Three Essays in Operations Management

by

Lifei Sheng

B.Sc. (Mathematics), Shanghai Jiao Tong University, 2012

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Abstract

This thesis comprises three independent essays in operations management. The first essay explores a specific issue encountered by mobile gaming companies. The remaining two essays address the contracting problem in a supply chain setting.

In the first essay, we study the phenomena of game companies offering to pay users in "virtual" benefits to take actions in-game that earn the game company revenue from third parties. Examples of such "incentivized actions" include paying users in "gold coins" to watch video advertising and speeding in-game progression in exchange for filling out a survey etc. We develop a dynamic optimization model that looks at the costs and benefits of offering incentivized actions to users as they progress in their engagement with the game. We find sufficient conditions for the optimality of a threshold strategy of offering incentivized actions to low-engagement users and then removing incentivized action to encourage real-money purchases once a player is sufficiently engaged. Our model also provides insights into what types of games can most benefit from offering incentivized actions.

In the second essay, we propose what we call a generalized price-only contract, which is a dynamic generalization of the simple wholesale price-only contract. We derive some interesting properties of this contract and relate them to well-known issues such as double marginalization, relative power in a supply chain due to Stackelberg leadership, contract structure and commitment issues.

In the third essay, we consider a supplier selling to a retailer with private inventory information over multiple periods. We focus on dynamic short-term contracts, where contracting takes place in every period. At the beginning of each period, with inventory or backlog kept privately by the retailer, the supplier offers a one-period contract and the retailer decides his order quantity in anticipation of uncertain customer demand. We cast the problem as a dynamic adverse-selection problem with Markovian dynamics. We show that the optimal short-term contract has a threshold structure, with possibly multiple thresholds. In certain cost regimes, the optimal contract entails a base-stock policy yet induces partial participation.

Lay Summary

In the first essay, we explore whether the mobile gaming company should pay users in "virtual" benefits to watch video advertising or fill out surveys so that the company earns revenue from third parties. We help the company to target what games benefit most from this practice, and we design the best way to implement it.

Essays 2 and 3 consider a supplier selling to a downstream retailer who faces random customer demand. The supplier determines the type and terms of the contract. In essay 2, we study a dynamic generalization of the simple wholesale price-only contract. We examine the impacts on the decisions and profits, if the two companies are allowed to trade multiple times. In essay 3, we characterize the optimal short-term contract in the case where the supplier needs to offer a new contract in every period, without knowing the retailer's beginning inventory or backorder.

Preface

Chapter 2 is co-authored with Christopher Ryan and Mahesh Nagarajan. Chapter 3 is co-authored with Daniel Granot, Tim Huh and Mahesh Nagarajan. Chapter 4 is co-authored with Mahesh Nagarajan and Hao Zhang.

In all chapters, I was responsible for developing the models, carrying out the analysis and presenting the results. My coauthors were involved in providing supervision and feedback in problem formulation, model analysis and manuscript edits. The three chapters will be modified and submitted for publication in academic peer reviewed journals.

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Dedication

To my parents, Fusen Sheng and Liping Shi

Chapter 1

Introduction

The area of operations management has found many applications and connections to other disciplines, such as economics, marketing and information systems. This thesis presents three essays in the domain of operations management. Although the topics are diverse, they can be sorted into two major streams. One is the application of operations management to study issues in the digital economy (Chapter 2). The other is the application of mechanism design in operations management (Chapters 3 and 4).

Operations Management in the Digital Economy

The digital economy, especially the emergence of smartphones, social media and cloud data service, is radically changing the ways in which people work, learn, entertain themselves and socialise. By 2015, the worth of the mobile content market alone reached 27.5 billion U.S. dollars worldwide, and the number of social network users exceeded 2 billion. The emergence of the digital economy also impacts the way people do business. In particular, as technical innovation spurs the era of big data, decision makers are able to collect personalized data and apply data driven analytics to understand customers better and improve decision-making. Moreover, a number of new business models have emerged. One example is freemium games that are free to download where revenue is generated only after use either through in-app transactions or from other third parties. The first essay (Chapter 2) explores a particular practice called incentivized actions that are commonly implemented in mobile games.

Mobile games are the fastest growing segment of the entertainment industry globally, which itself is dominated by freemium games. A recent innovation is to offer "incentives" for players by paying them with "virtual" benefits for clicking on banner ads, watching videos, or filling out surveys. These are collectively called incentivized actions, or shortened as incented actions. Such a new business model raises many interesting questions.

In Chapter 2, we take the perspective of a game publisher and explore the use of incented actions in mobile games. Specifically, we study the following questions: Should game publishers offer incented actions? If so, how to optimally design a policy for offering incented actions? If a gaming company offers several products, which of its games can most benefit from offering incented actions?

We model the publisher's problem as a Markov decision process where the underlying state is the player's engagement levels and the publisher's decision is whether or not to offer incented actions. We provide sufficient conditions for the optimality of a threshold strategy of offering incented actions to low-engaged players and then removing them to encourage real-money purchases once a player is sufficiently engaged. We also explore the settings where the optimality of the threshold policy breaks down. Moreover, we provide managerial insights and assist game publishers in targeting which types of games can take most advantage of delivering incented actions. For instance, we show analytically that social games that include player interactions as part of the design should offer incented actions more broadly. We also discuss different effects of the design of incented actions for attracting and engaging players, including their "strength", i.e. the power of their associated virtual benefits.

Mechanism Design in Operations Management

Mechanism design is a field pioneered by economists but has recently found important applications in operations management, especially the areas of supply chain management and healthcare management. Numerous studies have analyzed how contracts should be designed to mediate interactions among self-interested firms. The major part of the literature has focused on the static setting with complete information. In the real world, however, multi-period contracting is also (if not more) prevalent, with contracting parties having private information and decisions being made dynamically. Dynamic contracting is known to be a challenging problem due to a host of technical and expositional difficulties. Several researchers have exerted a significant effort to characterize the optimal mechanism in certain specific settings. For instance, Battaglini [4] characterizes the optimal long-term contract between a monopolist and a buyer whose private preferences evolve as a two-state Markov process. He finds that the optimal contract is contingent on the buyer's complete purchase history and once the buyer reveals himself to be the high-type, the supply will become efficient in all future periods. This is considered a significant finding in dynamic contracting, yet limited by the two-state assumption.

Chapters 3 and 4 focus on a two-echelon supply chain in which a retailer ("he") buys inventory from an upstream supplier ("she") dynamically in anticipation of uncertain customer demand. The supplier needs to determine the terms of the contract. We are interested in finding how dynamic interactions affect bilateral business relationships and whether it will lead to significantly different contracts than under one-shot interactions.

In Chapter 3, we look at a two-stage supply chain with symmetric information. We propose a generalized price-only contract that is a dynamic generalization of the simple wholesale price-only contract. The supplier first informs the retailer that n wholesale prices will be offered sequentially and dynamically. For each wholesale price proposed, the retailer chooses an order quantity at that price. At the end of the last offer, the retailer uses the total quantity cumulatively purchased to satisfy market demand. We examine how it will affect the firms' decisions if, instead of having one opportunity to trade, they are allowed to engage multiple times, still using a simple linear price-only contract.

It is well known that the classical wholesale price-only contract causes supply chain inefficiency due to the double marginalization effect. We show that the generalized price-only contract benefits both players. Moreover, as the number of price offers n approaches infinity, the supply chain profit approaches the first best profit. We also demonstrate that for a given contract with a specific n, the wholesale prices monotonically decrease. However, somewhat surprisingly, for a fixed n, the order quantities within the n periods may not be monotone. We provide necessary and sufficient conditions for the stationarity of the supplier's per period profit. Finally, we derive closed form solution for three settings in which the demand is exponential, uniform or constant.

In Chapter 4, we consider a supplier selling to a retailer with private inventory information over multiple periods. A few pioneering studies have explored the contracting problem in this setting. Zhang et al. [57] focus on dynamic short-term contract in the lost sales case and they show that the optimal contract is a batch-order contract under certain assumptions. Ilan and Xiao [27] study the optimal long-term contract and prove that it takes a simple form in both lost-sales and backlogging cases.

Filling a gap in the literature, our work focuses on dynamic short-term contracts, where contracting takes place in every period, with inventory or backlog kept privately by the retailer. We cast the problem as a dynamic adverse-selection problem with Markovian dynamics. Markovian adverse-selection models, in which the state and action in a period affect the state in the subsequent period, are theoretically challenging and much less understood. Our work contributes to a better understanding of such models, especially under short-term contracting.

We show that the optimal short-term contract has a threshold structure, with possibly multiple thresholds, under exponentially distributed demand. In a high cost regime, the optimal short-term contract may entail a base-stock order policy and an exclusion region. If the retailer's inventory (or backlog) falls in the exclusion region, the supplier terminates the relationship with the retailer. If not, the retailer participates and orders up to a constant base-stock level. It is drastically different from the lost sales setting.

Moreover, in the backlogging case, the supplier finds more sales opportunities in the retailer's backlog situation, which increases the retailer's bargaining power. As an interesting result, the information rent (profit yielded to the retailer) under the optimal contract may be non-monotone in the retailer's inventory (or backlog) level. The supplier would sometimes prefer to deal with retailers with high inventory, which is different from the lost-sales case where the supplier always wants to trade with retailers who have low inventory.

The rest of the thesis is organized as follows. Each essay is self-contained and is presented in one chapter, with a more exhaustive discussion of literature review, research question and main contributions. All proofs are relegated to Appendices.

Chapter 2

Incentivized Actions in Freemium Games

2.1 Introduction

Games represent the fastest growing sector of the entertainment industry globally, which includes music, movies and print publishing [39]. Moreover, the online/mobile space is the fastest growing segment within games, which itself is dominated by games employing a "freemium" business model. Freemium games are free to download and play and earn revenue through advertising or selling game enhancements to dedicated players. When accessed on 23 April 2015, Apple Inc.'s App Store showed 190 out of the 200 top revenue generating games (and all of the top 20) were free to download.[1] On Google Play, the other major mobile games platform, 297 out of the 300 top revenue generating games were freemium.[2] Moreover, games are the dominant revenue generators in the global app market. Revenues from mobile games account for 79% of total app revenue on Apple's App Store and 92% of revenue on Google Play [48].

The concept behind freemium is to attract large pools of players, many of whom might never "monetize"; that is, pay for an in-app purchase. The process by which a player begins to pay out-of-pocket for a freemium game is called *monetization*. In general, successful games have a monetization rate of between 2 and 10 per cent, with the average much closer to 2 per cent [36]. As for unsuccessful games, the monetization rate can be virtually zero.

When game publishers cannot earn directly from the pockets of consumers they turn to other sources of revenue. This is largely through earning revenue from third parties willing to pay publishers for delivering advertising content, have players download other apps, fill out surveys, or apply for services, such as credit cards. This stream of revenue is less lucrative per conversion than in-app purchases. For instance, delivering a video ad typically earns a fraction of a cent while an in-app purchase typically earns the publisher fifty cents or more.

Like most modern consumers, however, players can become irritated by advertising, especially when it interrupts the flow or breaks the fiction of a game. A recent innovation is to offer "incentives" for players to click on a banner ad, watch a video, or fill out a survey. These are collectively called *incentivized actions*, or as it is commonly shortened, *incented actions*. To get a clearer sense of the structure of an incented action and the value of the "incentive" to a

[1] `http://appshopper.com/bestsellers/games/gros/?device=iphone`
[2] `https://play.google.com/store/apps/collection/topgrossing?hl=en`

player, details of the mechanics and goals of a game are needed to provide context. We feel this is best achieved through the description of the following two concrete examples.

Crossy Road

Crossy Road is a freemium game developed by Hipster Whale that has recently (since 2014) seen great success with incented video advertising, earning over 10 million USD in the first three months after its launch [19]. In *Crossy Road*, the player controls a character who attempts to cross busy streets full of fatal obstacles. The main progression of the game is to collect additional characters to play, including animals, avatars of famous people, and many others. The characters must be unlocked through earning "coins". "Coins" are earned organically by playing the game at a slow rate. Periodically the player has an option to watch a video ad to earn a large bundle of coins all at once. Once a player collects one hundred coins she can use them to randomly draw a character. If the player is unlucky she may draw a character she previously unlocked. If the player wishes to purchase a specific character (of which there are now dozens) it will cost at least 0.99 USD. The incented action (watching an ad) accelerates the progression of the player by rewarding large bundles of coins, but the value of the incentive weakens as random draws are increasingly unlikely to unlock a new character as the player progresses. Moreover, there can be long stretches of time where video ads are not offered, forcing the player to either make progress organically or purchase characters with real money.

Candy Crush Saga

A second illustrative example is *Candy Crush Saga*, published by King. King was recently acquired by Activision-Blizzard for 5.9 billion USD based on the enduring popularity of *Candy Crush Saga* and its portfolio of successful games [37]. In *Candy Crush Saga*, a player attempts to solve a progression of increasingly challenging puzzles. At the higher levels it is typical for players to get stuck for extended periods of time on a single puzzle. Player progression is further hindered by a "lives" mechanic where each failed attempt at a puzzle consumes one of at most five total lives. Lives are regenerated either through waiting long periods of real time or by purchasing additional lives with real money. In addition to lives, players can also pay for items that enhance their chances of completing a puzzle.

