Applications of Harmonic Maass Forms

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Abstract

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In this thesis, we prove various results in the theory of modular forms and harmonic Maass forms, representation theory, elliptic curves and differential geometry. In particular, we give a broad framework of Rogers–Ramanujan identities and algebraic values; we prove that Ramanujan's mock theta functions satisfy his original conjectured definition; and we show that certain harmonic Maass forms which arise naturally from the arithmetic of elliptic curves encode central L-values and L-derivatives involved in the Birch and Swinnerton-Dyer conjecture. We also prove a conjecture of Moore and Witten connecting the regularized u-plane integral on the complex projective plane with Donaldson invariants for the SU(2)-gauge theory. In our final two applications, we turn to moonshine phenomena. Monstrous Moonshine relates the Fourier coefficients of certain modular functions to values of the irreducible characters of the Monster group–the largest of the sporadic simple groups. We give the asymptotic distribution of these character values, answering a question of Witten with applications to mathematical physics. The Umbral Moonshine conjectures relate the the values of irreducible characters of prescribed finite groups with the Fourier coefficients of distinguished mock modular forms. Gannon has proved this for the special case involving the largest sporadic simple Mathieu group. We complete the proof in the remaining cases.

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Chapter 1

Introduction

In 1913, The mathematician G. H. Hardy received a letter from a young and unknown Indian clerk named Srinivas Ramanujan. *"Dear Sir,"* the letter began, *"I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras... I am now about 23 years of age... I have not trodden through the conventional regular course which is followed in a University course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as 'startling'."* [25] Near the end of the letter, Ramanujan claimed that

$$\frac{1}{1+\frac{e^{-2\pi}}{1+\frac{e^{-4\pi}}{1+\frac{e^{-6\pi}}{1+\ddots}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right)\sqrt[5]{e^{2\pi}},$$

$$\frac{1}{1-\frac{e^{-\pi}}{1+\frac{e^{-2\pi}}{1-\frac{e^{-3\pi}}{1+\ddots}}}} = \left(\sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}\right)\sqrt[5]{e^{\pi}},$$

and that the function

$$\frac{1}{1+\frac{e^{-\pi\sqrt{n}}}{1+\frac{e^{-2\pi\sqrt{n}}}{1+\frac{e^{-3\pi\sqrt{n}}}{1+\ddots}}}} \tag{1.1}$$

"can be exactly found if n be any positive rational."

Speaking of these particular formulas, Hardy later said *"They defeated me completely. I had never seen anything in the least like this before... they could only be written down by a mathematician of the highest class. They must be true because no one would have the imagination to invent them."*

The heart of Ramanujan's evaluations of 1.1 are the Rogers–Ramanujan identities which show that this is essentially a *modular function*. Modular functions and more general *modular forms* have been studied since the early nineteenth century in connection with elliptic curves. They have since been shown to connect broadly across mathematics from fields such as number theory, class field theory, and representation theory, to combinatorics, mathematical physics and more.

Many modern advances in the study of modular forms are rooted in another letter from Ramanujan to Hardy–his last one. After returning to India, Ramanujan wrote to Hardy and introduced an enigmatic collection of functions which he called the *mock theta functions*. These functions would eventually lead to the theory of *harmonic Maass forms*, special functions which generalize the notion of a modular form and provide a foundational framework which has lead to a deeper understanding of these important mathematical objects. Through the course of this thesis, we will examine several new results and applications of the mathematics rooted in Ramanujan's first and last letters.

A framework of Rogers–Ramanujan identities (Joint work with K. Ono and S. O. Warnaar)

Returning to Ramanujan's first letter, the Rogers–Ramanujan (RR) identities [263] are given by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \tag{1.2}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \tag{1.3}$$

These are essentially modular functions, and their ratio $H(q)/G(q)$ is the Rogers–Ramanujan q-continued fraction

$$\frac{H(q)}{G(q)} = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\ddots}}}}. \tag{1.4}$$

Note that when $q = e^{-\pi\sqrt{n}}$, this is (1.1).

The *golden ratio* ϕ satisfies $H(1)/G(1) = 1/\phi = (-1+\sqrt{5})/2$. Ramanujan computed further values such as

$$\mathrm{e}^{-\frac{2\pi}{5}} \cdot \frac{H(\mathrm{e}^{-2\pi})}{G(\mathrm{e}^{-2\pi})} = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}. \tag{1.5}$$

The minimal polynomial of this value is

$$x^4 + 2x^3 - 6x^2 - 2x + 1,$$

which shows that it is an algebraic integral unit. All of Ramanujan's evaluations are such units.

Ramanujan's evaluations inspired early work by Watson [234, 296, 297] and Ramanathan [257]. Then in 1996, Berndt, Chan and Zhang [24][1] finally obtained general theorems concerning such values. The theory pertains to values at $q := \mathrm{e}^{2\pi i\tau}$, where the τ are quadratic irrational points in the upper-half of the complex plane. We refer to such a point τ as a *CM point* with

[1] Cais and Conrad [60] and Duke [102] later revisited these results from the perspective of arithmetic geometry and the symmetries of the regular icosahedron respectively.

discriminant $-D<0$, where $-D$ is the discriminant of the minimal polynomial of τ. The corresponding evaluation is known as a *singular value*. Berndt, Chan and Zhang proved that the singular values $q^{-1/60}G(q)$ and $q^{11/60}H(q)$ are algebraic numbers in abelian extensions of $\mathbb{Q}(\tau)$ which enjoy the surprising property (see [24, Theorem 6.2]) that their ratio $q^{1/5}H(q)/G(q)$ is an algebraic integral unit which generates specific abelian extensions of $\mathbb{Q}(\tau)$.

Remark. The individual values of $q^{-1/60}G(q)$ and $q^{11/60}H(q)$ generically are not algebraic integers. For example, in (1.5) we have $\tau = \mathrm{i}$, and the numerator and denominator

$$q^{-\frac{1}{60}}G(q) = -\sqrt[4]{\frac{1+3\sqrt{5}-2\sqrt{10+2\sqrt{5}}}{10}} \quad \text{and} \quad q^{\frac{11}{60}}H(q) = -\sqrt[4]{\frac{1+3\sqrt{5}+2\sqrt{10+2\sqrt{5}}}{10}}$$

share the minimal polynomial $625x^{16}-250x^{12}-1025x^{8}-90x^{4}+1$.

In addition to the deep algebraic properties described above, (1.2) and (1.3) have been related to a large number of different areas of mathematics. They were were first recognized by MacMahon and Schur as identities for integer partitions [220,266], but have since been linked to algebraic geometry [59, 140], K-theory [111], conformal field theory [22, 186, 209], group theory [128], Kac–Moody, Virasoro, vertex and double affine Hecke algebras [76,120, 206, 207, 211–214], knot theory [159, 160], modular forms [38–40, 43, 46, 250], orthogonal polynomials [14,35,138], statistical mechanics [10,20], probability [129] and transcendental number theory [262].

In 1974 Andrews [7] extended (1.2) and (1.3) to an infinite family of Rogers–Ramanujan-type identities by proving that

$$\sum_{r_1\geq\cdots\geq r_m\geq 0} \frac{q^{r_1^2+\cdots+r_m^2+r_i+\cdots+r_m}}{(q)_{r_1-r_2}\cdots(q)_{r_{m-1}-r_m}(q)_{r_m}} = \frac{(q^{2m+3};q^{2m+3})_\infty}{(q)_\infty}\cdot\theta(q^i;q^{2m+3}), \tag{1.6}$$

where $1 \leq i \leq m+1$. As usual, here we have that

$$(a)_k = (a;q)_k := \begin{cases} (1-a)(1-aq)\cdots(1-aq^{k-1}) & \text{if } k \geq 0, \\ \prod_{j=0}^{\infty}(1-aq^j) & \text{if } k = \infty, \end{cases}$$

and

$$\theta(a;q) := (a;q)_\infty(q/a;q)_\infty$$

is a modified theta function. The identities (1.6), which can be viewed as the analytic counterpart of Gordon's partition theorem [139], are now commonly referred to as the Andrews–Gordon (AG) identities.

Remark. The specializations of $\theta(a;q)$ in (1.6) are (up to powers of q) modular functions, where $q := \mathrm{e}^{2\pi\mathrm{i}\tau}$ and τ is any complex point with $\mathrm{Im}(\tau) > 0$. It should be noted that this differs from our use of q and τ above where we required τ to be a quadratic irrational point. Such infinite product modular functions were studied extensively by Klein and Siegel.

There are numerous algebraic interpretations of the Rogers–Ramanujan and Andrews–Gordon identities. For example, the above-cited papers by Milne, Lepowsky and Wilson show that they arise as principally specialized characters of integrable highest-weight modules of the affine Kac–Moody algebra $\mathbb{A}_1^{(1)}$. Similarly, Feigin and Frenkel proved the Rogers–Ramanujan and Andrews–Gordon identities by considering certain irreducible minimal representations of the Virasoro algebra [120]. We should also mention the much larger program by Lepowsky and others on combinatorial and algebraic extensions of Rogers–Ramanujan-type identities, leading to the introduction of Z-algebras for all affine Lie algebras, vertex-operator-theoretic proofs of the Rogers–Ramanujan identities, and Rogers–Ramanujan identities for arbitrary affine Lie algebras in which the sum side is replaced by a combinatorial sum, see e.g., [126, 205, 236] and references therein.

Here we have a similar but distinct aim, namely to find a concrete framework of Rogers–Ramanujan type identities in the q-series sense of "Infinite sum = Infinite product", where the infinite products arise as specialized characters of appropriately chosen affine Lie algebras $\mathrm{X}_N^{(r)}$ for arbitrary N. Such a general framework would give new connections between Lie algebras and the theory of modular functions.

In [11] (see also [119, 292]) some partial results concerning the above question were obtained, resulting in Rogers–Ramanujan-type identities for $\mathbb{A}_2^{(1)}$. Unfortunately, the approach of [11] does not in any obvious manner extend to $\mathbb{A}_n^{(1)}$ for all n, and this paper aims to give a more complete answer. By using a level-m Rogers–Selberg identity for the root system $\mathbb{C}_n$ as recently obtained by Bartlett and Warnaar [19], we show that the Rogers–Ramanujan and Andrews–Gordon identities are special cases of a doubly-infinite family of q-identities arising from the Kac–Moody algebra $\mathbb{A}_{2n}^{(2)}$ for arbitrary n. In their most compact form, the "sum-sides" are expressed in terms of Hall–Littlewood polynomials $P_\lambda(x;q)$ evaluated at infinite geometric progressions (see Section 3.1 for definitions and further details), and the "product-sides" are essentially products of modular theta functions. We shall present four pairs (a,b) such that for all $m,n \geq 1$ we have an identity of the form

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{a|\lambda|} P_{2\lambda}(1,q,q^2,\dots;q^{2n+b}) = \text{"Infinite product modular function".}$$

To make this precise, we fix notation for *integer partitions*, nonincreasing sequences of nonnegative integers with at most finitely many nonzero terms. For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we let $|\lambda| := \lambda_1 + \lambda_2 + \cdots$, and we let $2\lambda := (2\lambda_1, 2\lambda_2, \dots)$. We also require λ', the *conjugate* of λ, the partition which is obtained by transposing the Ferrers–Young diagram of λ. Finally, for convenience we let

$$\theta(a_1,\dots,a_k;q) := \theta(a_1;q)\cdots\theta(a_k;q). \tag{1.7}$$

Example 1.1. If $\lambda = (5,3,3,1)$, then we have that $|\lambda| = 12$, $2\lambda = (10,6,6,2)$ and $\lambda' = (4,3,3,1,1)$.

Using this notation, we have the following pair of doubly-infinite Rogers–Ramanujan type identities which correspond to specialized characters of $\mathbb{A}_{2n}^{(2)}$.

Theorem 1.2 ($\mathbb{A}_{2n}^{(2)}$ RR and AG identities)**.** *If m and n are positive integers and $\kappa := 2m+2n+1$, then we have that*

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{|\lambda|} P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-1}\big) \tag{1.8a}$$
$$= \frac{(q^\kappa;q^\kappa)_\infty^n}{(q)_\infty^n} \cdot \prod_{i=1}^n \theta\big(q^{i+m};q^\kappa\big) \prod_{1\le i<j\le n} \theta\big(q^{j-i},q^{i+j-1};q^\kappa\big)$$
$$= \frac{(q^\kappa;q^\kappa)_\infty^m}{(q)_\infty^m} \cdot \prod_{i=1}^m \theta\big(q^{i+1};q^\kappa\big) \prod_{1\le i<j\le m} \theta\big(q^{j-i},q^{i+j+1};q^\kappa\big),$$

and

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{2|\lambda|} P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-1}\big) \tag{1.8b}$$
$$= \frac{(q^\kappa;q^\kappa)_\infty^n}{(q)_\infty^n} \cdot \prod_{i=1}^n \theta\big(q^{i};q^\kappa\big) \prod_{1\le i<j\le n} \theta\big(q^{j-i},q^{i+j};q^\kappa\big)$$
$$= \frac{(q^\kappa;q^\kappa)_\infty^m}{(q)_\infty^m} \cdot \prod_{i=1}^m \theta\big(q^{i};q^\kappa\big) \prod_{1\le i<j\le m} \theta\big(q^{j-i},q^{i+j};q^\kappa\big).$$

Four remarks.

(1) When $m = n = 1$, Theorem 1.2 gives the Rogers–Ramanujan identities (1.2) and (1.3). The summation defining the series is over the empty partition, $\lambda = 0$, and partitions consisting of n copies of 1, i.e., $\lambda = (1^n)$. Since

$$q^{(\sigma+1)|(1^n)|} P_{(2^n)}(1,q,q^2,\dots;q) = \frac{q^{n(n+\sigma)}}{(1-q)\cdots(1-q^n)},$$

identities (1.2) and (1.3) thus follow from Theorem 1.2 by letting $\sigma = 0, 1$.

(2) When $n = 1$, Theorem 1.2 gives the $i = 1$ and the $i = m + 1$ instances of the Andrews–Gordon identities in a representation due to Stembridge [278] (see also Fulman [128]). The equivalence with (1.6) follows from the specialization formula [219, p. 213]

$$q^{(\sigma+1)|\lambda|} P_{2\lambda}(1, q, q^2, \dots; q) = \prod_{i \geq 1} \frac{q^{r_i(r_i+\sigma)}}{(q)_{r_i - r_{i+1}}},$$

where $r_i := \lambda'_i$. Note that $\lambda_1 \leq m$ implies that $\lambda'_i = r_i = 0$ for $i > m$.

(3) We note the beautiful level-rank duality exhibited by the products on the right-hand sides of the expressions in Theorem 1.2 (especially those of (1.8b)).

(4) In the next section we shall show that the more general series

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{(\sigma+1)|\lambda|} P_{2\lambda}\bigl(1, q, q^2, \dots; q^n\bigr) \tag{1.9}$$

are also expressible in terms of q-shifted factorials, allowing for a formulation of Theorem 1.2 (see Lemma 3.3) which is independent of Hall–Littlewood polynomials.

Example 1.3. Here we illustrate Theorem 1.2 when $m = n = 2$. Then (1.8a) is

$$\sum_{\substack{\lambda \\ \lambda_1 \leq 2}} q^{|\lambda|} P_{2\lambda}\bigl(1, q, q^2, \dots; q^3\bigr) = \prod_{n=1}^{\infty} \frac{(1 - q^{9n})}{(1 - q^n)},$$

giving another expression for the q-series in Dyson's favorite identity, as recalled in his "A walk through Ramanujan's Garden" [112]:

"The end of the war was not in sight. In the evenings of that winter I kept sane by wandering in Ramanujan's garden. ... I found a lot of identities of the sort that Ramanujan would have enjoyed. My favorite one was this one:

$$\sum_{n=0}^{\infty} x^{n^2+n} \cdot \frac{(1 + x + x^2)(1 + x^2 + x^4) \cdots (1 + x^n + x^{2n})}{(1 - x)(1 - x^2) \cdots (1 - x^{2n+1})} = \prod_{n=1}^{\infty} \frac{(1 - x^{9n})}{(1 - x^n)}.$$

In the cold dark evenings, while I was scribbling these beautiful identities amid the death and destruction of 1944, I felt close to Ramanujan. He had been scribbling even more beautiful identities amid the death and destruction of 1917."

The series in (1.8b) is

$$\sum_{\substack{\lambda \\ \lambda_1\le 2}} q^{2|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^3\big) = \prod_{n=1}^{\infty}\frac{(1-q^{9n})(1-q^{9n-1})(1-q^{9n-8})}{(1-q^n)(1-q^{9n-4})(1-q^{9n-5})}.$$

We also have an even modulus analog of Theorem 1.2. Surprisingly, the $a=1$ and $a=2$ cases correspond to dual affine Lie algebras, namely $\mathbb{C}_n^{(1)}$ and $\mathbf{D}_{n+1}^{(2)}$.

Theorem 1.4 ($\mathbb{C}_n^{(1)}$ RR and AG identities)**.** *If m and n are positive integers and $\kappa := 2m+2n+2$, then we have that*

$$\sum_{\substack{\lambda \\ \lambda_1\le m}} q^{|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n}\big) \tag{1.10}$$

$$= \frac{(q^2;q^2)_\infty (q^{\kappa/2};q^{\kappa/2})_\infty (q^\kappa;q^\kappa)_\infty^{n-1}}{(q)_\infty^{n+1}} \cdot \prod_{i=1}^{n}\theta\big(q^i;q^{\kappa/2}\big) \prod_{1\le i<j\le n}\theta\big(q^{j-i},q^{i+j};q^\kappa\big)$$

$$= \frac{(q^\kappa;q^\kappa)_\infty^m}{(q)_\infty^m}\cdot \prod_{i=1}^{m}\theta\big(q^{i+1};q^\kappa\big)\prod_{1\le i<j\le m}\theta\big(q^{j-i},q^{i+j+1};q^\kappa\big).$$

Theorem 1.5 ($\mathbf{D}_{n+1}^{(2)}$ RR and AG identities)**.** *If m and n are positive integers such that $n\ge 2$, and $\kappa := 2m+2n$, then we have that*

$$\sum_{\substack{\lambda \\ \lambda_1\le m}} q^{2|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-2}\big) \tag{1.11}$$

$$= \frac{(q^\kappa;q^\kappa)_\infty^n}{(q^2;q^2)_\infty (q)_\infty^{n-1}}\cdot\prod_{1\le i<j\le n}\theta\big(q^{j-i},q^{i+j-1};q^\kappa\big)$$

$$= \frac{(q^\kappa;q^\kappa)_\infty^m}{(q)_\infty^m}\cdot\prod_{i=1}^{m}\theta\big(q^i;q^\kappa\big)\prod_{1\le i<j\le m}\theta\big(q^{j-i},q^{i+j};q^\kappa\big).$$

Two remarks.

(1) The $(m,n)=(1,2)$ case of (1.11) is equivalent to Milne's modulus 6 Rogers–Ramanujan identity [238, Theorem 3.26].

(2) If we take $m=1$ in (1.10) (with $n \mapsto n-1$) and (1.11), and apply formula (3.7) below (with $\delta=0$), we obtain the $i=1,2$ cases of Bressoud's even modulus identities [36]

$$\sum_{r_1\geq\cdots\geq r_n\geq 0} \frac{q^{r_1^2+\cdots+r_n^2+r_i+\cdots+r_n}}{(q)_{r_1-r_2}\cdots(q)_{r_{n-1}-r_n}(q^2;q^2)_{r_n}} = \frac{(q^{2n+2};q^{2n+2})_\infty}{(q)_\infty} \cdot \theta(q^i;q^{2n+2}). \tag{1.12}$$

By combining (1.8)–(1.11), we obtain an identity of "mixed" type.

Corollary 1.6. *If m and n are positive integers and $\kappa := 2m+n+2$, then for $\sigma=0,1$ we have that*

$$\sum_{\substack{\lambda \\ \lambda_1\leq m}} q^{(\sigma+1)|\lambda|} P_{2\lambda}\big(1,q,q^2,\dots;q^n\big) \tag{1.13}$$

$$= \frac{(q^\kappa;q^\kappa)_\infty^m}{(q)_\infty^m} \cdot \prod_{i=1}^m \theta\big(q^{i-\sigma+1};q^\kappa\big) \prod_{1\leq i<j\leq m} \theta\big(q^{j-i},q^{i+j-\sigma+1};q^\kappa\big). \tag{1.14}$$

Identities for $\mathbb{A}_{n-1}^{(1)}$ also exist, although their formulation is perhaps slightly less satisfactory. We have the following "limiting" Rogers–Ramanujan type identities.

Theorem 1.7 ($\mathbb{A}_{n-1}^{(1)}$ RR and AG identities)**.** *If m and n are positive integers and $\kappa := m+n$, then we have that*

$$\begin{aligned}\lim_{r\to\infty} q^{-m\binom{r}{2}} P_{(m^r)}(1,q,q^2,\dots;q^n) &= \frac{(q^\kappa;q^\kappa)_\infty^{n-1}}{(q)_\infty^n} \cdot \prod_{1\leq i<j\leq n} \theta(q^{j-i};q^\kappa) \\ &= \frac{(q^\kappa;q^\kappa)_\infty^{m-1}}{(q)_\infty^m} \cdot \prod_{1\leq i<j\leq m} \theta(q^{j-i};q^\kappa).\end{aligned}$$

Now we turn to the question of whether the new q-series appearing in these theorems, which arise so simply from the Hall–Littlewood polynomials, enjoy the same deep algebraic properties as (1.2), (1.3), and the Rogers–Ramanujan continued fraction. As it turns out they do: their singular values are algebraic numbers. Moreover, we can characterize those ratios which simplify to algebraic integral units.

To make this precise, we recall that $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ for $\mathrm{Im}(\tau) > 0$, and that m and n are arbitrary positive integers. The auxiliary parameter $\kappa = \kappa_*(m,n)$ in Theorems 1.2, 1.4 and 1.5 is defined as follows:

$$\kappa = \begin{cases} \kappa_1(m,n) := 2m + 2n + 1 & \text{for } \mathbb{A}_{2n}^{(2)} \\ \kappa_2(m,n) := 2m + 2n + 2 & \text{for } \mathbb{C}_{n}^{(1)} \\ \kappa_3(m,n) := 2m + 2n & \text{for } \mathbf{D}_{n+1}^{(2)}. \end{cases} \tag{1.15}$$

Remark. The parameter κ has a representation theoretic interpretation arising from the corresponding affine Lie algebra $X_N^{(r)}$ (see Section 3.2). It turns out that

$$\kappa_*(m,n) = \frac{2}{r}\big(\,\mathrm{lev}(\Lambda) + h^\vee\big),$$

where $\mathrm{lev}(\Lambda)$ is the level of the corresponding representation, $h^\vee$ is the dual Coxeter number and r is the tier number.

To obtain algebraic values, we require certain normalizations of these series. The subscripts below correspond to the labelling in the theorems. In particular, Φ_{1a} and Φ_{1b} appear in Theorem 1.2, Φ_2 is in Theorem 1.4, and

Φ_3 is in Theorem 1.5. Using this notation, the series are

$$\Phi_{1a}(m,n;\tau) := q^{\frac{mn(4mn-4m+2n-3)}{12\kappa}} \sum_{\lambda\colon \lambda_1\le m} q^{|\lambda|} P_{2\lambda}(1,q,q^2,\ldots;q^{2n-1}) \quad (1.16a)$$

$$\Phi_{1b}(m,n;\tau) := q^{\frac{mn(4mn+2m+2n+3)}{12\kappa}} \sum_{\lambda\colon \lambda_1\le m} q^{2|\lambda|} P_{2\lambda}(1,q,q^2,\ldots;q^{2n-1}) \quad (1.16b)$$

$$\Phi_{2}(m,n;\tau) := q^{\frac{m(2n+1)(2mn-m+n-1)}{12\kappa}} \sum_{\lambda\colon \lambda_1\le m} q^{|\lambda|} P_{2\lambda}(1,q,q^2,\ldots;q^{2n}) \quad (1.16c)$$

$$\Phi_{3}(m,n;\tau) := q^{\frac{m(2n-1)(2mn+n+1)}{12\kappa}} \sum_{\lambda\colon \lambda_1\le m} q^{2|\lambda|} P_{2\lambda}(1,q,q^2,\ldots;q^{2n-2}). \quad (1.16d)$$

Two remarks.

(1) We note that $\Phi_3(m,n;\tau)$ is not well defined when $n=1$.

(2) We note that the $\kappa_*(m,n)$ are odd in the $\mathbb{A}_{2n}^{(2)}$ cases, and are even for the $\mathbb{C}_n^{(1)}$ and $\mathbf{D}_{n+1}^{(2)}$ cases. This dichotomy will be important when seeking pairs of Φ_* whose singular values have ratios that are algebraic integral units.

Our first result concerns the algebraicity of these values and their Galois theoretic properties. We show that these values are in specific abelian extensions of imaginary quadratic fields (see [33,85] for background on the explicit class field theory of imaginary quadratic fields). For convenience, if $-D<0$ is a discriminant, then we define

$$D_0 := \begin{cases} \frac{D}{4} & \text{if } D\equiv 0 \pmod 4, \\ \frac{-D-1}{4} & \text{if } -D\equiv 1 \pmod 4. \end{cases}$$

Theorem 1.8. *Assume the notation above, and let $\kappa := \kappa_*(m,n)$. If $\kappa\tau$ is a CM point with discriminant $-D<0$, then the following are true:*

1. *The singular value $\Phi_*(m,n;\tau)$ is an algebraic number.*
2. *The multiset*

$$\left\{\Phi_*(m,n,\tau_Q/\kappa)^{12\kappa}_{(\gamma\cdot\delta_Q(\tau))} : (\gamma,Q)\in W_{\kappa,\tau}\times\mathcal{Q}_D\right\}$$

(see Section 3.4 for definitions) consists of multiple copies of a Galois orbit over $\mathbb{Q}$.

3. *If* $\kappa > 10$, $|-D| > \kappa^4/2$, *and* $\gcd(D_0, \kappa) = 1$, *then the multiset in (2) is a Galois orbit over* $\mathbb{Q}$.

Four remarks.

(1) For each pair of positive integers m and n, the inequality in Theorem 1.8 (3) holds for all but finitely many discriminants.

(2) In Section 3.4 we will show that the values $\Phi_*(m, n; \tau)^{12\kappa}$ are in a distinguished class field over the ring class field $\mathbb{Q}(j(\kappa^2\tau))$, where $j(\tau)$ is the usual Klein j-function.

(3) The Φ_* singular values do not in general contain full sets of Galois conjugates. In particular, the singular values in the multiset in Theorem 1.8 (2) generally require q-series which are not among the four families Φ_*. For instance, only the $i = 1$ and $i = m + 1$ cases of the Andrews–Gordon identities arise from specializations of Φ_{1a} and Φ_{1b} respectively. However, the values associated to the other AG identities arise as Galois conjugates of these specializations. One then naturally wonders whether there are even further families of identities, perhaps those which can be uncovered by the theory of complex multiplication.

(4) Although Theorem 1.8 (3) indicates that the multiset in (2) is generically a single orbit of Galois conjugates, it turns out that there are indeed situations where the set is more than a single copy of such an orbit. Indeed, the two examples in Section 3.6 will be such accidents.

We now address the question of singular values and algebraic integral units. Although the singular values of $q^{-1/60}G(q)$ and $q^{11/60}H(q)$ are not generally algebraic integers, their denominators can be determined exactly, and their ratios always are algebraic integral units. The series Φ_* exhibit similar behavior. The following theorem determines the integrality properties of the singular values. Moreover, it gives algebraic integral unit ratios in the case of the $\mathbb{A}_{2n}^{(2)}$ identities, generalizing the case of the Rogers–Ramanujan continued fraction.

Theorem 1.9. *Assume the notation and hypotheses in Theorem 1.8. Then the following are true:*

1. *The singular value* $1/\Phi_*(m, n; \tau)$ *is an algebraic integer.*
2. *The singular value* $\Phi_*(m, n; \tau)$ *is a unit over* $\mathbb{Z}[1/\kappa]$.
3. *The ratio* $\Phi_{1a}(m, n; \tau)/\Phi_{1b}(m, n; \tau)$ *is an algebraic integral unit.*

Two remarks.

(1) We have that $\Phi_{1a}(1, 1; \tau) = q^{-1/60}G(q)$ and $\Phi_{1b}(1, 1; \tau) = q^{11/60}H(q)$. Therefore, Theorem 1.15 (3) implies the theorem of Berndt, Chan, and Zhang that the ratios of these singular values—the singular values of the Rogers–Ramanujan continued fraction—are algebraic integral units.

(2) It is natural to ask whether Theorem 1.15 (3) is a special property enjoyed only by the $\mathbb{A}_{2n}^{(2)}$ identities. More precisely, are ratios of singular values of further pairs of Φ_* series algebraic integral units? By Theorem 1.15 (2), it is natural to restrict attention to cases where the $\kappa_*(m, n)$ integers agree. Indeed, in these cases the singular values are already integral over the common ring $\mathbb{Z}[1/\kappa]$. Due to the parity of the $\kappa_*(m, n)$, the only other cases to consider are pairs involving Φ_2 and Φ_3. In Section 3.6 we give an example illustrating that such ratios for Φ_2 and Φ_3 are not generically units.

Example 1.10. In Section 3.6 we shall consider the q-series $\Phi_{1a}(2, 2; \tau)$ and $\Phi_{1b}(2, 2; \tau)$. For $\tau = \mathrm{i}/3$, the first 100 coefficients of the q-series respectively give the numerical approximations

$$\Phi_{1a}(2,2;\mathrm{i}/3) = 0.577350\cdots \stackrel{?}{=} \frac{1}{\sqrt{3}}$$
$$\Phi_{1b}(2,2;\mathrm{i}/3) = 0.217095\ldots$$

Here we have that $\kappa_1(2,2) = 9$. Indeed, these values are not algebraic integers. Respectively, they are roots of

$$3x^2 - 1$$
$$19683x^{18} - 80919x^{12} - 39366x^9 + 11016x^6 - 486x^3 - 1.$$

However, Theorem 1.15 (2) applies, and we find that $\sqrt{3}\Phi_{1a}(2,2;\mathrm{i}/3)$ and $\sqrt{3}\Phi_{1b}(2,2;\mathrm{i}/3)$ are units. Respectively, they are roots of

$$x - 1$$
$$x^{18} + 6x^{15} - 93x^{12} - 304x^9 + 420x^6 - 102x^3 + 1.$$

Lastly, Theorem 1.15 (3) applies, and so their ratio

$$\frac{\Phi_{1a}(2,2;\mathrm{i}/3)}{\Phi_{1b}(2,2;\mathrm{i}/3)} = 4.60627\ldots$$

is a unit. Indeed, it is a root of

$$x^{18} - 102x^{15} + 420x^{12} - 304x^9 - 93x^6 + 6x^3 + 1.$$

Ramanujan's mock theta functions (Joint work with K. Ono and L. Rolen)

Ramanujan's collaborations with Hardy were fruitful, but were cut short. After falling deathly ill, he returned to India in 1919. Near the end of his life, Ramanujan wrote a final letter [25] to Hardy which included a list of 17 enigmatic functions which he referred to *mock theta functions*. Thanks to Zwegers [311, 312], it is now known that these functions are essentially the holomorphic parts of weight 1/2 *harmonic Maass forms* (see section 2.1) whose nonholomorphic parts are period integrals of weight 3/2 unary theta functions. This realization has many applications (e.g. [250, 306]).

Here we revisit Ramanujan's original definition from his deathbed letter [25]. After a discussion of the asymptotics of certain modular forms which are given as *Eulerian series*, he writes:

"...Suppose there is a function in the Eulerian form and suppose that all or an infinity of points $q = e^{2i\pi m/n}$ are exponential singularities and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is: - is the function taken the sum of two functions one of which is an ordinary theta function and the other a (trivial) function which is $O(1)$ at all the points $e^{2i\pi m/n}$? The answer is it is not necessarily so. When it is not so I call the function Mock ϑ-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singularities of the original function."

Remark. By *ordinary theta function*, Ramanujan meant a meromorphic modular form with $k \in \frac{1}{2}\mathbb{Z}$ on some $\Gamma_1(N)$ (see [249] for background), whose poles (if any) are supported at cusps. We refer to such forms as *weakly holomorphic modular forms.*

Little attention has been given to Ramanujan's original definition, prompting Berndt to remark [23] that "*it has not been proved that any of Ramanujan's mock theta functions are really mock theta functions according to his definition.*" The following fact fills in this gap.

Theorem 1.11. *Suppose that $f(z) = f^-(z) + f^+(z)$ is a harmonic Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_1(N)$, where $f^-(z)$ (resp. $f^+(z)$) is the nonholomorphic (resp. holomorphic) part of $f(z)$. If $f^-(z)$ is nonzero and $g(z)$ is a weight k weakly holomorphic modular form on any $\Gamma_1(N')$, then $f^+(z) - g(z)$*

has exponential singularities as q approaches infinitely many roots of unity ζ.

As a corollary, we obtain the following fitting conclusion to Ramanujan's enigmatic question by proving that his alleged examples indeed satisfy his original definition (Note. Throughout, we let $q := e^{2\pi i z}$). More precisely, we prove the following.

Corollary 1.12. *Suppose that $M(z)$ is one of Ramanujan's mock theta functions, and let γ and δ be integers for which $q^\gamma M(\delta z)$ is the holomorphic part of a weight 1/2 harmonic Maass form. Then there does not exist a weakly holomorphic modular form $g(z)$ of any weight $k \in \frac{1}{2}\mathbb{Z}$ on any congruence subgroup $\Gamma_1(N')$ such that for every root of unity ζ we have*

$$\lim_{q \to \zeta} \left(q^\gamma M(\delta z) - g(z)\right) = O(1).$$

Remark. The limits in Corollary 4.1 are *radial* limits taken from within the unit disk.

Example 1.13. Although a weakly holomorphic modular and a mock theta function cannot cut out each other's singularities, Ramanujan discusses a *near miss.* He considers his mock theta function

$$f(q) := 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \ldots, \tag{1.17}$$

and he compares it to a q-series $b(q)$ which is essentially a weight 1/2 weakly holomorphic modular form. He then conjectures, as q approaches an even order $2k$ primitive root of unity ζ, that

$$\lim_{q \to \zeta} \left(f(q) - (-1)^k b(q)\right) = O(1).$$

Watson confirmed this in [294], and Folsom, Ono, and Rhoades went further by deriving formulas for the $O(1)$ numbers as explicit numbers in $\mathbb{Z}[\zeta]$.

Weierstrass mock modular forms (Joint work C. Alfes, K. Ono, and L. Rolen)

The theory of *mock modular forms* provides the underlying theoretical framework for Ramanujan's enigmatic mock theta functions [31, 44, 309, 311]. Here we consider mock modular forms and the arithmetic of elliptic curves.

There is a canonical weight 0 harmonic Maass form which arises from the analytic realization of an elliptic curve $E/\mathbb{Q}$. This was first observed by Guerzhoy [151, 152]. To define it we recall that $E \cong \mathbb{C}/\Lambda_E$, where Λ_E is a 2-dimensional lattice in $\mathbb{C}$. The parameterization of E is given by $\mathfrak{z} \mapsto P_{\mathfrak{z}} = (\wp(\Lambda_E;\mathfrak{z}), \wp'(\Lambda_E;\mathfrak{z}))$, where

$$\wp(\Lambda_E;\mathfrak{z}) := \frac{1}{\mathfrak{z}^2} + \sum_{w\in\Lambda_E\setminus\{0\}} \left(\frac{1}{(\mathfrak{z}-w)^2} - \frac{1}{w^2}\right)$$

is the usual Weierstrass $\wp$-function for Λ_E. Here E is given by the Weierstrass equation

$$E\colon\ y^2 = 4x^3 - 60G_4(\Lambda_E)x - 140G_6(\Lambda_E),$$

where $G_{2k}(\Lambda_E) := \sum_{w\in\Lambda_E\setminus\{0\}} w^{-2k}$ is the classical weight $2k$ Eisenstein series. The canonical harmonic Maass form arises from the Weierstrass zeta-function

$$\zeta(\Lambda_E;\mathfrak{z}) := \frac{1}{\mathfrak{z}} + \sum_{w\in\Lambda_E\setminus\{0\}} \left(\frac{1}{\mathfrak{z}-w} + \frac{1}{w} + \frac{z}{w^2}\right) = \frac{1}{\mathfrak{z}} - \sum_{k=1}^{\infty} G_{2k+2}(\Lambda_E)\mathfrak{z}^{2k+1}. \tag{1.18}$$

This function already plays important roles in the theory of elliptic curves. The first role follows from the well-known "addition law"

$$\zeta(\Lambda_E;\mathfrak{z}_1+\mathfrak{z}_2) = \zeta(\Lambda_E;\mathfrak{z}_1) + \zeta(\Lambda_E;\mathfrak{z}_2) + \frac{1}{2}\frac{\wp'(\Lambda_E;\mathfrak{z}_1) - \wp'(\Lambda_E;\mathfrak{z}_2)}{\wp(\Lambda_E;\mathfrak{z}_1) - \wp(\Lambda_E;\mathfrak{z}_2)}, \tag{1.19}$$

which can be interpreted in terms of the "group law" of E.

To obtain the canonical forms from $\zeta(\Lambda_E;\mathfrak{z})$, we make use of the modularity of elliptic curves over $\mathbb{Q}$, which gives the modular parameterization

$$\phi_E : X_0(N_E) \to \mathbb{C}/\Lambda_E \cong E,$$

where N_E is the conductor of E. For convenience, we suppose throughout that E is a strong Weil curve. Let $F_E(z) = \sum_{n=1}^{\infty} a_E(n)q^n \in S_2(\Gamma_0(N_E))$ be the associated newform, and let $\mathcal{E}_E(z)$ be its *Eichler integral*

$$\mathcal{E}_E(z) := -2\pi i \int_z^{i\infty} F_E(\tau)d\tau = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} \cdot q^n. \tag{1.20}$$

Using an observation of Eisenstein, we define the function $\mathcal{Z}_E^+(\mathfrak{z})$ by

$$\mathcal{Z}_E^+(\mathfrak{z}) := \zeta(\Lambda_E; \mathfrak{z}) - S(\Lambda_E)\mathfrak{z}, \tag{1.21}$$

where

$$S(\Lambda_E) := \lim_{s\to 0^+} \sum_{w\in\Lambda_E\setminus\{0\}} \frac{1}{w^2|w|^{2s}}. \tag{1.22}$$

We define the nonholomorphic function $\mathcal{Z}_E(\mathfrak{z})$ by

$$\mathcal{Z}_E(\mathfrak{z}) := \mathcal{Z}_E^+(\mathfrak{z}) - \frac{\deg(\phi_E)}{4\pi||F_E||^2} \cdot \bar{\mathfrak{z}}, \tag{1.23}$$

where $||F_E||$ is the Petersson norm of F_E. Finally, we define the nonholomorphic function $\widehat{\mathfrak{Z}}_E(z)$ on $\mathbb{H}$ by the specialization of this function at $\mathfrak{z} = \mathcal{E}_E(z)$ given by

$$\widehat{\mathfrak{Z}}_E(z) = \widehat{\mathfrak{Z}}_E^+(z) + \widehat{\mathfrak{Z}}_E^-(z) := \mathcal{Z}_E(\mathcal{E}_E(z)). \tag{1.24}$$

In particular, the holomorphic part of $\widehat{\mathfrak{Z}}_E(z)$ is $\widehat{\mathfrak{Z}}_E^+(z) = \mathcal{Z}_E^+(\mathcal{E}_E(z))$.

Theorem 1.14. *Assume the notation and hypotheses above. The following are true:*

(1) The poles of $\widehat{\mathfrak{Z}}_E^+(z)$ are precisely those points z for which $\mathcal{E}_E(z) \in \Lambda_E$.

(2) If $\widehat{\mathfrak{Z}}_E^+(z)$ has poles in $\mathbb{H}$, then there is a canonical modular function $M_E(z)$ with algebraic coefficients on $\Gamma_0(N_E)$ for which $\widehat{\mathfrak{Z}}_E^+(z) - M_E(z)$ is holomorphic on $\mathbb{H}$.

(3) We have that $\widehat{\mathfrak{Z}}_E(z) - M_E(z)$ is a weight 0 harmonic Maass form on $\Gamma_0(N_E)$. In particular, $\widehat{\mathfrak{Z}}_E^+(z)$ is a weight 0 mock modular form.

Remark. Guerzhoy [152] has used such harmonic Maass forms in his work on the Kaneko-Zagier hypergeometric differential equation, and in [151] he studies their p-adic properties.

Remark. We refer to $\widehat{\mathfrak{Z}}_E^+(z)$ as the *Weierstrass mock modular form* for E. It is a simple task to compute this mock modular form. Using the two Eisenstein numbers $G_4(\Lambda_E)$ and $G_6(\Lambda_E)$, one then computes the remaining Eisenstein numbers using the recursion

$$G_{2n}(\Lambda_E) := \sum_{j=2}^{n-2} \frac{3(2j-1)(2n-2j-1)}{(2n+1)(2n-1)(n-3)} \cdot G_{2j}(\Lambda_E) G_{2n-2j}(\Lambda_E).$$

Armed with the Fourier expansion of $F_E(z)$ and $S(\Lambda_E)$, one then simply applies (1.20)-(1.24).

Remark. The number $\deg(\phi_E)$, which appears in (1.23), gives information about modular form congruences. The *congruence number* for E is the largest integer, say r_E, with the property that there is a $g \in S_2(\Gamma_0(N_E)) \cap \mathbb{Z}[[q]]$, which is orthogonal to F_E with respect to the Petersson inner product, which also satisfies $F_E \equiv g \pmod{r_E}$. A theorem of Ribet asserts that $\deg(\phi_E) \mid r_E$ (see Theorem 2.2 of [2]).

Many applications require the explicit Fourier expansions of harmonic Maass forms at cusps. The following theorem gives such expansions for the forms $\widehat{\mathfrak{Z}}_E(z)$ in Theorem 1.14 at certain cusps. These expansions follow from the fact that these forms transform nicely under $\Gamma_0^*(N_E)$, the extension of $\Gamma_0(N_E)$ by the Atkin-Lehner involutions. For each positive integer $q|N_E$ we have a determinant q^α matrix

$$W_q := \begin{pmatrix} q^\alpha a & b \\ N_E c & q^\alpha d \end{pmatrix}, \qquad (1.25)$$

where $q^\alpha || N_E$. By Atkin-Lehner Theory, there is a $\lambda_q \in \{\pm 1\}$ for which $F_E|_2 W_q = \lambda_q F_E$. The following result uses these involutions to give the

Fourier expansions of $\widehat{\mathfrak{Z}}_E(z)$ at cusps. When the level N is squarefree, the next theorem gives the expansion at all cusps of $\Gamma_0(N)$, which can be explicitly computed using (1.19).

Theorem 1.15. *If $q|N_E$, then*

$$\widehat{\mathfrak{Z}}_E(z)|_0 W_q = \mathcal{Z}_E^+(\lambda_q(\mathcal{E}_E(z) - \Omega_q(F_E))) - \frac{\deg(\phi_E)}{4\pi||F_E||^2} \cdot \overline{\lambda_q(\mathcal{E}_E(z) - \Omega_q(F_E))},$$

where we have

$$\Omega_q(F_E) := -2\pi i \int_{W_q^{-1} i\infty}^{i\infty} F_E(z) dz.$$

Remark. In particular, we have $\Omega_{N_E}(F_E) = L(F_E, 1)$. By the modular parameterization, we have that $\wp(\Lambda_E; \mathcal{E}_E(z))$ is a modular function on $\Gamma_0(N_E)$. We then have for each $q|N_E$ that $\Omega_q(F_E) \in r\Lambda_E$, where r is a rational number. This can be seen by considering the constant term of $\wp(\Lambda_E; \mathcal{E}_E(z))$ at cusps. The constant term of $\wp(\Lambda_E; \mathcal{E}_E(z))$ is $\wp(\Lambda_E; \Omega_q(F_E))$ (see Section 5.1.2 for more details). More generally, if N_E is square free, then $\Omega_q(F_E)$ maps to a rational torsion point of E.

As these facts illustrate, the harmonic Maass form $\widehat{\mathfrak{Z}}_E(z)$ and the mock modular form $\widehat{\mathfrak{Z}}_E^+(z)$ encode the degree of the modular parameterization ϕ_E, which in turns gives information about the congruence number r_E, and it encodes information about $\mathbb{Q}$-rational torsion.

By the work of Bruinier, Ono, and Rhoades [50] and Candelori [61], the coefficients of $\widehat{\mathfrak{Z}}_E^+(z)$ are $\mathbb{Q}$-rational when E has complex multiplication. For example, consider the elliptic curve $E\colon y^2 + y = x^3 - 38x + 90$ of conductor 361 with CM in the field $K = \mathbb{Q}(\sqrt{-19})$. We find

$$F_E(z) = q - 2q^4 - q^5 + 3q^7 - 3q^9 - 5q^{11} + 4q^{16} - 7q^{17} + \dots$$

and

$$\zeta(\Lambda_E; \mathcal{E}_E(z)) = q^{-1} + \frac{1}{2}q^2 - \frac{7}{3}q^3 + \frac{12}{5}q^5 + 4q^6 - \frac{6}{7}q^7 - \frac{27}{4}q^8 - \frac{13}{3}q^9 + \frac{17}{2}q^{10} + \dots.$$

As an illustration of this $\mathbb{Q}$-rationality, we find that $S(\Lambda_E) = -2$, which in turns gives

$$\widehat{\mathfrak{Z}}^+_E(z) = q^{-1} + 2q + \frac{1}{2}q^2 - \frac{7}{3}q^3 - q^4 + 2q^5 + 4q^6 - \frac{27}{4}q^8 - 5q^9 + \frac{17}{2}q^{10} + 14q^{11} - \dots.$$

This power series enjoys some deep p-adic properties with respect to Hecke operators. For example, it turns out that

$$\lim_{n\to+\infty} \frac{\left[q\frac{d}{dq}\zeta(\Lambda_E;\mathcal{E}_E(z))\right]|T(5^n)}{a_E(5^n)} = -2F_E(z)$$

as a 5-adic limit. To illustrate this phenomenon we offer:

$$\frac{\left[q\frac{d}{dq}\zeta(\Lambda_E;\mathcal{E}_E(z))\right]|T(5)}{a_E(5)} + 2F_E(z) \qquad = 5q^{-5} - 20q - 85q^2 - 430q^3 - \dots \qquad \equiv 0 \pmod{5}$$

$$\frac{\left[q\frac{d}{dq}\zeta(\Lambda_E;\mathcal{E}_E(z))\right]|T(5^2)}{a_E(5^2)} + 2F_E(z) \qquad = \tfrac{25}{4}q^{-25} - \tfrac{9525}{4}q - 2031975q^2 - \dots \qquad \equiv 0 \pmod{5^2}$$

$$\frac{\left[q\frac{d}{dq}\zeta(\Lambda_E;\mathcal{E}_E(z))\right]|T(5^3)}{a_E(5^3)} + 2F_E(z) \quad = -\tfrac{125}{9}q^{-125} - 89698470642375q + \dots \quad \equiv 0 \pmod{5^3}.$$

Our next result explains this phenomenon. There are such p-adic formulas for every E provided that $p \nmid N_E$ has the property that $p \nmid a_E(p)$ (i.e. p is ordinary). In analogy with recent work of Guerzhoy, Kent, and Ono [153], we obtain the following formulas.

Theorem 1.16. *If $p \nmid N_E$ is ordinary, then there is a constant $\mathfrak{S}_E(p)$ for which*

$$\lim_{n\to+\infty} \frac{\left[q\frac{d}{dq}\zeta(\Lambda_E;\mathcal{E}_E(z))\right]|T(p^n)}{a_E(p^n)} = \mathfrak{S}_E(p)F_E(z).$$

Remark. If E has CM in Theorem 1.16, then $\mathfrak{S}_E(p) = S(\Lambda_E)$ as rational numbers. In other cases $S(\Lambda_E)$ is expected to be transcendental, and one can interpret $\mathfrak{S}_E(p)$ as its p-adic expansion.

The harmonic Maass forms $\widehat{\mathfrak{Z}}_E(z)$ also encode much information about Hasse-Weil L-functions. The seminal works by Birch and Swinnerton-Dyer [26, 27] give an indication of this role in the case of CM elliptic curves. They obtained beautiful formulas for $L(E, 1)$, for certain CM elliptic curves, as finite sums of numbers involving special values of $\zeta(\Lambda_E, s)$. Such formulas have been generalized by many authors for CM elliptic curves (for example, see the famous papers by Damerell [89, 90]), and these generalizations have played a central role in the study of the arithmetic of CM elliptic curves.

Here we obtain results which show that the arithmetic of Weierstrass zeta-functions gives rise to deep information which hold for all elliptic curves $E/\mathbb{Q}$, not just those with CM. We prove that the canonical harmonic Maass forms $\widehat{\mathfrak{Z}}_E(z)$ "encode" the vanishing and nonvanishing of the central values $L(E_D, 1)$ and central derivatives $L'(E_D, 1)$ for the quadratic twist elliptic curves E_D of all modular elliptic curves.

The connection between these values and the theory of harmonic Maass forms was first made by Bruinier and Ono [47]. Their work proved that there are weight $1/2$ harmonic Maass forms whose coefficients give exact formulas for $L(E_D, 1)$, and which also encode the vanishing of $L'(E_D, 1)$. For central L-values their work relied on deep previous results of Shimura and Waldspurger. In the case of central derivatives, they made use of the theory of generalized Borcherds products and the Gross-Zagier Theorem. Bruinier [52] has recently refined this work by obtaining exact formulas involving periods of algebraic differentials.

The task of computing these weight $1/2$ harmonic Maass forms has been nontrivial. Natural difficulties arise (see [51]). These weight $1/2$ forms are preimages under $\xi_{1/2}$ of certain weight $3/2$ cusp forms, and as mentioned earlier, there are infinitely many such preimages. Secondly, the methods implemented to date for constructing such forms have relied on the theory of Poincaré series, forms whose coefficients are described as infinite sums of

Kloosterman sums weighted by Bessel functions. Establishing the convergence of these expressions can already pose difficulties. Moreover, there are infinitely many linear relations among Poincaré series.

Here we circumvent these issues. We construct canonical weight 1/2 harmonic Maass forms by making use of the canonical weight 0 harmonic Maass form $\widehat{\mathfrak{Z}}_E(z)$. More precisely, we define a twisted theta lift using the usual Siegel theta function modified by a simple polynomial. This function was studied by Hövel [166] in his Ph.D. thesis. The twisted lift $\mathcal{I}_{\Delta,r}(\bullet; z)$ (see Section 5.3) then maps weight 0 harmonic Maass forms to weight 1/2 harmonic Maass forms. Here Δ is a fundamental discriminant and r is an integer satisfying $r^2 \equiv \Delta \pmod{4N_E}$. For simplicity, we drop the dependence on Δ and r in the introduction. The canonical weight 1/2 harmonic Maass form we define is

$$f_E(z) := \mathcal{I}\left(\widehat{\mathfrak{Z}}^*_E(z) - M^*_E(z); z\right), \tag{1.26}$$

where $\widehat{\mathfrak{Z}}^*_E(z)$ and $M^*_E(z)$ denote a suitable normalization of $\widehat{\mathfrak{Z}}_E(z)$ and $M_E(z)$ (see Section 5.4). The normalization originates from the fact that we need the rationality of the principal part of f_E and we need to substract constant terms from the input. Following (2.1), we let

$$f_E(z) = f^+_E(z) + f^-_E(z) = \sum_{n \gg -\infty} c^+_E(n) q^n + \sum_{n<0} c^-_E(n) \Gamma\left(\frac{1}{2}, 4\pi|n|y\right) q^n. \tag{1.27}$$

Although we treat the general case in this paper (see Theorem 5.13), to simplify exposition, in the remainder of the introduction we shall assume that $N_E = p$ is prime, and we shall assume that the sign of the functional equation of $L(E,s)$ is $\epsilon(E) = -1$. Therefore, we have that $L(E,1) = 0$. The coefficients of f_E then satisfy the following theorem.

Theorem 1.17. *Suppose that $N_E = p$ is prime and that $\epsilon(E) = -1$. Then we have that $f_E(z)$ is a weight 1/2 harmonic Maass form on $\Gamma_0(4p)$. Moreover, the following are true:*

(1) If $d < 0$ is a fundamental discriminant for which $\left(\frac{d}{p}\right) = 1$, then

$$L(E_d, 1) = 0 \text{ if and only if } c_E^-(d) = 0.$$

(2) If $d > 0$ is a fundamental discriminant for which $\left(\frac{d}{p}\right) = 1$, then

$$L'(E_d, 1) = 0 \text{ if and only if } c_E^+(d) \text{ is in } \mathbb{Q}.$$

Remark. Assume that E is as in Theorem 1.17. By work of Kolyvagin [195] and Gross and Zagier [149] on the Birch and Swinnerton-Dyer Conjecture, we then have the following for fundamental discriminants d:

1. If $d < 0$, $\left(\frac{d}{p}\right) = 1$, and $c_E^-(d) \neq 0$, then the rank of $E_d(\mathbb{Q})$ is 0.
2. If $d > 0$, $\left(\frac{d}{p}\right) = 1$, and $c_E^+(d)$ is transcendental, then the rank of $E_d(\mathbb{Q})$ is 1.

Criterion (1) is analogous to Tunnell's [289] work on the *Congruent Number Problem.*

Remark. Theorem 1.17 follows from exact formulas. In particular, Theorem 1.17 (1) follows from the exact formula

$$L(E_d, 1) = 8\pi^2 ||F_E||^2 \cdot ||g_E||^2 \cdot \sqrt{\frac{|d|}{p}} \cdot c_E^-(d)^2.$$

Here g_E is the weight 3/2 cusp form which is the image of $f_E(z)$ under the differential operator $\xi_{\frac{1}{2}}$ (see (5.9)). More precisely, we require that $\xi_{1/2}(f_E) = ||g_E||^{-2} g_E$ (resp. $\xi_{1/2}(f_E) \in \mathbb{R} \cdot g_E$). Theorem 1.17 (2) is also related to exact formulas, ones involving periods of algebraic differentials. Recent work by Bruinier [52] establishes that

$$c_E^+(d) = \frac{\Re \int_{C_{F_E}} \zeta_d(f_E)}{\sqrt{d} \int_{C_{F_E}} \omega_{F_E}},$$

where $\zeta_d(f_E)$ is the normalized differential of the third kind for a certain divisor associated to f_E and $\omega_{F_E} = 2\pi i F_E(z)dz$. Here C_{F_E} is a generator of the F_E-isotypical component of the first homology of X. The interested reader should consult [52] for further details.

Theorem 1.17 follows from a general result on the theta lift $\mathcal{I}(\bullet, z)$ we define in Section 5.3. Earlier work of Bruinier and Funke [55], Alfes and Ehlen [6], and more recent work of Alfes [4] and of Bruinier and Ono [57], consider similar theta lifts which implement the Kudla-Millson theta function as the kernel function. Those works give lifts which map weight $-2k$ forms to weight $3/2+k$ forms when k is even. For odd k, these lifts map to weight $1/2-k$ forms. The new theta lift here makes use of the usual Siegel theta kernel which is modified with a simple polynomial. Using this weight 1/2 function Hövel [166] defined a theta lift going in the direction "opposite" to ours, i.e. from forms for the symplectic group to forms for the orthogonal group.

We prove that the lift we consider maps weight 0 forms to weight 1/2 forms. Moreover, it satisfies Hecke equivariant commutative diagrams, involving $\xi_0, \xi_{1/2}$ and the Shintani lift, of the form:

$$\begin{array}{ccc} \widehat{\mathfrak{Z}}^*_E(z) - M_E(z) & \xrightarrow{\xi_0} & F_E \\ \downarrow \mathcal{I} & & \downarrow \text{Shin} \\ \mathcal{I}(\widehat{\mathfrak{Z}}^*_E(z) - M^*_E(z);\tau) & \xrightarrow{\xi_{1/2}} & \mathbb{R}\cdot g_E. \end{array}$$

Here g_E is the weight 3/2 cusp form in Remark 8.

Remark. It turns out that the coefficients $c_E^+(n)$ of $f_E(\tau)$ are "twisted traces" of the singular moduli for the weight 0 harmonic Maass form $\widehat{\mathfrak{Z}}^*_E(z) - M^*_E(z)$. This is Theorem 5.10. This phenomenon is not new. Seminal works by Zagier [308] and Katok and Sarnak [185], followed by subsequent works by Bringmann, Bruinier, Duke, Funke, Imamoḡlu, Jenkins, Miller, Pixton, and Tóth [42, 55, 56, 102–105, 237], among many others, give situations where

Fourier coefficients are such traces. In particular, we obtain (vector valued versions of) the generating functions for the twisted traces of the j-invariant that Zagier called f_d, where d is a fundamental discriminant, in [308]. We explain this in more detail in Example 5.16.

Example 1.18. In Section 5.5 we shall consider the conductor 37 elliptic curve

$$E: \quad y^2 - y = x^3 - x.$$

The sign of the functional equation of $L(E, s)$ is -1, and $E(\mathbb{Q})$ has rank 1.

The table below illustrates Theorem 1.17, and its implications for ranks of elliptic curves.

d	$c^+(d)$	$L'(E_d, 1)$	$\mathrm{rk}(E_d(\mathbb{Q}))$
1	$-0.2817617849\ldots$	$0.3059997738\ldots$	1
12	$-0.4885272382\ldots$	$4.2986147986\ldots$	1
21	$-0.1727392572\ldots$	$9.0023868003\ldots$	1
28	$-0.6781939953\ldots$	$4.3272602496\ldots$	1
33	$0.5663023201\ldots$	$3.6219567911\ldots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1489	9	0	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4393	66	0	3

For the d in the table we have that the sign of the functional equation of $L(E_d, s)$ is -1. Therefore, if $L'(E_d, 1) \neq 0$, then we have that $\mathrm{ord}_{s=1}(L(E_d, s)) = 1$, which then implies that $\mathrm{rk}(E_d(\mathbb{Q})) = 1$ by Kolyvagin's Theorem. For such d, Theorem 1.17 asserts that $L'(E_d, 1) = 0$ if and only if $c_E^+(d) \in \mathbb{Q}$. Therefore, for these d the Birch and Swinnerton-Dyer Conjecture implies that $\mathrm{rk}(E_d(\mathbb{Q})) \geq 3$ is odd if and only if $c_E^+(d) \in \mathbb{Q}$. We note that for $d \in \{1489, 4393\}$, we find[2] that the curves have rank 3.

SU(2)-**Donaldson invariants** (Joint work with A. Malmendier and K. Ono)

Maass forms have also found deep applications to the study of differential topology. Donaldson invariants of smooth simply connected four-dimensional manifolds [97] are diffeomorphism class invariants which play a central role in differential topology and mathematical physics. There are two families of Donaldson invariants, corresponding to the SU(2)-gauge theory and the SO(3)-gauge theory with non-trivial Stiefel-Whitney class. In each case, the invariants are graded homogeneous polynomials on the homology $H_0(\mathbb{C}\mathrm{P}^2) \oplus H_2(\mathbb{C}\mathrm{P}^2)$, where $H_i(\mathbb{C}\mathrm{P}^2)$ is considered to have degree $(4-i)/2$, defined using the fundamental homology classes of the corresponding moduli spaces of anti-selfdual instantons arising in gauge theory. These invariants are typically very difficult to calculate.

Here we consider the simplest manifold to which Donaldson's definition applies, the complex projective plane $\mathbb{C}\mathrm{P}^2$ with the Fubini-Study metric. In earlier work, Göttsche and Zagier [144] gave a formula for the Donaldson invariants of rational surfaces in terms of theta functions of indefinite lattices. As an application, Göttsche [141] derived closed expressions for the two families of the Donaldson invariants of $\mathbb{C}\mathrm{P}^2$ assuming the truth of the Kotschick-Morgan conjecture. Recently, Göttsche, Nakajima, Hiraku, and Yoshioka [143] have unconditionally proved these formulas.

Deep conjectures exist which relate these formulas to constructions in theoretical physics. From the viewpoint of theoretical physics [301], these two families of Donaldson invariants and the related Seiberg-Witten invariants are the correlation functions of a supersymmetric topological gauge theory with gauge group SU(2) and SO(3). Witten [300] argued that one should be

[2] These computations were done using `Sage` [277] by Bruinier and Strömberg in [51]. Stephan Ehlen obtained the same numbers using our results (also using `Sage`).

able to compute these correlation functions in a so called *low energy effective field theory*. This theory has the advantage of being an *abelian* $\mathcal{N} = 2$ supersymmetric topological gauge theory, and the data required to define the theory only involves line bundles of even (resp. odd) first Chern class on $\mathbb{C}\mathrm{P}^2$ if the gauge group is SU(2) (resp. SO(3)). The vacua of the low energy effective field theory are parametrized by the u-plane which Seiberg and Witten [268] describe in terms of the classical modular curve $\mathbb{H}/\Gamma_0(4)$, together with a meromorphic one-form. Finally, Moore and Witten [241] obtained the correlation functions as regularized integrals over the u-plane, where the integrands are modular functions which are determined by the gauge group. These regularized u-plane integrals define a way of extracting certain contributions for each boundary component near the cusps at $\tau = 0, 2, \infty$ of the modular curve, and Moore and Witten observed [241] that the cuspidal contributions at $\tau = 0, 2$ vanish trivially. This vanishing corresponds to the mathematical statement that the Seiberg-Witten invariants on $\mathbb{C}\mathrm{P}^2$ vanish due to the presence of a Fubini-Study metric of positive scalar curvature [302].

Concerning the contribution from the cusp $\tau = \infty$, Moore and Witten made the following deep conjecture which relates the u-plane integral to Donaldson invariants.

Conjecture 1.19 (Moore and Witten [241]). The contribution at $\tau = \infty$ to the regularized u-plane integral is the generating function for the Donaldson invariants of $\mathbb{C}\mathrm{P}^2$.

As evidence for this conjecture, in the case of the gauge group SU(2), Moore and Witten [241] computed the first 40 invariants and found them to be in agreement with the results of Ellingsrud and Göttsche [115]. In recent work, Ono and Malmendier [227] proved this conjecture in the SO(3) case. Here we complete the proof of the conjecture by confirming the claim for the gauge group SU(2). We prove the following theorem.

Theorem 1.20. *The conjecture of Moore and Witten in the case of the* SU(2)*-gauge theory on* $\mathbb{C}\mathrm{P}^2$ *is true.*

Moonshine (Joint work with J. F. Duncan and K. Ono)

Our final application of the theory of harmonic Maass forms is to the theory of monstrous moonshine which once again connects the theory of modular and mock modular forms with representation theory, and in recent years, with important applications to mathematical physics.

The classification of finite simple groups [13] distinguishes twenty-six examples above the others; namely, the *sporadic* simple groups, which are those that belong to none of the naturally occurring infinite families: cyclic groups, alternating groups, or finite groups of Lie type. Distinguished amongst the sporadic simple groups is the *Fischer–Griess monster* $\mathbb{M}$, on account of its size, which is

$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \quad (1.28)$$

(cf. [145]). Note that the margin is not small, for the order of the monster is

$$2^5 \cdot 3^7 \cdot 5^3 \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 29 \cdot 41 \cdot 59 \cdot 71 \quad (1.29)$$

times that of the next largest sporadic simple group, the baby monster (cf. [203]).

Fischer and Griess independently produced evidence for the monster group in 1973 (cf. [145]). Well before it was proven to exist, Tits gave a lecture on its conjectural properties at the Collège de France in 1975. In particular, he described its order (1.28). Around this time, Ogg had been considering the automorphism groups of certain algebraic curves, and had arrived at the set of primes

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\} \quad (1.30)$$

in a purely geometric way (cf. the Corollaire of [248]). Making what may now be identified as the first observation of monstrous moonshine, Ogg offered a bottle of Jack Daniels [3] for an explanation of this coincidence (cf. Remarque 1 of [248]).

Ogg's observation would ultimately be recognized as reflecting another respect in which the monster is distinguished amongst finite simple groups: as demonstrated by the pioneering construction of Frenkel–Lepowsky–Meurman [125–127], following the astonishing work of Griess [146, 147], the "most natural" representation of the monster, is infinite-dimensional.

The explanation of this statement takes us back to McKay's famous observation, that

$$196884 = 1 + 196883 \tag{1.31}$$

(cf. [81, 284]), and the generalizations of this observed by Thompson [284], including

$$\begin{aligned} 21493760 &= 1 + 196883 + 21296876, \\ 864299970 &= 2 \times 1 + 2 \times 196883 + 21296876 + 842609326, \\ 20245856256 &= 3 \times 1 + 3 \times 196883 + 21296876 + 2 \times 842609326 + 18538750076. \end{aligned} \tag{1.32}$$

Of course the left hand sides of (1.31) and (1.32) are familiar to number theorists and algebraic geometers, as coefficients in the Fourier coefficients of the *normalized elliptic modular invariant*

$$\begin{aligned} J(\tau) &:= \frac{1728 g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2} - 744 \\ &= q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \ldots \end{aligned} \tag{1.33}$$

[3] We refer the reader to [109] for a recent analysis of the Jack Daniels problem.

Here $q := e^{2\pi i\tau}$, and we set $g_2(\tau) := 60G_4(\tau)$ and $g_3(\tau) := 140G_6(\tau)$, where $G_{2k}(\tau)$ denotes the Eisenstein series of weight $2k$,

$$G_{2k}(\tau) := \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-2k}, \tag{1.34}$$

for $k \geq 2$. The functions g_2 and g_3 serve to translate between the two most common parameterizations of a complex elliptic curve: as a quotient $\mathrm{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ for τ in the upper-half plane, $\mathbb{H} := \{\tau \in \mathrm{C} \mid \Im(\tau) > 0\}$, and as the locus of a Weierstrass equation, $y^2 = 4x^3 - g_2x - g_3$.

The fundamental property of $J(\tau)$, from both the number theoretic and algebro-geometric points of view, is that it is a modular function for $SL_2(\mathbb{Z})$. In fact, and importantly for the monster's natural infinite-dimensional representation, $J(\tau)$ is a generator for the field of $SL_2(\mathbb{Z})$-invariant holomorphic functions on $\mathbb{H}$ that have at most exponential growth as $\Im(\tau) \to \infty$.

The right hand sides of (1.31) and (1.32) are familiar to finite group theorists, as simple sums of dimensions of irreducible representations of the monster $\mathbb{M}$. In fact, the irreducible representations appearing in (1.31) and (1.32) are just the first five, of a total of 194, in the character table of $\mathbb{M}$ (cf. [80]), when ordered by size. We have that

$$\begin{aligned}
\chi_1(e) &= 1 \\
\chi_2(e) &= 196883 \\
\chi_3(e) &= 21296876 \\
\chi_4(e) &= 842609326 \\
\chi_5(e) &= 18538750076 \\
&\vdots \\
\chi_{194}(e) &= 258823477531055064045234375.
\end{aligned} \tag{1.35}$$

Here e denotes the identity element of $\mathbb{M}$, so $\chi_i(e)$ is just the dimension of the irreducible representation of $\mathbb{M}$ with character χ_i.

The coincidences (1.31) and (1.32) led Thompson to make the following conjecture [284] which realizes the natural representation of the monster alluded to above.

Conjecture 1.21 (Thompson). There is a naturally defined graded infinite-dimensional monster module, denoted $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$, which satisfies

$$\dim(V_n^\natural) = c(n) \tag{1.36}$$

for $n \geq -1$ (Cf. (1.33)), such that the decompositions into irreducible representations of the monster satisfy (1.31) and (1.32) for $n = 1, 2, 3$ and 4 (and a similar condition for $n = 5$).

At the time that Thompson's conjecture was made, the monster had not yet been proven to exist, but Griess [145], and Conway–Norton [81], had independently conjectured the existence of a faithful representation of dimension 196883, and Fischer–Livingstone–Thorne had constructed the character table of $\mathbb{M}$, by assuming the validity of this claim (cf. [81]) together with conjectural statements (cf. [145]) about the structure of $\mathbb{M}$.

Thompson also suggested [283] to investigate the properties of the graded-trace functions

$$T_g(\tau) := \sum_{n=-1}^{\infty} \operatorname{tr}(g|V_n^\natural) q^n, \tag{1.37}$$

for $g \in \mathbb{M}$, now called the *monstrous McKay–Thompson series*, that would arise from the conjectural monster module $V^\natural$. Using the character table constructed by Fischer–Livingstone–Thorne, it was observed [81, 283] that the functions T_g are in many cases directly similar to J in the following respect: the first few coefficients of each T_g coincide with those of a generator for the function field of a discrete group[4] $\Gamma_g < SL_2(\mathbb{R})$, commensurable with

[4] The relevant groups Γ_g shall be discussed in detail in Section 7.6.1.

$SL_2(\mathbb{Z})$, containing $-I$, and having *width one at infinity*, meaning that the subgroup of upper-triangular matrices in Γ_g coincides with

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}, \tag{1.38}$$

for all $g \in \mathbb{M}$.

This observation was refined and developed by Conway–Norton [81], leading to their famous *monstrous moonshine conjectures*:.

Conjecture 1.22 (Monstrous Moonshine: Conway–Norton). For each $g \in \mathbb{M}$ there is a specific group $\Gamma_g < SL_2(\mathbb{R})$ such that T_g is the unique *normalized principal modulus*[5] for Γ_g.

This means that each T_g is the unique Γ_g-invariant holomorphic function on $\mathbb{H}$ which satisfies

$$T_g(\tau) = q^{-1} + O(q), \tag{1.39}$$

as $\Im(\tau) \to \infty$, and remains bounded as τ approaches any non-infinite cusp of Γ_g. We refer to this feature of the T_g as the *principal modulus property* of monstrous moonshine.

The hypothesis that T_g is Γ_g-invariant, satisfying (1.39) near the infinite cusp of Γ_g but having no other poles, implies that T_g generates the field of Γ_g-invariant holomorphic functions on $\mathbb{H}$ that have at most exponential growth at cusps, in direct analogy with J. In particular, the natural Riemann surface structure on $\Gamma_g \backslash \mathbb{H}$ (cf. e.g. [272]) must be that of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with finitely many points removed, and for this reason the groups Γ_g are said to have *genus zero*, and the principal modulus property is often referred to as the *genus zero property* of monstrous moonshine.

The reader will note the astonishing predictive power that the principal modulus property of monstrous moonshine bestows: the fact that a normalized principal modulus for a genus zero group Γ_g is unique, means that we

[5] A principal modulus is also referred to as a *Hauptmodul*.

can compute the trace of an element $g \in \mathbb{M}$, on any homogeneous subspace of the monster's natural infinite-dimensional representation $V^\natural$, without any information about the monster, as soon as we can guess correctly the discrete group Γ_g. The analysis of Conway–Norton in [81] establishes very strong guidelines for the determination of Γ_g, and once Γ_g has been chosen, the "theory of replicability" (cf. [3, 81, 245]) allows for efficient computation of the coefficients of the normalized principal modulus T_g, given the knowledge of just a few of them (cf. [122], or (7.6)).

It was verified by Atkin–Fong–Smith [275], using results of Thompson [283] (cf. also [261]), that a graded (possibly virtual) infinite-dimensional monster module $V^\natural$, such that the functions T_g of (1.37) are exactly those predicted by Conway–Norton in [81], exists.

Theorem 1.23 (Atkin–Fong–Smith)**.** *There exists a (possibly virtual) graded* $\mathbb{M}$*-module* $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ *such that if* T_g *is defined by (1.37), then* T_g *is the Fourier expansion of the unique* Γ_g*-invariant holomorphic function on* $\mathbb{H}$ *that satisfies* $T_g(\tau) = q^{-1} + O(q)$ *as* τ *approaches the infinite cusp, and has no poles at any non-infinite cusps of* Γ_g*, where* Γ_g *is the discrete subgroup of* $SL_2(\mathbb{R})$ *specified by Conway–Norton in [81].*

Thus Thompson's conjecture was confirmed, albeit indirectly. By this point in time, Griess, in an astonishing tour de force, had constructed the monster explicitly, by hand, by realizing it as the automorphism group of a commutative but non-associative algebra of dimension 196884 [146,147]. (See also [79, 285].) Inspired by Griess' construction, and by the representation theory of affine Lie algebras, which also involves graded infinite-dimensional vector spaces whose graded dimensions enjoy good modular properties (cf. e.g. [178–180, 184]), Frenkel–Lepowsky–Meurman established Thompson's conjecture in a strong sense.

Theorem 1.24 (Frenkel–Lepowsky–Meurman)**.** *Thompson's Conjecture is* *true. In particular, the* moonshine module $V^\natural$ *is constructed in [125, 127].*

Frenkel–Lepowsky–Meurman generalized the homogeneous realization of the basic representation of an affine Lie algebra $\hat{\mathfrak{g}}$ due, independently, to Frenkel–Kac [124] and Segal [267], in such a way that *Leech's lattice* Λ [201, 202]—the unique [78] even self-dual positive-definite lattice of rank 24 with no roots—could take on the role played by the root lattice of $\mathfrak{g}$ in the Lie algebra case. In particular, their construction came equipped with rich algebraic structure, furnished by vertex operators, which had appeared first in the physics literature in the late 1960's.

We refer to [124], and also the introduction to [126] for accounts of the role played by vertex operators in physics (up to 1988) along with a detailed description of their application to the representation theory of affine Lie algebras. The first application of vertex operators to affine Lie algebra representations was obtained by Lepowsky–Wilson in [210].

Borcherds described a powerful axiomatic formalism for vertex operators in [29]. In particular, he introduced the notion of a *vertex algebra*, which can be regarded as similar to a commutative associative algebra, except that multiplications depend upon formal variables z_i, and can be singular, in a certain formal sense, along the canonical divisors $\{z_i = 0\}$, $\{z_i = z_j\}$ (cf. [32, 123]). This lead eventually to Borcherds' proof of monstrous moonshine.

Theorem 1.25 (Borcherds)**.** *Let $V^\natural$ be the moonshine module vertex operator algebra constructed by Frenkel–Lepowsky–Meurman, whose automorphism group is $\mathbb{M}$. If T_g is defined by (1.37) for $g \in \mathbb{M}$, and if Γ_g is the discrete subgroup of $SL_2(\mathbb{R})$ specified by Conway–Norton in [81], then T_g is the unique normalized principal modulus for Γ_g.*

For more on vertex operator algebras and the proof of Frenkel–Lepowsky–Meurman theorem and Borcherd's theorem, see section 7.1.

