that multiplicity is possible. In this environment, autarky is always an equilibrium, since there is no incentive to repay debt when a zero price for future issuance is expected and there is no additional penalty for default. But, under certain conditions, incentives to repay emerge endogenously in an alternative equilibrium where positive debt prices are expected and the government repays to preserve its debt market access. Interestingly, this multiplicity embodies the intuition that propositions 53 and 59 reject whenever any level of government savings is allowed: in Passadore and Xandri (2014), expectations of high bond prices *can* be self-sustaining, encouraging the low default rates needed to justify high prices.

3.3.2 Stochastic market reentry

In the literature, a very common departure from the benchmark model of Section 3.1 is an assumption that market reaccess is possible after default. This makes the value of default depend on the equilibrium value of borrowing, implying that lemma 52 and proposition 53 do not directly apply. Nevertheless, we will be able to rule out the most commonly hypothesized form of multiplicity—the existence of distinct "favorable" and "adverse" equilibria, in which the favorable equilibrium offers uniformly better debt prices Q.

To be concrete, suppose that it is possible to re-access markets with zero debt after a stochastic period of exclusion, which has independent probability $1-\lambda$ of ending in each period. That is, replace (3.3.1) by[8]

$$V^d(s) = v^d(s) + \beta\lambda \mathbb{E}_{s'}\left[V^d(s')\right] + \beta(1-\lambda)\mathbb{E}_{s'}\left[V^o(0,s')\right] \tag{3.3.5}$$

In this framework, we can now prove the following specialized analog of proposition 53.

Proposition 60. *In the model with stochastic reentry, there do not exist two distinct equilibria (V, Q) and $(\widetilde{V}, \widetilde{Q})$ such that $Q(b,s) \geq \widetilde{Q}(b,s)$ for all b and s.*

Proof. In appendix 3.5.4. □

In general, the endogeneity of $V^d(s)$ in (3.3.5) makes it difficult to analytically characterize equilibria. In the particular case examined by proposition 60, however, the proof strategy from proposition 53 still applies with some modification. The core insight is that if $Q \geq \widetilde{Q}$, then $V^d \geq \widetilde{V}^d$, because a government facing a uniformly better price schedule after reentry is better off. Furthermore, if Q and $\widetilde{Q}$ are distinct and $Q \geq \widetilde{Q}$, there must be some s for which $b^*(s) > \widetilde{b}^*(s)$. We then can apply the argument from lemma 52 and proposition 53, having a government in the $(\widetilde{V}, \widetilde{Q})$ equilibrium mimic the strategy of a government

[7] Recall that we do not consider cases where $\underline{b} > 0$. Indeed, our proposition does not rule out multiplicity in such cases, where active governments must maintain some positive minimum debt level, and default frees the government from the cost of servicing this debt. Since $\underline{b} > 0$ is rare in the literature and a positive minimum debt level can be difficult to interpret, we emphasize $\underline{b} \leq 0$ instead.

[8] This formulation is the one used by Arellano (2008) and Aguiar and Gopinath (2006). It does not encompass the possibility of recovery on defaulted debt or debt renegotiation (see for example Yue, 2010).

in the (V, Q) equilibrium. The fact that $\widetilde{V}^d \leq V^d$ only helps our argument, since it is further reason why government in the $(\widetilde{V}, \widetilde{Q})$ equilibrium will prefer the mimicking strategy to default.

In short, when there is reentry, uniformly higher bond prices defeat themselves: they make default and eventual reentry more attractive, raising the probability of default and pushing bond prices back down.

Although we cannot prove uniqueness more generally, this result does rule out the popular hypothesis—as discussed in the introduction—that sovereign debt markets can vary between self-sustaining "favorable" and "adverse" equilibria. Instead, if multiplicity exists, we know that it must be a surprising kind of multiplicity: among any two equilibria, each must offer cheaper borrowing in some places and more expensive borrowing in others.

Special case with iid exogenous state. It is possible to demonstrate full uniqueness in one special case. Suppose now that s follows an iid process with probability $\pi(s)$. It follows that the expected value from reentry $\mathbb{E}_{s'}[V^o(0, s')]$ in (3.3.5) is independent of the states preceding s', and we can denote this expectation by V^{re}. The iid assumption also implies that the debt price schedule Q depends only on the debt amount b', not the current state s, as (3.1.5) reduces to

$$Q(b') = \frac{b'}{R} \sum_{\{s' : b' \leq b^*(s')\}} \pi(s') \tag{3.3.6}$$

Proposition 61. *In the model with iid states and stochastic market reentry, Markov perfect equilibrium has a unique value function $V(b, s)$ and debt price schedule $Q(b)$.*

Proof. In appendix 3.5.5. □

Proposition 61 follows for reasons similar to proposition 60. For any distinct equilibria (V, Q) and $(\widetilde{V}, \widetilde{Q})$, the only difference between the default value functions V^d and $\widetilde{V}^d$ arises from the expected reentry value, which is now just a scalar V^{re}. Whichever equilibrium has the higher reentry value must have a more favorable bond price schedule, meaning that at least one of its default thresholds is higher. As with proposition 60, we can then invoke a mimicking argument to show that the equilibrium with a higher default value cannot also have a higher default threshold for some s.

This result further emphasizes how subtle any multiplicity in the model with reentry, if it exists, must be: it must rely, in some way, on the transition probabilities of the Markov process being non-iid.

3.3.3 Other variations on the model and multiplicity results

We have showed that the benchmark Eaton-Gersovitz model of sovereign debt with default does not admit multiple equilibria, and that this uniqueness result partly extends to the more complex environments of subsections 3.3.1 and 3.3.2. Nevertheless, multiplicity arises in several other sovereign debt models in the literature. This section reviews the ways in which these models sidestep the uniqueness result present in the benchmark framework.

Markov perfect equilibrium in the model we studied includes a price function $Q(b', s)$, which depends only on the current state s and the bond payment b' promised tomorrow. After observing s, in each period the government can choose either to default or to repay and sell some quantity b' of bonds for next period. Once the government chooses to repay and selects some b', there is no uncertainty about the amount $Q(b', s)$ that will be raised; no further choices are made until the next period, when the next state s' is realized and the process repeats itself. As presented in appendix 3.5.2, this process can be explicitly written as a game between governments and risk-neutral investors. It is possible to define subgame perfect equilibria in this game, and proposition 54 shows that uniqueness still holds for these equilibria in the benchmark model.

Our uniqueness result can disappear if the timing and action space of the game are modified. For instance, in the model of Cole and Kehoe (2000), the government has the option to default after observing the outcome of the current period's bond auction. If it defaults, it can keep the proceeds of the auction but avoid repayment on its maturing debt. Given enough risk aversion, this option is preferable when the current period's auction yields little revenue, and the cost of repaying maturing debt out of current-period resources is prohibitively high. A coordination problem among creditors thus emerges, leading to multiple equilibria: they might either offer high prices, in which case the government will repay, or offer low prices, in which case the government will default and thereby justify the low prices. The literature sometimes refers to this phenomenon as "rollover multiplicity". It is absent in the model we study, which excludes the option to default after revenue from the auction comes in; but it captures an important intuition, which is that rolling over large amounts of short-term debt can be a source of fragility.