Early versions of *Candy Crush Saga* had incented actions, including advertising. A player could take an incented actions to earn lives or items without using real money. However, in June of 2013, six months after *Candy Crush Saga* launched on Apple iOS, King decided to drop all forms of in-game advertising in the game [18].

King's choice was surprising to many observers. What was the logic for removing a potential revenue stream? How did this move affect the monetization rate? The ramifications from such decisions vary depending on the game and can potentially have significant financial consequences. To get a sense of this, note that an in-app-purchase can be between a dollar to around $5 and Supercell earned approximately 2.3 billion in revenue in 2015 purely through monetization of its three games [51]. Our two examples of games that have experimented with the use of incented actions also raise several related important questions about the impact of incented actions. For example, when is it best to offer incented actions? If offered, is it optimal to offer them to certain players at certain times, but not others? Also, if a gaming company offers several products, which of its games are better suited to offering incented actions? Our paper develops a framework for answering some of these important questions.

Our contributions

In this paper we present an analytical model to explore the use of incented actions. In particular, we are interested in a game publisher's decision of when to offer incented actions to players, and when to remove this option. Our model emphasizes the connection of incented actions to two other useful concepts often discussed in the game industry – engagement and retention. The *engagement* of a player measures their commitment. Highly engaged players are more likely to make in-app purchases and less likely to quit. *Retention* refers to a game's effectiveness at keeping players from quitting. Intuitively, the longer a player is retained in the game, the more likely they are to become engaged and monetize. Clearly, these two concepts are interrelated. Analytically, player engagement levels are modeled as states in a Markov chain and retention is captured as the time a player stays in the system before being absorbed into a "quit" state.

The main insights from our model deal with the relationship between engagement, retention and incented actions. We identify, and provide analytical characterizations for, three main effects of incented actions. These effects are described in with greater precision below, but we mention them here at a conceptual level.

First is the *revenue effect*: by offering incented actions game publishers open up another channel of revenue. However, the net revenue of offering incented actions may nonetheless be *negative* if one accounts for the opportunity costs of players not making in-app purchases. That is, this captures the possibility that a player would have made an in-app purchase if an incented action was not available. For instance, in *Crossy Road* a player may collect characters entirely through watching video ads, but if this option were removed a player may begin to purchase characters with real money.

The *retention effect* measures how effective an incented action is at keeping players from quitting. Again, in the example of *Crossy Road*, at some point the organic accumulation of "coins" may feel prohibitively slow to a player. If the option of watching video ads were removed, a player may prefer to quit rather than start to use real money to purchase characters. In other words, incented actions can delay a player's decision to quit the game.

Finally, the *progression effect* refers the effectiveness of an incented action in deepening the engagement level of the player. It refers to an incented actions ability to increase the player's attachment to the game. In *Crossy Road* video ads allow players to collect characters, potentially deepening their engagement. These three effects are intuitively understood by game developers and the topic of much discussion and debate in the gaming industry.[3] Gaming companies grapple with the issue of understanding how these effects interact with each other in the context of specific games. As we shall see in concrete examples below, all three effects can act to either improve or erode the overall revenue available to the publisher. Each effect is clearly connected and they often move in similar directions as players progress. Part of our analysis is to describe situations where the effects move in different, sometimes counter-intuitive, directions.

We are able to analytically characterize each effect, allowing us to gain insights into how to optimally design a policy for offering incented actions. To understand the interactions between these effects and to capture the dynamics in a game, we use *Markov chains* to model player engagement and how they transition from one level of engagement to another. Then, using a Markov Decision Process (MDP) model we study the effect of specific decisions or policies of the game publisher. For example, we provide sufficient conditions for when a *threshold* policy is optimal. In a threshold policy incented actions are offered until a player reaches a target engagement level, after which incented actions are removed. The intuition of these policies is clear. By offering incented actions, the retention effect and progression effect keep the player in for longer by providing a non-monetizing option for progression. However, once a player is sufficiently engaged, the revenue effect becomes less beneficial and the retention effect less significant because highly engaged players are more likely to buy in-app purchases and keep playing the game. This suggests that it is optimal to remove incented actions and attempt to extract revenue directly from the player through monetization. Our sufficient conditions provide justification for this logic, but we also explore settings where this basic intuition breaks down. For instance, it is possible that the retention effect remains a dominant concern even at higher engagement levels. Indeed, a highly engaged player may be quite likely to monetize and so there is a strong desire on the part of the publisher to keep the player in the system for longer by offering incented actions to bolster retention.

MDPs are used to study dynamics in systems such as ours and are popular in the economics, operations management and marketing literatures. There are several advantages to using MDPs to model and study settings such as ours. First of all, they are an effective tool for theoretical analyses, such as the one we are interested in. This is because MDP theory is rich and allows one to prove formal results on the interactions between different variables of interest. Second, with the availability of player level data as is the case with games, it is relatively easy to validate these models and perform "what if" scenarios using simulations to test different scenarios of interest. We believe ours is the first formal model and study using these ideas in a gaming setting and we anticipate that the results and modeling approach will be useful to researchers in this area as well as practitioners.

Clearly, the relative strengths of these three effects depend on the characteristics the game, including all the parameters in our MDP model. We examine this dependence by tracking

[3] Discussion of issues is a regular occurrence on gaming industry forums, such as `gaminginsiders.com`.

how the threshold in an optimal threshold policy changes with the parameters. This analysis provides insights into the nature of optimal incented action policies.

For instance, we show analytically that the more able players are at attracting their friends into playing the game, the greater should be the threshold for offering incented actions. This suggests that social games that include player interaction as part of their design should offer incented actions more broadly, particularly when the retention effect is strongly positive, since keeping players in the game for longer gives them more opportunities to invite friends. Indeed, a common incented action is to contact friends in your social network or to build a social network to earn in-game rewards. This managerial insight can assist game publishers in targeting what types of games in a portfolio of game projects can take most advantage of delivering incented actions.

We also discuss the different effects of the design of incented actions, in particular their "strength" at attracting and engaging players. "Strength" here refers to how powerful the reward of the incented action is in the game. For instance, the number of "coins" given to the player when an incented action is taken. If this reward is powerful, in comparison to in-app purchases, then it can help players progress, strengthening the progression effect. On the other hand, a stronger incented action may dissuade players further from monetizing, strengthening cannibalization. Through numerical examples we illustrate a variety of possible effects that tradeoff the behavioral effects of players responding to the nature of the incented action reward and show that whether or not to offer incented actions to highly engaged players depends in a nonmonotonic way on the parameters of our model that indicate the strength of incented actions.

The rest of the paper is organized as follows. In Section 2.2 we review related work, paying close attention to contributions from the information systems and marketing literatures. Section 2.3 presents our model, first developing a stochastic model of player behavior and then formulating the game publisher's decision problem as an MDP. In Section 2.4 we formally define the three effects mentioned above and characterize them analytically. These effects are leveraged to provide sufficient conditions for an optimal threshold policy in Section 2.5. Section 2.6 draws out policy implications and managerial insights that arise from studying optimal threshold policies. Section 2.7 concludes. Proofs of all results are in the appendix.

2.2 Related Literature

As freemium business models have grown in prominence, so has interest in studying various aspects of freemium in the management literature. While papers in the marketing literature on freemium business models have been largely empirical (see for instance Gupta et al. [24] and Lee et al. [33]), our work connects most directly to a stream of analytical studies in the information systems literature that explore how various approaches to the concept of "free" have been used in the software industry. Two important papers for our context are Niculescu and Wu [43] and Cheng et al. [13] that together establish a taxonomy of different freemium strategies and examine in what situations a given strategy is most advantageous. *Seeding* is a strategy where a number of products are given away entirely for free, to build a user base that attracts new users through word-of-mouth and network effects. Previous studies explored the seeding strategy by adapting the Bass model [3] to the software setting (see for instance Jiang and Sarkar [28]). Another strategy is *time-limited freemium* where all users are given access to a complete product for a limited time, after which access is restricted (see Cheng and Liu [12] for more details). Our setting is best captured by the *feature-limited freemium* category where a functional base product can always be accessed by users, with additional features available for purchase by users. In freemium mobile games, a base game is available freely for download with additional items and features for sale through accumulated virtual currency or real-money purchases.

Our work departs from this established literature in at least two dimensions. First, we focus on how to tactically implement a freemium strategy, in particular, when and how to offer incented actions to drive player retention and monetization. By contrast, the existing literature has largely focused on comparing different freemium strategies and their advantage over conventional software sales. This previous work is, of course, essential in understanding the business case for freemium. Our work contributes to a layer of tactical questions of interest to firms committed to a freemium strategy in need of further insights in how it should be deployed.

Second, games present a specific context that may be at odds with some common conceptualizations of a freemium software product. For a productivity-focused product, such as a PDF editor, a typical implementation of freemium is to put certain advanced features behind a pay wall, such as the ability to make handwritten edits on files using a stylus. Once purchased, features are typically unlocked either in perpetuity or for a fixed duration by the paying player. By contrast, in games what is often purchased are virtual items or currency that may enhance the in-game experience, speed progression, or provide some competitive advantage. These purchases are often *consumables*, meaning that they are depleted through use. This is true, for instance, of all purchases in *Candy Crush Saga*. Our model allows for a player to make repeated purchases and the degree of intensity of monetization to evolve over the course of play.

Other researchers have examined the specific context offered by games, as opposed to general software products, and have adapted specialized theory to this specific context. Guo et al. [23] examine how the sale of virtual currencies in digital games can create a win-win scenario for players and publishers from a social-welfare perspective. They make a strong case for the value created by games offering virtual currency systems. Our work adds an additional layer by examining how virtual currencies can be used to incentivize players to take actions that are profitable to the firm that does not involve a real-money exchange. A third-party, such as an advertiser, can create a mutually beneficial situation where the player earns additional virtual currency, the publisher earns revenue from the advertiser, and the advertiser promotes their product. Also, Guo et al. [23] develop a static model where players decide on how to allocate a budget between play and purchasing virtual currency. We relate a player's willingness to take incented actions or monetize as their engagement with the game evolves, necessitating the use of a dynamic model. This allows us to explore how a freemium design can respond to the actions of players over time. This idea of progression in games has been explored empirically in Albuquerque and Nevskaya [1] and we adapt similar notions to derive analytical insights in our setting.

The dynamic nature of our model also shares similarities with threads of the vast customer relationship management (CRM) literature in marketing. In this literature, researchers are interested in how firms balance acquisition, retention and monetization of players through the pricing and design of their product or service over time. For example, Libai et al. [35] adapt Bass's model to the diffusion of services where player retention is an essential ingredient in the spread of the popularity of a platform. Fruchter and Sigué [21] provide insight into how a service can be priced to maximize revenue over its lifespan. Both studies employ continuous-time and continuous-state models that are well-suited to examine the overall flow of player population. Our focus of analysis is at the player level and asks how to design the game (i.e. service) to balance retention and monetization through offering incented actions for a given acquired player. Indeed, game designs on mobile platforms can, in principle, be specialized down to a specific player. With the increasing availability of individual player level data, examination of how to tailor design with more granularity is worthy of exploration. By contrast, existing continuous models treat a single player's choice with measure zero significance.

Finally, our modeling approach of using a discrete time Markov decision process model in search of threshold policies is a standard-bearer of analysis in the operations management literature. We have mentioned the advantages of this approach earlier. Threshold policies, which we work to establish, have the benefit of being easily implementable and thus draw favor in studies of tactical decision-making that is common in multiple areas including the economics and operations management literature. The intuition for their ease of use is somewhat easy to understand. The simplest type of threshold policies allows the system designer to simply keep track of nothing but the threshold (target) level and monitor the state of the system and take the appropriate action to reap the benefits of optimality. This is in contrast to situations where the optimal policy can be complex and has nontrivial state and parameter dependencies. Examples of this policy being effectively used in dynamic settings include inventory and capacity management and control [58], revenue management [52] and adaptive learning and pricing [50].

2.3 Model

We take the perspective of a game publisher who is deciding how to optimally deploy incented actions in its game. Incented actions can be offered (or not) at different times during a player's experience with the game. For example, a novice player may be able to watch video ads for rewards during the first few hours of game play, only later to have this option removed.

Our model has two agents: the game publisher and a single player. This assumes that the game publisher has the ability to offer a customized policy to each its player, or at least customized policies to different classes of players. In other words, the "player" in our model can be seen as the representative of a class of players who behave similarly. The publisher may need to decide on several different policies for different classes of players for an overall optimal design.

We assume that the player in our two-agent model behaves stochastically according to the options presented to her by the game publisher. The player model is a Markov chain with engagement level as the state variable. The game publisher's decision problem is a Markov Decision Problem (MDP) where the stochasticity is a function of the underlying player model and the publisher decision whether or not to offer incented actions. The player model is described in detail in the next subsection. The publisher's problem is detailed in Section 2.3.2.