Witten was the first to predict a role for the monster in quantum gravity. In [304] Witten considered *pure quantum gravity* in three dimensions with negative cosmological constant, and presented evidence that the moonshine module $V^\natural$ is a chiral half of the conformal field theory dual to such a quantum gravity theory, at the most negative possible value of the cosmological constant.

To explain some of the content of this statement, note that the action in pure three-dimensional quantum gravity is given explicitly by

$$I_{\mathrm{EG}} := \frac{1}{16\pi G}\int \mathrm{d}^3x\sqrt{-g}(R-2\Lambda), \tag{1.40}$$

where G is the *Newton* or *gravitational constant*, R denotes the Ricci scalar, and the *cosmological constant* is the scalar denoted by Λ.

The case that the cosmological constant Λ is negative is distinguished, since then there exist black hole solutions to the action (1.40), as was discovered by Bañados–Teitelboim–Zanelli [18]. These black hole solutions—the *BTZ black holes*—are locally isomorphic to three-dimensional anti-de Sitter space [17], which is a Lorentzian analogue of hyperbolic space, and may be realized explicitly as the universal cover of a hyperboloid

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 + X_3^3 = -\ell^2 \tag{1.41}$$

in $\mathbb{R}^{2,3}$ (cf. e.g. [222]). The parameter ℓ in (1.41) is called the *radius of curvature*. For a locally anti-de Sitter (AdS) solution to (1.40), the radius of curvature is determined by the cosmological constant, according to

$$\ell^2 = -1/\Lambda. \tag{1.42}$$

In what has become the most cited[6] paper in the history of high energy physics, Maldacena opened the door on a new, and powerful approach to

[6] Maldacena's groundbreaking paper [223] on the gauge/gravity duality has over 10,000 citations at the time of writing, according to `inspirehep.net`.

quantum gravity in [223], by presenting evidence for a gauge/gravity duality, in which gauge theories serve as duals to gravity theories in one dimension higher. (See [222] for a recent review.) In the simplest examples, the gauge theories are conformal field theories, and the gravity theories involve locally AdS spacetimes. The gauge/gravity duality for these cases is now known as the *AdS/CFT correspondence*.

Maldacena's duality furnishes a concrete realization of the *holographic principle*, introduced by 't Hooft [12], and elaborated on by Susskind [282]. For following refinements to Maldacena's proposal due to Gubser–Klebanov–Polyakov [150], and Witten [303], it is expected that gravity theories with $(d+1)$-dimensional locally AdS spacetimes can be understood through the analysis of d-dimensional conformal field theories defined on the boundaries of these AdS spaces. Thus in the case of AdS solutions to three-dimensional quantum gravity, a governing role may be played by two-dimensional conformal theories, which can be accessed mathematically via vertex operator algebras (as we have mentioned previously).

The conjecture of [304] is that the two-dimensional conformal field theory corresponding to a tensor product of two copies of the moonshine module $V^\natural$ (one "left-moving," the other "right-moving") is the holographic dual to pure three-dimensional quantum gravity with $\ell = 16G$, and therefore

$$\Lambda = -\frac{1}{256G^2}. \tag{1.43}$$

It is also argued that the only physically consistent values of ℓ are $\ell = 16Gm$, for m a positive integer, so that (1.43) is the most negative possible value for Λ, by force of (1.42).

Shortly after this conjecture was formulated, problems with the quantum mechanical interpretation were identified by Maloney–Witten in [229]. Moreover, Gaiotto [134] and Höhn [163] cast doubt on the relevance of the monster to gravity by demonstrating that it cannot act on a holographically dual conformal field theory corresponding to $\ell = 32G$ (i.e. $m = 2$), at least under the hypotheses (namely, an extremality condition, and holomorphic factorization) presented in [304].

Interestingly, the physical problems with the analysis of [304] seem to disappear in the context of *chiral three-dimensional gravity*, which was introduced and discussed in detail by Li–Song–Strominger in [215] (cf. also [228, 279]). This is the gravity theory which motivates much of the discussion in §7 of [110].

In order to define chiral three-dimensional gravity, we first describe *topologically massive gravity*, which was introduced in 1982 by Deser–Jackiw–Templeton [92, 265]. (See also [91].) The action for topologically massive gravity is given by

$$I_{\mathrm{TMG}} := I_{\mathrm{EG}} + I_{\mathrm{CSG}}, \tag{1.44}$$

where I_{EG} is the Einstein–Hilbert action (cf. (1.40)) of pure quantum gravity, and I_{CSG} denotes the *gravitational Chern–Simons term*

$$I_{\mathrm{CSG}} := \frac{1}{32\pi G\mu}\int \mathrm{d}^3x\sqrt{-g}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}{}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}{}_{\mu\tau}\Gamma^{\tau}{}_{\nu\rho}\right). \tag{1.45}$$

The $\Gamma^{*}{}_{**}$ are Christoffel symbols, and the parameter μ is called the *Chern–Simons coupling constant*.

Chiral three-dimensional gravity is the special case of topologically massive gravity in which the Chern–Simons coupling constant is set to $\mu = 1/\ell = \sqrt{-\Lambda}$. It is shown in [215] that at this special value of μ, the left-moving central charges of the boundary conformal field theories vanish, and the right-moving central charges are

$$c = \frac{3\ell}{2G} = 24m, \tag{1.46}$$

for m a positive integer, $\ell = 16Gm$.

Much of the analysis of [304] still applies in this setting, and the natural analogue of the conjecture mentioned above states that $V^\natural$ is holographically dual to chiral three-dimensional quantum gravity at $\ell = 16G$, i.e. $m = 1$. However, as argued in detail in [228], the problem of quantizing chiral three-dimensional gravity may be regarded as equivalent to the problem of constructing a sequence of extremal chiral two-dimensional conformal field theories (i.e. vertex operator algebras), one for each central charge $c = 24m$, for m a positive integer. Here, a vertex operator algebra $V = \bigoplus_n V_n$ with central charge $c = 24m$ is called *extremal*, if its graded[7] dimension function satisfies

$$\sum_{n\in\mathbb{Z}} \dim(V_n)q^n = q^{-m}\frac{1}{\prod_{n>1}(1-q^n)} + O(q). \tag{1.47}$$

The moonshine module is the natural candidate for $m = 1$ (indeed, it is the only candidate if we assume the uniqueness conjecture of [126]), as the right hand side of (1.47) reduces to $q^{-1} + O(q)$ in this case, but the analysis of [134, 163] also applies here, indicating that the monster cannot act non-trivially on any candidate[8] for $m = 2$. Thus the role of the monster in quantum gravity is still unclear, even in the more physically promising chiral gravity setting.

Nonetheless, the moonshine module $V^\natural$ may still serve as the holographic dual to chiral three-dimensional quantum gravity at $\ell = 16G$, $m = 1$. In this interpretation, the graded dimension, or *genus one partition function* for $V^\natural$—namely, the elliptic modular invariant J—serves as the exact spectrum of physical states of chiral three-dimensional gravity at $\mu = \sqrt{-\Lambda} = 1/16G$, in spacetime asymptotic to the three-dimensional anti-de Sitter space (cf. (1.41)).

[7]We regard all vertex operator algebras as graded by $L(0) - \mathbf{c}/24$. Cf. (7.1).

[8]The existence of extremal vertex operator algebras with central charge $c = 24m$ for $m > 1$ remains an open question. We refer to [133, 135, 163, 305] for analyses of this problem.

Recall that if V is a representation of the Virasoro algebra $\mathcal{V}$ (cf. (7.1)), then $v \in V$ is called a *Virasoro highest weight vector* with *highest weight* $h \in \mathrm{C}$ if $L(m)v = h\delta_{m,0}v$ whenever $m \geq 0$. A *Virasoro descendant* is a vector of the form

$$L(m_1)\cdots L(m_k)v, \tag{1.48}$$

where v is a Virasoro highest weight vector, and $m_1 \leq \cdots \leq m_k \leq -1$.

Assuming that $V^\natural$ is dual to chiral three-dimensional gravity at $m = 1$, the Virasoro highest weight vectors in $V^\natural$ define operators that create black holes, and the Virasoro descendants of a highest weight vector describe black holes embellished by boundary excitations. In particular, the 196883-dimensional representation of the monster which is contained in the 196884-dimensional homogenous subspace $V_1^\natural < V^\natural$ (cf. (1.31) and (1.36)), represents an 196883-dimensional space of black hole states in the chiral gravity theory.

More generally, the black hole states in the theory are classified, by the monster, into 194 different kinds, according to which monster irreducible representation they belong to.

Question 1.26. Assuming that the moonshine module $V^\natural$ serves as the holographic dual to chiral three-dimensional quantum gravity at $m = 1$, how are the 194 different kinds of black hole states distributed amongst the homogeneous subspaces $V_n^\natural < V^\natural$. Are some kinds of black holes more or less common than others?

This question will be answered precisely by Corollary 1.28. Monstrous moonshine implies that the McKay–Thompson series can be written as

$$T_g(\tau) = q^{-1} + \sum_{n=1}^{\infty}\sum_{i=1}^{194} \mathbf{m}_i(n)\chi_i(g)q^n,$$

where the χ_i range over the 194 irreducible characters for the monster. In section 7.6, we find exact formulas for the multiplicities $\mathbf{m}_i(n)$, which lead to the following asymptotics.

Theorem 1.27. *If* $1 \leq i \leq 194$*, then as* $n \to +\infty$ *we have*

$$\mathbf{m}_i(n) \sim \frac{\dim(\chi_i)|m|^{1/4}}{\sqrt{2}|n|^{3/4}|\mathbb{M}|} \cdot e^{4\pi\sqrt{|mn|}}$$

These asymptotics immediately imply that the following limits are well-defined

$$\delta\left(\mathbf{m}_i\right) := \lim_{n \to +\infty} \frac{\mathbf{m}_i(n)}{\sum_{i=1}^{194} \mathbf{m}_i(n)} \tag{1.49}$$

Corollary 1.28. *In particular, we have that*

$$\delta\left(\mathbf{m}_i\right) = \frac{\dim(\chi_i)}{\sum_{j=1}^{194} \dim(\chi_j)} = \frac{\dim(\chi_i)}{5844076785304502808013602136}.$$

Remark. Theorem 1.27 and Corollary 1.28 are the $m = 1$ cases of Theorem 7.10 and Corollary 7.11 respectively.

We illustrate these asymptotics explicitly, for χ_1, χ_2, and χ_{194} in Table 1.1. The precise values given in the bottom row of Table 1.1 admit the following decimal approximations:

$$\begin{aligned} \delta\left(\mathbf{m}_1\right) &= \frac{1}{5844076785304502808013602136} \approx 1.711\ldots \times 10^{-28} \\ \delta\left(\mathbf{m}_2\right) &= \frac{196883}{5844076785304502808013602136} \approx 3.368\ldots \times 10^{-23} \\ \delta\left(\mathbf{m}_{194}\right) &= \frac{258823477531055064045234375}{5844076785304502808013602136} \approx 4.428\ldots \times 10^{-2} \end{aligned} \tag{1.50}$$

Umbral Moonshine (Joint work with J. F. Duncan and K. Ono)

In 2010, Eguchi, Ooguri, and Tachikawa reignited moonshine with their observation [114] that dimensions of some representations of M_{24}, the largest sporadic simple Mathieu group (cf. e.g. [80, 83]), are multiplicities of superconformal algebra characters in the K3 elliptic genus. This observation

Table 1.1

n	$\delta\left(\mathbf{m}_1(-1,n)\right)$	$\delta\left(\mathbf{m}_2(-1,n)\right)$	$\cdots$	$\delta\left(\mathbf{m}_{194}(-1,n)\right)$
-1	1	0	$\cdots$	0
0	0	0	$\cdots$	0
1	$1/2$	$1/2$	$\cdots$	0
2	$1/3$	$1/3$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
40	$4.011\ldots \times 10^{-4}$	$2.514\ldots \times 10^{-3}$	$\cdots$	$0.00891\ldots$
60	$2.699\ldots \times 10^{-9}$	$2.732\ldots \times 10^{-8}$	$\cdots$	$0.04419\ldots$
80	$4.809\ldots \times 10^{-14}$	$7.537\ldots \times 10^{-13}$	$\cdots$	$0.04428\ldots$
100	$4.427\ldots \times 10^{-18}$	$1.077\ldots \times 10^{-16}$	$\cdots$	$0.04428\ldots$
120	$1.377\ldots \times 10^{-21}$	$5.501\ldots \times 10^{-20}$	$\cdots$	$0.04428\ldots$
140	$1.156\ldots \times 10^{-24}$	$1.260\ldots \times 10^{-22}$	$\cdots$	$0.04428\ldots$
160	$2.621\ldots \times 10^{-27}$	$3.443\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
180	$1.877\ldots \times 10^{-28}$	$3.371\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
200	$1.715\ldots \times 10^{-28}$	$3.369\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
220	$1.711\ldots \times 10^{-28}$	$3.368\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
240	$1.711\ldots \times 10^{-28}$	$3.368\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
∞	$\frac{1}{5844076785304502808013602136}$	$\frac{196883}{5844076785304502808013602136}$	$\cdots$	$\frac{258823477531055064045234375}{5844076785304502808013602136}$

suggested a manifestation of moonshine for M_{24}: Namely, there should be an infinite-dimensional graded M_{24}-module whose McKay-Thompson series are holomorphic parts of *harmonic Maass forms*, or mock modular forms.

Following the work of Cheng [70], Eguchi and Hikami [113], and Gaberdiel, Hohenegger, and Volpato [132,217], Gannon established the existence of this infinite-dimensional graded M_{24}-module in [136].

It is natural to seek a general mathematical and physical setting for these results. Here we consider the mathematical setting, which develops from the close relationship between the monster group $\mathbb{M}$ and the Leech lattice Λ_{24}. Recall (cf. e.g. [83]) that the Leech lattice is even, unimodular, and positive-definite of rank 24. It turns out that M_{24} is closely related to another such lattice. Such observations led Cheng, Duncan and Harvey to further instances of moonshine within the setting of even unimodular positive-definite lattices of rank 24. In this way they arrived at the *Umbral Moonshine Conjectures* (cf. §5 of [75], §6 of [68], and §2 of [69]), predicting the existence of 22 further, graded infinite-dimensional modules, relating certain finite groups to distinguished mock modular forms.

To explain this prediction in more detail we recall Niemeier's result [244] that there are 24 (up to isomorphism) even unimodular positive-definite lattices of rank 24. The Leech lattice is the unique one with no root vectors (i.e. lattice vectors with norm-square 2), while the other 23 have root systems with full rank, 24. These *Niemeier root systems* are unions of simple simply-laced root systems with the same Coxeter numbers, and are given explicitly as

$$\begin{gathered} A_1^{24},\ A_2^{12},\ A_3^8, A_4^6,\ A_6^4,\ A_{12}^2, \\ A_5^4D_4,\ A_7^2D_5^2,\ A_8^3,\ A_9^2D_6, A_{11}D_7E_6,\ A_{15}D_9,\ A_{17}E_7,\ A_{24}, \\ D_4^6,\ D_6^4,\ D_8^3,\ D_{10}E_7^2, D_{12}^2,\ D_{16}E_8,\ D_{24}, E_6^4,\ E_8^3, \end{gathered} \tag{1.51}$$

in terms of the standard ADE notation. (Cf. e.g. [83] or [167] for more on root systems.)

For each Niemeier root system X let N^X denote the corresponding unimodular lattice, let W^X denote the (normal) subgroup of $\mathrm{Aut}(N^X)$ generated by reflections in roots, and define the *umbral group* of X by setting

$$G^X := \mathrm{Aut}(N^X)/W^X. \tag{1.52}$$

(See §A.2.1 for explicit descriptions of the groups G^X.)

Let m^X denote the Coxeter number of any simple component of X. An association of distinguished $2m^X$-vector-valued mock modular forms $H_g^X(\tau) = (H_{g,r}^X(\tau))$—called *umbral McKay-Thompson series*—to elements $g \in G^X$ is described and analyzed in [68, 69, 75].

For $X = A_1^{24}$ we have $G^X \simeq M_{24}$ and $m^X = 2$, and the functions $H_{g,1}^X(\tau)$ are precisely the mock modular forms assigned to elements $g \in M_{24}$ in the works [70, 113, 132, 217] mentioned above. Generalizing the M_{24} moonshine initiated by Eguchi, Ooguri and Tachikawa, we have the following conjecture of Cheng, Duncan and Harvey (cf. §2 of [69] or §9.3 of [106]).

Conjecture 1.29 (Umbral Moonshine Modules). Let X be a Niemeier root system X and set $m := m^X$. There is a naturally defined bi-graded infinite-dimensional G^X-module

$$\check{K}^X = \bigoplus_{r \in I^X} \bigoplus_{\substack{D \in \mathbb{Z},\, D \leq 0, \\ D = r^2 \,(\mathrm{mod}\, 4m)}} \check{K}^X_{r, -D/4m} \tag{1.53}$$

such that the vector-valued mock modular form $H_g^X = (H_{g,r}^X)$ is related[9] to

[9]In the statement of Conjecture 6.1 of [68] the function $H_{g,r}^X$ in (1.54) is replaced with $3H_{g,r}^X$ in the case that $X = A_8^3$. This is now known to be an error, arising from a misspecification of some of the functions H_g^X for $X = A_8^3$. Our treatment of the case $X = A_8^3$ in this work reflects the corrected specification of the corresponding H_g^X which is described and discussed in detail in [69].

the graded trace of g on $\check{K}^X$ by

$$H^X_{g,r}(\tau) = -2q^{-1/4m}\delta_{r,1} + \sum_{\substack{D\in\mathbb{Z},\ D\leq 0,\\ D=r^2 \pmod{4m}}} \operatorname{tr}(g|\check{K}^X_{r,-D/4m})q^{-D/4m} \tag{1.54}$$

for $r \in I^X$.

In (1.53) and (1.54) the set $I^X \subset \mathbb{Z}/2m\mathbb{Z}$ is defined in the following way. If X has an A-type component then $I^X := \{1, 2, 3, \ldots, m-1\}$. If X has no A-type component but does have a D-type component then $m = 2 \mod 4$, and $I^X := \{1, 3, 5, \ldots, m/2\}$. The remaining cases are $X = E_6^4$ and $X = E_8^3$. In the former of these, $I^X := \{1, 4, 5\}$, and in the latter case $I^X := \{1, 7\}$.

The functions $H^X_g(\tau)$ are described explicitly in §A.3.4.

Here we prove the following theorem.

Theorem 1.30. *The umbral moonshine modules exist.*

Two remarks.

1) Theorem 1.30 for $X = A_1^{24}$ is the main result of Gannon's work [136].

2) The vector-valued mock modular forms $H^X = (H^X_{g,r})$ have "minimal" *principal parts.* This minimality is analogous to the fact that the original McKay-Thompson series $T_g(\tau)$ for the Monster are hauptmoduln, and plays an important role in our proof.

Example 1.31. Many of Ramanujan's mock theta functions [258] are components of the vector-valued umbral McKay-Thompson series $H^X_g = (H^X_{g,r})$. For example, consider the root system $X = A_2^{12}$, whose umbral group is a double cover $2.M_{12}$ of the sporadic simple Mathieu group M_{12}. In terms of

Ramanujan's 3rd order mock theta functions

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$
$$\phi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})},$$
$$\chi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})}$$
$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2},$$
$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2n+1}+q^{4n+2})},$$

we have that

$$\begin{aligned} H^X_{2B,1}(\tau) = H^X_{2C,1}(\tau) = H^X_{4C,1}(\tau) &= -2q^{-\frac{1}{12}} \cdot f(q^2), \\ H^X_{6C,1}(\tau) = H^X_{6D,1}(\tau) &= -2q^{-\frac{1}{12}} \cdot \chi(q^2), \\ H^X_{8C,1}(\tau) = H^X_{8D,1}(\tau) &= -2q^{-\frac{1}{12}} \cdot \phi(-q^2), \\ H^X_{2B,2}(\tau) = -H^X_{2C,2}(\tau) &= -4q^{\frac{2}{3}} \cdot \omega(-q), \\ H^X_{6C,2}(\tau) = -H^X_{6D,2}(\tau) &= 2q^{\frac{2}{3}} \cdot \rho(-q). \end{aligned}$$

See §5.4 of [68] for more coincidences between umbral McKay-Thompson series and mock theta functions identified by Ramanujan almost a hundred years ago.

Our proof of Theorem 1.30 involves the explicit determination of each G^X-module $\check{K}^X$ by computing the multiplicity of each irreducible component for each homogeneous subspace. It guarantees the existence and uniqueness of a $\check{K}^X$ which is compatible with the representation theory of G^X and the Fourier expansions of the vector-valued mock modular forms $H^X_g(\tau) = (H^X_{g,r}(\tau))$.

At first glance our methods do not appear to shed light on any deeper algebraic properties of the $\check{K}^X$, such as might correspond to the vertex operator algebra structure on $V^\natural$, or the monster Lie algebra introduced by

Borcherds in [30]. However, we do determine, and utilize, specific recursion relations for the coefficients of the umbral McKay-Thompson series which are analogous to the replicability properties of monstrous moonshine formulated by Conway and Norton in §8 of [81] (cf. also [3]). More specifically, we use recent work [168] of Imamoḡlu, Raum and Richter, as generalized [235] by Mertens, to obtain such recursions. These results are based on the process of *holomorphic projection.*

Theorem 1.32. *For each $g \in G^X$ and $0 < r < m$, the McKay-Thompson series $H^X_{g,r}(\tau)$ is replicable in the mock modular sense.*

A key step in Borcherds' proof [30] of the monstrous moonshine conjecture is the reformulation of replicability in Lie theoretic terms. We may speculate that the *mock modular replicability* utilized in this work will ultimately admit an analogous algebraic interpretation. Such a result remains an important goal for future work.

In the statement of Theorem 1.32, replicable means that there are explicit recursion relations for the coefficients of the vector-valued mock modular form in question. For example, we recall the recurrence formula for Ramanujan's third order mock theta function $f(q) = \sum_{n=0}^{\infty} c_f(n)q^n$ that was obtained recently by Imamoḡlu, Raum and Richter [168]. If $n \in \mathbb{Q}$, then let

$$\sigma_1(n) := \begin{cases} \sum_{d|n} d & \text{if } n \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{sgn}^+(n) := \begin{cases} \mathrm{sgn}(n) & \text{if } n \neq 0, \\ 1 & \text{if } n = 0, \end{cases}$$

and then define

$$d(N, \widetilde{N}, t, \widetilde{t}) := \mathrm{sgn}^+(N) \cdot \mathrm{sgn}^+(\tilde{N}) \cdot \left(|N + t| - |\widetilde{N} + \widetilde{t}| \right).$$

Then for positive integers n, we have that

$$\sum_{\substack{m\in\mathbb{Z}\\ 3m^2+m\leq 2n}} \left(m+\frac{1}{6}\right)c_f\left(n-\frac{3}{2}m^2-\frac{1}{2}m\right)$$
$$=\frac{4}{3}\sigma(n)-\frac{16}{3}\sigma\left(\frac{n}{2}\right)-2\sum_{\substack{a,b\in\mathbb{Z}\\ 2n=ab}} d\left(N,\widetilde{N},\frac{1}{6},\frac{1}{6}\right),$$

where $N := \frac{1}{6}(-3a+b-1)$ and $\widetilde{N} := \frac{1}{6}(3a+b-1)$, and the sum is over integers a, b for which $N, \widetilde{N} \in \mathbb{Z}$. This is easily seen to be a recurrence relation for the coefficients $c_f(n)$. The replicability formulas for all of the $H^X_{g,r}(\tau)$ are similar (although some of these relations are slightly more complicated and involve the coefficients of weight 2 cusp forms).

It is important to emphasize that, despite the progress which is represented by our main results, Theorems 1.30 and 1.32, the following important question remains open in general.

Question 1.33. Is there a “natural" construction of $\check{K}^X$? Is $\check{K}^X$ equipped with a deeper algebra structure as in the case of the monster module $V^\natural$ of Frenkel, Lepowsky and Meurman?

We remark that this question has been answered positively, recently, in one special case: A vertex operator algebra structure underlying the umbral moonshine module $\check{K}^X$ for $X = E_8^3$ has been described explicitly in [107]. See also [74,108], where the problem of constructing algebraic structures that illuminate the umbral moonshine observations is addressed from a different point of view.

The proof of Theorem 1.30 is not difficult. It is essentially a collection of tedious calculations. We use the theory of mock modular forms and the character table for each G^X (cf. §A.2.2) to solve for the multiplicities of the irreducible G^X-module constituents of each homogeneous subspace in the alleged G^X-module $\check{K}^X$. To prove Theorem 1.30 it suffices to prove that these multiplicities are non-negative integers. To prove Theorem 1.32 we apply recent work [235] of Mertens on the holomorphic projection of weight $\frac{1}{2}$ mock modular forms, which generalizes earlier work [168] of Imamoğlu, Raum and Richter.

In §1.24 we recall the facts about mock modular forms that we require, and we prove Theorem 1.32. We prove Theorem 1.30 in §8.2. The appendices furnish all the data that our method requires. In particular, the umbral groups G^X are described in detail in §A.2, and the umbral McKay-Thompson series $H_g^X(\tau)$ are given explicitly in §A.3.

Chapter 2

Background

2.1 Harmonic Maass forms

We begin by briefly recalling the definition of a *harmonic Maass form* of weight $k \in \frac{1}{2}\mathbb{Z}$ and multiplier ν (a generalization of the notion of a Nebentypus). If $\tau = x + iy$ with x and y real, we define the weight k hyperbolic Laplacian by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \tag{2.1}$$

and if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, define

$$(\gamma : \tau) := (c\tau + d).$$

Suppose Γ is a subgroup of finite index in $SL_2(\mathbb{Z})$ and $\frac{3}{2} \leq k \in \frac{1}{2}\mathbb{Z}$. Then a real analytic function $F(\tau)$ is a *harmonic Maass form* of weight k on Γ with multiplier ν if:

(a) The function $F(\tau)$ satisfies the modular transformation with respect to the weight k slash operation,

$$F(\tau)|_k\gamma := (\gamma : \tau)^{-k} F(\gamma\tau) = \nu(\gamma) F(\tau)$$

for every matrix $\gamma \in \Gamma$, where if $k \in \mathbb{Z} + \frac{1}{2}$, the square root is taken to be the principal branch. In particular, if ν is trivial, then F is invariant under the action of the slash operator.

(b) We have that $\Delta_k F(\tau) = 0$,

(c) The singularities of F (if any) are supported at the cusps of Γ, and for each cusp ρ there is a polynomial $P_{F,\rho}(q^{-1}) \in \mathbb{C}[q^{-1/t_\rho}]$ and a constant $c > 0$ such that $F(\tau) - P_{F,\rho}(e^{-2\pi i \tau}) = O(e^{-cy})$ as $\tau \to \rho$ from inside a fundamental domain. Here t_ρ is the width of the cusp ρ. If ρ is not specified, we assume $\rho = \infty$.

Remark. The polynomial $P_{F,\rho}$ above is referred to as the *principal part of F at ρ.* In certain applications, condition (c) of the definition may be relaxed to admit larger classes of harmonic Maass forms. However, for our purposes we will only be interested in those satisfying the given definition, having a holomorphic principal part.

We denote the complex vector space of such functions by $H_k(\Gamma, \nu)$, and note that in order for $H_k(\Gamma, \nu)$ to be nonzero, ν must satisfy

$$(\gamma : \delta\tau)^k (\delta : \tau)^k \nu(\gamma)\nu(\delta) = (\gamma\delta : \tau)^k \nu(\gamma\delta)$$

for every $\gamma, \delta \in \Gamma$.

Let $\mathcal{S}(\Gamma)$ denote some fixed complete set of inequivalent representatives of the cusps of Γ. For each representative $\rho = \frac{\alpha}{\gamma}$ with $(\alpha, \gamma) = 1$, fix a matrix

$$L_\rho = \begin{pmatrix} -\delta & \beta \\ \gamma & -\alpha \end{pmatrix} \in SL_2(\mathbb{Z})$$

so that $\rho = L_\rho^{-1}\infty$. Following Rankin [259], let t_ρ be the cusp width and let κ_ρ be the cusp parameter, defined as the least nonnegative integer so that $\nu(L_\rho T^{t_\rho} L_\rho^{-1}) = e^{2\pi i \kappa_\rho}$, where $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The stabilizer of ρ in Γ is given by $\Gamma_\rho := \langle \pm T^{t_\rho} \rangle$, so for example $\Gamma_\infty = \langle \pm T \rangle$. Given $F(\tau) \in H_{2-k}(\Gamma, \nu)$, we refer to $F_\rho(\tau) := F(\tau)|_{2-k} L_\rho$ as the expansion of F at the cusp ρ. We note that this expansion depends on the choice of L_ρ. These facts imply that the expansion of F_ρ can be given as a Fourier series of the form

$$F_\rho(\tau) = \sum_n a(n, y) e^{2\pi i x (n+\kappa_\rho)/t_\rho}.$$

More precisely, we have the following. The Fourier expansion of harmonic Maass forms F at a cusp ρ (see Proposition 3.2 of [54]) splits into two components. As before, we let $q := e^{2\pi i \tau}$.

Lemma 2.1. *If $F(\tau)$ is a harmonic Maass form of weight $2-k$ for Γ where $\frac{3}{2} \leq k \in \frac{1}{2}\mathbb{Z}$, and if ρ is a cusp of Γ, then*

$$F_\rho(\tau) = F_\rho^+(\tau) + F_\rho^-(\tau)$$

where F_ρ^+ is the holomorphic part of F_ρ, given by

$$F_\rho^+(\tau) := \sum_{n \gg -\infty} c_{F,\rho}^+(n) q^{(n+\kappa_\rho)/t_\rho},$$

and F_ρ^- is the nonholomorphic part, given by

$$F_\rho^-(\tau) + \sum_{n<0} c_{F\rho}^-(n) \Gamma(k-1, 4\pi y |(n+\kappa_\rho)/t_\rho|) q^{(n+\kappa_\rho)/t_\rho}.$$

The holomorphic part of a harmonic Maass form is referred to as a *mock modular form.* By inspection, we see that weakly holomorphic modular forms are themselves harmonic Maass forms. In fact, under the given definition, all harmonic Maass forms of positive weight are weakly holomorphic.

The ξ-operator is a differential operator on harmonic Maass forms which is useful for understanding the nonholomorphic part. It is defined as $\xi_k := 2iy^k \cdot \overline{\frac{\partial}{\partial \bar{z}}}$.

Zagier calls the image under ξ of f the *shadow* of f, and we note that the shadow is nonzero if and only if f has a nonzero non-holomorphic part. We further note that for our definition of a harmonic Maass form, the shadow is always a cusp form of weight $2-k$.

Suppose $F(\tau)$ is a harmonic Maass form of weight $2-k$ as in Lemma 2.1 with shadow $h(\tau) := \xi_{2-k}F(\tau)$. Then if ρ is a cusp, we have that

$$F_\rho^-(\tau) = \int_{-\overline{\tau}}^{i\infty} h_\rho(z)(-i(\tau+z))^{k-2}\mathrm{d}z. \tag{2.2}$$

Bruinier and Funke used the ξ operator to define a bilinear pairing $\{\cdot,\cdot\} : M_k \times H_{2-k} \to \mathbb{C}$ by

$$\{g, f\} := (g, \xi_{2-k}f)_k\,, \tag{2.3}$$

where $(\cdot,\cdot)_k$ is the regularized Petersson scalar product. Proposition (3.5) of [53] gives this pairing in terms of the Fourier coefficients of g and the holomorphic part of f. In particular, suppose at a cusp h, g has an expansion $\sum_n a(h,n)q^n$ and f has an expansion with holomorphic part $\sum_n b(h,n)q^n$. They then show that

$$\{g, f\} = \sum_h \sum_{n\leq 0} a(-n)b(n). \tag{2.4}$$

The first sum here is over the components of a vector-valued form. In their notation, all Maass forms are level 1, and higher level forms may be viewed as level 1 vector-valued forms if we sum over the cusps.

This pairing is important to us primarily because of the following observation: $\{\xi_k f, f\} \neq 0$. This follows from the properties of the Peterson scalar product. However, since $\xi_k f$ is a cusp form, (2.4) gives us the following theorem:

Theorem 2.2 (Bruinier, Funke)**.** *If $f(z)$ is a harmonic weak Maass form with a nonzero non-holomorphic part, then f must have a non-zero principal part at at least one cusp.*

2.2 Maass-Poincaré series

The Maass–Poincaré series define a basis for a space of harmonic Maass forms and provide exact formulae for their coefficients. The following construction of the Maass–Poincaré series follows the method and notation of Bringmann and Ono [45] which builds on the early work of Rademacher, followed by more contemporary work of Fay, Niebur, among many others [117, 242, 243]. The Poincaré series we construct in this section are modular for congruence subgroups $\Gamma_0(N)$.

For $s \in \mathbb{C}$, $w \in \mathbb{R} \setminus \{0\}$, and $k \geq 3/2$, $k \in \frac{1}{2}\mathbb{Z}$, let

$$\mathcal{M}_s(w) := |y|^{\frac{k}{2}-1} M_{\mathrm{sign}(w)(1-k/2), s-\frac{1}{2}}(|w|), \tag{2.5}$$

where $M_{\nu,\mu}(z)$ is the M-Whittaker function which is a solution to the differential equation

$$\frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) u = 0,$$

and (here and throughout this paper) $\tau = x + iy$. Using this function, let

$$\phi_s(\tau) := \mathcal{M}_s(4\pi y) e^{2\pi i x}. \tag{2.6}$$

Given a positive integer m and a cusp ρ, Maass–Poincaré series provide a form with principal part equal to $q^{(-m+\kappa_\rho)/t_\rho}$ plus a constant at the cusp ρ, and constant at all other cusps, thereby forming a basis for $H_{2-k}(\Gamma, \nu)$.

Suppose $m > 0$ and $L \in SL_2(\mathbb{Z})$ with $\rho = L^{-1}\infty$. Then we have the Maass–Poincaré series

$$\mathcal{P}_L(\tau, m, \Gamma, 2-k, s, \nu) := \sum_{M \in \Gamma_\rho \backslash \Gamma} \frac{\phi_s\left(\frac{-m+\kappa_\rho}{t_\rho} \cdot L^{-1} M \tau \right)}{(L^{-1} : M\tau)^{2-k} (M : \tau)^{2-k} \nu(M)}. \tag{2.7}$$

It is easy to check that $\phi_s(\tau)$ is an eigenfunction of Δ_{2-k} with eigenvalue

$$s(1-s) + \frac{k^2 - 2k}{4}.$$

The right hand side of (2.7) converges absolutely for $\Re(s) > 1$, however Bringmann and Ono establish conditional convergence when $s \geq 3/4$ [45], giving Theorem 2.3 below. The theorem is stated for the specific case $\Gamma = \Gamma_0(N)$ for some N and $k \geq 3/2$, in which case we modify the notation slightly and define

$$\mathcal{P}_L(\tau, m, N, 2-k, \nu) := \frac{1}{\Gamma(k)} \mathcal{P}_L(\tau, m, \Gamma_0(N), 2-k, \frac{k}{2}, \nu) \tag{2.8}$$

In the statement of the theorem below, K_c is a modified Kloosterman sum given by

$$K_c(2-k, L, \nu, m, n) := \sum_{\substack{0 \leq a < ct_\rho \\ a \equiv -\frac{c\cdot(\alpha,N)}{\alpha\gamma} \left(\bmod \frac{N}{(\gamma,N)}\right) \\ ad \equiv 1 \pmod{c}}} \frac{(S:\tau)^{2-k} \exp\left(2\pi i \left(\frac{a\cdot\frac{(-m+\kappa_\rho)}{t_\rho} + d\cdot\frac{(n+\kappa_\infty)}{t_\infty}}{c}\right)\right)}{(L^{-1}:LS\tau)^{2-k}(LS:\tau)^{2-k}\nu(LS)}, \tag{2.9}$$

where $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. If ν is trivial, we omit it from the notation. We also have that $\delta_{L,S}(m)$ is an indicator function for the cusps $\rho = L^{-1}\infty$ and $\mu = S^{-1}\infty$ given by

$$\delta_{L,S}(m) := \begin{cases} \nu(M)^{-1} e^{2\pi i r \frac{-m+\kappa_\rho}{t_\rho}} & \text{if } M = LT^r S^{-1} \in \Gamma_0(N), \\ 0 & \text{if } \mu \not\sim \rho \text{ in } \Gamma_0(N). \end{cases}$$

Using this notation, we have the following theorem which gives exact formulae for the coefficients and principal part of $\mathcal{P}_L(\tau, m, N, 2-k, \nu)$, which is a generalization of Theorem 3.2 of [45].

Theorem 2.3. *Suppose that* $\frac{3}{2} \leq k \in \frac{1}{2}\mathbb{Z}$, *and suppose* $\rho = L^{-1}\infty$ *is a cusp of* $\Gamma_0(N)$. *If* m *is a positive integer, then* $\mathcal{P}_L(\tau, m, N, 2-k, \nu)$ *is in* $H_{2-k}(\Gamma_0(N), \nu)$. *Moreover, the following are true:*

1. *We have*

$$\mathcal{P}_L^+(\tau, m, N, 2-k, \nu) = \delta_{\rho,I}(m) \cdot q^{-m+\kappa_\infty} + \sum_{n\geq 0} a^+(n) q^n.$$

Moreover, if $n > 0$*, then* $a^+(n)$ *is given by*

$$-\, i^k 2\pi \left| \frac{-m+\kappa_\rho}{t_\rho(n+\kappa_\infty)} \right|^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ (c,N)=(\gamma,N)}} \frac{K_c(2-k, L, \nu, -m, n)}{c} \cdot$$

$$I_{k-1}\left(\frac{4\pi}{c} \sqrt{\frac{|-m+\kappa_\rho|\,|n+\kappa_\infty|}{t_\rho}} \right),$$

where I_k *is the usual* I*-Bessel function.*

2. *If* $S \in SL_2(\mathbb{Z})$*, then there is some* $c \in \mathbb{C}$ *so that the principal part of* $\mathcal{P}_L(\tau, m, N, 2-k)$ *at the cusp* $\mu = S^{-1}\infty$ *is given by*

$$\delta_{L,S}(m) q^{\frac{-m+\kappa_\rho}{t_\rho}} + c$$

Sketch of the proof. Writing $\rho = \frac{\alpha}{\gamma}$, Bringmann and Ono prove this theorem for the case that $\gamma \mid N$ and $(\alpha, N) = 1$, along with the assumption that μ and ρ are in a fixed complete set of inequivalent cusps, so that $\delta_{\mu,\rho} = 1$ or 0. This general form is useful to us particularly since it works equally well for the cusps ∞ with L taken to be the identity, and for 0 with L taken to be $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Here and throughout, we let $\mathcal{S}_N$ denote any complete set of inequivalent cusps of $\Gamma_0(N)$, and for each $\rho \in \mathcal{S}_N$, we fix some L_ρ with $\rho = L_\rho^{-1}\infty$. Rankin notes [259] (Proof of Theorem 4.1.1(iii)) that given some choice of $\mathcal{S}_N$, each right coset of $\Gamma_0(N)\backslash SL_2(\mathbb{Z})$ is in $\Gamma_0(N) \cdot L_\rho T^r$ for some unique $\rho \in \mathcal{S}_N$. Moreover, the r in the statement is unique modulo t_ρ, so the function $\delta_{L,S}(m)$ given above is well-defined on all matrices in $SL_q(\mathbb{Z})$.

In the proof given by Bringmann and Ono, the sum of Kloosterman sums $\sum_{\substack{c>0 \\ (c,N)=(\gamma,N)}} \frac{K_c(2-k,L,\nu,-m,n)}{c}\dots$ is written as a sum over representatives of the double coset $\Gamma_\rho\backslash L^{-1}\Gamma_0(N)/\Gamma_\infty$ (omitting the identity if present). Following similar arguments, but without the assumptions on α and γ, we find the indices of summation given in (2.9) and Theorem 2.3. As in their case, we find that the principal part of $\mathcal{P}_{L_\rho}(\tau,m,N,2-k)$ at a cusp μ is constant if $\mu \not\sim \rho$ and is $\delta_{L_\rho,L_\mu} q^{\frac{-m}{t_\rho}} + c$ for some constant if $\mu = \rho$. Therefore, if μ is a cusp with $L_\mu = M^{-1}L_\rho T^r$ for some $M \in \Gamma_0(N,)$, then the principal part of $\mathcal{P}_{L_\rho}(\tau,m,N,2-k,\nu)$ at μ is clearly $\nu(M)^{-1}e^{2\pi i r\frac{-m+\kappa}{t_\rho}} q^{\frac{-m+\kappa_\rho}{t_\rho}} + c.$ □

Since harmonic Maass forms with a nonholomorphic part have a nonconstant principal part at some cusp, we have the following theorem.

Theorem 2.4. *[45, Theorem 1.1] Assuming the notation above, if $\frac{3}{2} \leq k \in \frac{1}{2}\mathbb{Z}$, and $F(\tau) \in H_{2-k}(\Gamma_0(N),\nu)$ has principal part*

$$P_\rho(\tau) = \sum_{m\geq 0} a_\rho(-m) q^{\frac{-m+\kappa_\rho}{t_\rho}}$$

for each cusp $\rho \in \mathcal{S}_N$, then

$$F(\tau) = \sum_{\rho\in\mathcal{S}_N} \sum_{m>0} a_\rho(-m)\mathcal{P}_\rho(\tau,m,N,2-k,\nu) + g(\tau),$$

where $g(\tau)$ is a holomorphic modular form. Moreover, we have that $c = 0$ whenever $k > 2$, and is a constant when $k = 2$.

2.3 Holomorphic projection

The theory of holomorphic projections provides a way to explicitly relate the coefficients of harmonic Maass forms and other nonholomorphic modular forms to classical modular or quasimodular forms[1]. For instance, given a weight 1/2 harmonic Maass form, we may multiply by its shadow to obtain a weight 2 nonholomorphic modular form with simple transformation properties. The holomorphic projection provides a simple explicit correction term to convert this nonholomorphic modular form into a weight 2 quasimodular form with the same transformation properties. In this way, we may essentially reduce many questions about the coefficients of weight $\frac{1}{2}$ mock modular forms to questions about weight 2 holomorphic modular forms.

Although holomorphic projections may be applied more generally, we will restrict our attention in this section to weight 1/2 Harmonic Maass forms multiplied by weight 3/2 theta series. The following theorem is a special case of a more general theorem due to Mertens (cf. Theorem 6.3 of [235]).

Theorem 2.5 (Mertens)**.** *Suppose $g(\tau)$ and $h(\tau)$ are both theta functions of weight $\frac{3}{2}$ contained in $S_{\frac{3}{2}}(\Gamma, \nu_g)$ and $S_{\frac{3}{2}}(\Gamma, \nu_h)$ respectively, with Fourier expansions*

$$g(\tau) := \sum_{i=1}^{s} \sum_{n\in\mathbb{Z}} n\chi_i(n) q^{n^2},$$

$$h(\tau) := \sum_{j=1}^{t} \sum_{n\in\mathbb{Z}} n\psi_j(n) q^{n^2},$$

where each χ_i and ψ_i is a Dirichlet character. Moreover, suppose $h(\tau)$ is the shadow of a weight $\frac{1}{2}$ harmonic Maass form $f(\tau) \in H_{\frac{1}{2}}(\Gamma, \overline{\nu_h})$. Define the function

$$D^{f,g}(\tau) := 2 \sum_{r=1}^{\infty} \sum_{\chi_i, \psi_j} \sum_{\substack{m,n\in\mathbb{Z}^+ \\ m^2-n^2=r}} \chi_i(m)\overline{\psi_j(n)}(m-n)q^r.$$

If $f(\tau)g(\tau)$ has no singularity at any cusp, then $f^+(\tau)g(\tau) + D^{f,g}(\tau)$ is a weight 2 quasimodular form. In other words, it lies in the space $\mathbb{C}E_2(\tau) \oplus M_2(\Gamma, \nu_g\overline{\nu_h})$.

[1]By quasimodular forms here, we mean spaces of the form $\mathbb{C}E_2(\tau) \oplus M_2(\Gamma, \nu)$.

Two Remarks.

1) These identities give recurrence relations for the weight $\frac{1}{2}$ mock modular form f^+ in terms of the weight 2 quasimodular form which equals $f^+(\tau)g(\tau)+D^{f,g}(\tau)$. The example after Theorem 1.32 for Ramanujan's third order mock theta function f is an explicit example of such a relation.

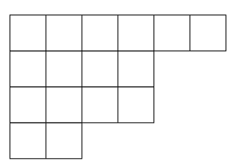
2) Theorem 2.5 extends to vector-valued mock modular forms in the natural way by acting on components.

Chapter 3

Rogers–Ramanujan identities

3.1 The Hall–Littlewood polynomials

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be an *integer partition* [9], a nonincreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \dots$ with only finitely nonzero terms. The positive λ_i are called the *parts* of λ, and the number of parts, denoted $l(\lambda)$, is the *length* of λ. The *size* $|\lambda|$ of λ is the sum of its parts. The Ferrers–Young diagram of λ consists of $l(\lambda)$ left-aligned rows of squares such that the ith row contains λ_i squares. For example, the Ferrers–Young diagram of $\nu = (6, 4, 4, 2)$ of length 4 and size 16 is



The *conjugate* partition λ' corresponds to the transpose of the Ferrers–Young diagram of λ. For example, we have $\nu' = (4, 4, 3, 3, 1, 1)$. We define nonnegative integers $m_i = m_i(\lambda)$, for $i \geq 1$, to be the *multiplicities* of parts of size i, so that $|\lambda| = \sum_i i m_i$. It is easy to see that $m_i = \lambda'_i - \lambda'_{i+1}$. We say that a partition is *even* if its parts are all even. Note that λ' is even if all multiplicities $m_i(\lambda)$ are even. The partition ν above is an even partition. Given two partitions λ, μ we write $\mu \subseteq \lambda$ if the diagram of μ is contained in the diagram of λ, or, equivalently, if $\mu_i \leq \lambda_i$ for all i. To conclude our discussion of partitions, we define the *generalized q-shifted factorial*

$$b_\lambda(q) := \prod_{i\geq 1}(q)_{m_i} = \prod_{i\geq 1}(q)_{\lambda'_i-\lambda'_{i+1}}. \tag{3.1}$$

Hence, for ν as above we have $b_\nu(q) = (q)_1^2(q)_2$.

For a fixed positive integer n, let $x = (x_1, \dots, x_n)$. Given a partition λ such that $l(\lambda) \leq n$, write x^λ for the monomial $x_1^{\lambda_1} \dots x_n^{\lambda_n}$, and define

$$v_\lambda(q) = \prod_{i=0}^{n} \frac{(q)_{m_i}}{(1-q)^{m_i}}, \tag{3.2}$$

where $m_0 := n - l(\lambda)$. The *Hall–Littlewood polynomial* $P_\lambda(x; q)$ is defined as the symmetric function [219]

$$P_\lambda(x;q) = \frac{1}{v_\lambda(q)} \sum_{w\in\mathfrak{S}_n} w\left(x^\lambda \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j}\right), \tag{3.3}$$

where the symmetric group $\mathfrak{S}_n$ acts on x by permuting the x_i. It follows from the definition that $P_\lambda(x; q)$ is a homogeneous polynomial of degree $|\lambda|$, a fact used repeatedly in the rest of this paper. $P_\lambda(x; q)$ is defined to be identically 0 if $l(\lambda) > n$. The Hall–Littlewood polynomials may be extended in the usual way to symmetric functions in countably-many variables, see [219].

Here we make this precise when x is specialized to an infinite geometric progression. For $x = (x_1, x_2, \dots)$ not necessarily finite, let p_r be the r-th power sum symmetric function

$$p_r(x) = x_1^r + x_2^r + \cdots,$$

and $p_\lambda = \prod_{i\geq 1} p_{\lambda_i}$. The power sums $\{p_\lambda(x_1, \dots, x_n)\}_{l(\lambda)\leq n}$ form a $\mathbb{Q}$-basis of the ring of symmetric functions in n variables. If ϕ_q denotes the ring homomorphism $\phi_q(p_r) = p_r/(1-q^r)$, then the *modified Hall–Littlewood polynomials*

$P'_\lambda(x;q)$ are defined as the image of the $P_\lambda(x;q)$ under ϕ_q:

$$P'_\lambda = \phi_q\big(P_\lambda\big).$$

We note that as stated, this homomorphism is well defined only if n is at least the degree of P_λ (i.e. $|\lambda|$). Otherwise nontrivial relations exist among products of the $p_r(x)$ with degree greater than n. However if $n \geq |\lambda|$, then P_λ can be expressed uniquely as terms of the p_r. Moreover this expression is otherwise independent of n. Therefore for the purposes of this homomorphism, we identify P_λ with this expansion.

We also require the Hall–Littlewood polynomials Q_λ and Q'_λ defined by

$$Q_\lambda(x;q) := b_\lambda(q)P_\lambda(x;q) \quad \text{and} \quad Q'_\lambda(x;q) := b_\lambda(q)P'_\lambda(x;q). \tag{3.4}$$

Clearly, $Q'_\lambda = \phi_q\big(Q_\lambda\big)$.

Up to the point where the x-variables are specialized, our proof of Theorems 1.2–1.5 will make use of the modified Hall–Littlewood polynomials, rather than the ordinary Hall–Littlewood polynomials. Through specialization, we arrive at P_λ evaluated at a geometric progression thanks to

$$P_\lambda(1,q,q^2,\dots;q^n) = P'_\lambda(1,q,\dots,q^{n-1};q^n), \tag{3.5}$$

which readily follows from

$$\phi_{q^n}\big(p_r(1,q,\dots,q^{n-1})\big) = \frac{1-q^{nr}}{1-q^r}\cdot\frac{1}{1-q^{nr}} = p_r(1,q,q^2,\dots).$$

Example 3.1. Let $\lambda = (2)$. If we take $n=1$, then $P_\lambda(x;q) = x_1^2$, whereas if $n=2$, then $P_\lambda(x;q) = x_1^2 + (1-q)\cdot x_1x_2 + x_2^2$, from which we can find

$$P_\lambda(x;q) = \frac{1-q}{2}p_1(x)^2 + \left(1-\frac{1-q}{2}\right)p_2(x).$$

In order to find $P_\lambda(1,q,q^2,\dots;q)$ we replace $p_1(x)$ in the expression above with $\frac{1}{1-q}$, and $p_2(x)$ with $\frac{1}{1-q^2}$ to get

$$P_\lambda(1,q,q^2,\dots;q) = \frac{1-q}{2}\frac{1}{(1-q)^2} + \left(1-\frac{1-q}{2}\right)\frac{1}{1-q^2} = \frac{1}{1-q}.$$

Example 3.2. Let $\lambda = (2,2)$. The table below gives values of $P_\lambda(1,q,q^2,\ldots q^{n-1};q)$ for various choices of n.

n	$P_\lambda(1,q,q^2,\ldots q^{n-1};q)$
2	q^2
3	$q^2+q^3+q^4$
4	$q^2+q^3+2q^4+q^5+q^6$
5	$q^2+q^3+2q^4+2q^5+2q^6+q^7+q^8.$

The limiting value $P_\lambda(1,q,q^2,\ldots;q)$ can be found exactly as described above. We find that

$$P_\lambda(x;q) = \frac{2-3q+q^3}{24}p_1(x)^4 + \frac{q-q^3}{4}p_2(x)p_1(x)^2 + \frac{2+q+q^3}{8}p_2(x)^2 - \frac{1-q^3}{3}p_3(x)p_1(x) - \frac{q+q^3}{4}p_4(x),$$

which implies

$$P_\lambda(1,q,q^2,\ldots;q) = \frac{q^2}{(1-q)(1-q^2)}.$$

From [187, 293] we may infer the following combinatorial formula for the modified Hall–Littlewood polynomials:

$$Q'_\lambda(x;q) = \sum \prod_{i=1}^{\lambda_1}\prod_{a=1}^{n} x_a^{\mu_i^{(a-1)}-\mu_i^{(a)}} q^{\binom{\mu_i^{(a-1)}-\mu_i^{(a)}}{2}} \begin{bmatrix} \mu_i^{(a-1)}-\mu_{i+1}^{(a)} \\ \mu_i^{(a-1)}-\mu_i^{(a)} \end{bmatrix}_q,$$

where the sum is over partitions $0 = \mu^{(n)} \subseteq \cdots \subseteq \mu^{(1)} \subseteq \mu^{(0)} = \lambda'$ and

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \dfrac{(q)_n}{(q)_m(q)_{n-m}} & \text{if } m \in \{0,1,\ldots,n\} \\ 0 & \text{otherwise} \end{cases}$$

is the usual q-binomial coefficient. Therefore, by (3.1)–(3.5), we have obtained the following combinatorial description of the q-series we have assembled from the Hall–Littlewood polynomials.

Lemma 3.3. *If m and n are positive integers, then*

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{(\sigma+1)|\lambda|} P_{2\lambda}(1,q,q^2,\dots;q^n)$$
$$= \sum \prod_{i=1}^{2m} \left\{ \frac{q^{\frac{1}{2}(\sigma+1)\mu_i^{(0)}}}{(q^n;q^n)_{\mu_i^{(0)}-\mu_{i+1}^{(0)}}} \prod_{a=1}^{n} q^{\mu_i^{(a)}+n\binom{\mu_i^{(a-1)}-\mu_i^{(a)}}{2}} \begin{bmatrix} \mu_i^{(a-1)}-\mu_{i+1}^{(a)} \\ \mu_i^{(a-1)}-\mu_i^{(a)} \end{bmatrix}_{q^n} \right\}, \quad (3.6)$$

where the sum on the right is over partitions $0=\mu^{(n)} \subseteq \cdots \subseteq \mu^{(1)} \subseteq \mu^{(0)}$ such that $(\mu^{(0)})'$ is even and $l(\mu^{(0)}) \le 2m$.

Lemma 3.3 may be used to express the sum sides of (1.8)–(1.13) combinatorially. Moreover, we have that (3.6) generalizes the sums in (1.2), (1.3), and (1.6). To see this, we note that the above simplifies for $n=1$ to

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{(\sigma+1)|\lambda|} P_{2\lambda}(1,q,q^2,\dots;q) = \sum \prod_{i=1}^{2m} \frac{q^{\frac{1}{2}\mu_i(\mu_i+\sigma)}}{(q)_{\mu_i-\mu_{i+1}}}$$

summed on the right over partitions μ of length at most $2m$ whose conjugates are even. Such partitions are characterized by the restriction $\mu_{2i}=\mu_{2i-1}=: r_i$ so that we get

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{(\sigma+1)|\lambda|} P_{2\lambda}(1,q,q^2,\dots;q) = \sum_{r_1 \ge \cdots \ge r_m \ge 0} \prod_{i=1}^{m} \frac{q^{r_i(r_i+\sigma)}}{(q)_{r_i-r_{i+1}}}$$

in accordance with (1.6).

If instead we consider $m=1$ and replace $\mu^{(j)}$ by (r_j,s_j) for $j \ge 0$, we find

$$\sum_{r=0}^{\infty} q^{(\sigma+1)r} P_{(2^r)}(1,q,q^2,\dots;q^n)$$
$$= \sum \frac{q^{(\sigma+1)r_0}}{(q^n;q^n)_{r_0}} \prod_{j=1}^{n} q^{r_j+s_j+n\binom{r_{j-1}-r_j}{2}+n\binom{s_{j-1}-s_j}{2}} \begin{bmatrix} r_{j-1}-s_j \\ r_{j-1}-r_j \end{bmatrix}_{q^n} \begin{bmatrix} s_{j-1} \\ s_j \end{bmatrix}_{q^n}$$
$$= \frac{(q^{n+4};q^{n+4})_\infty}{(q)_\infty} \cdot \theta\big(q^{2-\sigma};q^{n+4}\big),$$

where the second sum is over $r_0, s_0, \ldots, r_{n-1}, s_{n-1}$ such that $r_0 = s_0$, and $r_n = s_n := 0$.

We conclude this section with a remark about Theorem 1.7. Due to the occurrence of the limit, the left-hand side does not take the form of the usual sum-side of a Rogers–Ramanujan-type identity. For special cases it is, however, possible to eliminate the limit. For example, for partitions of the form (2^r) we found that

$$P_{(2^r)}(1, q, q^2, \ldots; q^{2n+\delta}) = \sum_{r \geq r_1 \geq \cdots \geq r_n \geq 0} \frac{q^{r^2 - r + r_1^2 + \cdots + r_n^2 + r_1 + \cdots + r_n}}{(q)_{r-r_1}(q)_{r_1-r_2} \cdots (q)_{r_{n-1}-r_n}(q^{2-\delta}; q^{2-\delta})_{r_n}} \tag{3.7}$$

for $\delta = 0, 1$. This turns the $m = 2$ case of Theorem 1.7 into

$$\sum_{r_1 \geq \cdots \geq r_n \geq 0} \frac{q^{r_1^2 + \cdots + r_n^2 + r_1 + \cdots + r_n}}{(q)_{r_1-r_2} \cdots (q)_{r_{n-1}-r_n}(q^{2-\delta}; q^{2-\delta})_{r_n}} = \frac{(q^{2n+2+\delta}; q^{2n+2+\delta})_\infty}{(q)_\infty} \cdot \theta(q; q^{2n+2+\delta}).$$

For $\delta = 1$ this is the $i = 1$ case of the Andrews–Gordon identity (1.6) (with m replaced by n). For $\delta = 0$ it corresponds to the $i = 1$ case of (1.12). We do not know how to generalize (3.7) to arbitrary rectangular shapes.

3.2 Proof of Theorems 1.2–1.5

Here we prove Theorems 1.2–1.5. We begin by recalling key aspects of the classical works of Andrews and Watson which give hints of the generalizations we obtain.

3.2.1 The Watson–Andrews approach

In 1929 Watson proved the Rogers–Ramanujan identities (1.2) and (1.3) by first proving a new basic hypergeometric series transformation between a terminating balanced ${}_4\phi_3$ series and a terminating very-well-poised ${}_8\phi_7$ series

[295]

$$\frac{(aq,aq/bc)_N}{(aq/b,aq/c)_N}\sum_{r=0}^{N}\frac{(b,c,aq/de,q^{-N})_r}{(q,aq/d,aq/e,bcq^{-N}/a)_r}\,q^r$$
$$=\sum_{r=0}^{N}\frac{1-aq^{2r}}{1-a}\cdot\frac{(a,b,c,d,e,q^{-N})_r}{(q,aq/b,aq/c,aq/d,aq/e)_r}\cdot\left(\frac{a^2q^{N+2}}{bcde}\right)^r. \quad (3.8)$$

Here a, b, c, d, e are indeterminates, N is a nonnegative integer and

$$(a_1,\dots,a_m)_k := (a_1,\dots,q_m;q) = (a_1;q)_k\cdots(a_m;q)_k.$$

By letting b, c, d, e tend to infinity and taking the nonterminating limit $N \to \infty$, Watson arrived at what is known as the Rogers–Selberg identity [264, 269][1]

$$\sum_{r=0}^{\infty}\frac{a^rq^{r^2}}{(q)_r}=\frac{1}{(aq)_\infty}\sum_{r=0}^{\infty}\frac{1-aq^{2r}}{1-a}\cdot\frac{(a)_r}{(q)_r}\cdot(-1)^ra^{2r}q^{5\binom{r}{2}+2r}. \quad (3.9)$$

For $a = 1$ or $a = q$ the sum on the right can be expressed in product-form by the Jacobi triple-product identity

$$\sum_{r=-\infty}^{\infty}(-1)^rx^rq^{\binom{r}{2}}=(q)_\infty\cdot\theta(x;q),$$

resulting in (1.2) and (1.3).

Almost 50 years after Watson's work, Andrews showed that the Andrews–Gordon identities (1.6) for $i = 1$ and $i = m + 1$ follow in a similar way from a multiple series generalization of (3.8) in which the ${}_8\phi_7$ series on the right is replaced by a terminating very-well-poised ${}_{2m+6}\phi_{2m+5}$ series depending on

[1]Here and elsewhere in the paper we ignore questions of convergence. From an analytic point of view, the transition from (3.8) to (3.9) requires the use of the dominated convergence theorem, imposing the restriction $|q| < 1$ on the Rogers–Selberg identity. We however choose to view this identity as an identity between formal power series in q, in line with the combinatorial and representation-theoretic interpretations of Rogers–Ramanujan-type identities.

$2m+2$ parameters instead of b, c, d, e [8]. Again the key steps are to let all these parameters tend to infinity, to take the nonterminating limit, and to then express the $a = 1$ or $a = q$ instances of the resulting sum as a product by the Jacobi triple-product identity.

Recently, Bartlett and Warnaar obtained an analog of Andrews' multiple series transformation for the C_n root system [19, Theorem 4.2]. Apart from the variables $(x_1, \dots, x_n)$—which play the role of a in (3.8), and are related to the underlying root system—the C_n Andrews transformation again contains $2m+2$ parameters. Unfortunately, simply following the Andrews–Watson procedure is no longer sufficient. In [238] Milne already obtained the C_n analogue of the Rogers–Selberg identity (3.9) (the $m = 1$ case of (3.10) below) and considered specializations along the lines of Andrews and Watson. Only for C_2 did this result in a Rogers–Ramanujan-type identity: the modulus 6 case of (1.11) mentioned previously.

The first two steps towards a proof of (1.8)–(1.13), however, are the same as those of Watson and Andrews: we let all $2m+2$ parameters in the C_n Andrews transformation tend to infinity and take the nonterminating limit. Then, as shown in [19], the right-hand side can be expressed in terms of modified Hall–Littlewood polynomials, resulting in the level-m C_n Rogers–Selberg identity

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{|\lambda|} P'_{2\lambda}(x;q) = L_m^{(0)}(x;q), \tag{3.10}$$

where

$$L_m^{(0)}(x;q) := \sum_{r \in \mathbb{Z}_+^n} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^n x_i^{2(m+1)r_i} q^{(m+1)r_i^2 + n\binom{r_i}{2}} \cdot \prod_{i,j=1}^n \Big(-\frac{x_i}{x_j}\Big)^{r_i} \frac{(x_i x_j)_{r_i}}{(q x_i/x_j)_{r_i}}.$$

Here we have that

$$\Delta_{\mathrm{C}}(x) := \prod_{i=1}^n (1 - x_i^2) \prod_{1 \le i < j \le n} (x_i - x_j)(x_i x_j - 1)$$

is the C_n Vandermonde product, and $f(xq^r)$ is shorthand for $f(x_1 q^{r_1}, \dots, x_n q^{r_n})$.

Remark. As mentioned previously, (3.10) for $m = 1$ is Milne's C_n Rogers–Selberg formula [238, Corollary 2.21].

The strategy for the proofs of Theorems 1.2–1.5 is now simple to describe. By comparing the left-hand side of (3.10) with that of (1.8)–(1.11), it follows that we should make the simultaneous substitutions

$$q \mapsto q^n, \qquad x_i \mapsto q^{(n+\sigma+1)/2-i} \ \ (1 \le i \le n). \tag{3.11}$$

Then, by the homogeneity and symmetry of the (modified) Hall–Littlewood polynomials and (3.5), we have

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{|\lambda|} P'_{2\lambda}(x;q) \longmapsto \sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{(\sigma+1)|\lambda|} P_{2\lambda}(1,q,q^2,\dots;q^n).$$

Therefore, we wish to carry out these maneuvers and prove that the resulting right-hand side can be described as a product of modified theta functions in the four families in the theorems. The problem we face is that making the substitutions (3.11) in the right-hand side of (3.10) and then writing the resulting q-series in product form is very difficult.

To get around this problem, we take a rather different route and (up to a small constant) first double the rank of the underlying C_n root system and then take a limit in which products of pairs of x-variables tend to one. To do so we require another result from [19].

First we extend our earlier definition of the q-shifted factorial to

$$(a)_k = (a)_\infty/(aq^k)_\infty. \tag{3.12}$$

Importantly, we note that $1/(q)_k = 0$ for k a negative integer. Then, for $x = (x_1, \dots, x_n)$, p an integer such that $0 \le p \le n$ and $r \in \mathbb{Z}^n$, we have

$$L_m^{(p)}(x;q) := \sum_{r \in \mathbb{Z}^n} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^n x_i^{2(m+p+1)r_i} q^{(m+1)r_i^2+(n+p)\binom{r_i}{2}} \\ \times \prod_{i=1}^n \prod_{j=p+1}^n \left(-\frac{x_i}{x_j}\right)^{r_i} \frac{(x_i x_j)_{r_i}}{(q x_i/x_j)_{r_i}}. \tag{3.13}$$

Note that the summand of $L_m^{(p)}(x;q)$ vanishes if one of $r_{p+1},\dots,r_n<0$.

The following key lemma will be crucial for our strategy to work.

Lemma 3.4 ([19, Lemma A.1])**.** *For* $1\le p\le n-1$,

$$\lim_{x_{p+1}\to x_p^{-1}} L_m^{(p-1)}(x;q)=L_m^{(p)}(x_1,\dots,x_{p-1},x_{p+1},\dots,x_n;q). \tag{3.14}$$

This will be the key to the proof of all four generalized Rogers–Ramanujan identities, although the level of difficulty varies considerably from case to case. We begin with the simplest proof, that of Theorem 1.4 (i.e.equation (1.10)).

3.2.2 Proof of Theorem 1.4

Here we carry out the strategy described in the previous section by making use of the C_n and B_n Weyl denominator formulas, and the $\mathbf{D}_{n+1}^{(2)}$ Macdonald identity.

Proof of Theorem 1.4. By iterating (3.14), we have

$$\lim_{y_1\to x_1^{-1}}\cdots\lim_{y_n\to x_n^{-1}} L_m^{(0)}(x_1,y_1,\dots,x_n,y_n)=L_m^{(n)}(x_1,\dots,x_n).$$

Hence, after replacing $x\mapsto(x_1,y_1,\dots,x_n,y_n)$ in (3.10) (which corresponds to the doubling of the rank mentioned previously) and taking the $y_i\to x_i^{-1}$ limit for $1\le i\le n$, we find

$$\sum_{\substack{\lambda\\ \lambda_1\le m}} q^{|\lambda|}P'_{2\lambda}(x^{\pm};q)=\frac{1}{(q)_\infty^n\prod_{i=1}^n\theta(x_i^2;q)\prod_{1\le i<j\le n}\theta(x_i/x_j,x_ix_j;q)}\\ \times\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{C}}(xq^r)\prod_{i=1}^n x_i^{\kappa r_i-i+1}q^{\frac12\kappa r_i^2-nr_i}, \tag{3.15}$$

where $\kappa=2m+2n+2$ and $f(x^{\pm})=f(x_1,x_1^{-1},\dots,x_n,x_n^{-1})$. Next we make the simultaneous substitutions

$$q\mapsto q^{2n},\qquad x_i\mapsto q^{n-i+1/2}=:\hat{x}_i\ \ (1\le i\le n), \tag{3.16}$$

which corresponds to (3.11) with $(n,\sigma)\mapsto(2n,0)$. By the identity

$$(q^{2n};q^{2n})_\infty^n\cdot\prod_{i=1}^n\theta(q^{2n-2i+1};q^{2n})\cdot\prod_{1\le i<j\le n}\theta(q^{j-i},q^{2n-i-j+1};q^{2n})=\frac{(q)_\infty^{n+1}}{(q^2;q^2)_\infty},$$

and

$$\begin{aligned}
&q^{2n|\lambda|}P'_{2\lambda}(q^{n-1/2},q^{1/2-n},\dots,q^{1/2},q^{-1/2};q^{2n})\\
&\quad=q^{2n|\lambda|}P'_{2\lambda}(q^{1/2-n},q^{3/2-n},\dots,q^{n-1/2};q^{2n}) &&\text{by symmetry}\\
&\quad=q^{|\lambda|}P'_{2\lambda}(1,q,\dots,q^{2n-1};q^{2n}) &&\text{by homogeneity}\\
&\quad=q^{|\lambda|}P_{2\lambda}(1,q,q^2,\dots;q^{2n}) &&\text{by (3.5),}
\end{aligned}$$

we obtain

$$\sum_{\substack{\lambda\\ \lambda_1\le m}}q^{|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n}\big)=\frac{(q^2;q^2)_\infty}{(q)_\infty^{n+1}}\,\mathcal{M},\tag{3.17}$$

where

$$\mathcal{M}:=\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{C}}(\hat{x}q^{2nr})\prod_{i=1}^n\hat{x}_i^{\kappa r_i-i+1}q^{nr_i^2-2n^2r_i}.$$

We must express $\mathcal{M}$ in product form. As a first step, we use the C_n Weyl denominator formula [198, Lemma 2]

$$\Delta_{\mathrm{C}}(x)=\det_{1\le i,j\le n}\big(x_i^{j-1}-x_i^{2n-j+1}\big),\tag{3.18}$$

as well as multilinearity, to write $\mathcal{M}$ as

$$\mathcal{M}=\det_{1\le i,j\le n}\left(\sum_{r\in\mathbb{Z}}\hat{x}_i^{\kappa r-i+1}q^{n\kappa r^2-2n^2r}\Big((\hat{x}_iq^{2nr})^{j-1}-(\hat{x}_iq^{2nr})^{2n-j+1}\Big)\right).\tag{3.19}$$

We now replace $(i,j)\mapsto(n-j+1,n-i+1)$ and, viewing the resulting determinant as being of the form $\det\big(\sum_r u_{ij;r}-\sum_r v_{ij;r}\big)$, we change the summation index $r\mapsto -r-1$ in the sum over $v_{ij;r}$. Then we find that

$$\mathcal{M}=\det_{1\le i,j\le n}\left(q^{a_{ij}}\sum_{r\in\mathbb{Z}}y_i^{2nr-i+1}q^{2n\kappa\binom{r}{2}+\frac{1}{2}\kappa r}\Big((y_iq^{\kappa r})^{j-1}-(y_iq^{\kappa r})^{2n-j}\Big)\right),\tag{3.20}$$

where $y_i=q^{\kappa/2-i}$ and $a_{ij}=j^2-i^2+(i-j)(\kappa+1)/2$. Since the factor $q^{a_{ij}}$ does not contribute to the determinant, we can apply the B_n Weyl denominator formula [198]

$$\det_{1\le i,j\le n}\bigl(x_i^{j-1}-x_i^{2n-j}\bigr)=\prod_{i=1}^n(1-x_i)\prod_{1\le i<j\le n}(x_i-x_j)(x_ix_j-1)=:\Delta_{\mathrm{B}}(x) \quad (3.21)$$

to obtain

$$\mathcal{M}=\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{B}}(yq^{\kappa r})\prod_{i=1}^n y_i^{2nr_i-i+1}q^{2n\kappa\binom{r_i}{2}+\frac{1}{2}\kappa r_i}.$$

By the $\mathbf{D}_{n+1}^{(2)}$ Macdonald identity [218]

$$\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{B}}(xq^r)\prod_{i=1}^n x_i^{2nr_i-i+1}q^{2n\binom{r_i}{2}+\frac{1}{2}r_i}$$
$$=(q^{1/2};q^{1/2})_\infty(q)_\infty^{n-1}\prod_{i=1}^n\theta(x_i;q^{1/2})_\infty\prod_{1\le i<j\le n}\theta(x_i/x_j,x_ix_j;q)$$

with $(q,x)\mapsto(q^\kappa,y)$, we obtain

$$\mathcal{M}=(q^{\kappa/2};q^{\kappa/2})_\infty(q^\kappa;q^\kappa)_\infty^{n-1}\prod_{i=1}^n\theta\bigl(q^i;q^{\kappa/2}\bigr)\prod_{1\le i<j\le n}\theta\bigl(q^{j-i},q^{i+j};q^\kappa\bigr), \quad (3.22)$$

where we have also used the simple symmetry $\theta(q^{a-b};q^a)=\theta(q^b;q^a)$. Substituting (3.22) into (3.17) proves the first equality of (1.10).

Establishing the second equality is a straightforward exercise in manipulating infinite products, and we omit the details. □

There is a somewhat different approach to (1.10) based on the representation theory of the affine Kac–Moody algebra $\mathrm{C}_n^{(1)}$ [183]. Let $I=\{0,1,\dots,n\}$, and α_i, $\alpha_i^\vee$ and Λ_i for $i\in I$ the simple roots, simple coroots and fundamental weights of $\mathrm{C}_n^{(1)}$. Let $\langle\cdot,\cdot\rangle$ denote the usual pairing between the Cartan subalgebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$, so that $\langle\Lambda_i,\alpha_j^\vee\rangle=\delta_{ij}$. Finally, let $V(\Lambda)$ be the integrable highest-weight module of $\mathrm{C}_n^{(1)}$ of highest weight Λ with character $\operatorname{ch} V(\Lambda)$.

The homomorphism

$$F_{\mathbb{1}}:\ \mathbb{C}[[\mathrm{e}^{-\alpha_0},\dots,\mathrm{e}^{-\alpha_n}]]\to\mathbb{C}[[q]],\qquad F_{\mathbb{1}}(\mathrm{e}^{-\alpha_i})=q\quad\text{for all } i\in I \tag{3.23}$$

is known as principal specialization [204]. Subject to this specialization, $\operatorname{ch} V(\Lambda)$ admits a simple product form as follows. Let ρ be the Weyl vector (that is $\langle\rho,\alpha_i^\vee\rangle=1$ for $i\in I$) and $\operatorname{mult}(\alpha)$ the multiplicity of α. Then [179, 205] we have

$$F_{\mathbb{1}}\big(\mathrm{e}^{-\Lambda}\operatorname{ch} V(\Lambda)\big)=\prod_{\alpha\in\Delta_+^\vee}\left(\frac{1-q^{\langle\Lambda+\rho,\alpha\rangle}}{1-q^{\langle\rho,\alpha\rangle}}\right)^{\operatorname{mult}(\alpha)}, \tag{3.24}$$

where $\Delta_+^\vee$ is the set of positive coroots. This result, which is valid for all types $\mathrm{X}_N^{(r)}$, can be rewritten in terms of theta functions. Assuming $\mathrm{C}_n^{(1)}$ and setting

$$\Lambda=(\lambda_0-\lambda_1)\Lambda_0+(\lambda_1-\lambda_2)\Lambda_1+\dots+(\lambda_{n-1}-\lambda_n)\Lambda_{n-1}+\lambda_n\Lambda_n,$$

for $\lambda=(\lambda_0,\lambda_1,\dots,\lambda_n)$ a partition, this rewriting takes the form

$$\begin{aligned}F_{\mathbb{1}}\big(\mathrm{e}^{-\Lambda}\operatorname{ch} V(\Lambda)\big)&=\frac{(q^2;q^2)_\infty(q^{\kappa/2};q^{\kappa/2})_\infty(q^\kappa;q^\kappa)_\infty^{n-1}}{(q;q)_\infty^{n+1}}\\&\times\prod_{i=1}^n\theta\big(q^{\lambda_i+n-i+1};q^{\kappa/2}\big)\prod_{1\le i<j\le n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n+2-i-j};q^\kappa\big),\end{aligned} \tag{3.25}$$

where $\kappa=2n+2\lambda_0+2$.