In the model of Calvo (1988), multiplicity arises because of the way the bond revenue-raising process works. In the Calvo model, a government borrows an exogenous amount b at date 0 and inherits a liability of $R_b b$ at date 1. It then uses a mix of distortionary taxation and debt repudiation to finance a given level of government spending. Since a higher interest rate R_b tilts the balance towards more repudiation at date 1, and since investors need to break even when lending to the government, there exist two rational expectations equilibria: one with high R_b and high repudiation, and one with low R_b and low repudiation. This is sometimes called "Laffer curve multiplicity" in reference to the shape of the bond revenue curve that arises in this model (the function that gives bond revenue b as a function of promised repayment $R_b b$ has an inverted-V shape). In the model we study, the government directly announces the amount it will owe tomorrow, allowing it to avoid the downward-sloping part of the bond revenue curve.[9] Lorenzoni and Werning (2014) make a forceful argument that such an assumption requires a form of commitment that governments are unlikely to have: in practice, if they raise less auction revenue than expected, they may auction additional debt rather than making the burdensome fiscal adjustments that are otherwise necessary.

In effect, both the rollover multiplicity of Cole and Kehoe (2000) and the Laffer curve multiplicity of Calvo (1988) emerge from a more elaborate game between governments and investors. They create self-fulfilling alternate equilibria by allowing governments to act in ways ruled out by the game-theoretic formulation of

[9] Interestingly, the setup of the original Eaton and Gersovitz (1981) model does not let the government choose on the bond revenue curve *a priori*, although their analysis focuses on equilibria in which it effectively does.

the benchmark Eaton-Gersovitz model: when auction revenue is insufficient, governments can either take the revenue and then default (as in Cole and Kehoe, 2000) or dilute investors by issuing more debt in the same period (as in Lorenzoni and Werning's interpretation of Calvo, 1988). Since the Eaton-Gersovitz model alone cannot produce multiplicity, these modifications to the game may prove important to interpreting any multiplicity we see in practice. More generally, they suggest that a detailed look at institutions, and the practical options available to sovereign debtors when they raise funds in debt markets, is necessary to understand when the Eaton-Gersovitz model succeeds and when it fails as a benchmark.

Finally, another important strand of the literature considers long-term debt, as in Hatchondo and Martinez (2009). Here, uniqueness of equilibrium remains uncertain: multiplicity has not been explicitly demonstrated, but our mimicking-based proof of uniqueness breaks down when bond prices are influenced by the likelihood of endogenous default in the arbitrarily distant future. In a related continuous time environment, Lorenzoni and Werning (2014) find multiple equilibria: in their model, an adverse shift in the bond price schedule forces the government onto a path of increasing debt, justifying the initial shift. Although their analysis does not adapt directly to the Hatchondo and Martinez (2009) model, it does suggest that multiple equilibria may be present.

3.4 Conclusion

While there is a prevalent view that multiplicity is a general feature of infinite-horizon models with sovereign debt, we proved that the Eaton-Gersovitz model and several of its variants have a unique equilibrium. Our results settle an important outstanding question in the literature, and invite a renewed focus on the assumptions that sovereign debt models need to generate multiplicity. Our replication-based proof may also be applicable more generally to establish uniqueness in infinite-horizon games.

3.5 Appendix: proofs

3.5.1 Existence of Markov perfect equilibrium

We prove existence of Markov perfect equilibrium constructively, following a fixed point procedure similar to the one typically used by the sovereign debt literature to find an equilibrium. Section 3.5.1.1 defines a functional $\mathcal{B}(V)$ mapping value functions to default thresholds, and proves properties of this mapping. Section 3.5.1.2 defines a functional $\mathcal{V}(B)$ mapping default thresholds to value functions, and proves properties of that mapping. Finally, section 3.5.1.3 shows that iterating on the operator $T = \mathcal{B} \circ \mathcal{V}$, starting from thresholds identically equal to zero, produces a limit set of default thresholds that constitute an equilibrium. Throughout we will need the following additional, technical assumptions.

Assumption 4. $u(c, s)$ is continuous in c for every s

Assumption 5. $u(0, s) = -\infty$

Assumption 6. There exist $\gamma > 0$ and $\kappa > 0$ such that $u(c, s) \leq \gamma c^{\kappa}$ for all c, s; and $\beta R^{\kappa} < 1$

Assumption 7. There exists an upper bound $\bar{b}$ such that $b \leq \bar{b}$

Assumption 8. $-\infty < V^{d}(s) \leq V^{nb}(0, s)$, where V^{nb} is defined in (3.2.5)

Assumptions 4-7 guarantee that the value function is well-defined and finite, and that default thresholds are uniquely defined. Assumption 8 is needed to ensure that governments with positive assets are not tempted to default.

3.5.1.1 Default thresholds for given V: $\mathcal{B}(V)$

Consider a set of S strictly decreasing, continuous functions $V(b, s)$. For each state s, define the threshold $b^{*}(s)$ as $-\infty$ if $\sup_{b} V(b, s) < V^{d}(s)$, or$+\infty$ if $\inf_{b} V(b, s) > V^{d}(s)$. In other cases, let $b^{*}(s)$ be equal to the unique solution to

$$V(b^{*}(s), s) = V^{d}(s)$$

This defines a functional $\mathcal{B}(V)$. The following shows that this is a monotone mapping, and provides conditions on V under which $\mathcal{B}(V)$ is positive and bounded.

Lemma 62. *The following propositions hold for every s.*

a) If $V(0, s) \geq V^{nb}(0, s)$, then $b^{*}(s) \geq 0$

b) If $V(b_{s}, s) = -\infty$, then $b^{*}(s) < b_{s}$

c) If $V^{A}(b, s) \geq V^{B}(b, s)$ for all b, then the respective default thresholds satisfy $b^{*A}(s) \geq b^{*B}(s)$

Proof. The proof follows because V is continuous and strictly decreasing. Assumption 8 guarantees that $V(0, s) \geq V^{nb}(0, s) \geq V^{d}(s) = V(b^{*}(s), s)$, so a) holds. Assumption 8 also guarantees that $V^{d}(s)$ is finite, so $V(b_{s}, s) < V(b^{*}(s), s)$, and b) holds. Finally, $V^{B}\left(b^{*B}(s), s\right) = V^{d}(s) = V^{A}\left(b^{*A}(s), s\right) \geq V^{B}\left(b^{*A}(s), s\right)$, so c) holds. □