2.3.1 Player Model

The player can take three actions while playing the game. The first is to *monetize* (denoted M) by making an in-app purchase with real money. The second is to *quit* (denoted Q). Once a player takes the quit action she never returns to playing the game. Third, the player can take an *incented action* (denoted I). The set of available actions is determined by whether the publisher offers an incented action or not. We let $A_1 = \{M, I, Q\}$ denote the set of available actions when an incented action is offered and $A_0 = \{M, Q\}$ otherwise.

The probability that the player takes a particular action depends on her *engagement level*. Engagement level is a general concept that can be understood in different ways depending on the specifics of the game. For example, in *Crossy Road* engagement level may be a function of the number of characters that have been collected, in *Candy Crush Saga* the level of the puzzle the player is currently on. Let $E = \{0, 1, \ldots, N\}$ be the set of possible engagement levels of the player.

The probability that the player takes an action also depends on what actions are available to her. We used the letter "p" to denote probabilities when an incented action is available and write $p_a(e)$ to denote the probability of taking action $a \in A_1$ at engagement level $e \in E$. For example, $p_M(2)$ is the probability of monetizing at engagement level 2 while $p_I(0)$ is the probability of taking an incented action at engagement level 0. We use the letter "q" to denote action probabilities when the incented action is unavailable and write $q_a(e)$ for the probability of taking action $a \in A_0$ at engagement level $e \in E$. By definition $p_M(e) + p_I(e) + p_Q(e) = 1$ and $q_M(e) + q_Q(e) = 1$ for all $e \in E$.

There is a relationship between $p_a(e)$ and $q_a(e)$. When an incentivized action is not available the probability $p_I(e)$ is allocated to the remaining two actions M and Q. We assume this probability is allocated as follows.

Assumption 2.3.1. *For each $e \in E$ there exists a parameter $\alpha(e) \in [0,1]$ such that:*

$$q_M(e) = p_M(e) + \alpha(e)p_I(e) \tag{2.1}$$

$$q_Q(e) = p_Q(e) + (1-\alpha(e))p_I(e). \tag{2.2}$$

Note that $\alpha(e)$ must be such that $p_M(e)+p_I(e)+p_Q(e) = 1$ and $q_M(e)+q_Q(e) = 1$ for all $e \in E$.

We call $\alpha(e)$ the *cannibalization parameter at engagement level e*, since $\alpha(e)$ measures the impact of removing an incented action on the probability of monetizing and thus captures the degree to which incented actions cannibalize demand for in-app purchases. A large $\alpha(e)$ (close to 1) implies strong cannibalization whereas a small $\alpha(e)$ (close to 0) signifies weak cannibalization.

It remains to consider how a player transitions from one engagement level to another engagement level. We must first describe the time epochs where actions and transitions take place. The decision epochs where actions are undertaken occur when the player is assessing whether they want or not to continue playing the game. For example, in *Crossy Road* a player must choose to monetize, watch a video ad, or quit once she can no longer tolerate the organic rate of progression. The real elapsed time between decision epochs is not constant, since it depends on the behavior of the player between sessions of play. Some players may play frequently, others only for a few minutes per day. A player might be highly engaged but nonetheless have little time to play due to other life obligations. This reality underscores that the elapsed time between decision epochs should not be a critical factor in our model. We denote the engagement level at decision epoch t by e_t and the action at decision epoch t by a_t.

Returning to the question of transitioning from engagement level to engagement level, in principle we would need to determine individually each transition probability $\mathbb{P}(e_{t+1} = e'|e_t = e \text{ and } a_t = a)$ (or more simply, $\mathbb{P}(e'|e,a)$ since we will assume that transition probabilities are stationary over time). However, we make the following simplifying assumption about state transitions: (i) engagement increases by at most one level at every decision epoch and never goes down, (ii) the transition probability is independent of the current engagement level and depends only on the action taken by the player, (iii) the impact on transitioning to a higher engagement level when taking the monetize action M is independent of whether an incented action was offered or not. This implies the following structure.

Assumption 2.3.2. *The engagement level transition probabilities satisfy the following conditions:*

$$\mathbb{P}(e'|e,a) = \begin{cases} \tau_a & \text{if } e' = e+1 \text{ and } e < N \\ 1-\tau_a & \text{if } e' = e < N \\ 1 & \text{if } e = e' = N \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

for $a \in \{M, I\}$. For $a = Q$ the player transitions with probability one to a quit state denoted

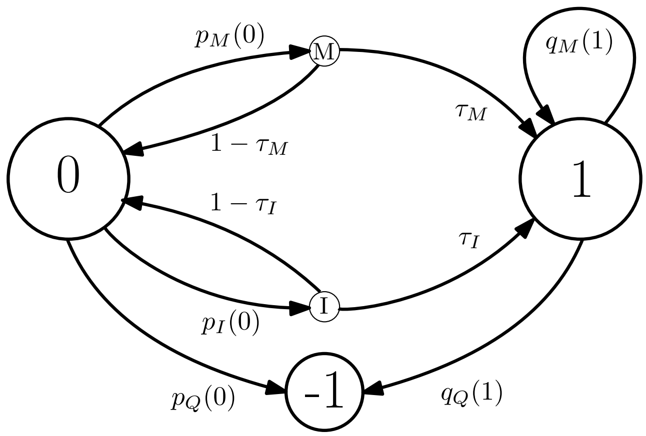


Figure 2.1: A visual representations of the Markov chain model of player behavior with two engagement levels and incented actions available at engagement level 0.

-1.

The overall state transition probabilities are:

$$\mathbb{P}_1(e'|e) = \begin{cases} p_M(e)\tau_M + p_I(e)\tau_I & \text{if } e' = e+1 \text{ and } e < N \\ p_M(e)(1-\tau_M) + p_I(e)(1-\tau_I) & \text{if } e' = e < N \\ p_M(e) + p_I(e) & \text{if } e = e' = N \\ p_Q(e) & \text{if } e' = -1 \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

when an incented action is available to the player and

$$\mathbb{P}_0(e'|e) = \begin{cases} q_M(e)\tau_M & \text{if } e' = e+1 \text{ and } e < N \\ q_M(e)(1-\tau_M) & \text{if } e' = e < N \\ q_M(e) & \text{if } e = e' = N \\ q_Q(e) & \text{if } e' = -1 \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

when incented actions are not offered. Figure 2.1 provides a visual representation of the Markov chain describing player behavior when there are two engagement levels, with incented action only offered at engagement level 0. Our assumptions make the structure of state transitions relatively simple, but nonetheless still capture the complexity of first having a probabilistic realization of an action followed by a random transition depending on the action taken. Indeed, despite the simplicity of these assumptions they still turn out to be insufficient to ensure some seemingly intuitive properties of incented actions (see further discussion below, in particular the need for several additional assumptions to drive analysis).

We close this subsection by providing some additional basic assumptions on the Markov chain data. These assumptions ensure that the model is consistent with what we mean by player engagement and our understanding of what holds in practice.

Assumption 2.3.3. *We make the following assumptions:*

(A3.1) $p_M(e)$ *and* $q_M(e)$ *increase in* e*,*

(A3.2) $p_Q(e)$ *and* $q_Q(e)$ *decrease in* e*,*

(A3.3) $p_Q(e), q_Q(e) > 0$ *for all* $e \in E$*,*

(A3.4) $p_I(e)$ *decreases in* e*,*

(A3.5) $\tau_M > \tau_I$*, and*

(A3.6) $\alpha(e)$ *is increasing in* e*.*

Assumptions (A3.1) and (A3.2) ensure that more engaged players are more likely to make in-app purchases and less likely to quit. This is precisely how we understand the concept of engagement – the more invested a player is in a game the more likely they are to spend and the less likely they are to quit. Assumption (A3.3) ensures that there is always a nonzero probability of quitting, no matter the level of engagement. This acknowledges the fact that games are entertainment activities, and there are numerous reasons for a player to quit due to factors in their daily lives, even when engrossed in the game. Moreover, this turns out to be an important technical assumption that allows us to consider a total reward criterion for the publisher's decision problem that avoids mathematical complexities (see Section 2.3.2 below).

Assumption (A3.4) ensures that players are less likely to take an incented action as their engagement level increases. One interpretation of this is that the rewards associated with an incented action are less valuable as a player progresses, decreasing the probability of taking such an action. Observe that (A3.1)–(A3.4) put implicit assumptions on the cannibalization parameter $\alpha(e)$ via (2.1) and (2.2).

Assumption (A3.5) implies that a player is more likely to increase their engagement when monetizing than taking an incented action. This assumption is well-justified for two reasons. The first is that players may view the making an in-app purchase as a kind of investment and become more committed to playing to ensure their investment pays off. Second, the rewards for incented actions are typically less powerful than what can be purchased for real money. The example of *Crossy Road* is illustrative: specific characters can be directly bought with real money, but watching video ads only contributes to random draws for characters.

Finally, (A3.6) implies that a greater share of the probability of taking an incented actions when offered is allocated to monetization when an incented ad is removed (see (2.1)). This assumption is intuitive and consistent again with our concept of engagement – as a player becomes more engaged the monetization option becomes relatively more attractive than quitting when the incented action is removed.

2.3.2 The Publisher's Problem

We model the publisher's problem as an infinite horizon Markov decision process under a total reward criterion (for details see Puterman [46]). A Markov decision process is specified by a set of states, controls in each state, transition probabilities under pairs of states and controls, and rewards for each transition.

Specifically in our setting based on the description of the dynamics we have laid out thus far, the set of states is $\{-1\} \cup E$ and the set of controls $U = \{0, 1\}$ is independent of the state, where 1 represents offering an incented action and 0 not offering an incented action. The transition probabilities are given by (2.4) when $u = 1$ and (2.5) when $u = 0$. The reward depends on the action of the player. When the player quits, the publisher earns no revenue, denoted by $\mu_Q = 0$. When the player takes an incented action the publisher earns μ_I, while a monetization actions earns μ_M.

Assumption 2.3.4. *We assume* $\mu_I < \mu_M$.

This assumption is in concert with practice, as discussed in the introduction.

The expected reward in state e under control u is:

$$r(e, u) = \begin{cases} p_M(e)\mu_M + p_I(e)\mu_I & \text{if } e \in E \text{ and } u = 1 \\ q_M(e)\mu_M & \text{if } e \in E \text{ and } u = 0 \\ 0 & \text{if } e = -1. \end{cases} \tag{2.6}$$

Note that expected rewards do not depend on whether the player transitions to a higher engagement level and so the probabilities τ_M and τ_I do not appear in (2.6).

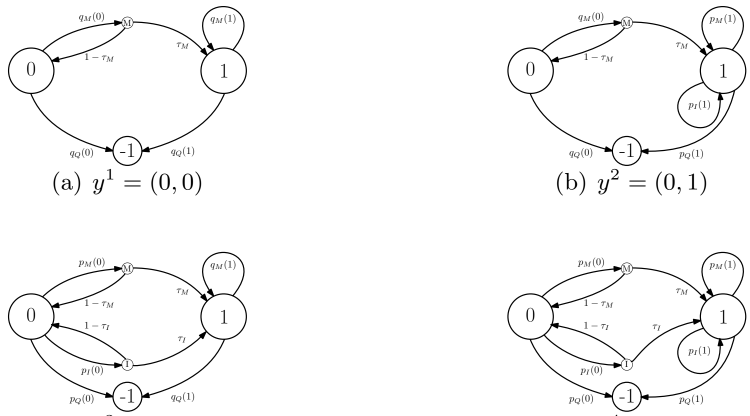
A *policy* y for the publisher is a mapping from E to U. Figure 2.1 illustrates the policy $y(0) = 1$ and $y(1) = 0$. Each policy y induces a stochastic process over rewards, allowing us to write its value as:

$$W^y(e) := \mathbb{E}_e^y \left\{ \sum_{t=1}^{\infty} r(e_t, y(e_t)) \right\} \tag{2.7}$$

where e is the player's initial engagement level and the expectation is from the induced stochastic process. In many examples of Markov decision processes, the sum in (2.7) does not converge, but under our assumptions (in particular, (A3.3)) the expected total reward does converge for every policy y. In fact, our problem has a special structure that we can exploit to derive a convenient analytical form for (2.7) as follows:

$$W^y(e) = \sum_{e' \geq e} n_{e,e'}^y r(e', y(e')) \tag{2.8}$$

where $n_{e,e'}^y$ is the expected number of visits to engagement level e' starting in engagement level e. We derive closed-form expressions for $n_{e,e'}$ that facilitate analysis. For details see Appendix A.



(a) $y^1 = (0,0)$ (b) $y^2 = (0,1)$

(c) $y^3 = (1,0)$ (d) $y^4 = (1,1)$

Figure 2.2: Induced absorbing Markov chains for alternate policies in the two-engagement level case.