The earlier product form now arises by recognizing (see e.g., [19, Lemma 2.1]) the right-hand side of (3.15) as

$$\mathrm{e}^{-m\Lambda_0}\operatorname{ch} V(m\Lambda_0)$$

upon the identification

$$q=\mathrm{e}^{-\alpha_0-2\alpha_1-\dots-2\alpha_{n-1}-\alpha_n}\quad\text{and}\quad x_i=\mathrm{e}^{-\alpha_i-\dots-\alpha_{n-1}-\alpha_n/2}\ \ (1\le i\le n).$$

Since (3.16) corresponds exactly to the principal specialization (3.23), it follows from (3.25) with $\lambda = (m, 0^n)$, that

$$F_{\mathbb{1}}\big(\mathrm{e}^{-m\Lambda_0}\,\mathrm{ch}\,V(m\Lambda_0)\big) = \frac{(q^2;q^2)_\infty (q^{\kappa/2};q^{\kappa/2})_\infty (q^\kappa;q^\kappa)_\infty^{n-1}}{(q;q)_\infty^{n+1}} \times \prod_{i=1}^n \theta\big(q^{n-i+1};q^{\kappa/2}\big) \prod_{1\le i<j\le n} \theta\big(q^{j-i},q^{i+j};q^\kappa\big).$$

We should remark that this representation-theoretic approach is not essentially different from our earlier q-series proof. Indeed, the principal specialization formula (3.25) itself is an immediate consequence of the $\mathbf{D}_{n+1}^{(2)}$ Macdonald identity, and if, instead of the right-hand side of (3.15), we consider the more general

$$\mathrm{e}^{-\Lambda}\,\mathrm{ch}\,V(\Lambda) = \frac{1}{(q)_\infty^n \prod_{i=1}^n \theta(x_i^2;q) \prod_{1\le i<j\le n} \theta(x_i/x_j, x_i x_j;q)} \times \sum_{r\in\mathbb{Z}^n} \det_{1\le i,j\le n} \Big((x_i q^{r_i})^{j-\lambda_i-1} - (x_i q^{r_i})^{2n-j+\lambda_i+1} \Big) \prod_{i=1}^n x_i^{\kappa r_i+\lambda_i-i+1} q^{\frac{1}{2}\kappa r_i^2 - n r_i}$$

for $\kappa = 2n + 2\lambda_0 + 2$, then all of the steps carried out between (3.15) and (3.22) carry over to this more general setting. The only notable changes are that (3.19) generalizes to

$$\mathcal{M} = \det_{1\le i,j\le n} \bigg(\sum_{r\in\mathbb{Z}} \hat{x}_i^{\kappa r+\lambda_i-i+1} q^{n\kappa r^2 - 2n^2 r} \cdot \Big((\hat{x}_i q^{2nr})^{j-\lambda_i-1} - (\hat{x}_i q^{2nr})^{2n-j+\lambda_i+1} \Big) \bigg),$$

and that in (3.20) we have to redefine y_i as $q^{\kappa/2-\lambda_{n-i+1}-i}$, and a_{ij} as

$$j^2 - i^2 + (i-j)(\kappa+1)/2 + (j-1/2)\lambda_{n-j+1} - (i-1/2)\lambda_{n-i+1}.$$

3.2.3 **Proof of Theorem 1.2** (1.8a)

Here we prove (1.8a) by making use of the $\mathrm{B}_n^{(1)}$ Macdonald identity.

Proof of Theorem 1.2(1.8a). Again we iterate (3.14), but this time the variable x_n, remains unpaired:

$$\lim_{y_1\to x_1^{-1}} \cdots \lim_{y_{n-1}\to x_{n-1}^{-1}} L_m^{(0)}(x_1,y_1,\dots,x_{n-1},y_{n-1},x_n) = L_m^{(n-1)}(x_1,\dots,x_n).$$

Therefore, if we replace $x \mapsto (x_1,y_1,\dots,x_{n-1},y_{n-1},x_n)$ in (3.10) (changing the rank from n to $2n-1$) and take the $y_i \to x_i^{-1}$ limit for $1 \le i \le n-1$, we obtain

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\le m}} q^{|\lambda|} P'_{2\lambda}\big(x_1^{\pm},\dots,x_{n-1}^{\pm},x_n;q\big) \\ = \frac{1}{(q)_\infty^{n-1}(qx_n^2)_\infty \prod_{i=1}^{n-1}(qx_i^{\pm}x_n, qx_i^{\pm 2})_\infty \prod_{1\le i<j\le n-1}(qx_i^{\pm}x_j^{\pm})_\infty} \\ \times \sum_{r\in\mathbb{Z}^n} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^{n}\left(-\frac{x_i^{\kappa}}{x_n}\right)^{r_i} q^{\frac{1}{2}\kappa r_i^2-\frac{1}{2}(2n-1)r_i} \frac{(x_ix_n)_{r_i}}{(qx_i/x_n)_{r_i}},\end{aligned} \tag{3.26}$$

where $\kappa = 2m+2n+1$, $(ax_i^{\pm})_\infty := (ax_i)_\infty(ax_i^{-1})_\infty$ and

$$(ax_i^{\pm}x_j^{\pm})_\infty := (ax_ix_j)_\infty(ax_i^{-1}x_j)_\infty(ax_ix_j^{-1})_\infty(ax_i^{-1}x_j^{-1})_\infty.$$

Recalling the comment immediately after (3.13), the summand of (3.26) vanishes unless $r_n \ge 0$.

Let $\hat{x} := (-x_1,\dots,-x_{n-1},-1)$ and

$$\phi_r = \begin{cases} 1 & \text{if } r=0 \\ 2 & \text{if } r=1,2,\dots. \end{cases} \tag{3.27}$$

Letting x_n tend to 1 in (3.26), and using

$$\lim_{x_n\to 1} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathbb{C}}(x)} \prod_{i=1}^{n} \frac{(x_ix_n)_{r_i}}{(qx_i/x_n)_{r_i}} = \phi_{r_n} \frac{\Delta_{\mathrm{B}}(\hat{x}q^r)}{\Delta_{\mathrm{B}}(\hat{x})},$$

we find that

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{|\lambda|} P'_{2\lambda}\big(x_1^{\pm}, \dots, x_{n-1}^{\pm}, 1; q\big)$$

$$= \frac{1}{(q)_\infty^n \prod_{i=1}^{n-1}(qx_i^{\pm}, qx_i^{\pm 2})_\infty \prod_{1\leq i<j\leq n-1}(qx_i^{\pm}x_j^{\pm})_\infty}$$

$$\times \sum_{r_1,\dots,r_{n-1}=-\infty}^{\infty} \sum_{r_n=0}^{\infty} \phi_{r_n} \frac{\Delta_{\mathrm{B}}(\hat{x}q^r)}{\Delta_{\mathrm{B}}(\hat{x})} \prod_{i=1}^{n} \hat{x}_i^{\kappa r_i} q^{\frac{1}{2}\kappa r_i^2 - \frac{1}{2}(2n-1)r_i}.$$

It is easily checked that the summand on the right (without the factor ϕ_{r_n}) is invariant under the variable change $r_n \mapsto -r_n$. Using the elementary relations

$$\theta(-1;q) = 2(-q)_\infty^2, \quad (-q)_\infty(q;q^2)_\infty = 1, \quad \theta(z,-z;q)\theta(qz^2;q^2) = \theta(z^2), \tag{3.28}$$

we can then simplify the above to obtain

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{|\lambda|} P'_{2\lambda}\big(x_1^{\pm}, \dots, x_{n-1}^{\pm}, 1; q\big) \tag{3.29}$$

$$= \frac{1}{(q)_\infty^n \prod_{i=1}^{n} \theta(\hat{x}_i;q)\theta(q\hat{x}_i^2;q^2) \prod_{1\leq i<j\leq n} \theta(\hat{x}_i/\hat{x}_j, \hat{x}_i\hat{x}_j;q)}$$

$$\times \sum_{r\in\mathbb{Z}^n} \Delta_{\mathrm{B}}(\hat{x}q^r) \prod_{i=1}^{n} \hat{x}_i^{\kappa r_i - i+1} q^{\frac{1}{2}\kappa r_i^2 - \frac{1}{2}(2n-1)r_i}.$$

The remainder of the proof is similar to that of (1.10). We make the simultaneous substitutions

$$q \mapsto q^{2n-1}, \qquad x_i \mapsto q^{n-i} \ \ (1 \leq i \leq n), \tag{3.30}$$

so that from here on $\hat{x}_i := -q^{n-i}$. By the identity

$$(q^{2n-1};q^{2n-1})_\infty^n \prod_{i=1}^{n} \theta(-q^{n-i};q^{2n-1})\theta(q^{2n-2i+1};q^{4n-2})$$

$$\times \prod_{1\leq i<j\leq n} \theta(q^{j-i}, q^{2n-i-j}; q^{2n-1}) = 2(q)_\infty^n$$

and (3.5), we find that

$$\sum_{\substack{\lambda \\ \lambda_1\le m}} q^{|\lambda|} P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-1}\big) = \frac{\mathcal{M}}{2(q)_\infty^n},$$

where we have that

$$\mathcal{M} := \sum_{r\in\mathbb{Z}^n} \Delta_{\mathrm{B}}\big(\hat{x}q^{(2n-1)r}\big) \prod_{i=1}^n \hat{x}_i^{\kappa r_i - i+1} q^{\frac{1}{2}(2n-1)\kappa r_i^2 - \frac{1}{2}(2n-1)^2 r_i}.$$

By (3.21) and multilinearity, $\mathcal{M}$ can be rewritten in the form

$$\mathcal{M} = \det_{1\le i,j\le n}\Bigg(\sum_{r\in\mathbb{Z}} \hat{x}_i^{\kappa r-i+1} q^{\frac{1}{2}(2n-1)\kappa r^2 - \frac{1}{2}(2n-1)^2 r} \times\Big(\big(\hat{x}_i q^{(2n-1)r}\big)^{j-1} - \big(\hat{x}_i q^{(2n-1)r}\big)^{2n-j}\Big)\Bigg).$$

Following the same steps that led from (3.19) to (3.20), we obtain

$$\mathcal{M} = \det_{1\le i,j\le n}\Bigg((-1)^{i-j} q^{b_{ij}} \sum_{r\in\mathbb{Z}} (-1)^r y_i^{(2n-1)r-i+1} q^{(2n-1)\kappa\binom{r}{2}} \times\Big((y_i q^{\kappa r})^{j-1} - (y_i q^{\kappa r})^{2n-j}\Big)\Bigg), \quad (3.31)$$

where

$$y_i = q^{\frac{1}{2}(\kappa+1)-i} \quad \text{and} \quad b_{ij} := j^2 - i^2 + \frac{1}{2}(i-j)(\kappa+3). \quad (3.32)$$

Again, the factor $(-1)^{i-j}q^{b_{ij}}$ does not contribute, and so (3.21) then gives

$$\mathcal{M} = \sum_{r\in\mathbb{Z}^n} \Delta_{\mathrm{B}}(y_i q^{\kappa r}) \prod_{i=1}^n (-1)^{r_i} y_i^{(2n-1)r_i-i+1} q^{(2n-1)\kappa\binom{r_i}{2}}.$$

To complete the proof, we apply the following variant of the $\mathrm{B}_n^{(1)}$ Macdonald identity[2]

$$\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{B}}(xq^r)\prod_{i=1}^n(-1)^{r_i}x_i^{(2n-1)r_i-i+1}q^{(2n-1)\binom{r_i}{2}}$$
$$=2(q)_\infty^n\prod_{i=1}^n\theta(x_i;q)\prod_{1\le i<j\le n}\theta(x_i/x_j,x_ix_j;q),\quad (3.33)$$

with $(q,x)\mapsto(q^\kappa,y)$. □

Identity (1.8a) can be understood representation-theoretically, but this time the relevant Kac–Moody algebra is $\mathbb{A}_{2n}^{(2)}$. According to [19, Lemma 2.3] the right-hand side of (3.29), with $\hat{x}$ interpreted as

$$\hat{x}_i=\mathrm{e}^{-\alpha_0-\cdots-\alpha_{n-i}}\quad(1\le i\le n)$$

and q as

$$q=\mathrm{e}^{-2\alpha_0-\cdots-2\alpha_{n-1}-\alpha_n},\qquad (3.34)$$

is the $\mathbb{A}_{2n}^{(2)}$ character

$$\mathrm{e}^{-m\Lambda_n}\operatorname{ch}V(m\Lambda_n).$$

The substitution (3.30) corresponds to

$$\mathrm{e}^{-\alpha_0}\mapsto -1\quad\text{and}\quad \mathrm{e}^{-\alpha_i}\mapsto q\quad(1\le i\le n).\qquad (3.35)$$

Denoting this by F, we have the general specialization formula

$$F\big(\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)\big)=\frac{(q^\kappa;q^\kappa)_\infty^n}{(q)_\infty^n}\prod_{i=1}^n\theta\big(q^{\lambda_0-\lambda_i+i};q^\kappa\big)\prod_{1\le i<j\le n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j-i-j+2n+1};q^\kappa\big),\qquad (3.36)$$

[2]The actual $\mathrm{B}_n^{(1)}$ Macdonald identity has the restriction $|r|\equiv 0 \pmod 2$ in the sum over $r\in\mathbb{Z}^n$, which eliminates the factor 2 on the right. To prove the form used here it suffices to take the $a_1,\dots,a_{2n-1}\to 0$ and $a_{2n}\to -1$ limit in Gustafson's multiple ${}_6\psi_6$ summation for the affine root system $\mathbb{A}_{2n-1}^{(2)}$, see [154].

where $\kappa = 2n + 2\lambda_0 + 1$ and

$$\Lambda = 2\lambda_n\Lambda_0 + (\lambda_{n-1} - \lambda_n)\Lambda_1 + \cdots + (\lambda_1 - \lambda_2)\Lambda_{n-1} + (\lambda_0 - \lambda_1)\Lambda_n$$

for $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ a partition. If we let $\lambda = (m, 0^n)$ (so that $\Lambda = m\Lambda_n$), then this is in accordance with (1.8a).

3.2.4 Proof of Theorem 1.2 (1.8b)

Here we prove the companion result to (1.8a).

Proof of Theorem 1.2 (1.8b). In (3.26) we set $x_n = q^{1/2}$ so that

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{|\lambda|} P'_{2\lambda}\big(x_1^{\pm}, \ldots, x_{n-1}^{\pm}, q^{1/2}; q\big)$$
$$= \frac{1}{(q)_\infty^{n-1}(q^2)_\infty \prod_{i=1}^{n-1}(q^{3/2}x_i^{\pm}, qx_i^{\pm 2})_\infty \prod_{1\le i<j\le n-1}(qx_i^{\pm}x_j^{\pm})_\infty}$$
$$\times \sum_{r_1,\ldots,r_{n-1}=-\infty}^{\infty} \sum_{r_n=0}^{\infty} \frac{\Delta_{\mathrm{C}}(\hat{x}q^r)}{\Delta_{\mathrm{C}}(\hat{x})} \prod_{i=1}^{n} (-1)^{r_i} \hat{x}_i^{\kappa r_i} q^{\frac{1}{2}\kappa r_i^2 - nr_i},$$

where $\kappa = 2m + 2n + 1$ and $\hat{x} = (x_1, \ldots, x_{n-1}, q^{1/2})$. The r_n-dependent part of the summand is

$$(-1)^{r_n} q^{\kappa\binom{r_n+1}{2} - nr_n} \frac{1 - q^{2r_n+1}}{1-q} \prod_{i=1}^{n-1} \frac{x_i q^{r_i} - q^{r_n+1/2}}{x_i - q^{1/2}} \cdot \frac{x_i q^{r_n+r_i+1/2} - 1}{x_i q^{1/2} - 1},$$

which is readily checked to be invariant under the substitution $r_n \mapsto -r_n - 1$. Hence

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{|\lambda|} P'_{2\lambda}\big(x_1^{\pm}, \ldots, x_{n-1}^{\pm}, q^{1/2}; q\big)$$
$$= \frac{1}{2(q)_\infty^n \prod_{i=1}^{n-1}(-1)\theta(q^{1/2}x_i, x_i^2; q) \prod_{1\le i<j\le n-1} \theta(x_i/x_j, x_i x_j; q)}$$
$$\times \sum_{r\in\mathbb{Z}^n} \Delta_{\mathrm{C}}(\hat{x}q^r) \prod_{i=1}^{n} (-1)^{r_i} \hat{x}_i^{\kappa r_i - i} q^{\frac{1}{2}\kappa r_i^2 - nr_i + \frac{1}{2}}.$$

Our next step is to replace $x_i \mapsto x_{n-i+1}$ and $r_i \mapsto r_{n-i+1}$. By $\theta(x;q) = -x\theta(x^{-1};q)$ and (3.28), this leads to

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{|\lambda|} P'_{2\lambda}\big(q^{1/2}, x_2^{\pm}, \dots, x_n^{\pm}; q\big) \tag{3.37}$$
$$= \frac{1}{(q)_\infty^n \prod_{i=1}^n \theta(-q^{1/2}\hat{x}_i;q)\theta(\hat{x}_i^2;q^2) \prod_{1\le i<j\le n} \theta(\hat{x}_i/\hat{x}_j, \hat{x}_i\hat{x}_j;q)}$$
$$\times \sum_{r\in\mathbb{Z}^n} \Delta_{\mathrm{C}}(\hat{x}q^r) \prod_{i=1}^n (-1)^{r_i} \hat{x}_i^{\kappa r_i - i+1} q^{\frac{1}{2}\kappa r_i^2 - nr_i},$$

where now $\hat{x} = (q^{1/2}, x_2, \dots, x_n)$. Again we are at the point where we can specialize, letting

$$q \mapsto q^{2n-1}, \qquad x_i \mapsto q^{n-i+1/2} =: \hat{x}_i \;\; (1 \le i \le n). \tag{3.38}$$

This is consistent, since $x_1 = q^{1/2} \mapsto q^{n-1/2}$. By the identity

$$(q^{2n-1};q^{2n-1})_\infty^n \prod_{i=1}^n \theta(-q^{2n-i};q^{2n-1})\theta(q^{2n-2i+1};q^{4n-2})$$
$$\times \prod_{1\le i<j\le n} \theta(q^{j-i}, q^{2n-i-j+1}; q^{2n-1}) = 2(q)_\infty^n,$$

we obtain

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{2|\lambda|} P_{2\lambda}\big(1, q, q^2, \dots; q^{2n-1}\big) = \frac{\mathcal{M}}{2(q)_\infty^n},$$

where

$$\mathcal{M} := \sum_{r\in\mathbb{Z}^n} \Delta_{\mathrm{C}}(\hat{x}q^{(2n-1)r}) \prod_{i=1}^n (-1)^{r_i} \hat{x}_i^{\kappa r_i - i+1} q^{\frac{1}{2}(2n-1)\kappa r_i^2 - (2n-1)nr_i}.$$

Expressing $\mathcal{M}$ in determinantal form using (3.18) yields

$$\mathcal{M} = \det_{1\le i,j\le n} \Bigg(\sum_{r\in\mathbb{Z}} (-1)^r \hat{x}_i^{\kappa r - i+1} q^{\frac{1}{2}(2n-1)\kappa r^2 - (2n-1)nr}$$
$$\times \Big(\big(\hat{x}_i q^{(2n-1)r}\big)^{j-1} - \big(\hat{x}_i q^{(2n-1)r}\big)^{2n-j+1} \Big) \Bigg).$$

We now replace $(i,j) \mapsto (j,i)$ and, viewing the resulting determinant as of the form $\det\big(\sum_r u_{ij;r} - \sum_r v_{ij;r}\big)$, we change the summation index $r \mapsto -r$ in the sum over $u_{ij;r}$. The expression for $\mathcal{M}$ we obtain is exactly (3.31) except that $(-1)^{i-j}q^{b_{ij}}$ is replaced by $q^{c_{ij}}$ and y_i is given by q^{n-i+1} instead of $q^{(\kappa+1)/2-i}$. Following the previous proof results in (1.8b). □

To interpret (1.8b) in terms of $\mathbb{A}_{2n}^{(2)}$, we note that by [19, Lemma 2.2] the right-hand side of (3.37) in which $\hat{x}$ is interpreted as

$$\hat{x}_i = -q^{1/2}\mathrm{e}^{\alpha_0+\cdots+\alpha_{i-1}} \quad (1 \le i \le n)$$

(and q again as (3.34)) corresponds to the $\mathbb{A}_{2n}^{(2)}$ character

$$\mathrm{e}^{-2m\Lambda_0}\,\mathrm{ch}\,V(2m\Lambda_0).$$

The specialization (3.38) is then again consistent with (3.35). From (3.36) with $\lambda = (m^{n+1})$, the first product-form on the right of (1.8b) immediately follows. By level-rank duality, we can also identify (1.8b) as a specialization of the $\mathbb{A}_{2m}^{(2)}$ character $\mathrm{e}^{-2n\Lambda_0}\,\mathrm{ch}\,V(2n\Lambda_0)$.

3.2.5 Proof of Theorem 1.5

This proof, which uses the $\mathbf{D}_n^{(1)}$ Macdonald identity, is the most complicated of the four.

Proof of Theorem 1.5. Once again we iterate (3.14), but now both x_{n-1} and x_n remain unpaired:

$$\lim_{y_1 \to x_1^{-1}} \cdots \lim_{y_{n-2} \to x_{n-2}^{-1}} L_m^{(0)}(x_1, y_1, \dots, x_{n-2}, y_{n-2}, x_{n-1}, x_n) = L_m^{(n-2)}(x_1, \dots, x_n).$$

Accordingly, if we replace $x \mapsto (x_1, y_1, \dots, x_{n-2}, y_{n-2}, x_{n-1}, x_n)$ in (3.10) (thereby changing the rank from n to $2n-2$) and take the $y_i \to x_i^{-1}$ limit, for $1 \le i \le n-2$, we obtain

$$\begin{aligned}
&\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{|\lambda|} P'_{2\lambda}\big(x_1^{\pm}, \dots, x_{n-2}^{\pm}, x_{n-1}, x_n; q\big) \\
&\quad = \frac{1}{(q)_\infty^{n-2}(qx_{n-1}^2, qx_{n-1}x_n, qx_n^2)_\infty} \\
&\qquad \times \frac{1}{\prod_{i=1}^{n-2}(qx_i^{\pm 2}, qx_i^{\pm}x_{n-1}, qx_i^{\pm}x_n)_\infty \prod_{1 \le i < j \le n-2}(qx_i^{\pm}x_j^{\pm})_\infty} \\
&\qquad \times \sum_{r \in \mathbb{Z}^n} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^n \left(\frac{x_i^\kappa}{x_{n-1}x_n}\right)^{r_i} q^{\frac{1}{2}\kappa r_i^2 - (n-1)r_i} \frac{(x_i x_{n-1}, x_i x_n)_{r_i}}{(qx_i/x_{n-1}, qx_i/x_n)_{r_i}},
\end{aligned}$$

where $\kappa = 2m + 2n$. It is important to note that the summand vanishes unless r_{n-1} and r_n are both nonnegative. Next we let (x_{n-1}, x_n) tend to $(q^{1/2}, 1)$ using

$$\lim_{(x_{n-1},x_n) \to (q^{1/2},1)} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^n \frac{(x_i x_{n-1}, x_i x_n)_{r_i}}{(qx_i/x_{n-1}, qx_i/x_n)_{r_i}} = \phi_{r_n} \frac{\Delta_{\mathrm{B}}(\hat{x}q^r)}{\Delta_{\mathrm{B}}(\hat{x})},$$

with ϕ_r as in (3.27) and $\hat{x} := (-x_1, \dots, -x_{n-2}, -q^{1/2}, -1)$. Hence we find that

$$\begin{aligned}
&\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{|\lambda|} P'_{2\lambda}\big(x_1^{\pm}, \dots, x_{n-2}^{\pm}, q^{1/2}, 1; q\big) \\
&\quad = \frac{1}{(q)_\infty^{n-1}(q^{3/2}; q^{1/2})_\infty \prod_{i=1}^{n-2}(qx_i^{\pm}; q^{1/2})_\infty (qx_i^{\pm 2})_\infty \prod_{1 \le i < j \le n-2}(qx_i^{\pm}x_j^{\pm})_\infty} \\
&\qquad \times \sum_{r_1,\dots,r_{n-2}=-\infty}^{\infty} \; \sum_{r_{n-1},r_n=0}^{\infty} \phi_{r_n} \frac{\Delta_{\mathrm{B}}(\hat{x}q^r)}{\Delta_{\mathrm{B}}(\hat{x})} \prod_{i=1}^n \hat{x}_i^{\kappa r_i} q^{\frac{1}{2}\kappa r_i^2 - \frac{1}{2}(2n-1)r_i}.
\end{aligned}$$

Since the summand (without the factor ϕ_{r_n}) is invariant under the variable change $r_n \mapsto -r_n$, as well as the change $r_{n-1} \mapsto -r_{n-1} - 1$, we can rewrite this as

$$\sum_{\substack{\lambda \\ \lambda_1\le m}} q^{|\lambda|} P'_{2\lambda}\big(x_1^{\pm},\dots,x_{n-2}^{\pm},q^{1/2},1;q\big)$$
$$=\frac{1}{(q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty \prod_{i=1}^n \theta(\hat{x}_i;q^{1/2}) \prod_{1\le i<j\le n}\theta(\hat{x}_i/\hat{x}_j,\hat{x}_i\hat{x}_j)}$$
$$\times \sum_{r\in\mathbb{Z}^n} \Delta_{\mathrm{B}}(\hat{x}q^r) \prod_{i=1}^n \hat{x}_i^{\kappa r_i-i+1} q^{\frac{1}{2}\kappa r_i^2-\frac{1}{2}(2n-1)r_i},$$

where, once again, we have used (3.28) to clean up the infinite products. Before we can carry out the usual specialization, we need to relabel $x_1,\dots,x_{n-2}$ as $x_2,\dots,x_{n-1}$ and, accordingly, we redefine $\hat{x}$ as

$$(-q^{1/2},-x_2,\dots,-x_{n-1},-1.$$

For $n\ge 2$, we then find that

$$\sum_{\substack{\lambda \\ \lambda_1\le m}} q^{|\lambda|} P'_{2\lambda}\big(q^{1/2},x_2^{\pm},\dots,x_{n-1}^{\pm},1;q\big) \tag{3.39}$$
$$=\frac{1}{(q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty \prod_{i=1}^n \theta(\hat{x}_i;q^{1/2}) \prod_{1\le i<j\le n}\theta(\hat{x}_i/\hat{x}_j,\hat{x}_i\hat{x}_j)}$$
$$\times \sum_{r\in\mathbb{Z}^n} \Delta_{\mathrm{B}}(\hat{x}q^r) \prod_{i=1}^n \hat{x}_i^{\kappa r_i-i+1} q^{\frac{1}{2}\kappa r_i^2-\frac{1}{2}(2n-1)r_i}.$$

We are now ready to make the substitutions

$$q\mapsto q^{2n-2},\qquad x_i\mapsto q^{n-i}\ \ (2\le i\le n-1), \tag{3.40}$$

so that $\hat{x}_i:=-q^{n-i}$ for $1\le i\le n$. By the identity

$$(q^{2n-2};q^{2n-2})_\infty^{n-1}(q^{n-1};q^{n-1})_\infty \prod_{i=1}^n \theta(-q^{n-i};q^{n-1})$$
$$\times \prod_{1\le i<j\le n} \theta(q^{j-i},q^{2n-i-j};q^{2n-2}) = 4(q^2;q^2)_\infty(q)_\infty^{n-1}$$

and (3.5), we obtain

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{2|\lambda|} P_{2\lambda}\bigl(1, q, q^2, \dots ; q^{2n-3}\bigr) = \frac{\mathcal{M}}{4(q^2;q^2)_\infty (q)_\infty^{n-1}},$$

where $\mathcal{M}$ is given by

$$\mathcal{M} := \sum_{r \in \mathbb{Z}^n} \Delta_{\mathrm{B}}(\hat{x} q^{2(n-1)r}) \prod_{i=1}^{n} \hat{x}_i^{\kappa r_i - i + 1} q^{(n-1)\kappa r_i^2 - (n-1)(2n-1) r_i}.$$

By the B_n determinant (3.21), we find that

$$\mathcal{M} = \det_{1 \le i,j \le n} \biggl(\sum_{r \in \mathbb{Z}} \hat{x}_i^{\kappa r - i + 1} q^{(n-1)\kappa r^2 - (n-1)(2n-1) r} \cdot \Bigl(\bigl(\hat{x}_i q^{2(n-1)r}\bigr)^{j-1} - \bigl(\hat{x}_i q^{2(n-1)r}\bigr)^{2n-j} \Bigr) \biggr).$$

By the same substitutions that transformed (3.19) into (3.20), we obtain

$$\mathcal{M} = \det_{1 \le i,j \le n} \biggl((-1)^{i-j} q^{b_{ij}} \sum_{r \in \mathbb{Z}} y_i^{2(n-1)r - i + 1} q^{2(n-1)\kappa \binom{r}{2}} \cdot \Bigl(\bigl(y_i q^{\kappa r}\bigr)^{j-1} + \bigl(y_i q^{\kappa r}\bigr)^{2n-j-1} \Bigr) \biggr),$$

where y_i and b_{ij} are as in (3.32). Recalling the Weyl denominator formula for $\mathbf{D}_n$ [198]

$$\frac{1}{2} \det_{1 \le i,j \le n} \bigl(x_i^{j-1} + x_i^{2n-j-1} \bigr) = \prod_{1 \le i < j \le n} (x_i - x_j)(x_i x_j - 1) =: \Delta_{\mathbf{D}}(x)$$

we can rewrite $\mathcal{M}$ in the form

$$\mathcal{M} = 2 \sum_{r \in \mathbb{Z}^n} \Delta_{\mathbf{D}}(x q^r) \prod_{i=1}^{n} y_i^{2(n-1)r_i - i + 1} q^{2(n-1)\kappa \binom{r_i}{2}}.$$

Taking the $a_1, \dots, a_{2n-2} \to 0$, $a_{2n-1} \to 1$ and $a_{2n} \to -1$ limit in Gustafson's multiple ${}_6\psi_6$ summation for the affine root system $\mathbb{A}_{2n-1}^{(2)}$ [154] leads to the following variant of the $\mathbf{D}_n^{(1)}$ Macdonald identity[3]

$$\sum_{r\in\mathbb{Z}^n}\Delta_{\mathbf{D}}(xq^r)\prod_{i=1}^n x_i^{2(n-1)r_i-i+1}q^{2(n-1)\binom{r_i}{2}}=2(q)_\infty^n\prod_{1\le i<j\le n}\theta(x_i/x_j,x_ix_j;q).$$

This implies the claimed product form for $\mathcal{M}$ and completes our proof. □

Identity (1.11) has a representation-theoretic interpretation. By [19, Lemma 2.4], the right-hand side of (3.39) in which $\hat{x}$ is interpreted as

$$\hat{x}_i=\mathrm{e}^{-\alpha_i-\cdots-\alpha_n}\quad(1\le i\le n)$$

and q as

$$q=\mathrm{e}^{-2\alpha_0-\cdots-2\alpha_n}$$

yields the $\mathbf{D}_{n+1}^{(2)}$ character

$$\mathrm{e}^{-2m\Lambda_0}\,\mathrm{ch}\,V(2m\Lambda_0).$$

The specialization (3.40) then corresponds to

$$\mathrm{e}^{-\alpha_0},\,\mathrm{e}^{-\alpha_n}\mapsto -1\quad\text{and}\quad \mathrm{e}^{-\alpha_i}\mapsto q\quad(2\le i\le n-1).$$

Denoting this by F, we have

$$F\big(\mathrm{e}^{-\Lambda}\,\mathrm{ch}\,V(\Lambda)\big)=\frac{(q^\kappa;q^\kappa)_\infty^n}{(q^2;q^2)_\infty(q)_\infty^{n-1}}\prod_{1\le i<j\le n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j-i-j+2n+1};q^\kappa\big),$$

where $\kappa=2n+2\lambda_0$ and

$$\Lambda=2(\lambda_0-\lambda_1)\Lambda_0+(\lambda_1-\lambda_2)\Lambda_1+\cdots+(\lambda_{n-1}-\lambda_n)\Lambda_{n-1}+2\lambda_n\Lambda_n,$$

for $\lambda=(\lambda_0,\lambda_1,\dots,\lambda_n)$ a partition or half-partition (i.e., all $\lambda_i\in\mathbb{Z}+1/2$). For $\lambda=(m,0^n)$ this agrees with (1.11).

---

[3] As in the $\mathrm{B}_n^{(1)}$ case, the actual $\mathbf{D}_n^{(1)}$ Macdonald identity contains the restriction $|r|\equiv 0\pmod 2$ on the sum over r.

3.3 Proof of Theorem 1.7

For integers k and m, where $0 \leq k \leq m$, we denote the the *nearly-rectangular* partition $(\underbrace{m, \dots, m}_{r \text{ times}}, k)$ as (m^r, k). Using these partitions, we have the following "limiting" Rogers–Ramanujan-type identities, which imply Theorem 1.7 when $k = 0$ or $k = m$.

Theorem 3.5 ($\mathbb{A}_{n-1}^{(1)}$ RR and AG identities)**.** *If m and n are positive integers and $0 \leq k \leq m$, then we have*

$$\begin{aligned}\lim_{r\to\infty} q^{-m\binom{r}{2}-kr} Q_{(m^r,k)}(1, q, q^2, \dots; q^n) \\ = \frac{(q^n; q^n)_\infty (q^\kappa; q^\kappa)_\infty^{n-1}}{(q)_\infty^n} \cdot \prod_{i=1}^{n-1} \theta(q^{i+k}; q^\kappa) \cdot \prod_{1\leq i<j\leq n-1} \theta(q^{j-i}; q^\kappa), \quad (3.41)\end{aligned}$$

where $\kappa = m + n$.

Remark. A similar calculation when $k \geq m$ gives

$$\begin{aligned}\lim_{r\to\infty} q^{-m\binom{r+1}{2}} Q_{(k,m^r)}(1, q, q^2, \dots; q^n) \\ = \begin{bmatrix} k-m+n-1 \\ n-1 \end{bmatrix}_q \frac{(q^n; q^n)_\infty (q^\kappa; q^\kappa)_\infty^{n-1}}{(q)_\infty^n} \prod_{1\leq i<j\leq n} \theta(q^{j-i}; q^\kappa).\end{aligned}$$

Proof of Theorem 3.5. It suffices to prove the identity for $0 \leq k < m$, and below assume that k satisfies this inequality.

The following identity for the modified Hall–Littlewood polynomials indexed by near-rectangular partitions is a special case of [19, Corollary 3.2]:

$$\begin{aligned}Q'_{(m^r,k)}(x; q) = (q)_r (q)_1 \sum_{\substack{u\in\mathbb{Z}_+^n \\ |u|=r+1}} \sum_{\substack{v\in\mathbb{Z}_+^n \\ |v|=r}} \prod_{i=1}^n x_i^{ku_i+(m-k)v_i} q^{k\binom{u_i}{2}+(m-k)\binom{v_i}{2}} \\ \times \prod_{i,j=1}^n \frac{(qx_i/x_j)_{u_i-u_j}}{(qx_i/x_j)_{u_i-v_j}} \cdot \frac{(qx_i/x_j)_{v_i-v_j}}{(qx_i/x_j)_{v_i}}.\end{aligned}$$

It is enough to compute the limit on the left-hand side of (3.41) for r a multiple of n. Hence we replace r by nr in the above expression, and then shift $u_i \mapsto u_i + r$ and $v_i \mapsto v_i + r$, for all $1 \le i \le n$, to obtain

$$Q'_{(m^{nr},k)}(x;q) = (x_1 \cdots x_n)^{mr} q^{mn\binom{r}{2}+kr} (q)_{nr} (q)_1 \\ \times \sum_{\substack{u\in\mathbb{Z}^n \\ |u|=1}} \sum_{\substack{v\in\mathbb{Z}^n \\ |v|=0}} \prod_{i=1}^{n} x_i^{ku_i+(m-k)v_i} q^{k\binom{u_i}{2}+(m-k)\binom{v_i}{2}} \prod_{i,j=1}^{n} \frac{(qx_i/x_j)_{u_i-u_j}}{(qx_i/x_j)_{u_i-v_j}} \cdot \frac{(qx_i/x_j)_{v_i-v_j}}{(qx_i/x_j)_{r+v_i}}.$$

Since the summand vanishes unless $u_i \ge v_i$ for all i and $|u| = |v| + 1$, it follows that $u = v + \epsilon_\ell$, for some $\ell = 1, \dots, n$, where $(\epsilon_\ell)_i = \delta_{\ell i}$. Hence we find that

$$Q'_{(m^{nr},k)}(x;q) = (x_1 \cdots x_n)^{mr} q^{mn\binom{r}{2}+kr} (q)_{nr} \\ \times \sum_{\substack{v\in\mathbb{Z}^n \\ |v|=0}} \prod_{i=1}^{n} x_i^{mv_i} q^{m\binom{v_i}{2}} \prod_{i,j=1}^{n} \frac{(qx_i/x_j)_{v_i-v_j}}{(qx_i/x_j)_{r+v_i}} \sum_{\ell=1}^{n} \left(x_\ell q^{v_\ell}\right)^k \prod_{\substack{i=1 \\ i\ne k}}^{n} \frac{1}{1-q^{v_i-v_\ell}x_i/x_\ell}.$$

Next we use

$$\prod_{i,j=1}^{n} (qx_i/x_j)_{v_i-v_j} = \frac{\Delta(xq^v)}{\Delta(x)} (-1)^{(n-1)|v|} q^{-\binom{|v|}{2}} \prod_{i=1}^{n} x_i^{nv_i-|v|} q^{n\binom{v_i}{2}+(i-1)v_i},$$

where $\Delta(x) := \prod_{1\le i<j\le n}(1 - x_i/x_j)$, and

$$\sum_{\ell=1}^{n} x_\ell^k \prod_{\substack{i=1 \\ i\ne k}}^{n} \frac{1}{1-x_i/x_\ell} = \sum_{1\le i_1\le i_2\le\cdots\le i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k} = h_k(x) = s_{(k)}(x),$$

where h_k and s_λ are the complete symmetric and Schur function, respectively. Thus we have

$$Q'_{(m^{nr},k)}(x;q) = (x_1 \cdots x_n)^{mr} q^{mn\binom{r}{2}+kr} (q)_{nr} \\ \times \sum_{\substack{v\in\mathbb{Z}^n \\ |v|=0}} s_{(k)}(xq^v) \frac{\Delta(xq^v)}{\Delta(x)} \prod_{i=1}^{n} x_i^{\kappa v_i} q^{\frac{1}{2}\kappa v_i^2 + i v_i} \prod_{i,j=1}^{n} \frac{1}{(qx_i/x_j)_{r+v_i}},$$

where $\kappa := m+n$. Note that the summand vanishes unless $v_i \geq -r$ for all i. This implies the limit

$$\lim_{r\to\infty} q^{-mn\binom{r}{2}-kr}\frac{Q'_{(m^{nr},k)}(x;q)}{(x_1\cdots x_n)^{mr}} = \frac{1}{(q)_\infty^{n-1}\prod_{1\leq i<j\leq n}\theta(x_i/x_j;q)}\sum_{\substack{v\in\mathbb{Z}^n\\|v|=0}} s_{(k)}(xq^v)\Delta(xq^v)\prod_{i=1}^n x_i^{\kappa v_i}q^{\frac{1}{2}\kappa v_i^2+iv_i}.$$

The expression on the right is exactly the Weyl–Kac formula for the level-m $\mathbb{A}_{n-1}^{(1)}$ character [183]

$$\mathrm{e}^{-\Lambda}\,\mathrm{ch}\,V(\Lambda), \quad \Lambda=(m-k)\Lambda_0+k\Lambda_1,$$

provided we identify

$$q=\mathrm{e}^{-\alpha_0-\alpha_1-\cdots-\alpha_{n-1}} \quad \text{and} \quad x_i/x_{i+1}=\mathrm{e}^{-\alpha_i} \;\; (1\leq i\leq n-1).$$

Hence

$$\lim_{r\to\infty} q^{-mn\binom{r}{2}-kr}\frac{Q'_{(m^{nr},k)}(x;q)}{(x_1\cdots x_n)^{mr}} = \mathrm{e}^{-\Lambda}\,\mathrm{ch}\,V(\Lambda),$$

with Λ as above. For $m=1$ and $k=0$ this was obtained in [187] by more elementary means. The simultaneous substitutions $q\mapsto q^n$ and $x_i\mapsto q^{n-i}$ correspond to the principal specialization (3.23). From (3.24) we can then read off the product form claimed in (3.41). □

3.4 Siegel Functions

The normalizations for the series Φ_* were chosen so that the resulting q-series are modular functions on the congruence subgroups $\Gamma(N)$, where

$$\Gamma(N) := \left\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in SL_2(\mathbb{Z}):\; a\equiv d\equiv 1 \,(\mathrm{mod}\, N)\,,\; b\equiv c\equiv 0\,(\mathrm{mod}\,N)\,\right\}.$$

These groups act on $\mathbb{H}$, the upper-half of the complex plane, by $\gamma\tau := \frac{a\tau+b}{c\tau+d}$, where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. If f is a meromorphic function on $\mathbb{H}$ and $\gamma \in SL_2(Z)$, then we define

$$(f|_k\gamma)(\tau) := (c\tau+d)^{-k} f(\gamma\tau).$$

Modular functions are meromorphic functions which are invariant with respect to this action. More precisely, a meromorphic function f on $\mathbb{H}$ is a *modular function* on $\Gamma(N)$ if for every $\gamma \in \Gamma(N)$ we have

$$f(\gamma\tau) = (f|_0\gamma)(\tau) = f(\tau).$$

The set of such functions form a field. We let $\mathcal{F}_N$ denote the canonical subfield of those modular functions on $\Gamma(N)$ whose Fourier expansions are defined over $\mathbb{Q}(\zeta_N)$, where $\zeta_N := \mathrm{e}^{2\pi \mathrm{i}/N}$.

The important work of Kubert and Lang [199] plays a central role in the study of these modular function fields. Their work, which is built around the Siegel g_a functions and the Klein $\mathfrak{t}_a$ functions, allows us to understand the fields $\mathcal{F}_N$, as well as the Galois theoretic properties of the extensions $\mathcal{F}_N/\mathcal{F}_1$. These results will be fundamental tools in the proofs of Theorems 1.8 and 1.15.

3.4.1 Basic Facts about Siegel functions

We begin by recalling the definitions of the Siegel and Klein functions. Let $\mathbf{B}_2(x) := x^2 - x + \frac{1}{6}$ be the second Bernoulli polynomial and $\mathrm{e}(x) := \mathrm{e}^{2\pi \mathrm{i} x}$. If $a = (a_1, a_2) \in \mathbb{Q}^2$, then *Siegel function* g_a is defined as

$$g_a(\tau) := -q^{\frac{1}{2}\mathbf{B}_2(a_1)} \mathrm{e}\big(a_2(a_1-1)/2\big) \prod_{n=1}^{\infty} \big(1 - q^{n-1+a_1}\mathrm{e}(a_2)\big)\big(1 - q^{n-a_1}\mathrm{e}(-a_2)\big), \tag{3.42}$$

and the *Klein function* $\mathfrak{t}_a$ is defined as

$$\mathfrak{t}_a(\tau) := \frac{g_a(\tau)}{\eta(\tau)^2} \tag{3.43}$$

where $\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the *Dedekind η-function.*

Neither g_a nor $\mathfrak{t}_a$ are modular on $\Gamma(N)$, however if $N\cdot a\in\mathbb{Z}^2$, then $\mathfrak{t}_a^{2N}$ is on $\Gamma(N)$ (or $\mathfrak{t}_a^N$ if N is odd). Therefore if $g_a^{\mathrm{lcm}(12,2N)}\in\mathcal{F}_N$, and if $N\cdot a'\in\mathbb{Z}^2$, then $\left(\frac{g_a(\tau)}{g_{a'}(\tau)}\right)^{2N}\in\mathcal{F}_N$ if N is even and $\left(\frac{g_a(\tau)}{g_{a'}(\tau)}\right)^{N}\in\mathcal{F}_N$ if N is odd. Given $a\in\mathbb{Q}^2$, we denote the smallest $N\in\mathbb{N}$ such that $N\cdot a\in\mathbb{Z}^2$ by $\mathrm{Den}(a)$.

Theorem 3.6 ([199, Ch. 2 of K1 and K2]). *Assuming the notation above, the following are true:*

1. *If $\gamma\in SL_2(\mathbb{Z})$, then*
$$(\mathfrak{t}_a|_{-1}\gamma)(\tau)=\mathfrak{t}_{a\gamma}(\tau).$$

2. *If $b=(b_1,b_2)$, then*
$$\mathfrak{t}_{a+b}(\tau)=e\big((b_1b_2+b_1+b_2-b_1a_2+b_2a_1)\big)\mathfrak{t}_{a\gamma}(\tau).$$

These properties for $\mathfrak{t}_a$, (3.42), and the fact that $\eta(\tau)^{24}=\Delta(\tau)$ is modular on $SL_2(\mathbb{Z})$, lead to the following properties for g_a.

Theorem 3.7 ([199, Ch. 2, Thm 1.2]). *If $a\in\mathbb{Z}^2/N$ and $\mathrm{Den}(a)=N$, then the following are true:*

1. *If $\gamma\in SL_2(\mathbb{Z})$, then*
$$(g_a^{12}|_0\gamma)(\tau)=g_{a\gamma}^{12}(\tau).$$

2. *If $b=(b_1,b_2)\in\mathbb{Z}^2$, then*
$$g_{a+b}(\tau)=e\big(1/2\cdot(b_1b_2+b_1+b_2-b_1a_2+b_2a_1)\big)g_a(\tau).$$

3. *We have that $g_{-a}(\tau)=-g_a(\tau)$.*

4. *The $g_a(\tau)^{12N}$ are modular functions on $\Gamma(N)$. Moreover, if $\gamma\in SL_2(\mathbb{Z})$, then we have*
$$(g_a^{12}|_0\gamma)(\tau)=g_{a\gamma}^{12}(\tau).$$

The following theorem addresses the modularity properties of products and quotients of Siegel functions.

Theorem 3.8 ([199, Ch. 3, Lemma 5.2, Thm 5.3])**.** *Let $N \geq 2$ be an integer, and let $\{m(a)\}_{r\in\frac{1}{N}\mathbb{Z}^2/\mathbb{Z}^2}$ be a set of integers. Then the product of Siegel functions*

$$\prod_{a\in\frac{1}{N}\mathbb{Z}^2/\mathbb{Z}^2} g_a^{m(a)}(\tau)$$

belongs to $\mathcal{F}_N$ if $\{m(a)\}$ satisfies the following:

1. *We have that $\sum_a m(a)(Na_1)^2 \equiv \sum_a m(a)(Na_2)^2 \equiv 0 \,(\mathrm{mod}\; \gcd(2,N)\cdot N)$.*
2. *We have that $\sum_a m(a)(Na_1)(Na_2) \equiv 0 \,(\mathrm{mod}\, N)$.*
3. *We have that $\gcd(12,N)\cdot\sum_a m(a) \equiv 0 \,(\mathrm{mod}\, 12)$.*

Additionally, we have the following important results about the algebraicity of the singular values of the Siegel functions in relation to the singular values of the $SL_2(\mathbb{Z})$ modular function

$$\begin{aligned} j(\tau) &:= \frac{\left(1+240\sum_{n=1}^{\infty}\sum_{d|n} d^3q^n\right)^3}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} \\ &= \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} + 3\cdot 2^8 + 3\cdot 2^{16}\,\frac{\eta(2\tau)^{24}}{\eta(\tau)^{24}} + 2^{24}\,\frac{\eta(2\tau)^{48}}{\eta(\tau)^{48}} \\ &= q^{-1} + 744 + 196884q + \cdots, \end{aligned}$$

which are well known to be algebraic by the theory of complex multiplication (for example, see [33, 85]).

Theorem 3.9 ([199, Ch. 1, Thm. 2.2])**.** *If τ is a CM point and $N = \mathrm{Den}(a)$, then the following are true:*

1. *We have that $g_a(\tau)$ is an algebraic integer.*

2. If N has at least two prime factors, then $g_a(\tau)$ is a unit over $\mathbb{Z}[j(\tau)]$.

3. If $N = p^r$ is a prime power, then $g_a(\tau)$ is a unit over $\mathbb{Z}[1/p][j(\tau)]$.

4. If $c \in \mathbb{Z}$ and $(c, N) = 1$, then (g_{ca}/g_a) is a unit over $\mathbb{Z}[j(\tau)]$.

3.4.2 Galois theory of singular values of Siegel functions

We now recall the Galois-theoretic properties of extensions of modular function fields, and we then relate these properties to the Siegel and Klein functions.

The Galois group $\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$ is isomorphic to $GL_2(N)/\{\pm I\} = GL_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\}$ (see [199, Ch. 3, Lemma 2.1]), where I is the identity matrix. This group factors naturally as

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \ : \ d \in (\mathbb{Z}/N\mathbb{Z})^\times \right\} \times SL_2(N)/\{\pm I\},$$

where an element $\left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)$ acts on the Fourier coefficients by sending $\zeta_N \to \zeta_N^d$, and a matrix $\gamma \in SL_2(\mathbb{Z})$ acts by the standard fractional linear transformation on τ. If $f(\tau) \in \mathcal{F}_N$ and $\gamma \in GL_2(N)$, then we use the notation $f(\tau)_{(\gamma)} := (\gamma \circ f)(\tau)$. Applying these facts to the Siegel functions, we obtain the following.

Proposition 3.10. *If $a \in \mathbb{Q}^2$, and $\mathrm{Den}(a)$ divides N, then the multiset*

$$\left\{ g_a^{12N}(\tau)(\gamma) := g_{a\gamma}^{12N}(\tau) : \ \gamma \in GL_2(N) \right\}$$

is a union of Galois orbits for $g_a^{12N}(\tau)$ over $\mathcal{F}_1$.

If θ is a CM point of discriminant $-D$, we define the field

$$K_{(N)}(\theta) := \mathbb{Q}(\theta)\big(f(\theta) : \ f \in \mathcal{F}_N \text{ s.t. } f \text{ is defined and finite at } \theta\big),$$

and $\mathcal{H} := \mathbb{Q}(\theta, j(\theta))$ be the Hilbert class field over $\mathbb{Q}(\theta)$. The Galois group $K_{(N)}(\theta)/\mathcal{H}$ is isomorphic to the matrix group $W_{N,\theta}$ (see [274]) defined by

$$W_{N,\theta} = \left\{ \begin{pmatrix} t - sB & -sC \\ sA & t \end{pmatrix} \in GL_2(\mathbb{Z}/N\mathbb{Z}) \right\} \Big/ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

where $Ax^2 + Bx + C$ is a minimal polynomial for θ over $\mathbb{Z}$. The Galois group $\mathrm{Gal}(\mathcal{H}/\mathbb{Q})$ is isomorphic to the group $\mathcal{Q}_D$ of primitive reduced positive-definite integer binary quadratic forms of negative discriminant $-D$. For each $Q = ax^2 + bxy + cy^2 \in \mathcal{Q}_D$, we define the corresponding CM point $\tau_Q = \frac{-b+\sqrt{-D}}{2a}$. In order to define the action of this group, we must also define corresponding matrices $\beta_Q \in GL_2(\mathbb{Z}/N\mathbb{Z})$ which we may build up by way of the Chinese Remainder Theorem and the following congruences. For each prime p dividing N, we have the following congruences which hold $\left(\mathrm{mod}\, p^{\mathrm{ord}_p(N)}\right)$:

$$\beta_Q \equiv \begin{cases} \begin{pmatrix} a & \frac{b}{2} \\ 0 & 1 \end{pmatrix} & \text{if } p \nmid a \\ \begin{pmatrix} -\frac{b}{2} & -c \\ 1 & 0 \end{pmatrix} & \text{if } p|a, \text{ and } p \nmid c \\ \begin{pmatrix} -\frac{b}{2} - a & -\frac{b}{2} - c \\ 1 & -1 \end{pmatrix} & \text{if } p|a, \text{ and } p|c \end{cases}$$

if $-D \equiv 0 \pmod 4$, and

$$\beta_Q \equiv \begin{cases} \begin{pmatrix} a & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} & \text{if } p \nmid a \\ \begin{pmatrix} -\frac{b+1}{2} & -c \\ 1 & 0 \end{pmatrix} & \text{if } p|a, \text{ and } \ p \nmid c \\ \begin{pmatrix} -\frac{b+1}{2} - a & \frac{1-b}{2} - c \\ 1 & -1 \end{pmatrix} & \text{if } p|a, \ \text{ and } p|c \end{cases}$$

if $-D \equiv 1 \pmod 4$. Then given $\theta = \tau_{Q'}$ for some $Q' \in \mathcal{Q}_D$, define $\delta_Q(\theta) := \beta_{Q'}^{-1}\beta_Q$. The Galois group $\mathrm{Gal}(\mathcal{H}/\mathbb{Q})$ can be extended into $\mathrm{Gal}(K_{(N)}(\theta)/\mathbb{Q})$ by taking the action of a quadratic form Q on the element $f(\theta) \in K_{(N)}(\theta)$ to be given by

$$Q \circ f(\theta) = f(\tau_Q)_{(\delta_Q(\theta))}.$$

We combine these facts into the following theorem.

Theorem 3.11. *Let $F(\tau)$ be in $\mathcal{F}_N$ and let θ be a CM point of discriminant $-D < 0$. Then the multiset*

$$\left\{F(\tau_Q)_{(\gamma\cdot\delta_Q(\theta))} : (\gamma, Q) \in W_{\kappa,\tau} \times \mathcal{Q}_D\right\}$$

is a union of the Galois orbits of $F(\theta)$ over $\mathbb{Q}$.

3.5 Proofs of Theorems 1.8 and 1.15

Here we prove Theorems 1.8 and 1.15. We shall prove these theorems using the results of the previous section.

3.5.1 Reformulation of the $\Phi_*(m, n; \tau)$ series

To ease the proofs of Theorems 1.8 and 1.15, we begin by reformulating each of the $\Phi_*(m, n; \tau)$ series, as well as

$$\frac{\Phi_{1a}(m, n; \tau)}{\Phi_{1b}(m, n; \tau)},$$

as pure products of modified theta functions. These factorizations will be more useful for our purposes. In order to ease notation, for a fixed κ, if $1 \leq j < \kappa/2$, then we let

$$\theta_{j,\kappa} := \theta(q^j; q^\kappa),$$

If κ is even, then we let

$$\theta_{\kappa/2,\kappa} := (q^{\kappa/2}; q^{\kappa})_\infty = \theta(q^{\kappa/2}; q^{2\kappa}),$$

which is a square root of $\theta(q^{\kappa/2}; q^{\kappa})$.

The reformulations below follow directly from (1.16) by making use of the fact that

$$\frac{(q^\kappa; q^\kappa)_\infty}{(q)_\infty} = \prod_{j=1}^{\lfloor \kappa/2 \rfloor} \theta_{j,\kappa}.$$

Lemma 3.12. *Let m and n be positive integers and $\kappa_* = \kappa_*(m,n)$ as in* (1.15)*. Then the following are true:*

(1a) With $\kappa = \kappa_1$,

$$\Phi_{1a}(m,n;\tau) = q^{\frac{mn(4mn-4m+2n-3)}{12\kappa}} \prod_{j=1}^{m} \theta_{j,\kappa}^{-1} \prod_{j=1}^{m+n} \theta_{j,\kappa}^{-\min(m,n-1,\lceil j/2 \rceil -1)}.$$

(1a) With $\kappa = \kappa_1$,

$$\Phi_{1b}(m,n;\tau) = q^{\frac{mn(4mn+2m+2n+3)}{12\kappa}} \prod_{j=1}^{m+n} \theta_{j,\kappa}^{-\min(m,n,\lfloor j/2 \rfloor)}.$$

(2) With $\kappa = \kappa_2$,

$$\Phi_2(m,n;\tau) = q^{\frac{m(2n+1)(2mn-m+n-1)}{12\kappa}} \prod_{j=1}^{m} \theta_{j,\kappa}^{-1} \prod_{j=1}^{m+n+1} \theta_{j,\kappa}^{-\min(m,n-1,\lceil j/2 \rceil -1)} \prod_{j=n}^{\lfloor (m+n)/2 \rfloor} \theta_{2j+1,\kappa}^{-1}.$$

(3) For $n \geq 2$ and $\kappa = \kappa_3$,

$$\Phi_3(m,n;\tau) = q^{\frac{m(2n-1)(2mn+n+1)}{12\kappa}} \prod_{j=1}^{m+n} \theta_{j,\kappa}^{-\min(m,n-1,\lfloor j/2 \rfloor)} \prod_{j=n}^{\lfloor (m+n)/2 \rfloor} \theta_{2j,\kappa}^{-1}.$$

Moreover, with $\kappa = \kappa_1(m,n)$,

$$\Psi_1(m,n;\tau) := \frac{\Phi_{1a}(m,n;\tau)}{\Phi_{1b}(m,n;\tau)} = q^{-\frac{mn(m+1)}{2\kappa}} \prod_{j=1}^{m} \frac{\theta(q^{2j};q^\kappa)}{\theta(q^j;q^\kappa)},$$

and with $\kappa = \kappa_2(m,n) = \kappa_3(m,n+1)$,

$$\Psi_2(m,n;\tau) := \frac{\Phi_2(m,n;\tau)}{\Phi_3(m,n;\tau)} = q^{-\frac{m(m+1)(2n+1)}{4\kappa}} \prod_{j=1}^{m} \frac{\theta(q^{2j};q^\kappa)}{\theta(q^j;q^\kappa)}.$$

Proof. Since the proofs of the four cases are essentially the same, we only prove Lemma 3.12 (1a).

Let $\varphi = mn(4mn - 4m + 2n - 3)/(12\kappa_1)$. By Theorem 1.2, we have that

$$\begin{aligned}\Phi_{1a}(m,n;\tau) &= q^\varphi \cdot \frac{(q^\kappa;q^\kappa)_\infty^n}{(q)_\infty^n} \prod_{i=1}^{n} \theta(q^{i+m};q^\kappa) \prod_{1\le i<j\le n} \theta(q^{j-i},q^{i+j-1};q^\kappa)\\ &= q^\varphi \cdot \frac{(q^\kappa;q^\kappa)_\infty^m}{(q)_\infty^m} \prod_{i=1}^{m} \theta(q^{i+1};q^\kappa) \prod_{1\le i<j\le m} \theta(q^{j-i},q^{i+j+1};q^\kappa).\end{aligned}$$

Using the simple identity

$$\frac{(q^\kappa;q^\kappa)_\infty}{(q)_\infty} = \prod_{j=1}^{m+n} \theta(q^j;q^\kappa)^{-1},$$

we can rewrite these two forms as

$$\begin{aligned}\Phi_{1a}(m,n;\tau) &= q^\varphi \cdot \frac{\prod_{i=1}^{n} \theta(q^{i+m};q^\kappa)}{\prod_{i=1}^{m+n} \theta(q^j;q^\kappa)} \cdot \prod_{j=2}^{n} \frac{\prod_{i=1}^{j-1} \theta(q^{j-i},q^{i+j-1};q^\kappa)}{\prod_{i=1}^{m+n} \theta(q^i,q^\kappa)}\\ &= q^\varphi \cdot \frac{\prod_{i=1}^{m} \theta(q^{i+1};q^\kappa)}{\prod_{i=1}^{m+n} \theta(q^j;q^\kappa)} \cdot \prod_{j=2}^{m} \frac{\prod_{i=1}^{j-1} \theta(q^{j-i},q^{i+j+1};q^\kappa)}{\prod_{i=1}^{m+n} \theta(q^i;q^\kappa)}.\end{aligned}$$

If $m \ge n-1$ then the first identity reduces to

$$\begin{aligned}\Phi_{1a}(m,n;\tau) &= q^\varphi \cdot \left(\prod_{j=1}^{m} \theta(q^j;q^\kappa)\right)^{-1} \prod_{j=2}^{n} \left(\prod_{i=2j-1}^{m+n} \theta(q^i;q^\kappa)\right)^{-1}\\ &= q^\varphi \cdot \left(\prod_{j=1}^{m} \theta(q^j;q^\kappa)\right)^{-1} \prod_{j=1}^{n-1} \left(\prod_{i=2j+1}^{m+n} \theta(q^i;q^\kappa)\right)^{-1}.\end{aligned}$$

If $m \leq n-1$ then the second identity reduces to

$$\begin{aligned}\Phi_{1a}(m,n;\tau) &= q^{\varphi} \cdot \left(\theta(q;q^{\kappa}) \prod_{j=m+2}^{m+n} \theta(q^j;q^{\kappa})\right)^{-1} \\ &\qquad \times \prod_{j=2}^{m} \left(\theta(q^j,q^{j+1};q^{\kappa}) \prod_{i=2j+1}^{m+n} \theta(q^i;q^{\kappa})\right)^{-1} \\ &= q^{\varphi} \cdot \left(\prod_{j=1}^{m} \theta(q^j;q^{\kappa})\right)^{-1} \prod_{j=1}^{m} \left(\prod_{i=2j+1}^{m+n} \theta(q^i;q^{\kappa})\right)^{-1}.\end{aligned}$$

Together these imply Lemma 3.12 (1a). □

Since the modified θ-functions $\theta(q^{\ell};q^{\kappa})$ are essentially Siegel functions (up to powers of q), we can immediately rewrite Lemma 3.12 in terms of modular functions. We shall omit the proofs for brevity.

Lemma 3.13. *Let m and n be positive integers and $\kappa_* = \kappa_*(m,n)$ as in* (1.15)*. Then the following are true:*

(1a) With $\kappa = \kappa_1$,

$$\Phi_{1a}(m,n;\tau) = \prod_{j=1}^{m} g_{j/\kappa,0}(\kappa\tau)^{-1} \prod_{j=1}^{m+n} g_{j/\kappa,0}(\kappa\tau)^{-\min(m,n-1,\lceil j/2\rceil-1)}.$$

(1b) With $\kappa = \kappa_1$,

$$\Phi_{1b}(m,n;\tau) = \prod_{j=1}^{m+n} g_{j/\kappa,0}(\kappa\tau)^{-\min(m,n,\lfloor j/2\rfloor)}.$$

(2) With $\kappa = \kappa_2$,

$$\begin{aligned}\Phi_2(m,n;\tau) &= g_{\frac{1}{4},0}(2\kappa\tau)^{-\min(m,n-1)-\delta_1} \prod_{j=1}^{m} g_{\frac{j}{\kappa},0}(\kappa\tau)^{-1} \\ &\times \prod_{j=1}^{m+n} g_{\frac{j}{\kappa},0}(\kappa\tau)^{-\min(m,n-1,\lceil j/2\rceil-1)} \prod_{j=n}^{\lfloor (m+n-1)/2\rfloor} g_{\frac{(2j+1)}{\kappa},0}(\kappa\tau)^{-1}.\end{aligned}$$

(3) For $n \geq 2$ and $\kappa = \kappa_3$,

$$\Phi_3(m,n;\tau) = g_{\frac{1}{4},0}(2\kappa\tau)^{-\min(m,n-1)-\delta_2}$$
$$\times \prod_{j=1}^{m+n-1} g_{\frac{j}{\kappa},0}(\kappa\tau)^{-\min(m,n-1,\lfloor j/2 \rfloor)} \prod_{j=n}^{\lfloor (m+n-1)/2 \rfloor} g_{\frac{2j}{\kappa},0}(\kappa\tau)^{-1}.$$

(4) With $\kappa = \kappa_1$,

$$\Psi_1(m,n;\tau) := \frac{\Phi_{1a}(m,n;\tau)}{\Phi_{1b}(m,n;\tau)} = \prod_{j=1}^{m} \frac{g_{\frac{2j}{\kappa},0}(\kappa\tau)}{g_{\frac{j}{\kappa},0}(\kappa\tau)}.$$

(5) With $\kappa = \kappa_2(m,n) = \kappa_3(m,n+1)$,

$$\Psi_2(m,n;\tau) := \frac{\Phi_2(m,n;\tau)}{\Phi_3(m,n;\tau)} = \prod_{j=1}^{m} \frac{g_{\frac{2j}{\kappa},0}(\kappa\tau)}{g_{\frac{j}{\kappa},0}(\kappa\tau)}.$$

3.5.2 Proofs of Theorems 1.8 and 1.15

We now apply the results in Section 3.4 to prove Theorems 1.8 and 1.15.

Proof of Theorem 1.8 (1) and (2). Lemma 3.13 shows that each of the $\Phi_*(m,n;\tau)$ is exactly a pure product of Siegel functions. Therefore, we may apply Theorem 3.7 directly to each of the Siegel function factors, and as a consequence to each $\Phi_*(m,n;\tau)$.

Since by Theorem 3.7 (4), $g_a(\tau)^{12N}$ is in $\mathcal{F}_N$ if $N = \mathrm{Den}(a)$, we may take $N = \kappa_*(m,n)$, and so we have that $\Phi_*(m,n;\tau)^{12\kappa} \in \mathcal{F}_{\kappa_*(m,n)}$. We now apply Theorem 3.11 to obtain Theorem 1.8 (1) and (2). □

Sketch of the Proof of Theorem 1.8 (3). By Theorem 1.8 (2), we have that this multiset consists of multiple copies of a single Galois orbit of conjugates over $\mathbb{Q}$. Therefore to complete the proof, it suffices to show that the given conditions imply that there are singular values which are not repeated. To this end, we focus on those CM points with maximal imaginary parts. Indeed, because each $\Phi_*(m,n;\tau)$ begins with a negative power of q, one generically expects that these corresponding singular values will be the one with maximal complex absolute value.

To make this argument precise requires some cumbersome but unenlightening details (which we omit)[4]. One begins by observing why the given conditions are necessary. For small κ it can happen that the matrices in $W_{\kappa,\tau}$ permute the Siegel functions in the factorizations of $\Phi_*(m,n;\tau)$ obtained in Lemma 3.13. However, if $\kappa > 9$, then this does not happen. The condition that $\gcd(D_0,\kappa) = 1$ is required for a similar reason. More precisely, the group does not act faithfully. However, under these conditions, the only obstruction to the conclusion would be a nontrivial identity between the evaluations of two different modular functions. In particular, under the given assumptions, we may view these functions as a product of distinct Siegel functions. Therefore, the proof follows by studying the asymptotic properties of the CM values of individual Siegel functions, and then considering the Φ_* functions as a product of these values.

The relevant asymptotics arise by considering, for each $-D$, a canonical CM point with discriminant $-D$. Namely, we let

$$\tau_* := \begin{cases} \frac{\sqrt{-D}}{2} & \text{if } -D \equiv 0 \pmod{4}, \\ \frac{1+\sqrt{-D}}{2} & \text{if } -D \equiv 1 \pmod{4}. \end{cases}$$

By the theory of reduced binary quadratic forms, these points are the CM points with maximal imaginary parts corresponding to reduced forms with discriminant $-D$. Moreover, every other CM point with discriminant $-D$ has imaginary part less than $|\sqrt{-D}|/3$. Now the singular values of each Siegel function then essentially arise from the values of the second Bernoulli polynomial. The point is that one can uniformly estimate the infinite product portion of each singular value, and it turns out that they are exponentially

[4] A similar analysis is carried out in detail by Jung, Koo, and Shin in [172, Sec. 4].

close to the number 1. By assembling these estimates carefully, one obtains the result. □

Proof of Theorem 1.15. Lemma 3.13 reformulates each Φ_* function in terms of products of negative powers of Siegel functions of the form $g_{j/\kappa,0}(\kappa\tau)$, where $1 \le j \le \kappa/2$, and $g_{1/4,0}(2\kappa\tau)$, when κ is even. Theorem 3.9 (1) then implies Theorem 1.15 (1).

Since $\mathrm{Den}(j/\kappa, 0)$ may be any divisor of κ, and since $j(\tau)$ is an algebraic integer [33, 85], Theorem 3.9 (2) and (3) imply Theorem 1.15 (2).

Using Theorem 3.13 (5), we have that

$$\frac{\Phi_{1a}(m,n;\tau)}{\Phi_{1b}(m,n;\tau)} = \prod_{j=1}^{m} \frac{g_{\frac{2j}{\kappa},0}(\kappa\tau)}{g_{\frac{j}{\kappa},0}(\kappa\tau)},$$

where $\kappa = \kappa_1 = 2m + 2n + 1$. Since κ is odd, Theorem 3.9 (4) implies that each term

$$\frac{g_{\frac{2j}{\kappa},0}(\kappa\tau)}{g_{\frac{j}{\kappa},0}(\kappa\tau)}$$

in the product is a unit. Therefore, Theorem 1.15 (3) follows. □

3.6 Examples

Here we give two examples of the main results in this paper.

Example 3.14. This is a detailed discussion of the example in the introduction.

Consider the q-series

$$\begin{aligned}\Phi_{1a}(2,2;\tau) &= q^{1/3} \prod_{n=1}^{\infty} \frac{(1-q^{9n})}{(1-q^n)} \\ &= q^{1/3} + q^{4/3} + 2q^{7/3} + 3q^{10/3} + 5q^{13/3} + 7q^{16/3} + \cdots,\end{aligned}$$

and

$$\Phi_{1b}(2,2;\tau) = q\prod_{n=1}^{\infty}\frac{(1-q^{9n})(1-q^{9n-1})(1-q^{9n-8})}{(1-q^{n})(1-q^{9n-4})(1-q^{9n-5})}$$
$$= q + q^3 + q^4 + 3q^5 + 3q^6 + 5q^7 + 6q^8 + \cdots$$

For $\tau = \mathrm{i}/3$, the first 100 coefficients of the q-series respectively give the numerical approximations

$$\Phi_{1a}(2,2;\mathrm{i}/3) = 0.577350\cdots \stackrel{?}{=} \frac{1}{\sqrt{3}}$$
$$\Phi_{1b}(2,2;\mathrm{i}/3) = 0.217095\ldots$$

Here we have that $\kappa_1(2,2) = 9$. Theorem 3.8 tells us that $\Phi_{1a}(2,2;\tau)^3$ and $\Phi_{1b}(2,2;\tau)^3$ are in $\mathcal{F}_9$, so we may use Theorem 3.11 to find the conjugates of the values of the functions at $\tau = \mathrm{i}/3$. We have $\kappa_1(2,2)\cdot \mathrm{i}/3 = 3\mathrm{i}$ and

$$W_{9,3i} = \left\{ \begin{pmatrix} t & 0 \\ s & t \end{pmatrix} \in GL_2(\mathbb{Z}/9\mathbb{Z}) \right\} \Big/ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

which has 27 elements. However each of these acts like the identity on $\Phi_{1a}(2,2;\tau)$, and the group has an orbit of size three when acting on $\Phi_{1b}(2,2;\tau)$. The set $\mathcal{Q}_{36}$ has two elements $Q_1 = x^2+9y^2$ and $Q_2 = 2x^2+2xy+5y^2$. These qive us β_{Q_1} which is the identity, and $\beta_{Q_2} = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore $\Phi_{1a}(2,2;\mathrm{i}/3)^3$ has only one other conjugate,

$$\left(g_{2/9,1/9}\Big(\frac{-1+3\mathrm{i}}{2}\Big)g_{4/9,2/9}\Big(\frac{-1+3\mathrm{i}}{2}\Big)g_{6/9,3/9}\Big(\frac{-1+3\mathrm{i}}{2}\Big)g_{8/9,4/9}\Big(\frac{-1+3\mathrm{i}}{2}\Big)\right)^{-3},$$

although the multiset described in Theorem 1.8 (2) contains 27 copies of these two numbers. On the other hand, $\Phi_{1b}(2,2;\mathrm{i}/3)^3$ has an orbit of six conjugates, and the multiset from Theorem 1.8 (2) contains nine copies of this orbit. Theorem 1.15 (2) tells us that $\Phi_{1a}(2,2;\mathrm{i}/3)$ and $\Phi_{1b}(2,2;\mathrm{i}/3)$ may have denominators which are powers of three, whereas Theorem 1.15 (1) tells us that their inverses are algebraic integers. Therefore, we find the minimal polynomials for the inverses and then invert the polynomials. In this way, we find that $\Phi_{1a}(2,2;\mathrm{i}/3)$ and $\Phi_{1b}(2,2;\mathrm{i}/3)$ are roots of the irreducible polynomials

$$3x^2 - 1$$
$$19683x^{18} - 80919x^{12} - 39366x^9 + 11016x^6 - 486x^3 - 1.$$

The full polynomials whose roots are the elements of the multisets corresponding to $\Phi_{1a}(2,2;\mathrm{i}/3)^3$ and $\Phi_{1b}(2,2;\mathrm{i}/3)^3$, counting multiplicity are

$$(27x^2 - 1)^{27}$$
$$(19683x^6 - 80919x^4 - 39366x^3 + 11016x^2 - 486x^2 - 1)^9.$$

Applying Theorem 1.15(2), we find that $\sqrt{3}\Phi_{1a}(2,2;\mathrm{i}/3)$ and $\sqrt{3}\Phi_{1b}(2,2;\mathrm{i}/3)$ are units and roots of the polynomials

$$x - 1$$
$$x^{18} + 6x^{15} - 93x^{12} - 304x^9 + 420x^6 - 102x^3 + 1.$$

Lastly, Theorem 1.15 (3) applies, and we know that the ratio

$$\begin{aligned}\frac{\Phi_{1a}(2,2;\tau)}{\Phi_{1b}(2,2;\tau)} &= q^{-2/3} \prod_{n=1}^{\infty} \frac{(1-q^{9n-4})(1-q^{9n-5})}{(1-q^{9n-1})(1-q^{9n-8})} \\ &= q^{-2/3}(1 + q + q^2 + q^3 - q^5 - q^6 - q^7 + \cdots)\end{aligned}$$

evaluates to a unit at $\tau = \mathrm{i}/3$. In fact we find that

$$\frac{\Phi_{1a}(2,2;\mathrm{i}/3)}{\Phi_{1b}(2,2;\mathrm{i}/3)} = 4.60627\ldots$$

is a unit. Indeed, it is a root of

$$x^{18} - 102x^{15} + 420x^{12} - 304x^9 - 93x^6 + 6x^3 + 1.$$

Example 3.15. Here we give an example which illustrates the second remark after Theorem 1.15. This is the discussion concerning ratios of singular values of Φ_2 and Φ_3 with the same κ_*. Here we show that these ratios are not generically algebraic integral units as Theorem 1.15(3) guarantees for the $\mathbb{A}_{2n}^{(2)}$ cases.