3.5.1.2 Value functions V given default thresholds: $\mathcal{V}(B)$

Consider now a set of positive default thresholds $B = \{b^*(s)\}$, $b^*(s) \geq 0$. Define V as the solution to

$$V(b, s; \{b^*(s')\}) \equiv \max_{b(s^t), p(s^t) \in \{0,1\}} \left\{ \sum_{s^t} \beta^t \Pi(s^t) u(c(s^t), s_t) \mathbf{1}_{\{p(s^t)=1\}} + \sum_{s^t} \beta^t \Pi(s^t) V^d(s_t) \mathbf{1}_{\{p(s^t)=0, p(s^{t-1})=1\}} \right\}$$

$$\text{s.t. } c(s^t) = y(s_t) + \frac{b(s^t)}{R}\left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \cdot \mathbf{1}_{\{b(s^t) \leq b^*(s_{t+1})\}}\right) - b(s^{t-1}) \tag{3.5.1}$$

$$p(s^t) \leq p(s^{t-1})$$

$$p(s_0) = 1$$

$$b(s^{-1}) = b$$

$$s_0 = s$$

We now show that this generates a well-defined mapping $\mathcal{V}(B)$, and prove properties of this mapping, including monotonicity and continuity.

Lemma 63. *The following propositions hold for every s.*

a) The maximum in (3.5.1) is attained for any b, and $V(b, s) < \infty$

b) $V(b, s)$ is strictly decreasing in b

c) $V(b, s)$ is continuous in b for every s

d) $V(b, s; \{b^*(s')\})$ is increasing in $\{b^*(s')\}$

e) $V(0, s) \geq V^{nb}(0, s)$

f) $V\left(\frac{\bar{b}}{R} + y(s), s\right) = -\infty$

g) $V(b, s; \{b^*(s')\})$ is continuous in $\{b^*(s')\}$

Proof. We prove each of the propositions in turn

a) We restrict ourselves to cases where such that $V(b, s) > -\infty$, otherwise the proposition is trivial. We prove that the maximum is attained by showing that the problem in (3.5.1) is the maximization of an upper semicontinuous function on a compact set, and exhibit an upper bound to show $V(b, s) < \infty$. First, assumption 5 guarantees that $c(s^t) > 0$, which (given that assets receive the risk-free rate) bounds the rate of growth of assets: there exists $D > 0$ such that $b(s^t) \geq -DR^{t+1}$. Together with assumption 7, this guarantees that $b(s^t)$ must be chosen on a compact interval $[-DR^{t+1}, \bar{b}]$, and hence that the set of all arguments $\{b(s^t), p(s^t)\}$ is compact. Second, these bounds on $b(s^t)$ place an upper bound on $c(s^t)$ which, together with assumption 6, yields a bound on flow utility, $\beta^t u(c(s^t), s_t) \leq (\beta R^\kappa)^t \overline{u}$ where $\overline{u} < \infty$. Third, the presence of the default option implies that flow utility along the no-default path is bounded below in all periods, $\beta^t u(c(s^t), s_t) \geq \beta^t \underline{u}$ for $\overline{u} > -\infty$. Summing up, we have bounds on flow utility:

$$\beta^t \underline{u} \leq \beta^t u(c(s^t), s_t) \leq (\beta R^\kappa)^t \overline{u} \tag{3.5.2}$$

Next, all partial sums in the maximand (3.5.1) are upper semicontinuous in the argument. This follows from the fact that they consist entirely of continuous functions except $\mathbf{1}_{\{b(s^t) \leq b^*(s_{t+1})\}}$, which is upper semicontinuous. Inequality (3.5.2) together with $\beta < 1$ and $\beta R^\kappa < 1$ allows one to apply the Weierstrass M-test to conclude that the sum converges uniformly, and hence that the limit is also upper semicontinous in the argument. Hence the maximum in (3.5.1) is attained. Finally, (3.5.2) together with the fact that default values are finite guarantee that the objective in (3.5.1) is uniformly bounded from above, and hence the maximum $V(b, s) < \infty$ as well.

b) Fix s and consider $\widetilde{b} > b$. Consider the optimal plan $\left\{\widetilde{b}(s^t), \widetilde{p}(s^t)\right\}$ starting at $\left(\widetilde{b}, s\right)$. Then the plan $\left\{\widetilde{b}(s^t), \widetilde{p}(s^t)\right\}$ is also feasible starting at (b, s), so that, letting $Q = \frac{\widetilde{b}(s^0)}{R} \sum_{\{s': \widetilde{b}(s^0) \leq b^*(s')\}} \pi(s'|s)$, we have

$$\begin{aligned} V(b,s) - V\left(\widetilde{b}, s\right) &\geq u\left(y(s) + Q - b, s\right) - u\left(y(s) + Q - \widetilde{b}, s\right) \\ &> u\left(y(s) + Q - \widetilde{b}, s\right) - u\left(y(s) + Q - \widetilde{b}, s\right) = 0 \end{aligned}$$

c) Fix (b, s) and let $\epsilon > 0$. We show that (i) there exists $\delta_1 > 0$ such that for any $b < \widetilde{b} < b + \delta_1$, $V(\widetilde{b}, s) > V(b, s) - \epsilon$, and (ii) there exists $\delta_2 > 0$ such that for any $b > \widetilde{b} > b - \delta_2$, $V(\widetilde{b}, s) < V(b, s) + \epsilon$. Together with V being strictly decreasing, (i) and (ii) establish continuity.

For (i), consider the optimal plan $\{b(s^t), p(s^t)\}$ starting at (b, s). This plan is also feasible starting at $(\widetilde{b}, s)$ and delivers the same consumption at every point except $t = 0$, where consumption is $\widetilde{b} - b$ lower. Hence letting $c(s^0)$ be the $t = 0$ consumption level for the optimal plan starting at (b, s), we know

$$V(b, s) - V(\widetilde{b}, s) = u(c(s^0)) - u(c(s^0) - \delta_1)$$

will be $< \epsilon$ as desired if $\delta_1 > 0$ is defined via continuity of u such that $|u(c) - u(c(s^0))| < \epsilon$ for all $|c - c(s^0)| < \delta_1$.