Policy	$W^y(0)$	$W^y(1)$
$y^1 = (0,0)$	$\frac{q_M(0)\mu_M}{1-q_M(0)(1-\tau_M)} + \frac{q_M(0)\tau_M}{(1-q_M(0)(1-\tau_M))q_Q(1)} q_M(1)\mu_M$	$\frac{q_M(1)\mu_M}{q_Q(1)}$
$y^2 = (0,1)$	$\frac{q_M(0)\mu_M}{1-q_M(0)(1-\tau_M)} + \frac{q_M(0)\tau_M}{(1-q_M(0)(1-\tau_M))p_Q(1)} (p_M(1)\mu_M + p_I(1)\mu_I)$	$\frac{p_M(1)\mu_M+p_I(1)\mu_I}{p_Q(1)}$
$y^3 = (1,0)$	$\frac{p_M(0)\mu_M+p_I(0)\mu_I}{1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I)} + \frac{p_M(0)\tau_M+p_I(0)\tau_I}{(1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I))q_Q(1)} q_M(1)\mu_M$	$\frac{q_M(1)\mu_M}{q_Q(1)}$
$y^4 = (1,1)$	$\frac{p_M(0)\mu_M+p_I(0)\mu_I}{1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I)} + \frac{p_M(0)\tau_M+p_I(0)\tau_I}{(1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I))p_Q(1)} (p_M(1)\mu_M + p_I(1)\mu_I)$	$\frac{p_M(1)\mu_M+p_I(1)\mu_I}{p_Q(1)}$

Table 2.1: Total expected profit for Example 1.

For concrete expressions in a special case see Example 1 below.

The game publisher's decision is to choose a policy to solve the optimization problem: given a starting player's engagement level e solve:

$$\max_{y\in\{0,1\}^E} W^y(e). \tag{2.9}$$

In the discussion in later sections, we will often refer to a simple setting with two engagement levels. Here we can use (2.8) to derive clean expressions for the total expected reward of all four possible policies.

Example 1. *Consider the case where $N = 1$ and there are two engagement levels $E = \{0, 1\}$. There are four possible policies: $y^1 = (0,0)$, $y^2 = (0,1)$, $y^3 = (1,0)$ and $y^4 = (1,1)$. Figure 2.2 gives a visual representation of these four policies.*

Table 2.1 shows the total expected reward functions for all four policies. Details of their derivation are in Appendix A. Which of the four policies $y^1, \ldots, y^4$ is optimal depends on the values of the parameters in the model. Most of the numerical examples in the paper refer to the formulas in Table 2.1.

2.4 Understanding the Effects of Incented Actions

In this section we show how our analytical model helps us sharpen our insight into the costs and benefits of offering incented actions in games. In particular, we give precise analytical definitions of the *revenue*, *retention* and *progression* effects of offering of incented actions to a player.

Let $y^1_{\bar{e}}$ be a given policy with $y^1_{\bar{e}}(\bar{e}) = 0$ for some engagement level $\bar{e}$. Consider a local change to a new policy $y^2_{\bar{e}}$ where $y^2_{\bar{e}}(\bar{e}) = 1$ but $y^2_{\bar{e}}(e) = y^1_{\bar{e}}(e)$ for $e \neq \bar{e}$. We call $y^1_{\bar{e}}$ and $y^2_{\bar{e}}$ *paired policies with a local change at* $\bar{e}$. Analyzing this local change at the target engagement level $\bar{e}$ gives insight into the effect of starting to offer an incented action at a given engagement level. Moreover, this flavor of analysis suffices to determine an optimal threshold policy, as discussed in Section 2.5 below. For ease of notation, let $W^1(e) = W^{y^1_{\bar{e}}}(e)$ and $W^2(e) = W^{y^2_{\bar{e}}}(e)$.

Our goal is to understand the change in expected revenue moving from policy $y^1_{\bar{e}}$ to policy $y^2_{\bar{e}}$ where the player starts (or has reached) engagement level $\bar{e}$. Indeed, because the engagement does not decrease (before the player quits) if the player has reached engagement level $\bar{e}$ the result is the same as if the player just started at engagement level $\bar{e}$ by the Markovian property of the player model. Understanding when, and for what reasons, this change has a positive impact on revenue provides insights into the value of incented actions.

The change in total expected revenue from the policy change from $y^1_{\bar{e}}$ to $y^2_{\bar{e}}$ at engagement level $\bar{e}$ is:

$$\begin{aligned} W^2(\bar{e}) - W^1(\bar{e}) &= \underbrace{n^2_{\bar{e},\bar{e}} r(\bar{e},1) - n^1_{\bar{e},\bar{e}} r(\bar{e},0)}_{(i)} + \underbrace{\sum_{e>\bar{e}} (n^2_{\bar{e},e} - n^1_{\bar{e},e}) r(e, y(e))}_{(ii)} \\ &= C(\bar{e}) + F(\bar{e}) \end{aligned} \tag{2.10}$$

Term (i), denoted $C(\bar{e})$, is the change of revenue accrued from visits to the current engagement level $\bar{e}$. We may think of $C(\bar{e})$ as denoting the *current* benefits of offering an incented action in state $\bar{e}$, where "current" means the current level of engagement. Term (ii), denoted $F(\bar{e})$, captures the change due to visits to all other engagement levels. We may think of $F(\bar{e})$ as denoting the *future* benefits of visiting higher ("future") states of engagement. We can give explicit formulas for $C(\bar{e})$ and $F(\bar{e})$ for $e < N$ (after some work detailed in the Appendix A) as follows:

$$C(\bar{e}) = \tfrac{p_M(\bar{e})\mu_M + p_I(\bar{e})\mu_I}{1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)} - \tfrac{q_M(\bar{e})\mu_M}{1 - q_M(\bar{e})(1-\tau_M)} \tag{2.11}$$

and

$$F(\bar{e}) = \{\tfrac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I}{1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)} - \tfrac{q_M(\bar{e})\tau_M}{1 - q_M(\bar{e})(1-\tau_M)}\}\{\sum_{e'>\bar{e}} n^{y^1}_{\bar{e}+1,e'} r(e', y(e'))\}. \tag{2.12}$$

One interpretation of the formula $C(\bar{e})$ is that the two terms in (2.11) are conditional expected revenues associated with progressing to engagement level $\bar{e}+1$ conditioned on the event that the player does not stay in engagement level e (by either quitting or advancing). Thus, $C(\bar{e})$ is the change in conditional expected revenue from offering incented actions. There is a similar interpretation of the expression

$$\tfrac{p_M(\bar{e})\tau_M+p_I(\bar{e})\tau_I}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} - \tfrac{q_M(\bar{e})\tau_M}{1-q_M(\bar{e})(1-\tau_M)} \tag{2.13}$$

in the definition of $F(\bar{e})$. Both terms in (2.13) are conditional probabilities of progressing from engagement level $\bar{e}$ to engagement level $\bar{e}+1$ conditioned on the event that the player does not stay in engagement level $\bar{e}$ (by either quitting or advancing). Thus, $F(\bar{e})$ can be seen as the product of a term representing the increase in the conditional probability of progressing to engagement level $\bar{e}$ and the sum of revenues from expected visits from state $\bar{e}+1$ to the higher engagement levels.

These expressions turn out to be quite useful in later development and numerical examples. For now, we want to provide some intuition behind what drives the benefits of offering incented action, both current and future, that is not easily gleaned from these detailed formulas. In particular, we provide precise identification of three effects of incented actions that were discussed informally in the introduction. To this end, we introduce the notation:

$$\Delta_r(e|\bar{e}) := r(e, y^2_{\bar{e}}(e)) - r(e, y^1_{\bar{e}}(e)), \tag{2.14}$$

which expresses the change in the expected revenue per visit to engagement level e and

$$\Delta_n(e|\bar{e}) = n^2_{\bar{e},e} - n^1_{\bar{e},e}, \tag{2.15}$$

which expresses the change in the number of expected visits to engagement level e (starting at engagement level $\bar{e}$) before quitting.

Note that $\Delta_r(e|\bar{e}) = 0$ for $e \neq \bar{e}$ since we are only considering a local change in policy at engagement level $\bar{e}$. On the other hand,

$$\Delta_r(\bar{e}|\bar{e}) = -(q_M(\bar{e}) - p_M(\bar{e}))\mu_M + p_I(\bar{e})\mu_I. \tag{2.16}$$

The latter value is called the *revenue effect* as it expresses the change in the revenue per visit to the starting engagement level $\bar{e}$. The *retention effect* is the value $\Delta_n(\bar{e}|\bar{e})$ and expresses the change in the number of visits to the starting engagement level $\bar{e}$. Lastly, we refer to the value $\Delta_n(e|\bar{e})$ for $e > \bar{e}$ as the *progression effect at engagement level* e. At first blush it may seem possible for the progression effect to have different in sign at different engagement levels, but the following result shows that the progression effect is, in fact, uniform in sign.

Proposition 2.4.1. *Under Assumptions 2.3.1–2.3.4, the progression effect is uniform in sign; that is, either* $\Delta_n(e|\bar{e}) \geq 0$ *for all* $e \neq \bar{e}$ *or* $\Delta_n(e|\bar{e}) \leq 0$ *for all* $e \neq \bar{e}$.

The intuition for the above result is simple. There is only a policy change at the starting engagement level $\bar{e}$. Thus, the probability of advancing from engagement level e to engagement level $e+1$ is the same for policy $y^1_{\bar{e}}$ and $y^2_{\bar{e}}$ for $e > \bar{e}$. Hence, if $\Delta_n(\bar{e}+1|\bar{e})$ is positive then $\Delta_n(e|\bar{e})$ is positive for $e > \bar{e}+1$ since there will be more visits to engagement level $\bar{e}+1$ and thus more visits to higher engagement levels since the transition probabilities at higher engagement levels are unchanged. In light this proposition we may refer to the *progression effect* generally (without reference to a particular engagement level).

If both the revenue effect and retention effects are positive $C(\bar{e})$ in (2.10) is positive and there is a net increase in revenue due to visits to engagement level $\bar{e}$. Similarly, if both effects are negative then $C(\bar{e})$ is negative. When one effect is positive and the other is negative, the sign of $C(\bar{e})$ is unclear. The sign of $F(\bar{e})$ is completely determined by the direction of the progression effect.

One practical motivation for incented actions is that relatively few players monetize in practice, and so opening up another channel of revenue the publisher is able to earn more from its players. Indeed, if $q_M(\bar{e})$ and $p_M(\bar{e})$ are small (say in the order of 2%) then the first term in the revenue effect (2.16) is insignificant when compared to the second term $p_I(\bar{e})\mu_I$ and so most likely to be positive at low engagement levels. Moreover, having an incented action as an alternative to monetizing also suggests that it will keep players from quitting and build their commitment to playing the game. This motivation suggests that the retention and progression effects are also likely to be positive, particularly at early engagement levels when players are most likely to quit and least likely to invest money into playing a game.

However, our current assumptions do not fully capture the above logic. It is straightforward to construct specific scenarios that satisfy Assumptions 2.3.1–2.3.4 where the revenue and progression effects are negative even at low engagement levels (see Example 3 below). Further refinements are needed (see Section 2.5 for further assumptions). This complexity is somewhat unexpected, given the parsimony on the model and structure already placed on the problem. Indeed, the assumptions do reveal a certain structure as demonstrated in the following result.

Proposition 2.4.2. *Under Assumptions 2.3.1–2.3.4, the retention effect is always nonnegative; that is* $\Delta_n(\bar{e}|\bar{e}) \geq 0$.

There are two separate reasons for why offering incented actions at engagement level $\bar{e}$ changes the number of visits to $\bar{e}$. This first comes from the fact that the quitting probability at engagement level $\bar{e}$ goes down from $q_Q(\bar{e})$ to $p_Q(\bar{e})$. The second is that the probability of progressing to a higher level engagement also changes from $q_M(\bar{e})\tau_M$ to $p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I$ when offering an ad. Intuitively, the overall affect is somewhat unclear. However, the proposition reveals that the net effect is always nonnegative as a consequence of our assumptions. Observe that the probability of staying in engagement level e always improves when an incented action is offered:

$$p_M(\bar{e})(1-\tau_M) + p_I(\bar{e})(1-\tau_I) - q_M(\bar{e})(1-\tau_M) = p_I(e)(-\alpha(e)(1-\tau_M) + (1-\tau_I)) > 0.$$

However, it is not necessarily desirable for there to be more visits to engagement e if it is primarily at the expense of visits to more lucrative engagement levels. We must, in addition, consider the future benefits of the change in policy. The following examples illustrate how these current and future benefits can be in opposing directions, making it qualitatively more difficult to decide whether to offer an incented action.

Example 2. *Consider the following two engagement level example. Assume* $\mu_M = 1$, $\mu_I = 0.25$, $\tau_M = 0.8$, $\tau_I = 0.1$. *At level* 0, $p_M(0) = 0.05$, $p_I(0) = 0.65$, $\alpha(0) = 0.5$ *and thereby* $q_M(0) = 0.375$. *At level* 1, $p_M(1) = 0.2$, $p_I(1) = 0.6$, $\alpha(1) = 0.75$ *and thereby* $q_M(1) = 0.65$.