We consider $\Phi_2(1,1;\tau)$ and $\Phi_3(1,2;\tau)$, with $\tau = \sqrt{-1/3}$. For these example we have $\kappa_2(1,1) = \kappa_3(1,2) = 6$. A short computation by way of the q-series shows that

$$\Phi_2\big(1,1;\sqrt{-1/3}\big) = 0.883210\ldots,$$

and

$$\Phi_3\big(1,2;\sqrt{-1/3}\big) = 0.347384\ldots.$$

Since $\Phi_2(1,1;\tau)^{24}$ and $\Phi_3(1,2;\tau)^{24}$ are in $\mathcal{F}_{12}$, we find that

$$\Phi_2\big(1,1;\sqrt{-1/3}\big)^{24} \quad \text{and} \quad \Phi_3\big(1,2;\sqrt{-1/3}\big)^{24}$$

each have one other conjugate, namely

$$\Big(g_{1/2,1/3}\big(\sqrt{-4/3}\big)\cdot g_{1/4,0}\big(2\sqrt{-4/3}\big)\Big)^{-24} \quad \text{and} \quad \Big(g_{0,1/3}\big(\sqrt{-4/3}\big)\cdot g_{1/2,0}\big(2\sqrt{-4/3}\big)\Big)^{-24}$$

respectively, and the corresponding multisets described in Theorem 1.8 (2) each contain six copies of the respective orbits. In this way we find that $\Phi_2\big(1,1;\sqrt{-1/3}\big)$ is a root of

$$2^{20}x^{48} - 2^{12}\cdot 13x^{24} + 1$$

and $\Phi_3(1,2;\sqrt{-1/3})$ is a root of

$$2^{20}3^{12}x^{48} - 12^{6}\cdot 35113x^{24} + 1.$$

Therefore, their ratio

$$\frac{\Phi_2(1,1;\sqrt{-1/3})}{\Phi_3(1,2;\sqrt{-1/3})} = 2.542459\ldots$$

is not a unit. Its minimal polynomial is

$$x^4 - 6x^2 - 3.$$

Chapter 4

Ramanujan's mock theta functions

4.1 Proof of Theorem 1.11

Proof. Suppose that g is such weakly holomorphic modular form on $\Gamma_1(m)$ which cuts out the singularities of a mock modular form F^+ on $\Gamma_1(n)$ which has a non-zero shadow. Moreover, suppose that both are of weight $k \in \frac{1}{2}\mathbb{Z}$. Then the difference $F - g$ is also a Maass form of weight k on $\Gamma_1(\mathrm{GCD}(m, n))$ with non-zero shadow. Due to the growth condition required in by the definition of a harmonic Maass form given in 2.1, the non-holomorphic part of F^- exhibits exponential decay as $\tau \to \infty$. Suppose $h(\tau) = \xi_k F$, and pick $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$, and let $\zeta = e(a/c)$ and ρ be the cusp d/c. The growth of $F^-(\tau)$ as $q \to \zeta$ is bounded, as can be seen by taking the following limit.

$$\begin{aligned}
\lim_{\tau\to+i\infty}\left(F^{-}\right)(\gamma\tau) &= \lim_{\tau\to+i\infty} F^{-}(\gamma\tau) - F_{\rho}^{-}(\tau) \\
&= \lim_{\tau\to+i\infty} \int_{-\overline{\gamma\tau}}^{i\infty} h(z)(-i(\gamma\tau+z))^{k-2}\mathrm{d}z - F_{\rho}^{-}(\tau) \\
&= \lim_{\tau\to+i\infty} \int_{-\overline{\tau}}^{\frac{d}{c}} h(\gamma^{*}z)(-i(\gamma\tau+\gamma^{*}z))^{k-2}\mathrm{d}(\gamma^{*}z) - F_{\rho}^{-}(\tau) \\
&= \lim_{\tau\to+i\infty} -\int_{\frac{d}{c}}^{i\infty} h_{\rho}(z)\left(\frac{-i(\tau+z)}{c\tau+d}\right)^{k-2}\mathrm{d}z.
\end{aligned}$$

Here $\gamma^{*}=\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. Thus $F^{+}-g$ is $O(1)$ if and only if $F-g$ is as well.

Suppose that $F-g$ has a non-trivial principal part at the expansion at infinity (a similar arguments works if we have a non-trivial principal part at any other cusp). Then take $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_{1}(\mathrm{GCD}(m,n))$, with ζ as above. The radial limit $\lim_{q\to\zeta} F-g$ may be computed by taking

$$\lim_{z\to+i\infty}(F-g)\left(\frac{az+b}{cz+d}\right)=(cz+d)^{k}(F-g)(z).$$

Since the principal part is nontrivial, we observe exponential growth on $(cz+d)^{k}(F^{+}-g)(z)$, while $(cz+d)^{k}F^{-}(z)$ must approach 0. Thus, neither $F-g$ nor $F^{+}-g$ is $O(1)$ as $q\to\zeta$.

Therefore, if g cuts out the singularities at roots of unity, it must also cut out the principal part of F at all cusps. Then $F-g$ is a weak Maass form with no principal part at any cusp, but has a non-zero non-holomorphic part. This, however, contradicts Theorem 2.2.

□

4.2 Proof of Corollary 4.1

The corollary follows immediately from a stronger corollary which we state here.

Corollary 4.1. *Suppose that $F(\tau) \in H_{k_1}(\Gamma_1(N))$. Then there does not exist a weakly holomorphic modular form $g(z)$ of any weight $k_2 \in \frac{1}{2}\mathbb{Z}$ on any congruence subgroup $\Gamma_1(N')$ such that for every root of unity ζ we have*

$$\lim_{q \to \zeta} \left(F^+(\tau) - g(\tau) \right) = O(1).$$

Proof. As noted in the previous section, $F^-(\tau)$ is $O(1)$ as τ approaches cusps, therefore this is equivalent to the statement that no such $g(\tau)$ exists such that

$$\lim_{q \to \zeta} (F(\tau) - g(\tau)) = O(1).$$

Suppose such a $g(\tau)$ existed. Following the argument in the previous section, if such a $g(\tau)$ existed then both $F(\tau)$ and $g(\tau)$ must have the same principal parts at all cusps. Due to Theorem 2.2, at least one of these must be non constant. Without loss of generality, suppose there is a non-constant principal part at the cusp infinity. Consider the function $h(\tau) := F(\tau) - g(\tau)$. By hypothesis, $h(\tau)$ has bounded radial limits as q approaches every root of unity. We note that $F(\tau)$ and $g(\tau)$ are modular on some common subgroup $\Gamma_1(M)$. If we take $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_1(M)$ with $c \neq 0$, then we have

$$h\left(\tfrac{a\tau+b}{c\tau+d}\right) = F\left(\tfrac{a\tau+b}{c\tau+d}\right) - g\left(\tfrac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k_1} F(\tau) - (c\tau+d)^{k_2} g(\tau). \quad (4.1)$$

Letting $z \to i\infty$, we see that $g(\tau)$ cannot cut out the exponential singularity of $F(\tau)$ due to the difference between the weights, thus contradicting the assumption. $\square$

4.3 The $f(q)$ example

Having proven that the poles of a mock theta function cannot mirror those of a weakly holomorphic modular form at every cusp, we return to Ramanujan's example of a subtler way in which a modular form may imitate a mock theta function at the cusps. Namely, Ramanujan conjectures for the mock theta function

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots$$

and for the q-series $b(q)$ (which is a weakly holomorphic modular form up to a $q^{-\frac{1}{24}}$)

$$b(q) := (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\dots)$$

that as q approaches an even $2k$ order root of unity ζ, $b(q)$ is a "near miss" to $f(q)$, in that half the time one must subtract, and half of the time one must add the values. More specifically, he conjectures that

$$f(q) - (-1)^k b(q) = O(1).$$

Folsom, Ono, and Rhoades show this conjecture and moreover prove the following:

Theorem 4.2 (Theorem 1.1 of [121]). *If ζ is a primitive, even order $2k$ root of unity, then, as q approaches ζ radially in the unit disk, we have that*

$$\lim_{q\to\zeta}\left(f(q) - (-1)^k b(q)\right) = -4 \cdot \sum_{n=0}^{k-1} (1+\zeta)^2(1+\zeta^2)^2 \cdots (1+\zeta^n)^2 \zeta^{n+1}.$$

Chapter 5

Weierstrass mock modular forms

5.1 Weierstrass Theory and the proof of Theorems 1.14, 1.15 and 1.16

Here we recall the essential features of the Weierstrass theory of elliptic curves. After recalling these facts, we then prove Theorems 1.14 and 1.15.

5.1.1 Basic facts about Weierstrass theory

As noted in the introduction, the analytic parameterization $\mathbb{C}/\Lambda_E \cong E$ of an elliptic curve is given by $\mathfrak{z} \to P_{\mathfrak{z}} = (\wp(\Lambda_E; \mathfrak{z}), \wp'(\Lambda_E; \mathfrak{z}))$. By evaluating the Weierstrass $\wp$-function at the Eichler integral given in (1.20), this analytic parameterization becomes the modular parameterization. The Eichler integral is not modular, however its obstruction to modularity is easily characterized. The map $\Psi_E \colon \Gamma_0(N) \to \mathbb{C}$ given by

$$\Psi_E(\gamma) := \mathcal{E}_E(z) - \mathcal{E}_E(\gamma z) \tag{5.1}$$

is a homomorphism of groups. Its image in $\mathbb{C}$ turns out to be the lattice Λ_E. Hence, since $\wp(\Lambda_E; \mathfrak{z})$ is invariant on the lattice, the map $\wp(\Lambda_E; \mathcal{E}_E(z))$ parameterizes E and is also a modular function.

Theorems 1.14 and 1.15 rely on a similar observation, but in this case involving the Weierstrass ζ-function. Unlike the Weierstrass $\wp$-function, the ζ-function itself is not lattice-invariant. However, Eisenstein [298] observed that it could be modified to become lattice-invariant but this modification necessarily sacrifices holomorphicity.

5.1.2 Proofs of Theorems 1.14 and 1.15

We now prove Theorems 1.14 and 1.15.

Proof of Theorem 1.14. Eisenstein's modification to the ζ-function is given by

$$\zeta(\Lambda_E;\mathfrak{z}) - S(\Lambda_E)\mathfrak{z} - \frac{\pi}{a(\Lambda_E)}\bar{\mathfrak{z}}. \tag{5.2}$$

Here S is as in (1.22) and $a(\Lambda_E)$ is the area of a fundamental parallelogram for Λ_E.

Using the formula

$$a(\Lambda_E) = \frac{4\pi^2||F_E||^2}{\deg(\phi_E)}, \tag{5.3}$$

we have that the function $\mathfrak{Z}_E(\mathfrak{z})$ defined in (1.23) above is Eisenstein's corrected ζ-function and is lattice-invariant. Formula (5.3) was first given by Zagier [307] for prime conductor and generalized by Cremona for general level [86]. Since $\mathfrak{Z}_E(\mathfrak{z})$ is lattice-invariant, $\widehat{\mathfrak{Z}}_E(z)$, defined by (1.24), is modular.

Part (1) of Theorem 1.14 follows by noting that $\mathfrak{Z}_E(\mathfrak{z})$ diverges precisely for $\mathfrak{z} \in \Lambda_E$. This divergence must result from a pole in the holomorphic part, $\mathfrak{Z}_E^+(\mathfrak{z})$.

In order to establish part (2), we consider the modular function $\wp(\Lambda_E;\mathcal{E}_E(z))$. We observe that $\wp(\Lambda_E;\mathcal{E}_E(z))$ is meromorphic with poles precisely for those z such that $\mathcal{E}_E(z) \in \Lambda_E$. Therefore $\wp(\Lambda_E;\mathcal{E}_E(z))$ may be decomposed into modular functions with algebraic coefficients, each with only a simple pole at one such z and possibly at cusps. These simple modular functions may be combined appropriately to construct the function $M_E(z)$ to cancel the poles of $\hat{\mathfrak{Z}}_E^+(z)$.

The proof of (3) follows from straightforward calculations. $\square$

Using the theory of Atkin-Lehner involutions, we now prove Theorem 1.15.

Proof of Theorem 1.15. Recall that by classical theory of Atkin-Lehner, every newform of level N_E is an eigenform of the Atkin-Lehner involution

$$W_q = \begin{pmatrix} q^\alpha a & b \\ Nc & q^\alpha d \end{pmatrix},$$

for every prime power $q||N_E$, with eigenvalue ± 1. We note that

$$\widehat{\mathfrak{Z}}_E(z)|_0 W_q = \mathfrak{Z}_E(\Lambda_E; \mathcal{E}_E(z)|_0 W_q).$$

It suffices to show $\mathcal{E}_E(z) - \lambda_q \mathcal{E}_E(z)|W_q$ is equal to $\Omega_q(F_E)$. To this end note that

$$\begin{aligned} \mathcal{E}_E(z) - \lambda_q \mathcal{E}_E(z)|W_q &= -2\pi i \left[\int_z^{i\infty} F_E(z)dz - \lambda_q \int_{W_q z}^{i\infty} F_E(z)dz \right] \qquad (5.4) \\ &= -2\pi i \left[\int_z^{i\infty} F_E(z)dz - \lambda_q \int_z^{W_q^{-1} i\infty} \frac{\det(W_q)}{(Ncz + q^\alpha d)^2} F_E(W_q z)dz \right] \\ &= -2\pi i \left[\int_z^{i\infty} F_E(z)dz + \lambda_q^2 \int_{W_q^{-1} i\infty}^{z} F_E(z)dz \right] \\ &= -2\pi i \int_{W_q^{-1} i\infty}^{i\infty} F_E(z)dz. \end{aligned}$$

We note that if $\Omega_q(F_E)$ is in the lattice, then we may ignore this term, and we see that $\widehat{\mathfrak{Z}}_E(z)$ is an eigenfunction for the involution W_q. Otherwise, $\widehat{\mathfrak{Z}}_E(z)|_0 W_q$ has a constant term equal to $\mathfrak{Z}_E(\Omega_q(F_E))$. $\square$

5.1.3 Proof of Theorem 1.16

The proof of Theorem 1.16 is similar to recent work of Guerzhoy, Kent, and Ono [153]. We will need the following proposition.

Proposition 5.1. *Suppose that $R(z)$ is a meromorphic modular function on $\Gamma_0(N)$ with $\mathbb{Q}$-rational coefficients. If $p \nmid N$ is prime, then there is an A such that*

$$\mathrm{ord}_p\left(q\frac{d}{dq}R|T(p^n)\right) \geq n - A.$$

Proof. For convenience, we let $R(z) = \sum_{n\gg-\infty} a(n)q^n$. We first show that the coefficients $a(n)$ of R have bounded denominators. In other words, we have that $A := \inf_n(\mathrm{ord}_p(a(n))) < \infty$. Indeed, we can always multiply R with an appropriate power of (z) and a monic polynomial in $j(z)$ with rational coefficients to obtain a cusp form of positive integer weight and rational coefficients. The resulting Fourier coefficients will have bounded denominators by Theorem 3.52 of [274]. One easily checks that dividing by the power of $\Delta(z)$ and this polynomial in $j(z)$ preserves the boundedness. The proposition now follows easily from

$$\left(q\frac{d}{dq}R\right)|T(p^n) = \sum_{m\gg\infty}\sum_{j=0}^{\min\{\mathrm{ord}_p(m),n\}} p^{n-j}ma(p^{n-2j}m)q^m.$$

□

Remark. Proposition 5.1 is analogous to Proposition 2.1 of [153] which concerns Atkin's $U(p)$ operator.

Proof of Theorem 1.16. We first consider the case where E has CM. Suppose $D < 0$ is the discriminant of the imaginary quadratic field K. The nonzero coefficients of $F_E(z)$ are supported on powers q^n with $\chi_D(n) := \left(\frac{D}{n}\right) \neq -1$. Let φ_D be the trivial character modulo $|D|$. We construct the modular function

$$\mathcal{Z}_E(z) = \frac{1}{2}\left(\widehat{\mathfrak{Z}}_E|\varphi_D + \widehat{\mathfrak{Z}}_E|\chi_D\right). \tag{5.5}$$

Since the coefficients of the nonholomorphic part of $\widehat{\mathfrak{Z}}_E(z)$ are supported on powers q^{-n} with $\chi_D(-n) \neq 1$, we see that the twisting action in the definition of $\mathcal{Z}_E(z)$ kills the nonholomorphic part. Therefore, $\mathcal{Z}_E(z)$ is a meromorphic modular function on $\Gamma_0(ND^2)$ whose nonzero coefficients are supported on q^m where $\chi_D(m) = 1$, and are equal to the original coefficients of $\widehat{\mathfrak{Z}}^+_E(z)$.

We now aim to prove the following p-adic limits:

$$\lim_{n\to+\infty} \left[q\frac{d}{dq}(\widehat{\mathfrak{Z}}_E(z))\right] |T(p^n) = \lim_{n\to+\infty} \left[q\frac{d}{dq}(\widehat{\mathfrak{Z}}_E(z) - \mathcal{Z}_E(z))\right] |T(p^n) = 0. \tag{5.6}$$

By Proposition 5.1, the two limits are equal, and so it suffices to prove the vanishing of the second limit.

Since $\chi_D(p^n) = 1$, it follows that the coefficients of q^{p^n} (including q^1) in $\widehat{\mathfrak{Z}}^+_E(z) - \mathcal{Z}_E(z)$ all vanish. Therefore the coefficient of q^1 for each n in the second limit of (5.6) is zero. Since the principal part of $\widehat{\mathfrak{Z}}_E(z) - \mathcal{Z}_E(z)$ is q^{-1}, the principal parts in the second limit p-adically tend to 0 thanks to the definition of the Hecke operators $T(p^n)$.

Suppose that $m > 1$ is coprime to N_E. Then note that F_E is an eigenfunction for the Hecke operator $T(m)$ with eigenvalue $a_E(m)$. Since the nonholomorphic part of $\widehat{\mathfrak{Z}}_E(z)$ is the period integral of $F_E(z)$, it follows that $Q_m(z) := m\widehat{\mathfrak{Z}}_E(z)|T(m) - a_E(m)\widehat{\mathfrak{Z}}_E(z) = m\widehat{\mathfrak{Z}}^+_E(z)|T(m) - a_E(m)\widehat{\mathfrak{Z}}^+_E(z)$ is a meromorphic modular function. Note that the functions $q\frac{d}{dq}Q_m(z)$ have denominators that are bounded independently of m. This follows from the proof of Proposition 5.1 and the fact that (see Theorem 1.1 of [50]) $q\frac{d}{dq}\widehat{\mathfrak{Z}}_E(z)$ is a weight 2 meromorphic modular form. Since Hecke operators commute, we have

$$\left[q\frac{d}{dq}\widehat{\mathfrak{Z}}^+_E(z)\right] |T(p^n)T(m) = \left[q\frac{d}{dq}(a_E(m)\widehat{\mathfrak{Z}}^+_E(z) + Q_m(z))\right] |T(p^n).$$

Modulo any fixed power of p, say p^t, Proposition 5.1 then implies that

$$\left[q\frac{d}{dq}\widehat{\mathfrak{Z}}^+_E(z)\right] |T(p^n)T(m) \equiv a_E(m) \cdot \left[q\frac{d}{dq}\widehat{\mathfrak{Z}}^+_E(z)\right] |T(p^n) \left(\bmod p^t\right),$$

for sufficiently large n. In other words, we have that $\left[q\frac{d}{dq}\widehat{\mathfrak{Z}}^{+}_{E}(z)\right]|T(p^n)$ is congruent to a Hecke eigenform for $T(m)$ modulo p^t for sufficiently large n. By Proposition 5.1 again, we have that $\left[q\frac{d}{dq}(\widehat{\mathfrak{Z}}^{+}_{E}(z) - \mathcal{Z}_E(z))\right]|T(p^n)$ is an eigenform of $T(m)$ modulo p^t for sufficiently large n. Obviously, this conclusion holds uniformly in n for all $T(m)$ with $\gcd(m, N_E) = 1$.

Generalizing this argument in the obvious way to incorporate Atkin's U-operators (as in [153]), we conclude that these forms are eigenforms of all the Hecke operators. By the discussion above, combined with the fact that the constant terms vanish after applying $q\frac{d}{dq}$, these eigenforms are congruent to $0+O(q^2) \pmod{p^t}$. Such an eigenform must be identically $0 \pmod{p^t}$, thereby establishing (5.6).

To complete the proof in this case, we observe that $p \nmid a_E(p^n)$ for any n. This follows from the recurrence relation on $a_E(p^n)$ in n, combined with the fact that $p \nmid a_E(p)$ since p is split in K. By (5.6) we have that

$$\lim_{n\to+\infty} \frac{\left[q\frac{d}{dq}(\widehat{\mathfrak{Z}}^{+}_{E}(z))\right]|T(p^n)}{a_E(p^n)} = 0. \tag{5.7}$$

The proof now follows from the identities

$$\widehat{\mathfrak{Z}}^{+}_{E}(z) = \zeta(\Lambda_E; \mathcal{E}_E(z)) - S(\Lambda_E)\mathcal{E}_E(z) \quad \text{and} \quad F_E(z) = q\frac{d}{dq}\mathcal{E}_E(z).$$

The proof for E without CM is nearly identical. We replace $\widehat{\mathfrak{Z}}^{+}_{E}(z)$ by $\widehat{\mathfrak{Z}}^{+}_{E}(z) + S(\Lambda_E)\mathcal{E}_E(z)$, which has $\mathbb{Q}$-rational coefficients. In (5.7) the limiting value of 0 is replaced by a constant multiple of $F_E(z)$. □

5.2 Vector valued harmonic Maass forms

To ease exposition, the results in the introduction were stated using the classical language of half-integral weight modular forms. To treat the case of general levels and functional equations, it will be more convenient to work with vector-valued forms and certain Weil representations. Here we recall this framework, and we discuss important theta functions which will be required in the section to define the theta lift $\mathcal{I}(\bullet;\tau)$. We shall let $q := e^{2\pi i\tau}$. The modular parameter will always be clear in context. The need for multiple modular variables arises from the structure of the theta lift. As a rule of thumb, τ shall be the modular variable for all the half-integral weight forms in the remainder of this paper.

For a positive integer N we consider the rational quadratic space of signature $(1,2)$ given by

$$V := \left\{ \lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & -\lambda_1 \end{pmatrix}; \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q} \right\}$$

and the quadratic form $Q(\lambda) := N\det(\lambda)$. The associated bilinear form is $(\lambda, \mu) = -N\mathrm{tr}(\lambda\mu)$ for $\lambda, \mu \in V$.

We let $G = \mathrm{Spin}(V) \simeq SL_2$, viewed as an algebraic group over $\mathbb{Q}$ and write $\overline{\Gamma}$ for its image in $\mathrm{SO}(V) \simeq \mathrm{PSL}_2$. By D we denote the associated symmetric space. It can be realized as the Grassmannian of lines in $V(\mathbb{R})$ on which the quadratic form Q is positive definite,

$$D \simeq \{z \subset V(\mathbb{R});\ \dim z = 1 \text{ and } Q|_z > 0\}.$$

Then the group $SL_2(\mathbb{Q})$ acts on V by conjugation

$$g.\lambda := g\lambda g^{-1},$$

for $\lambda \in V$ and $g \in SL_2(\mathbb{Q})$. In particular, $G(\mathbb{Q}) \simeq SL_2(\mathbb{Q})$.

We identify the symmetric space D with the upper-half of the complex plane $\mathbb{H}$ in the usual way, and obtain an isomorphism between $\mathbb{H}$ and D by

$$z \mapsto \mathbb{R}\lambda(z),$$

where, for $z = x + iy$, we pick as a generator for the associated positive line

$$\lambda(z) := \frac{1}{\sqrt{N}y} \begin{pmatrix} -(z+\bar{z})/2 & z\bar{z} \\ -1 & (z+\bar{z})/2 \end{pmatrix}.$$

The group G acts on $\mathbb{H}$ by linear fractional transformations and the isomorphism above is G-equivariant. Note that $Q(\lambda(z)) = 1$ and $g.\lambda(z) = \lambda(gz)$ for $g \in G(\mathbb{R})$. Let $(\lambda, \lambda)_z = (\lambda, \lambda(z))^2 - (\lambda, \lambda)$. This is the minimal majorant of $(\cdot, \cdot)$ associated with $z \in D$.

We can view $\Gamma_0(N)$ as a discrete subgroup of $\mathrm{Spin}(V)$ and we write $M = \Gamma_0(N) \setminus D$ for the attached locally symmetric space.

We identify the set of isotropic lines $\mathrm{Iso}(V)$ in $V(\mathbb{Q})$ with $P^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ via

$$\psi : P^1(\mathbb{Q}) \to \mathrm{Iso}(V), \quad \psi((\alpha : \beta)) = \mathrm{span}\left(\begin{pmatrix} \alpha\beta & \alpha^2 \\ -\beta^2 & -\alpha\beta \end{pmatrix}\right).$$

The map ψ is a bijection and $\psi(g(\alpha : \beta)) = g.\psi((\alpha : \beta))$. Thus, the cusps of M (i.e. the $\Gamma_0(N)$-classes of $P^1(\mathbb{Q})$) can be identified with the $\Gamma_0(N)$-classes of $\mathrm{Iso}(V)$.

If we set $\ell_\infty := \psi(\infty)$, then ℓ_∞ is spanned by$\lambda_\infty = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. For $\ell \in \mathrm{Iso}(V)$ we pick $\sigma_\ell \in SL_2(\mathbb{Z})$ such that $\sigma_\ell \ell_\infty = \ell$.

Heegner points are given as follows. For $\lambda \in V(\mathbb{Q})$ with $Q(\lambda) > 0$ we let

$$D_\lambda = \mathrm{span}(\lambda) \in D.$$

For $Q(\lambda) \leq 0$ we set $D_\lambda = \emptyset$. We denote the image of D_λ in M by $Z(\lambda)$.

5.2.1 A lattice related to $\mathbf{\Gamma_0(N)}$

We consider the lattice

$$L := \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix}; \quad a, b, c \in \mathbb{Z} \right\}.$$

The dual lattice corresponding to the bilinear form $(\cdot, \cdot)$ is given by

$$L' := \left\{ \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix}; \quad a, b, c \in \mathbb{Z} \right\}.$$

We identify the discriminant group $L'/L =: \mathcal{D}$ with $\mathbb{Z}/2N\mathbb{Z}$, together with the $\mathbb{Q}/\mathbb{Z}$ valued quadratic form $x \mapsto -x^2/4N$. The level of L is $4N$.

For a fundamental discriminant $\Delta \in \mathbb{Z}$ we will consider the rescaled lattice ΔL together with the quadratic form $Q_\Delta(\lambda) := \frac{Q(\lambda)}{|\Delta|}$. The corresponding bilinear form is then given by $(\cdot,\cdot)_\Delta = \frac{1}{|\Delta|}(\cdot,\cdot)$. The dual lattice of ΔL with respect to $(\cdot,\cdot)_\Delta$ is equal to L'. We denote the discriminant group $L'/\Delta L$ by $\mathcal{D}(\Delta)$.

For $m \in \mathbb{Q}$ and $h \in \mathcal{D}$, we let

$$L_{m,h} = \{\lambda \in L + h; Q(\lambda) = m\}.$$

By reduction theory, if $m \neq 0$ the group $\Gamma_0(N)$ acts on $L_{m,h}$ with finitely many orbits.

We will also consider the one-dimensional lattice $K = \mathbb{Z}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \subset L$. We have $L = K + \mathbb{Z}\ell + \mathbb{Z}\ell'$ where ℓ and ℓ' are the primitive isotropic vectors

$$\ell = \begin{pmatrix} 0 & 1/N \\ 0 & 0 \end{pmatrix}, \qquad \ell' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Then $K'/K \simeq L'/L$.

5.2.2 The Weil representation and vector-valued automorphic forms

By $\mathrm{Mp}_2(\mathbb{Z})$ we denote the integral metaplectic group. It consists of pairs (γ, ϕ), where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $\phi^2(\tau) = c\tau + d$. The group $\widetilde{\Gamma} = \mathrm{Mp}_2(\mathbb{Z})$ is generated by $S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$ and $T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$. We let $\widetilde{\Gamma}_\infty := \langle T \rangle \subset \widetilde{\Gamma}$. We consider the Weil representation ρ_Δ of $\mathrm{Mp}_2(\mathbb{Z})$ corresponding to the discriminant group $\mathcal{D}(\Delta)$ on the group ring $\mathbb{C}[\mathcal{D}(\Delta)]$, equipped with the standard scalar product $\langle \cdot, \cdot \rangle$, conjugate-linear in the second variable. We simply write ρ for ρ_1.

Let $e(a) := e^{2\pi i a}$. We write $\mathfrak{e}_\delta$ for the standard basis element of $\mathbb{C}[\mathcal{D}(\Delta)]$ corresponding to $\delta \in \mathcal{D}(\Delta)$. The action of ρ_Δ on basis vectors of $\mathbb{C}[\mathcal{D}(\Delta)]$ is given by the following formulas for the generators S and T of $\mathrm{Mp}_2(\mathbb{Z})$

$$\rho_\Delta(T)\mathfrak{e}_\delta = e(Q_\Delta(\delta))\mathfrak{e}_\delta,$$

and

$$\rho_\Delta(S)\mathfrak{e}_\delta = \frac{\sqrt{i}}{\sqrt{|\mathcal{D}(\Delta)|}} \sum_{\delta' \in \mathcal{D}(\Delta)} e(-(\delta', \delta)_\Delta)\mathfrak{e}_{\delta'}.$$

Let $k \in \frac{1}{2}\mathbb{Z}$, and let A_{k,ρ_Δ} be the vector space of functions $f : \mathbb{H} \to \mathbb{C}[\mathcal{D}(\Delta)]$, such that for $(\gamma, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ we have

$$f(\gamma\tau) = \phi(\tau)^{2k}\rho_\Delta(\gamma, \phi)f(\tau).$$

A smooth function $f \in A_{k,\rho_\Delta}$ is called a *harmonic (weak) Maass form of weight k with respect to the representation* ρ_Δ if it satisfies in addition (see [53, Section 3]):

1. $\Delta_k f = 0$,

2. the singularity at ∞ is locally given by the pole of a meromorphic function.

Here we write $\tau = u + iv$ with $u, v \in \mathbb{R}$, and

$$\Delta_k = -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right) \tag{5.8}$$

is the weight k Laplace operator. We denote the space of such functions by H_{k,ρ_Δ}.

By $M^!_{k,\rho_\Delta} \subset H_{k,\rho_\Delta}$ we denote the subspace of weakly holomorphic modular forms. Recall that weakly holomorphic modular forms are meromorphic modular forms whose poles (if any) are supported at cusps.

Similarly, we can define scalar-valued analogs of these spaces of automorphic forms. In this case, we require analogous conditions at all cusps of $\Gamma_0(N)$ in (ii). We denote these spaces by $H_k^+(N)$ and $M_k^!(N)$.

Note that the Fourier expansion of any harmonic Maass form uniquely decomposes into a holomorphic and a nonholomorphic part [53, Section 3]

$$f^+(\tau) = \sum_{h\in L'/L} \sum_{\substack{n\in\mathbb{Q} \\ n\gg-\infty}} c^+(n,h)q^n\mathfrak{e}_h$$

$$f^-(\tau) = \sum_{h\in L'/L} \sum_{n\in\mathbb{Q}} c^-(n,h)\Gamma(1-k, 4\pi|n|v)q^n\mathfrak{e}_h,$$

where $\Gamma(a,x)$ denotes the incomplete Γ-function. The first summand is called the holomorphic part of f, the second one the nonholomorphic part.

We define a differential operator ξ_k by

$$\xi_k(f) := -2iv^k \overline{\frac{\partial}{\partial\bar{\tau}} f}. \tag{5.9}$$

We then have the following exact sequence [53, Corollary 3.8]

$$0 \longrightarrow M^!_{k,\rho_\Delta} \longrightarrow H_{k,\rho_\Delta} \xrightarrow{\xi_k} S_{2-k,\bar{\rho}_\Delta} \longrightarrow 0.$$

5.2.3 Poincaré series and Whittaker functions

We recall some facts on Poincaré series with exponential growth at the cusps following Section 2.6 of [57].

We let $k \in \frac{1}{2}\mathbb{Z}$, and $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ denote the usual Whittaker functions (see p. 190 of [1]). For $s \in \mathbb{C}$ and $y \in \mathbb{R}_{>0}$ we put

$$\mathcal{M}_{s,k}(y) = y^{-k/2} M_{-\frac{k}{2}, s-\frac{1}{2}}(y).$$

We let Γ_∞ be the subgroup of $\Gamma_0(N)$ generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. For $k \in \mathbb{Z}$, $m \in \mathbb{N}$, $z = x + iy \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\Re(s) > 1$, we define

$$F_m(z,s,k) = \frac{1}{2\Gamma(2s)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} [\mathcal{M}_{s,k}(4\pi my)e(-mx)] \,|_k\, \gamma. \tag{5.10}$$

This Poincaré series converges for $\Re(s) > 1$, and it is an eigenfunction of Δ_k with eigenvalue $s(1-s) + (k^2 - 2k)/4$. Its specialization at $s_0 = 1 - k/2$ is a harmonic Maass form [48, Proposition 1.10]. The principal part at the cusp ∞ is given by $q^{-m} + C$ for some constant $C \in \mathbb{C}$. The principal parts at the other cusps are constant.

We now define $\mathbb{C}[L'/L]$-valued analogs of these series. Let $h \in L'/L$ and $m \in \mathbb{Z} - Q(h)$ be positive. For $k \in \left(\mathbb{Z} - \frac{1}{2}\right)_{<1}$ we let

$$\mathcal{F}_{m,h}(\tau, s, k) = \frac{1}{2\Gamma(2s)} \sum_{\gamma \in \widetilde{\Gamma}_\infty \backslash \widetilde{\Gamma}} [\mathcal{M}_{s,k}(4\pi m y) e(-mx) \mathfrak{e}_h]|_{k,\rho}\ \gamma.$$

The series $\mathcal{F}_{m,h}(\tau, s, k)$ converges for $\Re(s) > 1$ and it defines a harmonic Maass form of weight k for $\widetilde{\Gamma}$ with representation ρ. The special value at $s = 1 - k/2$ is harmonic [48, Proposition 1.10]. For $k \in \mathbb{Z} - \frac{1}{2}$ the principal part is given by $q^{-m}\mathfrak{e}_h + q^{-m}\mathfrak{e}_{-h} + C$ for some constant $C \in \mathbb{C}[L'/L]$.

Remark. If we let (in the same setting as above)

$$\mathcal{F}_{m,h}(\tau, s, k) = \frac{1}{2\Gamma(2s)} \sum_{\gamma \in \widetilde{\Gamma}_\infty \backslash \widetilde{\Gamma}} [\mathcal{M}_{s,k}(4\pi m y) e(-mx) \mathfrak{e}_h]|_{k,\bar{\rho}}\ \gamma,$$

then this has the same convergence properties. But for the special value at $s = 1 - k/2$, the principal part is given by $q^{-m}\mathfrak{e}_h - q^{-m}\mathfrak{e}_{-h} + C$ for some constant $C \in \mathbb{C}[L'/L]$.

5.2.4 Twisted theta series

We define a generalized genus character for $\delta = \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N \end{smallmatrix}\right) \in L'$. From now on let $\Delta \in \mathbb{Z}$ be a fundamental discriminant and $r \in \mathbb{Z}$ such that $\Delta \equiv r^2 \pmod{4N}$.

Then

$$\chi_\Delta(\delta) = \chi_\Delta([a,b,Nc]) := \begin{cases} \left(\frac{\Delta}{n}\right), & \text{if } \Delta | b^2-4Nac \text{ and } (b^2-4Nac)/\Delta \text{ is a} \\ & \text{square mod } 4N \text{ and } \gcd(a,b,c,\Delta)=1, \\ 0, & \text{otherwise.} \end{cases}$$

Here $[a, b, Nc]$ is the integral binary quadratic form corresponding to δ, and n is any integer prime to Δ represented by $[a, b, Nc]$.

The function χ_Δ is invariant under the action of $\Gamma_0(N)$ and under the action of all Atkin-Lehner involutions. It can be computed by the following formula [148, Section I.2, Proposition 1]: If $\Delta = \Delta_1\Delta_2$ is a factorization of Δ into discriminants and $N = N_1N_2$ is a factorization of N into positive factors such that $(\Delta_1, N_1a) = (\Delta_2, N_2c) = 1$, then

$$\chi_\Delta([a,b,Nc]) = \left(\frac{\Delta_1}{N_1a}\right)\left(\frac{\Delta_2}{N_2c}\right).$$

If no such factorizations of Δ and N exist, we have $\chi_\Delta([a, b, Nc]) = 0$.

Since $\chi_\Delta(\delta)$ depends only on $\delta \in L'$ modulo ΔL, we can view it as a function on the discriminant group $\mathcal{D}(\Delta)$.

We now let

$$\varphi^0_\Delta(\lambda, z) = p_z(\lambda)e^{-2\pi R(\lambda,z)/|\Delta|}, \tag{5.11}$$

where $p_z(\lambda) = (\lambda, \lambda(z))$ and $R(\lambda, z) := \frac{1}{2}(\lambda, \lambda(z))^2 - (\lambda, \lambda)$. This function was recently studied extensively by Hövel [166]. From now on, if $\Delta = 1$, we omit the index Δ and simply write $\varphi^0(\lambda, z)$. Let $\varphi(\lambda, \tau, z) = e^{2\pi i Q_\Delta(\lambda)\tau}\varphi^0_\Delta(\sqrt{v}\lambda, z)$ (for notational purposes we drop the dependence on *Delta*). By π we denote the canonical projection $\pi : \mathcal{D}(\Delta) \to \mathcal{D}$.

Moreover, we let $\tilde{\rho} = \rho$, if $\Delta > 0$, and $\tilde{\rho} = \bar{\rho}$, if $\Delta < 0$.

Theorem 5.2. *The theta function*

$$\Theta_{\Delta,r}(\tau, z, \varphi) := v^{1/2}\sum_{h\in\mathcal{D}} \sum_{\substack{\delta\in\mathcal{D}(\Delta)\\ \pi(\delta)=rh \\ Q_\Delta(\delta)\equiv \mathrm{sgn}(\Delta)Q(h)\,(\mathbb{Z})}} \chi_\Delta(\delta) \sum_{\lambda\in\Delta L+\delta} \varphi(\lambda, \tau, z)\mathfrak{e}_h \tag{5.12}$$

is a nonholomorphic $\mathbb{C}[\mathcal{D}]$-valued modular form of weight $1/2$ for the representation $\widetilde{\rho}$ in the variable τ. Furthermore, it is a nonholomorphic automorphic form of weight 0 for $\Gamma_0(N)$ in the variable $z \in D$.

Proof. This follows from [166, Satz 2.8] and the results in [6]. □

We use the following representation for $\Theta_{\Delta,r}(\tau, z, \varphi)$ as a Poincaré series using the lattice K. We let $\epsilon = 1$, when $\Delta > 0$, and $\epsilon = i$, when $\Delta < 0$. The following proposition can be found in [166, Satz 2.22].

Proposition 5.3. *We have*

$$\Theta_{\Delta,r}(\tau, z, \varphi) = -\frac{Ny^2\bar{\epsilon}}{2i}\sum_{n=1}^{\infty} n\left(\frac{\Delta}{n}\right)$$
$$\times \sum_{\gamma\in\widetilde{\Gamma}_\infty\backslash\widetilde{\Gamma}} \left[\frac{1}{v^{1/2}} e\left(-\frac{Nn^2y^2}{2i|\Delta|v}\right)\sum_{\lambda\in K'} e\left(\frac{\lambda^2}{2}|\Delta|\bar{\tau} - 2nN\lambda x\right)\mathfrak{e}_{r\lambda}\right]|_{1/2,\widetilde{\rho_K}}\,\gamma.$$

Now we define the theta kernel of the Shintani lift. Recall that for a lattice element $\lambda \in L'/L$ we write $\lambda = \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N \end{smallmatrix}\right)$. Let

$$\varphi_{\mathrm{Sh}}(\lambda, \tau, z) = -\frac{cN\bar{z}^2 - b\bar{z} + a}{4Ny^2} e^{-2\pi v R(\lambda,z)/|\Delta|} e^{2\pi i Q_\Delta(\lambda)\tau}.$$

The Shintani theta function then transforms as follows.

Theorem 5.4. *The theta function*

$$\Theta_{\Delta,r}(\tau, z, \varphi_{Sh}) = v^{1/2}\sum_{h\in\mathcal{D}} \sum_{\substack{\delta\in\mathcal{D}(\Delta)\\ \pi(\delta)=rh\\ Q_\Delta(\delta)\equiv\mathrm{sgn}(\Delta)Q(h)\,(\mathbb{Z})}} \chi_\Delta(\delta)\sum_{\lambda\in\Delta L+\delta}\varphi_{Sh}(\lambda,\tau,z)\mathfrak{e}_h \tag{5.13}$$

is a nonholomorphic automorphic form of weight 2 for $\Gamma_0(N)$ in the variable $z \in D$. Moreover, $\overline{\Theta_{\Delta,r,h}(\tau, z, \varphi_{Sh})}$ is a nonholomorphic $\mathbb{C}[\mathcal{D}]$-valued modular form of weight $3/2$ for the representation $\overline{\widetilde{\rho}}$ in the variable τ.

Proof. This follows from the results in [58, p. 142] and the results in [6]. □

We have the following relation between the two theta functions. This was already investigated in [53] and [37].

Lemma 5.5. *We have*

$$\xi_{1/2,\tau}\Theta_{\Delta,r}(\tau,z,\varphi) = 4i\sqrt{N}y^2\frac{\partial}{\partial z}\overline{\Theta_{\Delta,r}(\tau,z,\varphi_{Sh})}.$$

Proof. We first compute

$$\xi_{1/2,\tau}v^{1/2}\varphi(\lambda,\tau,z) = -v^{1/2}p_z(\lambda)e^{-2\pi vR(\lambda,z)/|\Delta|}e(-Q_\Delta(\lambda)\bar{\tau})\left(1-2\pi v\frac{R(\lambda,z)}{|\Delta|}\right).$$

For the derivative of complex conjugate of the Shintanti theta kernel we obtain

$$\begin{aligned}
&-\frac{1}{4N}v^{1/2}e^{-2\pi vR(\lambda,z)/|\Delta|}e(-Q_\Delta(\lambda)\bar{\tau}) \\
&\quad\times\left(\frac{\partial}{\partial z}y^{-2}(cNz^2-bz+a)+y^{-2}(cNz^2-bz+a)(-2\pi v)\frac{1}{|\Delta|}\frac{\partial}{\partial z}R(\lambda,z)\right) \\
&=\frac{i}{4\sqrt{N}y^2}v^{1/2}p_z(\lambda)e^{-2\pi vR(\lambda,z)/|\Delta|}e(-Q_\Delta(\lambda)\bar{\tau})\left(1-2\pi v\frac{R(\lambda,z)}{|\Delta|}\right),
\end{aligned}$$

using that

$$\begin{aligned}
&\frac{\partial}{\partial z}y^{-2}(cNz^2-bz+a) = -i\sqrt{N}y^{-2}p_z(\lambda), \\
&\frac{\partial}{\partial z}R(\lambda,z) = -\frac{i}{2\sqrt{N}}y^{-2}p_z(\lambda)(cN\bar{z}^2-b\bar{z}+a), \\
&y^{-2}(cNz^2-bz+a)(cN\bar{z}^2-b\bar{z}+a) = 2NR(\lambda,z).
\end{aligned}$$

□

5.3 Theta lifts of harmonic Maass forms

Recall that Δ is a fundamental discriminant and that $r\in\mathbb{Z}$ is such that $r^2\equiv\Delta \pmod{4N}$. Let F be a harmonic Maass form in $H_0^+(N)$. We define the twisted theta lift of F as follows

$$\mathcal{I}_{\Delta,r}(\tau, F) = \int_M F(z)\Theta_{\Delta,r}(\tau, z, \varphi) d\mu(z).$$

Theorem 5.6. *Let $\Delta \neq 1$ and let F be a harmonic Maass form in $H_0^+(N)$ with vanishing constant term at all cusps. Then $\mathcal{I}_{\Delta,r}(\tau, F)$ is a harmonic Maass form of weight $1/2$ transforming with respect to the representation $\tilde{\rho}$. Moreover, the theta lift is equivariant with respect to the action of $O(L'/L)$.*

To prove the theorem we establish a couple of results. Note that the transformation properties of the twisted theta function $\Theta_{\Delta,r}(\tau, z, \varphi)$ directly imply that the lift transforms with representation $\widetilde{\rho}$. The equivariance follows from [166, Proposition 2.7]. First we show that the lift is annihilated by the Laplace operator. Together with a result relating this theta lift to the Shintani lift, these results imply Theorem 5.6. We also compute the lift of Poincaré series and the constant function since this will be useful in Section 5.4. Further properties of this lift will be investigated in a forthcoming paper [5].

Proposition 5.7. *Let F be a harmonic Maass form in $H_0^+(N)$. Then $\mathcal{I}_{\Delta,r}(\tau, F)$ is well-defined and*

$$\Delta_{1/2,\tau}\, \mathcal{I}_{\Delta,r}(\tau, F) = 0.$$

Proof. We first investigate the growth of the theta function $\Theta_{\Delta,r}(\tau, z, \varphi) = \sum_{h \in L'/L} \theta_h(\tau, z, \varphi)$ in the cusps of M. For simplicity we let $\Delta = N = 1$. Then $L = \mathbb{Z}^3$ and $h = \left(\begin{smallmatrix} h' & 0 \\ 0 & h' \end{smallmatrix}\right)$ with $h' = 0$ or $h' = 1/2$. So we consider

$$\theta_h(\tau, z, \varphi) = \sum_{\substack{a,c \in \mathbb{Z} \\ b \in \mathbb{Z}+h'}} -\frac{v}{y}(c|z|^2 - bx + a) e^{-\frac{\pi v}{y}(c|z|^2 - bx + a)^2} e^{2\pi i \bar{\tau}(-b^2/4 + ac)}.$$

We apply Poisson summation on the sum over a. We consider the summands as a function of a and compute the Fourier transform, i.e.

$$\begin{aligned}
&-\int_{-\infty}^{\infty} \frac{v}{y}(c|z|^2 - bx + a)e^{-\frac{\pi v}{y}(c|z|^2-bx+a)^2} e^{2\pi i\bar{\tau}(-b^2/4+ac)} e^{2\pi i w a} da \\
&= -ye^{-\pi i\bar{\tau}b^2/2} e^{2\pi i(c\bar{\tau}+w)(bx-c|z|^2)} \int_{-\infty}^{\infty} te^{-\pi t^2} e^{2\pi i t \frac{y}{\sqrt{v}}(c\bar{\tau}+w)} dt,
\end{aligned}$$

where we set $t = \frac{\sqrt{v}}{y}(c|z|^2 - bx + a)$. Since the Fourier transform of $xe^{-\pi x^2}$ is $ixe^{-\pi x^2}$ this equals

$$\begin{aligned}
&- i\frac{y^2}{\sqrt{v}} e^{-\pi i\bar{\tau}b^2/2} e^{2\pi i(c\bar{\tau}+w)(bx-c|z|^2)} (c\bar{\tau} + w) e^{-\frac{\pi y^2}{v}(c\bar{\tau}+w)^2} \\
&= -i\frac{y^2}{\sqrt{v}}(c\bar{\tau} + w) e^{-2\pi i\bar{\tau}(b/2-cx)^2} e^{2\pi i(bxw-cx^2w)} e^{-\frac{\pi y^2}{v}|c\tau+w|^2}.
\end{aligned}$$

We obtain that

$$\theta_h(\tau, z, \varphi) = -\frac{y^2}{\sqrt{v}} \sum_{\substack{w,c\in\mathbb{Z} \\ b\in\mathbb{Z}+h'}} (c\bar{\tau} + w) e^{-2\pi i\bar{\tau}(b/2-cx)^2} e^{2\pi i(bxw-cx^2w)} e^{-\frac{\pi y^2}{v}|c\tau+w|^2}.$$

If c and w are non-zero this decays exponentially, and if $c = w = 0$ it vanishes.

In general we obtain for $h \in L'/L$ and at each cusp ℓ

$$\theta_h(\tau, \sigma_\ell z, \varphi) = O(e^{-Cy^2}), \quad \text{as } y \to \infty,$$

uniformly in x, for some constant $C > 0$.

Thus, the growth of $\Theta_{\Delta,r}(\tau, z, \varphi)$ offsets the growth of F and the integral converges. By [166, Proposition 3.10] we have

$$\begin{aligned}
\Delta_{1/2,\tau}\mathcal{I}_{\Delta,r}(\tau, F) &= \int_M F(z)\Delta_{1/2,\tau}\Theta_{\Delta,r}(\tau, z, \varphi)d\mu(z) \\
&= \frac{1}{4}\int_M F(z)\Delta_{0,z}\Theta_{\Delta,r}(\tau, z, \varphi)d\mu(z).
\end{aligned}$$

By the rapid decay of the theta function we may move the Laplacian to F. Since $F \in H_0^+(N)$ we have $\Delta_{0,z}F = 0$, which implies the vanishing of the integral. □

By $\mathcal{I}^{\mathrm{Sh}}_{\Delta,r}(\tau,G)$ we denote the Shintani lifting of a cusp form G of weight 2 for $\Gamma_0(N)$. It is defined as

$$\mathcal{I}^{\mathrm{Sh}}_{\Delta,r}(\tau,G) = \int_M G(z)\overline{\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{Sh}})}y^2 d\mu(z).$$

We then have the following relation between the two theta lifts.

Theorem 5.8. *Let $F \in H_0^+(N)$ with vanishing constant term at all cusps. Then we have that*

$$\xi_{1/2,\tau}\left(\mathcal{I}_{\Delta,r}(\tau,F)\right) = \frac{1}{2\sqrt{N}}\mathcal{I}^{Sh}_{\Delta,r}(\tau,\xi_{0,z}(F)).$$

Proof. By Stokes' theorem we have that

$$\begin{aligned}\mathcal{I}^{\mathrm{Sh}}_{\Delta,r}(\tau,\xi_{0,z}(F)) &= \int_M \xi_0(F(z))\overline{\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{Sh}})}y^2 d\mu(z)\\ &= -\int_M \overline{F(z)}\xi_{2,z}(\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{Sh}}))d\mu(z) + \lim_{t\to\infty}\int_{\partial\mathcal{F}_t}\overline{F(z)\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{Sh}})}d\bar{z},\end{aligned}$$

where $\mathcal{F}_t = \{z \in \mathbb{H} : \Im(z) \leq t\}$ denotes the truncated fundamental domain. Lemma 5.5 implies that

$$\begin{aligned}&-\int_M \overline{F(z)}\xi_{2,z}(\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{Sh}}))d\mu(z)\\ &= \frac{1}{2\sqrt{N}}\int_M \overline{F(z)}\xi_{1/2,\tau}(\Theta_{\Delta,r}(\tau,z,\varphi))d\mu(z) = \frac{1}{2\sqrt{N}}\xi_{1/2,\tau}\left(\mathcal{I}_{\Delta,r}(\tau,F)\right).\end{aligned}$$

It remains to show that

$$\lim_{t\to\infty}\int_{\partial\mathcal{F}_t}\overline{F(z)\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{Sh}})}d\bar{z} = 0.$$

As in the proof of Proposition 5.7 we have to investigate the growth of the theta function in the cusps. We have (again, $\Delta = N = 1$, $L = \mathbb{Z}^3$, and $h' = 0, 1/2$)

$$\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{Sh}}) = \sum_{\substack{a,c\in\mathbb{Z}\\ b\in\mathbb{Z}+h'}} -\frac{c\bar{z}^2 - b\bar{z} + a}{4y^2}e^{-\frac{\pi v}{y^2}(c|z|^2 - bx + a)}e^{2\pi i\bar{\tau}(-b^2/4+ac)},$$

and apply Poisson summation to the sum on a. Thus, we consider

$$\int_{-\infty}^{\infty} -\frac{c\bar{z}^2 - b\bar{z} + a}{4y^2} e^{-\frac{\pi v}{y^2}(c|z|^2 - bx + a)} e^{2\pi i \bar{\tau}(-b^2/4+ac)} e^{2\pi i w a} da.$$

Proceeding as before, we obtain

$$\theta_h(\tau, z, \varphi_{\mathrm{Sh}}) = -\frac{1}{4\sqrt{v}y} \sum_{\substack{w,c\in\mathbb{Z}\\ b\in\mathbb{Z}+h'}} e^{-2\pi i\bar{\tau}(b/2-cx)^2} e^{2\pi i(bxw - cx^2 w)} \times \left(c\bar{z}^2 + biy - c|z|^2 + i\frac{y^2}{v}(c\bar{\tau} + w)\right) e^{-\frac{\pi y^2}{v}|c\tau + w|^2}.$$

If c and w are not both equal to 0 this vanishes in the limit as $y \to \infty$. In this case, the whole integral vanishes. But if $c = w = 0$ we have

$$-\frac{i}{4\sqrt{v}} \sum_{b\in\mathbb{Z}+h'} b e^{\pi i\bar{\tau}b^2/2}.$$

Thus, we are left with (the complex conjugate of)

$$\int_{\partial\mathcal{F}_T} F(z)\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{Sh}})dz = \frac{i}{4\sqrt{v}} \sum_{b\in\mathbb{Z}+h'} b e^{\pi i\bar{\tau}b^2/2} \int_1^T \int_0^1 F(z)dxdy.$$

We see that

$$\lim_{T\to\infty} \int_1^T \int_0^1 F(z)dxdy = 0,$$

since the constant coefficient of F vanishes. Therefore,

$$\lim_{T\to\infty} \int_{\partial M_T} \overline{F(z)\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{Sh}})} d\bar{z} = 0.$$

Generalizing to arbitrary N, a similar result holds for the other cusps of M.

□

For a cusp form $G = \sum_{n=1}^{\infty} b(n)q^n \in S_2^{\mathrm{new}}(N)$ we let $L(G, \Delta, s)$ be its twisted L-function

$$L(G, \Delta, s) = \sum_{n=1}^{\infty} \left(\frac{\Delta}{n}\right) b(n)n^{-s}.$$

The relation to the Shintani lifting directly implies

Proposition 5.9. *Let $F \in H_0^+(N)$ with vanishing constant term at all cusps and let $\xi_{0,z}(F) = F_E \in S_2^{new}(N)$. The lift $\mathcal{I}_{\Delta,r}(\tau, F)$ is weakly holomorphic if and only if*

$$L(F_E, \Delta, 1) = 0.$$

In particular, this happens if F is weakly holomorphic.

Proof. Clearly, the lift is weakly holomorphic if and only if the Shintani lifiting of F_E vanishes. This is trivially the case when $F_E = \xi_0(F) = 0$, i.e. when F is weakly holomorphic. In the other case, the coefficients of the Shintani lifting are given by (in terms of Jacobi forms; for the definition of Jacobi forms and the cycle integral r see [148])

$$\mathcal{I}^{\mathrm{Sh}}_{\Delta,r}(\tau, \xi_{0,z}(F)) = \sum_{\substack{n, r_0 \in \mathbb{Z} \\ r_0^2 < 4nN}} r_{1,N,\Delta(r_0^2-4nN),rr_0,\Delta}(F_E) q^n \zeta^{r_0}.$$

Now by the Theorem and Corollary in Section II.4 in [148] we have

$$|r_{1,N,\Delta(r_0^2-4nN),rr_0,\Delta}(F_E)|^2 = \frac{1}{4\pi^2} |\Delta|^{1/2} |r_0^2 - 4nN|^{1/2} \, L(F_E, \Delta, 1) \, L(F_E, r_0^2 - 4nN, 1).$$

Since r_0 and n vary this expression vanishes if and only if $L(F_E, \Delta, 1)$ vanishes. □

Proof of Theorem 5.6. Proposition 5.7 implies that an $F \in H_0^+(N)$ with vanishing constant term at all cusps maps to a form of weight 1/2 transforming with representation $\tilde{\rho}$ that is annihilated by the Laplace operator $\Delta_{1/2,\tau}$. Theorem 5.8 then implies, that the lift satisfies the correct growth conditions at all cusps. □

5.3.1 Fourier expansion of the holomorphic part

Now we turn to the computation of the Fourier coefficients of positive index of the holomorphic part of the theta lift.

Let $h \in L'/L$ and $m \in \mathbb{Q}_{>0}$ with $m \equiv \operatorname{sgn}(\Delta)Q(h)$ $(\mathbb{Z})$. We define a twisted Heegner divisor on M by

$$Z_{\Delta,r}(m,h) = \sum_{\lambda \in \Gamma_0(N) \backslash L_{rh,m|\Delta|}} \frac{\chi_\Delta(\lambda)}{|\overline{\Gamma}_\lambda|} Z(\lambda).$$

Here $\overline{\Gamma}_\lambda$ denotes the stabilizer of λ in $\overline{\Gamma_0(N)}$.

Let F be a harmonic Maass form of weight 0 in $H_0^+(N)$. Then the twisted modular trace function is defined as follows

$$\operatorname{tr}_{\Delta,r}(F;m,h) = \sum_{z \in Z_{\Delta,r}(m,h)} F(z) = \sum_{\lambda \in \Gamma \backslash L_{|\Delta|m,rh}} \frac{\chi_\Delta(\lambda)}{|\bar{\Gamma}_\lambda|} f(D_\lambda). \tag{5.14}$$

Here we need to define a refined modular trace function. We let

$$L^+_{|\Delta|m,rh} = \left\{ \lambda = \begin{pmatrix} \frac{b}{2N} & -\frac{a}{N} \\ c & -\frac{b}{2N} \end{pmatrix} \in L_{|\Delta|m,rh}\,;\; a \geq 0 \right\},$$

and similarly

$$L^-_{|\Delta|m,rh} = \left\{ \lambda = \begin{pmatrix} \frac{b}{2N} & -\frac{a}{N} \\ c & -\frac{b}{2N} \end{pmatrix} \in L_{|\Delta|m,rh}\,;\; -a > 0 \right\},$$

and define modular trace functions

$$\operatorname{tr}^+_{\Delta,r}(F;m,h) = \sum_{\lambda \in \Gamma \backslash L^+_{|\Delta|m,rh}} \frac{\chi_\Delta(\lambda)}{|\bar{\Gamma}_\lambda|} f(D_\lambda)$$

and

$$\operatorname{tr}^-_{\Delta,r}(F;m,h) = \sum_{\lambda \in \Gamma \backslash L^-_{|\Delta|m,rh}} \frac{\operatorname{sgn}(\Delta)\chi_\Delta(\lambda)}{|\bar{\Gamma}_\lambda|} f(D_\lambda).$$

Theorem 5.10. *Let F be a harmonic Maass form of weight 0 in $H_0^+(N)$, $m > 0$, and $h \in L'/L$. The coefficients of index (m,h) of the holomorphic part of the lift $\mathcal{I}_{\Delta,r}(\tau,F)$ are given by*

$$\frac{\sqrt{\Delta}}{2\sqrt{m}} \left(\operatorname{tr}^+_{\Delta,r}(F;m,h) - \operatorname{tr}^-_{\Delta,r}(F;m,h) \right). \tag{5.15}$$

Proof. To ease notation we start proving the result when $\Delta = 1$. Using the arguments of the proof of Theorem 5.5 in [6] it is straightforward to later generalize to the case $\Delta \neq 1$.

We consider the Fourier expansion of $\int_M F(z)\Theta(\tau, z, \varphi)d\mu(z)$, namely

$$\sum_{h\in L'/L}\sum_{m\in\mathbb{Q}}\left(\sum_{\lambda\in L_{m,h}}\int_M F(z)v^{1/2}\varphi^0(\sqrt{v}\lambda, z)d\mu(z)\right)e^{2\pi i m\tau}. \tag{5.16}$$

We denote the (m, h)-th coefficient of the holomorphic part of (5.16) by $C(m, h)$. Using the usual unfolding argument implies that

$$\begin{aligned}C(m,h) &= \sum_{\lambda\in\Gamma\backslash L_{m,h}}\frac{1}{|\bar{\Gamma}_\lambda|}\int_D F(z)v^{1/2}\varphi^0(\sqrt{v}\lambda, z)d\mu(z)\\ &= \sum_{\lambda\in\Gamma\backslash L^+_{m,h}}\frac{1}{|\bar{\Gamma}_\lambda|}\int_D F(z)v^{1/2}\varphi^0(\sqrt{v}\lambda, z)d\mu(z)\\ &\quad + \sum_{\lambda\in\Gamma\backslash L^-_{m,h}}\frac{1}{|\bar{\Gamma}_\lambda|}\int_D F(z)v^{1/2}\varphi^0(\sqrt{v}\lambda, z)d\mu(z).\end{aligned}$$

Since $\varphi^0(-\sqrt{v}\lambda, z) = -\varphi^0(\sqrt{v}\lambda, z)$ the latter summand equals

$$-\sum_{\lambda\in\Gamma\backslash L^-_{m,h}}\frac{1}{|\bar{\Gamma}_{-\lambda}|}\int_D F(z)v^{1/2}\varphi^0(-\sqrt{v}\lambda, z)d\mu(z).$$

As in [185] and [57] we rewrite the integral over D as an integral over $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$. We normalize the Haar measure such that the maximal compact subgroup $\mathrm{SO}(2)$ has volume 1. We then have

$$\int_D F(z)\varphi^0(\sqrt{v}\lambda, z)d\mu(z) = \int_{G(\mathbb{R})} F(gi)\varphi^0(\pm\sqrt{v}\lambda, gi)dg, \qquad \text{for } \lambda\in\Gamma\setminus L^{\pm}_{m,h}.$$

Note that in [185] it is assumed that $SL_2(\mathbb{R})$ acts transitively on vectors of the same norm. This is not true. However, $SL_2(\mathbb{R})$ acts transitively on vectors of the same norm satisfying $a > 0$. Therefore, there is a $g_1 \in \mathrm{SL}_2(\mathbb{R})$ such that $g_1^{-1}.\lambda = \sqrt{m}\lambda(i)$ for $\lambda \in L_{m,h}^+$. Similarly, there is a $g_1 \in \mathrm{SL}_2(\mathbb{R})$ such that $g_1^{-1}.(-\lambda) = \sqrt{m}\lambda(i)$ for $\lambda \in L_{m,h}^-$. So, we have

$$C(m,h) = \sum_{\lambda \in \Gamma\backslash L_{m,h}^+} \frac{1}{|\bar{\Gamma}_\lambda|} v^{1/2} \int_{G(\mathbb{R})} F(gg_1 i)\varphi^0\left(\sqrt{v}\sqrt{m}g^{-1}.\lambda(i), i\right) dg$$
$$- \sum_{\lambda \in \Gamma\backslash L_{m,h}^-} \frac{1}{|\bar{\Gamma}_{-\lambda}|} v^{1/2} \int_{G(\mathbb{R})} F(gg_1 i)\varphi^0\left(\sqrt{v}\sqrt{m}g^{-1}.\lambda(i), i\right) dg.$$

Using the Cartan decomposition of $SL_2(\mathbb{R})$ we find proceeding as in [185] that

$$C(m,h) = \sum_{\lambda \in \Gamma\backslash L_{m,h}^+} \frac{1}{|\bar{\Gamma}_\lambda|} F(D_\lambda) v^{1/2} Y(\sqrt{mv}) - \sum_{\lambda \in \Gamma\backslash L_{m,h}^-} \frac{1}{|\bar{\Gamma}_{-\lambda}|} F(D_{-\lambda}) v^{1/2} Y(\sqrt{mv}), \tag{5.17}$$

where

$$Y(t) = 4\pi \int_1^\infty \varphi^0(t\alpha(a)^{-1}.\lambda(i), i) \frac{a^2 - a^{-2}}{2} \frac{da}{a}. \tag{5.18}$$

Here $\alpha(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$. We have that

$$\varphi^0(t\alpha(a)^{-1}.\lambda(i), i) = t(a^2 + a^{-2})e^{-\pi t^2 (a^2 - a^{-2})^2}.$$

Substituting $a = e^{r/2}$ we obtain that (5.18) equals

$$4\pi t \int_0^\infty \cosh(r)\sinh(r) e^{-4\pi t^2 \sinh(r)^2} dr = \frac{1}{2t}.$$

Thus, we have $Y(\sqrt{mv}) = \frac{1}{2\sqrt{mv}}$ which implies that

$$C(m,h) = \frac{1}{2\sqrt{m}} \left(\sum_{\lambda \in \Gamma\backslash L_{m,h}^+} \frac{1}{|\bar{\Gamma}_\lambda|} F(D_\lambda) - \sum_{\lambda \in \Gamma\backslash L_{m,h}^-} \frac{1}{|\bar{\Gamma}_\lambda|} F(D_\lambda) \right),$$

since $|\bar{\Gamma}_\lambda| = |\bar{\Gamma}_{-\lambda}|$ and $D_\lambda = D_{-\lambda}$.

Using the methods of [6] it is not hard to see that the (m, h)-th coefficient of the twisted lift is equal to

$$\frac{\sqrt{\Delta}}{2\sqrt{m}} \left(\sum_{\lambda \in \Gamma \backslash L^+_{m|\Delta|,rh}} \frac{\chi_\Delta(\lambda)}{|\bar{\Gamma}_\lambda|} F(D_\lambda) - \sum_{\lambda \in \Gamma \backslash L^-_{m|\Delta|,rh}} \frac{\chi_\Delta(-\lambda)}{|\bar{\Gamma}_\lambda|} F(D_\lambda) \right).$$

We have that $\chi_\Delta(-\lambda) = \operatorname{sgn}(\Delta)\chi_\Delta(\lambda)$ which implies the result.

□

5.3.2 Lift of Poincaré series and constants

In this section we compute the lift of Poincaré series and the constant function in the case $\Delta \neq 1$. This will be useful for the computation of the principal part of the theta lift.

Theorem 5.11. *We have*

$$\mathcal{I}_{\Delta,r}(\tau, F_m(z, s, 0)) = \frac{2^{-s+1} i}{\Gamma(s/2)} \sqrt{\pi N|\Delta|}\bar{\epsilon} \sum_{n|m} \left(\frac{\Delta}{n}\right) \mathcal{F}_{\frac{m^2}{4Nn^2}|\Delta|, -\frac{m}{n}r} \left(\tau, \frac{s}{2} + \frac{1}{4}, \frac{1}{2}\right).$$

Remark. In particular, we have

$$\mathcal{I}_{\Delta,r}(\tau, F_m(z, 1, 0)) = i\bar{\epsilon}\sqrt{N|\Delta|} \sum_{n|m} \left(\frac{\Delta}{n}\right) \mathcal{F}_{\frac{m^2}{4Nn^2}|\Delta|, -\frac{m}{n}r} \left(\tau, \frac{3}{4}, \frac{1}{2}\right).$$

Proof. The proof follows the one in [47, Theorem 3.3] or [4, Theorem 4.3]. Using the definition of the Poincaré series (5.10) and an unfolding argument we obtain

$$\mathcal{I}_{\Delta,r}(\tau, F_m(z, s, 0)) = \frac{1}{\Gamma(2s)} \int_{\Gamma_\infty \backslash \mathbb{H}} \mathcal{M}_{s,0}(4\pi m y) e(-mx) \Theta_{\Delta,r}(\tau, z, \varphi) d\mu(z).$$

By Proposition 5.3 this equals

$$-\frac{\bar{\epsilon} N}{\Gamma(2s) 2i} \sum_{n=1}^{\infty} \left(\frac{\Delta}{n}\right) n \sum_{\gamma \in \widetilde{\Gamma}_\infty \backslash \widetilde{\Gamma}} I(\tau, s, m, n)|_{1/2, \widetilde{\rho}_K}\ \gamma,$$

where

$$I(\tau, s, m, n) = \int_{y=0}^{\infty} \int_{x=0}^{1} y^2 \mathcal{M}_{s,0}(4\pi my) e(-mx) \exp\left(-\frac{\pi n^2 N y^2}{|\Delta| v}\right) \\ \times v^{-1/2} \sum_{\lambda \in K'} e\left(|\Delta| Q(\lambda)\bar{\tau} - 2N\lambda nx\right) \mathfrak{e}_{r\lambda} \frac{dxdy}{y^2}.$$

Identifying $K' = \mathbb{Z}\begin{pmatrix} 1/2N & 0 \\ 0 & -1/2N \end{pmatrix}$ we find that

$$\sum_{\lambda \in K'} e\left(|\Delta| Q(\lambda)\bar{\tau} - 2N\lambda nx\right) \mathfrak{e}_{r\lambda} = \sum_{b \in \mathbb{Z}} e\left(-|\Delta| \frac{b^2}{4N}\bar{\tau} - nbx\right) \mathfrak{e}_{rb}.$$

Inserting this in the formula for $I(\tau, s, m, n)$, and integrating over x, we see that $I(\tau, s, m, n)$ vanishes whenever $n \nmid m$ and the only summand occurs for $b = -m/n$, when $n \mid m$. Thus, $I(\tau, s, m, n)$ equals

$$v^{-1/2} e\left(-|\Delta| \frac{m^2}{4Nn^2}\bar{\tau}\right) \cdot \int_{y=0}^{\infty} \mathcal{M}_{s,0}(4\pi my) \exp\left(-\frac{\pi n^2 N y^2}{|\Delta| v}\right) dy \ \mathfrak{e}_{-rm/n}. \tag{5.19}$$

To evaluate the integral in (5.19) note that (see for example (13.6.3) in [1])

$$\mathcal{M}_{s,0}(4\pi my) = 2^{2s-1}\Gamma\left(s + \frac{1}{2}\right) \sqrt{4\pi my} \cdot I_{s-1/2}(2\pi my).$$

Substituting $t = y^2$ yields

$$\begin{aligned} &\int_{y=0}^{\infty} \mathcal{M}_{s,0}(4\pi my) \exp\left(-\frac{\pi n^2 N y^2}{|\Delta| v}\right) dy \\ &= 2^{2s-1}\Gamma\left(s + \frac{1}{2}\right) \int_{y=0}^{\infty} \sqrt{4\pi my}\ I_{s-1/2}(2\pi my) \exp\left(-\frac{\pi n^2 N y^2}{|\Delta| v}\right) dy \\ &= 2^{2s-1}\Gamma\left(s + \frac{1}{2}\right) \sqrt{m\pi} \int_{t=0}^{\infty} t^{-1/4} I_{s-1/2}(2\pi m t^{1/2}) \exp\left(-\frac{\pi n^2 N t}{|\Delta| v}\right) dt. \end{aligned}$$

The last integral is a Laplace transform and is computed in [116] (see (20) on p. 197). It equals

$$\frac{\Gamma\left(\frac{s}{2} + \frac{1}{2}\right)}{\Gamma\left(s + \frac{1}{2}\right)} (\pi m)^{-1} \left(\frac{\pi n^2 N}{|\Delta| v}\right)^{-1/4} \exp\left(\frac{\pi m^2 |\Delta| v}{2n^2 N}\right) M_{-\frac{1}{4}, \frac{s}{2} - \frac{1}{4}}\left(\frac{\pi m^2 |\Delta| v}{n^2 N}\right).$$

Therefore, we have that $I(\tau, s, m, n)$ equals

$$2^{2s-1}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\sqrt{\frac{|\Delta|}{\pi N n^2}}e\left(-\frac{m^2|\Delta|u}{4n^2N}\right)\mathcal{M}_{s/2+1/4,1/2}\left(\frac{\pi m^2|\Delta|v}{n^2N}\right)\mathfrak{e}_{-rm/n}.$$

Putting everything together we obtain the following for the lift of $F_m(z,s,0)$

$$\begin{aligned}
&-\frac{2^{2s-2}\Gamma(s/2+1/2)\bar{\epsilon}}{\Gamma(2s)i}\sqrt{\frac{N|\Delta|}{\pi}}\sum_{n|m}\left(\frac{\Delta}{n}\right)\\
&\times\sum_{\gamma\in\widetilde{\Gamma}_\infty\backslash\widetilde{\Gamma}}\left[e\left(-\frac{m^2|\Delta|u}{4Nn^2}\right)\mathcal{M}_{s/2+1/4,1/2}\left(\frac{\pi m^2|\Delta|v}{n^2N}\right)\mathfrak{e}_{-rm/n}\right]|_{1/2,\widetilde{\rho}_K}\ \gamma\\
&=-\frac{2^{-s+1}}{i\Gamma(s/2)}\sqrt{\pi N|\Delta|}\bar{\epsilon}\sum_{n|m}\left(\frac{\Delta}{n}\right)\mathcal{F}_{\frac{m^2}{4Nn^2}|\Delta|,-\frac{m}{n}r}\left(\tau,\frac{s}{2}+\frac{1}{4},\frac{1}{2}\right).
\end{aligned}$$

□

We define

$$\Theta_K(\tau)=\sum_{\lambda\in K'}e(Q(\lambda)\tau)\mathfrak{e}_{\lambda+K}.$$

Theorem 5.12. *Let $N=1$ and $\Delta<0$ (for $\Delta>0$ and $N=1$ the lift vanishes), $\epsilon_\Delta(n)=\left(\frac{\Delta}{n}\right)$ and $L(\epsilon_\Delta,s)$ be the Dirichlet L-series associated with ϵ_Δ. We have*

$$\mathcal{I}_{\Delta,r}(\tau,1)=\frac{\bar{\epsilon}\,i}{\pi}|\Delta|L(\epsilon_\Delta,1)\,\Theta_K(\tau).$$

Proof. This result follows analogously to [55, Theorem 7.1, Corollary 7.2] and [6, Theorem 6.1]. We compute the lift of the nonholomorphic weight 0 Eisenstein series and then take residues at $s=1/2$. Let $z\in\mathbb{H}$, $s\in\mathbb{C}$ and

$$\mathcal{E}_0(z,s)=\frac{1}{2}\zeta^*(2s+1)\sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})}(\Im(\gamma z))^{s+\frac{1}{2}},$$

where $\zeta^*(s)$ is the completed Riemann Zeta function. The Eisenstein series $\mathcal{E}_0(z,s)$ has a simple pole at $s=\frac{1}{2}$ with residue $\frac{1}{2}$. Using the standard unfolding trick we obtain

$$\mathcal{I}_{\Delta,r}(\tau,\mathcal{E}_0(z,s)) = \zeta^*(2s+1)\int_{\Gamma_\infty\backslash\mathbb{H}} \Theta_{\Delta,r}(\tau,z,\varphi) y^{s+\frac{1}{2}} d\mu(z).$$

By Proposition 5.3 we have that this equals

$$\begin{aligned} &-\zeta^*(2s+1)\frac{\bar{\epsilon}}{2i}\sum_{n\geq 1} n\left(\frac{\Delta}{n}\right)\sum_{\gamma\in\widetilde{\Gamma}_\infty\backslash\widetilde{\Gamma}} \phi(\tau)^{-1}\tilde{\rho}_K^{-1}(\gamma)\frac{1}{\Im(\gamma\tau)^{1/2}} \\ &\times\int_{y=0}^{\infty} y^{s+\frac{1}{2}}\exp\left(-\frac{\pi n^2 y^2}{|\Delta|\Im(\gamma\tau)}\right)dy \\ &\times\int_{x=0}^{1}\sum_{\lambda\in K'} e\left(\frac{\lambda^2\bar{\tau}}{2|\Delta|}-2\lambda n x\right)\mathfrak{e}_{r\lambda}dx. \end{aligned}$$

The integral over x equals $\mathfrak{e}_0$ and the one over y equals

$$\frac{1}{2}\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)(|\Delta|\Im(\gamma\tau))^{\frac{s}{2}+\frac{3}{4}}\pi^{-\frac{s}{2}-\frac{3}{4}}n^{-s-\frac{3}{2}}.$$

Thus, we have

$$\begin{aligned} \mathcal{I}_{\Delta,r}(\tau,\mathcal{E}_0(z,s)) = &-\zeta^*(2s+1)\frac{\bar{\epsilon}}{2i}\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)|\Delta|^{\frac{s}{2}+\frac{3}{4}}\pi^{-\frac{s}{2}-\frac{3}{4}} \\ &\times L\left(\epsilon_\Delta, s+\frac{1}{2}\right)\frac{1}{2}\sum_{\gamma\in\widetilde{\Gamma}_\infty\backslash\widetilde{\Gamma}}(v^{\frac{1}{2}(s+\frac{1}{2})}\mathfrak{e}_0)|_{1/2,K}\gamma. \end{aligned}$$

We now take residues at $s=1/2$ on both sides. Note that the residue of the weight 1/2 Eisenstein series is given by (see [171, Proof of Proposition 5.14])

$$\mathrm{res}_{s=1/2}\left(\frac{1}{2}\sum_{\gamma\in\widetilde{\Gamma}_\infty\backslash\widetilde{\Gamma}}(v^{\frac{1}{2}(s+\frac{1}{2})}\mathfrak{e}_0)|_{1/2,K}\gamma\right) = \frac{6}{\pi}\Theta_K(\tau).$$

We have $\zeta^*(2)=\pi/6$ which concludes the proof of the theorem. □

5.4 General version of Theorem 1.17 and its proof

Here we give the general version of Theorem 1.17, give its proof, and then conclude with some numerical examples.