For (ii), we must appeal to a uniform continuity argument to choose δ_2. We first find a compact set $[\underline{c}, \overline{c}]$ such that any optimal plan with $\widetilde{b} < b$ (and hence $V(\widetilde{b}, s) > V(b, s)$) has first period consumption $\widetilde{c}(s^0) \in [\underline{c}, \overline{c}]$. To do this, recall from section 3.5.1.2 that the sum of all terms in (3.5.1) for $t \geq 1$ is bounded from above by an upper bound $\overline{V} < \infty$. Hence the initial consumption level $\widetilde{c}(s^0)$ associated with an optimum $V(\widetilde{b}, s) > V(b, s)$ must be such that

$$u(\widetilde{c}(s^0), s_0) + \overline{V} \geq V(b, s) \tag{3.5.3}$$

From assumption 5, $u(\widetilde{c}(s^0), s_0) \to -\infty$ as $\widetilde{c}(s^0) \to 0$, and hence for (3.5.3) to be satisfied we must have $\widetilde{c}(s^0) \geq \underline{c} > 0$ for some lower bound $\underline{c}$. We also know that $\widetilde{c}(s^0) \leq \overline{b}/R + \overline{y} - b(s^0) \equiv \overline{c}$, giving us an upper bound. Since u is continuous and $[\underline{c}, \overline{c}]$ is a compact interval, we can pick a single $\delta_2 > 0$ such that $|u(c_A, s_0) - u(c_B, s_0)| < \epsilon$ for all $c_A \in [\underline{c}, \overline{c}]$ and $|c_B - c_A| < \delta_2$.

Now consider the optimal plan $\{\widetilde{b}(s^t), \widetilde{p}(s^t)\}$ starting at $(\widetilde{b}, s)$. This plan is also feasible starting at (b, s) and delivers the same consumption at every point except $t = 0$, where consumption is $b - \widetilde{b}$ lower. Hence we have

$$V(\widetilde{b}, s) - V(b, s) = u(\widetilde{c}(s^0), s_0) - u(\widetilde{c}(s^0) - \delta_2, s_0) < \epsilon$$

as desired.

d) Since $b^*(s') \geq 0$, increasing $b^*(s')$ always weakly increases $\frac{b(s^t)}{R}\left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \cdot \mathbf{1}_{\{b(s^t) \leq b^*(s_{t+1})\}}\right)$ when $b(s^t) \geq 0$ and leaves it unchanged when $b(s^t) \leq 0$, which completes the proof.

e) Follows from d), since $V^{nb}(0, s)$ is the value with default thresholds all equal to zero, as shown in section 3.2.2.

f) Assumption 7 ensures that for any $b' > 0$, $\frac{b'}{R} \sum_{\{s': b' \leq b^*(s')\}} \pi(s'|s) < \frac{\overline{b}}{R}$. Hence feasible consumption at date 0 is $c(s^0) < y(s) + \frac{\overline{b}}{R} - \left(\frac{\overline{b}}{R} + y(s)\right) = 0$. Given that the continuation value for any b' is finite, f) follows from assumption 5.

g) Fix b and s^0. Let $\epsilon > 0$ and let $\{b^*(s')\}$ be a set of default thresholds. In an argument similar to the proof of c), we show that (i) there exists δ_1 such that, for any alternative set of default thresholds $\{\widetilde{b}^*(s')\}$ such that $|b^*(s') - \widetilde{b}^*(s')| < \delta_1$ for all s', we have $V(b, s, \{\widetilde{b}^*(s')\}) > V(b, s, \{b^*(s')\}) - \epsilon$, and (ii) there exists δ_2 such that, for any $\{\widetilde{b}^*(s')\}$ such that $|b^*(s') - \widetilde{b}^*(s')| < \delta_2$ for all s', we have $V(b, s, \{b^*(s')\}) > V(b, s, \{\widetilde{b}^*(s')\}) - \epsilon$. Combining (i) and (ii) then proves continuity. In both cases, we use the fact that a government facing debt thresholds that are lower by at most δ can guarantee itself a consumption plan that is only δ below that of a government with reference debt thresholds at date 0—and above at every other date—using a mimicking strategy, as embodied in the following claim.

Claim. Assume that $|b^*(s') - \widetilde{b}^*(s')| < \delta$. Let $\{b(s^t), p(s^t)\}$ be a plan that achieves consumption $c(s^t)$ subject to the default thresholds $\{b^*(s')\}$ starting from (b, s^0). Then there is another plan $\{\widetilde{b}(s^t), p(s^t)\}$ that achieves consumption $\widetilde{c}(s^t)$ subject to the default thresholds $\{\widetilde{b}^*(s')\}$ such that $\widetilde{c}(s^t) > c(s^t)$ for all $t \geq 1$ and $\widetilde{c}(s^0) > c(s^0) - \delta$.

Proof of claim. Define $\widetilde{b}(s^t) \equiv b(s^t) - \delta$ for all $t \geq 0$ and $\widetilde{b}(s^{-1}) \equiv b(s^{-1}) = b$. Then compute

$$\widetilde{c}(s^t) = y(s_t) + \frac{b(s^t) - \delta}{R}\left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \cdot \mathbf{1}_{\{b(s^t) - M \leq \widetilde{b}^*(s_{t+1})\}}\right) - (b(s^{t-1}) - \delta)$$

$$\geq y(s_t) + \frac{b(s^t)}{R}\left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \cdot \mathbf{1}_{\{b(s^t) \leq b^*(s_{t+1})\}}\right) - b(s^{t-1}) + \left(1 - \frac{1}{R}\right)\delta > c(s^t)$$

and

$$\widetilde{c}(s^0) = y(s_0) + \frac{b(s^0) - \delta}{R}\left(\sum_{s_1} \pi(s_1|s_0) \cdot \mathbf{1}_{\{b(s^0) - M \leq \widetilde{b}^*(s_1)\}}\right) - b(s^{-1})$$

$$\geq y(s_0) + b(s^0)\left(\sum_{s_1} \pi(s_1|s_0) \cdot \mathbf{1}_{\{b(s^0) \leq b^*(s_1)\}}\right) - b(s^{-1}) - \frac{\delta}{R} > c(s^0) - \delta$$

□

To prove (i), consider the plan $\{c(s^t), b(s^t), p(s^t)\}$ that achieves $V(b, s, \{b^*(s')\})$. Using the continuity of $u(c, s^0)$, let δ_1 be such that $|u(c', s_0) - u(c(s^0), s_0)| < \epsilon$ for all $|c' - c(s^0)| < \delta_1$. Then, whenever the thresholds $\left\{\widetilde{b}^*(s')\right\}$ are such that $|b^*(s') - \widetilde{b}^*(s')| < \delta_1$ for all s', it follows from the claim that there is a consumption plan $\{\widetilde{b}(s^t), p(s^t)\}$ for these thresholds that achieves consumption above $c(s^0) - \delta$ in the first period and above $c(s^t)$ everywhere else, and hence value greater than $V(b, s, \{b^*(s')\}) - \epsilon$.
To prove (ii), suppose for some $\{\widetilde{b}^*(s')\}$ that $V(b, s, \{\widetilde{b}^*(s')\}) \geq V(b, s, \{b^*(s')\})$ (otherwise, the desired inequality is immediate), and let $\{\widetilde{b}(s^t), p(s^t)\}$ be the plan attaining the optimum for $V(b, s, \{\widetilde{b}^*(s')\})$. We can establish using the argument from the proof of c) we can pick a single $\delta_2 > 0$ such that $|u(c, s_0) - u(\widetilde{c}(s_0), s_0)| < \epsilon$ for $|c - \widetilde{c}(s_0)| < \delta_2$. It follows from the claim that there is a plan $\{\widetilde{\widetilde{b}}(s^t), p(s^t)\}$ that (subject to the default thresholds $\{b^*(s')\}$) achieves consumption $\widetilde{\widetilde{c}}(s^t)$ that is strictly greater than $\widetilde{c}(s^t)$ for all $t \geq 1$ and strictly greater than $\widetilde{c}(s^0) - \delta_2$ for $t = 0$. From the choice of δ_2 we know that $|u(\widetilde{\widetilde{c}}(s^0), s_0) - u(\widetilde{c}(s^0), s_0)| < \epsilon$, and hence that the proposed plan $\{\widetilde{\widetilde{b}}(s^t), p(s^t)\}$ gives value strictly greater than $V(b, s, \{\widetilde{b}^*(s')\}) - \epsilon$. It follows that $V(b, s, \{b^*(s')\}) > V(b, s, \{\widetilde{b}^*(s')\}) - \epsilon$ as desired.