At level 0, *the revenue effect is* $\Delta_r(0|0) = p_M(0)\mu_M + p_I(0)\mu_I - q_M(0)\mu_M = 0.05(1) + 0.65(0.25) - 0.375(1) = -0.1625$ *while the retention effect is* $\Delta_n(0|0) = 1/(1 - p_M(0)(1-\tau_M) - p_I(0)(1-\tau_I)) - 1/(1 - q_M(0)(1-\tau_M)) = 1/(0.405) - 1/(0.925) = 1.388$. *Therefore,*

$$C(0) = \tfrac{p_M(0)\mu_M + p_I(0)\mu_I}{1 - p_M(0)(1-\tau_M) - p_I(0)(1-\tau_I)} - \tfrac{q_M(0)\mu_M}{1 - q_M(0)(1-\tau_M)} = \tfrac{0.2125}{0.405} - \tfrac{0.375}{0.925} = 0.1193 > 0 \tag{2.17}$$

Suppose $y^1(1) = y^2(1) = 0$, *the progression effect is*

$$\Delta_n(1|0) = \tfrac{p_M(0)\tau_M + p_I(0)\tau_I}{(1 - p_M(0)(1-\tau_M) - p_I(0)(1-\tau_I))q_Q(1)} - \tfrac{q_M(0)\tau_M}{(1 - q_M(0)(1-\tau_M))q_Q(1)} \tag{2.18}$$

$$= \tfrac{0.105}{0.405(0.35)} - \tfrac{0.3}{0.925(0.35)} = -0.1859 \tag{2.19}$$

as a result,

$$F(0) = \Delta_n(1|0)q_M(1)\mu_M = -0.1859(0.65) = -0.1207 < 0 \tag{2.20}$$

Notice that $\tau_I/\tau_M = 0.125$ *but* $\mu_I/\mu_M = 0.25$ *and therefore* τ_I *is much smaller than* τ_M. *This implies that the progression effect is negative and so* $F(0)$ *is negative. But* $C(0)$ *is positive since the retention effect is dominant. In other words, although the "current" benefits of offering an incented action at engagement level* 0 *are positive, these gains out outweighed by losses in "future" benefits.*

Example 3. *Consider the following two engagement level example. Assume* $\mu_M = 1$, $\mu_I = 0.05$, $\tau_M = 0.8$, $\tau_I = 0.3$. *At level* 0, $p_M(0) = 0.05$, $p_I(0) = 0.65$, $\alpha(0) = 0.5$ *and thereby* $q_M(0) = 0.375$. *At level* 1, $p_M(1) = 0.2$, $p_I(1) = 0.6$, $\alpha(1) = 0.75$ *and thereby* $q_M(1) = 0.65$.

At level 0, *the revenue effect is* $\Delta_r(0|0) = p_M(0)\mu_M + p_I(0)\mu_I - q_M(0)\mu_M = 0.05(1) + 0.65(0.05) - 0.375(1) = -0.2925$ *while the retention effect is* $\Delta_n(0|0) = 1/(1 - p_M(0)(1-\tau_M) - p_I(0)(1-\tau_I)) - 1/(1 - q_M(0)(1-\tau_M)) = 1/(0.535) - 1/(0.925) = 0.788$. *Therefore,*

$$C(0) = \tfrac{p_M(0)\mu_M + p_I(0)\mu_I}{1 - p_M(0)(1-\tau_M) - p_I(0)(1-\tau_I)} - \tfrac{q_M(0)\mu_M}{1 - q_M(0)(1-\tau_M)} = \tfrac{0.0825}{0.535} - \tfrac{0.375}{0.925} = -0.2512 < 0 \tag{2.21}$$

Suppose $y^1(1) = y^2(1) = 0$*, the progression effect is*

$$\Delta_n(1|0) = \tfrac{p_M(0)\tau_M + p_I(0)\tau_I}{(1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I))q_Q(1)} - \tfrac{q_M(0)\tau_M}{(1-q_M(0)(1-\tau_M))q_Q(1)} \tag{2.22}$$

$$= \tfrac{0.235}{0.535(0.35)} - \tfrac{0.3}{0.925(0.35)} = 0.3284 \tag{2.23}$$

hence

$$F(0) = \Delta_n(1|0)q_M(1)\mu_M = 0.3284(0.65) = 0.2134 > 0 \tag{2.24}$$

In contrast to the previous example, there are "current" losses and "future" gains to be had by offering incented actions at engagement level 0 *but, similar to that example, the overall verdict is that it is better* not *to offer incented actions.*

These example underscore that it is a nontrivial task to assess the optimality of an incented action policy. Whether and how to offer incented actions depends on the specifics of the game and must weigh how the current and future benefits of incented actions, described in terms of the three effects, change as the engagement level evolves. This is the task of the next section.

2.5 Optimal Policies for The Publisher

Recall the publisher's problem described in (2.9). This is a dynamic optimization problem where the publisher must decide on whether to deploy incented actions at *each* engagement level, with the knowledge that a change in policy at one engagement level can effect the behavior of the player at subsequent engagement levels. This "forward-looking" nature adds a great deal of complexity to the problem. A much simpler task would be to examine each engagement level in isolation, implying that the publisher need only consider term (i) of (2.10) at engagement level e to decide if $y(e) = 1$ or $y(e) = 0$ provides more revenue. A policy built in this way is called *myopically optimal*. More precisely, policy y is myopically optimal if $y(e) = 1$ when $C(e) > 0$ and $y(e) = 0$ when $C(e) < 0$.

A myopically optimal policy need not be optimal because it fails to consider future impacts, which can be significant (see Example 2). However, the next result gives a sufficient condition for a myopically-optimal policy to be optimal.

Proposition 2.5.1. *Under Assumptions 2.3.1–2.3.4, if* $\frac{\mu_I}{\mu_M} = \frac{\tau_I}{\tau_M}$ *then a myopically-optimal policy is optimal.*

This result is best understood by looking at the two terms in the change in revenue formula (2.10) discussed in the previous section. It is straightforward to see from (2.11) and (2.12) that when $\tau_I = \mu_I$ and $\tau_M = \mu_M$ that the sign of $C(\bar{e})$ and $F(\bar{e})$ are identical. That is, if the current benefit of offering the incented action has the same sign as the future benefit of offering an action then it suffices to consider the term first $C(\bar{e})$ only when determining an optimal policy. Given our interpretation of $C(\bar{e})$ and $F(\bar{e})$, the conditions of Proposition 2.5.1 imply that the conditional expected revenue from progressing one engagement level precisely equals the conditional probability of progressing one engagement level. This is a rather restrictive condition.

Since we know of only the above strict condition under which an optimal policy is myopic, in general we are in search of forward-looking optimal policies. Since the game publisher's problem is a Markov decision process, an optimal forward-looking policy y must satisfy the optimality equations for $e = 0, \ldots, N-1$

$$W^y(e) = \begin{cases} r(e,1) + \mathbb{P}_1(e|e)W(e) + \mathbb{P}_1(e+1|e)W(e+1) & \text{if } y(e) = 1 \\ r(e,0) + \mathbb{P}_0(e|e)W(e) + \mathbb{P}_0(e+1|e)W(e+1) & \text{if } y(e) = 0 \end{cases} \tag{2.25}$$

and for $e = N$

$$W^y(N) = \begin{cases} r(N,1) + \mathbb{P}_1(N|N)W^y(N) & \text{if } y(N) = 1 \\ r(N,0) + \mathbb{P}_0(N|N)W^y(N) & \text{if } y(N) = 0, \end{cases} \tag{2.26}$$

where $\mathbb{P}_1$ and $\mathbb{P}_0$ are as defined in (2.4) and (2.5) respectively. The above structure shows that an optimal policy can be constructed by backwards induction (for details see Chapter 4 of Puterman [46]): first determine an optimal choice of $y(N)$ and then successively find optimal choices for $y(N-1), \ldots, y(1)$ and finally $y(0)$. We use the notation $W(e)$ to denote the *optimal* revenue possible with a player starting at engagement level e, called the *optimal value function*. In addition we use the notation $W(e, y = 1)$ to denote the optimal expected total revenue possible when an incented action is offered at starting engagement level e. Similarly, we let $W(e, y = 0)$ denote the optimal expected revenue possible when an incented action is *not* offered at starting engagement level e. Then $W(e)$ must satisfy *Bellman's equation* for $e = 0, \ldots, N-1$:

$$\begin{aligned} W(e) &= \max\{W(e, y=1), W(e, y=0)\} \\ &= \max\{r(e,1) + \mathbb{P}_1(e|e)W(e) + \mathbb{P}_1(e+1|e)W(e+1), \\ &\qquad r(e,0) + \mathbb{P}_0(e|e)W(e) + \mathbb{P}_0(e+1|e)W(e+1)\}. \end{aligned} \tag{2.27}$$

A key fact that we leverage throughout our development is the following.

Theorem 2.5.2. *Under Assumptions 2.3.1–2.3.4, $W(e)$ is a nondecreasing function of e.*

This result underscores the value of having players progress in engagement with the game. The higher the engagement of a player, the more revenue can be extracted from them. This result has a natural intuition. Indeed, if the theorem were not true it would even suggest that our use of the word "engagement" to describe the states would be ill-placed. However, this result serves as an important reality check and goes towards establishing the validity of our modeling approach.

The focus of our discussion is on optimal *forward threshold* policies that start by offering incented action. Such a threshold policy y is determined by a single engagement level $\bar{e}$ where

$y(e') = 1$ for $e' \leq \bar{e}$ and $y(e) = 0$ for $e' > \bar{e}$. According to (2.27) this happens when $W(\bar{e}+1, y = 1) \leq W(\bar{e}+1, y = 0)$ implies $W(e', y = 1) \geq W(e', y = 0)$ for all $e' \leq \bar{e}$ and $W(e', y = 0) < W(e', y = 0)$ for all $e' > \bar{e} + 1$. In the general nomenclature of Markov decision processes other policies would be classified as threshold policies. This includes policies that start with not offering the incented action until some point and thereafter offering the incented action. We call these policies *backward threshold*.

Our interest in forward threshold policies comes from the following appealing practical logic, already hinted at in the introduction. When players start out playing a game their engagement level is low and they are likely to quit. Indeed, Theorem 2.5.2 says we get more value out of players at higher levels of engagement. Hence, retaining players at early stages and progressing them to higher levels of engagement is important for overall revenue. In Proposition 2.4.2, we see the retention effect of offering incented actions is always positive, and intuitively, the revenue and progression effects are largest at low levels of engagement because players are unlikely to monetize early on and the benefits derived from increasing player engagement are likely to be at their greatest. This suggests it is optimal to offer incented actions at low levels of engagement. However, once players are sufficiently engaged it might make sense to removed incented actions to focus their attention on the monetization option. If sufficiently engaged and $\alpha(e)$ is sufficiently large, most of the probability of taking the incented action shifts to monetizing which drives greater revenue.

Despite this appealing logic, the following example shows that our current set of assumptions (Assumptions 2.3.1–2.3.4) are insufficient to guarantee the existence an optimal forward threshold policy.

Example 4. *Consider the following two engagement level example. Assume* $\mu_M = 1$, $\mu_I = 0.05$, $\tau_M = 0.5$, $\tau_I = 0.4$. *At level* 0, $p_M(0) = 0.05$, $p_I(0) = 0.65$, $\alpha(0) = 0.5$ *and thereby* $q_M(0) = 0.375$. *At level* 1, $p_M(1) = 0.2$, $p_I(1) = 0.6$, $\alpha(1) = 0.55$ *and thereby* $q_M(1) = 0.53$.

We solve the optimal policy by backward induction. At level 1, $W(1, y = 1) = \frac{p_M(1)\mu_M + p_I(1)\mu_I}{1 - p_M(1) - p_I(1)} = \frac{0.23}{0.2} = 1.15$ *while* $W(1, y = 0) = \frac{q_M(1)\mu_M}{1 - q_M(1)} = \frac{0.53}{0.47} \approx 1.13$. *Therefore,* $y^*(1) = 1$ *and* $W(1) = \max\{W(1, y = 1), W(1, y = 0)\} = 1.15$.

At level 0,

$$W(0, y = 1) = \frac{p_M(0)\mu_M + p_I(0)\mu_I}{1 - p_M(0)(1 - \tau_M) - p_I(0)(1 - \tau_I)} + \frac{p_M(0)\tau_M + p_I(0)\tau_I}{(1 - p_M(0)(1 - \tau_M) - p_I(0)(1 - \tau_I))} \frac{q_M(1)\mu_M}{q_Q(1)} \tag{2.28}$$

$$= \frac{0.0825}{0.585} + \frac{0.285}{0.585}(1.15) = 0.141 + 0.56 = 0.701 \tag{2.29}$$

$$W(0, y = 0) = \frac{q_M(0)\mu_M}{1 - q_M(0)(1 - \tau_M)} + \frac{q_M(0)\tau_M}{1 - q_M(0)(1 - \tau_M)} \frac{q_M(1)\mu_M}{q_Q(1)} \tag{2.30}$$

$$= \frac{0.375}{0.8125} + \frac{0.1875}{0.8125}(1.15) = 0.462 + 0.265 = 0.727 \tag{2.31}$$

hence $y^*(0) = 0$ *and* $W(0) = \max\{W(0, y = 1), W(0, y = 0)\} = 0.727$.

Next we show that $y^* = (0, 1)$ *is the only optimal policy. In fact, we compute* $W^y(0)$ *and* $W^y(1)$ *under all possible policies in the following table. We observe that none of* $(0, 0)$, $(1, 0)$ *and*

Policy	$W^y(0)$	$W^y(1)$
$y = (0,0)$	0.723	1.13
$y = (1,0)$	0.691	1.13
$y = (1,1)$	0.701	1.15
$y^* = (0,1)$	0.727	1.15

Table 2.2: Total expected profit for Example 4.