We begin with some notation. Let L be the lattice of discriminant $2N$ defined in Section 5.2.1 and let $\rho = \rho_1$ be as in Section 5.2.2. Let $F_E \in S_2^{new}(\Gamma_0(N_E))$ be a normalized newform of weight 2 associated to the elliptic curve $E/\mathbb{Q}$. Let $\epsilon \in \{\pm 1\}$ be the eigenvalue of the Fricke involution on F_G. If $\epsilon = 1$, we put $\rho = \bar{\rho}$ and assume that Δ is a negative fundamental disriminant. If $\epsilon = -1$ we put $\rho = \rho$ and assume that Δ is a positive fundamental discriminant. There is a newform $g_E \in S_{3/2,\rho}^{new}$ mapping to F_E under the Shimura correspondence. We may normalize g_E such that all its coefficients are contained in $\mathbb{Q}$.

Recall that

$$\widehat{\mathfrak{Z}}_E(z) = \zeta(\Lambda_E; \mathcal{E}_E(z)) - S(\Lambda_E)\mathcal{E}_E(z) - \frac{\deg(\phi_E)}{4\pi||F_E||^2}\overline{\mathcal{E}_E(z)},$$

and $M_E(z)$ is chosen such that $\widehat{\mathfrak{Z}}_E(z) - M_E(z)$ is holomorphic on $\mathbb{H}$. By $a_{\ell,\widehat{\mathfrak{Z}}_E}(0)$ and $a_{\ell,M_E}(0)$ we denote the constant terms of these two functions at the cusp ℓ.

We then let

$$\widehat{\mathfrak{Z}}_E^*(z) = \frac{1}{\sqrt{|\Delta|N}}\left(\widehat{\mathfrak{Z}}_E(z) - \sum_{\ell \in \Gamma\backslash \mathrm{Iso}(V)} a_{\ell,\widehat{\mathfrak{Z}}_E}(0)\right).$$

Analogously, we let

$$M_E^*(z) = \frac{1}{\sqrt{|\Delta|N}}\left(M_E(z) - \sum_{\ell \in \Gamma\backslash \mathrm{Iso}(V)} a_{\ell,M_E}(0))\right).$$

Then $\widehat{\mathfrak{Z}}_E^*(z) - M_E^*(z)$ is a harmonic Maass form of weight 0.

By $f_{E,\Delta,r} = f_E$ we denote the twisted theta lift of $\widehat{\mathfrak{Z}}^*_E(z) - M^*_E(z)$ as in Section 5.3.

We begin with some notation. Let L be the lattice of discriminant $2N$ defined in Section 5.2.1 and let $\rho = \rho_1$ be as in Section 5.2.2. Let $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. The space of vector-valued holomorphic modular forms $M_{k,\bar{\rho}}$ is isomorphic to the space of skew holomorphic Jacobi forms $J^{skew}_{k+1/2,N}$ of weight $k + 1/2$ and index N. Moreover, $M_{k,\rho}$ is isomorphic to the space of holomorphic Jacobi forms $J_{k+1/2,N}$. The subspace $S^{new}_{k,\bar{\rho}}$ of newforms of the cusp forms $S_{k,\bar{\rho}}$ is isomorphic as a module over the Hecke algebra to the space of newforms $S^{new,+}_{2k-1}(\Gamma_0(N))$ of weight $2k-1$ for $\Gamma_0(N)$ on which the Fricke involution acts by multiplication with $(-1)^{k-1/2}$. The isomorphism is given by the Shimura correspondence [273]. Similarly, the subspace $S^{new}_{k,\rho}$ of newforms of $S_{k,\rho}$ is isomorphic as a module over the Hecke algebra to the space of newforms $S^{new,-}_{2k-1}(\Gamma_0(N))$ of weight $2k-1$ for $\Gamma_0(N)$ on which the Fricke involution acts by multiplication with $(-1)^{k+1/2}$ [148]. Let ϵ be the eigenvalue of the Fricke involution on G.

The Hecke L-series of any $G \in S^{new,\pm}_{2k-1}(\Gamma_0(N))$ satisfies a functional equation under $s \mapsto 2k - 1 - s$ with root number $-\epsilon$. If $G \in S^{new,\pm}_{2k-1}(\Gamma_0(N))$ is a normalized newform (in particular a common eigenform of all Hecke operators), we denote by F_G the number field generated by the Hecke eigenvalues of G. It is well known that we may normalize the preimage of G under the Shimura correspondence such that all its Fourier coefficients are contained in F_G.

Theorem 5.13. *Assume that $E/\mathbb{Q}$ is an elliptic curve of square free conductor N_E, and suppose that $F_E|_2 W_{N_E} = \epsilon F_E$. Denote the coefficients of $f_E(\tau)$ by $c^{\pm}_E(h,n)$. Then the following are true:*

(i) If $d \neq 1$ is a fundamental discriminant and $r \in \mathbb{Z}$ such that $d \equiv$

$r^2 \pmod{4N_E}$, *and* $\epsilon d < 0$, *then*

$$L(E_d, 1) = 8\pi^2 ||F_E||^2 ||g_E||^2 \sqrt{\frac{|d|}{N_E}} \cdot c_E^-(\epsilon d, r)^2.$$

(ii) If $d \neq 1$ *is a fundamental discriminant and* $r \in \mathbb{Z}$ *such that* $d \equiv r^2 \pmod{4N_E}$ *and* $\epsilon d > 0$, *then*

$$L'(E_d, 1) = 0 \iff c_E^+(\epsilon d, r) \in \overline{\mathbb{Q}} \iff c_E^+(\epsilon d, r) \in \mathbb{Q}.$$

Remark. In contrast to Bruinier and Ono in [47] we are able to relate the weight 1/2 form to the elliptic curve in a direct way.

Proof. To prove Theorem 5.13, we shall employ the results in Section 7 in [47]. It suffices to prove that f_E can be taken for f in Theorem 7.6 and 7.8 in [47]. Therefore, we need to prove that f_E has rational principal part and that $\xi_{1/2}(f_E) \in \mathbb{R}g$, where g is the preimage of F_E under the Shimura lift. (In the case we consider it suffices to require that $\xi_{1/2}(f) \in \mathbb{R}g$ in [47, Theorem 7.6].)

We first prove that f_E has rational principal part at the cusp ∞. We write $\widehat{\mathfrak{Z}}_E^*(z) - M_E^*(z)$ as a linear combination of Poincaré series and constants, i.e.

$$\widehat{\mathfrak{Z}}_E^*(z) - M_E^*(z) = C + \frac{1}{\sqrt{|\Delta| N}} \sum_{m>0} a_{\widehat{\mathfrak{Z}}_E}(-m) F_m(z, 1, 0) + \frac{1}{\sqrt{|\Delta| N}} \sum_{k>0} a_{M_E}(-k) F_k(z, 1, 0).$$

Here C is a constant and the coefficients $a_{\widehat{\mathfrak{Z}}_E}(-m)$ and $a_{M_E}(-k)$ are rational by construction.

Then, by Theorem 5.11 and Theorem 5.12 the coefficients of the principal part of f_E are rational. For the other cusps of $\Gamma_0(N)$ this follows by the equivariance of the theta lift under $O(L'/L)$ and the fact that we can identify $O(L'/L)$ with the group generated by the Atkin-Lehner involutions.

By construction we have

$$\xi_0\left(\widehat{\mathfrak{Z}}^*_E(z) - M^*_E(z)\right) = \frac{-\deg(\phi_E)}{\sqrt{|\Delta|N}||F_E||^2}F_E.$$

At the same time Theorem 5.8 implies that

$$\mathcal{I}^{\mathrm{Sh}}_{\Delta,r}\left(\frac{-\deg(\phi_E)}{\sqrt{|\Delta|N}||F_E||^2}F_E\right) = 2\sqrt{N}\xi_{1/2}(f_E).$$

Thus, we have that $\xi_{1/2}(f_E) \in \mathbb{R}g$. □

5.5 Examples

Here we give examples which illustrate the results proved in this paper.

Example 5.14. For $X_0(11)$, we have a single isogeny class. The strong Weil curve

$$E\colon y^2 + y = x^3 - x^2 - 10x - 20,$$

has sign of the functional equation equal to +1 and the Mordell-Weil group $E(\mathbb{Q})$ has rank 0. In terms of Dedekind's eta-function, we have that

$$F_E(z) = \eta^2(z)\eta^2(11z) = q-2q^2-q^3+2q^4+q^5+2q^6-2q^7-2q^9-2q^{10}+q^{11}-\dots.$$

We find that the corresponding mock modular form $\widehat{\mathfrak{Z}}^+_E(z)$ is

$$\widehat{\mathfrak{Z}}^+_E(z) = q^{-1} + 1 + 0.9520...q + 1.5479...q^2 + 0.3493...q^3 + 1.9760...q^4 - O(q^5).$$

The apparent transcendence of these coefficients arise from $S(\Lambda_E) = 0.381246\dots$. We find that $\Omega_{11}(F_E) = 0.2538418...$ which is 1/5 of the real period of E. This 1/5 is related to the fact that the Mordell-Weil group has a cyclic torsion subgroup of order 5. A short calculation shows that the expansion of $\mathfrak{Z}_E(z)$ at the cusp zero is given by

$$\widehat{\mathfrak{Z}}^+_E(z)|_0\begin{pmatrix}0 & -1\\ 11 & 0\end{pmatrix} = \widehat{\mathfrak{Z}}^+_E(z)|U(11) + \frac{12}{5}.$$

In particular, the constant term is 17/5.

We see that $p=5$ is ordinary for $X_0(11)$. Here we illustrate Theorem 1.16. As a 5-adic expansion we have that

$$\mathfrak{S}_E(5) = 4 + 2\cdot 5^2 + 4\cdot 5^3 + \dots$$

which can be thought of as a 5-adic expansion of $S(\Lambda_E)$ given above. It turns out that

$$\lim_{n\to+\infty} \frac{\left[q\frac{d}{dq}\zeta(\Lambda_E;\mathcal{E}_E(z))\right]|T(5^n)}{a_E(5^n)} = \mathfrak{S}_E(5)F_E(z)$$

as a 5-adic limit. To illustrate this phenomenon, we let

$$T_n(E,z) := \frac{\left[q\frac{d}{dq}\zeta(\Lambda_E;\mathcal{E}_E(z))\right]|T(5^n)}{a_E(5^n)}.$$

We then have that

$$\begin{aligned}
T_1(E,z) - 4F_E(z) &= -5q^{-5} - \tfrac{50}{3}q - \tfrac{65}{3}q^2 + \dots &&\equiv 0 \pmod{5}\\
T_2(E,z) - (4+0\cdot 5)F_E(z) &= \tfrac{25}{4}q^{-25} - \tfrac{25}{6}q + \tfrac{925}{3}q^2 - \dots &&\equiv 0 \pmod{5^2}\\
&\vdots\\
T_4(E,z) - (4+2\cdot 5^2+4\cdot 5^3)F_E(z) &= -\tfrac{625}{11}q^{-625} + \tfrac{5^4\cdot 61301717918}{33}q + \dots &&\equiv 0 \pmod{5^4}.
\end{aligned}$$

Example 5.15. Here we illustrate Theorem 1.17 using the following numerical example computed by Strömberg [51]. We consider the elliptic curve $37a1$ given by the Weierstrass model

$$E: \quad y^2 + y = x^3 - x.$$

The sign of the functional equation of $L(E,s)$ is -1, and $E(\mathbb{Q})$ has rank 1. The q-expansion of $F_E(z)$ begins with the terms

$$F_E(z) = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + 4q^{10} - \dots \in S_2^{new}(\Gamma_0(37)).$$

Using Remark 3, we find that the corresponding mock modular form is

$$\widehat{\mathfrak{Z}}_E^+(z) = q^{-1}+1+2.1132...q+2.3867...q^2+4.2201...q^3+5.5566...q^4+8.3547...q^5+O(q^6).$$

It turns out that the weight $1/2$ harmonic Maass form $f_E(z) = \mathcal{I}_{-3}(\tau, \widehat{\mathfrak{Z}}_E^+(z))$ corresponds to the Poincaré series $\mathcal{M}_{-3/148,21}$ (see Section 5.2.3)). Using `Sage` [277], Strömberg and Bruinier computed all values of $L'(E_d, 1)$ for fundamental discriminants $d > 0$ such that $\left(\frac{d}{37}\right) = 1$ and $|d| \leq 15000$. The following table illustrates Theorem 1.17.

d	$c^+(d)$	$L'(E_d, 1)$	$\mathrm{rk}(E_d(\mathbb{Q}))$
1	$-0.2817617849\ldots$	$0.3059997738\ldots$	1
12	$-0.4885272382\ldots$	$4.2986147986\ldots$	1
21	$-0.1727392572\ldots$	$9.0023868003\ldots$	1
28	$-0.6781939953\ldots$	$4.3272602496\ldots$	1
33	$0.5663023201\ldots$	$3.6219567911\ldots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1489	9	0	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4393	66	0	3

Stephan Ehlen numerically confirmed that $c^+(d) = \frac{1}{2\sqrt{d}}\left(\mathrm{tr}^+_{-3}(\widehat{\mathfrak{Z}}_E^+(z); d) - \mathrm{tr}^-_{-3}(\widehat{\mathfrak{Z}}_E^+(z); d)\right)$ using `Sage` [277].

Example 5.16. In [308] Zagier defines the generating functions for the twisted traces of the modular invariant. For coprime fundamental discriminants $d < 0$ and $D > 1$, he sets

$$f_d = q^{-d} + \sum_{D>0} \left(\frac{1}{\sqrt{D}} \sum_{Q \in \mathcal{Q}_{dD} \backslash \Gamma} \chi(Q) j(\alpha_Q) \right) q^D,$$

where $\mathcal{Q}_{dD}$ are the quadratic forms of discriminant dD, $\chi(Q) = \left(\frac{D}{p}\right)$, where p is a prime represented by Q and α_Q is the corresponding CM-point.

With $d = -\Delta$ and $D = m$ we rediscover a vector-valued version of his results. For example

$$\mathcal{I}_{-3}(\tau, j-744) = f_3 = q^{-3}-248q+26752q^4-85995q^5+1707264q^8-4096248q^9+\cdots .$$

Chapter 6

SU(2)-Donaldson Invariants

6.1 Some relevant classical functions

Here we fix notation concerning theta functions, and we recall a few standard facts about Dedekind's eta-function and the *nearly* modular Eisenstein series $E_2(\tau)$. We use the following normalization for the Jacobi theta function

$$\vartheta_{ab}(v|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{(2n+a)^2}{8}} \; e^{\pi i\,(2n+a)(v+\frac{b}{2})}, \tag{6.1}$$

where $a, b \in \{0, 1\}$, $v \in \mathbb{C}$, $q = \exp(2\pi i\tau)$, $\tau = x + iy \in \mathbb{H}$, and $\mathbb{H}$ is the complex upper half-plane. The relation to the standard Jacobi theta functions is summarized in the following table:

$\vartheta_1(v\|\tau) = \vartheta_{11}(v\|\tau)$	$\vartheta_1(0\|\tau) = 0$	$\vartheta_1'(0\|\tau) = -2\pi\eta^3(\tau)$
$\vartheta_2(v\|\tau) = \vartheta_{10}(v\|\tau)$	$\vartheta_2(0\|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{(2n+1)^2}{8}}$	$\vartheta_2'(0\|\tau) = 0$
$\vartheta_3(v\|\tau) = \vartheta_{00}(v\|\tau)$	$\vartheta_3(0\|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}$	$\vartheta_3'(0\|\tau) = 0$
$\vartheta_4(v\|\tau) = \vartheta_{01}(v\|\tau)$	$\vartheta_4(0\|\tau) = \sum_{n\in\mathbb{Z}} (-1)^n\, q^{\frac{n^2}{2}}$	$\vartheta_4'(0\|\tau) = 0$

(6.2)

Here $\eta(\tau)$ is the Dedekind eta-function with

$$\eta^3(\tau) = \sum_{n=0}^{\infty} (-1)^n \, (2n+1) \, q^{\frac{(2n+1)^2}{8}} \ . \tag{6.3}$$

We will also use the notation $\vartheta_j(\tau) = \vartheta_j(0|\tau)$ for $j = 2, 3, 4$, and

$$\vartheta_2(\tau) = 2\,\Theta_2\left(\frac{\tau}{8}\right) , \quad \vartheta_3(\tau) = \Theta_3\left(\frac{\tau}{8}\right) , \quad \vartheta_4(\tau) = \Theta_4\left(\frac{\tau}{8}\right) \ . \tag{6.4}$$

Also, we have that $E_2(\tau)$ is the normalized *nearly modular* weight 2 Eisenstein series

$$E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sum_{d|n} d \, q^n \ . \tag{6.5}$$

6.2 SU(2)-Donaldson invariants on $\mathbb{CP}^2$

Here we recall a closed formula expression for these Donaldson invariants which is due to Göttsche and his collaborators [141, 143], and we recall the conjecture of Moore and Witten in this case. We then conclude this section with Theorem 6.5 which we shall use to prove Theorem 1.20.

The Donaldson invariants of a smooth, compact, oriented, simply connected Riemannian four-manifold (X, g) without boundary are defined by using intersection theory on the moduli space of anti-self-dual instantons for the gauge groups SU(2) and SO(3) [142]. Given a homology orientation some cohomology classes on the instanton moduli space can be associated to homology classes of X through the slant product and then evaluated on a fundamental class. We define

$$\mathbf{A}(X) := \mathrm{Sym}(H_0(X, \mathbb{Z}) \oplus H_2(X, \mathbb{Z})),$$

and we regard the Donaldson invariants as the functional

$$\mathcal{D}^{X,g}_{w_2(E)} : \mathbf{A}(X) \to \mathbb{Q} \ , \tag{6.6}$$

where $w_2(E) \in H^2(X, \mathbb{Z}_2)$ is the second Stiefel-Whitney class of the gauge bundles which are considered. Since X is simply connected, there is an integer class $2\lambda_0 \in H^2(\mathbb{C}\mathrm{P}^2, \mathbb{Z})$ that is not divisible by two and whose mod-two reduction is $w_2(E)$. Let $\{\mathrm{s}_i\}_{i=1,\ldots,b_2}$ be a basis of the two-cycles of X. We introduce the formal sum $S = \sum_{i=1}^{b_2} \kappa^i\, \mathrm{s}_i$, where κ^i are complex numbers. The generator of the zero-class of X will be denoted by $\mathrm{x} \in H_0(X, \mathbb{Z})$. The Donaldson-Witten generating function is

$$Z_{\mathrm{DW}}(p, \kappa) = \mathcal{D}^{X,g}_{w_2(E)}(e^{p\,\mathrm{x}+S}) \ , \tag{6.7}$$

so that the Donaldson invariants are read off from the expansion of (6.7) as the coefficients of powers of p and $\kappa = (\kappa^1, \ldots, \kappa^{b_2})$.

In the case of the complex projective plane $\mathbb{C}\mathrm{P}^2$, we have $b_2 = b_2^+ = 1$. The Fubini-Study metric g on $\mathbb{C}\mathrm{P}^2$ is Kähler with the Kähler form $K = \frac{i}{2} g_{a\bar{b}}\, dz^a \wedge dz^{\bar{b}}$. We denote the first Chern class of the dual of the hyperplane bundle over $\mathbb{C}\mathrm{P}^2$ by $\mathrm{H} = K/\pi$, so that $\int_{\mathbb{C}\mathrm{P}^2} \mathrm{H}^2 = 1$, $c_1(\mathbb{C}\mathrm{P}^2) = 3\,\mathrm{H}$, and $p_1(\mathbb{C}\mathrm{P}^2) = 3\,\mathrm{H}^2$. The Poincaré dual h of H is a generator of the rank-one homology group $H_2(\mathbb{C}\mathrm{P}^2, \mathbb{Z})$. The SO(3)-bundles on four-dimensional manifolds are classified by the second Stiefel-Whitney class $w_2(E) \in H^2(\mathbb{C}\mathrm{P}^2, \mathbb{Z}_2)$, and the first Pontrjagin class $p_1(E) \in H^4(\mathbb{C}\mathrm{P}^2, \mathbb{Z})$, such that

$$p_1(E)[\mathbb{C}\mathrm{P}^2] \equiv w_2^2(E)[\mathbb{C}\mathrm{P}^2]\, (\mathrm{mod}\, 4) \ .$$

Since $\mathbb{C}\mathrm{P}^2$ is simply connected, there is an integer class $2\lambda_0 \in H^2(\mathbb{C}\mathrm{P}^2, \mathbb{Z})$ whose mod-two reduction is $w_2(E)$. Then, there is a smooth complex two-dimensional vector bundle $\xi \to \mathbb{C}\mathrm{P}^2$ with the Chern classes $c_1(\xi) = 2\lambda_0$ and $c_2(\xi)$, such that $c_1^2(\xi) - 4\, c_2(\xi) = p_1(E)$. We denote by $\mathfrak{M}(c_1, c_2)$ the moduli space of rank-two vector bundles ξ over $\mathbb{C}\mathrm{P}^2$ with Chern classes c_1, c_2. It is known that $\mathfrak{M}(c_1, c_2)$ only depends on the discriminant $c_1^2 - 4c_2$ with the discriminant being negative for stable bundles. The bundle ξ can be reduced to an SU(2)-bundle if and only if $c_1(\xi) = 0$ and a SO(3)-bundle, which does not arise as the associated bundle for the adjoint representation of a SU(2)-bundle, satisfies $w_2(E) \neq 0$.

From now on, we will restrict ourselves to $w_2(E) = 0$, i.e., the case of SU(2)-bundles and where $c_1(\xi) = 0$ and $c_2(\xi) = k\,\mathrm{H}^2$ with $k \in \mathbb{N}$. The moduli space of anti-selfdual irreducible SU(2)-connections with $c_2(\xi)[\mathbb{C}\mathrm{P}^2] = k$ modulo gauge transformations is then the smooth, projective variety $\mathfrak{M}(0,k)$ of dimension $2d_k = 8k - 6$ [280]. The generating function (6.7) can be described as follows

$$Z_{\mathrm{DW}}(p,\kappa) = \sum_{m,n \geq 0} \Phi_{m,n} \, \frac{p^m}{m!} \, \frac{\kappa^n}{n!} \ , \tag{6.8}$$

where $S = \kappa$ h. Here $\Phi_{m,n}$ is the intersection number obtained by evaluating the top-dimensional cup product of the mth power of a universal four-form and the nth power of a two-form on the fundamental class of the Uhlenbeck compactification of $\mathfrak{M}(0,k)$ such that $4m + 2n = 8k - 6$ with $k \in \mathbb{N}$. Thus, for dimensional reasons we have $\Phi_{m,n} = 0$ for $2m + n \not\equiv 1 \pmod 4$

6.2.1 The work of Göttsche and his collaborators

The work of Göttsche and his collaborators [141,143] gives a closed expression for the SU(2) Donaldson invariants for the complex projective plane. Using the blowup formula for the Donaldson invariants, Göttsche [141] derived a closed formula expression for $\Phi_{m,n}$ assuming the truth of the Kotschick-Morgan Conjecture. Recently, Göttsche, Nakajima, Hiraku, and Yoshioka [143] have unconditionally proved these formulas. His work was based on earlier work with Ellingsrud [115] and it extended the results previously obtained by Kotschick and Lisca [197] up to an overall sign convention. His work with Zagier [144] was an application of [141]. We state [141, Thm. 3.5, (1)] using the original sign convention of [115, 197]. We write the result in terms of the Jacobi theta-functions $\vartheta_2, \vartheta_3, \vartheta_4$. In this way, we obtain a closed formula expression for $\Phi_{m,n}$, which we shall later show equals the the

Moore-Witten prediction based on the u-plane integral.

Theorem 6.1 (Göttsche [141])**.** *Assuming the notation and hypotheses above, then we have that the only non-vanishing coefficients in the generating function in (6.8) satisfy*

$$\begin{aligned}\Phi_{m,2n+1} = \sum_{l=0}^{n}\sum_{j=0}^{l} \frac{(-1)^{n+j+1}\,2^{2n-3l+4}}{3^l}\;\frac{(2n+1)!}{(2n-2l+1)!\;j!\;(l-j)!} \\ \times \mathrm{Coeff}_{q^0}\left(\frac{\vartheta_4^8(\tau)\;[\vartheta_2^4(\tau)+\vartheta_3^4(\tau)]^{m+j}}{[\vartheta_2(\tau)\,\vartheta_3(\tau)]^{2m+2n+5}}\;E_2^{l-j}(\tau)\;K_{2(n-l)}(\tau)\right)\;,\end{aligned} \tag{6.9}$$

where $m, n \in \mathbb{N}_0$*,* Coeff_{q^0} *is the constant term of a series expansion in* $q = \exp(2\pi i\tau)$*. The series* $K_t(\tau)$ *is*

$$K_t(\tau) := q^{\frac{1}{8}}\sum_{\beta=1}^{\infty}\sum_{\alpha=\beta}^{\infty}(-1)^{\alpha+\beta}\;(2\alpha+1)\;\beta^{t+1}\;q^{\frac{\alpha(\alpha+1)-\beta^2}{2}}\;. \tag{6.10}$$

Proof. The following table summarizes the quantities used by Göttsche [141, Thm. 3.5, (1)] and in this article:

Göttsche	Present Paper
z	S
x	p
n	$2\beta+1,\ \beta \geq 0$
a	$2\alpha,\ \alpha \geq \beta+1$
τ	$\frac{\tau-1}{2}$
q	$-q^{\frac{1}{2}}$

Göttsche	Present Paper
$\theta(\tau)$	$\vartheta_4(\tau)$
$f(\tau)$	$\frac{1}{2\sqrt{i}}\vartheta_2(\tau)\,\vartheta_3(\tau)$
$\frac{\Delta^2(2\tau)}{\Delta(\tau)\,\Delta(4\tau)}$	$-16\frac{\vartheta_4^8(\tau)}{[\vartheta_2(\tau)\,\vartheta_3(\tau)]^4}$
$G_2(2\tau)$	$-\frac{1}{24}\,E_2(\tau)$
$e_3(2\tau)$	$\frac{1}{12}\left[\vartheta_2^4(\tau)+\vartheta_3^4(\tau)\right]$
$\frac{-3i\,e_3(2\tau)}{f(\tau)^2}$	$\frac{\vartheta_2^4(\tau)+\vartheta_3^4(\tau)}{[\vartheta_2(\tau)\,\vartheta_3(\tau)]^2}$

We use

$$\left(\frac{n}{2}\,\frac{\sqrt{i}}{f(\tau)}\right)^{2(n-l)}\;\left(-\frac{i}{2\,f(\tau)^2}(2\,G_2(2\tau)+e_3(2\tau))\right)^{l}$$
$$= \quad \frac{(-1)^{n+l}}{2^l\,3^l}\,(2\beta+1)^{2(n-l)}\,\frac{\left(-E_2(\tau)+[\vartheta_2^4(\tau)+\vartheta_3^4(\tau)]\right)^l}{[\vartheta_2(\tau)\,\vartheta_3(\tau)]^{2n}}\;.$$

An expansion of the exponential in [141, Thm. 3.5, (1)] then yields (6.9). □

6.2.2 The u-plane integral and the work of Moore and Witten

Here we recall the theory of the u-plane and the work of Moore and Witten.

From now on we will assume that (X, g) is a smooth, compact, oriented, simply connected Riemannian four-manifold without boundary and $b_2^+ = 1$. The u-plane integral Z is a generating function in the variables p and κ whose coefficients are the integrals of certain modular forms over the fundamental domain of the group $\Gamma_0(4)$. It depends on the period point ω, the lattice $H_2(X, \mathbb{Z})$ together with the intersection form $(.\,,.)$, the second Stiefel-Whitney classes of the gauge bundle $w_2(E)$, and the tangent bundle $w_2(X)$, whose integral liftings are denoted by $2\lambda_0$ and w_2 respectively. The u-plane integral is non-vanishing only for manifolds with $b_2^+ = 1$. The explicit form of Z for simply connected four-manifolds was first introduced in [241]. For the reader's convenience, we quickly review the explicit construction of the u-plane in this chapter. Our approach to the u-plane integral, as well as its normalization, closely follows the approach in [200, 216, 232].

We will denote the self-dual and anti-self-dual projections of any two-form $\lambda \in H^2(X,\, \mathbb{Z}) + \lambda_0$ by $\lambda_+ = (\lambda, \omega)\omega$ and $\lambda_- = \lambda - \lambda_+$ respectively. We first introduce the integral

$$\mathcal{G}(\rho) = \int_{\Gamma_0(4)\backslash\mathbb{H}}^{\mathrm{reg}} \frac{dxdy}{y^{\frac{3}{2}}}\, \widehat{f}(p,\kappa)\, \bar{\Theta}(\xi)\ . \tag{6.11}$$

In this expression $\widehat{f}(p,\kappa)$ is the almost holomorphic modular form given by

$$\widehat{f}(p,\kappa) = \frac{\sqrt{2}}{64\pi}\frac{\vartheta_4^\sigma}{h^3\cdot f_2}\; e^{2\,p\,u+S^2\widehat{T}}, \tag{6.12}$$

where σ is the signature of X and $S^2 = (S,S) = \sum_{i,j}\kappa^i\kappa^j(\mathrm{s}_i,\mathrm{s}_j)$. Also, $\bar{\Theta}$ is the Siegel-Narain theta function

$$\begin{aligned}\bar{\Theta}(\xi) &= \exp\left[\frac{\pi}{2\,y}\Big(\bar{\xi}_+^2 - \bar{\xi}_-^2\Big)\right]\\ &\times \sum_{\lambda\in H^2+\lambda_0} \exp\Big[-i\pi\bar{\tau}(\lambda_+)^2 - i\pi\tau(\lambda_-)^2 - 2\pi i\,(\lambda,\bar{\xi}) + \pi i\,(\lambda,w_2)\Big],\end{aligned} \tag{6.13}$$

where $\bar{\xi} = \bar{\xi}_+ + \bar{\xi}_-$, $\bar{\xi}_+ = \rho\,y\,h\,\omega$, $\bar{\xi}_- = S_-/(2\pi h)$, and $\rho\in\mathbb{R}$. The Siegel-Narain theta function only depends on the lattice data $(H^2(X),\omega,\lambda_0,w_2)$. We have denoted the intersection form in two-cohomology by $(.\,,.)$, and we used Poincaré duality to convert cohomology classes into homology classes. In the above expressions, u, T, h, and f_2 are the modular forms defined as follows:

$$\begin{aligned} u &= \frac{\vartheta_2^4+\vartheta_3^4}{2\,(\vartheta_2\vartheta_3)^2}\ , & h &= \tfrac{1}{2}\,\vartheta_2\,\vartheta_3\ ,\\ T &= -\frac{1}{24}\left(\frac{E_2}{h^2} - 8\,u\right)\ , & f_2 &= \frac{\vartheta_2\,\vartheta_3}{2\,\vartheta_4^8}\ .\end{aligned} \tag{6.14}$$

Note that T does not transform well under modular transformations, due to the presence of the second normalized Eisenstein series $E_2 = E_2(\tau)$. Therefore, in (6.12) we have used the related form $\widehat{T} = T + 1/(8\pi y h^2)$ which is not holomorphic but transforms well under modular transformations. We also define the related holomorphic function $f(p,\kappa)$ as in (6.12), but with T instead of $\widehat{T}$. The u-plane integral is defined to be

$$Z\Big(X,\omega,\lambda_0,w_2\Big) = \left[(S,\omega) + 2\frac{d}{d\rho}\right]\Bigg|_{\rho=0} \mathcal{G}(\rho)\ . \tag{6.15}$$

If there is no danger of confusion we suppress the arguments (X,ω,λ_0,w_2) of Z.

Two remarks.

1) This regularized u-plane integral can be thought of as the regularized Peterson inner product of two half-integral weight modular forms on $\Gamma_0(4)$, where the regularization is obtained by integrating over the truncated fundamental domain for $\Gamma_0(4)$ where neighborhoods of the cusps are removed.

2) Definition (6.15) agrees with the definition given in [216]. However, compared to the original definition in [241], a factor of $\exp\left[2\pi i(\lambda_0,\lambda_0)+\pi i(\lambda_0,w_2)\right]$ is missing. For the case considered in this article, this factor is equal to one.

The regularization procedure applied in the definition of the integral (6.11) was described in detail in [241]. It defines a way of extracting certain contributions for each boundary component near the cusps of $\Gamma_0(4)\backslash\mathbb{H}$. Since the cusps are located at $\tau=\infty$, $\tau=0$, and $\tau=2$, we obtain Z_{u} as the sum of these contributions from the cusps:

$$Z_{\mathrm{u}} = Z_{\tau=0} + Z_{\tau=2} + Z_{\tau=\infty}\ . \tag{6.16}$$

We now apply the construction of the u-plane integral to $X=\mathbb{C}\mathrm{P}^2$. We denote the integral lifting of $w_2(E)$ by $2\lambda_0 = a\,\mathrm{H}\in H^2(\mathbb{C}\mathrm{P}^2,\mathbb{Z})$, and the integral lifting of $w_2(\mathbb{C}\mathrm{P}^2)$ by $w_2=-b\,\mathrm{H}\in H^2(\mathbb{C}\mathrm{P}^2,\mathbb{Z})$. The following lemma then follows immediately from the definition:

Lemma 6.2. *On $X=\mathbb{C}\mathrm{P}^2$ let $\omega=\mathrm{H}$ be the period point of the metric. Let $2\,\lambda_0=a\,\mathrm{H}$ with $a\in\{0,1\}$ be an integral lifting of $w_2(E)$. For $(X,\omega,\lambda_0,w_2=-\mathrm{H})$, the Siegel-Narain theta function is*

$$\bar{\Theta} = \exp\left(\frac{\pi}{2\,y}\,\bar{\xi}_+^2\right)\,\overline{\vartheta_{a1}\Big((\xi_+,\mathrm{H})\Big|\tau\Big)}, \tag{6.17}$$

where $\bar{\xi}=\bar{\xi}_+=\rho\,y\,h\,\omega$.

It was shown in [241] that for $\sigma = b_2^+ - b_2^- = 1$ and any value of a, we have

$$Z_{\tau=0} = Z_{\tau=2} = 0 \,, \tag{6.18}$$

and so $Z_{\rm u} = Z_{\tau=\infty}$. We restrict ourselves to the case $a = 0$, i.e., the case of SU(2)-bundles on $\mathbb{CP}^2$. The u-plane integral in (6.15) can be expanded as follows

$$Z_{\tau=\infty} = \sum_{m,n\in\mathbb{N}_0} \frac{p^m}{m!} \, \frac{\kappa^{2n+1}}{(2n+1)!} \, \mathrm{D}_{m,n}, \tag{6.19}$$

where

$$\mathrm{D}_{m,n} := -\frac{\sqrt{2}}{32\pi} \sum_{l=0}^{n} \int_{\Gamma_0(4)\backslash\mathbb{H}}^{\mathrm{reg}} \frac{dx\, dy}{y^{\frac{3}{2}}} \, R_{mnl} \, \widehat{E}_2^l \, \overline{\vartheta_{01}(0|\tau)} \,. \tag{6.20}$$

For $m, n \in \mathbb{N}_0$ and $0 \leq l \leq n$ we have set

$$R_{mnl} := (-1)^{l+1} \, \frac{(2n+1)!}{l!\,(n-l)!} \, \frac{2^{m-3l-1}}{3^n} \, \frac{\vartheta_4 \cdot u^{m+n-l}}{h^{3+2l} \cdot f_2}, \tag{6.21}$$

where u, h, and f_2 were defined in (6.14). To evaluate the regularized u-plane integral we introduce the non-holomorphic modular form $Q_{ab}(\tau) = Q_{ab}^+(\tau) + Q_{ab}^-(\tau)$ of weight 3/2 such that

$$8\sqrt{2}\pi\, i \, \frac{d}{d\bar{\tau}} \, Q_{ab}\,(\tau) = y^{-\frac{3}{2}} \, \overline{\vartheta_{ab}(0|\tau)}, \tag{6.22}$$

where a or b must be zero. These non-holomorphic modular forms were constructed by Zagier [306] and reviewed in [241]. The holomorphic parts of Zagier's weight 3/2 Maass-Eisenstein series, which first arose [161] in connection with intersection theory for certain Hilbert modular surfaces, are generating functions for Hurwitz class numbers. The holomorphic part of Zagier's weight 3/2 Maass-Eisenstein series is the generating function for Hurwitz class numbers. They have series expansions of the form

$$\begin{aligned} Q_{10}^+\,(\tau) &= \frac{1}{q^{\frac{1}{8}}} \, \sum_{l>0} \mathcal{H}_{4l-1} \, q^{\frac{l}{2}} \,, \\ Q_{00}^+\,(\tau) &= \sum_{l\geq 0} \mathcal{H}_{4l} \, q^{\frac{l}{2}}, \end{aligned} \tag{6.23}$$

where $\mathcal{H}_\alpha$ are the Hurwitz class numbers. The first nonvanishing Hurwitz class numbers are as follows:

$\mathcal{H}_0$	$\mathcal{H}_3$	$\mathcal{H}_4$	$\mathcal{H}_7$	$\mathcal{H}_8$	$\mathcal{H}_{11}$	$\mathcal{H}_{12}$	...
$-1/12$	$1/3$	$1/2$	1	1	1	$4/3$	...

The non-holomorphic parts have series expansions of the form

$$\begin{aligned} Q_{10}^{-}(\tau) &= \frac{1}{8\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} (l+\frac{1}{2}) \cdot \Gamma\left(-\frac{1}{2}, 2\pi \left(l+\frac{1}{2}\right)^2 y\right) q^{-\frac{(l+1/2)^2}{2}} , \\ Q_{00}^{-}(\tau) &= \frac{1}{8\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} l \cdot \Gamma\left(-\frac{1}{2}, 2\pi l^2 y\right) q^{-\frac{l^2}{2}} , \end{aligned} \tag{6.24}$$

where $\Gamma(3/2, x)$ is the incomplete gamma function

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t}\, t^{\alpha-1}\, dt \,. \tag{6.25}$$

The forms Q_{10} and Q_{00} combine to form a weight 3/2 form for the modular group. As explained in [241] the form $Q_{01}(\tau) = Q_{00}(4\tau) - Q_{10}(4\tau) + \frac{1}{2}Q_{00}(\tau+1)$ is modular for $\Gamma_0(4)$ of weight 3/2. We write the holomorphic part as

$$Q_{01}^{+}(\tau) = \sum_{n\geq 0} \mathcal{R}_n\, q^{\frac{n}{2}} \,. \tag{6.26}$$

The first nonvanishing coefficients in the series expansion are as follows:

$\mathcal{R}_0$	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	...
$-1/8$	$-1/4$	$1/2$	-1	$5/4$	...

All non-holomorphic parts have an exponential decay since

$$\Gamma(\alpha, t) = t^{\alpha-1}\, e^{-t}\, \left(1 + O(t^{-1})\right) \qquad (t \to \infty) \,. \tag{6.27}$$

The following lemma was proved in [241, (9.18)]:

Lemma 6.3. *The weakly holomorphic function*

$$\mathcal{E}^l\left[Q_{01}\right] = \sum_{j=0}^{l} (-1)^j \binom{l}{j} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+j\right)} \, 2^{2j} \, 3^j \, E_2^{l-j}(\tau) \left(q\,\frac{d}{dq}\right)^j Q_{01}\left(\tau\right) \tag{6.28}$$

is modular for $\Gamma_0(4)$ *of weight* $2l+3/2$ *and satisfies*

$$8\sqrt{2}\pi\, i \,\frac{d}{d\bar{\tau}}\, \mathcal{E}^l\left[Q_{01}\right] = y^{-\frac{3}{2}}\, \widehat{E}_2^l(\tau)\, \overline{\vartheta_{01}(0|\tau)}\,. \tag{6.29}$$

The evaluation of the u-plane integral.

It was shown in [241] that the cusp contribution at $\tau=\infty$ to the regularized u-plane integral can be evaluated as follows: in (6.31) we integrate by parts using the modular forms constructed in Lemma 6.3, i.e., we rewrite an integrand f as a total derivative using

$$dx\wedge dy\;\partial_{\bar\tau} f = \frac{1}{2}\, dx\wedge dy\;\left(\partial_x + i\,\partial_y\right)\, f = -\frac{i}{2}\, d\Big(f\, dx + i\, f\, dy\Big)\,.$$

We carry out the integral along the boundary $x=\mathrm{Re}(\tau)\in[0,4]$ and $y\gg 1$ fixed. This extracts the constant term coefficient. We then take the limit $y\to\infty$. Since all non-holomorphic parts have an exponential decay, the non-holomorphic dependence drops out. The following expression for the u-plane integral was obtained for the gauge group SU(2) in [241]. Additional information about the evaluation of the u-plane integral as well as the geometry of the Seiberg-Witten curve can be found in [225, 226].

Theorem 6.4. *On* $X=\mathbb{C}\mathrm{P}^2$, *let* $\omega=\mathrm{H}$ *be the period point of the metric. For* $(X,\omega,\lambda_0=0,w_2=-\mathrm{H})$, *the* u*-plane integral in the variables* $p\,\mathrm{x}\in H_0(X,\mathbb{Z})$, $S=\kappa\,\mathrm{h}\in H_2(X,\mathbb{Z})$ *is*

$$Z_{\mathrm{u}} = Z_{\tau=\infty} = \sum_{m,n\in\mathbb{N}_0} \frac{p^m}{m!}\frac{\kappa^{2n+1}}{(2n+1)!}\,\mathrm{D}_{m,n}, \tag{6.30}$$

where

$$\mathrm{D}_{m,n} = \sum_{l=0}^{n}\ \mathrm{Coeff}_{q^0}\Big(R_{mnl}\ \ \mathcal{E}^l[Q_{01}^{+}(\tau)]\Big) \tag{6.31}$$

and where R_{mnl} and $\mathcal{E}^l[Q_{01}(\tau)]$ are defined in (6.21) and (6.28) respectively.

For concreteness, we list the first nonvanishing coefficients of the generating function in Theorem 6.4, i.e., if $m+n = 2(k-1)$ for some $k \in \mathbb{N}$.

k	m	n	$\mathrm{D}_{m,n}$	$\mathrm{D}_{m,n}$
1	0	0	$-\frac{3}{2}$	$-\frac{1}{2}\,\mathcal{R}_1 + 13\,\mathcal{R}_0$
2	0	2	1	$-2\,\mathcal{R}_2 + 7\,\mathcal{R}_1 - 30\,\mathcal{R}_0$
2	1	1	-1	$-\frac{1}{4}\,\mathcal{R}_2 + \frac{1}{2}\,\mathcal{R}_1 + 6\,\mathcal{R}_0$
2	2	0	$-\frac{13}{8}$	$-\frac{1}{32}\,\mathcal{R}_2 - \frac{7}{16}\,\mathcal{R}_1 + \frac{55}{4}\,\mathcal{R}_0$

6.2.3 Criterion for proving Theorem 1.20

Here we combine the results of the previous two subsections to obtain a criterion for proving Theorem 1.20.

From a physics point of view, at a high energy scale, the SU(2)-Donaldson theory is described by the low energy effective field theory. Thus, the cuspidal contributions to the generating function of the low energy effective field theory should be equal to the generating function of the SU(2)-Donaldson theories The conjecture is equivalent to the assertion that the generating functions Z_{DW} in (6.8) and Z_{u} in (6.30) are equal. This amounts to proving that for all $m, n \in \mathbb{N}_0$ we have

$$\Phi_{m,2n+1} = \mathrm{D}_{m,n}\ . \tag{6.32}$$

In particular, the coefficients in (6.32) vanish for $m+n \equiv 1 \pmod 2$. We will prove (6.32) by proving:

Theorem 6.5. *Theorem 1.20 is equivalent to the vanishing of constant terms,* *for every pair of non-negative integers m and n, of the series*

$$\sum_{l=0}^{n}\sum_{j=0}^{l}(-1)^{j+1}\,\frac{(2n+1)!}{(n-l)!\,j!\,(l-j)!}\frac{\vartheta_4^8(\tau)\,\left[\vartheta_2^4(\tau)+\vartheta_3^4(\tau)\right]^m}{\left[\vartheta_2(\tau)\,\vartheta_3(\tau)\right]^{2m+2n+4}}\,E_2^{l-j}(\tau)$$
$$\times\left[\frac{(-1)^n\,2^{2n-3l+4}}{3^l}\,\frac{(n-l)!}{(2n-2l+1)!}\,\frac{\left[\vartheta_2^4(\tau)+\vartheta_3^4(\tau)\right]^j}{\vartheta_2(\tau)\,\vartheta_3(\tau)}\,K_{2(n-l)}(\tau)\right.$$
$$\left.-\,\frac{(-1)^l\,2^{2j-n+3}}{3^{n-j}}\,\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(j+\frac{3}{2}\right)}\,\vartheta_4(\tau)\,\left[\vartheta_2^4(\tau)+\vartheta_3^4(\tau)\right]^{n-l}\,\left(q\frac{d}{dq}\right)^j Q_{01}^+(\tau)\right], \tag{6.33}$$

where the series $K_t(\tau)$ are defined in (6.10).

6.3 The proof of Theorem 1.20

Here we prove Theorem 1.20 by using the theory of non-holomorphic modular forms and meromorphic Jacobi forms to check the condition in Theorem 6.5. To this end, we recall the important q-series

$$K_t(\tau) := q^{\frac{1}{8}}\sum_{\beta=1}^{\infty}\sum_{\alpha=\beta}^{\infty}(-1)^{\alpha+\beta}\,(2\alpha+1)\,\beta^{t+1}\,q^{\frac{\alpha(\alpha+1)-\beta^2}{2}} \tag{6.34}$$

from (6.10). In the following section we relate $K_{2t}(\tau)$ to derivatives of important power series.

6.3.1 q-series identities

Here we begin with the following elementary identity.

Proposition 6.6. *Let $\rho = e^{2\pi i u}$ and $\omega = e^{2\pi i v}$, and let $D_z = \frac{1}{2\pi i}\frac{d}{dz}$, where z is one of u, v, or τ. Then we have that*

$$K_{2t}(8\tau) = 2^{-2t-1}\,D_u^{2t+1}D_v\sum_{n\in\mathbb{Z}}\frac{(-1)^n\omega^{2n+1}q^{(2n+1)^2}}{1-\rho^2\omega^2q^{8n+4}}\Bigg|_{u=v=0}.$$

Proof. By rearranging terms, it is not difficult to see that the summation on the right (prior to taking derivatives) is equal to

$$\sum_{n\geq 0}\frac{(-1)^n\omega^{2n+1}q^{(2n+1)^2}}{1-\rho^2\omega^2q^{4(2n+1)}}-\rho^{-2}\omega^{-2}q^{4(2n+1)}\frac{(-1)^n\omega^{-(2n+1)}q^{(2n+1)^2}}{1-\rho^{-2}\omega^{-2}q^{4(2n+1)}}$$
$$=\sum_{n\geq 0}(-1)^n\omega^{2n+1}q^{(2n+1)^2}$$
$$+\sum_{n\geq 0}\sum_{m\geq 1}(-1)^n\left(\omega^{2n+1+2m}\rho^{2m}+\omega^{-(2n+1+2m)}\rho^{-2m}\right)q^{(2n+1)^2+4m(2n+1)}. \tag{6.35}$$

We then set $\alpha = n+m$, and $\beta = m$. After applying the derivatives, evaluating at $u = v = 0$, and factoring out the powers of 2, this becomes

$$\sum_{\beta=1}^{\infty}\sum_{\alpha=\beta}^{\infty}(-1)^{\alpha+\beta}\,(2\alpha+1)\,\beta^{2t+1}\,q^{4\alpha^2+4\alpha-4\beta^2+1}=K_{2t}(8\tau). \tag{6.36}$$

□

The summation in the right hand side of the equation in Proposition 6.6 is in the form of an Appell-Lerch function. In the next section, we show how to write this in terms of Zwegers's μ-function, from which we can infer its modularity properties.

6.3.2 Work of Zwegers

In his Ph.D. thesis on mock theta functions [311], Zwegers constructs weight 1/2 harmonic weak Maass forms by making use of the transformation properties of functions which were investigated earlier by Appell and Lerch. Here we briefly recall some of his results.

For τ in $\mathbb{H}$ and $u, v \in \mathbb{C}\setminus(\mathbb{Z}\tau+\mathbb{Z})$, Zwegers defines the function

$$\mu(u,v;\tau) := \frac{\rho^{1/2}}{\theta(v;\tau)}\cdot\sum_{n\in\mathbb{Z}}\frac{(-\omega)^n q^{n(n+1)/2}}{1-\rho q^n}, \tag{6.37}$$

where $\rho = e^{2\pi i u}$ and $\omega = e^{2\pi i v}$ as above, and

$$\theta(v;\tau) := \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} (-1)^{\nu-\frac{1}{2}}\omega^{\nu}q^{\nu^2/2}. \tag{6.38}$$

Zwegers's (see Section 1.3 of [311]) proves that $\mu(u,v,\tau)$ satisfies the following important properties.

Lemma 6.7. *Assuming the notation above, we have that*

(1) $$\mu(u,v;\tau) = \mu(v,u,\tau),$$

(2) $$\mu(u+1,v,\tau) = -\mu(u,v;\tau),$$

(3) $$\rho^{-1}\omega q^{-\frac{1}{2}}\mu(u+\tau,v;\tau) = -\mu(u,v;\tau)+\rho^{-\frac{1}{2}}\omega^{\frac{1}{2}}q^{-\frac{1}{8}},$$

(4) $$\mu(u,v;\tau+1) = \zeta_8^{-1}\mu(u,v;\tau) \quad (\zeta_N := e^{2\pi i/N})$$

(5) $$(\tau/i)^{-\frac{1}{2}}e^{\pi i(u-v)^2/\tau}\mu(\tfrac{u}{\tau},\tfrac{v}{\tau};-\tfrac{1}{\tau}) = -\mu(u,v;\tau)+\tfrac{1}{2}h(u-v;\tau),$$

where

$$h(z;\tau) := \int_{-\infty}^{\infty}\frac{e^{\pi i x^2\tau-2\pi x z}dx}{\cosh \pi x}$$

Remark. The integral $h(z,\tau)$ is known as a *Mordell integral.*

Lemma 6.7 shows that $\mu(u,v;\tau)$ is nearly a weight $1/2$ Jacobi form, where τ is the modular variable. Zwegers then uses μ to construct weight $1/2$ harmonic weak Maass forms. He achieves this by modifying μ to obtain a function $\widehat{\mu}$ which he then uses as a building block for such Maass forms. To make this precise, for $\tau\in\mathbb{H}$ and $u\in\mathbb{C}$, let

$$c := \mathrm{Im}(u)/\mathrm{Im}(\tau),$$

and let

$$R(u;\tau) := \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} (-1)^{\nu-\frac{1}{2}} \left\{\operatorname{sgn}(\nu) - E\left((v+c)\sqrt{2\operatorname{Im}(\tau)}\right)\right\} e^{-2\pi i\nu u} q^{-\nu^2/2}, \tag{6.39}$$

where $E(z)$ is the odd function

$$E(z) := 2\int_0^z e^{-\pi u^2} du.$$

Using μ and R, we let

$$\widehat{\mu}(u,v;\tau) := \mu(u,v;\tau) - \frac{1}{2}R(u-v;\tau). \tag{6.40}$$

Zwegers's construction of weight $1/2$ harmonic weak Maass forms depends on the following theorem (see Section 1.4 of [311]).

Theorem 6.8. *Assuming the notation above, we have that*

(1) $\widehat{\mu}(u,v;\tau) = \widehat{\mu}(v,u,\tau)$,

(2) $\widehat{\mu}(u+1,v,\tau) = \rho^{-1}\omega q^{-\frac{1}{2}}\mu(u+\tau,v;\tau) = -\widehat{\mu}(u,v;\tau)$,

(3) $\zeta_8^{-1}\widehat{\mu}(u,v;\tau+1) = -(\tau/i)^{-\frac{1}{2}}e^{\pi i(u-v)^2/\tau}\widehat{\mu}(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}) = \widehat{\mu}(u,v;\tau)$,

(4) $\widehat{\mu}\left(\frac{u}{c\tau+d},\frac{v}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right) = \chi(A)^{-3}(c\tau+d)^{\frac{1}{2}}e^{-\pi ic(u-v)^2/(c\tau+d)}\cdot\widehat{\mu}(u,v;\tau)$,

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *and* $\chi(A) := \eta(A\tau)/\left((c\tau+d)^{\frac{1}{2}}\eta(\tau)\right)$.

Theorem 6.8 gives the modular transformation properties for $\widehat{\mu}$. In the following section we will write K_{2t} in terms of μ, and we then use its properties to complete $K_0(\tau)$ as a nonholomorphic modular form on $\Gamma_0(8)$.

6.3.3 Modularity Properties of $K_0(\tau)$

We begin with the following proposition.

Proposition 6.9. *We have that*

$$K_{2t}(8\tau) = 2^{-2t-1}\ D_u^{2t+1} D_v \left(\rho^{-1}q^{-1}\mu(2u+2v+4\tau, 2v; 8\tau)\theta(2v; 8\tau)\right)\Big|_{u=v=0}.$$

Moreover, $\frac{K_0(8\tau)}{\eta^3(8\tau)}$ *is the holomorphic part of a weight* $3/2$ *weak Maass which is modular on* $\Gamma_0(8)$*, and whose non-holomorphic part is the period integral of* $\Theta_4(\tau)$*.*

Proof. The first statement follows directly from Proposition 6.6 and the definition of the μ function defined in (6.37).

To prove the remainder of the proposition, let

$$\widehat{K_0}(\tau) = 2^{-1}\ D_u D_v \left(\rho^{-1}q^{-1}\widehat{\mu}(2u+2v+4\tau, 2v; 8\tau)\theta(2v; 8\tau)\right)\Big|_{u=v=0}, \quad (6.41)$$

so that the holomorphic part of $\widehat{K_0}(\tau)$ is $K_0(8\tau)$. Suppose $A = \begin{pmatrix} a & b \\ 8c & d \end{pmatrix} \in \Gamma_0(8)$. Then we note $\overline{A} = \begin{pmatrix} a & 8b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Using the transformation laws for $\widehat{\mu}$ found in Lemma 6.8, we have

$$\begin{aligned} &e^{2\pi i \frac{-u-a\tau-b}{8c\tau+d}}\widehat{\mu}\left(\frac{2u+2v+4(a\tau+b)}{8c\tau+d}, \frac{2v}{8c\tau+b}; \frac{a8\tau+8b}{c8\tau+d}\right) \\ &\qquad =\chi(\overline{A})^{-3}(-1)^{\frac{a-1}{2}}(8c\tau+d)^{1/2}e^{(2\pi i)\frac{-1}{4}\frac{8cu^2}{8c\tau+d}}\cdot\widehat{\mu}(2u+2v+4\tau, 2v, 8\tau), \end{aligned} \quad (6.42)$$

which is obtained by substituting $u \to \frac{u}{8c\tau+d}$, $v \to \frac{v}{8c\tau+d}$, and $\tau \to \frac{a\tau+b}{8c\tau+d}$ into the expression on the right hand side of (6.41), before taking derivatives. With a little more algebra, we find that

$$\widehat{K_0}\left(\frac{a\tau+b}{8c\tau+d}\right) = \chi(\overline{A})^{-3}(-1)^{\frac{a-1}{2}}(8c\tau+d)^3\widehat{K_0}(\tau). \quad (6.43)$$

Therefore $\widehat{K_0}(\tau)$ is modular on $\Gamma_0(8)$ with weight 3/2. The non-holomorphic part of $\widehat{K_0}(\tau)$ is

$$\frac{-1}{4}\, D_u D_v R(2u+4\tau;8\tau)\theta(2v;8\tau)|_{u=v=0} = \frac{-1}{4}\eta^3(8\tau)\; D_u R(2u+4\tau;8\tau)|_{u=0}\,. \tag{6.44}$$

After factoring out $\eta^3(8\tau)$, a straightforward calculation gives us that

$$\frac{\partial}{\partial\bar{\tau}}\frac{-1}{4}\, D_u R(2u+4\tau;8\tau)|_{u=0} = \frac{-1}{32\pi y^{\frac{3}{2}}}\overline{\Theta_4(\tau)}. \tag{6.45}$$

□

6.3.4 The proof of Theorem 1.20

Thanks to Theorem 6.5, it suffices to prove that the differences between certain q-series have vanishing constant term. We shall derive these conclusions by using differential operators, using methods very similar to those found in Section 8.1 of [227]. For brevity, we describe the $n=0$ cases in detail, and then provide general remarks which are required to justify the remaining cases.

By (6.26) and Proposition 6.9, we have

$$8Q_{01}^{+}(8q) = -1 - 2q^4 + 4q^8 - 8q^{12} + 10q^{16} + \ldots, \tag{6.46}$$

and

$$8\frac{\widehat{K_0}(\tau)}{\eta^3(8\tau)} = 24q^4 + 80q^8 + 240q^{12} + 528q^{16}\ldots. \tag{6.47}$$

Comparing (6.22) (with 8τ substituted for τ) and (6.45), we see that both of these are the holomorphic parts of weight 3/2 harmonic weak Maass forms with equal non-holomorphic parts. Therefore, it follows that

$$8\frac{\widehat{K_0}(\tau)}{\eta^3(8\tau)} - 8Q_{01}(8\tau) = 1 + 26q^4 + 76q^8 + 248q^{12} + 518q^{16} + \ldots \tag{6.48}$$

is a modular form. A short calculation shows that

$$\frac{\widehat{K_0}(\tau)}{\eta^3(8\tau)} - Q_{01}(8\tau) = \frac{E^*(4\tau)}{8\Theta_4(\tau)} \tag{6.49}$$

where $E^*(\tau)$ is the weight 2 Eisenstein series

$$E^*(\tau) := -E_2(\tau) + 2E_2(2\tau) = 1 + 24\sum_{n=1}^{\infty}\sigma_{\text{odd}}(n)q^n, \tag{6.50}$$

and $\sigma_{\text{odd}}(n)$ denotes the sum of the positive odd divisors of n. Noting that

$$\eta^3(8\tau) = \Theta_2(\tau)\Theta_3(\tau)\Theta_4(\tau), \tag{6.51}$$

where Θ_1, Θ_2, and Θ_3 are defined in (6.4), we can rewrite this as

$$\frac{8\widehat{K_0}(\tau)}{\Theta_2(\tau)\Theta_3(\tau)} - 8\Theta_4(\tau)Q_{01}(8\tau) = E^*(4\tau). \tag{6.52}$$

For $n = 0$, Theorem 1.20 is equivalent to the claim, for every $m \geq 0$, that the constant term vanishes in the expression

$$\frac{\Theta_4(\tau)^8(16\Theta_2(\tau)^4 + \Theta_3(\tau)^4)^m E^*(4\tau)}{\Theta_2(\tau)^{2m+4}\Theta_3(\tau)^{2m+4}}. \tag{6.53}$$

In order to verify this claim, we will find if helpful to define

$$Z(q) := \frac{E^*(4\tau)}{\Theta_2(\tau)^2\Theta_3(\tau)^2}. \tag{6.54}$$

which has the derivative

$$q\frac{d}{dq}Z(q) = \frac{-2\ \Theta_4(\tau)^8}{\Theta_2(\tau)^2\Theta_3(\tau)^2}. \tag{6.55}$$

Here $Z(q)$ is the same as $\widehat{Z_0}(q)$ defined in Section 8.1 of [227]. We also note that

$$16\Theta_2(\tau)^4 + \Theta_3(\tau)^4 = 1 + 24q^4 + 24q^2 + \cdots = E^*(4\tau).$$

Using this notation, (6.53) becomes

$$\frac{-1}{2(m+2)}\; q\frac{d}{dq}Z(q)^{m+2},$$

which has a vanishing constant term.

In fact, for each $m, n \geq 0$, we find a similar phenomenon. For every non-negative k, define

$$\begin{aligned}\mathcal{G}_\ell(q) := \sum_{j=0}^{\ell}\binom{\ell}{j}\frac{(-12)^j E_2(8\tau)^{\ell-j}\Gamma(\frac{3}{2})}{\left(\Theta_2(\tau)\Theta_3(\tau)\right)^{2\ell+2} 8^j\Gamma\left(\frac{3}{2}+j\right)} \\ \times\left[\frac{(-4)^j 8K_{2j}(8\tau)}{\Theta_2(\tau)\Theta_3(\tau)} - 8\Theta_4(\tau)\left(q\frac{d}{dq}\right)^j Q_{01}^{+}(8\tau)\right].\end{aligned} \tag{6.56}$$

Using this notation, the criterion given in Theorem 6.5 is equivalent to the claim that the constant coefficient of

$$\left(q\frac{d}{dq}Z(q)\right)Z(q)^m\sum_{\ell=0}^{n}\binom{n}{\ell}\left(-Z(q)\right)^{n-\ell}\mathcal{G}_\ell(\tau) \tag{6.57}$$

is zero for each non-negative m and n. It suffices to show that $\mathcal{G}_\ell(q)$ is a polynomial in $Z(q)$. We define $M_0^*(\Gamma_0(8))$ to be the space of modular functions on $\Gamma_0(8)$ which are holomorphic away from infinity, and is a subspace of $\mathbb{C}((q^2))$. One can easily verify that $M_0^*(\Gamma_0(8))$ is precisely the set of polynomials in $Z(q)$. In order to show that $\mathcal{G}_\ell(\tau)$ is in $M_0^*(\Gamma_0(8))$, we first show that a similar function, $\mathcal{H}_\ell(q)$ is in $M_0^*(\Gamma_0(8))$. We define the function

$$\mathcal{H}_\ell(q) := \frac{\Theta_4(\tau)}{\left(\Theta_2(\tau)\Theta_3(\tau)\right)^{2\ell+2}}\sum_{j=0}^{\ell}\binom{\ell}{j}\frac{\Gamma(\frac{3}{2})(-12)^j E_2(8\tau)^{\ell-j}}{\Gamma\left(\frac{3}{2}+j\right)8^j}\left(q\frac{d}{dq}\right)^j\frac{E^*(8\tau)}{\Theta_4(\tau)}. \tag{6.58}$$

We can observe that $\mathcal{H}_\ell(q)$ is modular on $\Gamma_0(8)$ with weight 0 by comparing the summation to the expression $\mathcal{E}^\ell\left[\frac{E^*(8\tau)}{\Theta_4(\tau)}\right]$, where the bracket operator

$$\mathcal{E}^\ell[f] := \sum_{j=0}^{\ell}\binom{\ell}{j}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+j\right)}(-12)^j E_2^{\ell-j}(\tau)\left(q\frac{d}{dq}\right)^j f(\tau)$$

is defined as in equation (9.18) of [241] (See also [77]). This is the bracket operator used in Lemma 6.3 and, as noted, preserves modularity, but changes the weight from $\frac{3}{2}$ to $\frac{3}{2}+2\ell$. A calculation shows that $(\Theta_2(\tau)\Theta_3(\tau))^{-2}$ and $\Theta_4(\tau)^{-1}$ are holomorphic away from infinity, which, combined with the fact that $\Theta_2(\tau)\Theta_3(\tau) \in \mathbb{Z}[[q^2]]$, shows that $\mathcal{H}_\ell(\tau)$ is in $M_0^*(\Gamma_0(8))$. Hence it suffices to show that $\mathcal{G}_\ell(q) - \mathcal{H}_\ell(q)$ is in $M_0^*(\Gamma_0(8))$ as well. From (6.49), we see that

$$\mathcal{G}_\ell(q) - \mathcal{H}_\ell(q) = \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{\Gamma(\frac{3}{2})(-12)^j E_2(8\tau)^{\ell-j}\Theta_4(\tau)}{(\Theta_2(\tau)\Theta_3(\tau))^{2\ell+2}\,\Gamma\left(\frac{3}{2}+j\right)8^j} \left[\frac{(-4)^j 8K_{2j}(8\tau)}{\eta^3(8\tau)} - \left(q\frac{d}{dq}\right)^j \frac{8K_0(8\tau)}{\eta^3(8\tau)}\right]. \tag{6.59}$$

Using Theorem 6.9, this can be written as

$$\begin{aligned}\mathcal{G}_\ell(q)-\mathcal{H}_\ell(q) &= \frac{\Theta_4(\tau)}{(\Theta_2(\tau)\Theta_3(\tau))^{2\ell+2}} \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{\Gamma(\frac{3}{2})(-12)^j E_2(8\tau)^{\ell-j}}{\Gamma\left(\frac{3}{2}+j\right)8^{j-1}} \\ &\quad \times \left[(-1)^j D_u^{2j+1} D_v - D_\tau^j D_u D_v\right] \left.\frac{\rho^{-1}q^{-1}\mu(2u+2v+4\tau, 2v; 8\tau)\theta(2v;4\tau)}{2\eta^3(8\tau)}\right|_{u=v=0}.\end{aligned} \tag{6.60}$$

Paying particular attention to the derivatives of the μ-function above, we use the transformation laws for μ found in Lemma 6.7, and observe that the Mordel integrals that arise as obstructions to the modular transformation of (6.60) cancel directly. Therefore an argument similar to the proof that the bracket operator preserves modularity suffices to show that $\mathcal{G}_\ell(q) - \mathcal{H}_\ell(q)$ is modular with respect to $\Gamma_0(8)$. Some simple accounting shows that $\mathcal{G}_\ell(q) - \mathcal{H}_\ell(q)$ is supported on even exponents of q, and hence $\mathcal{G}_\ell(q) - \mathcal{H}_\ell(q)$ is in $M_0^*(\Gamma_0(8))$. This completes the proof.

6.3.5 Examples

In the table below, we give the polynomial $P_n(x)$ such that the expression in the statement of Theorem 6.5 can be written as

$$\left(q\frac{d}{dq}Z(q^{1/8})\right)\left(\frac{Z(q^{1/8})}{2}\right)^m P_n(Z(q^{1/8})). \tag{6.61}$$

n	$P_n(x)$
0	$\frac{1}{32}x$
1	$-1/2$
2	$\frac{13}{16}x$
3	$-\frac{11}{16}x^2 - 87$
4	$\frac{13}{16}x^3 + \frac{4175}{8}x$
5	$-\frac{11}{16}x^4 - \frac{9607}{4}x^2 - 80662$
6	$\frac{13}{16}x^5 + \frac{80153}{8}x^3 + \frac{5958039}{4}$

Chapter 7

Moonshine

We briefly return to a discussion of the history of classical moonshine, building up to the moonshine towers and the proofs of Theorem 7.10 and Corollary 7.11 which generalize Theorem 1.27 and Corollary 1.28 form the introduction.

7.1 Vertex operators and the proof of classical moonshine

In order to prove Thompson's conjecture, Frenkel–Lepowsky–Meurman generalized the homogeneous realization of the basic representation of an affine Lie algebra $\hat{\mathfrak{g}}$ due, independently, to Frenkel–Kac [124] and Segal [267], in such a way that *Leech's lattice* Λ [201, 202]—the unique [78] even self-dual positive-definite lattice of rank 24 with no roots—could take on the role played by the root lattice of $\mathfrak{g}$ in the Lie algebra case. In particular, their construction came equipped with rich algebraic structure, furnished by vertex operators, which had appeared first in the physics literature in the late 1960's.

We refer to [124], and also the introduction to [126] for accounts of the role played by vertex operators in physics (up to 1988) along with a detailed description of their application to the representation theory of affine

Lie algebras. The first application of vertex operators to affine Lie algebra representations was obtained by Lepowsky–Wilson in [210].

Borcherds described a powerful axiomatic formalism for vertex operators in [29]. In particular, he introduced the notion of a *vertex algebra*, which can be regarded as similar to a commutative associative algebra, except that multiplications depend upon formal variables z_i, and can be singular, in a certain formal sense, along the canonical divisors $\{z_i = 0\}$, $\{z_i = z_j\}$ (cf. [32, 123]).

The appearance of affine Lie algebras above, as a conceptual ingredient for the Frenkel–Lepowsky–Meurman construction of $V^\natural$ hints at an analogy between complex Lie groups and the monster. Borcherds' vertex algebra theory makes this concrete, for Borcherds showed [29] that both in the case of the basic representation of an affine Lie algebra, and in the case of the moonshine module $V^\natural$, the vertex operators defined by Frenkel–Kac, Segal, and Frenkel–Lepowsky–Meurmann, extend naturally to vertex algebra structures.

In all of these examples the *Virasoro algebra*, $\mathcal{V} = \bigoplus_n \mathbb{C}L(n) \oplus \mathbb{C}\mathbf{c}$, being the unique universal central extension of the Lie algebra $\mathbb{C}[t, t^{-1}]\frac{\mathrm{d}}{\mathrm{d}t}$ of polynomial vector fields on the circle,

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}\mathbf{c}, \quad [L(m), \mathbf{c}] = 0, \quad (7.1)$$

acts naturally on the underlying vector space. (See [181] for a detailed analysis of $\mathcal{V}$. The generator $L(m)$ lies above the vector field $-t^{m+1}\frac{\mathrm{d}}{\mathrm{d}t}$.) This Virasoro structure, which has powerful applications, was axiomatized in [126], with the introduction of the notion of a *vertex operator algebra*. If V is a vertex operator algebra and the central element $\mathbf{c}$ of the Virasoro algebra acts as c times the identity on V, for some $c \in \mathbb{C}$, then V is said to have *central charge* c.

For the basic representation of an affine Lie algebra $\hat{\mathfrak{g}}$, the group of vertex operator algebra automorphisms—i.e. those vertex algebra automorphisms that commute with the Virasoro action—is the adjoint complex Lie group associated to $\mathfrak{g}$. For the moonshine module $V^\natural$, it was shown by Frenkel–Lepowsky–Meurman in [126], that the group of vertex operator algebra automorphisms is precisely the monster.

Theorem 7.1 (Frenkel–Lepowsky–Meurman). *The moonshine module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ is a vertex operator algebra of central charge* 24 *whose graded dimension is given by $J(\tau)$, and whose automorphism group is $\mathbb{M}$.*

Vertex operator algebras are of relevance to physics, for we now recognize them as "chiral halves" of two-dimensional conformal field theories (cf. [130, 131]). From this point of view, the construction of $V^\natural$ by Frenkel–Lepowsky–Meurman constitutes one of the first examples of an orbifold conformal field theory (cf. [94–96]). In the case of $V^\natural$, the underlying geometric orbifold is the quotient

$$\left(\mathbb{R}^{24}/\Lambda\right)/(\mathbb{Z}/2\mathbb{Z}), \tag{7.2}$$

of the 24-dimensional torus $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}/\Lambda \simeq \mathbb{R}^{24}/\Lambda$ by the Kummer involution $x \mapsto -x$, where Λ denotes the Leech lattice. So in a certain sense, $V^\natural$ furnishes a "24-dimensional" construction of $\mathbb{M}$. We refer to [123,126,182,208] for excellent introductions to vertex algebra, and vertex operator algebra theory.