□

3.5.1.3 Existence of equilibrium

Using the operators defined in section 3.5.1.1-3.5.1.2, we can define the operator $T = \mathcal{B} \circ \mathcal{V}$.

Lemma 64. *The operator T is monotone increasing and maps the set $\prod_s \left[0, y(s) + \frac{\bar{b}}{R}\right]$ onto itself*

Proof. Monotonicity follows by combining lemmas 62c) and 63d). By combining lemmas 62a) and 63e), we obtain that $Tb^*(s) \geq 0$ whenever $b^*(s) \geq 0$. By combining lemmas 62b) and 63f), we obtain that $Tb^*(s) \leq y(s) + \frac{\bar{b}}{R}$ for each s. □

Let $b^{*0}(s) = 0$ for every s. For $n \geq 1$ define the sequence

$$b^{*n} = Tb^{*n-1}$$

By lemma 64, the sequences $b^{*n}(s)$ are increasing and bounded for every s. Hence they converge to form a set of thresholds $\{b^{*\infty}\}$. Define $V^n = \mathcal{V}(b^{*n})$ and $V^\infty = \mathcal{V}(b^{*\infty})$. From Lemma 63g) it follows that $V^\infty(b,s) = \lim_{n\to\infty} V^n(b,s)$. Next, because V^n is a sequence of continuous bijective functions with continuous inverses, whose limit V^∞ is continuous and bijective, and since by definition $\mathcal{B}(V^n)(s) = (V^n)^{-1}(V^d(s), s)$, we have that

$$\mathcal{B}\left(\lim_{n\to\infty} V^n\right) = \lim_{n\to\infty} \mathcal{B}(V^n)$$

and therefore

$$\mathcal{B}(V^\infty) = \lim_{n\to\infty} Tb^{*n} = b^{*\infty} \tag{3.5.4}$$

So $(V^\infty, b^{*\infty})$ constitutes an equilibrium, as we set out to prove. To map these objects to those in the main text, define$V = V^\infty$ and the bond price schedule Q as

$$Q(b', s) = \frac{b'}{R}\mathbb{P}_{s'|s}\left[b' \leq b^{*\infty}(s')\right] = \frac{b'}{R} \sum_{\{s': b' \leq b^{*\infty}(s')\}} \pi(s'|s)$$

then (V, Q) is a Markov perfect equilibrium, since (3.1.1)-(3.1.2) is the recursive formulation of the problem in (3.5.1) for the schedule Q generated by the thresholds $\mathcal{B}(V)$, and (3.5.4) guarantees that (3.1.3) holds.

3.5.2 Uniqueness of subgame perfect equilibrium

This appendix proves uniqueness of the subgame perfect equilibrium in the game of section 3.1. In order to define the game explicitly, we assume that there exist overlapping generations of two-period lived international investors. The set of investors born at time t is denoted by $\mathcal{I}_t$. We assume that $\mathcal{I}_t$ is finite, that $|\mathcal{I}_t| \geq 2$, and that all investors are risk-neutral with preferences given by

$$-q_t a^i_{t+1} + \frac{1}{R}\mathbb{E}_t\left[a^i_{t+1} p_{t+1}\right] \tag{3.5.5}$$

where $R > 1$. We next describe the sequence of actions.

Every period, with incoming history h^{t-1}, after Nature realizes the exogenous state s_t, the government chooses repayment p_t. If it chooses $p_t = 0$ (default), it obtains value $V^d(s_t)$, investors receive zero, and the game ends.

If it chooses $p_t = 1$, the government receives income $y(s_t) \geq 0$ and chooses next period debt b_{t+1}. Next, every investor i simultaneously bids a price $q^i_t \geq 0$ for the government's debt. Given bids q^i_t, an auctioneer allocates the bonds a_{t+1}^i according to the following rule:

$$a_{t+1}^i = \begin{cases} \frac{b_{t+1}}{J} & \text{if} \quad q_t^i = \max_{i'} q_t^{i'} \\ 0 & \text{otherwise} \end{cases}$$

where J is the number of investors bidding the maximum price. History for period t is now $h^t = \left(h^{t-1}, s_t, b_{t+1}, \{q_t^i\}\right)$.

The government receives $Q_t = q_t b_{t+1}$ where $q_t = \max_{i'} q_t^{i'}$ and repays debt b_t to previous investors. Its consumption is then

$$c_t = y(s_t) - b_t + q_t b_{t+1}$$

for which it receives flow utility $u(c_t, s_t)$, and expected value

$$V\left(h^{t-1}, s_t\right) = \begin{cases} u(c_t, s_t) + \beta \mathbb{E}_t\left[V\left(h^t, s_{t+1}\right)\right] & \text{if} \quad p_t = 1 \\ V^d(s_t) & \text{if} \quad p_t = 0 \end{cases} \tag{3.5.6}$$

Definition 2. A government strategy is $p\left(h^{t-1}, s_t\right)$, $b'\left(h^{t-1}, s_t\right)$ specifying the repayment and next period debt decision after each history h^{t-1} and state s_t. A strategy of investor i born at time t is a price bid $q^i\left(h^{t-1}, s_t, b_{t+1}\right)$.

Together, investor strategies imply a bond revenue function $Q\left(h^{t-1}, s_t, b_{t+1}\right)$.