$(1,1)$ *are optimal. This implies* y^* *is the only optimal policy. Since* y^* *is not a forward threshold policy, this implies there is no optimal forward threshold policy. Thus we see it is optimal to offer incented actions at the higher engagement level because of the dramatic reduction in the quitting probability when offering incented actions to* 0.2 *quitting probability compared to a* 0.47 *quitting probability when not offering incented actions. Although the expected revenue per period the player stays at the highest engagement level is lower when incented actions are offered (*0.23 *as compared to* 0.47*) the player will stay longer and thus earn additional revenue. However, at the lowest engagement level the immediate reward of not offering incented actions (*0.462 *versus* 0.141*) outweighed the losses due to a lower chance of advancing to the higher engagement level.*

The goal for the remainder of this section is to devise additional assumptions that are relevant to the settings of interest to our paper and that guarantee the existence of an optimal forward threshold policy. The previous example shows how α plays a key role in determining whether an threshold policy is optimal or not. When incentives actions are removed the probability $p_I(e)$ is distributed to the monetization and quitting actions according to $\alpha(e)$. The associated increase in the probability of monetizing from $p_M(e)$ to $q_M(e)$ makes removing incented actions attractive, since the player is more likely to pay. However, the quitting probability increases from $p_Q(e)$ to $q_Q(e)$, a downside of removing incented actions. Intuitively speaking, if $\alpha(e)$ grows sufficiently quickly, the benefits will outweigh the costs of removing incented actions. From Assumption (A3.6) we know that $\alpha(e)$ increases, but this alone is insufficient. Just how quickly we require $\alpha(e)$ to grow to ensure a threshold policy requires careful analysis. This analysis results in lower bounds on the growth of $\alpha(e)$ that culminates in Theorem 2.5.8 below.

Our first assumption on $\alpha(e)$ is a basic one:

Assumption 2.5.3. $\alpha(N) = 1$*; that is,* $q_Q(N) = p_Q(N)$ *and* $q_M(N) = p_M(N) + p_I(N)$*.*

It is straightforward to see that under this assumption it is never optimal to offer incented action at the highest engagement level. This assumption also serves as an interpretation of what it means to be in the highest engagement level, simply that players who are maximally engaged are no more likely to quit when the incented action is removed. Under this assumption, and by Bellman's equation (2.27), every optimal policy y^* has $y^*(N) = 0$. Note that this excludes the scenario in Example 4 and also implies that backwards threshold policies are not optimal (except possibly the policy that $y(e) = 0$ for all $e \in E$ that is both a backward and forward threshold). Given this, we restrict attention to forward threshold policies and drop the modifier "forward" in the rest of our development.

The next step is to establish further sufficient conditions on the data that ensure that once the revenue, retention and progression are negative, they stay negative. As in Section 2.4, we consider paired policies $y^1_{\bar{e}}$ and $y^2_{\bar{e}}$ with a local change at $\bar{e}$. Recall the notation $\Delta_r(e|\bar{e})$ and $\Delta_n(e|\bar{e})$ defined in (2.14) and (2.15), respectively. We are concerned with how $\Delta_r(e|\bar{e})$ and $\Delta_n(e|\bar{e})$ change with the starting engagement level $\bar{e}$. It turns out that the revenue effect $\Delta_r(e|\bar{e})$ always behaves in a way that is consistent with a threshold policy, without any additional assumptions.

Proposition 2.5.4. *Suppose Assumptions 2.3.1–2.3.4 hold. For every engagement level* $\bar{e}$ *let* $y^1_{\bar{e}}$ *and* $y^2_{\bar{e}}$ *be paired policies with a local change at* $\bar{e}$*. Then the revenue effect* $\Delta_r(\bar{e}|\bar{e})$ *is nonincreasing in* $\bar{e}$ *when* $\Delta_r(\bar{e}|\bar{e}) \geq 0$*. Moreover, if* $\Delta_r(\bar{e}|\bar{e}) < 0$ *for some* $\bar{e}$ *then* $\Delta_r(e'|e') < 0$ *for all* $e' \geq \bar{e}$*.*

This proposition says that the net revenue gain per visit to engagement level $\bar{e}$ is likely to only be positive (if it is ever positive) at lower engagement levels, confirming our basic intuition that incented actions can drive revenue from low engagement levels, but less so from highly engaged players. To show a similar result for the progression effect we make the following assumption.

Assumption 2.5.5. $\alpha(e+1) - \alpha(e) > q_M(e+1) - q_M(e)$ *for all* $e \in E$*.*

This provides our first general lower bound on the growth of $\alpha(e)$. It says that $\alpha(e)$ must grow faster than the probability $q_M(e)$ of monetizing when the incented action is not offered.

Proposition 2.5.6. *Suppose Assumptions 2.3.1–2.5.5 hold. For every engagement level* $\bar{e}$ *let* $y^1_{\bar{e}}$ *and* $y^2_{\bar{e}}$ *be paired policies with a local change at* $\bar{e}$ *such that* $y^1_{\bar{e}}$ *and* $y^2_{\bar{e}}$ *are identical to some fixed policy* y *(fixed in the sense that* y *is not a function of* $\bar{e}$*) except at engagement level* $\bar{e}$*. Then*

(a) If $\Delta_n(e|\bar{e}) < 0$ *for some* $\bar{e}$ *then* $\Delta_n(e|e') < 0$ *for all* $e' \geq \bar{e}$*.*

(b) If $C(\bar{e}) < 0$ *for some* $\bar{e}$ *then* $C(e') < 0$ *for all* $e' \geq \bar{e}$*, where* C *is as defined in* (2.10)*.*

This result implies that once the current and future benefits of offering an incented action are negative, they stay negative for higher engagement levels. Indeed, Proposition 2.5.6(a) ensures that the future benefits F in (2.10) stay negative once negative, while (b) ensures the current benefits C stay negative once negative. In other words, once the game publisher stops offering incented actions it is never optimal for them to return. Note that Proposition 2.5.4 does not immediately imply Proposition 2.5.6(b), Assumption 2.5.5 is needed to ensure the retention effect has similar properties, as guaranteed by Proposition 2.5.6(a) for $e' = \bar{e}$.

As mentioned above, the conditions established in Proposition 2.5.6 are necessary for the existence of an optimal threshold policy, but does not imply that an threshold policy exists. This is due to the fact that C and F in (2.10) may not switch sign from positive to negative at the same engagement level. This is illustrated in the following example.

Example 5. *Consider the following two engagement level example. Assume* $\mu_M = 1$, $\mu_I = 0.2$, $\tau_M = 0.91$, $\tau_I = 0.47$. *At level* 0, $p_M(0) = 0.03$, $p_I(0) = 0.51$, $\alpha(0) = 0.59$ *and thereby* $q_M(0) = 0.3309$. *At level* 1, $p_M(1) = 0.05$, $p_I(1) = 0.5$, $\alpha(1) = 0.62$ *and thereby* $q_M(1) = 0.36$. *At level* 2, $p_M(2) = 0.34$, $p_I(2) = 0.45$, $\alpha(2) = 1$ *and thereby* $q_M(1) = 0.79$.

The optimal policy is $y^* = (0, 1, 0)$. *We use backward induction. At the highest level* 2, *we have* $y^*(2) = 0$ *and* $W(2) = 0.79/0.21 = 3.7619$. *At level* 1,

$$\begin{aligned}
W(1, y = 1) &= \tfrac{p_M(1)\mu_M + p_I(1)\mu_I}{1-p_M(1)(1-\tau_M)-p_I(1)(1-\tau_I)} + \tfrac{p_M(1)\tau_M + p_I(1)\tau_I}{(1-p_M(1)(1-\tau_M)-p_I(1)(1-\tau_I))} \frac{q_M(2)\mu_M}{q_Q(2)} \\
&= \tfrac{0.15}{0.7305} + \tfrac{0.2805}{0.7305}(3.7619) = 0.2053 + 0.3840(3.7619) = 1.6498 \\
W(1, y = 0) &= \tfrac{q_M(1)\mu_M}{1-q_M(1)(1-\tau_M)} + \tfrac{q_M(1)\tau_M}{1-q_M(1)(1-\tau_M)} \frac{q_M(2)\mu_M}{q_Q(2)} \\
&= \tfrac{0.36}{0.9676} + \tfrac{0.3276}{0.9676}(3.7619) = 0.3721 + 0.3386(3.7619) = 1.6459
\end{aligned}$$

therefore $y^*(1) = 1$ *and* $W(1) = W(1, y = 1) = 1.6498$. *Moreover,* $C(1) = 0.2053 - 0.3721 = -0.1668$ *and* $F(1) = (0.3840 - 0.3386)(3.7619) = 0.0454(3.7619) = 0.1708$. *Finally, we look at level* 0.

$$\begin{aligned}
W(0, y = 1) &= \tfrac{p_M(0)\mu_M + p_I(0)\mu_I}{1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I)} + \tfrac{p_M(0)\tau_M + p_I(0)\tau_I}{(1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I))} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.1320}{0.7270} + \tfrac{0.2670}{0.7270}(1.6498) = 0.1816 + 0.3673(1.6498) = 0.7876 \\
W(0, y = 0) &= \tfrac{q_M(0)\mu_M}{1-q_M(0)(1-\tau_M)} + \tfrac{q_M(0)\tau_M}{1-q_M(0)(1-\tau_M)} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.3309}{0.9702} + \tfrac{0.3011}{0.9702}(1.6498) = 0.3411 + 0.3104(1.6498) = 0.8532
\end{aligned}$$

as we can see $y^*(0) = 0$ *and* $W(0) = W(0, y = 0) = 0.8532$. *Besides,* $C(0) = 0.1816 - 0.3411 = -0.1595$ *and* $F(0) = (0.3673 - 0.3104)(1.6498) = 0.0569(1.6498) = 0.0939$. *The optimal policy is not a threshold policy.*

In fact, Assumption 2.5.5 is satisfied because $\alpha(1) - \alpha(0) = 0.62 - 0.59 = 0.03$ *while* $q_M(1) - q_M(0) = 0.3600 - 0.3309 = 0.0291$.

We thus require one additional assumption:

Assumption 2.5.7. $1-\alpha(e+1) \leq (1-\alpha(e)) \frac{p_M(e+1)\tau_M + p_I(e+1)\tau_I}{p_Q(e+1)+p_M(e+1)\tau_M+p_I(e+1)\tau_I}$ *for* $e = 1, 2, \ldots, N-1$.

Note that the fractional term in the assumption is the probability of advancing from engagement level $e+1$ to $e+2$ conditioned on leaving engagement level $e+1$ and is thus less than one. Hence, this is yet another lower bound on the rate of growth in $\alpha(e)$, complementing Assumptions 2.5.3 and 2.5.5. Which of the bounds in Assumptions 2.5.5 or 2.5.7 is tighter depends on the data specifications that arise from specific game settings. In Example 5 we noted that Assumption 2.5.5 holds but observe that Assumption 2.5.7 fails, since $(1 - \alpha(1)) = 1 - 0.62 = 0.38$ and $(1 - \alpha(0)) \frac{p_M(1)\tau_M + p_I(1)\tau_I}{(1-p_M(1)(1-\tau_M)-p_I(1)(1-\tau_I))} = (1 - 0.59)\frac{0.2805}{0.7305} = 0.1574$.

Theorem 2.5.8. *Suppose Assumptions 2.3.1–2.5.7 hold. Then there exists an optimal threshold policy with threshold engagement level* e^**. That is, there exists an optimal policy* y^* *with* $y^*(e) = 1$ *for any* $e \leq e^*$ *and* $y^*(e) = 0$ *for any* $e > e^*$*.*

The existence of an optimal threshold is the cornerstone analytical result of this paper. From our development above, it should be clear that obtaining a sensible threshold policy is far from a trivial task. Indeed, in many MDP models great effort is put into establishing their existence. We believe our assumptions are reasonable based on our understanding of the games, given the difficult standard of guaranteeing the existence of a threshold policy. Of course, such policies will be welcomed in practice, precisely because of their simplicity and (relatively) intuitive justification. We also remark that none of these assumptions are superfluous. In the Appendix A we show that if you drop Assumptions 2.5.5 then a threshold policy may no longer be optimal. Example 5 shows that the same is true if Assumption 2.5.7 is dropped. As we see in some examples in the next section, our assumptions are sufficient but not necessary conditions for an optimal threshold policy to exist.

To simplify matters further, we also take the convention that when there is a tie in Bellman's equation (2.27) whether to offer an incented action or not, the publisher always chooses not to offer. This is consistent with the fact that is a cost to offering incented actions. Although we do not model costs formally, we will use reasoning to break ties. Under this tie-breaking rule there is, in fact, a *unique* optimal threshold policy guaranteed by Theorem 2.5.8. This unique threshold policy is our object of study in the remainder of the paper.