Affine Lie algebras are special cases of Kac–Moody algebras, first considered by Kac [177] and Moody [239, 240], independently. Roughly speaking, a Kac–Moody algebra is "built" from copies of $\mathfrak{sl}_2$, in such a way that most examples are infinite-dimensional, but much of the finite-dimensional theory carries through (cf. [183]). Borcherds generalized this further, allowing also copies of the three-dimensional Heisenberg Lie algebra to serve as building blocks, and thus arrived [28] at the notion of *generalized Kac–Moody alge-*

bra, or *Borcherds–Kac–Moody (BKM) algebra*, which has subsequently found many applications in mathematics and mathematical physics (cf. [164, 260]).

One of the most powerful such applications occurred in moonshine, when Borcherds introduced a particular example—the *monster Lie algebra* $\mathfrak{m}$—and used it to prove [30] the moonshine conjectures of Conway–Norton. His method entailed using monster-equivariant versions of the denominator identity for $\mathfrak{m}$ to verify that the coefficients of the McKay–Thompson series T_g, defined by (1.37) according to the Frenkel–Lepowsky–Meurman construction of $V^\natural$, satisfy the replication formulas conjectured by Conway–Norton in [81]. This powerful result reduced the proof of the moonshine conjectures to a small, finite number of identities, that he could easily check by hand.

Theorem 7.2 (Borcherds). *Let $V^\natural$ be the moonshine module vertex operator algebra constructed by Frenkel–Lepowsky–Meurman, whose automorphism group is $\mathbb{M}$. If T_g is defined by (1.37) for $g \in \mathbb{M}$, and if Γ_g is the discrete subgroup of $SL_2(\mathbb{R})$ specified by Conway–Norton in [81], then T_g is the unique normalized principal modulus for Γ_g.*

Recall that an even self-dual lattice of signature (m, n) exists if and only if $m - n = 0 \pmod 8$ (cf. e.g. [83]). Such a lattice is unique up to isomorphism if $mn > 0$, and is typically denoted $II_{m,n}$. In the case that $m = n = 1$ we may take

$$II_{1,1} := \mathbb{Z}e + \mathbb{Z}f, \tag{7.3}$$

where e and f are isotropic, $\langle e, e\rangle = \langle f, f\rangle = 0$, and $\langle e, f\rangle = 1$. Then $me + nf \in II_{1,1}$ has square-length $2mn$. Note that $II_{25,1}$ and $\Lambda \oplus II_{1,1}$ are isomorphic, for Λ the Leech lattice, since both lattices are even and self-dual, with signature $(25, 1)$.

In physical terms the monster Lie algebra $\mathfrak{m}$ is (“about half” of) the space of "physical states" of a bosonic string moving in the quotient

$$\left(\mathbb{R}^{24}/\Lambda \oplus \mathbb{R}^{1,1}/II_{1,1}\right)/(\mathbb{Z}/2\mathbb{Z}) \tag{7.4}$$

of the 26-dimensional torus $II_{25,1} \otimes_{\mathbb{Z}} \mathbb{R}/II_{25,1} \simeq \mathbb{R}^{24}/\Lambda \oplus \mathbb{R}^{1,1}/II_{1,1}$ by the Kummer involution $x \mapsto -x$. The monster Lie algebra $\mathfrak{m}$ is constructed in a functorial way from $V^\natural$ (cf. [66]), inherits an action by the monster from $V^\natural$, and admits a monster-invariant grading by $II_{1,1}$.

The denominator identity for a Kac–Moody algebra $\mathfrak{g}$ equates a product indexed by the positive roots of $\mathfrak{g}$ with a sum indexed by the Weyl group of $\mathfrak{g}$. A BKM algebra also admits a denominator identity, which for the case of the monster Lie algebra $\mathfrak{m}$ is the beautiful *Koike–Norton–Zagier formula*

$$p^{-1} \prod_{\substack{m,n \in \mathbb{Z} \\ m>0}} (1 - p^m q^n)^{c(mn)} = J(\sigma) - J(\tau), \tag{7.5}$$

where $\sigma \in \mathbb{H}$ and $p = e^{2\pi i \sigma}$ (and $c(n)$ is the coefficient of q^n in $J(\tau)$, cf. (1.33)). Since the right hand side of (7.5) implies that the left hand side has no terms $p^m q^n$ with $mn \neq 0$, this identity imposes many non-trivial polynomial relations upon the coefficients of $J(\tau)$. Among these is

$$c(4n+2) = c(2n+2) + \sum_{k=1}^{n} c(k)c(2n-k+1), \tag{7.6}$$

which was first found by Mahler [221] by a different method, along with similar expressions for $c(4n)$, $c(4n+1)$, and $c(4n+3)$, which are also entailed in (7.5). Taken together these relations allow us to compute the coefficients of $J(\tau)$ recursively, given just the values

$$\begin{aligned} c(1) &= 196884, \\ c(2) &= 21493760, \\ c(3) &= 864299970, \\ c(5) &= 333202640600. \end{aligned} \tag{7.7}$$

To recover the replication formulas of [81, 245] we require to replace J with T_g, and $c(n)=\dim(V^\natural_n)$ with $\mathrm{tr}(g|V^\natural_n)$ in (7.5), and for this we require a categorification of the denominator identity, whereby the positive integers $c(mn)$ are replaced with $\mathbb{M}$-modules of dimension $c(mn)$.

A categorification of the denominator formula for a finite-dimensional simple complex Lie algebra was obtained by Kostant [196], following an observation of Bott [34]. This was generalized to Kac–Moody algebras by Garland–Lepowsky [137], and generalized further to BKM algebras by Borcherds in [30]. In its most compact form, it is the identity of virtual vector spaces

$$\bigwedge\nolimits_{-1}(\mathfrak{e}) = H(\mathfrak{e}), \tag{7.8}$$

where $\mathfrak{e}$ is the sub Lie algebra of a BKM algebra corresponding to its positive roots (cf. [174, 175, 183]).

In (7.8) we understand $\bigwedge_{-1}(\mathfrak{e})$ to be the specialization of the formal series

$$\bigwedge\nolimits_t(\mathfrak{e}) := \sum_{k\geq 0} \wedge^k(\mathfrak{e})t^k \tag{7.9}$$

to $t=-1$, where $\wedge^k(\mathfrak{e})$ is the k-th exterior power of $\mathfrak{e}$, and we write

$$H(\mathfrak{e}) := \sum_{k\geq 0}(-1)^k H_k(\mathfrak{e}) \tag{7.10}$$

for the alternating sum of the Lie algebra homology groups of $\mathfrak{e}$.

In the case of the monster Lie algebra $\mathfrak{m}$, the spaces $\wedge^k(\mathfrak{e})$ and $H_k(\mathfrak{e})$ are graded by $II_{1,1}$, and acted on naturally by the monster. If we use the variables p and q to keep track of the $II_{1,1}$-gradings, then the equality of (7.8) holds in the ring $R(\mathbb{M})[[p,q]][q^{-1}]$ of formal power series in p and q (allowing finitely many negative powers of q), with coefficients in the (integral) representation ring of $\mathbb{M}$. More precisely, (7.8) becomes

$$\bigwedge\nolimits_{-1}\left(\sum_{\substack{m,n\in\mathbb{Z}\\ m>0}} V^\natural_{mn}p^m q^n\right) = \sum_{m\in\mathbb{Z}} V^\natural_m p^{m+1} - \sum_{n\in\mathbb{Z}} V^\natural_n p q^n, \tag{7.11}$$

which returns (7.5), once we replace $V_k^\natural$ everywhere with $\dim(V_k^\natural) = c(k)$, and divide both sides by p. More generally, replacing $V_k^\natural$ with $\mathrm{tr}(g|V_k^\natural)$ for $g \in \mathbb{M}$, the identity (7.11) implies

$$p^{-1} \exp\left(-\sum_{k>0} \sum_{\substack{m,n\in\mathbb{Z} \\ m>0}} \frac{1}{k} \,\mathrm{tr}(g^k|V_{mn}^\natural) p^{mk} q^{nk}\right) = T_g(\sigma) - T_g(\tau) \tag{7.12}$$

(cf. [30], and also [173]), which, in turn, implies the replication formulas formulated in [81, 245]. Taking $g = e$ in (7.12) we recover (7.5), so (7.12) furnishes a natural, monster-indexed family of analogues of the identity (7.5).

7.2 Modularity

Despite the power of the BKM algebra theory developed by Borcherds, and despite some conceptual improvements (cf. [87, 175, 176]) upon Borcherds' original proof of the moonshine conjectures, a conceptual explanation for the principal modulus property of monstrous moonshine is yet to be established. Indeed, there are generalizations and analogs of the notion of replicability which hold for generic modular functions and forms (for example, see [49]), not just those modular functions which are principal moduli.

Zhu explained the modularity of the graded dimension $\sum_n \dim(V_n^\natural)q^n$ of $V^\natural$ in [310], by proving that this is typical for vertex operator algebras satisfying quite general hypotheses, and Dong–Li–Mason extended Zhu's work in [100], obtaining modular invariance results for graded trace functions arising from the action of a finite group of automorphisms.

To prepare for a statement of the results of Zhu and Dong–Li–Mason, we mention that the module theory for vertex operators algebras includes so-called *twisted modules*, associated to finite order automorphisms. If g is a finite order automorphism of V, then V is called *g-rational* in case every

g-twisted V-module is a direct sum of simple g-twisted V-modules. Dong–Li–Mason proved [99] that a g-rational vertex operator algebra has finitely many simple g-twisted modules up to isomorphism. So in particular, a rational vertex operator algebra has finitely many simple (untwisted) modules.

Theorem 7.3 (Zhu, Dong–Li–Mason). *Let V be rational C_2-cofinite vertex operator algebra. Then the generating functions $\sum_n \dim(M^i_n)q^n$, of the graded dimensions of its simple modules $M^i = \bigoplus_n M^i_n$, span a finite-dimensional representation of $SL_2(\mathbb{Z})$. More generally, if G is a finite subgroup of $\mathrm{Aut}(V)$ and V is g-rational for every $g \in G$, then the graded trace functions $\sum_n \mathrm{tr}(\tilde{h}|M_n)q^n$, attached to the triples $(g, \tilde{h}, M)$, where $g, h \in G$ commute, M is a simple h-stable g-twisted module for V, and $\tilde{h}$ is a lift of h to $GL(M)$, span a finite-dimensional representation of $SL_2(\mathbb{Z})$.*

We refer to the Introduction of [100] (see also §2 of [101]) for a discussion of h-stable twisted modules, and the relevant notion of *lift*. Note that any two lifts for h differ only up to multiplication by a non-zero scalar, so $\sum_n \mathrm{tr}(\tilde{h}|M_n)q^n$ is uniquely defined by (g, h, M), up to a non-zero scalar.

In the case of $V^\natural$, there is a unique simple g-twisted module $V^\natural_g = \bigoplus_n (V^\natural_g)_n$ for every $g \in \mathbb{M} = \mathrm{Aut}(V^\natural)$ (cf. Theorem 1.2 of [99]), and $V^\natural_g$ is necessarily h-stable for any $h \in \mathbb{M}$ that commutes with g. Therefore, Theorem 7.3 suggests that the functions

$$T_{(g,\tilde{h})}(\tau) := \sum_n \mathrm{tr}(\tilde{h}|(V^\natural_g)_n)q^n, \tag{7.13}$$

associated to pairs (g, h) of commuting elements of $\mathbb{M}$, may be of interest.

Indeed, this was anticipated a decade earlier by Norton, following observations of Conway–Norton [81] and Queen [252], which associated principal moduli to elements of groups that appear as centralizers of cyclic subgroups in the monster. Norton formulated his *generalized moonshine* conjectures in [246] (cf. also [247], and the Appendix to [233]).

Conjecture 7.4 (Generalized Moonshine: Norton). There is an assignment of holomorphic functions $T_{(g,\tilde{h})} : \mathbb{H} \to \mathrm{C}$ to every pair (g, h) of commuting elements in the monster, such that the following are true:

1. For every $x \in \mathbb{M}$ we have $T_{(x^{-1}gx, x^{-1}\tilde{h}x)} = T_{(g,\tilde{h})}$.
2. For every $\gamma \in SL_2(\mathbb{Z})$ we have that $T_{(g,\tilde{h})\gamma}(\tau)$ is a scalar multiple of $T_{(g,\tilde{h})}(\gamma\tau)$.
3. The coefficient functions $\tilde{h} \mapsto \mathrm{tr}(\tilde{h}|(V^\natural_g)_n)$, for fixed g and n, define characters of a projective representation of the centralizer of g in $\mathbb{M}$,
4. We have that $T_{(g,\tilde{h})}$ is either constant or a generator for the function field of a genus zero group $\Gamma_{(g,h)} < SL_2(\mathbb{R})$.
5. We have that $T_{(g,\tilde{h})}$ is a scalar multiple of J if and only if $g = h = e$.

Remark. In Conjecture 7.4 (2) above, the right-action of $SL_2(\mathbb{Z})$ on commuting pairs of elements of the monster is given by

$$(g, h)\gamma := (g^a h^c, g^b h^d) \tag{7.14}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The (slightly ambiguous) $T_{(g,\tilde{h})\gamma}$ denotes the graded trace of a lift of $g^b h^d$ to $GL(V^\natural_{g^a h^c})$. Norton's generalized moonshine conjectures reduce to the original Conway–Norton moonshine conjectures of [81] when $g = e$.

Conjecture 7.4 is yet to be proven in full, but has been established for a number of special cases. Theorem 7.3 was used by Dong–Li–Mason in [100], following an observation of Tuite (cf. [98], and [286–288] for broader context), to prove Norton's conjecture for the case that g and h generate a cyclic subgroup of $\mathbb{M}$, and this approach, via twisted modules for $V^\natural$, has been extended by Ivanov–Tuite in [169, 170]. Höhn obtained a generalization of Borcherds' method by using a particular twisted module for $V^\natural$ to construct a BKM algebra adapted to the case that g is in the class named $2A$ in [80]—the smaller of the two conjugacy class of involutions in $\mathbb{M}$—and in so doing established [162] generalized moonshine for the functions $T_{(g,\tilde{h})}$ with $g \in 2A$. So far the most general results in generalized moonshine have been obtained by Carnahan [64–66]. (See [67] for a recent summary.)

Theorem 7.3 explains why the McKay–Thompson series $T_g(\tau)$ of (1.37), and the $T_{(g,\tilde{h})}(\tau)$ of (7.13) more generally, should be invariant under the actions of (finite index) subgroups of $SL_2(\mathbb{Z})$, but it does not explain the surprising predictive power of monstrous moonshine. That is, it does not explain why the full invariance groups Γ_g of the T_g should be so large that they admit normalized principal moduli, nor does it explain why the T_g should actually be these normalized principal moduli.

A program to establish a conceptual foundation for the principal modulus property of monstrous moonshine, via *Rademacher sums* and *three-dimensional gravity*, was initiated in [110] by Duncan and Frenkel.

7.3 Rademacher Sums

To explain the conjectural connection between gravity and moonshine, we first recall some history. The roots of the approach of [110] extend back almost a hundred years, to Einstein's theory of general relativity, formulated in 1915, and the introduction of the circle method in analytic number theory, by Hardy–Ramanujan [158]. At the same time that pre-war efforts to quantize Einstein's theory of gravity were gaining steam (see [276] for a review), the circle method was being refined and developed, by Hardy–Littlewood (cf. [157]), and Rademacher [254], among others. (See [290] for a detailed account of what is now known as the *Hardy–Littlewood circle method*.) Despite being contemporaneous, these works were unrelated in science until this century: as we will explain presently, Rademacher's analysis led to a

Poincaré series-like expression—the prototypical Rademacher sum—for the elliptic modular invariant $J(\tau)$. It was suggested first in [253] (see also [231]) that this kind of expression might be useful for the computation of partition functions in quantum gravity.

Rademacher "perfected" the circle method introduced by Hardy–Ramanujan, and he obtained an exact convergent series expression for the combinatorial partition function $p(n)$. In 1938 he generalized this work [255] and obtained such exact formulas for the Fourier coefficients of general modular functions. For the elliptic modular invariant $J(\tau) = \sum_n c(n)q^n$ (cf. (1.33)), Rademacher's formula (which was obtained earlier by Petersson [251], via a different method) may be written as

$$c(n) = 4\pi^2 \sum_{c>0} \sum_{\substack{0<a<c \\ (a,c)=1}} \frac{e^{-2\pi i \frac{a}{c}} e^{2\pi i n \frac{d}{c}}}{c^2} \sum_{k\geq 0} \frac{(4\pi^2)^k}{c^{2k}} \frac{1}{(k+1)!} \frac{n^k}{k!}, \tag{7.15}$$

where d, in each summand, is a multiplicative inverse for a modulo c, and (a,c) is the greatest common divisor of a and c. Having established the formula (7.15), Rademacher sought to reverse the process, and use it to derive the modular invariance of $J(\tau)$. That is, he set out to prove directly that $J_0(\tau+1) = J_0(-1/\tau) = J_0(\tau)$, when $J_0(\tau)$ is defined by setting $J_0(\tau) = q^{-1} + \sum_{n>0} c(n)q^n$, with $c(n)$ defined by (7.15).

Rademacher achieved this goal in [256], by reorganizing the summation

$$\sum_{n>0} c(n)q^n = 4\pi^2 \sum_{n>0} \sum_{c>0} \sum_{\substack{0<a<c \\ (a,c)=1}} \frac{e^{-2\pi i \frac{a}{c}} e^{2\pi i n(\tau + \frac{d}{c})}}{c^2} \sum_{k\geq 0} \frac{(4\pi^2)^k}{c^{2k}} \frac{1}{(k+1)!} \frac{n^k}{k!} \tag{7.16}$$

into a Poincaré series-like expression for J. More precisely, Rademacher proved that

$$J(\tau) + 12 = e^{-2\pi i \tau} + \lim_{K\to\infty} \sum_{\substack{0<c<K \\ -K^2<d<K^2 \\ (c,d)=1}} e^{-2\pi i \frac{a\tau+b}{c\tau+d}} - e^{-2\pi i \frac{a}{c}}, \tag{7.17}$$

where $a, b \in \mathbb{Z}$ are chosen arbitrarily, in each summand, so that $ad - bc = 1$. We call the right hand side of (7.17) the first *Rademacher sum.*

Rademacher's expression (7.17) for the elliptic modular invariant J is to be compared to the formal sum

$$\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} e^{-2\pi i m\frac{a\tau+b}{c\tau+d}}, \tag{7.18}$$

for m a positive integer, which we may regard as a (formal) Poincaré series of weight zero for $SL_2(\mathbb{Z})$. In particular, (7.18) is (formally) invariant for the action of $SL_2(\mathbb{Z})$, as we see by recognizing the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ as representatives for the right coset space $\Gamma_\infty\backslash\Gamma$, where $\Gamma = SL_2(\mathbb{Z})$ and Γ_∞ is defined in (1.38): for a fixed bottom row (c, d) of matrices in $SL_2(\mathbb{Z})$, any two choices for the top row (a, b) are related by left-multiplication by some element of Γ_∞.

The formal sum (7.18) does not converge for any $\tau \in \mathbb{H}$, so a regularization procedure is required. Rademacher's sum (7.17) achieves this, for $m = 1$, by constraining the order of summation, and subtracting the limit as $\Im(\tau) \to \infty$ of each summand $e^{-2\pi i\frac{a\tau+b}{c\tau+d}}$, whenever this limit makes sense. Rademacher's method has by now been generalized in various ways by a number of authors. The earliest generalizations are due to Knopp [189–192], and a very general negative weight version of the Rademacher construction was given by Niebur in [243]. We refer to [71] for a detailed review and further references. A nice account of the original approach of Rademacher appears in [188].

We note here that one of the main difficulties in establishing formulas like (7.17) is the demonstration of convergence. When the weight w of the Rademacher sum under consideration lies in the range $0 \le w \le 2$, then one requires non-trivial estimates on *sums of Kloosterman sums*, like

$$\sum_{c>0} \sum_{\substack{0<a<c\\(a,c)=1}} \frac{e^{-2\pi i m\frac{a}{c}} e^{2\pi i n\frac{d}{c}}}{c^2} \tag{7.19}$$

(for the case that $w = 0$ or $w = 2$). The demonstration of convergence generally becomes more delicate as w approaches 1.

In [110] the convergence of a weight zero Rademacher sum $R_\Gamma^{(-m)}(\tau)$ is shown, for m a positive integer and Γ an arbitrary subgroup of $SL_2(\mathbb{R})$ that is commensurable with $SL_2(\mathbb{Z})$. Assuming that Γ contains $-I$ and has width one at infinity (cf. (1.38)), we have

$$R_\Gamma^{(-m)}(\tau) = e^{-2\pi i m\tau} + \lim_{K\to\infty} \sum_{(\Gamma_\infty\backslash\Gamma)^\times_{<K}} e^{-2\pi i m\frac{a\tau+b}{c\tau+d}} - e^{-2\pi i m\frac{a}{c}}, \tag{7.20}$$

where the summation, for fixed K, is over non-trivial right cosets of Γ_∞ in Γ (cf. (1.38)), having representatives $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ such that $0 < c < K$ and $|d| < K^2$.

The modular properties of the $R_\Gamma^{(-m)}$ are also considered in [110], and it is at this point that the significance of Rademacher sums in monstrous moonshine appears. To state the relevant result we give the natural generalization (cf. §3.2 of [110]) of the Rademacher–Petersson formula (7.15) for $c(n)$, which is

$$c_\Gamma(-m,n) = 4\pi^2 \lim_{K\to\infty} \sum_{(\Gamma_\infty\backslash\Gamma/\Gamma_\infty)^\times_{<K}} \frac{e^{-2\pi i m\frac{a}{c}} e^{2\pi i n\frac{d}{c}}}{c^2} \sum_{k\geq 0} \frac{(4\pi^2)^k}{c^{2k}} \frac{m^{k+1}}{(k+1)!} \frac{n^k}{k!}, \tag{7.21}$$

where the summation, for fixed K, is over non-trivial double cosets of Γ_∞ in Γ (cf. (1.38)), having representatives $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ such that $0 < c < K$. Note that this formula simplifies for $n = 0$, to

$$c_\Gamma(-m,0) = 4\pi^2 m \lim_{K\to\infty} \sum_{(\Gamma_\infty\backslash\Gamma/\Gamma_\infty)^\times_{<K}} \frac{e^{-2\pi i m\frac{a}{c}}}{c^2}. \tag{7.22}$$

The value $c_\Gamma(-1,0)$ is the *Rademacher constant* attached to Γ. (Cf. §6 of [245] and §5.1 of [110].)

A *normalized Rademacher sum* $T_\Gamma^{(-m)}(\tau)$ is defined in §4.1 of [110] by introducing an extra complex variable and taking a limit. It is shown in §4.4

of [110] that

$$T_\Gamma^{(-m)}(\tau) = R_\Gamma^{(-m)}(\tau) - \frac{1}{2}c_\Gamma(-m,0) \tag{7.23}$$

for any group $\Gamma < SL_2(\mathbb{R})$ that is commensurable with $SL_2(\mathbb{Z})$. If Γ has width one at infinity (cf. (1.38)), then also

$$T_\Gamma^{(-m)}(\tau) = q^{-m} + \sum_{n>0} c_\Gamma(-m,n)q^n, \tag{7.24}$$

so in particular, $T_\Gamma^{(-m)}(\tau) = q^{-m} + O(q)$ as $\Im(\tau) \to \infty$. The following theorem by Duncan and Frenkel summarizes the central role of Rademacher sums and the principal modulus property.

Theorem 7.5 (Duncan–Frenkel [110]). *Let Γ be a subgroup of $SL_2(\mathbb{R})$ that is commensurable with $SL_2(\mathbb{Z})$. Then the normalized Rademacher sum $T_\Gamma^{(-m)}$ is Γ-invariant if and only if Γ has genus zero. Furthermore, if Γ has genus zero then $T_\Gamma^{(-1)}$ is the normalized principal modulus for Γ.*

In the case that the normalized Rademacher sum $T_\Gamma^{(-1)}$ is not Γ-invariant, $T_\Gamma^{(-m)}$ is an *abelian integral of the second kind* for Γ, in the sense that it has at most exponential growth at the cusps of Γ, and satisfies $T_\Gamma^{(-m)}(\gamma\tau) = T_\Gamma^{(-m)}(\tau) + \omega(\gamma)$ for $\gamma \in \Gamma$, for some function $\omega : \Gamma \to \mathrm{C}$ (depending on m).

Theorem 7.5 is used as a basis for the formulation of a characterization of the discrete groups Γ_g of monstrous moonshine in terms of Rademacher sums in §6.5 of [110], following earlier work [84] of Conway–McKay–Sebbar. It also facilitates a proof of the following result, which constitutes a uniform construction of the McKay–Thompson series of monstrous moonshine.

Theorem 7.6 (Duncan–Frenkel [110]). *Let $g \in \mathbb{M}$. Then the McKay–Thompson series T_g coincides with the normalized Rademacher sum $T_{\Gamma_g}^{(-1)}$.*

Proof. Theorem 7.2 states that T_g is a normalized principal modulus for Γ_g, and in particular, all the Γ_g have genus zero. Given this, it follows from

Theorem 7.5 that $T_{\Gamma_g}^{(-1)}$ is also a normalized principal modulus for Γ_g. A normalized principal modulus is unique if it exists, so we conclude $T_g = T_{\Gamma_g}^{(-1)}$ for all $g \in \mathbb{M}$, as we required to show. $\square$

Perhaps most importantly, Theorem 7.5 is an indication of how the principal modulus property of monstrous moonshine can be explained conceptually. For if we can develop a mathematical theory in which the underlying objects are graded with graded traces that are provably

1. modular invariant, for subgroups of $SL_2(\mathbb{R})$ that are commensurable with $SL_2(\mathbb{Z})$, and

2. given explicitly by Rademacher sums, such as (7.20),

then these graded trace functions are necessarily normalized principal moduli, according to Theorem 7.5.

We are now led to ask: what kind of mathematical theory can support such results? As we have alluded to above, Rademacher sums have been related to quantum gravity by articles in the physics literature. Also, a possible connection between the monster and three-dimensional quantum gravity was discussed in [304]. This suggests the possibility that three-dimensional quantum gravity and moonshine are related via Rademacher sums, and was a strong motivation for the work [110]. In the next section we will give a brief review of quantum gravity, since it is an important area of physical inquiry which has played a role in the development of moonshine, but we must first warn the reader: problems have been identified with the existing conjectures that relate the monster to gravity, and the current status of this connection is uncertain.

7.4 Quantum Gravity

A positive solution to the conjecture that $V^\natural$ is dual to chiral three-dimensional gravity at $m = 1$ may furnish a conceptual explanation for why the graded dimension of $V^\natural$ is the normalized principal modulus for $SL_2(\mathbb{Z})$. For on the one hand, modular invariance is a consistency requirement of the physical theory—the genus one partition function function is really defined on the moduli space $SL_2(\mathbb{Z})\backslash\mathbb{H}$ of genus one curves, rather than on $\mathbb{H}$—and on the other hand, the genus one partition function of chiral three-dimensional gravity is given by a Rademacher sum, as explained by Manschot–Moore [231], following earlier work [253] by Dijkgraaf–Maldacena–Moore–Verlinde. (Cf. also [228–230].) So, as we discussed in §7.3, the genus one partition function must be the normalized principal modulus $J(\tau)$ for $SL_2(\mathbb{Z})$, according to Theorem 7.5.

In the analysis of [230, 231], the genus one partition function of chiral three-dimensional gravity is a Rademacher sum (7.20), because it is obtained as a sum over three-dimensional hyperbolic structures on a solid torus with genus one boundary, and such structures are naturally parameterized by the coset space $\Gamma_\infty \backslash SL_2(\mathbb{Z})$ (cf. (1.38)), as explained in [224] (see also §5.1 of [253]). The terms $e^{-2\pi i m \frac{a\tau+b}{c\tau+d}}$ in (7.20) are obtained by evaluating $e^{-I_{\mathrm{TMG}}}$, with $\mu = \sqrt{-\Lambda} = 1/16Gm$, on a solution with boundary curve $\mathrm{C}/(\mathbb{Z}+\tau\mathbb{Z})$, and the subtraction of $e^{-2\pi i m \frac{a}{c}}$ represents quantum corrections to the classical action.

In [110], the above conjecture is extended so as to encompass the principal modulus property for all elements of the monster, with a view to establishing a conceptual foundation for monstrous moonshine. More specifically, the first main conjecture of [110] states the following.

Conjecture 7.7 (Duncan–Frenkel). There exists a monster-indexed family of *twisted chiral three-dimensional gravity* theories, whose genus one partition functions at

$$\mu = \sqrt{-\Lambda} = 1/16G \tag{7.25}$$

are given by $T_{\Gamma_g}^{(-1)}(-1/\tau)$, where $T_{\Gamma_g}^{(-1)}(\tau)$ is the normalized Rademacher sum attached to Γ_g, satisfying (7.23).

From the point of view of vertex operator algebra theory, $T_g(-1/\tau)$—which coincides with $T_{\Gamma_g}^{(-1)}(-1/\tau)$ according to Theorems 7.2 and 7.5—is the graded dimension of the unique simple g-twisted $V^\natural$-module $V_g^\natural$ (cf. §7.2). This non-trivial fact about the functions $T_g(-1/\tau)$ is proven by Carnahan in Theorem 5.1.4 of [66].

Geometrically, the twists of the above conjecture are defined by imposing (generalized) spin structure conditions on solutions to the chiral gravity equations, and allowing orbifold solutions of certain kinds. See §7.1 of [110] for a more complete discussion. The corresponding sums over geometries are then indexed by coset spaces $\Gamma_\infty \backslash \Gamma$, for various groups $\Gamma < SL_2(\mathbb{R})$, commensurable with $SL_2(\mathbb{Z})$. According to Theorem 7.5, the genus one partition function corresponding to such a twist, expected to be a Rademacher sum on physical grounds, will only satisfy the basic physical consistency condition of Γ-invariance if Γ is a genus zero group. One may speculate that a finer analysis of physical consistency will lead to the list of conditions given in §6.5 of [110], which characterize the groups Γ_g for $g \in \mathbb{M}$, according to Theorem 6.5.1 of [110]. Thus the discrete groups Γ_g of monstrous moonshine may ultimately be recovered as those defining physically consistent twists of chiral three-dimensional gravity.

On the other hand, it is reasonable to expect that twisted chiral gravity theories are determined by symmetries of the underlying untwisted theory. Conceptually then, but still conjecturally, the monster group appears as the symmetry group of chiral three-dimensional gravity, for which the corresponding twists exist. The principal modulus property of monstrous moonshine may then be explained: as a consequence of Theorem 7.5, together with the statement that the genus one partition function of a twisted theory is $T_\Gamma^{(-1)}(-1/\tau)$, where $T_\Gamma^{(-1)}(\tau)$ is the normalized Rademacher sum attached to the subgroup $\Gamma < SL_2(\mathbb{R})$ that parameterizes the geometries of the twist.

For more background on the mathematics and physics of black holes we refer the reader to [88]. We refer to [62, 63] for reviews that focus on the particular role of conformal field theory in understanding quantum gravity.

7.5 Moonshine Tower

An optimistic view on the relationship between moonshine and gravity is adopted in §7 of [110]. In particular, in §7.2 of [110] the consequences of Conjecture 7.7 for the *second quantization* of chiral three-dimensional gravity are explored. (We warn the reader that the notion of second quantized gravity is very speculative at this stage.)

Motivated in part by the results on second quantized string theory in [93], the existence of a tower of monster modules

$$V^{(-m)} = \bigoplus_{n=-m}^{\infty} V_n^{(-m)}, \tag{7.26}$$

parameterized by positive integer values of m, is predicted in §7.2 of [110]. Moreover, it is suggested that the graded dimension of $V^{(-m)}$ should be given by

$$J^{(-m)} := m\hat{T}(m)J, \tag{7.27}$$

where $\hat{T}(m)$ denotes the *(order m) Hecke operator*, acting on $SL_2(\mathbb{Z})$-invariant holomorphic functions on $\mathbb{H}$ according to the rule

$$(\hat{T}(m)f)(\tau) := \frac{1}{m} \sum_{\substack{ad=m \\ 0\leq b<d}} f\left(\frac{a\tau+b}{d}\right). \tag{7.28}$$

Standard calculations (cf. e.g. Chp.VII, §5 of [270]) determine that $m\hat{T}(m)J$ is an $SL_2(\mathbb{Z})$-invariant holomorphic function on $\mathbb{H}$, whose Fourier coefficients

$$J^{(-m)}(\tau) = \sum_n c(-m,n)q^n \tag{7.29}$$

are expressed in terms of those of $J(\tau) = \sum_{n=-1}^{\infty} c(n)q^n$, by $c(-m,n) = \delta_{-m,n}$ for $n \leq 0$, and

$$c(-m,n) = \sum_{\substack{k>0 \\ k|(m,n)}} \frac{m}{k} c(mn/k^2), \tag{7.30}$$

for $n > 0$, where (m,n) denotes the greatest common divisor of m and n. In particular, $J^{(-m)}(\tau) = q^{-m} + O(q)$ as $\Im(\tau) \to \infty$. There is only one such $SL_2(\mathbb{Z})$-invariant holomorphic function on $\mathbb{H}$, so we have

$$J^{(-m)}(\tau) = \sum_{n=-m}^{\infty} \dim(V^{(-m)})q^n = T_\Gamma^{(-m)}(\tau) \tag{7.31}$$

according to (7.24) and Theorem 7.5, when $\Gamma = SL_2(\mathbb{Z})$. So the graded dimension of $V^{(-m)}$ is also a normalized Rademacher sum.

We would like to investigate the higher order analogues of the McKay–Thompson series T_g (cf. (1.37)), encoding the graded traces of monster elements on $V^{(-m)}$, but for this we must first determine the $\mathbb{M}$-module structure on each homogeneous subspace $V_n^{(-m)}$.

A solution to this problem is entailed in Borcherds' proof [30] of the monstrous moonshine conjectures, and the identity (7.11), in particular. To explain this, recall the *Adams operation* ψ^k on virtual G-modules, defined, for $k \geq 0$ and G a finite group, by requiring that

$$\mathrm{tr}(g|\psi^k(V)) = \mathrm{tr}(g^k|V) \tag{7.32}$$

for $g \in G$. (Cf. [16, 193] for more details on Adams operations.) Using the ψ^k we may equip $V^{(-m)}$ with a virtual $\mathbb{M}$-module structure (we will see momentarily that it is actually an $\mathbb{M}$-module, cf. Proposition 7.9) by defining $V^{(-m)}_{-m} := \mathrm{C}$ to be the one-dimensional trivial $\mathbb{M}$-module, $V^{(-m)}_n := 0$ for $-m < n \leq 0$, and

$$V^{(-m)}_n := \bigoplus_{\substack{k>0 \\ k|(m,n)}} \mathrm{C}^{m/k} \otimes \psi^k(V^\natural_{mn/k^2}) \tag{7.33}$$

for $n > 0$, where $\mathrm{C}^{m/k}$ denotes the trivial $\mathbb{M}$-module of dimension m/k. For convenience later on, we also define $V^{(0)} = V^{(0)}_0 := \mathrm{C}$ to be the trivial, one-dimensional $\mathbb{M}$-module, regarded as graded, with grading concentrated in degree $n = 0$.

Evidently ψ^k preserves dimension, so the graded dimension of $V^{(-m)}$ is still given by $J^{(-m)}$, according to (7.30). Define the *order m McKay–Thompson series $T^{(-m)}_g$*, for $m \geq 0$ and $g \in \mathbb{M}$, by setting

$$T^{(-m)}_g(\tau) := q^{-m} + \sum_{n>0} \mathrm{tr}(g|V^{(-m)}_n)q^n. \tag{7.34}$$

Then $T^{(0)}_g = 1$ for all $g \in \mathbb{M}$, and $T^{(-1)}_g$ is the original McKay–Thompson series T_g. More generally, we have the following result, which constructs the $T^{(-m)}_g$ uniformly and explicitly as Rademacher sums.

Theorem 7.8. *For $m > 0$ and $g \in \mathbb{M}$ we have $T^{(-m)}_g(\tau) = T^{(-m)}_{\Gamma_g}(\tau)$, where Γ_g is the invariance group of $T_g(\tau)$, and $T^{(-m)}_\Gamma$ denotes the normalized Rademacher sum of order m attached to Γ, as in (7.23). In particular, $T^{(-m)}_g(\tau)$ is a monic integral polynomial of degree m in $T_g(\tau)$.*

Proof. We will use Borcherds' identity (7.11). To begin, note that $T^{(-m)}_g$ is given explicitly in terms of traces on $V^\natural$ by

$$T^{(-m)}_g(\tau) = q^{-m} + \sum_{n>0} \sum_{k|(m,n)} \frac{m}{k} \mathrm{tr}(g|\psi^k(V^\natural_{mn/k^2}))q^n \tag{7.35}$$

according to (7.33) and (7.34). Recall that $R(G)$ denotes the integral representation ring of a finite group G. Extend the ψ^k from $R(G)$ to $R(G)[[p, q]][q^{-1}]$, by setting $\psi^k(Mp^mq^n) = \psi^k(M)p^{km}q^{km}$ for $M \in R(G)$. Then it is a general property of the Adams operations (cf. §5.2 of [173]) that

$$\log \bigwedge\nolimits_{-1}(X) = -\sum_{k>0} \frac{1}{k}\psi^k(X) \tag{7.36}$$

in $R(G)[[p,q]][q^{-1}]\otimes_{\mathbb{Z}}\mathbb{Q}$, for $X \in R(G)[[p,q]][q^{-1}]$. So taking $X = \sum_{\substack{m,n\in\mathbb{Z}\\ m>0}} V^\natural_{mn}p^mq^n$ we obtain

$$\begin{aligned}\log \bigwedge\nolimits_{-1}\left(\sum_{\substack{m,n\in\mathbb{Z}\\ m>0}} V^\natural_{mn}p^mq^n\right) &= -\sum_{k>0}\sum_{\substack{m,n\in\mathbb{Z}\\ m>0}} \frac{1}{k}\psi^k(V^\natural_{mn})p^{km}q^{kn}\\ &= -\sum_{\substack{m,n\in\mathbb{Z}\\ m>0}}\sum_{k|(m,n)} \frac{1}{k}\psi^k(V^\natural_{mn/k^2})p^mq^n\end{aligned} \tag{7.37}$$

for the logarithm of the left hand side of (7.11). If we now define $V^{(-m)}(q) := \sum_n V^{(-m)}_n q^n$, an element of $R(\mathbb{M})[[q]][q^{-1}]$, then the generating series $\sum_{m>0} p^m V^{(-m)}(q)$ is obtained when we apply $-p\partial_p$ to (7.37), according to the definition (7.33) of the $V^{(-m)}_n$ as elements of $R(\mathbb{M})$. So apply $-p\partial_p \log(\,\cdot\,)$ to both sides of (7.11) to obtain the identity

$$\sum_{m>0} V^{(-m)}(q)p^m = -1 - (p\partial_p V^\natural(p))\sum_{k\geq 0} V^\natural(q)^k V^\natural(p)^{-k-1} \tag{7.38}$$

in $R(\mathbb{M})[[p,q]][q^{-1}]$, where $V^\natural(q) = V^{(-1)}(q) = q^{-1} + \sum_{n>0} V^\natural_n q^n$. The right hand side of (7.38) really is a taylor series in p, for we use $V^\natural(p)^{-1}$ as a short hand for $\sum_{k\geq 0}(-1)^k p^{k+1} V^\natural_+(p)$, where $V^\natural_+(p) := \sum_{n>0} V^\natural_n p^n$ is the regular part of $V^\natural(p)$.

The McKay–Thompson series $T^{(-m)}_g(\tau)$ is just the trace of g on $V^{(-m)}(q)$, so an application of $\mathrm{tr}(g|\,\cdot\,)$ to (7.38) replaces $V^{(-m)}(q)$ with $T^{(-m)}_g(\tau)$, and $V^\natural(q)$ with $T_g(\tau)$, etc. and shows that $T^{(-m)}_g$ is indeed a polynomial in T_g, of degree m since the leading term of $T^{(-m)}_g$ is q^{-m} by definition. In particular, $T^{(-m)}_g$ is a modular function for Γ_g, with no poles away from the infinite cusp. Since Γ_g has genus zero, such a function is uniquely determined (up to an additive constant) by the polar terms in its Fourier expansion. The McKay–Thompson series $T_g^{(-m)}$ and the Rademacher sum $T_{\Gamma_g}^{(-m)}$ both satisfy $q^{-m} + O(q)$ as $\Im(\tau) \to \infty$ (cf. (7.24)), and neither have poles away from the infinite cusp, so they must coincide. This completes the proof. □

Remark. The identity obtained by taking the trace of $g \in \mathbb{M}$ on (7.38) may be compactly rewritten

$$\sum_{m\geq 0} T_g^{(-m)}(\tau)p^m = \frac{p\partial_p T_g(\sigma)}{T_g(\tau) - T_g(\sigma)}, \tag{7.39}$$

where $p = e^{2\pi i\sigma}$ and $T_g(\sigma) = \sum_m \operatorname{tr}(g|V_m^\natural)p^m$. This expression (7.39) is proven for some special cases by a different method in [21].

Recall that the monster group has 194 irreducible ordinary representations, up to equivalence. Let us denote these by M_i, for $1 \leq i \leq 194$, where the ordering is as in [80], so that the character of M_i is the function denoted χ_i in [80]. Define $\mathbf{m}_i(-m,n)$ to be the multiplicity of M_i in $V_n^{(-m)}$, so that

$$V_n^{(-m)} \approx \bigoplus_{i=1}^{194} M_i^{\oplus \mathbf{m}_i(-m,n)} \tag{7.40}$$

as $\mathbb{M}$-modules, and $c(-m,n) = \sum_{i=1}^{194} \mathbf{m}_i(-m,n)\chi_i(e)$.

A priori, the $\mathbb{M}$-modules $V_n^{(-m)}$ may be virtual, meaning that some of the integers $\mathbf{m}_i(-m,n)$ are negative.

Proposition 7.9. *The $V_n^{(-m)}$ are all (non-virtual) modules for the monster. In particular, the integers $\mathbf{m}_i(-m,n)$ are all non-negative.*

Proof. The claim follows from the modification of Borcherds' proof of Theorem 7.2 presented by Jurisich–Lepowsky–Wilson in [173]. In [173] a certain free Lie sub algebra $\mathfrak{u}^-$ of the monster Lie algebra $\mathfrak{m}$ is identified, for which the identity $\Lambda(\mathfrak{u}^-) = H(\mathfrak{u}^-)$ (or rather, the logarithm of this) yields

$$\sum_{m,n>0}\sum_{k|(m,n)} \frac{1}{k}\psi^k(V^\natural_{mn/k^2})p^mq^n = \sum_{k>0}\frac{1}{k}\left(\sum_{m,n>0} V^\natural_{m+n-1}p^mq^n\right)^k \tag{7.41}$$

in $R(\mathbb{M})[[p,q]][q^{-1}]\otimes\mathbb{Q}$. (Notice the different range of summation, compared to (7.11).) We apply $p\partial_p$ to (7.41), and recall the definition (7.33) of $V_n^{(-m)}$ to obtain

$$\sum_{m,n>0} V_n^{(-m)}p^mq^n = \sum_{k>0}\left(\sum_{m,n>0} mV^\natural_{m+n-1}p^mq^n\right)\left(\sum_{m,n>0} V^\natural_{m+n-1}p^mq^n\right)^{k-1}. \tag{7.42}$$

The coefficient of p^mq^n in the right hand side of (7.42) is evidently a non-negative integer combination of the $\mathbb{M}$-modules $V_n^\natural$, so the proof of the claim is complete. □

In §7.6 we will determine the behavior of the multiplicity functions $\mathbf{m}_i(-m,n)$ (cf. (7.40)) as $n\to\infty$. For applications to gravity a slightly different statistic is more relevant. Recall from §7.4 that it is the Virasoro highest weight vectors—i.e. those $v\in V_n^\natural$ with $L(k)v=0$ for $k>0$—that represent black hole states in chiral three-dimensional gravity at $m=1$. Such vectors generate *highest weight modules* for $\mathcal{V}$, the structure of which has been determined by Feigin–Fuchs in [118]. (See [15] for an alternative treatment.) Specializing to the case that the central element $\mathbf{c}$ (cf. (7.1)) acts as $c=24m$ times the identity, for some positive integer m, we obtain from the results of [118] that the isomorphism type of an irreducible highest weight module for $\mathcal{V}$ depends only on the $L(0)$-eigenvalue of a generating highest weight vector, v, and if $L(0)v=hv$ for h a non-negative integer, then

$$\sum_n \dim(L(h,c)_n)q^n = \begin{cases} q^{-m}(1-q)(q)_\infty^{-1} & \text{if } h=0,\\ q^{h-m}(q)_\infty^{-1} & \text{if } h>0,\end{cases} \tag{7.43}$$

where $L(h,c)$ denotes the irreducible highest weight $\mathcal{V}$-module generated by v. We write $L(h,c)_n$ for the subspace of $L(h,c)$ with $L(0)$-eigenvalue $h-m$ in (7.43), and

$$(q)_\infty := \prod_{n>0}(1-q^n). \tag{7.44}$$

(See [156] for details of the calculation that returns (7.43) in the case that $m=1$.)

Remark. We may now recognize the leading terms in (1.47) as exactly those of the graded dimension of the Virasoro module $L(0,24m)$.

It is known that $V^\natural$ is a direct sum of highest weight modules for the Virasoro algebra (cf. e.g. [156]). Since the Virasoro and monster actions on $V^\natural$ commute, we have an isomorphism

$$V^\natural \simeq L(0,24)\otimes W^\natural_{-1} \oplus \bigoplus_{n>0} L(n+1,24)\otimes W^\natural_n \tag{7.45}$$

of modules for $\mathcal{V}\times\mathbb{M}$, where $W^\natural_n$ denotes the subspace of $V^\natural_n$ spanned by Virasoro highest weight vectors. To investigate how the black hole states in $V^\natural$ are organized by the representation theory of the monster, we define non-negative integers $\mathbf{n}_i(n)$ by requiring that

$$W^\natural_n \simeq \bigoplus_{i=1}^{194} M_i^{\oplus \mathbf{n}_i(n)}, \tag{7.46}$$

for $n\geq -1$.

Evidently $\mathbf{n}_i(n)\leq \mathbf{m}_i(-1,n)$ for all i and n since $W^\natural_n$ is a subspace of $V^\natural_n$. To determine the precise relationship between the $\mathbf{n}_i(n)$ and $\mathbf{m}_i(-1,n)$, define $U_g(\tau)$ for $g\in\mathbb{M}$ by setting

$$U_g(\tau) := \sum_{n=-1}^{\infty} \operatorname{tr}(g|W^\natural_n)q^n, \tag{7.47}$$

so that $U_g(\tau) = q^{-1} + \sum_{n>0}\sum_{i=1}^{194} \mathbf{n}_i(n)\chi_i(g)q^n$ (cf. (7.46)). Combining (7.43), (7.45) and (7.46), together with the definitions (1.37) of T_g and (7.47) of U_g, we obtain

$$T_g(\tau) = q^{-1}\frac{(1-q)}{(q)_\infty} + \sum_{n>0} q^n \frac{1}{(q)_\infty}\sum_{i=1}^{194} \mathbf{n}_i(n)\chi_i(g), \tag{7.48}$$

or equivalently,

$$U_g(\tau) = (q)_\infty T_g(\tau) + 1 \tag{7.49}$$

for all $g \in \mathbb{M}$. (This computation also appears in [156].)

In §7.6 we will use (7.49) to determine the asymptotic behavior of the $\mathbf{n}_i(n)$ (cf. Theorem 7.10), and thus the statistics of black hole states, at $\ell = 16G$, in the conjectural chiral three-dimensional gravity theory dual to $V^\natural$.

Remark. Note that we may easily construct modules for $\mathcal{V} \times \mathbb{M}$ satisfying the extremal condition (1.47), for each positive integer m, by considering direct sums of the monster modules $V^{(-m)}$ constructed in Proposition 7.9. A very slight generalization of the argument just given will then yield formulas for the graded traces of monster elements on the corresponding Virasoro highest weight spaces. Since it has been shown [134, 163] that such modules cannot admit vertex operator algebra structure, we do not pursue this here.

7.6 Monstrous Moonshine's Distributions

We now address the problem of determining exact formulas and asymptotic distributions of irreducible components. This work will rely heavily on the modularity of the underlying McKay–Thompson series (i.e. Theorems 1.23 and 7.8).

We prove formulas for the multiplicities $\mathbf{m}_i(-m, n)$ and $\mathbf{n}_i(n)$ which in turn imply the following asymptotics.

Theorem 7.10. *If m is a positive integer and $1 \le i \le 194$, then as $n \to +\infty$ we have*

$$\mathbf{m}_i(-m,n) \sim \frac{\dim(\chi_i)|m|^{1/4}}{\sqrt{2}|n|^{3/4}|\mathbb{M}|} \cdot e^{4\pi\sqrt{|mn|}}$$
$$\mathbf{n}_i(n) \sim \frac{\sqrt{12}\ \dim(\chi_i)}{|24n+1|^{1/2}|\mathbb{M}|} \cdot e^{\frac{\pi}{6}\sqrt{23|24n+1|}}$$

These asymptotics immediately imply that the following limits are well-defined

$$\begin{aligned} \delta\left(\mathbf{m}_i(-m)\right) &:= \lim_{n\to+\infty} \frac{\mathbf{m}_i(-m,n)}{\sum_{i=1}^{194} \mathbf{m}_i(-m,n)} \\ \delta\left(\mathbf{n}_i\right) &:= \lim_{n\to+\infty} \frac{\mathbf{n}_i(n)}{\sum_{i=1}^{194} \mathbf{n}_i(n)}. \end{aligned} \tag{7.50}$$

Corollary 7.11. *In particular, we have that*

$$\delta\left(\mathbf{m}_i(-m)\right) = \delta\left(\mathbf{n}_i\right) = \frac{\dim(\chi_i)}{\sum_{j=1}^{194} \dim(\chi_j)} = \frac{\dim(\chi_i)}{5844076785304502808013602136}.$$

7.6.1 The modular groups in monstrous moonshine

To obtain exact formulas, we begin by recalling the modular groups which arise in monstrous moonshine. Suppose $\Gamma_* < GL_2(\mathbb{R})$ is a discrete group which is commensurable with $SL_2(\mathbb{Z})$. If Γ_* defines a genus zero quotient of $\mathbb{H}$, then the field of modular functions which are invariant under Γ_* is generated by a single element, the principal modulus (cf. (1.39)). Theorem 7.2 implies that the T_g (defined by (1.37)) are principal moduli for certain groups Γ_g. We can describe these groups in terms of groups E_g which in turn may be described in terms of the congruence subgroups

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ : \ c \equiv 0 \,(\mathrm{mod}\, N) \right\}, \tag{7.51}$$

and the Atkin–Lehner involutions W_e for $\Gamma_0(N)$ given by

$$W_e := \begin{pmatrix} ae & b \\ cN & de \end{pmatrix}, \tag{7.52}$$

where e is an exact divisor of N (i.e. $e|N$, and $(e, N/e) = 1$), and a, b, c, and d are integers chosen so that W_e has determinant e.

Following Conway–Norton [81] and Conway–McKay–Sebbar [84], we denote the groups E_g by symbols of the form $\Gamma_0(N|h) + e, f, \dots$ (or simply $N|h + e, f, \dots$), where h divides $(N, 24)$, and each of e, f, etc. exactly divide N/h. This symbol represents the group

$$\Gamma_0(N|h) + e, f, \dots := \begin{pmatrix} 1/h & 0 \\ 0 & 1 \end{pmatrix} \langle \Gamma_0(N/h), W_e, W_f, \dots \rangle \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix},$$

where W_e, W_f, etc. are representative of Atkin–Lehner involutions on $\Gamma_0(N/h)$. We use the notation $\mathcal{W}_g := \{1, e, f, \dots\}$ to denote this list of Atkin–Lehner involutions contained in E_g. We also note that $\Gamma_0(N|h) + e, f, \dots$ contains $\Gamma_0(Nh)$.

The groups E_g are eigengroups for the T_g, so that if $\gamma \in E_g$, then $T_g(\gamma\tau) = \sigma_g(\gamma)T_g$, where σ_g is a multiplicative group homomorphism from E_g to the group of h-th roots of unity. Conway and Norton [81] give the following values for σ_g evaluated on generators of $N|h + e, f, \dots$.

Lemma 7.12 (Conway–Norton). *Assuming the notation above, the following are true:*

(a) $\sigma_g(\gamma) = 1$ *if* $\gamma \in \Gamma_0(Nh)$
(b) $\sigma_g(\gamma) = 1$ *if* γ *is an Atkin–Lehner involution of* $\Gamma_0(Nh)$ *inside* E_g
(c) $\sigma_g(\gamma) = e^{\frac{-2\pi i}{h}}$ *if* $\gamma = \begin{pmatrix} 1 & 1/h \\ 0 & 1 \end{pmatrix}$
(d) $\sigma_g(\gamma) = e^{-\lambda_g \frac{2\pi i}{h}}$ *if* $\gamma = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}$,

where λ_g *in (d) is* -1 *if* $N/h \in \mathcal{W}_g$, *and* $+1$ *otherwise.*

This information is sufficient to properly describe the modularity of the series $T_g^{(-m)}(\tau)$ on E_g. In section 7.6.2, we give an explicit procedure for evaluating σ_g. The invariance group Γ_g, denoted by $\Gamma_0(N||h)+e,f,\ldots$ (or by the symbol $N||h+e,f,\ldots$), is defined as the kernel of σ_g. A complete list of the groups Γ_g can be found in the Appendix (§A.1) of this paper, or in table 2 of [81].

Theorems 1.23 and 7.8 are summarized by the following uniform statement.

Theorem 7.13. *Let $g \in \mathbb{M}$ and $m \geq 1$. Then $T_g^{(-m)}$ is the unique weakly holomorphic modular form of weight zero for Γ_g that satisfies $T_g^{(-m)} = q^{-m}+O(q)$ as τ approaches the infinite cusp, and has no poles at any cusps inequivalent to the infinite one.*

Remark. A *weakly holomorphic* modular form is a meromorphic modular form whose poles (if any) are supported at cusps.

7.6.2 Exact formulas for $T_g^{(-m)}$

Using Theorems 2.3 and 2.4, we can write exact formulas for the coefficients of the $T_g^{(-m)}$ provided we know its principal parts at all cusps of $\Gamma_0(Nh)$. With this in mind, we now regard T_g as a modular function on $\Gamma_0(Nh)$ with trivial multiplier. The location and orders of the poles were determined by Harada and Lang [155].

Lemma 7.14. *[155, Lemma 7, 9] Suppose the Γ_g is given by the symbol $N||h+e,f...$, and let $L = \begin{pmatrix} -\delta & \beta \\ \gamma & -\alpha \end{pmatrix} \in SL_2(\mathbb{Z})$. Then $T_g|_0 L$ has a pole if and only if $\left(\frac{\gamma}{(\gamma,h)}, \frac{N}{h}\right) = \frac{N}{eh}$, for some $e \in \mathcal{W}_g$ (Note, here we allow $e=1$). The order of the pole is given by $\frac{(h,\gamma)^2}{eh^2}$.*

Harada and Lang prove this lemma by showing that if u is an integer chosen such $\frac{u\gamma-\alpha\cdot(h,\gamma)}{h}$ is integeral and divisible by e and $U = \begin{pmatrix} \frac{e\cdot h}{(h,\gamma)} & \frac{u}{h} \\ 0 & \frac{(h,\gamma)}{h} \end{pmatrix}$, then LU is an Atkin–Lehner involution $W_e \in E_g$. Therefore, we have that

$$T_g|_0 L = \sigma_g(LU) T_g \left(\frac{(h,\gamma)^2}{eh^2}\tau - \frac{u\cdot(h,\gamma)}{eh^2} \right). \tag{7.53}$$

Harada and Lang do not compute $\sigma_g(LU)$, however we will need these values in order to apply Theorem 2.4. Using Lemma 7.12, the following procedure allows us to compute $\sigma_g(M)$ for any matrix $M \in E_g$.

Given a matrix $M \in E_g$, we may write M as $M = \begin{pmatrix} ae & \frac{b}{h} \\ cN & de \end{pmatrix}$ with $e \in \mathcal{W}_g$ and $ade - bc\frac{N}{eh} = 1$. We may also write $h = h_e \cdot h_{\overline{e}}$, where $h_{\overline{e}}$ is the largest divisor of h co-prime to e. Since $c\frac{N}{eh}$ is co-prime to both d and e, we may chose integers A, B, and $C \pmod{h}$ such that:

- $c\frac{N}{eh}A + d$ is co-prime to $h_{\overline{e}}$ but is divisible by h_e,
- $B \equiv -(eaA + b)(c\frac{N}{h}A + ed)^{-1} \pmod{h_{\overline{e}}}$ and $Bc\frac{N}{eh} + b \equiv 0 \pmod{h_e}$,
- $C \equiv -c(c\frac{N}{h}A + ed)^{-1} \pmod{h_{\overline{e}}}$, and $C \equiv 0 \pmod{h_e}$.

A calculation shows that $\widehat{M} := \begin{pmatrix} 1 & \frac{B}{h} \\ 0 & 1 \end{pmatrix} M \begin{pmatrix} 1 & \frac{A}{h} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ CN & 1 \end{pmatrix} \begin{pmatrix} h_e & 0 \\ 0 & h_e \end{pmatrix}$ is an Atkin–Lehner involution W_E for $\Gamma_0(Nh)$ where $E = e\cdot h_e^2$. By Lemma 7.12, this implies $\sigma_g(\widehat{M}) = 1$, and therefore

$$\sigma_g(M) = exp\left(\frac{2\pi i}{h}(A + B + \lambda_g C) \right).$$

Combined with Lemma 7.14, this leads us to the following proposition.

Proposition 7.15. *Given a matrix* $L = \begin{pmatrix} -\delta & \beta \\ \gamma & -\alpha \end{pmatrix} \in SL_2(\mathbb{Z})$, *let* u *and* U *be chosen as above, and define*

$$\epsilon_g(L) := \sigma_g(LU) \cdot e^{2\pi i \frac{u\cdot(h,\gamma)}{eh^2}}.$$

Then by (7.53), we have that

$$T_g|_0 L = \epsilon_g(L) q^{-\frac{(h,\gamma)^2}{eh^2}} + O(q).$$

Using this notation, we are equipped to find exact formulas for the $T_g^{(-m)}$.

Theorem 7.16. *Let $g \in \mathbb{M}$, with $\Gamma_g = N|h + e, f, \ldots$, and let $\mathcal{S}_{Nh}$ and $\mathcal{W}_g$ be as above. if m and n are positive integers, then there is a constant c for which*

$$T_g^{(-m)}(\tau) = c + \sum_{e \in \mathcal{W}_g} \sum_{\substack{\frac{\alpha}{\gamma} \in \mathcal{S}_{Nh} \\ \left(\frac{\gamma}{(\gamma,h)}, \frac{N}{h}\right) = \frac{N}{eh}}} \epsilon_g(L_\rho)^m \mathcal{P}^+_{\alpha/\gamma}(\tau, m, Nh, 0).$$

The n-th coefficient of $T_g^{(-m)}(\tau)$ is given by

$$\sum_{e \in \mathcal{W}_g} \sum_{\substack{\rho = \frac{\alpha}{\gamma} \in \mathcal{S}_{Nh} \\ \left(\frac{\gamma}{(\gamma,h)}, \frac{N}{h}\right) = \frac{N}{eh}}} \epsilon_g(L_\rho)^m 2\pi \left| \frac{-m}{n} \cdot \frac{(h,\gamma)^2}{eh^2} \right|^{\frac{1}{2}} \times$$

$$\sum_{\substack{c > 0 \\ (c, Nh) = (\gamma, Nh)}} \frac{K_c(0, L, -m, n)}{c} \cdot I_1\left(\frac{4\pi}{c} \sqrt{\left| \frac{-mn \cdot (h,\gamma)^2}{eh^2} \right|} \right).$$

Proof. Every modular function is a harmonic Maass form. Therefore, the idea is to exhibit a linear combination of Maass–Poincaré series with exactly the same principal parts at all cusps as $T_g^{(-m)}$. By Lemma 2.2 and Theorem 2.4, this form equals $T_g^{(-m)}$ up to an additive constant. Lemma 7.12 (c) implies that the coefficients $c_g(n)$ of T_g are supported on the arithmetic progression $n \equiv -1 \pmod{h}$. The function $T_g^{(-m)}$ is a polynomial in T_g, and as such must be the sum of powers of T_g each of which is congruent to $m \pmod{h}$. Therefore, if $M \in \Gamma_g$, then $T^{(-m)}|_0 M = \sigma_g(M)^m T^{(-m)}$. Given $L \in SL_2(\mathbb{Z})$, let U be a matrix as in (7.53) so that $LU \in \Gamma_g$. By applying Proposition 7.15, we find

$$T_g^{(-m)}|_0 L = \sigma_g(LU)^m T_g^{(-m)}|_0 U^{-1} = \epsilon_g(L)^m q^{-m\frac{(h,\gamma)^2}{eh^2}} + O(q).$$

Theorem 2.4, along with the observations that $t_\rho = \frac{(h,\gamma)^2}{eh^2}$ and $\kappa_\rho = 0$ for every $\rho = \frac{\alpha}{\gamma} = L_\rho^{-1}\infty$, implies the first part of the theorem. The formula for the coefficients follows by Theorem 2.3. □

7.6.3 Exact formulas for U_g up to a theta function

Following a similar process to that in the previous section, we may construct a series $\widehat{U}_g(\tau) = q^{-\frac{23}{24}} + O(q^{\frac{1}{24}})$ with principal parts matching those of $\eta(\tau)T_g(\tau)$ at all cusps. Then according to (7.49), the difference $q^{\frac{1}{24}}(U_g - 1) - \widehat{U}_g$ is a weight $\frac{1}{2}$ holomorphic modular form, which by a celebrated result [271] of Serre–Stark, is a finite linear combination of unary theta functions. This will not affect the asymptotics in Theorem 7.10. The functions T_g and $\widehat{U}_g$ differ primarily in their weight, and in that $\widehat{U}_g$ has a non-trivial multiplier $\nu_\eta : M \to \frac{\eta(M\tau)}{(M:\tau)^{1/2}\eta(\tau)}$. They also have slightly different orders of poles, which is accounted for by the fact that the multiplier ν_η implies that $\kappa_\rho = t_\rho/24$ at every cusp ρ for the $\widehat{U}_g$, rather than 0 for the T_g. The proof of the following theorem is the same as that of Theorem 7.16, *mutatis mutandis.*

Theorem 7.17. *Let* $g \in \mathbb{M}$, *with* $\Gamma_g = N|h + e, f, \ldots$, *and let* $\mathcal{S}_{Nh}$ *and* $\mathcal{W}_g$ *be as above. If* m *is a positive integer then*

$$\widehat{U}_g = \sum_{e \in \mathcal{W}_g} \sum_{\substack{\rho = \frac{\alpha}{\gamma} \in \mathcal{S}_{Nh} \\ \left(\frac{\gamma}{(\gamma,h)}, \frac{N}{h}\right) = \frac{N}{eh}}} \epsilon_g(L_\rho)\mathcal{P}^+_{L_\rho}(\tau, 1, Nh, 1/2, \nu_\eta).$$

For n *a non-negative integer, the coefficient of* $q^{n+\frac{1}{24}}$ *in* $\widehat{U}_g$ *is given by*

$$\sum_{e \in \mathcal{W}_g} \sum_{\substack{\rho = \frac{\alpha}{\gamma} \in \mathcal{S}_{Nh} \\ \left(\frac{\gamma}{(\gamma,h)}, \frac{N}{h}\right) = \frac{N}{eh}}} \epsilon_g(L_\rho)\frac{1-i}{\sqrt{2}} 2\pi \left| \frac{-\frac{(h,\gamma)^2}{eh^2} + \frac{1}{24}}{n + \frac{1}{24}} \right|^{\frac{1}{4}} \times$$

$$\sum_{\substack{c > 0 \\ (c, Nh) = (\gamma, Nh)}} \frac{K_c(\frac{1}{2}, L, \nu_\eta, -1, n)}{c} \cdot I_{\frac{1}{2}}\left(\frac{4\pi}{c} \sqrt{\left| -\frac{(h,\gamma)^2}{eh^2} + \frac{1}{24} \right| \left| n + \frac{1}{24} \right|} \right).$$

This immediately admits the following corollary.

Corollary 7.18. Given the notation above, there is a weight $\frac{1}{2}$ linear combination of theta functions $h_g(\tau)$ for which the coefficient q^n in $U_g(\tau) - q^{-\frac{1}{24}} h_g(\tau)$ coincides with the coefficient of $q^{n+\frac{1}{24}}$ in $\widehat{U}_g$, given explicitly in Theorem 7.17.

7.6.4 Proof of Theorem 7.10

Proof of Theorem 7.10. Following Harada and Lang [155], we begin by defining the functions

$$T_{\chi_i}^{(-m)}(\tau) := \frac{1}{|\mathbb{M}|} \sum_{g \in \mathbb{M}} \chi_i(g) T_g^{(-m)}(\tau). \tag{7.54}$$

The orthogonality of characters imply that for g and $h \in \mathbb{M}$,

$$\sum_{i=1}^{194} \overline{\chi_i(g)} \chi_i(h) = \begin{cases} |C_{\mathbb{M}}(g)| & \text{if } g \text{ and } h \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases} \tag{7.55}$$

Here $|C_{\mathbb{M}}(g)|$ is the order of the centralizer of g in $\mathbb{M}$. Since the order of the centralizer times the order of the conjugacy class of an element is the order of the group, (7.55) and (7.54) together imply the inverse relation

$$T_g^{(-m)}(\tau) = \sum_{i=1}^{194} \overline{\chi_i(g)} T_{\chi_i}^{(-m)}(\tau).$$

In particular we have that $T_e^{(-m)}(\tau) = \sum_{i=1}^{194} \dim(\chi_i) T_{\chi_i}^{(-m)}(\tau)$, and therefore we can identify the $\mathbf{m}_i(-m,n)$ as the Fourier coefficients of the $T_{\chi_i}^{(-m)}(\tau) = \sum_{n=-m}^{\infty} \mathbf{m}_i(-m,n) q^n$.

Using Theorem 7.16, we obtain exact formulas for the coefficients of $T_{\chi_i}^{(-m)}(\tau)$. Let $g \in \mathbb{M}$ with $\Gamma_g = N_g || h_g + e_g, f_g, \ldots$. If m and n are positive integers, then the nth coefficient is given exactly by

$$\frac{1}{|\mathbb{M}|}\sum_{g\in\mathbb{M}}\chi_i(g)\sum_{e\in\mathcal{W}_g}\sum_{\substack{\frac{\alpha}{\gamma}\in\mathcal{S}_{N_gh_g}\\ \left(\frac{\gamma}{(\gamma,h_g)},\frac{N_g}{h_g}\right)=\frac{N_g}{eh_g}}}\epsilon_g(L_\rho)^m 2\pi\left|\frac{-m}{n}\cdot\frac{(h_g,\gamma)^2}{eh_g^2}\right|^{\frac{1}{2}}$$

$$\sum_{\substack{c>0\\(c,N_gh_g)=(\gamma,N_gh_g)}}\frac{K_c(2-k,L,\nu,-m,n)}{c}\cdot I_1\left(\frac{4\pi}{c}\sqrt{\left|\frac{-mn\cdot(h_g,\gamma)^2}{eh_g^2}\right|}\right),$$

where $\mathcal{S}_{N_gh_g}$ and $\mathcal{W}_g$ are given as above.

Using the well-known asymptotics for the I-Bessel function

$$I_k(x)\sim\frac{e^x}{\sqrt{2\pi x}}\left(1-\frac{4k^2-1}{8x}+\dots\right),$$

we see that the formula for $\mathbf{m}_i(-m,n)$ is dominated by the $c=1$ term which appears only for $g=\mathrm{e}$ (so that $N_\mathrm{e}=h_\mathrm{e}=1$). This term yields the asymptotic

$$\mathbf{m}_i(-m,n)\sim\frac{\chi_i(\mathrm{e})\cdot|m|^{1/4}}{\sqrt{2}n^{3/4}|\mathbb{M}|}\cdot e^{4\pi\sqrt{|mn|}}$$

as in the statement of the theorem.

The asymptotics for $\mathbf{n}_i(n)$ follows similarly, using the formula

$$U_{\chi_i}(\tau):=\frac{1}{|\mathbb{M}|}\sum_{g\in\mathbb{M}}\chi_i(g)U_g^{(-m)}(\tau).$$

We note that the coefficients of the theta functions $h_g(\tau)$ in Corollary 7.18 are bounded by constants and so do not affect the asymptotics. This yields

$$\mathbf{n}_i(n)\sim\frac{\sqrt{12}\ \chi_i(\mathrm{e})}{|24n+1|^{1/2}|\mathbb{M}|}\cdot e^{\frac{\pi}{6}\sqrt{23|24n+1|}}$$

as in the theorem. $\square$

7.6.5 Examples of the exact formulas

We conclude with a few examples illustrating the exact formulas for the McKay–Thompson series. These formulas for the coefficients generally converge rapidly. However the rate of convergence is not uniform and often requires many more terms to converge to a given precision.

Example 7.19. We first consider the example $g = \mathrm{e}$. In this case we have $\Gamma_g = SL_2(\mathbb{Z})$, which has only the cusp infinity. In this case Theorem 7.16 reduces to the well known expansion

$$T_{\mathrm{e}} = J(\tau) - 744 = q^{-1} + \sum_{n\geq 1} \frac{2\pi}{\sqrt{n}} \cdot \sum_{c>0} \frac{K_c(\infty, -m, n)}{c} \cdot I_1\left(\frac{4\pi\sqrt{n}}{c}\right) q^n.$$

Table 7.1 below contains several approximations made by bounding the size of the c term in the summation.

Table 7.1

	$n = 1$	$n = 5$	$n = 10$
$c \leq 25$	$196883.661\ldots$	$333202640598.254\ldots$	$22567393309593598.047\ldots$
≤ 50	$196883.881\ldots$	$333202640599.429\ldots$	$22567393309593598.660\ldots$
≤ 75	$196883.840\ldots$	$333202640599.828\ldots$	$22567393309593599.369\ldots$
≤ 100	$196883.958\ldots$	$333202640599.827\ldots$	$22567393309593599.681\ldots$
∞	196884	333202640600	22567393309593600

Example 7.20. The second example we consider is g in the conjugacy class 4B. In this case we have $\Gamma_g = 4||2+2 \supset \Gamma_0(8)$. The function T_g has a pole at each of the four cusps of $\Gamma_0(8)$:

1. The cusp ∞ has $e = 1$, width $t = 1$, and coefficient $\epsilon(L_\infty) = 1$.

2. The cusp 0 has $e = 2$, width $t = 8$, and coefficient $\epsilon(L_0) = 1$.

3. The cusp $1/2$ has $e = 2$, width $t = 2$, and coefficient $\epsilon(L_{1/2}) = i$.

4. The cusp $1/4$ has $e = 1$, width $t = 1$, and $\epsilon(L_{1/4}) = -1$.

Table 7.2 below contains several approximations as in Table 7.1.

Table 7.2

	$n = 1$	$n = 5$	$n = 10$
$c \leq 25$	$51.975\ldots$	$4760.372\ldots$	$0.107\ldots$
≤ 50	$52.003\ldots$	$4759.860\ldots$	$0.117\ldots$
≤ 75	$52.041\ldots$	$4760.066\ldots$	$0.092\ldots$
≤ 100	$51.894\ldots$	$4760.049\ldots$	$0.040\ldots$
∞	52	4760	0

Chapter 8

Umbral Moonshine

8.1 Proof of Theorem 1.32

We now prove the replicability formula as described in Theorem 1.32.

Proof. Fix a Niemeier lattice and its root system X, and let $M = m^X$ denote its Coxeter number. Each $H^X_{g,r}(\tau)$ is the holomorphic part of a weight $\frac{1}{2}$ harmonic Maass form $\widehat{H}^X_{g,r}(\tau)$. To simplify the exposition in the following section, we will emphasize the case that the root system X is of pure A-type. If the root system X is of pure A-type, the shadow function $S^X_{g,r}(\tau)$ is given by $\hat{\chi}^{X_A}_{g,r} S_{M,r}(\tau)$ (see §A.3.2), where

$$S_{M,r}(\tau) = \sum_{\substack{n \in \mathbb{Z} \\ m \equiv r \pmod{2M}}} n\ q^{\frac{n^2}{4M}},$$

and $\hat{\chi}^{X_A}_{g,r} = \chi^{X_A}_g$ or $\bar{\chi}^{X_A}_g$ depending on the parity of r is the twisted Euler character given in the appropriate table in §A.2.3, a character of G^X. (If X is not of pure A-type, then the shadow function $S^X_{g,r}(\tau)$ is a linear combination of similar functions as described in §A.3.2.)

Given X and g, the symbol $n_g|h_g$ given in the corresponding table in §A.2.3 defines the modularity for the vector-valued function $(\widehat{H}^X_{g,r}(\tau))$. In particular, if the shadow $(S^X_{g,r}(\tau))$ is nonzero, and if for $\gamma \in \Gamma_0(n_g)$ we have that

$$(S^X_{g,r}(\tau))|_{3/2}\gamma = \sigma_{g,\gamma}(S^X_{g,r}(\tau)),$$

then

$$(\widehat{H}^X_{g,r}(\tau))|_{1/2}\gamma = \overline{\sigma_{g,\gamma}}(\widehat{H}^X_{g,r}(\tau)).$$

Here, for $\gamma \in \Gamma_0(n_g)$, we have $\sigma_{g,\gamma} = \nu_g(\gamma)\sigma_{e,\gamma}$ where $\nu_g(\gamma)$ is a multiplier which is trivial on $\Gamma_0(n_g h_g)$. This identity holds even in the case that the shadow $S^X_{g,r}$ vanishes.

The vector-valued function $(H^X_{g,r}(\tau))$ has poles only at the infinite cusp of $\Gamma_0(n_g)$, and only at the component $H^X_{g,r}(\tau)$ where $r = 1$ if X has pure A-type, or at components where $r^2 \equiv 1 \pmod{2M}$ otherwise. These poles may only have order $\frac{1}{4M}$. This implies that the function $(\widehat{H}^X_{g,r}(\tau)S^X_{g,r}(\tau))$ has no pole at any cusp, and is therefore a candidate for an application of Theorem 2.5.