Definition 3. A subgame perfect equilibrium consists of strategies for the government and investors such that at each $\left(h^{t-1}, s_t\right)$:

a) $p\left(h^{t-1}, s_t\right)$, $b'\left(h^{t-1}, s_t\right)$ maximize (3.5.6)

b) For all $i \in \mathcal{I}_t$, $q^i\left(h^{t-1}, s_t, b_{t+1}\right)$ maximizes (3.5.5)

In any subgame perfect equilibrium, investor maximization leads to

$$q\left(h^{t-1}, s_t, b_{t+1}\right) = \frac{1}{R}\mathbb{E}_t\left[p\left(h^t, s_{t+1}\right)\right] \tag{3.5.7}$$

We retain the other assumptions from the model in section 3.1 on u, V^d, and the no-Ponzi bound on debt $\bar{b}$. These include assumption 1 and assumptions 4 through 8. Importantly, assumption 8 continues to imply that a government with debt $b < 0$ never finds it optimal to default, so $q(h, s, b') = \frac{1}{R}$ for any $b' < 0$.

The following lemma is crucial to the proof of unique equilibrium. It shows that in equilibrium, regardless of the history of play, a government with a strictly lower level of debt can always achieve a weakly higher value than a government with more debt in the same state, and is also weakly more likely to repay. Like the proof of lemma 52, it uses a mimicking-based argument, although here the proof is written in a recursive setting and must deal with technical complications that arise from the more general notion of equilibrium.

Lemma 65. *Consider two subgame perfect equilibria A and B. For any (h_A, h_B, s), if $b(h_A) > b(h_B)$ then $V_A(h_A, s) \leq V_B(h_B, s)$, and $p_B(h_B, s) = 1$ if $p_A(h_A, s) = 1$.*

Proof. Define

$$M \equiv \sup_{h_A, h_B, s} \{b(h_A) - b(h_B) \quad s.t. \quad V_A(h_A, s) \geq V_B(h_B, s) \quad and \quad p_A(h_A, s) = 1\}$$

Assume $M > 0$.[10] Let $0 < \epsilon < \frac{R-1}{R+1}M$, and let (h_A, h_B, s) be such that $V_A(h_A, s) \geq V_B(h_B, s)$, $p_A(h_A, s) = 1$ and $b(h_A) > b(h_B) + M - \epsilon$. Define

$$\widetilde{b}'_B = b'_A(h_A, s) - M - \epsilon \tag{3.5.8}$$

and continuation histories

$$\begin{aligned} h'_A &= \left(h_A, s, b'_A(h_A, s), \{q^i_A\}\right) \\ \widetilde{h}'_B &= \left(h_B, s, \widetilde{b}'_B, \{\widetilde{q}^i_B\}\right) \end{aligned}$$

This is a feasible choice for the B government at (h_B, s) because we assume that debt can be chosen at any level below some upper bound. We aim to prove that through this choice of $\widetilde{b}'_B$, the government in the B equilibrium achieves expected utility strictly greater than $V_A(h_A, s)$, thus establishing that $V_B(h_B, s) > V_A(h_A, s)$, a contradiction. We first establish that continuation utility for B is weakly greater in each future state, and then that current consumption is strictly greater, than their corresponding values for A.

We have, for all $s' \in \mathcal{S}$,

$$V_B\left(\widetilde{h}'_B, s'\right) \geq V_A(h'_A, s') \tag{3.5.9}$$

Indeed, if $p_A(h'_A, s') = 0$, then immediately $V_B\left(\widetilde{h}'_B, s'\right) \geq V^d(s') = V_A(h'_A, s')$. Moreover, if $p_A(h'_A, s') = 1$ then, since $b(h'_A) - b\left(\widetilde{h}'_B\right) > M$ by (3.5.8), we must have $V_B\left(\widetilde{h}'_B, s'\right) > V_A(h'_A, s') \geq V^d(s')$.

This last observation also implies that $p_B\left(\widetilde{h}'_B, s'\right) = 1$ whenever $p_A(h'_A, s') = 1$. Hence, using the pricing condition (3.5.7), we also have

$$q_B\left(h_B, s, \widetilde{b}'_B\right) \geq q_A(h_A, s, b'_A) \tag{3.5.10}$$

Using (3.5.10), we now show that the consumption achieved by B from the choice of $\widetilde{b}'_B$ is strictly greater than that achieved by A. Indeed, using the flow budget constraints of both governments, and dropping

[10] One can rule out the case $M = \infty$ through a more direct mimicking argument: whenever $b(h_A) - b(h_B) > \bar{b}$, where $\bar{b}$ is the upper bound on debt, then a government at (h_B, s) can mimic at distance $\bar{b}$ the strategy of a government at (h_A, s), with weakly more favorable prices (and hence strictly higher consumption due to its lower b) guaranteed because it will never be in debt.

dependence on history for ease of notation:

$$\begin{aligned} \widetilde{c}_B &= c_A + b_A - b_B + \widetilde{q}_B \widetilde{b}'_B - q_A b'_A \\ &\geq c_A + M - \epsilon + (\widetilde{q}_B - q_A) b'_A + \widetilde{q}_B \left(\widetilde{b}'_B - b'_A \right) \end{aligned} \tag{3.5.11}$$

where the inequality follows from the definition of A and B.

Now if $b'_A < 0$ then, since $\widetilde{b}'_B \leq b'_A < 0$ as well we have $q_A = \widetilde{q}_B = \frac{1}{R}$, and hence $(\widetilde{q}_B - q_A) b'_A = 0$. If $b'_A \geq 0$ then using (3.5.10), $(\widetilde{q}_B - q_A) b'_A \geq 0$.

Moreover, from (3.5.8), $\widetilde{b}'_B - b'_A = -M - \epsilon$, and using $\widetilde{q}_B \leq \frac{1}{R}$, $\widetilde{q}_B \left(\widetilde{b}'_B - b'_A \right) \geq -\frac{1}{R}(M + \epsilon)$. Using these inequalities in (3.5.11),

$$\begin{aligned} \widetilde{c}_B &\geq c_A + M - \epsilon - \frac{1}{R}(M + \epsilon) \\ &\geq c_A + \left(1 - \frac{1}{R}\right) M - \epsilon \left(1 + \frac{1}{R}\right) \\ &> c_A \end{aligned} \tag{3.5.12}$$

where the last line follows from the choice of ϵ.

Since the utility from choosing $\widetilde{b}'_B$ provides a lower bound on $V_B(h_B, s)$, we have

$$\begin{aligned} V_B(h_B, s) &\geq u(\widetilde{c}_B, s) + \beta \sum_{s'} V_B \left(\widetilde{h}'_B, s' \right) \\ &> u(c_A, s) + \beta \sum_{s'} V_A(h'_A, s') = V_A(h_A, s) \end{aligned}$$

where the second line follows from (3.5.9) and (3.5.12). This contradicts $M > 0$. Hence $M \leq 0$. We have proved that for (h_A, h_B, s), if $V_A(h_A, s) \geq V_B(h_B, s)$ and $p_A(h_A, s) = 1$ then $b(h_A) \leq b(h_B)$.