2.6 Game Design and Optimal Use of Incented Actions

So far we have provided a detailed analytical description of the possible benefits of offering incented actions (in Section 2.4) and the optimality of certain classes of policies (in Section 2.5). There remains the question of what types of games most benefit from offering incented actions and how different types of games may qualitatively differ in their optimal policies. We focus on optimal threshold policies and concern ourselves with how changes in the parameters of the model affect the optimal threshold e^* of an optimal threshold policy y^* that is guaranteed to exist under Assumptions 2.3.1–2.5.7 by Theorem 2.5.8. Of course, these are only sufficient conditions and so we do not restrict ourselves to that setting when conducting numerical experiments in this section.

We first consider how differences in the revenue parameters μ_I and μ_M affect e^*. Observe that only the revenue effect in (2.16) is impacted by changes in μ_I and μ_M, the retention and progression effects are unaffected. This suggests the following result:

Proposition 2.6.1. *The optimal threshold* e^* *is a nondecreasing function of the ratio* $\frac{\mu_I}{\mu_M}$*.*

Note that the revenue effect is nondecreasing in the ratio $\frac{\mu_I}{\mu_M}$. Since the other effects are unchanged, this implies that the benefit of offering incented actions at each engagement level is nondecreasing in $\frac{\mu_I}{\mu_M}$, thus establishing the monotonicity of e^* in μ_I/μ_M.

To interpret this result we consider what types of games have a large or small ratio $\frac{\mu_I}{\mu_M}$. From the introduction in Section 2.1 we know that incented actions typically deliver far less revenue to the publisher than in-app purchases. This suggests that the ratio is small, favoring a lower threshold. However, this conclusion ignores how players in the game may influence each other. Although our model is a single player model, one way we can include the interactions among players is through the revenue terms μ_I and μ_M. In many cases, a core value of a player to the game publisher is the word-of-mouth a player spreads to their contacts. Indeed, this is the value of non-paying players that other researchers have mostly focused on (see, for instance Lee et al. [33], Jiang and Sarkar [28], and Gupta et al. [24]). In cases where this "social effect" is significant it is plausible that the ratio of revenue terms is not so small. For instance, if δ is the revenue attributable to the word-of-mouth or network effects of a player, regardless of whether the player takes an incented actions or monetizes, then the ratio of interest is $\frac{\mu_I+\delta}{\mu_M+\delta}$. The larger is δ the larger is this ratio, and according to Proposition 2.6.1, the larger is the optimal threshold.

This analysis suggests that games with a significant social component should offer incented actions more broadly in social games. For instance, if a game includes cooperative or competitive multi-player features, then spreading the player base is of particular value to the company. Thought of another way, in a social game it is important to have a large player base to create positive externalities for new players to join, and so having players quit is of greater concern in more social games. Hence, it is best to offer incented action until higher levels of engagement are reached. All of this intuition is confirmed by Proposition 2.6.1.

Besides the social nature of the game, other factors can greatly impact the optimal threshold. Genre, intended audience, and structure of the game affect the other parameters of our model; particularly, τ_I, τ_M, and $\alpha(e)$. We first examine the progression probabilities τ_I and τ_M. As we did in the case of the revenue parameters, we focus on the ratio $\frac{\tau_I}{\tau_M}$. This ratio measures the relative probability of advancing through incented actions versus monetization. By Assumption (A3.5), $\frac{\tau_I}{\tau_M} \leq 1$ but its precise value can depend on several factors. One is the relative importance of the reward granted to the player when taking an incented action. Recall our discussion of *Crossy Road* in the introduction, one measure of engagement could be the number of unique characters accumulated by the player. Because characters can be purchased directly with real money, this makes $\tau_M = 1$. However, the reward for watching a video ad is only coins that can be used in a random draw for new characters. Depending on the odds of that draw, τ_I can be large or small.

Taking τ_M fixed (possibly at 1 as in the example of *Crossy Road*) we note that increasing τ_I decreases the "current" benefit of offering incented actions, as seen by examining term $C(\bar{e})$ in (2.10). Indeed, the revenue effect is unchanged by τ_I, but the retention effect is weakened. The impact on future benefits is less obvious. However, we know players are more likely to advance to a higher level of engagement with a larger τ_I. From Theorem 2.5.2 we also know higher engagement states are more valuable, and so we expect the future benefits of offering incented actions to be positive with a higher τ_I and even outweigh the loss in current benefits. This reasoning is confirmed in the next result.

Proposition 2.6.2. *The optimal threshold e^* is a nondecreasing function of the ratio $\frac{\tau_I}{\tau_M}$.*

One interpretation of this result is that the more effective an incented action is at increasing engagement of the player, the longer the incented action should be offered. This is indeed reasonable under the assumption that $p_I(e)$ and $p_M(e)$ are unaffected by changes in τ_I. However, if increasing τ_I necessarily increases $p_I(e)$ (for instance, if the reward of the incented action becomes more powerful and so drives the player to take the incented action with greater probability) the effect on the optimal threshold is less clear.

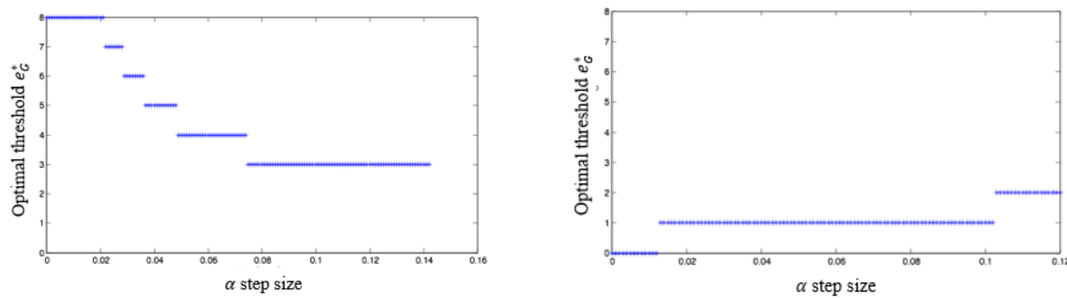
Example 6. *In this example we show that when the incented action is more effective it can lead to a* decrease *in the optimal threshold if $p_I(e)$ and $p_M(e)$ are affected by changes in τ_I. Consider the following two engagement level example. In the base case let $\mu_M = 1$, $\mu_I = 0.05$, $\tau_M = 0.8$, $\tau_I = 0.2$. At level 0, $p_M(0) = 0.3$, $p_I(0) = 0.5$, $\alpha(0) = 0.7$ and thereby $q_M(0) = 0.65$. At level 1, $p_M(1) = 0.5$, $p_I(1) = 0.4$, $\alpha(1) = 1$ and thereby $q_M(1) = 0.9$. One can show that the unique optimal policy is $y* = (0, 1)$ (for details see Appendix A).*

Now change the parameters as follows: increase τ_I to 0.25, which affects the decision-making of the player so that $p_M(0) = 0.1$, $p_I(0) = 0.7$, $p_M(1) = 0.3$ and $p_I(1) = 0.6$. In other words, the incented action became so attractive it reduces the probability of monetizing while increasing the probability of taking the incented action. One can show that the unique optimal policy in this setting is $y^* = (0, 0)$. Hence the optimal threshold has decreased. In conclusion, a change in the effectiveness of the incented action in driving engagement can lead to an increase or decrease in the optimal threshold policy, depending on the player's behavioral response.

This leads to an important investigation of the changes in the degree of cannibalization between incented actions and monetization. Recall that $\alpha(e)$ is the vector of parameters that indicates the degree of cannibalization at each engagement level. For sake of analysis, we assume that $\alpha(e)$ is an affine function of e with

$$\alpha(e) = \alpha(0) + \alpha_{\text{step}} e \tag{2.32}$$

where $\alpha(0)$ and α_{step} are nonnegative real numbers. A very high $\alpha(0)$ indicates a design where the reward of the incented action and the in-app purchase have a similar degree of attractiveness to the player so that when the incented action is removed the player is likely to monetize. This suggests that the cost-to-reward ratio of the incented action is similar to that of the in-app purchase. If one is willing, for instance, to endure the inconvenience of watching a video ad in order to get some virtual currency, they should be similarly willing to pay real money for a proportionate amount of virtual currency. A very low $\alpha(0)$ is associated with a very attractive cost-to-reward ratio for the incented action that makes monetization seem expensive in comparison.



(a) Initial value $\alpha(0) = 0$ (b) Initial value $\alpha(0) = 0.16$

Figure 2.3: Sensitivity of the optimal threshold to changes in α_{step}.

The rate of change α_{step} represents the strength of increase in cannibalization as the player advances in engagement. A fast rate of increase is associated with a design where the value of the reward of the incented action quickly diminishes. For instance, in *Crossy Road* the reward of earning coins to watch a video ad for a chance of randomly drawing a character naturally diminishes in value as the player accumulates more characters. Despite the reward weakening, this option still attracts a lot of attention from players, especially if they have formed a habit of advancing via this type of reward. If, however, the videos are removed, the value proposition of monetizing seems attractive in comparison to the diminished value of the reward for watching a video. Seen in this light, the rate at which the value of the reward diminishes is controlled by the parameter α_{step}.

Analysis of how different values for $\alpha(0)$ and α_{step} impact the optimal threshold is not straightforward. This is illustrated in the following two examples. The first considers sensitivity of the optimal threshold to α_{step}.

Example 7. *Consider the following example with nine engagement levels and the following data:* $\mu_m = 1$, $\mu_I = 1, 0.05$ $\tau_M = 0.8$, $\tau_I = 0.4$, $p_M(e) = 0.0001 + 0.00005e$ *and* $p_I(e) = 0.7 - 0.00001e$ *for* $e = 0, 1, \ldots, 8$. *We have not yet specified* $\alpha(e)$. *We examine two scenarios: (a) where* $\alpha(0) = 0$ *and we vary the value of* α_{step} *(see Figure 2.3(a)) and (b) where* $\alpha(0) = 0.16$ *and we vary the value of* α_{step} *(see Figure 2.3(b)). The vertical axis of these figures is the optimal threshold of the unique optimal threshold policy for that scenario. What is striking is that the threshold* e^* *is nonincreasing in* α_{step} *when* $\alpha(0) = 0$ *but nondecreasing in* α_{step} *when* $\alpha(0) = 0.16$.

One explanation in the difference in patterns between Figures 2.3(a) and 2.3(b) concerns whether it is optimal to include incented actions initially or not. In Figure 2.3(a) the initial degree of cannibalization is zero, making it costless to offer incented actions initially. When α_{step} *is very small cannibalization is never an issue and incented actions are offered throughout. However, as* α_{step} *increases, the degree of cannibalization eventually makes it optimal to stop offering incented actions to encourage monetization. This explains the nonincreasing pattern in Figure 2.3(a).*

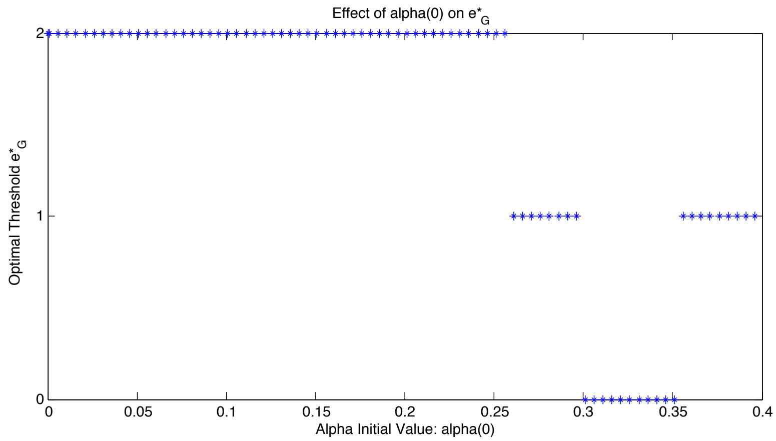


Figure 2.4: Sensitivity of the optimal threshold to changes in $\alpha(0)$.

By contrast, in Figure 2.3(b) the initial degree of cannibalization is already quite high, making it optimal to start by offering for low values of α_{step}. However, when α_{step} is sufficiently large, there are benefits to encouraging the player to advance. Recall, $\alpha(e)$ affects both the probability of monetization and the probability of quitting. In the case where α_{step} is sufficiently high there are greater benefits to the player progressing, making quitting early more costly. Hence it can be optimal to offer incented actions initially to discourage quitting and encourage progression. This explains the nondecreasing pattern in Figure 2.3(b).

As the following example illustrates, adjusting for changes in $\alpha(0)$ reveals a different type of complexity.

Example 8. *Consider the following two engagement level example. Assume* $\mu_M = 1$, $\mu_I = 0.0001$, $\tau_M = 0.01$, $\tau_I = 0.009$. *At level* 0, $p_M(0) = 0.05$, $p_I(0) = 0.68$. *At level* 1, $p_M(1) = 0.3$, $p_I(1) = 0.65$. *We set* α *step size be* 0.6, *i.e.* $\alpha(1) = \alpha(0) + 0.6$. *Figure 2.4 captures how changes in* $\alpha(0)$ *lead to different optimal thresholds. For complete details on the derivation of the figure see Appendix A.*

The striking feature of the figure is that the optimal threshold decreases, and then increases, as $\alpha(0)$ becomes larger. This "U"-shaped pattern reveals competing effects associated with changes in $\alpha(0)$. As $\alpha(0)$ increases, the benefit of increasing retention (at the cost of harming retention) weakens. This contributes to downward pressure on the optimal threshold. On the other hand, increasing $\alpha(0)$ also increases $\alpha(1)$. This increases the attractiveness of reaching a higher engagement level and dropping the incented action. Indeed, referring to Table 2.1, $W^y(1)$ is increasing in $\alpha(1)$ when $y(0) = 1$. This puts upward pressure on the optimal threshold. This latter "future" benefit is weak for lower levels of $\alpha(0)$, where it may be optimal to offer an incented action in the last period. This provides justification for the "U"-shaped pattern.