The modular transformation of $S_{M,r}(\tau)$ implies that

$$(\sigma_{e,S})^2 = (\sigma_{e,T})^{4M} = \mathbf{I}_{M-1},$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $\mathbf{I}_{M-1}$ is the $(M-1)\times(M-1)$ identity matrix. Therefore $S^X_{M,r}(\tau)$, viewed as a scalar-valued modular function, is modular on $\Gamma(4M)$, and so $(\widehat{H}^X_{g,r}(\tau)S^X_{g,r}(\tau))$ is a weight 2 nonholomorphic scalar-valued modular form for the group $\Gamma(4M) \cap \Gamma_0(n_g)$ with trivial multiplier.

Applying Theorem 2.5, we obtain a function $F^X_{g,r}(\tau)$—call it the holomorphic projection of $\widehat{H^X_{g,r}}(\tau)S^X_{e,r}(\tau)$—which is a weight 2 quasimodular form on $\Gamma(4M) \cap \Gamma_0(n_g)$. In the case that $S^X_{g,r}(\tau)$ is zero, we substitute $S^X_{e,r}(\tau)$ in its place to obtain a function $\widetilde{F}^X_{g,r}(\tau) = H^X_{g,r}(\tau)S^X_{e,r}(\tau)$ which is a weight 2 holomorphic scalar-valued modular form for the group $\Gamma(4M) \cap \Gamma_0(n_g)$ with multiplier ν_g (or alternatively, modular for the group $\Gamma(4M) \cap \Gamma_0(n_g h_g)$ with trivial multiplier).

The function $F^X_{g,r}(\tau)$ may be determined explicitly as the sum of Eisenstein series and cusp forms on $\Gamma(4M)\cap\Gamma_0(n_g h_g)$ using the standard arguments from the theory of holomorphic modular forms (i.e. the "first few" coefficients determine such a form). Therefore, we have the identity

$$F^X_{g,r}(\tau) = H^X_{g,r}(\tau) \cdot S^X_{g,r}(\tau) + D^X_{g,r}(\tau),$$

where the function $D^X_{g,r}(\tau)$ is the correction term arising in Theorem 2.5. If X has pure A-type, then

$$D^X_{g,r}(\tau) = (\hat{\chi}^{X_A}_{g,r})^2 \sum_{N=1}^{\infty} \sum_{\substack{m,n\in\mathbb{Z}_+ \\ m^2-n^2=N}} \phi_r(m)\phi_r(n)(m-n)q^{\frac{N}{4M}}, \tag{8.1}$$

where

$$\phi_r(\ell) = \begin{cases} \pm 1 & \text{if } \ell \equiv \pm r \pmod{2M} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $H^X_{g,r}(\tau) = \sum_{n=0}^{\infty} A^X_{g,r}(n)q^{n-\frac{D}{4M}}$ where $0 < D < 4M$ and $D \equiv r^2 \pmod{4M}$, and $F^X_{g,r}(\tau) = \sum_{N=0}^{\infty} B^X_{g,r}(n)q^n$. Then by Theorem 2.5, we find that

$$B^X_{g,r}(N) = \hat{\chi}^{X_A}_{g,r} \sum_{\substack{m\in\mathbb{Z} \\ m\equiv r \pmod{2M}}} m \cdot A^X_{g,r}\left(N + \frac{D-m^2}{4M}\right) + (\hat{\chi}^{X_A}_{g,r})^2 \sum_{\substack{m,n\in\mathbb{Z}^+ \\ m^2-n^2=N}} \phi_r(m)\phi_r(n)(m-n). \tag{8.2}$$

The function $F^X_{g,r}(\tau)$ may be found by considering its first few coefficients as determined using the explicit prescriptions given in §A.3.4. It may also be found exactly as a sum of Eisenstein series and cusp forms in the following manner. The Eisenstein component is determined by the constant terms at cusps. Since $D^X_{g,r}(\tau)$ (and the corresponding correction terms at other cusps) has no constant term, these are the same as the constant terms of $\widehat{H^X_{g,r}}(\tau)S^X_{g,r}(\tau)$, which are determined by the poles of $\widehat{H^X_{g,r}}$. The cuspidal component can be found by considering the order of vanishing of $\widehat{H^X_{g,r}}(\tau)S^X_{g,r}(\tau)$ at cusps.

Once the $B^X_{g,r}(N)$ are known, equation (8.2) provides a recursion relation which may be used to calculate the coefficients of $H^X_{g,r}(\tau)$. If the shadows $S^X_{g,r}(\tau)$ are zero, then we may apply a similar procedure in order to determine $\widetilde{F}^X_{g,r}(\tau)$. For example, suppose $\widetilde{F}^X_{g,r}(\tau) = \sum_{N=0}^{\infty} \widetilde{B}^X_{g,r}(n)q^n$, and X has pure A-type. Then we find that the coefficients $\widetilde{B}^X_{g,r}(N)$ satisfy

$$\widetilde{B}^X_{g,r}(N) = \hat{\chi}^{X_A}_{g,r} \sum_{\substack{m \in \mathbb{Z} \\ m \equiv s \pmod{2M}}} m \cdot A^X_{g,r}\left(N + \frac{D-m^2}{4M}\right) \tag{8.3}$$

Proceeding in this way we obtain the claimed results. □

8.2 Proof of Theorem 1.30

Here we prove Theorem 1.30. The idea is as follows. For each Niemeier root system X we begin with the vector-valued mock modular forms $(H^X_g(\tau))$ for $g \in G^X$. We use their q-expansions to solve for the q-series whose coefficients are the alleged multiplicities of the irreducible components of the alleged infinite-dimensional G^X-module

$$\check{K}^X = \bigoplus_{r \pmod{2m}} \bigoplus_{\substack{D\in\mathbb{Z},\, D\leq 0, \\ D=r^2 \pmod{4m}}} \check{K}^X_{r,-D/4m}.$$

These q-series turn out to be mock modular forms. The proof requires that we establish that these mock modular forms have non-negative integer coefficients.

Proof of Theorem 1.30. As in the previous section, we fix a root system X and set $M := m^X$, and we emphasize the case when X is of pure A-type.

The umbral moonshine conjecture asserts that

$$H^X_{g,r}(\tau) = \sum_{n=0}^{\infty} \sum_{\chi} m^X_{\chi,r}(n)\chi(g) q^{n-\frac{r^2}{4M}}$$

where the second sum is over the irreducible characters of G^X, and the $m^X_{\chi,r}(n)$ are non-negative integers which are the multiplicities of the irreducible G^X-modules in the graded components of the alleged G^X-module $\check{K}^X$. Moreover, the umbral moonshine conjecture is true if and only if the coefficients of certain weight $\frac{1}{2}$ mock modular forms are non-negative integers. Indeed, it turns out that the multiplicities $m^X_{\chi,r}(n)$ are the Fourier coefficients of

$$H^X_{\chi,r}(\tau) := \frac{1}{|G^X|} \sum_{g} \overline{\chi(g)} H^X_{g,r}(\tau) \tag{8.4}$$

if and only if the conjecture is true. To see this, we recall the orthogonality of characters. We have that for irreducible characters χ_i and χ_j,

$$\frac{1}{|G^X|} \sum_{g \in G^X} \overline{\chi_i(g)} \chi_j(g) = \begin{cases} 1 & \text{if } \chi_i = \chi_j, \\ 0 & \text{otherwise.} \end{cases} \tag{8.5}$$

We also have the relation for g and $h \in G^X$,

$$\sum_{\chi} \overline{\chi_i(g)} \chi_i(h) = \begin{cases} |C_{G^X}(g)| & \text{if } g \text{ and } h \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases} \tag{8.6}$$

Here $|C_{G^X}(g)|$ is the order of the centralizer of g in G^X. Since the order of the centralizer times the order of the conjugacy class of an element is the order of the group, (8.4) and (8.6) together imply the inverse relation

$$H^X_{g,r}(\tau) = \sum_{\chi} \chi(g) H^X_{\chi,r}(\tau).$$

These lead to the key identity

$$H^X_{\chi,r}(\tau) = \sum_{n=0}^{\infty} m^X_{\chi,r}(n) q^{n-\frac{r^2}{4M}}.$$

Therefore, in order to prove the theorem it suffices to prove that the coefficients of the mock modular forms $H^X_{\chi,r}(\tau)$ are all non-negative integers. For holomorphic modular forms, we may answer questions of this type by making use of Sturm's theorem [281] (see also Theorem 2.58 of [249]). This theorem provides a bound B associated to a space of modular forms such that a modular form $f(\tau)$ is uniquely identified by it's first B coefficients. This bound reduces many questions about the Fourier coefficients of modular forms to finite calculations. In particular, because of the existence of integral bases, this bound B may be used to show that if the first B coefficients of the form $f(\tau)$ are integral, then all coefficients of $f(\tau)$ must be integral.

Sturm's Theorem relies on the finite dimensionality of certain spaces of modular forms, and so it can not be applied directly to spaces of mock modular forms. However, by making use of holomorphic projection we can adapt Sturm's theorem to this setting.

Let $H^X_{\chi,r}(\tau)$ be defined as above. Recall that the transformation matrix for the vector-valued function $\widehat{H^X_{g,r}}(\tau))$ is $\overline{\sigma_{g,\gamma}}$, the conjugate of the transformation matrix for $(S^X_{e,r}(\tau))$ when $\gamma \in \Gamma_0(n_g h_g)$, and $\sigma_{g,\gamma}$ is the identity for $\gamma \in \Gamma(4M)$. Therefore if

$$N_\chi := \mathrm{lcm}\{n_g h_g \mid g \in G, \chi(g) \neq 0\},$$

then the scalar-valued functions $\widehat{H^X}_{\chi,r}(\tau)$ are modular on $\Gamma(4M) \cap \Gamma_0(N^X_\chi)$. Suppose that $H^X_{\chi,r}(\tau)$ has integral coefficients up to the Sturm bound for $\Gamma(4M) \cap \Gamma_0(N^X_\chi)$. Formulas for the shadow functions (cf. §A.3.2) show that the leading coefficient of $S^X_{e,1}(\tau)$ is 1 and has integral coefficients. This implies that the function

$$A_{\chi,r}(\tau) := H^X_{\chi,r}(\tau) S^X_{e,1}(\tau)$$

also has integral coefficients up to the Sturm bound for $\Gamma(4M)\cap\Gamma_0(N_\chi^X)$ and that every coefficient of $A_{\chi,r}(\tau)$ is integral if and only if every coefficient of $H^X_{\chi,r}$ is integral. The shadow of $H^X_{\chi,r}(\tau)$ is given by

$$S^X_{\chi,r}(\tau) := \frac{1}{|G^X|}\sum_g \overline{\chi(g)} S^X_{g,r}(\tau).$$

If X is pure A-type, then $S^X_{g,r}(\tau) = \chi^{X_A}_{g,r} S_{M,r}(\tau) = (\chi'(g) + \chi''(g))S_{M,r}(\tau)$ for some irreducible characters χ' and χ'', according to §A.2.3 and §A.3.2. Therefore,

$$S^X_{\chi,r}(\tau) = \begin{cases} S_{M,r}(\tau) & \text{if } \chi = \chi' \text{ or } \chi'', \\ 0 & \text{otherwise.} \end{cases}$$

When X is not of pure A-type the shadow is some sum of such functions, but in every case has integer coefficients, and so, applying Theorem 2.5 to $A_{\chi,r}(\tau)$, we find that the holomorphic projection of this function has only integer coefficients if and only if $A_{\chi,r}(\tau)$ has only integer coefficients. But the holomorphic projection is modular on $\Gamma(4M)\cap\Gamma_0(N_\chi^X)$ and has integer coefficients up to the Sturm bound for $\Gamma(4M)\cap\Gamma_0(N_\chi^X)$. Therefore, in order to check that $H^X_{\chi,r}(\tau)$ has only integer coefficients, it suffices to check up to the Sturm bound for $\Gamma(4M)\cap\Gamma_0(N_\chi)$. These calculations were carried out using the `sage` mathematical software [277].

To complete the proof, it suffices to check that the multiplicities $m^X_{\chi,r}(n)$ are non-negative. The proof of this claim follows easily by modifying step-by-step the argument in Gannon's proof of non negativity in the M_{24} case [136] (i.e. $X = A_1^{24}$). Here we describe how this is done.

Conjectural expressions for the alleged McKay-Thompson series $H^X_{g,r}(\tau)$ in terms of Rademacher sums and unary theta functions are given in §A.3.3. These expressions are known to hold in many cases, but in any case, the difference between $H^X_{g,r}(\tau)$ and the corresponding Rademacher sum (cf. (A.27), (A.28)) is a unary theta function of bounded level, according to the Serre–Stark theorem [271] on modular forms of weight $\frac{1}{2}$. The unary theta functions have bounded coefficients, and so the non-negativity depends on the asymptotic growth of the coefficients of the Rademacher sums.

Exact formulas are known for all the coefficients of Rademacher sums because they are defined by averaging the special function $\mathrm{r}^{[\alpha]}_{1/2}(\gamma,\tau)$ (see (A.23)) over cosets of a specific modular group modulo Γ_∞, the subgroup of translations. Therefore, Rademacher sums are standard Maass-Poincaré series, and as a result we have formulas for each of their coefficients as convergent infinite sums of Kloosterman-type sums weighted by values of the $I_{1/2}$ modified Bessel function. (For example, see [45] for the general theory, and [73] for the specific case that $X = A_1^{24}$.) More importantly, this means also that the generating function for the multiplicities $m^X_{\chi,r}(n)$ is a weight $\frac{1}{2}$ harmonic Maass form, which in turn means that exact formulas (modulo the unary theta functions) are also available in similar terms. For positive integers n, this then means that (cf. Theorem 1.1 of [45])

$$m^X_{\chi,r}(n) = \sum_{\rho}\sum_{m<0} \frac{a^X_\rho(m)}{n^{\frac{1}{4}}} \sum_{c=1}^{\infty} \frac{K^X_\rho(m,n,c)}{c} \cdot \mathbb{I}^X\left(\frac{4\pi\sqrt{|nm|}}{c}\right), \tag{8.7}$$

where the sums are over the cusps ρ of the group $\Gamma_0(N^X_g)$, and finitely many explicit negative rational numbers m. The constants $a^X_\rho(m)$ are essentially the coefficients which describe the generating function in terms of Maass-Poincaré series. Here $\mathbb{I}$ is a suitable normalization and change of variable for the standard $I_{1/2}$ modified Bessel-function.

The Kloosterman-type sums $K^X_\rho(m,n,c)$ are well known to be related to Salié-type sums (for example see Proposition 5 of [194]). These Salié-type sums are of the form

$$S^X_\rho(m,n,c) = \sum_{\substack{x \,(\mathrm{mod}\, c)\\ x^2 \equiv -D(m,n) (\mathrm{mod}\, c)}} \epsilon^X_\rho(m,n)\cdot e\left(\frac{\beta^X x}{c}\right),$$

where $\epsilon^X_\rho(m,n)$ is a root of unity, $-D(m,n)$ is a discriminant of a positive definite binary quadratic form, and β^X is a nonzero positive rational number.

These Salié sums may then be estimated using the equidistribution of CM points with discriminant $-D(m,n)$. This process was first introduced by Hooley [165], and it was first applied to the coefficients of weight $\frac{1}{2}$ mock modular forms by Bringmann and Ono [41]. Gannon explains how to make effective the estimates for sums of this shape in §4 of [136], thereby reducing the proof of the M_{24} case of umbral moonshine to a finite calculation. In particular, in equations (4.6-4.10) of [136] Gannon shows how to bound coefficients of the form (8.7) in terms of the Selberg–Kloosterman zeta function, which is bounded in turn in his proof of Theorem 3 of [136]. We follow Gannon's proof *mutatis mutandis*. We find, for each root system, that the coefficients of each multiplicity generating function are positive beyond the 390th coefficient. Moreover, the coefficients exhibit subexponential growth. A finite computer calculation in `sage` has verified the non-negativity of the finitely many remaining coefficients. □

Remark. It turns out that the estimates required for proving nonnegativity are the worst for the M_{24} case considered by Gannon.

Appendix

A.1 Monstrous Groups

The table below contains the symbols $\Gamma_g = N||h{+}e, f, \ldots,$ for each conjugacy class of the monster. Following [84], if $h = 1$, we omit the '$||1$' from the symbol. If $\mathcal{W}_g = \{1\}$, then we write $N||h$, whereas if it contains every exact divisor of N/h, we write $N||h+$.

1A	1	12C	$12\|\|2+$	21D	$21+21$
2A	$2+$	12D	$12\|\|3+$	22A	$22+$
2B	2	12E	$12+3$	22B	$22+11$
3A	$3+$	12F	$12\|\|2+6$	23AB	$23+$
3B	3	12G	$12\|\|2+2$	24A	$24\|\|2+$
3C	$3\|\|3$	12H	$12+12$	24B	$24+$
4A	$4+$	12I	12	24C	$24+8$
4B	$4\|\|2+$	12J	$12\|\|6$	24D	$24\|\|2+3$
4C	4	13A	$13+$	24E	$24\|\|6+$
4D	$4\|\|2$	13B	13	24F	$24\|\|4+6$
5A	$5+$	14A	$14+$	24G	$24\|\|4+2$
5B	5	14B	$14+7$	24H	$24\|\|2+12$
6A	$6+$	14C	$14+14$	24I	$24+24$
6B	$6+6$	15A	$15+$	24J	$24\|\|12$
6C	$6+3$	15B	$15+5$	25A	$25+$
6D	$6+2$	15C	$15+15$	26A	$26+$
6E	6	15D	$15\|\|3$	26B	$26+26$
6F	$6\|\|3$	16A	$16\|\|2+$	27A	$27+$
7A	$7+$	16B	16	27B	$27+$
7B	7	16C	$16+$	28A	$28\|\|2+$
8A	$8+$	17A	$17+$	28B	$28+$
8B	$8\|\|2+$	18A	$18+2$	28C	$28+7$

34A	34+	46CD	46+	68A	$68\|\|2+$
35A	35+	47AB	47+	69AB	69+
35B	$35+35$	48A	$48\|\|2+$	70A	70+
36A	36+	50A	50+	70B	$70+10,14,35$
36B	$36+4$	51A	51+	71AB	71+
36C	$36\|\|2+$	52A	$52\|\|2+$	78A	78+
36D	$36+36$	52B	$52\|\|2+26$	78BC	$78+6,26,39$
38A	38+	54A	54+	84A	$84\|\|2+$
39A	39+	55A	55+	84B	$84\|\|2+6,14,21$
39B	$39\|\|3+$	56A	56+	84C	$84\|\|3+$
39CD	$39+39$	56BC	$56\|\|4+14$	87AB	87+
40A	$40\|\|4+$	57A	$57\|\|3+$	88AB	$88\|\|2+$
40B	$40\|\|2+$	59AB	59+	92AB	92+
40CD	$40\|\|2+20$	60A	$60\|\|2+$	93AB	$93\|\|3+$
41A	41+	60B	60+	94AB	94+
42A	42+	60C	$60+4,15,60$	95AB	95+
42B	$42+6,14,21$	60D	$60+12,15,20$	104AB	$104\|\|4+$
42C	$42\|\|3+7$	60E	$60\|\|2+5,6,30$	105A	105+
42D	$42+3,14,42$	60F	$60\|\|6+10$	110A	110+
44AB	44+	62AB	62+	119AB	119+
45A	45+	66A	66+		
46AB	$46+23$	66B	$66+6,11,66$		

A.2 The Umbral Groups

In this section we present the facts about the umbral groups that we have used in establishing the main results of this paper. We recall (from [68]) their construction in terms of Niemeier root systems in §A.2.1, and we reproduce their character tables (appearing also in [68]) in §A.2.2. Note that we use the abbreviations $a_n := \sqrt{-n}$ and $b_n := (-1 + \sqrt{-n})/2$ in the tables of §A.2.2.

The root system description of the umbral groups (cf. §A.2.1) gives rise to certain characters called *twisted Euler characters* which we recall (from [68]) in §A.2.3. The data appearing in §A.2.3 plays an important role in §A.3.2, where we use it to describe the shadows S_g^X of the umbral McKay-Thompson series H_g^X explicitly.

A.2.1 Construction

As mentioned in the introduction, there are exactly 24 self-dual even positive-definite lattices of rank 24 up to isomorphism, according to the classification of Niemeier [244] (cf. also [82, 291]). Such a lattice L is determined up to isomorphism by its *root system* $L_2 := \{\alpha \in L \mid \langle \alpha, \alpha \rangle = 2\}$. The unique example without roots is the Leech lattice. We refer to the remaining 23 as the *Niemeier lattices*, and we call a root system X a *Niemeier root system* if it occurs as the root system of a Niemeier lattice.

The simple components of Niemeier root systems are root systems of ADE type, and it turns out that the simple components of a Niemeier root system X all have the same Coxeter number. Define m^X to be the Coxeter number of any simple component of X, and call this the *Coxeter number* of X.

For X a Niemeier root system write N^X for the corresponding Niemeier lattice. The *umbral group* attached to X is defined by setting

$$G^X := \mathrm{Aut}(N^X)/W^X \tag{A.1}$$

where W^X is the normal subgroup of $\mathrm{Aut}(N^X)$ generated by reflections in root vectors.

Observe that G^X acts as permutations on the simple components of X. In general this action is not faithful, so define $\overline{G}^X$ to be the quotient of G^X by its kernel. It turns out that the level of the mock modular form H_g^X attached to $g \in G^X$ is given by the order, denoted n_g, of the image of g in $\bar{G}^X$. (Cf. §A.2.3 for the values n_g.)

The Niemeier root systems and their corresponding umbral groups are described in Table A.1. The root systems are given in terms of their simple components of ADE type. Here $D_{10}E_7^2$, for example, means the direct sum of one copy of the D_{10} root system and two copies of the E_7 root system. The symbol ℓ is called the *lambency* of X, and the Coxeter number m^X appears as the first summand of ℓ.

In the descriptions of the umbral groups G^X, and their permutation group quotients $\bar{G}^X$, we write M_{24} and M_{12} for the sporadic simple groups of Mathieu which act quintuply transitively on 24 and 12 points, respectively. (Cf. e.g. [80].) We write $GL_n(q)$ for the general linear group of a vector space of dimension n over a field with q elements, and $SL_n(q)$ is the subgroup of linear transformations with determinant 1, &c. The symbols $AGL_3(2)$ denote the *affine general linear group*, obtained by adjoining translations to $GL_3(2)$. We write Dih_n for the dihedral group of order $2n$, and Sym_n denotes the symmetric group on n symbols. We use n as a shorthand for a cyclic group of order n.

We also use the notational convention of writing $A.B$ to denote the middle term in a short exact sequence $1 \to A \to A.B \to B \to 1$. This introduces some ambiguity which is nonetheless easily navigated in practice. For example, $2.M_{12}$ is the unique (up to isomorphism) double cover of M_{12} which is not $2 \times M_{12}$. The group $AGL_3(2)$ naturally embeds in $GL_4(2)$, which in turn admits a unique (up to isomorphism) double cover $2.GL_4(2)$ which is not a direct product. The group we denote $2.AGL_3(2)$ is the preimage of $AGL_3(2) < GL_4(2)$ in $2.GL_4(2)$ under the natural projection.

Table A.1: The Umbral Groups

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	$A_7^2D_5^2$
$\ll$	2	3	4	5	6	7	8
G^X	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4
$\bar{G}^X$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2

X	A_8^3	$A_9^2D_6$	$A_{11}D_7E_6$	A_{12}^2	$A_{15}D_9$	$A_{17}E_7$	A_{24}
$\ll$	9	10	12	13	16	18	25
G^X	Dih_6	4	2	4	2	2	2
$\bar{G}^X$	Sym_3	2	1	2	1	1	1

X	D_4^6	D_6^4	D_8^3	$D_{10}E_7^2$	D_{12}^2	$D_{16}E_8$	D_{24}
$\ll$	6+3	10+5	14+7	18+9	22+11	30+15	46+23
G^X	$3.Sym_6$	Sym_4	Sym_3	2	2	1	1
$\bar{G}^X$	Sym_6	Sym_4	Sym_3	2	2	1	1

X	E_6^4	E_8^3
$\ll$	12+4	30+6,10,15
G^X	$GL_2(3)$	Sym_3
$\bar{G}^X$	$PGL_2(3)$	Sym_3

A.2.2 Character Tables

Table A.2: Character table of $G^X \simeq M_{24}$, $X = A_1^{24}$

$[g]$	FS	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	23A	23B
$[g^2]$		1A	1A	1A	3A	3B	2A	2A	2B	5A	3A	3B	7A	7B	4B	5A	11A	6A	6B	7A	7B	15A	15B	21A	21B	23A	23B
$[g^3]$		1A	2A	2B	1A	1A	4A	4B	4C	5A	2A	2B	7B	7A	8A	10A	11A	4A	4C	14B	14A	5A	5A	7B	7A	23A	23B
$[g^5]$		1A	2A	2B	3A	3B	4A	4B	4C	1A	6A	6B	7B	7A	8A	2B	11A	12A	12B	14B	14A	3A	3A	21B	21A	23B	23A
$[g^7]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	1A	1A	8A	10A	11A	12A	12B	2A	2A	15B	15A	3B	3B	23B	23A
$[g^{11}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	1A	12A	12B	14A	14B	15B	15A	21A	21B	23B	23A
$[g^{23}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	1A	1A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	23	7	−1	5	−1	−1	3	−1	3	1	−1	2	2	1	−1	1	−1	−1	0	0	0	0	−1	−1	0	0
χ_3	∘	45	−3	5	0	3	−3	1	1	0	0	−1	b_7	$\overline{b_7}$	−1	0	1	0	1	$-b_7$	$-\overline{b_7}$	0	0	b_7	$\overline{b_7}$	−1	−1
χ_4	∘	45	−3	5	0	3	−3	1	1	0	0	−1	$\overline{b_7}$	b_7	−1	0	1	0	1	$-\overline{b_7}$	$-b_7$	0	0	$\overline{b_7}$	b_7	−1	−1
χ_5	∘	231	7	−9	−3	0	−1	−1	3	1	1	0	0	0	−1	1	0	−1	0	0	0	b_{15}	$\overline{b_{15}}$	0	0	1	1
χ_6	∘	231	7	−9	−3	0	−1	−1	3	1	1	0	0	0	−1	1	0	−1	0	0	0	$\overline{b_{15}}$	b_{15}	0	0	1	1
χ_7	+	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	−1	1	0	0	0	−1	−1	0	0	−1	−1
χ_8	+	253	13	−11	10	1	−3	1	1	3	−2	1	1	1	−1	−1	0	0	1	−1	−1	0	0	1	1	0	0
χ_9	+	483	35	3	6	0	3	3	3	−2	2	0	0	0	−1	−2	−1	0	0	0	0	1	1	0	0	0	0
χ_{10}	∘	770	−14	10	5	−7	2	−2	−2	0	1	1	0	0	0	0	0	−1	1	0	0	0	0	0	0	b_{23}	$\overline{b_{23}}$
χ_{11}	∘	770	−14	10	5	−7	2	−2	−2	0	1	1	0	0	0	0	0	−1	1	0	0	0	0	0	0	$\overline{b_{23}}$	b_{23}
χ_{12}	∘	990	−18	−10	0	3	6	2	−2	0	0	−1	b_7	$\overline{b_7}$	0	0	0	0	1	b_7	$\overline{b_7}$	0	0	b_7	$\overline{b_7}$	1	1
χ_{13}	∘	990	−18	−10	0	3	6	2	−2	0	0	−1	$\overline{b_7}$	b_7	0	0	0	0	1	$\overline{b_7}$	b_7	0	0	$\overline{b_7}$	b_7	1	1
χ_{14}	+	1035	27	35	0	6	3	−1	3	0	0	2	−1	−1	1	0	1	0	0	−1	−1	0	0	−1	−1	0	0
χ_{15}	∘	1035	−21	−5	0	−3	3	3	−1	0	0	1	$2b_7$	$2\overline{b_7}$	−1	0	1	0	−1	0	0	0	0	$-b_7$	$-\overline{b_7}$	0	0
χ_{16}	∘	1035	−21	−5	0	−3	3	3	−1	0	0	1	$2\overline{b_7}$	$2b_7$	−1	0	1	0	−1	0	0	0	0	$-\overline{b_7}$	$-b_7$	0	0
χ_{17}	+	1265	49	−15	5	8	−7	1	−3	0	1	0	−2	−2	1	0	0	−1	0	0	0	0	0	1	1	0	0
χ_{18}	+	1771	−21	11	16	7	3	−5	−1	1	0	−1	0	0	−1	1	0	0	−1	0	0	1	1	0	0	0	0
χ_{19}	+	2024	8	24	−1	8	8	0	0	−1	−1	0	1	1	0	−1	0	−1	0	1	1	−1	−1	1	1	0	0
χ_{20}	+	2277	21	−19	0	6	−3	1	−3	−3	0	2	2	2	−1	1	0	0	0	0	0	0	0	−1	−1	0	0
χ_{21}	+	3312	48	16	0	−6	0	0	0	−3	0	−2	1	1	0	1	1	0	0	−1	−1	0	0	1	1	0	0
χ_{22}	+	3520	64	0	10	−8	0	0	0	0	−2	0	−1	−1	0	0	0	0	0	1	1	0	0	−1	−1	1	1
χ_{23}	+	5313	49	9	−15	0	1	−3	−3	3	1	0	0	0	−1	−1	0	1	0	0	0	0	0	0	0	0	0
χ_{24}	+	5544	−56	24	9	0	−8	0	0	−1	1	0	0	0	0	−1	0	1	0	0	0	−1	−1	0	0	1	1

Table A.3: Character table of $G^X \simeq 2.M_{12}$, $X = A_2^{12}$

$[g]$	FS	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^2]$		1A	1A	2A	1A	1A	3A	3A	3B	3B	2B	2B	5A	5A	6B	3A	3A	4B	4B	4C	4C	10A	10A	11B	11B	11A	11A
$[g^3]$		1A	2A	4A	2B	2C	1A	2A	1A	2A	4B	4C	5A	10A	4A	2B	2C	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^5]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	1A	2A	12A	6C	6D	8B	8A	8D	8C	4A	4A	11A	22A	11B	22B
$[g^{11}]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20B	20A	1A	2A	1A	2A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	11	11	−1	3	3	2	2	−1	−1	−1	3	1	1	−1	0	0	−1	−1	1	1	−1	−1	0	0	0	0
χ_3	+	11	11	−1	3	3	2	2	−1	−1	3	−1	1	1	−1	0	0	1	1	−1	−1	−1	−1	0	0	0	0
χ_4	∘	16	16	4	0	0	−2	−2	1	1	0	0	1	1	1	0	0	0	0	0	0	−1	−1	b_{11}	b_{11}	$\overline{b_{11}}$	$\overline{b_{11}}$
χ_5	∘	16	16	4	0	0	−2	−2	1	1	0	0	1	1	1	0	0	0	0	0	0	−1	−1	$\overline{b_{11}}$	$\overline{b_{11}}$	b_{11}	b_{11}
χ_6	+	45	45	5	−3	−3	0	0	3	3	1	1	0	0	−1	0	0	−1	−1	−1	−1	0	0	1	1	1	1
χ_7	+	54	54	6	6	6	0	0	0	0	2	2	−1	−1	0	0	0	0	0	0	0	1	1	−1	−1	−1	−1
χ_8	+	55	55	−5	7	7	1	1	1	1	−1	−1	0	0	1	1	1	−1	−1	−1	−1	0	0	0	0	0	0
χ_9	+	55	55	−5	−1	−1	1	1	1	1	3	−1	0	0	1	−1	−1	−1	−1	1	1	0	0	0	0	0	0
χ_{10}	+	55	55	−5	−1	−1	1	1	1	1	−1	3	0	0	1	−1	−1	1	1	−1	−1	0	0	0	0	0	0
χ_{11}	+	66	66	6	2	2	3	3	0	0	−2	−2	1	1	0	−1	−1	0	0	0	0	1	1	0	0	0	0
χ_{12}	+	99	99	−1	3	3	0	0	3	3	−1	−1	−1	−1	−1	0	0	1	1	1	1	−1	−1	0	0	0	0
χ_{13}	+	120	120	0	−8	−8	3	3	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	−1	−1	−1	−1
χ_{14}	+	144	144	4	0	0	0	0	−3	−3	0	0	−1	−1	1	0	0	0	0	0	0	−1	−1	1	1	1	1
χ_{15}	+	176	176	−4	0	0	−4	−4	−1	−1	0	0	1	1	−1	0	0	0	0	0	0	1	1	0	0	0	0
χ_{16}	∘	10	−10	0	−2	2	1	−1	−2	2	0	0	0	0	0	1	−1	a_2	$\overline{a_2}$	a_2	$\overline{a_2}$	0	0	−1	1	−1	1
χ_{17}	∘	10	−10	0	−2	2	1	−1	−2	2	0	0	0	0	0	1	−1	$\overline{a_2}$	a_2	$\overline{a_2}$	a_2	0	0	−1	1	−1	1
χ_{18}	+	12	−12	0	4	−4	3	−3	0	0	0	0	2	−2	0	1	−1	0	0	0	0	0	0	1	−1	1	−1
χ_{19}	−	32	−32	0	0	0	−4	4	2	−2	0	0	2	−2	0	0	0	0	0	0	0	0	0	−1	1	−1	1
χ_{20}	∘	44	−44	0	4	−4	−1	1	2	−2	0	0	−1	1	0	1	−1	0	0	0	0	a_5	$\overline{a_5}$	0	0	0	0
χ_{21}	∘	44	−44	0	4	−4	−1	1	2	−2	0	0	−1	1	0	1	−1	0	0	0	0	$\overline{a_5}$	a_5	0	0	0	0
χ_{22}	∘	110	−110	0	−6	6	2	−2	2	−2	0	0	0	0	0	0	0	a_2	$\overline{a_2}$	$\overline{a_2}$	a_2	0	0	0	0	0	0
χ_{23}	∘	110	−110	0	−6	6	2	−2	2	−2	0	0	0	0	0	0	0	$\overline{a_2}$	a_2	a_2	$\overline{a_2}$	0	0	0	0	0	0
χ_{24}	+	120	−120	0	8	−8	3	−3	0	0	0	0	0	0	0	−1	1	0	0	0	0	0	0	−1	1	−1	1
χ_{25}	∘	160	−160	0	0	0	−2	2	−2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$-b_{11}$	b_{11}	$-\overline{b_{11}}$	$\overline{b_{11}}$

Table A.4: Character table of $G^X \simeq 2.AGL_3(2)$, $X = A_3^8$

$[g]$	FS	1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	7A	14A	7B	14B
$[g^2]$		1A	1A	1A	2A	2B	1A	3A	3A	3A	3A	4A	2C	7A	7A	7B	7B
$[g^3]$		1A	2A	2B	4A	4B	2C	1A	2A	2B	2B	8A	4C	7B	14B	7A	14A
$[g^7]$		1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	1A	2A	1A	2A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	∘	3	3	3	-1	-1	-1	0	0	0	0	1	1	b_7	b_7	$\overline{b_7}$	$\overline{b_7}$
χ_3	∘	3	3	3	-1	-1	-1	0	0	0	0	1	1	$\overline{b_7}$	$\overline{b_7}$	b_7	b_7
χ_4	+	6	6	6	2	2	2	0	0	0	0	0	0	-1	-1	-1	-1
χ_5	+	7	7	7	-1	-1	-1	1	1	1	1	-1	-1	0	0	0	0
χ_6	+	8	8	8	0	0	0	-1	-1	-1	-1	0	0	1	1	1	1
χ_7	+	7	7	-1	3	-1	-1	1	1	-1	-1	1	-1	0	0	0	0
χ_8	+	7	7	-1	-1	-1	3	1	1	-1	-1	-1	1	0	0	0	0
χ_9	+	14	14	-2	2	-2	2	-1	-1	1	1	0	0	0	0	0	0
χ_{10}	+	21	21	-3	1	1	-3	0	0	0	0	-1	1	0	0	0	0
χ_{11}	+	21	21	-3	-3	1	1	0	0	0	0	1	-1	0	0	0	0
χ_{12}	+	8	-8	0	0	0	0	2	-2	0	0	0	0	1	-1	1	-1
χ_{13}	∘	8	-8	0	0	0	0	-1	1	a_3	$\overline{a_3}$	0	0	1	-1	1	-1
χ_{14}	∘	8	-8	0	0	0	0	-1	1	$\overline{a_3}$	a_3	0	0	1	-1	1	-1
χ_{15}	∘	24	-24	0	0	0	0	0	0	0	0	0	0	$\overline{b_7}$	$-\overline{b_7}$	b_7	$-b_7$
χ_{16}	∘	24	-24	0	0	0	0	0	0	0	0	0	0	b_7	$-b_7$	$\overline{b_7}$	$-\overline{b_7}$

Table A.5: Character table of $G^X \simeq GL_2(5)/2$, $X = A_4^6$

$[g]$	FS	1A	2A	2B	2C	3A	6A	5A	10A	4A	4B	4C	4D	12A	12B
$[g^2]$		1A	1A	1A	1A	3A	3A	5A	5A	2A	2A	2C	2C	6A	6A
$[g^3]$		1A	2A	2B	2C	1A	2A	5A	10A	4B	4A	4D	4C	4B	4A
$[g^5]$		1A	2A	2B	2C	3A	6A	1A	2A	4A	4B	4C	4D	12A	12B
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
χ_3	+	4	4	0	0	1	1	-1	-1	2	2	0	0	-1	-1
χ_4	+	4	4	0	0	1	1	-1	-1	-2	-2	0	0	1	1
χ_5	+	5	5	1	1	-1	-1	0	0	1	1	-1	-1	1	1
χ_6	+	5	5	1	1	-1	-1	0	0	-1	-1	1	1	-1	-1
χ_7	+	6	6	-2	-2	0	0	1	1	0	0	0	0	0	0
χ_8	∘	1	-1	1	-1	1	-1	1	-1	a_1	$-a_1$	a_1	$-a_1$	a_1	$-a_1$

Table A.6: Character table of $G^X \simeq GL_2(3)$, $X \in \{A_5^4 D_4, E_6^4\}$

$[g]$	FS	1A	2A	2B	4A	3A	6A	8A	8B
$[g^2]$		1A	1A	1A	2A	3A	3A	4A	4A
$[g^3]$		1A	2A	2B	4A	1A	2A	8A	8B
χ_1	$+$	1	1	1	1	1	1	1	1
χ_2	$+$	1	1	-1	1	1	1	-1	-1
χ_3	$+$	2	2	0	2	-1	-1	0	0
χ_4	$+$	3	3	-1	-1	0	0	1	1
χ_5	$+$	3	3	1	-1	0	0	-1	-1
χ_6	$\circ$	2	-2	0	0	-1	1	a_2	$\overline{a_2}$
χ_7	$\circ$	2	-2	0	0	-1	1	$\overline{a_2}$	a_2
χ_8	$+$	4	-4	0	0	1	-1	0	0

Table A.7: Character table of $G^X \simeq 3.Sym_6$, $X = D_4^6$

$[g]$	FS	1A	3A	2A	6A	3B	3C	4A	12A	5A	15A	15B	2B	2C	4B	6B	6C
$[g^2]$		1A	3A	1A	3A	3B	3C	2A	6A	5A	15A	15B	1A	1A	2A	3B	3C
$[g^3]$		1A	1A	2A	2A	1A	1A	4A	4A	5A	5A	5A	2B	2C	4B	2B	2C
$[g^5]$		1A	3A	2A	6A	3B	3C	4A	12A	1A	3A	3A	2B	2C	4B	6B	6C
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1
χ_3	+	5	5	1	1	2	-1	-1	-1	0	0	0	3	-1	1	0	-1
χ_4	+	5	5	1	1	2	-1	-1	-1	0	0	0	-3	1	-1	0	1
χ_5	+	5	5	1	1	-1	2	-1	-1	0	0	0	-1	3	1	-1	0
χ_6	+	5	5	1	1	-1	2	-1	-1	0	0	0	1	-3	-1	1	0
χ_7	+	16	16	0	0	-2	-2	0	0	1	1	1	0	0	0	0	0
χ_8	+	9	9	1	1	0	0	1	1	-1	-1	-1	3	3	-1	0	0
χ_9	+	9	9	1	1	0	0	1	1	-1	-1	-1	-3	-3	1	0	0
χ_{10}	+	10	10	-2	-2	1	1	0	0	0	0	0	2	-2	0	-1	1
χ_{11}	+	10	10	-2	-2	1	1	0	0	0	0	0	-2	2	0	1	-1
χ_{12}	$\circ$	6	-3	-2	1	0	0	2	-1	1	b_{15}	$\overline{b_{15}}$	0	0	0	0	0
χ_{13}	$\circ$	6	-3	-2	1	0	0	2	-1	1	$\overline{b_{15}}$	b_{15}	0	0	0	0	0
χ_{14}	+	12	-6	4	-2	0	0	0	0	2	-1	-1	0	0	0	0	0
χ_{15}	+	18	-9	2	-1	0	0	2	-1	-2	1	1	0	0	0	0	0
χ_{16}	+	30	-15	-2	1	0	0	-2	1	0	0	0	0	0	0	0	0

Table A.8: Character table of $G^X \simeq SL_2(3)$, $X = A_6^4$

$[g]$	FS	1A	2A	4A	3A	6A	3B	6B
$[g^2]$		1A	1A	2A	3B	3A	3A	3B
$[g^3]$		1A	2A	4A	1A	2A	1A	2A
χ_1	$+$	1	1	1	1	1	1	1
χ_2	$\circ$	1	1	1	b_3	$\overline{b_3}$	$\overline{b_3}$	b_3
χ_3	$\circ$	1	1	1	$\overline{b_3}$	b_3	b_3	$\overline{b_3}$
χ_4	$+$	3	3	-1	0	0	0	0
χ_5	$-$	2	-2	0	-1	1	-1	1
χ_6	$\circ$	2	-2	0	$-\overline{b_3}$	b_3	$-b_3$	$\overline{b_3}$
χ_7	$\circ$	2	-2	0	$-b_3$	$\overline{b_3}$	$-\overline{b_3}$	b_3

Table A.9: Character table of $G^X \simeq Dih_4$, $X = A_7^2 D_5^2$

$[g]$	FS	1A	2A	2B	2C	4A
$[g^2]$		1A	1A	1A	1A	2A
χ_1	$+$	1	1	1	1	1
χ_2	$+$	1	1	-1	-1	1
χ_3	$+$	1	1	-1	1	-1
χ_4	$+$	1	1	1	-1	-1
χ_5	$+$	2	-2	0	0	0

Table A.10: Character table of $G^X \simeq Dih_6$, $X = A_8^3$

$[g]$	FS	1A	2A	2B	2C	3A	6A
$[g^2]$		1A	1A	1A	1A	3A	3A
$[g^3]$		1A	2A	2B	2C	1A	2A
χ_1	$+$	1	1	1	1	1	1
χ_2	$+$	1	1	-1	-1	1	1
χ_3	$+$	2	2	0	0	-1	-1
χ_4	$+$	1	-1	-1	1	1	-1
χ_5	$+$	1	-1	1	-1	1	-1
χ_6	$+$	2	-2	0	0	-1	1

Table A.11: Character table of $G^X \simeq 4$, for $X \in \{A_9^2 D_6, A_{12}^2\}$

$[g]$	FS	1A	2A	4A	4B
$[g^2]$		1A	1A	2A	2A
χ_1	$+$	1	1	1	1
χ_2	$+$	1	1	-1	-1
χ_3	$\circ$	1	-1	a_1	$\overline{a_1}$
χ_4	$\circ$	1	-1	$\overline{a_1}$	a_1

Table A.12: Character table of $G^X \simeq PGL_2(3) \simeq Sym_4$, $X = D_6^4$

$[g]$	FS	1A	2A	3A	2B	4A
$[g^2]$		1A	1A	3A	1A	2A
$[g^3]$		1A	2A	1A	2B	4A
χ_1	+	1	1	1	1	1
χ_2	+	1	1	1	−1	−1
χ_3	+	2	2	−1	0	0
χ_4	+	3	−1	0	1	−1
χ_5	+	3	−1	0	−1	1

Table A.13: Character table of $G^X \simeq 2$, for $X \in \{A_{11}D_7E_6, A_{15}D_9, A_{17}E_7, A_{24}, D_{10}E_7^2, D_{12}^2\}$

$[g]$	FS	1A	2A
$[g^2]$		1A	1A
χ_1	+	1	1
χ_2	+	1	−1

Table A.14: Character table of $G^X \simeq Sym_3$, $X \in \{D_8^3, E_8^3\}$

$[g]$	FS	1A	2A	3A
$[g^2]$		1A	1A	3A
$[g^3]$		1A	2A	1A
χ_1	+	1	1	1
χ_2	+	1	−1	1
χ_3	+	2	0	−1

A.2.3 Twisted Euler Characters

In this section we reproduce certain characters—the *twisted Euler characters*—which are attached to each group G^X, via its action on the root system X. (Their construction is described in detail in §2.4 of [68].)

To interpret the tables, write X_A for the (possibly empty) union of type A components of X, and interpret X_D and X_E similarly, so that if $m = m^X$ Then $X = A_{m-1}^d$ for some d, and $X = X_A \cup X_D \cup X_E$, for example. Then $g \mapsto \bar{\chi}_g^{X_A}$ denotes the character of the permutation representation attached to the action of $\bar{G}^X$ on the simple components of X_A. The characters $g \mapsto \bar{\chi}_g^{X_D}$ and $g \mapsto \bar{\chi}_g^{X_E}$ are defined similarly. The characters $\chi_g^{X_A}$, $\chi_g^{X_D}$, $\chi_g^{X_E}$ and $\check{\chi}_g^{X_D}$ incorporate outer automorphisms of simple root systems induced by the action G^X on X. We refer to §2.4 of [68] for full details of the construction. For the purposes of this work, it suffices to have the explicit descriptions in the tables in this section. The twisted Euler characters presented here will be used to specify the umbral shadow functions in §A.3.2.

The twisted Euler character tables also attach integers n_g and h_g to each $g \in G^X$. By definition, n_g is the order of the image of $g \in G^X$ in $\bar{G}^X$ (cf. §A.2.1). The integer h_g may be defined by setting $h_g := N_g/n_g$ where N_g is the product of the shortest and longest cycle lengths appearing in the cycle shape attached to g by the action of G^X on a (suitable) set of simple roots for X.

Table A.15: Twisted Euler characters at $\ll = 2$, $X = A_1^{24}$

$[g]$	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B
$n_g\|h_g$	1|1	2|1	2|2	3|1	3|3	4|2	4|1	4|4	5|1	6|1	6|6
$\bar{\chi}_g^{X_A}$	24	8	0	6	0	0	4	0	4	2	0
$[g]$	7AB	8A	10A	11A	12A	12B	14AB	15AB	21AB	23AB	
$n_g\|h_g$	7|1	8|1	10|2	11|1	12|2	12|12	14|1	15|1	21|3	23|1	
$\bar{\chi}_g^{X_A}$	3	2	0	2	0	0	1	1	0	1	

Table A.16: Twisted Euler characters at $\ll = 3$, $X = A_2^{12}$

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB
$n_g\|h_g$	1|1	1|4	2|8	2|1	2|2	3|1	3|4	3|3	3|12	4|2	4|1	5|1	5|4	6|24	6|1	6|2	8|4	8|1	10|8	11|1	11|4
$\bar{\chi}_g^{X_A}$	12	12	0	4	4	3	3	0	0	0	4	2	2	0	1	1	0	2	0	1	1
$\chi_g^{X_A}$	12	−12	0	4	−4	3	−3	0	0	0	0	2	−2	0	1	−1	0	0	0	1	−1

Table A.17: Twisted Euler characters at $\ll = 4$, $X = A_3^{8}$

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
$n_g\|h_g$	1|1	1|2	2|2	2|4	4|4	2|1	3|1	3|2	6|2	4|8	4|1	7|1	7|2
$\bar{\chi}_g^{X_A}$	8	8	0	0	0	4	2	2	0	0	2	1	1
$\chi_g^{X_A}$	8	−8	0	0	0	0	2	−2	0	0	0	1	−1

Table A.18: Twisted Euler characters at $\ll = 5$, $X = A_4^6$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
$n_g\|h_g$	1|1	1|4	2|2	2|1	3|3	3|12	5|1	5|4	2|8	4|1	6|24
$\bar{\chi}_g^{X_A}$	6	6	2	2	0	0	1	1	0	2	0
$\chi_g^{X_A}$	6	−6	−2	2	0	0	1	−1	0	0	0

Table A.19: Twisted Euler characters at $\ll = 6$, $X = A_5^4 D_4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
$n_g\|h_g$	1|1	1|2	2|1	2|2	3|1	3|2	4|2
$\bar{\chi}_g^{X_A}$	4	4	2	0	1	1	0
$\chi_g^{X_A}$	4	−4	0	0	1	−1	0
$\bar{\chi}_g^{X_D}$	1	1	1	1	1	1	1
$\chi_g^{X_D}$	1	1	−1	1	1	1	−1
$\check{\chi}_g^{X_D}$	2	2	0	2	−1	−1	0

Table A.20: Twisted Euler characters at $\ll = 6 + 3$, $X = D_4^6$

$[g]$	1A	3A	2A	6A	3B	6C	4A	12A	5A	15AB	2B	2C	4B	6B	6C
$n_g\|h_g$	1|1	1|3	2|1	2|3	3|1	3|3	4|2	4|6	5|1	5|3	2|1	2|2	4|1	6|1	6|6
$\bar{\chi}_g^{X_D}$	6	6	2	2	3	0	0	0	1	1	4	0	2	1	0
$\chi_g^{X_D}$	6	6	2	2	3	0	0	0	1	1	−4	0	−2	−1	0
$\check{\chi}_g^{X_D}$	12	−6	4	−2	0	0	0	0	2	−1	0	0	0	0	0

Table A.21: Twisted Euler characters at $\ll= 7$, $X = A_6^4$

$[g]$	1A	2A	4A	3AB	6AB
$n_g\|h_g$	1|1	1|4	2|8	3|1	3|4
$\bar{\chi}_g^{X_A}$	4	4	0	1	1
$\chi_g^{X_A}$	4	-4	0	1	-1

Table A.22: Twisted Euler characters at $\ll= 8$, $X = A_7^2 D_5^2$

$[g]$	1A	2A	2B	2C	4A
$n_g\|h_g$	1|1	1|2	2|1	2|1	2|4
$\bar{\chi}_g^{X_A}$	2	2	0	2	0
$\chi_g^{X_A}$	2	-2	0	0	0
$\bar{\chi}_g^{X_D}$	2	2	2	0	0
$\chi_g^{X_D}$	2	-2	0	0	0

Table A.23: Twisted Euler characters at $\ll= 9$, $X = A_8^3$

$[g]$	1A	2A	2B	2C	3A	6A
$n_g\|h_g$	1|1	1|4	2|1	2|2	3|3	3|12
$\bar{\chi}_g^{X_A}$	3	3	1	1	0	0
$\chi_g^{X_A}$	3	-3	1	-1	0	0

Table A.24: Twisted Euler characters at $\ll = 10$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
$n_g\|h_g$	1|1	1|2	2|2
$\bar{\chi}_g^{X_A}$	2	2	0
$\chi_g^{X_A}$	2	-2	0
$\bar{\chi}_g^{X_D}$	1	1	1
$\chi_g^{X_D}$	1	1	-1

Table A.25: Twisted Euler characters at $\ll = 10 + 5$, $X = D_6^4$

$[g]$	1A	2A	3A	2B	4A
$n_g\|h_g$	1|1	2|2	3|1	2|1	4|4
$\bar{\chi}_g^{X_D}$	4	0	1	2	0
$\chi_g^{X_D}$	4	0	1	-2	0

Table A.26: Twisted Euler characters at $\ll = 12$, $X = A_{11} D_7 E_6$

$[g]$	1A	2A
$n_g\|h_g$	1|1	1|2
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	-1
$\bar{\chi}_g^{X_E}$	1	1
$\chi_g^{X_E}$	1	-1

Table A.27: Twisted Euler characters at $\ll= 12+4$, $X = E_6^4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
$n_g\|h_g$	1|1	1|2	2|1	2|4	3|1	3|2	4|8
$\bar{\chi}_g^{X_E}$	4	4	2	0	1	1	0
$\chi_g^{X_E}$	4	-4	0	0	1	-1	0

Table A.28: Twisted Euler characters at $\ll= 13$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
$n_g\|h_g$	1|1	1|4	2|8
$\bar{\chi}_g^{X_A}$	2	2	0
$\chi_g^{X_A}$	2	-2	0

Table A.29: Twisted Euler characters at $\ll= 14+7$, $X = D_8^3$

$[g]$	1A	2A	3A
$n_g\|h_g$	1|1	2|1	3|3
$\bar{\chi}_g^{X_D}$	3	1	0
$\chi_g^{X_D}$	3	1	0

Table A.30: Twisted Euler characters at $\ll= 16$, $X = A_{15}D_9$

$[g]$	1A	2A
$n_g\|h_g$	1|1	1|2
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	-1

Table A.31: Twisted Euler characters at $\ll= 18$, $X = A_{17}E_7$

$[g]$	1A	2A
$n_g\|h_g$	1|1	1|2
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\chi}_g^{X_E}$	1	1

Table A.32: Twisted Euler characters at $\ll = 18+9$, $X = D_{10}E_7^2$

$[g]$	1A	2A
$n_g\|h_g$	1|1	2|1
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	−1
$\bar{\chi}_g^{X_E}$	2	0

Table A.33: Twisted Euler characters at $\ll = 22+11$, $X = D_{12}^2$

$[g]$	1A	2A
$n_g\|h_g$	1|1	2|2
$\bar{\chi}_g^{X_D}$	2	0
$\chi_g^{X_D}$	2	0

Table A.34: Twisted Euler characters at $\ll = 25$, $X = A_{24}$

$[g]$	1A	2A
$n_g\|h_g$	1|1	1|4
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	−1

Table A.35: Twisted Euler characters at $\ll = 30+6,10,15$, $X = E_8^3$

$[g]$	1A	2A	3A
$n_g\|h_g$	1|1	2|1	3|3
$\bar{\chi}_g^{X_E}$	3	1	0

A.3 The Umbral McKay-Thompson Series

In this section we describe the umbral McKay-Thompson series in complete detail. In particular, we present explicit formulas for all the McKay-Thompson series attached to elements of the umbral groups by umbral moonshine in §A.3.4. Most of these expressions appeared first in [68, 75], but some appear for the first time in this work.

In order to facilitate explicit formulations we recall certain standard functions in §A.3.1. We then, using the twisted Euler characters of §A.2.3, explicitly describe the shadow functions of umbral moonshine in §A.3.2. The Rademacher sum construction of the umbral McKay-Thompson series is described in §A.3.3.

A.3.1 Special Functions

The Dedekind eta function is $\eta(\tau) := q^{1/24}\prod_{n>0}(1-q^n)$, where $q = e^{2\pi i\tau}$. Write $\Lambda_M(\tau)$ for the function

$$\Lambda_M(\tau) := Mq\frac{\mathrm{d}}{\mathrm{d}q}\left(\log\frac{\mathbb{H}(M\tau)}{\mathbb{H}(\tau)}\right) = \frac{M(M-1)}{24} + M\sum_{k>0}\sum_{d|k} d\left(q^k - Mq^{Mk}\right),$$

which is a modular form of weight two for $\Gamma_0(N)$ if $M|N$.

Define the Jacobi theta function $\theta_1(\tau,z)$ by setting

$$\theta_1(\tau,z) := iq^{1/8}y^{-1/2}\sum_{n\in\mathbb{Z}}(-1)^n y^n q^{n(n-1)/2}. \tag{A.2}$$

According to the Jacobi triple product identity we have

$$\theta_1(\tau,z) = -iq^{1/8}y^{1/2}\prod_{n>0}(1-y^{-1}q^{n-1})(1-yq^n)(1-q^n). \tag{A.3}$$

The other Jacobi theta functions are

$$
\begin{aligned}
{}_2(\tau, z) &:= q^{1/8} y^{1/2} \prod_{n>0} (1 + y^{-1} q^{n-1})(1 + y q^n)(1 - q^n), \\
{}_3(\tau, z) &:= \prod_{n>0} (1 + y^{-1} q^{n-1/2})(1 + y q^{n-1/2})(1 - q^n), \\
{}_4(\tau, z) &:= \prod_{n>0} (1 - y^{-1} q^{n-1/2})(1 - y q^{n-1/2})(1 - q^n).
\end{aligned} \tag{A.4}
$$

Define $\Psi_{1,1}$ and $\Psi_{1,-1/2}$ by setting

$$
\begin{aligned}
\Psi_{1,1}(\tau, z) &:= -i \frac{\theta_1(\tau, 2z) \eta(\tau)^3}{\theta_1(\tau, z)^2}, \\
\Psi_{1,-1/2}(\tau, z) &:= -i \frac{\eta(\tau)^3}{\theta_1(\tau, z)}.
\end{aligned} \tag{A.5}
$$

These are meromorphic Jacobi forms of weight one, with indexes 1 and $-1/2$, respectively. Here, the term meromorphic refers to the presence of simple poles in the functions $z \mapsto \Psi_{1,*}(\tau, z)$, for fixed $\tau \in \mathbb{H}$, at lattice points $z \in \mathbb{Z}\tau + \mathbb{Z}$.

The standard index m theta functions, for m a positive integer, are defined by

$$
\theta_{m,r}(\tau, z) := \sum_{k \in \mathbb{Z}} y^{2mk+r} q^{(2mk+r)^2/4m}, \tag{A.6}
$$

where $r \in \mathbb{Z}$. Evidently, $\theta_{m,r}$ only depends on $r \mod 2m$. Set $S_{m,r}(\tau) := \frac{1}{2\pi i} \partial_z \theta_{m,r}(\tau, z)\big|_{z=0}$, so that

$$
S_{m,r}(\tau) = \sum_{k \in \mathbb{Z}} (2mk + r) q^{(2mk+r)^2/4m}. \tag{A.7}
$$

For a m a positive integer define

$$
\mu_{m,0}(\tau, z) = \sum_{k \in \mathbb{Z}} y^{2km} q^{mk^2} \frac{yq^k + 1}{yq^k - 1} = \frac{y+1}{y-1} + O(q). \tag{A.8}
$$

We recover $\Psi_{1,1}$ upon specializing to $m = 1$. Observe that

$$\mu_{m,0}(\tau, z+1/2) = \sum_{k\in\mathbb{Z}} y^{2km} q^{mk^2} \frac{yq^k - 1}{yq^k + 1} = \frac{y-1}{y+1} + O(q). \tag{A.9}$$

Define the even and odd parts of $\mu_{m,0}$ by setting

$$\mu^k_{m,0}(\tau, z) := \frac{1}{2}(\mu_{m,0}(\tau, z) + (-1)^k \mu_{m,0}(\tau, z+1/2)) \tag{A.10}$$

for $k \mod 2$.

For $m, r \in \mathbb{Z} + \frac{1}{2}$ with $m > 0$ define the half-integral index theta functions

$$\theta_{m,r}(\tau, z) := \sum_{k\in\mathbb{Z}} e(mk + r/2) y^{2mk+r} q^{(2mk+r)^2/4m}, \tag{A.11}$$

and define also $S_{m,r}(\tau) := \frac{1}{2\pi i} \partial_z \theta_{m,r}(\tau, z)|_{z=0}$, so that

$$S_{m,r}(\tau) = \sum_{k\in\mathbb{Z}} e(mk + r/2)(2mk + r) q^{(2mk+r)^2/4m}. \tag{A.12}$$

As in the integral index case, $\theta_{m,r}$ depends only on $r \mod 2m$. We recover $-\theta_1$ upon specializing $\theta_{m,r}$ to $m = r = 1/2$.

For $m \in \mathbb{Z} + 1/2$, $m > 0$, define

$$\mu_{m,0}(\tau, z) := i \sum_{k\in\mathbb{Z}} (-1)^k y^{2mk+1/2} q^{mk^2+k/2} \frac{1}{1 - yq^k} = \frac{-iy^{1/2}}{y-1} + O(q). \tag{A.13}$$

Given $\alpha \in \mathbb{Q}$ write $[\alpha]$ for the operator on q-series (in rational, possibility negative powers of q) that eliminates exponents not contained in $\mathbb{Z} + \alpha$, so that if $f = \sum_{\beta\in\mathbb{Q}} c(\beta) q^\beta$ then

$$[\alpha] f := \sum_{n\in\mathbb{Z}} c(n + \alpha) q^{n+\alpha} \tag{A.14}$$

A.3.2 Shadows

Let X be a Niemeier root system and let $m = m^X$ be the Coxeter number of X. For $g \in G^X$ we define the associated *shadow function* $S_g^X = (S_{g,r}^X)$ by setting

$$S_g^X := S_g^{X_A} + S_g^{X_D} + S_g^{X_E} \tag{A.15}$$

where the $S_g^{X_A}$, &c., are defined in the following way, in terms of the twisted Euler characters $\chi_g^{X_A}$, &c. given in §A.2.3, and the unary theta series $S_{m,r}$ (cf. (A.7)).

Note that if $m = m^X$ then $S_{g,r}^X = S_{g,r+2m}^X = -S_{g,-r}^X$ for all $g \in G^X$, so we need specify the $S_{g,r}^{X_A}$, &c., only for $0 < r < m$.

If $X_A = \emptyset$ then $S_g^{X_A} := 0$. Otherwise, we define $S_{g,r}^{X_A}$ for $0 < r < m$ by setting

$$S_{g,r}^{X_A} := \begin{cases} \chi_g^{X_A} S_{m,r} & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_A} S_{m,r} & \text{if } r = 1 \mod 2. \end{cases} \tag{A.16}$$

If $X_D = \emptyset$ then $S_g^{X_D} := 0$. If $X_D \neq \emptyset$ then m is even and $m \geq 6$. If $m = 6$ then set

$$S_{g,r}^{X_D} := \begin{cases} 0 & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_D} S_{6,r} + \chi_g^{X_D} S_{6,6-r} & \text{if } r = 1, 5 \mod 6, \\ \check{\chi}_g^{X_D} S_{6,r} & \text{if } r = 3 \mod 6. \end{cases} \tag{A.17}$$

If $m > 6$ and $m = 2 \mod 4$ then set

$$S_{g,r}^{X_D} := \begin{cases} 0 & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_D} S_{m,r} + \chi_g^{X_D} S_{m,m-r} & \text{if } r = 1 \mod 2. \end{cases} \tag{A.18}$$

If $m > 6$ and $m = 0 \mod 4$ then set

$$S_{g,r}^{X_D} := \begin{cases} \chi_g^{X_D} S_{m,m-r} & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_D} S_{m,r} & \text{if } r = 1 \mod 2. \end{cases} \tag{A.19}$$

If $X_E = \emptyset$ then $S_g^{X_E} := 0$. Otherwise, m is 12 or 18 or 30. In case $m = 12$ define $S_{g,r}^{X_E}$ for $0 < r < 12$ by setting

$$S_{g,r}^{X_E} = \begin{cases} \bar{\chi}_g^{X_E}(S_{12,1} + S_{12,7}) & \text{if } r \in \{1,7\}, \\ \bar{\chi}_g^{X_E}(S_{12,5} + S_{12,11}) & \text{if } r \in \{5,11\}, \\ \chi_g^{X_E}(S_{12,4} + S_{12,8}) & \text{if } r \in \{4,8\}, \\ 0 & \text{else.} \end{cases} \tag{A.20}$$

In case $m = 18$ define $S_{g,r}^{X_E}$ for $0 < r < 18$ by setting

$$S_{g,r}^{X_E} = \begin{cases} \bar{\chi}_g^{X_E}(S_{18,r} + S_{18,18-r}) & \text{if } r \in \{1,5,7,11,13,17\}, \\ \bar{\chi}_g^{X_E} S_{18,9} & \text{if } r \in \{3,15\}, \\ \bar{\chi}_g^{X_E}(S_{18,3} + S_{18,9} + S_{18,15}) & \text{if } r = 9, \\ 0 & \text{else.} \end{cases} \tag{A.21}$$

In case $m = 30$ define $S_{g,r}^{X_E}$ for $0 < r < 30$ by setting

$$S_{g,r}^{X_E} = \begin{cases} \bar{\chi}_g^{X_E}(S_{30,1} + S_{30,11} + S_{30,19} + S_{30,29}) & \text{if } r \in \{1,11,19,29\}, \\ \bar{\chi}_g^{X_E}(S_{30,7} + S_{30,13} + S_{30,17} + S_{30,23}) & \text{if } r \in \{7,13,17,23\}, \\ 0 & \text{else.} \end{cases} \tag{A.22}$$

A.3.3 Rademacher Sums

Let Γ_∞ denote the subgroup of upper-triangular matrices in $SL_2(\mathbb{Z})$. Given $\alpha \in \mathbb{R}$ and $\gamma \in SL_2(\mathbb{Z})$, define $\mathrm{r}_{1/2}^{[\alpha]}(\gamma,\tau) := 1$ if $\gamma \in \Gamma_\infty$. Otherwise, set

$$\mathrm{r}_{1/2}^{[\alpha]}(\gamma,\tau) := e(-\alpha(\gamma\tau - \gamma\infty)) \sum_{k\geq 0} \frac{(2\pi i\alpha(\gamma\tau - \gamma\infty))^{n+1/2}}{\Gamma(n+3/2)}, \tag{A.23}$$

where $e(x) := e^{2\pi i x}$. Let n be a positive integer, and suppose that ν is a multiplier system for vector-valued modular forms of weight $1/2$ on $\Gamma =$

$\Gamma_0(n)$. Assume that $\nu = (\nu_{ij})$ satisfies $\nu_{11}(T) = e^{\pi i/2m}$, for some basis $\{\mathfrak{e}_i\}$, for some positive integer m, where $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. To this data, attach the Rademacher sum

$$R_{\Gamma,\nu}(\tau) := \lim_{K\to\infty} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_{K,K^2}} \nu(\gamma)e\left(-\frac{\gamma\tau}{4m}\right)\mathfrak{e}_1\,\mathrm{j}(\gamma,\tau)^{1/2}\,\mathrm{r}_{1/2}^{[-1/4m]}(\gamma,\tau), \quad \text{(A.24)}$$

where $\Gamma_{K,K^2} := \{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma \mid 0\leq c<K,\ |d|<K^2\}$, and $\mathrm{j}(\gamma,\tau) := (c\tau+d)^{-1}$ for $\gamma = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. (See [299] for a more general and detailed discussion of vector-valued Rademacher sums.)

For the special case that $X = A_8^3$ we require 8-vector-valued functions $\check{t}_g^{(9)} = (\check{t}_{g,r}^{(9)})$ for $g\in G^X$ with order 3 or 6. For such g, define $\check{t}_{g,r}^{(9)}$, for $0<r<9$, by setting

$$\check{t}_{3A,r}^{(9)}(\tau) := \begin{cases} 0, & \text{if } r\neq 0 \mod 3,\\ -\theta_{3,3}(\tau,0), & \text{if } r=3,\\ \theta_{3,0}(\tau,0), & \text{if } r=6,\end{cases} \quad \text{(A.25)}$$

in the case that g has order 3, and

$$\check{t}_{6A,r}^{(9)}(\tau) := \begin{cases} 0, & \text{if } r\neq 0 \mod 3,\\ -\theta_{3,3}(\tau,0), & \text{if } r=3,\\ -\theta_{3,0}(\tau,0), & \text{if } r=6,\end{cases} \quad \text{(A.26)}$$

when $o(g)=6$. Here $\theta_{m,r}(\tau,z)$ is as defined in (A.6).

The following conjecture is formulated in [69].

Conjecture A.1. Let X be a Niemeier root system and let $g\in G^X$. If $X\neq A_8^3$ and $g\in G^X$, or if $X = A_8^3$ and $g\in G^X$ does not satisfy $o(g)=0 \mod 3$, then we have

$$\check{H}_g^X(\tau) = -2R^X_{\Gamma_0(n_g),\check{\nu}_g^X}. \quad \text{(A.27)}$$

If $X = A_8^3$ and $g \in G^X$ satisfies $o(g) = 0 \mod 3$ then

$$\check{H}^X_{g,r}(\tau) = -2R^X_{\Gamma_0(n_g),\check{\nu}^X_g}(\tau) + \check{t}^{(9)}_g(\tau). \tag{A.28}$$

Conjecture A.1 is a theorem in the case that $X = A_1^{24}$. This is the main result of [73]. A number of other cases of Conjecture A.1 are proved in [69].

A.3.4 Explicit Prescriptions

Here we give explicit expressions for all the umbral McKay-Thompson series H^X_g. Most of these appeared first in [68, 75]. The expressions in §§A.3.4, A.3.4, A.3.4, A.3.4 are taken from [108]. The expressions in §§A.3.4, A.3.4, A.3.4, A.3.4 are taken from [74]. The expressions for H^X_g with $X = E_8^3$ appeared first in [107]. The expression for $H^{(6+3)}_{2B,1}$ in §A.3.4, and the expressions for $H^{(12+4)}_{4A,r}$ and $H^{(12+4)}_{8AB,r}$ in §A.3.4, appear here for the first time.

The labels for conjugacy classes in G^X are as in §A.2.2.

$\ell = 2$, $X = A_1^{24}$

We have $G^{(2)} = G^X \simeq M_{24}$ and $m^X = 2$. So for $g \in M_{24}$, the associated umbral McKay-Thompson series $H^{(2)}_g = (H^{(2)}_{g,r})$ is a 4-vector-valued function, with components indexed by $r \in \mathbb{Z}/4\mathbb{Z}$, satisfying $H^{(2)}_{g,r} = -H^{(2)}_{g,-r}$, and in particular, $H^{(2)}_{g,r} = 0$ for $r = 0 \mod 2$. So it suffices to specify the $H^{(2)}_{g,1}$ explicitly.

Define $H^{(2)}_g = (H^{(2)}_{g,r})$ for $g = e$ by requiring that

$$-2\Psi_{1,1}(\tau,z)\varphi^{(2)}_1(\tau,z) = -24\mu_{2,0}(\tau,z) + \sum_{r \mod 4} H^{(2)}_{e,r}(\tau)\theta_{2,r}(\tau,z), \tag{A.29}$$

where

$$\varphi^{(2)}_1(\tau,z) := 4\left(\frac{{}_2(\tau,z)^2}{{}_2(\tau,0)^2} + \frac{{}_3(\tau,z)^2}{{}_3(\tau,0)^2} + \frac{{}_4(\tau,z)^2}{{}_4(\tau,0)^2}\right). \tag{A.30}$$

More generally, for $g \in G^{(2)}$ define

$$H^{(2)}_{g,1}(\tau) := \frac{\bar{\chi}^{(2)}_g}{24} H^{(2)}_{e,1}(\tau) - F^{(2)}_g(\tau) \frac{1}{S_{2,1}(\tau)}, \tag{A.31}$$

where $\bar{\chi}^{(2)}_g$ and $F^{(2)}_g$ are as specified in Table A.36. Note that $\bar{\chi}^{(2)}_g = \bar{\chi}^{X_A}_g$, the latter appearing in Table A.15. Also, $S_{2,1}(\tau) = \eta(\tau)^3$.

The functions $f_{23,a}$ and $f_{23,b}$ in Table A.36 are cusp forms of weight two for $\Gamma_0(23)$, defined by

$$\begin{aligned} f_{23,a}(\tau) &:= \frac{\eta(\tau)^3\eta(23\tau)^3}{\eta(2\tau)\eta(46\tau)} + 3\mathbb{H}(\tau)^2\mathbb{H}(23\tau)^2 + 4\eta(\tau)\eta(2\tau)\eta(23\tau)\eta(46\tau) + 4\eta(2\tau)^2\eta(46\tau)^2, \\ f_{23,b}(\tau) &:= \mathbb{H}(\tau)^2\mathbb{H}(23\tau)^2. \end{aligned} \tag{A.32}$$

Note that the definition of $F^{(2)}_g$ appearing here for $g \in 23A \cup 23B$ corrects errors in [72, 73].

$\ell = 3$, $X = A_2^{12}$

We have $G^{(3)} = G^X \simeq 2.M_{12}$ and $m^X = 3$. So for $g \in 2.M_{12}$, the associated umbral McKay-Thompson series $H^{(3)}_g = (H^{(3)}_{g,r})$ is a 6-vector-valued function, with components indexed by $r \in \mathbb{Z}/6\mathbb{Z}$, satisfying $H^{(3)}_{g,r} = -H^{(3)}_{g,-r}$, and in particular, $H^{(3)}_{g,r} = 0$ for $r = 0 \mod 3$. So it suffices to specify the $H^{(3)}_{g,1}$ and $H^{(3)}_{g,2}$ explicitly.