So if $b(h_A) > b(h_B)$, either $p_A(h_A, s) = 0$ so that $V^d(s) = V_A(h_A, s) \leq V_B(h_B, s)$, or $p_A(h_A, s) = 1$ and $V_A(h_A, s) < V_B(h_B, s)$. The lemma is proved. □

With lemma 65 in hand, the proof of proposition 54 requires only one additional step. We need to show that the value function is uniquely determined by b and s. If two governments start with the same levels of b and s, either one of them can mimic the other but choose ϵ less debt in the next period; lemma 65 implies that from this point forward, the mimicking government is weakly better off. The utility loss from paying down ϵ debt in the initial period can be made arbitrarily small by choosing arbitrarily small ϵ, and hence the mimicking government's value must be weakly higher. Since this argument works in both directions, we conclude that the value is indeed uniquely determined by b and s.

Proof of proposition 54. Consider (h_A, h_B) such that $b(h_A) = b(h_B) = b$. At (h_B, s) consider the feasible choice

$$\widetilde{b}'_B = b'_A(h_A, s) - \epsilon$$

for some $\epsilon > 0$. Define continuation histories

$$\begin{aligned} h'_A &= \left(h_A, s, b'_A\left(h_A, s\right), \{q^i_A\}\right) \\ \widetilde{h}'_B &= \left(h_B, s, \widetilde{b}'_B, \{\widetilde{q}^i_B\}\right) \end{aligned}$$

From Lemma 65,

$$V_B\left(\widetilde{h}'_B, s'\right) \geq V_A\left(h'_A, s'\right) \tag{3.5.13}$$

and B repays if A repays. Hence $\widetilde{q}_B \geq q_A$ by the pricing condition (3.5.7). Moreover,

$$\begin{aligned} \widetilde{c}_B &= c_A + \widetilde{q}_B \widetilde{b}'_B - q_A b'_A \\ &= c_A + \left(\widetilde{q}_B - q_A\right) b'_A + \widetilde{q}_B \left(\widetilde{b}'_B - b'_A\right) \\ &\geq c_A - \frac{1}{R}\epsilon \end{aligned} \tag{3.5.14}$$

where the inequality follows as in (3.5.12) in the proof of Lemma 65.

Now,

$$\begin{aligned} V_B\left(h_B, s\right) - V_A\left(h_A, s\right) &\geq u\left(\widetilde{c}_B\right) - u\left(c_A\right) + \beta \sum_{s'} \left(V_B\left(\widetilde{h}'_B, s'\right) - V_A\left(h'_A, s'\right)\right) \\ &\geq u\left(c_A - \frac{1}{R}\epsilon\right) - u\left(c_A\right) \end{aligned}$$

where inequality follows form (3.5.13) and (3.5.14). Taking the limit as $\epsilon \to 0$ and using continuity of u, we obtain $V_B\left(h_B, s\right) \geq V_A\left(h_A, s\right)$. The symmetric argument implies that $V_B\left(h_B, s\right) \leq V_A\left(h_A, s\right)$, which concludes the proof. □

3.5.3 Proof of lemma 58

Proof. First, note that for any x and s, we have

$$\begin{aligned} \widetilde{Q}(\lambda x + \underline{b}, s) = \frac{(\lambda x + \underline{b})}{R} & \sum_{\{s' : \lambda x \leq \widetilde{b}^*(s') - \underline{b}\}} \pi(s'|s) \\ & \geq (1-\lambda)\underline{b} + \frac{\lambda(x + \underline{b})}{R} \sum_{\{s' : x \leq b^*(s') - \underline{b}\}} \pi(s'|s) = \lambda Q(s, x + \underline{b}) + (1-\lambda)\underline{b} \end{aligned} \tag{3.5.15}$$

where there is strict inequality if $\underline{b} < 0$.

Now we can formally define the *mimicking at a distance* policy. Suppose that at time 0 we have state s and debt level b. The equilibrium (V, Q) induces an allocation $\left\{c\left(s^t\right), b\left(s^{t-1}\right), p\left(s^t\right)\right\}_{s^t \succeq s^0}$ at all histories following s^0. We construct a policy for the government in the equilibrium $\left(\widetilde{V}, \widetilde{Q}\right)$ starting at s^0 as follows.

For every history $s^t \succeq s^0$, let $\widetilde{p}(s^t) = p(s^t)$, and whenever $p(s^t) = 1$ define a plan for debt

$$\widetilde{b}(s^{t-1}) - \underline{b} = \lambda(b(s^{t-1}) - \underline{b})$$

The resulting consumption path, again for $p(s^t) = 1$, satisfies

$$\begin{aligned}
\widetilde{c}(s^t) &= y(s_t) - \widetilde{b}(s^{t-1}) + \widetilde{Q}(\widetilde{b}(s^t), s_t) \\
&= y(s_t) - \lambda b(s^{t-1}) - (1-\lambda)\underline{b} + Q(\lambda(b(s^t) - \underline{b}) + \underline{b}, s_t) \\
&\geq y(s_t) - \lambda b(s^{t-1}) - (1-\lambda)\underline{b} + \lambda Q(s_t, b(s^t)) + (1-\lambda)\underline{b} \\
&= (1-\lambda)y(s_t) + \lambda(y(s_t) - b(s^{t-1}) + Q(s_t, b(s^t))) \\
&= (1-\lambda)y(s_t) + \lambda c(s^t)
\end{aligned}$$

Using the concavity of u, whenever $p(s^t) = 1$ we have

$$u(\widetilde{c}(s^t), s_t) \geq (1-\lambda)u(y(s_t), s_t) + \lambda u(c(s^t), s_t) \geq (1-\lambda)v^d(s_t) + \lambda u(c(s^t), s_t) \tag{3.5.16}$$

where the strict inequality from (3.5.15) persists in (3.5.16) whenever $\underline{b} < 0$, and by assumption, $u(y(s_t), s_t) > v^d(s_t)$ gives strict inequality whenever $\underline{b} = 0$. Furthermore, in states where $p(s^t) = 0$, we have

$$u(\widetilde{c}(s^t), s_t) = u(c(s^t), s_t) = v^d(s_t) \tag{3.5.17}$$

Summing (3.5.16) and (3.5.17) across all times and states to obtain the expected discounted value, we obtain the result. □

3.5.4 Proof of proposition 60

Proof. Suppose to the contrary that there exist two distinct equilibria (V, Q) and $\left(\widetilde{V}, \widetilde{Q}\right)$, with associated default thresholds $\{b^*(s)\}_{s\in\mathcal{S}}$ and $\left\{\widetilde{b}^*(s)\right\}_{s\in\mathcal{S}}$, such that $Q(b,s) \geq \widetilde{Q}(b,s)$ for all b and s. It follows that $V(s,b) \geq \widetilde{V}(s,b)$ for all b and s as well, since a government facing the weakly higher debt schedule Q can always replicate the policy of the government facing $\widetilde{Q}$, achieving weakly higher consumption in the process.[11]

Since Q and $\widetilde{Q}$ are distinct, there exists some s' such that $b^*(s') > \widetilde{b}^*(s')$, and we define

$$M = \max_s b^*(s) - \widetilde{b}^*(s) > 0 \tag{3.5.18}$$

We first seek to prove that, for any s and $b \leq b^*(s)$

$$\widetilde{V}(b - M, s) - \widetilde{V}^d(s) > V(b,s) - V^d(s) \tag{3.5.19}$$

[11] Explicitly, it can set $b = \widetilde{b}$, $p = \widetilde{p}$, $c = \widetilde{c} + Q(s,b) - \widetilde{Q}(s,b) \geq 0$.

To do so, we use the same mimicking at a distance argument as in lemma 52, although the calculation becomes somewhat more complicated. Writing $s^0 \equiv s$, we continue to set $\widetilde{b}(s^t) = b(s^t) - M$ and $\widetilde{p}(s^t) = p(s^t)$, along with the consumption policy $\widetilde{c}(s^t)$ in (3.1.9). This strategy places a lower bound on $\widetilde{V}(b - M, s)$:

$$\widetilde{V}(b - M, s) \geq \sum_{p(s^t)=1} \beta^t \Pi\left(s^t\right) u\left(\widetilde{c}\left(s^t\right)\right) + \sum_{p(s^t)=0, p(s^{t-1})=1} \beta^t \Pi\left(s^t\right) \widetilde{V}^d(s_t) \tag{3.5.20}$$

Subtracting the corresponding expression for $V(b, s)$, and using $\widetilde{c}(s^t) > c(s^t)$, we have

$$\widetilde{V}(b - M, s) - V(b, s) > \sum_{p(s^t)=0, p(s^{t-1})=1} \beta^t \Pi(s^t) \left(\widetilde{V}^d(s_t) - V^d(s_t)\right) \tag{3.5.21}$$

Subtracting $\widetilde{V}^d(s) - V^d(s)$ from both sides we obtain

$$\begin{aligned} &\left(\widetilde{V}(b - M, s) - \widetilde{V}^d(s)\right) - \left(V(b, s) - V^d(s)\right) \\ &\qquad > -(\widetilde{V}^d(s) - V^d(s)) + \sum_{p(s^t)=0, p(s^{t-1})=1} \beta^t \Pi(s^t) \left(\widetilde{V}^d(s_t) - V^d(s_t)\right) \end{aligned} \tag{3.5.22}$$

and to prove (3.5.19) it suffices to show that the right side of (3.5.22) is nonnegative.

Expanding $\widetilde{V}^d(s_t) - V^d(s_t)$ gives

$$\widetilde{V}^d(s_t) - V^d(s_t) = \sum_{s_\tau} \beta^{\tau - t}(1 - \lambda)\lambda^{\tau - t - 1}\Pi(s_\tau | s_t) \left(\widetilde{V}^o(0, s_\tau) - V^o(0, s_\tau)\right) \tag{3.5.23}$$

Now, using (3.5.23), we can rewrite the right side of (3.5.21) as

$$\begin{aligned} &- \sum_{s^\tau \succ s^0} \beta^\tau (1 - \lambda)\lambda^{\tau - 1}\Pi(s^\tau) \left(\widetilde{V}^o(0, s_\tau) - V^o(0, s_\tau)\right) \\ &\qquad + \sum_{p(s^t)=0, p(s^{t-1})=1} \sum_{s^\tau \succ s^t} \beta^\tau (1 - \lambda)\lambda^{\tau - t - 1}\Pi(s^\tau) \left(\widetilde{V}^o(0, s_\tau) - V^o(0, s_\tau)\right) \end{aligned}$$

which can be rearranged as

$$\lambda^{-1}(1 - \lambda) \sum_{s^\tau \succ s^0} \beta^\tau \Pi(s^\tau) \left(V^o(0, s_\tau) - \widetilde{V}^o(0, s_\tau)\right) \cdot \left(1 - \sum_{s^\tau \succ s^t \succ s^0} \lambda^{\tau - t} \cdot \mathbf{1}_{\{p(s^t)=0, p(s^{t-1})=1\}}\right) \tag{3.5.24}$$

Since for any s^τ there exists at most one s^t such that $p(s^t) = 0$ and $p(s^{t-1}) = 1$, the rightmost factor in parentheses is nonnegative. Since in addition $V(0, s_\tau) \geq \widetilde{V}(0, s_\tau)$, the preceding factor is nonnegative as well, and hence (3.5.24) is nonnegative. (3.5.19) therefore follows.

Finally, suppose that the maximum in (3.5.18) is attained at $\bar{s}$, so that $b^*(\bar{s}) = \widetilde{b}^*(\bar{s}) + M$. Applying (3.5.19), we have

$$0 = \widetilde{V}(\widetilde{b}^*(\bar{s}), \bar{s}) - \widetilde{V}^d(\bar{s}) > V(b^*(\bar{s}), \bar{s}) - V^d(\bar{s}) = 0$$

which is a contradiction. □

3.5.5 Proof of proposition 61

Proof. Write $V^{re} = \mathbb{E}_{s'}[V^o(0, s')]$, and similarly $\widetilde{V^{re}}$ for a conjectured alternative equilibrium. First, observe that if $\widetilde{V}^{re} = V^{re}$, then the two equilibria have the same expected value from default V^d, and we can apply proposition 53 taking V^d as given to conclude that the two equilibria must be the same.

Otherwise, assume without loss of generality that $V^{re} > \widetilde{V}^{re}$. It cannot be that $\widetilde{Q}(b') \geq Q(b')$ for all b', since in that case a government starting with zero debt and facing the weakly higher debt schedule $\widetilde{Q}$ could always replicate the policy of the government facing Q, achieving weakly higher consumption in the process. This would imply $\widetilde{V}^{re} \geq V^{re}$, a contradiction. Hence $Q(b') > \widetilde{Q}(b')$ for some b'. From this point on, the proof is the same as the proof for proposition 60 starting with the definition of M in (3.5.18), except that we can replace (3.5.23) with simply

$$\widetilde{V}^d(s_t) - V^d(s_t) = \sum_{\tau > t} \beta^{\tau - t}(1 - \lambda)\lambda^{\tau - t - 1}\left(\widetilde{V}^{re} - V^{re}\right) \tag{3.5.25}$$

allowing us to replace (3.5.24) with

$$\lambda^{-1}(1 - \lambda)\sum_{\tau > 0} \beta^{\tau}(V^{re} - \widetilde{V}^{re}) \cdot \left(1 - \sum_{\tau > t > 0} \lambda^{\tau - t} \cdot \mathbf{1}_{\{p(s^t) = 0, p(s^{t-1}) = 1\}}\right) \tag{3.5.26}$$

again concluding that the expression is nonnegative, from which a contradiction follows. □

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