Define $H^{(3)}_g = (H^{(3)}_{g,r})$ for $g = e$ by requiring that

$$-2\Psi_{1,1}(\tau,z)\varphi^{(3)}_1(\tau,z) = -12\mu_{3,0}(\tau,z) + \sum_{r \mod 6} H^{(3)}_{e,r}(\tau)\theta_{3,r}(\tau,z), \tag{A.33}$$

where

$$\varphi^{(3)}_1(\tau,z) := 2\left(\frac{{}_3(\tau,z)^2}{{}_3(\tau,0)^2}\frac{{}_4(\tau,z)^2}{{}_4(\tau,0)^2} + \frac{{}_4(\tau,z)^2}{{}_4(\tau,0)^2}\frac{{}_2(\tau,z)^2}{{}_2(\tau,0)^2} + \frac{{}_2(\tau,z)^2}{{}_2(\tau,0)^2}\frac{{}_3(\tau,z)^2}{{}_3(\tau,0)^2}\right). \tag{A.34}$$

Table A.36: Character Values and Weight Two Forms for $\ell = 2$, $X = A_1^{24}$

$[g]$	$\bar{\chi}_g^{(2)}$	$F_g^{(2)}(\tau)$
$1A$	24	0
$2A$	8	$16\Lambda_2(\tau)$
$2B$	0	$2\eta(\tau)^8\eta(2\tau)^{-4}$
$3A$	6	$6\Lambda_3(\tau)$
$3B$	0	$2\eta(\tau)^6\eta(3\tau)^{-2}$
$4A$	0	$2\eta(2\tau)^8\eta(4\tau)^{-4}$
$4B$	4	$4(-\Lambda_2(\tau)+\Lambda_4(\tau))$
$4C$	0	$2\eta(\tau)^4\eta(2\tau)^2\eta(4\tau)^{-2}$
$5A$	4	$2\Lambda_5(\tau)$
$6A$	2	$2(-\Lambda_2(\tau)-\Lambda_3(\tau)+\Lambda_6(\tau))$
$6B$	0	$2\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^{-2}$
$7AB$	3	$\Lambda_7(\tau)$
$8A$	2	$-\Lambda_4(\tau)+\Lambda_8(\tau)$
$10A$	0	$2\eta(\tau)^3\eta(2\tau)\eta(5\tau)\eta(10\tau)^{-1}$
$11A$	2	$2(\Lambda_{11}(\tau)-11\eta(\tau)^2\eta(11\tau)^2)/5$
$12A$	0	$\frac{2\eta(\tau)^3\eta(4\tau)^2\eta(6\tau)^3}{\eta(2\tau)\eta(3\tau)\eta(12\tau)^2}$
$12B$	0	$2\eta(\tau)^4\eta(4\tau)\eta(6\tau)\eta(2\tau)^{-1}\eta(12\tau)^{-1}$
$14AB$	1	$(-\Lambda_2(\tau)-\Lambda_7(\tau)+\Lambda_{14}(\tau))/3$ $-14\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)/3$
$15AB$	1	$(-\Lambda_3(\tau)-\Lambda_5(\tau)+\Lambda_{15}(\tau))/4$ $-15\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)/4$
$21AB$	0	$\frac{7\eta(\tau)^3\eta(7\tau)^3}{3\eta(3\tau)\eta(21\tau)}-\frac{\eta(\tau)^6}{3\eta(3\tau)^2}$
$23AB$	1	$(\Lambda_{23}(\tau)-23f_{23,a}(\tau)-69f_{23,b}(\tau))/11$

More generally, for $g \in G^{(3)}$ define

$$H^{(3)}_{g,1}(\tau) := \frac{\bar{\chi}^{(3)}_g}{12} H^{(3)}_{e,1}(\tau) + \frac{1}{2}\left(F^{(3)}_g + F^{(3)}_{zg}\right)\frac{1}{S_{3,1}(\tau)}, \tag{A.35}$$

$$H^{(3)}_{g,2}(\tau) := \frac{\chi^{(3)}_g}{12} H^{(3)}_{e,1}(\tau) + \frac{1}{2}\left(F^{(3)}_g - F^{(3)}_{zg}\right)\frac{1}{S_{3,2}(\tau)}, \tag{A.36}$$

where $\chi^{(3)}_g$ and $F^{(3)}_g$ are as specified in Table A.37, and z is the non-trivial central element of $G^{(3)}$. The action of $g \mapsto zg$ on conjugacy classes can be read off Table A.37, for the horizontal lines indicate the sets $[g] \cup [zg]$.

Note the eta product identities, $S_{3,1}(\tau) = \eta(2\tau)^5/\eta(4\tau)^2$, and $S_{3,2}(\tau) = 2\eta(\tau)^2\eta(4\tau)^2/\eta(2\tau)$. Note also that $\bar{\chi}^{(3)}_g = \bar{\chi}^{X_A}_g$ and $\chi^{(3)}_g = \chi^{X_A}_g$, the latter appearing in Table A.16.

The function f_{44} is the unique new cusp form of weight 2 for $\Gamma_0(44)$, normalized so that $f_{44}(\tau) = q + O(q^3)$ as $\Im(\tau) \to \infty$. The coefficients $c_g(d)$ and $c'_g(d)$ for $g \in 10A \cup 22A \cup 22B$ are given by

$$c_{10A}(2) = -5,\ c_{10A}(4) = -\frac{5}{3},\ c_{10A}(5) = -\frac{2}{3},\ c_{10A}(10) = 1,\ c_{10A}(20) = -\frac{1}{3}, \tag{A.37}$$

$$c_{22AB}(2) = -\frac{11}{5},\ c_{22AB}(4) = \frac{11}{5},\ c_{22AB}(11) = -\frac{2}{15},\ c_{22AB}(22) = \frac{1}{5},\ c_{22AB}(44) = -\frac{1}{15}, \tag{A.38}$$

$$c'_{22AB}(1) = 1,\ c'_{22AB}(2) = 4,\ c'_{22AB}(4) = 8. \tag{A.39}$$

$\ell = 4$**,** $X = A_3^8$

We have $m^X = 4$, so the umbral McKay-Thompson series $H^{(4)}_g = (H^{(4)}_{g,r})$ associated to $g \in G^{(4)}$ is an 8-vector-valued function, with components indexed by $r \in \mathbb{Z}/8\mathbb{Z}$.

Table A.37: Character Values and Weight Two Forms for $\ell = 3$, $X = A_2^{12}$

$[g]$	$\bar{\chi}_g^{(3)}$	$\chi_g^{(3)}$	$F_g^{(3)}(\tau)$
1A	12	12	0
2A	12	-12	0
4A	0	0	$-2\mathbb{H}(\tau)^4\mathbb{H}(2\tau)^2/\mathbb{H}(4\tau)^2$
2B	4	4	$-16{}_2(\tau)$
2C	4	-4	$16{}_2(\tau) - \frac{16}{3}{}_4(\tau)$
3A	3	3	$-6{}_3(\tau)$
6A	3	-3	$-9{}_2(\tau) - 2{}_3(\tau) + 3{}_4(\tau) + 3{}_6(\tau) - {}_{12}(\tau)$
3B	0	0	$8{}_3(\tau) - 2{}_9(\tau) + 2\,\mathbb{H}^6(\tau)/\mathbb{H}^2(3\tau)$
6B	0	0	$-2\eta(\tau)^5\eta(3\tau)/\eta(2\tau)\eta(6\tau)$
4B	0	0	$-2\mathbb{H}(2\tau)^8/\mathbb{H}(4\tau)^4$
4C	4	0	$-8{}_4(\tau)/3$
5A	2	2	$-2{}_5(\tau)$
10A	2	-2	$\sum_{d\mid 20} c_{10A}(d){}_d(\tau) + \frac{20}{3}\eta(2\tau)^2\eta(10\tau)^2$
12A	0	0	$-2\mathbb{H}(\tau)\mathbb{H}(2\tau)^5\mathbb{H}(3\tau)/\mathbb{H}(4\tau)^2\mathbb{H}(6\tau)$
6C	1	1	$2({}_2(\tau) + {}_3(\tau) - {}_6(\tau))$
6D	1	-1	$-5{}_2(\tau) - 2{}_3(\tau) + \frac{5}{3}{}_4(\tau) + 3{}_6(\tau) - {}_{12}(\tau)$
8AB	0	0	$-2\mathbb{H}(2\tau)^4\mathbb{H}(4\tau)^2/\mathbb{H}(8\tau)^2$
8CD	2	0	$-2{}_2(\tau) + \frac{5}{3}{}_4(\tau) - {}_8(\tau)$
20AB	0	0	$-2\mathbb{H}(2\tau)^7\mathbb{H}(5\tau)/\mathbb{H}(\tau)\mathbb{H}(4\tau)^2\mathbb{H}(10\tau)$
11AB	1	1	$-\frac{2}{5}{}_{11}(\tau) - \frac{33}{5}\eta(\tau)^2\eta(11\tau)^2$
22AB	1	-1	$\sum_{d\mid 44} c_g(d){}_d(\tau) - \frac{11}{5}\sum_{d\mid 4} c'_g(d)\eta(d\tau)^2\eta(11d\tau)^2 + \frac{22}{3}f_{44}(\tau)$

Define $H_g^{(4)} = (H_{g,r}^{(4)})$ for $g \in G^{(4)}$, $g \notin 4C$, by requiring that

$$\psi_g^{(4)}(\tau,z) = -\chi_g^{(4)}\mu_{4,0}^0(\tau,z) - \bar{\chi}_g^{(4)}\mu_{4,0}^1(\tau,z) + \sum_{r \mod 8} H_{g,r}^{(4)}(\tau)\theta_{4,r}(\tau,z), \tag{A.40}$$

where $\chi_g^{(4)} := \chi_g^{X_A}$ and $\bar{\chi}_g^{(4)} := \bar{\chi}_g^{X_A}$ (cf. Table A.17), and the $\psi_g^{(4)}$ are meromorphic Jacobi forms of weight 1 and index 4 given explicitly in Table A.38.

Table A.38: Character Values and Meromorphic Jacobi Forms for $\ell = 4$, $X = A_3^8$

$[g]$	$\chi_g^{(4)}$	$\bar{\chi}_g^{(4)}$	$\psi_g^{(4)}(\tau,z)$
1A	8	8	$2i{}_1(\tau,2z)^3{}_1(\tau,z)^{-4}\eta(\tau)^3$
2A	-8	8	$2i{}_1(\tau,2z)^3{}_2(\tau,z)^{-4}\eta(\tau)^3$
2B	0	0	$-2i{}_1(\tau,2z)^3{}_1(\tau,z)^{-2}{}_2(\tau,z)^{-2}\eta(\tau)^3$
4A	0	0	$-2i{}_1(\tau,2z){}_2(\tau,2z)^2{}_2(2\tau,2z)^{-2}\eta(2\tau)^2\eta(\tau)^{-1}$
4B	0	0	$-2i{}_1(2\tau,2z){}_3(2\tau,2z)^2{}_4(2\tau,2z)\eta(2\tau)^2\eta(\tau)^{-2}\eta(4\tau)^{-2}$
2C	0	4	$2i{}_1(\tau,2z){}_2(\tau,2z)^2{}_1(\tau,z)^{-2}{}_2(\tau,z)^{-2}\eta(\tau)^3$
3A	2	2	$2i{}_1(3\tau,6z){}_1(\tau,z)^{-1}{}_1(3\tau,3z)^{-1}\eta(\tau)^3$
6A	-2	2	$-2i{}_1(3\tau,6z){}_2(\tau,z)^{-1}{}_2(3\tau,3z)^{-1}\eta(\tau)^3$
6BC	0	0	cf. (A.41)
8A	0	0	$-2i{}_1(\tau,2z){}_2(2\tau,4z){}_2(4\tau,4z)^{-1}\eta(\tau)\eta(4\tau)\eta(2\tau)^{-1}$
4C	0	2	$2i{}_1(\tau,2z){}_2(2\tau,4z){}_1(2\tau,2z)^{-2}\eta(2\tau)^7\eta(\tau)^{-3}\eta(4\tau)^{-2}$
7AB	1	1	cf. (A.41)
14AB	-1	1	cf. (A.41)

$$\psi^{(4)}_{6BC} := \left({}_1(\tau, z+\tfrac{1}{3}){}_1(\tau, z+\tfrac{1}{6}) - {}_1(\tau, z-\tfrac{1}{3}){}_1(\tau, z-\tfrac{1}{6})\right) \frac{-i{}_1(3\tau, 6z)}{{}_1(3\tau, 3z){}_2(3\tau, 3z)}\eta(3\tau)$$

$$\psi^{(4)}_{7AB} := \left(\prod_{j=1}^{3} {}_1(\tau, 2z+\tfrac{j^2}{7}){}_1(\tau, z-\tfrac{j^2}{7}) + \prod_{j=1}^{3} {}_1(\tau, 2z-\tfrac{j^2}{7}){}_1(\tau, z+\tfrac{j^2}{7})\right) \frac{-i}{{}_1(7\tau, 7z)}\frac{\eta(7\tau)}{\eta(\tau)^4}$$

$$\psi^{(4)}_{14AB} := \left(\prod_{j=1}^{3} {}_1(\tau, 2z+\tfrac{j^2}{7}){}_2(\tau, z-\tfrac{j^2}{7}) + \prod_{j=1}^{3} {}_1(\tau, 2z-\tfrac{j^2}{7}){}_2(\tau, z+\tfrac{j^2}{7})\right) \frac{i}{{}_2(7\tau, 7z)}\frac{\eta(7\tau)}{\eta(\tau)^4}$$

(A.41)

For use later on, note that $\psi^{(4)}_{1A} = -2\Psi_{1,1}\varphi^{(4)}_1$, where

$$\varphi^{(4)}_1(\tau, z) := \frac{{}_1(\tau, 2z)^2}{{}_1(\tau, z)^2}. \tag{A.42}$$

$\ell = 5$, $X = A_4^6$

We have $m^X = 5$, so the umbral McKay-Thompson series $H^{(5)}_g = (H^{(5)}_{g,r})$ associated to $g \in G^{(5)}$ is a 10-vector-valued function, with components indexed by $r \in \mathbb{Z}/10\mathbb{Z}$.

Define $H^{(5)}_g = (H^{(5)}_{g,r})$ for $g \in G^{(5)}$, $g \notin 5A \cup 10A$, by requiring that

$$\psi^{(5)}_g(\tau, z) = -\chi^{(5)}_g \mu^0_{5,0}(\tau, z) - \bar{\chi}^{(5)}_g \mu^1_{5,0}(\tau, z) + \sum_{r \bmod 10} H^{(5)}_{g,r}(\tau)\theta_{5,r}(\tau, z), \tag{A.43}$$

where $\chi^{(5)}_g := \chi^{X_A}_g$ and $\bar{\chi}^{(5)}_g := \bar{\chi}^{X_A}_g$ (cf. Table A.18), and the $\psi^{(5)}_g$ are meromorphic Jacobi forms of weight 1 and index 5 given explicitly in Table A.39.

Table A.39: Character Values and Meromorphic Jacobi Forms for $\ell = 5$, $X = A_4^6$

$[g]$	$\chi_g^{(5)}$	$\bar{\chi}_g^{(5)}$	$\psi_g^{(5)}(\tau, z)$
1A	6	6	$2i{}_1(\tau, 2z){}_1(\tau, 3z){}_1(\tau, z)^{-3}\eta(\tau)^3$
2A	-6	6	$-2i{}_1(\tau, 2z){}_2(\tau, 3z){}_2(\tau, z)^{-3}\eta(\tau)^3$
2B	-2	2	$-2i{}_1(\tau, 2z){}_1(\tau, 3z){}_1(\tau, z)^{-1}{}_2(\tau, z)^{-2}\eta(\tau)^3$
2C	2	2	$2i{}_1(\tau, 2z){}_2(\tau, 3z){}_1(\tau, z)^{-2}{}_2(\tau, z)^{-1}\eta(\tau)^3$
3A	0	0	$-2i{}_1(\tau, 2z){}_1(\tau, 3z){}_1(3\tau, 3z)^{-1}\eta(3\tau)$
6A	0	0	$-2i{}_1(\tau, 2z){}_2(\tau, 3z){}_2(3\tau, 3z)^{-1}\eta(3\tau)$
4AB	0	0	cf. (A.44)
4CD	0	2	cf. (A.44)
12AB	0	0	cf. (A.44)

$$
\begin{aligned}
\psi_{4AB}^{(5)}(\tau, z) &:= -i{}_2(\tau, 2z)\frac{{}_1(\tau, z+\frac{1}{4}){}_1(\tau, 3z+\frac{1}{4}) - {}_1(\tau, z-\frac{1}{4}){}_1(\tau, 3z-\frac{1}{4})}{{}_2(2\tau, 2z)^2}\frac{\eta(2\tau)^2}{\eta(\tau)} \\
\psi_{4CD}^{(5)}(\tau, z) &:= -i{}_2(\tau, 2z)\frac{{}_1(\tau, z+\frac{1}{4}){}_1(\tau, 3z-\frac{1}{4}) + {}_1(\tau, z-\frac{1}{4}){}_1(\tau, 3z+\frac{1}{4})}{{}_1(2\tau, 2z){}_2(2\tau, 2z)}\frac{\eta(2\tau)^2}{\eta(\tau)} \\
\psi_{12AB}^{(5)}(\tau, z) &:= i\frac{{}_2(\tau, 2z)}{{}_2(6\tau, 6z)}\left({}_1(\tau, z+\tfrac{1}{12}){}_1(\tau, z+\tfrac{1}{4}){}_1(\tau, z+\tfrac{5}{12}){}_1(\tau, 3z-\tfrac{1}{4})\right. \\
&\quad \left. - {}_1(\tau, z-\tfrac{1}{12}){}_1(\tau, z-\tfrac{1}{4}){}_1(\tau, z-\tfrac{5}{12}){}_1(\tau, 3z+\tfrac{1}{4})\right)\frac{\eta(6\tau)}{\eta(\tau)^3}
\end{aligned}
\tag{A.44}
$$

For $g \in 5A$ use the formulas of §A.3.4 to define

$$
H_{5A,r}^{(5)}(\tau) := H_{1A,r}^{(25)}(\tau/5) - H_{1A,10-r}^{(25)}(\tau/5) + H_{1A,10+r}^{(25)}(\tau/5) - H_{1A,20-r}^{(25)}(\tau/5) + H_{1A,20+r}^{(25)}(\tau/5).
\tag{A.45}
$$

For $g \in 10A$ set $H_{10A,r}^{(5)}(\tau) := -(-1)^r H_{5A,r}^{(5)}(\tau)$.

For use later on we note that $\psi^{(5)}_{1A} = -2\Psi_{1,1}\varphi^{(5)}_1$, where

$$\varphi^{(5)}_1(\tau,z) := \frac{{}_1(\tau,3z)}{{}_1(\tau,z)}. \tag{A.46}$$

$\ell = 6$**,** $X = A_5^4D_4$

We have $m^X = 6$, so the umbral McKay-Thompson series $H^{(6)}_g = (H^{(6)}_{g,r})$ associated to $g \in G^{(6)}$ is a 12-vector-valued function with components indexed by $r \in \mathbb{Z}/12\mathbb{Z}$. We have $H^{(6)}_{g,r} = -H^{(6)}_{g,-r}$, so it suffices to specify the $H^{(6)}_{g,r}$ for $r \in \{1,2,3,4,5\}$.

To define $H^{(6)}_g = (H^{(6)}_{g,r})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-2\Psi_{1,1}(\tau,z)\varphi^{(6)}_1(\tau,z) = -24\mu_{6,0}(\tau,z) + \sum_{r \mod 12} h_r(\tau)\theta_{6,r}(\tau,z), \tag{A.47}$$

where

$$\varphi^{(6)}_1(\tau,z) := \varphi^{(2)}_1(\tau,z)\varphi^{(5)}_1(\tau,z) - \varphi^{(3)}_1(\tau,z)\varphi^{(4)}_1(\tau,z). \tag{A.48}$$

(Cf. (A.30), (A.34), (A.42), (A.46).) Now define the $H^{(6)}_{1A,r}$ by setting

$$\begin{aligned}
H^{(6)}_{1A,1}(\tau) &:= \frac{1}{24}\left(5h_1(\tau) + h_5(\tau)\right),\\
H^{(6)}_{1A,2}(\tau) &:= \frac{1}{6}h_2(\tau),\\
H^{(6)}_{1A,3}(\tau) &:= \frac{1}{4}h_3(\tau),\\
H^{(6)}_{1A,4}(\tau) &:= \frac{1}{6}h_4(\tau),\\
H^{(6)}_{1A,5}(\tau) &:= \frac{1}{24}\left(h_1(\tau) + 5h_5(\tau)\right).
\end{aligned} \tag{A.49}$$

Define $H^{(6)}_{2A,r}$ by requiring

$$H^{(6)}_{2A,r}(\tau) := -(-1)^r H^{(6)}_{1A,r}(\tau). \tag{A.50}$$

For the remaining g, recall (A.14). The $H^{(6)}_{g,r}$ for $g \notin 1A \cup 2A$ are defined as follows for $r = 2$ and $r = 4$, noting that $H^{(3)}_{g,4} = H^{(3)}_{g,-2} = -H^{(3)}_{g,2}$.

$$
\begin{aligned}
H^{(6)}_{2B,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{4C,r}(\tau/2)\\
H^{(6)}_{4A,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{4B,r}(\tau/2)\\
H^{(6)}_{3A,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{6C,r}(\tau/2)\\
H^{(6)}_{6A,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{6D,r}(\tau/2)\\
H^{(6)}_{8AB,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{8CD,r}(\tau/2)
\end{aligned}
\tag{A.51}
$$

For the $H^{(6)}_{g,3}$ we define

$$
\begin{aligned}
H^{(6)}_{2B,3}(\tau), H^{(6)}_{4A,3}(\tau) &:= -[-\tfrac{9}{24}]H^{(2)}_{6A,1}(\tau/3),\\
H^{(6)}_{3A,3}(\tau), H^{(6)}_{6A,3}(\tau) &:= 0,\\
H^{(6)}_{8AB,3}(\tau) &:= -[-\tfrac{9}{24}]H^{(2)}_{12A,1}(\tau/3).
\end{aligned}
\tag{A.52}
$$

Noting that $H^{(2)}_{g,5} = H^{(2)}_{g,1}$ and $H^{(3)}_{g,5} = -H^{(3)}_{g,1}$, the $H^{(6)}_{g,1}$ and $H^{(6)}_{g,5}$ are defined for $o(g) \neq 0 \mod 3$ by setting

$$
\begin{aligned}
H^{(6)}_{2B,r}(\tau) &:= [-\tfrac{1}{24}]\frac{1}{2}\left(H^{(2)}_{6A,r}(\tau/3) + H^{(3)}_{4C,r}(\tau/2)\right)\\
H^{(6)}_{4A,r}(\tau) &:= [-\tfrac{1}{24}]\frac{1}{2}\left(H^{(2)}_{6A,r}(\tau/3) + H^{(3)}_{4B,r}(\tau/2)\right)\\
H^{(6)}_{8AB,r}(\tau) &:= [-\tfrac{1}{24}]\frac{1}{2}\left(H^{(2)}_{12A,r}(\tau/3) + H^{(3)}_{8CD,r}(\tau/2)\right)
\end{aligned}
\tag{A.53}
$$

It remains to specify the $H^{(6)}_{g,r}$ when $g \in 3A \cup 6A$ and r is 1 or 5. These cases are determined by using the formulas of §A.3.4 to set

$$
\begin{aligned}
H^{(6)}_{3A,1}(\tau), H^{(6)}_{6A,1}(\tau) &:= H^{(18)}_{1A,1}(3\tau) - H^{(18)}_{1A,11}(3\tau) + H^{(18)}_{1A,13}(3\tau),\\
H^{(6)}_{3A,5}(\tau), H^{(6)}_{6A,5}(\tau) &:= H^{(18)}_{1A,5}(3\tau) - H^{(18)}_{1A,7}(3\tau) + H^{(18)}_{1A,17}(3\tau).
\end{aligned}
\tag{A.54}
$$

$\ell = 6+3$, $X = D_4^6$

We have $m^X = 6$, so the umbral McKay-Thompson series $H^{(6+3)}_g = (H^{(6+3)}_{g,r})$ associated to $g \in G^{(6+3)}$ is a 12-vector-valued function with components indexed by $r \in \mathbb{Z}/12\mathbb{Z}$. In addition to the identity $H^{(6+3)}_{g,r} = -H^{(6+3)}_{g,-r}$, we have $H^{(6+3)}_{g,r} = 0$ for $r = 0 \mod 2$. Thus it suffices to specify the $H^{(6+3)}_{g,r}$ for $r \in \{1, 3, 5\}$.

Recall (A.14). For $r = 1$, define

$$\begin{aligned}
H^{(6+3)}_{1A,1}(\tau), H^{(6+3)}_{3A,1}(\tau) &:= H^{(6)}_{1A,1}(\tau) + H^{(6)}_{1A,5}(\tau),\\
H^{(6+3)}_{2A,1}(\tau), H^{(6+3)}_{6A,1}(\tau) &:= H^{(6)}_{2B,1}(\tau) + H^{(6)}_{2B,5}(\tau),\\
H^{(6+3)}_{3B,1}(\tau) &:= H^{(6)}_{3A,1}(\tau) + H^{(6)}_{3A,5}(\tau),\\
H^{(6+3)}_{3C,1}(\tau) &:= -2\frac{\eta(\tau)^2}{\eta(3\tau)},\\
H^{(6+3)}_{4A,1}(\tau), H^{(6+3)}_{12A,1}(\tau) &:= H^{(6)}_{8AB,1}(\tau) + H^{(6)}_{8AB,5}(\tau),\\
H^{(6+3)}_{5A,1}(\tau), H^{(6+3)}_{15A,1}(\tau) &:= [-\tfrac{1}{24}]H^{(2)}_{15AB,1}(\tau/3),\\
H^{(6+3)}_{2C,1}(\tau) &:= H^{(6)}_{4A,1}(\tau) - H^{(6)}_{4A,5}(\tau),\\
H^{(6+3)}_{4B,1}(\tau) &:= H^{(6)}_{8AB,1}(\tau) - H^{(6)}_{8AB,5}(\tau),\\
H^{(6+3)}_{6B,1}(\tau) &:= H^{(6)}_{6A,1}(\tau) - H^{(6)}_{6A,5}(\tau),\\
H^{(6+3)}_{6C,1}(\tau) &:= -2\frac{\eta(2\tau)\,\eta(3\tau)}{\eta(6\tau)}.
\end{aligned} \tag{A.55}$$

Then define $H^{(6+3)}_{2B,1}$ by setting

$$H^{(6+3)}_{2B,1}(\tau) := 2H^{(6+3)}_{4B,1}(\tau) + 2\frac{\eta(\tau)^3}{\eta(2\tau)^2}. \tag{A.56}$$

For $r=3$ set

$$\begin{aligned} H^{(6+3)}_{1A,3}(\tau) &:= 2H^{(6)}_{1A,3}(\tau),\\ H^{(6+3)}_{3A,3}(\tau) &:= -H^{(6)}_{1A,3}(\tau),\\ H^{(6+3)}_{2A,3}(\tau) &:= 2H^{(6)}_{2B,3}(\tau),\\ H^{(6+3)}_{6A,3}(\tau) &:= -H^{(6)}_{2B,3}(\tau),\\ H^{(6+3)}_{4A,3}(\tau) &:= 2H^{(6)}_{8AB,3}(\tau),\\ H^{(6+3)}_{12A,3}(\tau) &:= -H^{(6)}_{8AB,3}(\tau),\\ H^{(6+3)}_{5A,3}(\tau) &:= -2[-\tfrac{9}{24}]H^{(2)}_{15AB,1}(\tau),\\ H^{(6+3)}_{15A,3}(\tau) &:= [-\tfrac{9}{24}]H^{(2)}_{15AB,1}(\tau), \end{aligned} \tag{A.57}$$

and

$$H^{(6+3)}_{3B,3}(\tau), H^{(6+3)}_{3C,3}(\tau), H^{(6+3)}_{2B,3}(\tau), H^{(6+3)}_{2C,3}(\tau), H^{(6+3)}_{4B,3}(\tau), H^{(6+3)}_{6B,3}(\tau), H^{(6+3)}_{6C,3}(\tau) := 0. \tag{A.58}$$

For $r=5$ define $H^{(6+3)}_{g,5}(\tau) := H^{(6+3)}_{g,1}(\tau)$ for $[g] \in \{1A, 3A, 2A, 6A, 3B, 3C, 4A, 12A, 5A, 15AB$ and set $H^{(6+3)}_{g,5}(\tau) := -H^{(6+3)}_{g,1}(\tau)$ for the remaining cases, $[g] \in \{2B, 2C, 4B, 6B, 6C\}$.

$\ell = 7$, $X = A_6^4$

We have $m^X = 7$, so the umbral McKay-Thompson series $H^{(7)}_g = (H^{(7)}_{g,r})$ associated to $g \in G^{(7)} = G^X \simeq SL_2(3)$ is a 14-vector-valued function, with components indexed by $r \in \mathbb{Z}/14\mathbb{Z}$.

Define $H^{(7)}_g = (H^{(7)}_{g,r})$ for $g \in G^{(7)}$ by requiring that

$$\psi^{(7)}_g(\tau,z) = -\chi^{(7)}_g \mu^0_{7,0}(\tau,z) - \bar{\chi}^{(7)}_g \mu^1_{7,0}(\tau,z) + \sum_{r \mod 14} H^{(7)}_{g,r}(\tau)\theta_{7,r}(\tau,z), \tag{A.59}$$

where $\chi^{(7)}_g := \chi^{X_A}_g$ and $\bar{\chi}^{(7)}_g := \bar{\chi}^{X_A}_g$ (cf. Table A.21), and the $\psi^{(7)}_g$ are meromorphic Jacobi forms of weight 1 and index 7 given explicitly in Table A.40.

Table A.40: Character Values and Meromorphic Jacobi Forms for $\ell = 7$, $X = A_6^4$

$[g]$	$\chi_g^{(7)}$	$\bar{\chi}_g^{(7)}$	$\psi_g^{(7)}(\tau, z)$
1A	4	4	$2i{}_1(\tau, 4z){}_1(\tau, z)^{-2}\eta(\tau)^3$
2A	-4	4	$-2i{}_1(\tau, 4z){}_2(\tau, z)^{-2}\eta(\tau)^3$
4A	0	0	$-2i{}_1(\tau, 4z){}_2(2\tau, 2z)^{-1}\eta(2\tau)\eta(\tau)$
3A	1	1	cf. (A.60)
6A	-1	1	cf. (A.60)

$$\begin{aligned}\psi_{3A}^{(7)}(\tau, z) &:= -i\frac{{}_1(\tau, 4z + \frac{1}{3}){}_1(\tau, z - \frac{1}{3}) + {}_1(\tau, 4z - \frac{1}{3}){}_1(\tau, z + \frac{1}{3})}{{}_1(3\tau, 3z)}\eta(3\tau) \\ \psi_{6A}^{(7)}(\tau, z) &:= -i\frac{{}_1(\tau, 4z + \frac{1}{3}){}_1(\tau, z - \frac{1}{6}) - {}_1(\tau, 4z - \frac{1}{3}){}_1(\tau, z + \frac{1}{6})}{{}_2(3\tau, 3z)}\eta(3\tau)\end{aligned} \tag{A.60}$$

For use later on we note that $\psi_{1A}^{(7)} = -2\Psi_{1,1}\varphi_1^{(7)}$, where

$$\varphi_1^{(7)}(\tau, z) := \frac{{}_1(\tau, 4z)}{{}_1(\tau, 2z)}. \tag{A.61}$$

$\ell = 8$, $X = A_7^2 D_5^2$

We have $m^X = 8$, so the umbral McKay-Thompson series $H_g^{(8)} = (H_{g,r}^{(8)})$ associated to $g \in G^{(8)}$ is a 16-vector-valued function with components indexed by $r \in \mathbb{Z}/16\mathbb{Z}$. We have $H_{g,r}^{(8)} = -H_{g,-r}^{(8)}$, so it suffices to specify the $H_{g,r}^{(8)}$ for $r \in \{1, 2, 3, 4, 5, 6, 7\}$.

To define $H_g^{(8)} = (H_{g,r}^{(8)})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-2\Psi_{1,1}(\tau, z)\left(\varphi_1^{(8)}(\tau, z) + \frac{1}{2}\varphi_2^{(8)}(\tau, z)\right) = -24\mu_{8,0}(\tau, z) + \sum_{r \bmod 16} h_r(\tau)\theta_{8,r}(\tau, z), \tag{A.62}$$

where

$$\begin{aligned}\varphi_1^{(8)}(\tau,z) &:= \varphi_1^{(3)}(\tau,z)\varphi_1^{(6)}(\tau,z) - 5\varphi_1^{(4)}(\tau,z)\varphi_1^{(5)}(\tau,z),\\ \varphi_2^{(8)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_1^{(5)}(\tau,z) - \varphi_1^{(8)}(\tau,z).\end{aligned} \tag{A.63}$$

(Cf. (A.34), (A.42), (A.46), (A.48).) Now define the $H^{(8)}_{1A,r}$ by setting

$$H^{(8)}_{1A,r}(\tau) := \frac{1}{6}h_r(\tau), \tag{A.64}$$

for $r \in \{1,3,4,5,7\}$, and

$$H^{(8)}_{1A,2}(\tau), H^{(8)}_{1A,6}(\tau) := \frac{1}{12}\left(h_2(\tau) + h_6(\tau)\right). \tag{A.65}$$

Define $H^{(8)}_{2A,r}$ for $1 \le r \le 7$ by requiring

$$H^{(8)}_{2A,r}(\tau) := -(-1)^r H^{(8)}_{1A,r}(\tau). \tag{A.66}$$

For the remaining g, recall (A.14). The $H^{(8)}_{g,r}$ for $g \in 2B \cup 2C \cup 4A$ are defined as follows for $r \in \{1,3,5,7\}$, noting that $H^{(4)}_{g,7} = H^{(4)}_{g,-1} = -H^{(4)}_{g,1}$, &c.

$$\begin{aligned}H^{(8)}_{2BC,r}(\tau) &:= [-\tfrac{r^2}{32}]H^{(4)}_{4C,r}(\tau/2)\\ H^{(8)}_{4A,r}(\tau) &:= [-\tfrac{r^2}{32}]H^{(4)}_{4B,r}(\tau/2)\end{aligned} \tag{A.67}$$

The $H^{(8)}_{2BC,r}$ and $H^{(8)}_{4A,r}$ vanish for $r = 0 \mod 2$.

$\ell = 9$, $X = A_8^3$

We have $m^X = 9$, so for $g \in G^{(9)}$ the associated umbral McKay-Thompson series $H^{(9)}_g = (H^{(9)}_{g,r})$ is a 18-vector-valued function, with components indexed by $r \in \mathbb{Z}/18\mathbb{Z}$, satisfying $H^{(9)}_{g,r} = -H^{(9)}_{g,-r}$, and in particular, $H^{(9)}_{g,r} = 0$ for $r = 0 \mod 9$. So it suffices to specify the $H^{(9)}_{g,r}$ for $r \in \{1,2,3,4,5,6,7,8\}$.

Define $H^{(9)}_g = (H^{(9)}_{g,r})$ for $g = e$ by requiring that

$$-\Psi_{1,1}(\tau,z)\varphi_1^{(9)}(\tau,z) = -3\mu_{9,0}(\tau,z) + \sum_{r \bmod 18} H^{(9)}_{e,r}(\tau)\theta_{9,r}(\tau,z), \tag{A.68}$$

where

$$\varphi_1^{(9)}(\tau,z) := \varphi_1^{(3)}(\tau,z)\varphi_1^{(7)}(\tau,z) - \varphi_1^{(5)}(\tau,z)^2. \tag{A.69}$$

(Cf. (A.34), (A.46), (A.61).)

Recall (A.14). The $H^{(9)}_{2B,r}$ are defined for $r \in \{1,2,4,5,7,8\}$ by setting

$$H^{(9)}_{2B,r}(\tau) := [-\tfrac{r^2}{36}]H^{(3)}_{6C,r}(\tau/3), \tag{A.70}$$

where we note that $H^{(3)}_{g,4} = H^{(3)}_{g,-2} = -H^{(3)}_{g,2}$, &c. We determine $H^{(9)}_{2B,3}$ and $H^{(9)}_{2B,6}$ by using §A.3.4 to set

$$H^{(9)}_{2B,r}(\tau) := H^{(18)}_{1A,r}(2\tau) - H^{(18)}_{1A,18-r}(2\tau) \tag{A.71}$$

for $r \in \{3,6\}$.

The $H^{(9)}_{3A,r}$ are defined by the explicit formulas

$$\begin{aligned}
H^{(9)}_{3A,1}(\tau) &:= [-\tfrac{1}{36}]f_1^{(9)}(\tau/3),\\
H^{(9)}_{3A,2}(\tau) &:= [-\tfrac{4}{36}]f_2^{(9)}(\tau/3),\\
H^{(9)}_{3A,3}(\tau) &:= -\theta_{3,3}(\tau,0),\\
H^{(9)}_{3A,4}(\tau) &:= -[-\tfrac{16}{36}]f_2^{(9)}(\tau/3),\\
H^{(9)}_{3A,5}(\tau) &:= -[-\tfrac{25}{36}]f_1^{(9)}(\tau/3),\\
H^{(9)}_{3A,6}(\tau) &:= \theta_{3,0}(\tau,0),\\
H^{(9)}_{3A,7}(\tau) &:= [-\tfrac{13}{36}]f_1^{(9)}(\tau/3),\\
H^{(9)}_{3A,8}(\tau) &:= [-\tfrac{28}{36}]f_2^{(9)}(\tau/3),
\end{aligned} \tag{A.72}$$

where

$$\begin{aligned}
f_1^{(9)}(\tau) &:= -2\frac{\mathbb{H}(\tau)\mathbb{H}(12\tau)\mathbb{H}(18\tau)^2}{\mathbb{H}(6\tau)\mathbb{H}(9\tau)\mathbb{H}(36\tau)},\\
f_2^{(9)}(\tau) &:= \frac{\mathbb{H}(2\tau)^6\mathbb{H}(12\tau)\mathbb{H}(18\tau)^2}{\mathbb{H}(\tau)\mathbb{H}(4\tau)^4\mathbb{H}(6\tau)\mathbb{H}(9\tau)\mathbb{H}(36\tau)} - \frac{\mathbb{H}(\tau)\mathbb{H}(2\tau)\mathbb{H}(3\tau)^2}{\mathbb{H}(4\tau)^2\mathbb{H}(9\tau)}.
\end{aligned} \tag{A.73}$$

Finally, the $H^{(9)}_{g,r}$ are determined for $g \in 2A \cup 2C \cup 6A$ by setting

$$\begin{aligned} H^{(9)}_{2A,r}(\tau) &:= (-1)^{r+1} H^{(9)}_{1A,r}(\tau), \\ H^{(9)}_{2C,r}(\tau) &:= (-1)^{r+1} H^{(9)}_{2B,r}(\tau), \\ H^{(9)}_{6A,r}(\tau) &:= (-1)^{r+1} H^{(9)}_{3A,r}(\tau). \end{aligned} \tag{A.74}$$

$\ell = 10$, $X = A_9^2 D_6$

We have $m^X = 10$, so the umbral McKay-Thompson series $H^{(10)}_g = (H^{(10)}_{g,r})$ associated to $g \in G^{(10)}$ is a 20-vector-valued function with components indexed by $r \in \mathbb{Z}/20\mathbb{Z}$. We have $H^{(10)}_{g,r} = -H^{(10)}_{g,-r}$, so it suffices to specify the $H^{(10)}_{g,r}$ for $1 \leq r \leq 9$.

To define $H^{(10)}_g = (H^{(10)}_{g,r})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-6\Psi_{1,1}(\tau,z)\varphi^{(10)}_1(\tau,z) = -24\mu_{10,0}(\tau,z) + \sum_{r \bmod 20} h_r(\tau)\theta_{10,r}(\tau,z), \tag{A.75}$$

where

$$\varphi^{(10)}_1(\tau,z) := 5\varphi^{(4)}_1(\tau,z)\varphi^{(7)}_1(\tau,z) - \varphi^{(5)}_1(\tau,z)\varphi^{(6)}_1(\tau,z). \tag{A.76}$$

(Cf. (A.42), (A.46), (A.48), (A.61).) Now define the $H^{(10)}_{1A,r}$ for r odd by setting

$$\begin{aligned} H^{(10)}_{1A,1}(\tau) &:= \frac{1}{24}\left(3h_1(\tau) + h_9(\tau)\right), \\ H^{(10)}_{1A,3}(\tau) &:= \frac{1}{24}\left(3h_3(\tau) + h_7(\tau)\right), \\ H^{(10)}_{1A,5}(\tau) &:= \frac{1}{6}h_5(\tau), \\ H^{(10)}_{1A,3}(\tau) &:= \frac{1}{24}\left(h_3(\tau) + 3h_7(\tau)\right), \\ H^{(10)}_{1A,9}(\tau) &:= \frac{1}{24}\left(h_1(\tau) + 3h_9(\tau)\right). \end{aligned} \tag{A.77}$$

For $r = 0 \mod 2$ set

$$H^{(10)}_{1A,r}(\tau) := \frac{1}{12} h_r(\tau), \tag{A.78}$$

and define $H^{(10)}_{2A,r}$ for $1 \leq r \leq 9$ by requiring

$$H^{(10)}_{2A,r}(\tau) := -(-1)^r H^{(10)}_{1A,r}(\tau). \tag{A.79}$$

It remains to specify $H^{(10)}_{g,r}$ for $g \in 4A \cup 4B$. For $r = 0 \mod 2$ set

$$H^{(10)}_{4AB,r}(\tau) := 0. \tag{A.80}$$

For r odd, recall (A.14), and define

$$H^{(10)}_{4A,r}(\tau) := [-\tfrac{r^2}{40}]\frac{1}{2}\left(H^{(2)}_{10A,r}(\tau/5) + H^{(5)}_{4CD,r}(\tau/2)\right). \tag{A.81}$$

$\ell = 10+5$, $X = D_6^4$

We have $m^X = 10$, so the umbral McKay-Thompson series $H^{(10+5)}_g = (H^{(10+5)}_{g,r})$ associated to $g \in G^{(10+5)}$ is a 20-vector-valued function with components indexed by $r \in \mathbb{Z}/20\mathbb{Z}$. We have $H^{(10+5)}_{g,r} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H^{(10+5)}_{g,r}$ for r odd. Observing that $H^{(10+5)}_{g,r} = -H^{(10+5)}_{g,-r}$ we may determine $H^{(10+5)}_g$ by requiring that

$$\psi^{(5/2)}_g(\tau,z) = -2\chi^{(5/2)}_g i\mu_{5/2,0}(\tau,z) + \sum_{\substack{r\in\mathbb{Z}+1/2 \\ r \mod 5}} e(-r/2) H^{(10+5)}_{g,2r}(\tau)\theta_{5/2,r}(\tau,z), \tag{A.82}$$

where $\chi^{(5/2)}_g := \bar{\chi}^{X_D}_g$ as in Table A.25, and the $\psi^{(5/2)}_g$ are the meromorphic Jacobi forms of weight 1 and index 5/2 defined as follows.

$\ell = 12$, $X = A_{11}D_7E_6$

We have $m^X = 12$, so the umbral McKay-Thompson series $H^{(12)}_g = (H^{(12)}_{g,r})$ associated to $g \in G^{(12)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 24-vector-valued function with components indexed by $r \in \mathbb{Z}/24\mathbb{Z}$. We have $H^{(12)}_{g,r} = -H^{(12)}_{g,-r}$, so it suffices to specify the $H^{(12)}_{g,r}$ for $1 \leq r \leq 11$.

Table A.41: Character Values and Meromorphic Jacobi Forms for $\ell = 10+5$, $X = D_6^4$

$[g]$	$\bar{\chi}_g^{(5/2)}$	$\psi_g^{(5/2)}(\tau,z)$
1A	4	$2i\theta_1(\tau,2z)^2\theta_1(\tau,z)^{-3}\eta(\tau)^3$
2A	0	$-2i\theta_1(\tau,2z)^2\theta_1(\tau,z)^{-1}\theta_2(\tau,z)^{-2}\eta(\tau)^3$
3A	1	$2i\theta_1(3\tau,6z)\theta_1(\tau,2z)^{-1}\theta_1(3\tau,3z)^{-1}\eta(\tau)^3$
2B	2	$2i\theta_1(\tau,2z)\theta_2(\tau,2z)\theta_1(\tau,z)^{-2}\theta_2(\tau,z)^{-1}\eta(\tau)^3$
4A	0	$-2i\theta_1(\tau,2z)\theta_2(\tau,2z)\theta_2(2\tau,2z)^{-1}\eta(\tau)\eta(2\tau)$

To define $H_e^{(12)} = (H_{e,r}^{(12)})$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-2\Psi_{1,1}(\tau,z)\left(\varphi_1^{(12)}(\tau,z)+\varphi_2^{(12)}(\tau,z)\right) = -24\mu_{12,0}(\tau,z) + \sum_{r \bmod 24} h_r(\tau)\theta_{12,r}(\tau,z), \tag{A.83}$$

where

$$\begin{aligned}\varphi_1^{(12)}(\tau,z) &:= 3\varphi_1^{(3)}(\tau,z)\varphi_1^{(10)}(\tau,z) - 8\varphi_1^{(4)}(\tau,z)\varphi_1^{(9)}(\tau,z) + \varphi_1^{(5)}(\tau,z)\varphi_1^{(8)}(\tau,z),\\ \varphi_2^{(12)}(\tau,z) &:= 4\varphi_1^{(4)}(\tau,z)\varphi_1^{(9)}(\tau,z) - \varphi_1^{(5)}(\tau,z)\varphi_1^{(8)}(\tau,z) - \varphi_1^{(12)}(\tau,z).\end{aligned} \tag{A.84}$$

(Cf. (A.34), (A.42), (A.46), (A.61), (A.63), (A.69), (A.76).) Now define the $H_{1A,r}^{(12)}$ for $r \neq 0 \mod 3$ by setting

$$\begin{aligned} H_{1A,1}^{(12)}(\tau) &:= \frac{1}{24}\left(3h_1(\tau)+h_7(\tau)\right),\\ H_{1A,2}^{(12)}(\tau), H_{1A,10}^{(12)}(\tau) &:= \frac{1}{24}\left(h_2(\tau)+h_{10}(\tau)\right),\\ H_{1A,4}^{(12)}(\tau), H_{1A,8}^{(12)}(\tau) &:= \frac{1}{12}\left(h_4(\tau)+h_8(\tau)\right),\\ H_{1A,5}^{(12)}(\tau) &:= \frac{1}{24}\left(3h_5(\tau)+h_{11}(\tau)\right),\\ H_{1A,7}^{(12)}(\tau) &:= \frac{1}{24}\left(h_1(\tau)+3h_7(\tau)\right),\\ H_{1A,11}^{(12)}(\tau) &:= \frac{1}{24}\left(h_5(\tau)+3h_{11}(\tau)\right).\end{aligned} \tag{A.85}$$

For $r = 0 \mod 3$ set

$$H^{(12)}_{1A,r}(\tau) := \frac{1}{12} h_r(\tau), \tag{A.86}$$

and define $H^{(12)}_{2A,r}$ by requiring

$$H^{(12)}_{2A,r}(\tau) := -(-1)^r H^{(12)}_{1A,r}(\tau). \tag{A.87}$$

$\ell = 12 + 4$, $X = E_6^4$

We have $m^X = 12$, so the umbral McKay-Thompson series $H^{(12+4)}_g = (H^{(12+4)}_{g,r})$ associated to $g \in G^{(12+4)}$ is a 24-vector-valued function with components indexed by $r \in \mathbb{Z}/24\mathbb{Z}$. In addition to the identity $H^{(12+4)}_{g,r} = -H^{(12+4)}_{g,-r}$, we have $H^{(12+4)}_{g,r} = 0$ for $r \in \{2, 3, 6, 9, 10\}$, $H^{(12+4)}_{g,1} = H^{(12+4)}_{g,7}$, $H^{(12+4)}_{g,4} = H^{(12+4)}_{g,8}$, and $H^{(12+4)}_{g,5} = H^{(12+4)}_{g,11}$. Thus it suffices to specify the $H^{(12+4)}_{g,1}$, $H^{(12+4)}_{g,4}$ and $H^{(12+4)}_{g,5}$.

Recall (A.14). Also, set $S^{E_6}_1(\tau) := S_{12,1}(\tau) + S_{12,7}(\tau)$, and $S^{E_6}_5(\tau) := S_{12,5}(\tau) + S_{12,11}(\tau)$. For $r = 1$ define

$$\begin{aligned}
H^{(12+4)}_{1A,1}(\tau) &:= H^{(12)}_{1A,1}(\tau) + H^{(12)}_{1A,7}(\tau),\\
H^{(12+4)}_{2B,1}(\tau) &:= [-\tfrac{1}{48}] \left(H^{(6)}_{8AB,1}(\tau/2) - H^{(6)}_{8AB,5}(\tau/2) \right),\\
H^{(12+4)}_{4A,1}(\tau) &:= \frac{1}{S^{E_6}_1(\tau)^2 - S^{E_6}_5(\tau)^2} \left(-2\frac{\eta(2\tau)^8}{\eta(\tau)^4} S^{E_6}_1(\tau) + 8\frac{\eta(\tau)^4\eta(4\tau)^4}{\eta(2\tau)^4} S^{E_6}_5(\tau) \right),\\
H^{(12+4)}_{3A,1}(\tau) &:= [-\tfrac{1}{48}] \left(H^{(6)}_{3A,1}(\tau/2) - H^{(6)}_{3A,5}(\tau/2) \right),\\
H^{(12+4)}_{8AB,1}(\tau) &:= \frac{1}{S^{E_6}_1(\tau)^2 - S^{E_6}_5(\tau)^2} \left(-2F^{(12+4)}_{8AB,1}(\tau) S^{E_6}_1(\tau) + 12 F^{(12+4)}_{8AB,5}(\tau/2) S^{E_6}_5(\tau) \right).
\end{aligned} \tag{A.88}$$

In the expression for $g \in 8AB$, we write $F^{(12+4)}_{8AB,1}$ for the unique modular form of weight 2 for $\Gamma_0(32)$ such that

$$F^{(12+4)}_{8AB,1}(\tau) = 1 + 12q + 4q^2 - 24q^5 - 16q^6 - 8q^8 + O(q^9), \tag{A.89}$$

and we write $F^{(12+4)}_{8AB,5}$ for the unique modular form of weight 2 for $\Gamma_0(64)$ such that

$$F^{(12+4)}_{8AB,5}(\tau) = 3q + 4q^3 + 6q^5 - 8q^7 - 9q^9 + 12q^{11} - 18q^{13} - 24q^{15} + O(q^{17}). \tag{A.90}$$

For $r = 4$ define

$$\begin{aligned} H^{(12+4)}_{1A,4}(\tau) &:= H^{(12)}_{1A,4}(\tau) + H^{(12)}_{1A,8}(\tau), \\ H^{(12+4)}_{3A,4}(\tau) &:= H^{(6)}_{3A,2}(\tau/2) + H^{(6)}_{3A,4}(\tau/2), \end{aligned} \tag{A.91}$$

and set $H^{(12+4)}_{g,4}(\tau) := 0$ for $g \in 2B \cup 4A \cup 8AB$.

For $r = 5$ define

$$\begin{aligned} H^{(12+4)}_{1A,5}(\tau) &:= H^{(12)}_{1A,5}(\tau) + H^{(12)}_{1A,11}(\tau), \\ H^{(12+4)}_{2B,5}(\tau) &:= [-\tfrac{25}{48}]\left(H^{(6)}_{8AB,5}(\tau/2) - H^{(6)}_{8AB,1}(\tau/2)\right), \\ H^{(12+4)}_{4A,5}(\tau) &:= \frac{1}{S^{E_6}_1(\tau)^2 - S^{E_6}_5(\tau)^2}\left(2\frac{\eta(2\tau)^8}{\eta(\tau)^4}S^{E_6}_5(\tau) - 8\frac{\eta(\tau)^4\eta(4\tau)^4}{\eta(2\tau)^4}S^{E_6}_1(\tau)\right), \\ H^{(12+4)}_{3A,5}(\tau) &:= [-\tfrac{25}{48}]\left(H^{(6)}_{3A,5}(\tau/2) - H^{(6)}_{3A,1}(\tau/2)\right), \\ H^{(12+4)}_{8AB,5}(\tau) &:= \frac{1}{S^{E_6}_1(\tau)^2 - S^{E_6}_5(\tau)^2}\left(2F^{(12+4)}_{8AB,1}(\tau)S^{E_6}_5(\tau) - 12F^{(12+4)}_{8AB,5}(\tau/2)S^{E_6}_1(\tau)\right). \end{aligned} \tag{A.92}$$

Finally, define $H^{(12+4)}_{g,r}$ for $g \in 2A \cup 6A$ by setting

$$\begin{aligned} H^{(12+4)}_{2A,r}(\tau) &:= -(-1)^r H^{(12+4)}_{1A,r}(\tau), \\ H^{(12+4)}_{6A,r}(\tau) &:= -(-1)^r H^{(12+4)}_{3A,r}(\tau). \end{aligned} \tag{A.93}$$

$\ell = 13$**,** $X = A^2_{12}$

We have $m^X = 13$, so the umbral McKay-Thompson series $H^{(13)}_g = (H^{(13)}_{g,r})$ associated to $g \in G^{(13)} = G^X \simeq \mathbb{Z}/4\mathbb{Z}$ is a 26-vector-valued function, with components indexed by $r \in \mathbb{Z}/26\mathbb{Z}$.

Define $H_g^{(13)} = (H_{g,r}^{(13)})$ for $g \in G^{(13)}$ by requiring that

$$\psi_g^{(13)}(\tau, z) = -\chi_g^{(13)} \mu_{13,0}^0(\tau, z) - \bar{\chi}_g^{(13)} \mu_{13,0}^1(\tau, z) + \sum_{r \mod 26} H_{g,r}^{(13)}(\tau) \theta_{13,r}(\tau, z), \tag{A.94}$$

where $\chi_g^{(13)} := \chi_g^{X_A}$ and $\bar{\chi}_g^{(13)} := \bar{\chi}_g^{X_A}$ (cf. Table A.28), and the $\psi_g^{(13)}$ are meromorphic Jacobi forms of weight 1 and index 13 given explicitly in Table A.42.

Table A.42: Character Values and Meromorphic Jacobi Forms for $\ell = 13$, $X = A_{12}^2$

$[g]$	$\chi_g^{(13)}$	$\bar{\chi}_g^{(13)}$	$\psi_g^{(13)}(\tau, z)$
1A	2	2	$2i{}_1(\tau, 6z){}_1(\tau, z)^{-1}{}_1(\tau, 3z)^{-1}\eta(\tau)^3$
2A	-2	2	$-2i{}_1(\tau, 6z){}_2(\tau, z)^{-1}{}_2(\tau, 3z)^{-1}\eta(\tau)^3$
4A	0	0	cf. (A.95)

$$\psi_{4AB}^{(13)}(\tau, z) := -i{}_2(\tau, 6z) \frac{{}_1(\tau, z + \frac{1}{4}){}_1(\tau, 3z + \frac{1}{4}) - {}_1(\tau, z - \frac{1}{4}){}_1(\tau, 3z - \frac{1}{4})}{{}_2(2\tau, 2z){}_2(2\tau, 6z)} \frac{\eta(2\tau)^2}{\eta(\tau)} \tag{A.95}$$

For use later on we note that $\psi_{1A}^{(13)} = -2\Psi_{1,1}\varphi_1^{(13)}$, where

$$\varphi_1^{(13)}(\tau, z) := \frac{{}_1(\tau, z){}_1(\tau, 6z)}{{}_1(\tau, 2z){}_1(\tau, 3z)}. \tag{A.96}$$

$\ell = 14 + 7$**,** $X = D_8^3$

We have $m^X = 14$, so the umbral McKay-Thompson series $H_g^{(14+7)} = (H_{g,r}^{(14+7)})$ associated to $g \in G^{(14+7)}$ is a 28-vector-valued function with components indexed by $r \in \mathbb{Z}/28\mathbb{Z}$. We have $H_{g,r}^{(14+7)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H^{(14+7)}_{g,r}$ for r odd. Observing that $H^{(14+7)}_{g,r} = -H^{(14+7)}_{g,-r}$ we may determine $H^{(14+7)}_{g}$ by requiring that

$$\psi^{(7/2)}_g(\tau,z) = -2\bar{\chi}^{(7/2)}_g i\mu_{7/2,0}(\tau,z) + \sum_{\substack{r\in\mathbb{Z}+1/2\\ r \mod 7}} e(-r/2)H^{(14+7)}_{g,2r}(\tau)\theta_{7/2,r}(\tau,z), \tag{A.97}$$

where $\bar{\chi}^{(7/2)}_g := \bar{\chi}^{X_D}_g$ is the number of fixed points of $g \in G^{(14+7)} \simeq S_3$ in the defining permutation representation on 3 points. The $\psi^{(7/2)}_g$ are the meromorphic Jacobi forms of weight 1 and index 7/2 defined in Table A.43.

Table A.43: Character Values and Meromorphic Jacobi Forms for $\ell = 14+7$, $X = D_8^3$

$[g]$	$\bar{\chi}^{(7/2)}_g$	$\psi^{(7/2)}_g(\tau,z)$
1A	3	$2i\theta_1(\tau,3z)\theta_1(\tau,z)^{-2}\eta(\tau)^3$
2A	1	$2i\theta_2(\tau,3z)\theta_1(\tau,z)^{-1}\theta_2(\tau,z)^{-1}\eta(\tau)^3$
3A	0	$-2i_1(\tau,z)_1(\tau,3z)_1(3\tau,3z)^{-1}\eta(3\tau)$

$\ell = 16$, $X = A_{15}D_9$

We have $m^X = 16$, so the umbral McKay-Thompson series $H^{(16)}_g = (H^{(16)}_{g,r})$ associated to $g \in G^{(16)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 32-vector-valued function with components indexed by $r \in \mathbb{Z}/32\mathbb{Z}$. We have $H^{(16)}_{g,r} = -H^{(16)}_{g,-r}$, so it suffices to specify the $H^{(16)}_{g,r}$ for $1 \le r \le 15$.

To define $H^{(16)}_g = (H^{(16)}_{g,r})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-6\Psi_{1,1}(\tau,z)\left(\varphi^{(16)}_1(\tau,z) + \frac{1}{2}\varphi^{(16)}_2(\tau,z)\right) = -24\mu_{16,0}(\tau,z) + \sum_{r \mod 32} h_r(\tau)\theta_{16,r}(\tau,z), \tag{A.98}$$

where

$$\varphi_1^{(16)}(\tau,z) := 8\varphi_1^{(4)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(5)}(\tau,z)\varphi_1^{(12)}(\tau,z) + \varphi_1^{(7)}(\tau,z)\varphi_1^{(10)}(\tau,z),$$
$$\varphi_2^{(16)}(\tau,z) := 12\varphi_1^{(4)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(5)}(\tau,z)\varphi_1^{(12)}(\tau,z) - 3\varphi_1^{(16)}(\tau,z). \tag{A.99}$$

(Cf. (A.42), (A.46), (A.61), (A.76), (A.84), (A.96).) Now define the $H_{1A,r}^{(16)}$ by setting

$$H_{1A,r}^{(16)}(\tau) := \frac{1}{12}h_r(\tau) \tag{A.100}$$

for r odd. For r even, $2 \leq r \leq 14$, use

$$H_{1A,r}^{(16)}(\tau) := \frac{1}{24}\left(h_r(\tau) + h_{16-r}(\tau)\right). \tag{A.101}$$

Define $H_{2A,r}^{(16)}$ by requiring

$$H_{2A,r}^{(16)}(\tau) := -(-1)^r H_{1A,r}^{(16)}(\tau). \tag{A.102}$$

$\ell = 18$, $X = A_{17}E_7$

We have $m^X = 18$, so the umbral McKay-Thompson series $H_g^{(18)} = (H_{g,r}^{(18)})$ associated to $g \in G^{(18)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 36-vector-valued function with components indexed by $r \in \mathbb{Z}/36\mathbb{Z}$. We have $H_{g,r}^{(18)} = -H_{g,-r}^{(18)}$, so it suffices to specify the $H_{g,r}^{(18)}$ for $1 \leq r \leq 17$.

To define $H_g^{(18)} = (H_{g,r}^{(18)})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-24\Psi_{1,1}(\tau,z)\phi^{(18)}(\tau,z) = -24\mu_{18,0}(\tau,z) + \sum_{r \bmod 36} h_r(\tau)\theta_{18,r}(\tau,z), \tag{A.103}$$

where

$$\phi^{(18)} := \frac{1}{12}\left(\varphi_1^{(18)} + \frac{1}{3}\varphi_3^{(18)} + 4\frac{\overset{12}{1}}{\eta^{12}}\left(\varphi_1^{(12)} + 2\varphi_2^{(12)} + \frac{1}{3}\varphi_3^{(12)}\right)\right). \tag{A.104}$$

For the definition of $\phi^{(18)}$ we require

$$\begin{aligned}
\varphi_2^{(9)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_1^{(6)}(\tau,z) - 4\varphi_1^{(5)}(\tau,z)^2 - 4\varphi_1^{(9)}(\tau,z),\\
\varphi_1^{(11)}(\tau,z) &:= 3\varphi_1^{(5)}(\tau,z)\varphi_1^{(7)}(\tau,z) + 2\varphi_1^{(3)}(\tau,z)\varphi_1^{(9)}(\tau,z) - \varphi_1^{(4)}(\tau,z)\varphi_1^{(8)}(\tau,z),\\
\varphi_3^{(12)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_2^{(9)}(\tau,z),\\
\varphi_1^{(14)}(\tau,z) &:= 3\varphi_1^{(5)}(\tau,z)\varphi_1^{(10)}(\tau,z) + \varphi_1^{(3)}(\tau,z)\varphi_1^{(12)}(\tau,z) - 4\varphi_1^{(4)}(\tau,z)\varphi_1^{(11)}(\tau,z),\\
\varphi_1^{(15)}(\tau,z) &:= \varphi_1^{(5)}(\tau,z)\varphi_1^{(11)}(\tau,z) + 6\varphi_1^{(3)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(4)}(\tau,z)\varphi_1^{(12)}(\tau,z),\\
\varphi_2^{(15)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_1^{(12)}(\tau,z) - 2\varphi_1^{(5)}(\tau,z)\varphi_1^{(11)}(\tau,z) - 2\varphi_1^{(15)}(\tau,z),\\
\varphi_1^{(18)}(\tau,z) &:= \varphi_1^{(5)}(\tau,z)\varphi_1^{(14)}(\tau,z) + 3\varphi_1^{(3)}(\tau,z)\varphi_1^{(16)}(\tau,z) - 4\varphi_1^{(4)}(\tau,z)\varphi_1^{(15)}(\tau,z),\\
\varphi_3^{(18)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_2^{(15)}(\tau,z),
\end{aligned} \tag{A.105}$$

in addition to the other $\varphi_k^{(m)}$ that have appeared already. Now define the $H_{1A,r}^{(18)}$ by setting

$$H_{1A,r}^{(18)}(\tau) := \frac{1}{24}h_r(\tau) \tag{A.106}$$

for r even. For r odd, use

$$\begin{aligned}
H^{(18)}_{1A,1}(\tau) &:= \frac{1}{24}\left(2h_1(\tau)+h_{17}(\tau)\right),\\
H^{(18)}_{1A,3}(\tau) &:= \frac{1}{24}\left(h_3(\tau)+h_9(\tau)\right),\\
H^{(18)}_{1A,5}(\tau) &:= \frac{1}{24}\left(2h_5(\tau)+h_{13}(\tau)\right),\\
H^{(18)}_{1A,7}(\tau) &:= \frac{1}{24}\left(2h_7(\tau)+h_{11}(\tau)\right),\\
H^{(18)}_{1A,9}(\tau) &:= \frac{1}{24}\left(h_3(\tau)+2h_9(\tau)+h_{15}(\tau)\right),\\
H^{(18)}_{1A,11}(\tau) &:= \frac{1}{24}\left(h_7(\tau)+2h_{11}(\tau)\right),\\
H^{(18)}_{1A,13}(\tau) &:= \frac{1}{24}\left(h_5(\tau)+2h_{13}(\tau)\right),\\
H^{(18)}_{1A,15}(\tau) &:= \frac{1}{24}\left(h_{15}(\tau)+h_9(\tau)\right),\\
H^{(18)}_{1A,17}(\tau) &:= \frac{1}{24}\left(h_1(\tau)+2h_{17}(\tau)\right).
\end{aligned} \tag{A.107}$$

Define $H^{(18)}_{2A,r}$ in the usual way for root systems with a type A component, by requiring

$$H^{(18)}_{2A,r}(\tau) := -(-1)^r H^{(18)}_{1A,r}(\tau). \tag{A.108}$$

$\ell = 18+9$**,** $X = D_{10}E_7^2$

We have $m^X = 18$, so the umbral McKay-Thompson series $H^{(18+9)}_g = (H^{(18+9)}_{g,r})$ associated to $g \in G^{(18+9)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 36-vector-valued function with components indexed by $r \in \mathbb{Z}/36\mathbb{Z}$. We have $H^{(18+9)}_{g,r} = -H^{(18+9)}_{g,-r}$, $H^{(18+9)}_{g,r} = H^{(18+9)}_{g,18-r}$ for $1 \le r \le 17$, and $H^{(18+9)}_{g,r} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H^{(18+9)}_{g,r}$ for $r \in \{1,3,5,7,9\}$.

Define

$$\begin{aligned}
H^{(18+9)}_{1A,r}(\tau) &:= H^{(18)}_{1A,r}(\tau) + H^{(18)}_{1A,18-r}(\tau),\\
H^{(18+9)}_{2A,r}(\tau) &:= H^{(18)}_{1A,r}(\tau) - H^{(18)}_{1A,18-r}(\tau),
\end{aligned} \tag{A.109}$$

for $r \in \{1,3,5,7,9\}$.

$\ell = 22 + 11$, $X = D_{12}^2$

We have $m^X = 22$, so the umbral McKay-Thompson series $H_g^{(22+11)} = (H_{g,r}^{(22+11)})$ associated to $g \in G^{(22+11)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 44-vector-valued function with components indexed by $r \in \mathbb{Z}/44\mathbb{Z}$. We have $H_{g,r}^{(22+11)} = -H_{g,-r}^{(22+11)}$ and $H_{g,r}^{(22+11)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H_{g,r}^{(22+11)}$ for r odd. Observing that $H_{g,r}^{(22+11)} = -H_{g,-r}^{(22+11)}$ we may determine $H_g^{(22+11)}$ by requiring that

$$\psi_g^{(11/2)}(\tau, z) = -2\bar{\chi}_g^{(11/2)} i\mu_{11/2,0}(\tau, z) + \sum_{\substack{r\in\mathbb{Z}+1/2 \\ r \mod 11}} e(-r/2) H_{g,2r}^{(22+11)}(\tau)\theta_{11/2,r}(\tau, z), \tag{A.110}$$

where $\bar{\chi}_{1A}^{(11/2)} := 2$, $\bar{\chi}_{2A}^{(11/2)} := 0$, and the $\psi_g^{(11/2)}$ are the meromorphic Jacobi forms of weight 1 and index 11/2 defined as follows.

$$\begin{aligned} \psi_{1A}^{(11/2)}(\tau, z) &:= 2i\frac{\theta_1(\tau, 4z)}{\theta_1(\tau, z)\theta_1(\tau, 2z)}\eta(\tau)^3 \\ \psi_{2A}^{(11/2)}(\tau, z) &:= -2i\frac{\theta_1(\tau, 4z)}{\theta_2(\tau, z)\theta_2(\tau, 2z)}\eta(\tau)^3 \end{aligned} \tag{A.111}$$

$\ell = 25$, $X = A_{24}$

We have $m^X = 25$, so for $g \in G^{(25)} \simeq \mathbb{Z}/2\mathbb{Z}$, the associated umbral McKay-Thompson series $H_g^{(25)} = (H_{g,r}^{(25)})$ is a 50-vector-valued function, with components indexed by $r \in \mathbb{Z}/50\mathbb{Z}$, satisfying $H_{g,r}^{(25)} = -H_{g,-r}^{(25)}$, and in particular, $H_{g,r}^{(25)} = 0$ for $r = 0 \mod 25$. So it suffices to specify the $H_{g,r}^{(25)}$ for $1 \leq r \leq 24$.

Define $H_g^{(25)} = (H_{g,r}^{(25)})$ for $g = e$ by requiring that

$$-\Psi_{1,1}(\tau, z)\varphi_1^{(25)}(\tau, z) = -\mu_{25,0}(\tau, z) + \sum_{r \mod 50} H_{e,r}^{(25)}(\tau)\theta_{25,r}(\tau, z), \tag{A.112}$$

where

$$\varphi_1^{(25)}(\tau, z) := \frac{1}{2}\varphi_1^{(5)}(\tau, z)\varphi_1^{(21)}(\tau, z) - \varphi_1^{(7)}(\tau, z)\varphi_1^{(19)}(\tau, z) + \frac{1}{2}\varphi_1^{(13)}(\tau, z)^2. \tag{A.113}$$

For the definition of $\varphi_1^{(25)}$ we require

$$\begin{aligned}
\varphi_1^{(17)}(\tau,z) &:= 4\varphi_1^{(5)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(9)}(\tau,z)^2,\\
\varphi_1^{(19)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_1^{(16)}(\tau,z) + 2\varphi_1^{(7)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(5)}(\tau,z)\varphi_1^{(15)}(\tau,z),\\
\varphi_1^{(21)}(\tau,z) &:= \varphi_1^{(5)}(\tau,z)\varphi_1^{(17)}(\tau,z) - 2\varphi_1^{(9)}(\tau,z)\varphi_1^{(13)}(\tau,z),
\end{aligned} \tag{A.114}$$

in addition to the other $\varphi_k^{(m)}$ that have appeared already. Define $H_{2A,r}^{(25)}$ in the usual way for root systems with a type A component, by requiring

$$H_{2A,r}^{(18)}(\tau) := -(-1)^r H_{1A,r}^{(18)}(\tau). \tag{A.115}$$

$\ell = 30+15$**,** $X = D_{16}E_8$

We have $m^X = 30$, so the umbral McKay-Thompson series $H_g^{(30+15)} = (H_{g,r}^{(30+15)})$ associated to $g \in G^{(30+15)} = \{e\}$ is a 60-vector-valued function with components indexed by $r \in \mathbb{Z}/60\mathbb{Z}$. We have $H_{e,r}^{(30+15)} = -H_{e,-r}^{(30+15)}$, $H_{e,r}^{(30+15)} = H_{e,30-r}^{(30+15)}$ for $1 \le r \le 29$, and $H_{e,r}^{(30+15)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H_{e,r}^{(30+15)}$ for $r \in \{1,3,5,7,9,11,13,15\}$.

Define

$$\begin{aligned}
H_{1A,1}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,1}^{(30+6,10,15)} + [-\tfrac{1}{120}]H_{3A,1}^{(10+5)}(\tau/3)\right),\\
H_{1A,3}^{(30+15)}(\tau) &:= [-\tfrac{9}{120}]H_{3A,3}^{(10+5)}(\tau/3),\\
H_{1A,5}^{(30+15)}(\tau) &:= [-\tfrac{25}{120}]H_{3A,5}^{(10+5)}(\tau/3),\\
H_{1A,7}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,7}^{(30+6,10,15)} + [-\tfrac{49}{120}]H_{3A,3}^{(10+5)}(\tau/3)\right),\\
H_{1A,11}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,1}^{(30+6,10,15)} - [-\tfrac{1}{120}]H_{3A,1}^{(10+5)}(\tau/3)\right),\\
H_{1A,13}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,7}^{(30+6,10,15)} - [-\tfrac{49}{120}]H_{3A,3}^{(10+5)}(\tau/3)\right),\\
H_{1A,15}^{(30+15)}(\tau) &:= -[-\tfrac{105}{120}]H_{3A,5}^{(10+5)}(\tau/3).
\end{aligned} \tag{A.116}$$

$\ell = 30+6,10,15$**,** $X = E_8^3$

We have $m^X = 30$, and $G^{(30+6,10,15)} = G^X \simeq S_3$. The umbral McKay-Thompson series $H^{(30+6,10,15)}$ is a 60-vector-valued function with components indexed by $r \in \mathbb{Z}/60\mathbb{Z}$. We have

$$H_{g,r}^{(30+6,10,15)}(\tau) = \begin{cases} \pm H_{g,1}^{(30+6,10,15)} & \text{if } r = \pm 1, \pm 11, \pm 19, \pm 29 \mod 60, \\ \pm H_{g,7}^{(30+6,10,15)} & \text{if } r = \pm 7, \pm 13, \pm 17, \pm 27 \mod 60, \\ 0 & \text{else,} \end{cases} \tag{A.117}$$

so it suffices to specify the $H_{g,r}^{(30+6,10,15)}$ for $r = 1$ and $r = 7$. These functions may be defined as follows.

$$\begin{aligned}
H_{1A,1}^{(30+6,10,15)} &:= -2\frac{1}{\eta(\tau)^2}\left(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\right)(-1)^{k+l+m}q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+(k+l+m)/2+3/40}\\
H_{2A,1}^{(30+6,10,15)} &:= -2\frac{1}{\eta(2\tau)}\left(\sum_{k,m\geq 0} - \sum_{k,m<0}\right)(-1)^{k+m}q^{3k^2+m^2/2+4km+(2k+m)/2+3/40}\\
H_{3A,1}^{(30+6,10,15)} &:= -2\frac{\eta(\tau)}{\eta(3\tau)}\sum_{k\in\mathbb{Z}}(-1)^k q^{15k^2/2+3k/2+3/40}\\
H_{1A,7}^{(30+6,10,15)} &= -2\frac{1}{\eta(\tau)^2}\left(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\right)(-1)^{k+l+m}q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+3(k+l+m)/2+27/40}\\
H_{2A,7}^{(30+6,10,15)} &= 2\frac{1}{\eta(2\tau)}\left(\sum_{k,m\geq 0} - \sum_{k,m<0}\right)(-1)^{k+m}q^{3k^2+m^2/2+4km+3(2k+m)/2+27/40}\\
H_{3A,7}^{(30+6,10,15)} &= -2\frac{\eta(\tau)}{\eta(3\tau)}\sum_{k\in\mathbb{Z}}(-1)^k q^{15k^2/2+9k/2+27/40}
\end{aligned} \tag{A.118}$$

$\ell = 46+23$**,** $X = D_{24}$

We have $m^X = 22$, and $G^{(46+23)} = \{e\}$. The umbral McKay-Thompson series $H_e^{(46+23)} = (H_{e,r}^{(46+23)})$ is a 92-vector-valued function with components indexed by $r \in \mathbb{Z}/92\mathbb{Z}$. We have $H^{(46+23)}_{e,r} = -H^{(46+23)}_{e,-r}$ and $H^{(46+23)}_{e,r} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H^{(46+23)}_{e,r}$ for r odd. Observing that $H^{(46+23)}_{e,r} = -H^{(46+23)}_{e,-r}$ we may determine $H^{(46+23)}_{e}$ by requiring that

$$\psi^{(23/2)}_{e}(\tau, z) = -2i\mu_{23/2,0}(\tau, z) + \sum_{\substack{r\in\mathbb{Z}+1/2 \\ r \mod 23}} e(-r/2)H^{(46+23)}_{g,2r}(\tau)\theta_{23/2,r}(\tau, z), \tag{A.119}$$

where $\psi^{(23/2)}_{e}$ is the meromorphic Jacobi forms of weight 1 and index 23/2 defined by setting

$$\psi^{(23/2)}_{e}(\tau, z) := 2i\frac{\theta_1(\tau, 6z)}{\theta_1(\tau, 2z)\theta_1(\tau, 3z)}\eta(\tau)^3. \tag{A.120}$$

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