The scenarios in the above two examples provide a clear illustration of the complexity of our model. At different engagement levels, and with different prospects for the value of future benefits, the optimal strategy can be influenced in nonintuitive ways. This is particularly true for changes in $\alpha(e)$ as it impacts all three effects – revenue, retention, and progression. In some sense, cannibalization is the core issue in offering incented actions. This is evident in our examples and a careful examination of the definitions in Section 2.4 – the parameter $\alpha(e)$ is ubiquitous.

2.7 Conclusion

In this paper we investigated the use of incented actions in mobile games, a popular strategy for extracting additional revenue from players in freemium games where the vast majority of players are unlikely to monetize. We discussed the reasons for offering incented actions, and built an analytical model to assess the associated tradeoffs. This understanding lead us to define sufficient conditions for the optimality of threshold policies, which we later analyzed to provide managerial insights into what types of game designs are best suited to offering incented actions. Our approach of using an MDP has some direct benefits to practitioners. With player data and relevant game parameters that companies have access to in the age of big data, validating our model and using it to derive insights on the impact of certain policies is plausible.

Our analytical approach was to devise a parsimonious stylized model that abstracts a fair deal from reality and yet nonetheless maintained the salient features needed to assess the impact and effects of offering incented actions. For instance, we assume the publisher has complete knowledge about the player's transition probabilities and awareness of the engagement state. In the setting where transition probabilities are unknown, some statistical learning algorithm and classification of players into types would be required. Moreover, in the situation where engagement is difficult to define or measure, a partially observed Markov decision process (POMDP) model would be required, where only certain signals of the player's underlying engagement can be observed. There is also the question of microfounding the player model that we explore, asking what random utility model could give rise to the transition probabilities that we take as given in our model. All these questions are outside of our current scope but could nonetheless add realism to our approach. Of course, the challenge of establishing the existence of threshold policies in these extensions is likely to be prohibitive. Indeed, discovering analytical properties of optimal policies of any form in a POMDP is challenging [30]. It is likely that these extensions would produce studies that are more algorithmic and numerical in nature, whereas in the current study we were interested in deriving insights.

Finally, the current study ignores an important actor in the case of games hosted on mobile platforms – the platform holder. In the case of the iOS App Store, Apple has made several interventions that either limited or more closely monitored the practice of incented actions (see, for instance, Connelly [15]). In fact, the platform holder and game publisher have misaligned incentives when it comes to incented actions. Typically, the revenue derived from incented actions is not processed through the platform, whereas in-app purchases are. We feel that investigation of the incentive misalignment problem between platform and publisher, possibly as a dynamic contracting problem, is a promising area of future research. The model developed here is a building block for such a study.

Chapter 3

A Dynamic Price-Only Contract: Exact and Asymptotic Results

3.1 Introduction

Consider a simple supply chain with two firms with symmetric information, wherein a seller trades with a downstream buyer who faces customer demand for the product. This system is perhaps the most well understood decentralized model and has been analyzed in the extant literature in industrial organization and operations management among others. In this paper, we revisit this simple system under what we call a generalized price-only contract. We demonstrate several interesting properties of this system under this easy to understand contract, show interesting connections to established results in the literature and explore the implications of our findings to future research.

A price-only contract, otherwise known as a simple linear wholesale price contract, specifies a per unit price w at which the seller offers her product to the buyer who then buys some quantity q. The buyer uses this quantity (and potentially other levers such as selling price when relevant) to generate revenue from customer demand. Therefore, in the simplest setting where players are strategic, these decisions are arrived at as an equilibrium of a corresponding game. The common paradigm is one where the seller moves first, setting a wholesale price w, which is followed by the buyer's decision of purchasing a quantity q. This is referred to as a Stackelberg game (with the seller as the leader in this case). It is well known that the resulting vertical competition between the players in this model leads to inefficiencies referred to as double marginalization. This loss of efficiency (failing to get to the first best) can be addressed by numerous contracting recipes when there are no information asymmetries. Our interest is not in addressing this inefficiency, although one consequence of our analysis is related to achieving first best. Rather, we explore what would be the effect on the equilibrium decisions if instead of providing the two firms one opportunity to trade, they are allowed to engage dynamically, multiple times, still using a simple linear price-only contract. To be precise, for some n positive, the seller first informs the buyer that n wholesale prices will be offered sequentially and dynamically. Then she proposes the first price, w_1, and the retailer decides upon an order quantity, q_1, at this price. Thereafter, the supplier offers a new price, w_2, and the retailer orders a new quantity q_2, etc. At the end of the last offer, indexed by n, the buyer has cumulatively purchased $Q^n = q_1 + \cdots + q_n$, which is used to satisfy market demand. Figure 3.1 illustrates the sequence of events. Thus, transactions

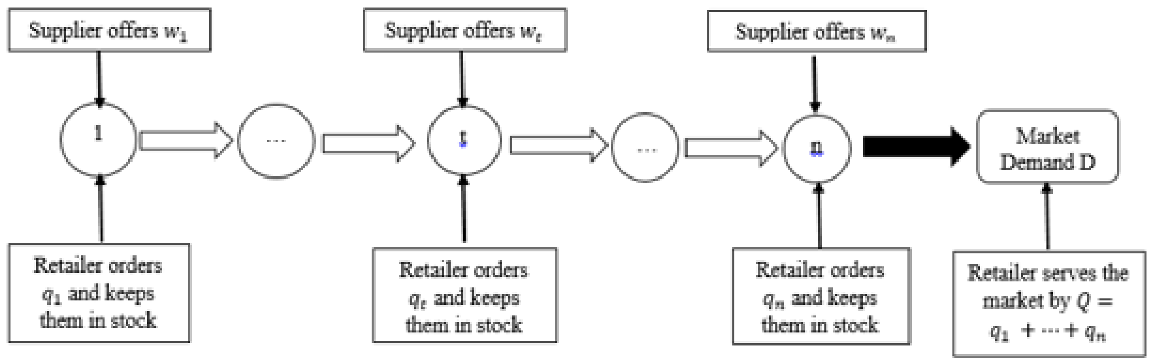
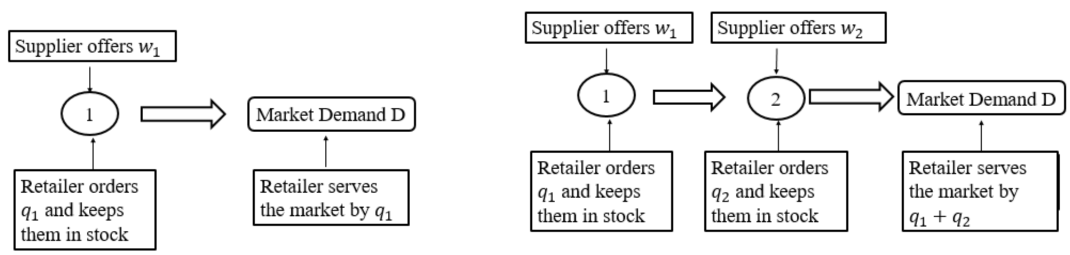


Figure 3.1: Sequence of events under n-stage generalized price-only contract



(a) $n = 1$

(b) $n = 2$

Figure 3.2: Illustration of generalized price-only contract

between the firms occur dynamically and incrementally, in anticipation of downstream revenues that will accrue by meeting demand after the n transactions were concluded. We refer to this contract as a *generalized price-only contract*, and we study some properties of this contract. Figure 3.2 gives two examples when $n = 1$ and $n = 2$.

Special instances of the generalized price-only contract were introduced and studied by Erhun et al. [20] and by Martínez de Albéniz and Simchi-Levi [38]. Specifically, Erhun et al. [20] study a dynamic model of a procurement between a supplier and a buyer, when demand is deterministic and linear with price. They prove that the supplier, the buyer and the end customers benefit from multiple trading opportunities versus a one-shot procurement agreement. Martínez de Albéniz and Simchi-Levi [38] extend the results to the newsvendor setting. They provide sufficient conditions for the existence and uniqueness of a well-behaved sub-game perfect equilibrium and they show that as the number of rounds increases, the profits of the supply chain increases towards the supply chain optimum.

In this paper, we consider a similar model, but we allow for fairly general demand settings. Similar to [20] and [38], we show that in our more general setting, both parties benefit from multiple trading opportunities. Indeed, the benefit increases with the value of n, the horizon specified in the contract. Moreover, as n approaches infinity, the sum of the profits of the two players approaches the first best profit. This, of course, implies that the total order quantity is non decreasing and converges to the corresponding first best order quantity as n increases. The results hold without specifying the demand function. We only require that the buyer's revenue function satisfies some mild conditions.

We also show that for a given generalized price-only contract with a specified n, the wholesale prices monotonically decrease. However, somewhat surprisingly, for a fixed n, the order quantities within the n periods may not be monotonic; that is, it can decrease or increase in $t = 1, \ldots, n$. Finally, we provide necessary and sufficient conditions for the supplier's revenue to increase, decrease or remain constant from one period to another, and we derive closed form solutions for three settings in which demand is exponential, uniform or constant.

Our paper is also related to three streams in the literature. The first deals with documenting and quantifying the loss to the supply chain of using a simple linear wholesale price contract. Lariviere and Porteus [31] study this problem in the newsvender setting and derive conditions on the demand distribution for the existence of an interior (equilibrium) solution. They also empirically calculate the potential efficiency losses for various demand distributions. Perakis and Roels [45] obtain a theoretical bound on the worst-case loss in this system, and extend the analysis to other related systems. One of our results demonstrates that as the number of transactions, n, grows, the efficiency loss decreases and approaches zero as n goes to infinity. In fact, this seems to be true for more general systems where the buyer may not be a newsvendor.

The second stream in the literature is from industrial organization and is related to the work of Coase [14] on durable goods. Coase's result (a conjecture to be precise) was that a duropolist (a monopolist selling a durable good) does not have monopoly power due to her inability to commit. The simple intuition behind the Coase conjecture is that if the duropolist charges a high price for the durable good, then consumers anticipate a future price reduction (as they expect the duropolist to later target consumers with lower valuations), and therefore they prefer to wait. The duropolist, anticipating such consumer behavior, will then drop prices down to the competitive level. Thus, a duropolist faces intense competition, not from other players but from future incarnations of herself. This intuition does not quite translate when customers are atomic and have impact on the outcome of the game. In this case, the subgame perfect equilibrium, known as the Pacman equilibrium, delivers a non zero profit to the duropolist and decreases the effect of the commitment problem. An excellent of discussion of this topic and bounds on the duropolist profits are given in Berbeglia et al. [6]. The results in our note are related as we look at a setting with two players, a buyer and a seller, trading with each other, and the buyer selling to the market. Our results indicate that in such a setting, the commitment problem is weakened and the first best solution emerges that distributes the profit between the players. We also note that our results cannot be recovered from this stream in the literature.

The third stream in the literature that is related to our paper is the one on negotiation power and contracting in supply chains. If one thinks of contracts as a way to achieve efficiency in a competitive supply chain, allocations of the first best solution is usually delegated to some sort of negotiation process. The outcome of this process depends on various factors, an important one being the negotiation power of the players. The results in our paper provide an organic division of the first best profits as n approaches infinity. The seller enjoys some power due to her role as a Stackelberg leader, but is unable to extract the entire surplus, both in the regular setting as well as in our dynamic and incremental setting. The exact division of the pie, in the dynamic setting, depends on several factors including the elasticity of demand, the shape of the demand distribution etc., which will be discussed later. Thus, our results provide an alternative way of looking at the notion of bargaining power in supply chains (see also Bernstein and Nagarajan [7] for a more extensive discussion of this topic). A related but somewhat different paper on this topic is the one by Anand et al. [2]. They study a simple dynamic problem where a buyer and a seller transact using wholesale prices and the buyer faces a demand which is price sensitive and deterministic. Unlike our model, the buyer faces customer demand in each period. Anand et al. [2] show that the buyer carries over inventory from one period to another purely for strategic reasons — as demand is certain, there are no fixed costs or other economies of scale. The main reason for carrying inventory is that by doing so, the buyer forces the seller to offer lower wholesale prices in future periods since the buyer is able to start future periods with a positive inventory. The monotonic nature of the wholesale prices in their model is similar to the monotonicity of wholesale prices that we derive in our model. Moreover, they similarly show that the dynamic interaction (and the presence of strategic interactions) is beneficial to both players despite the seller's channel relative power being eroded by the dynamic interaction.

3.2 Generalized Price-only Contracts with n Offers

We consider a seller (henceforth referred to as a supplier) who transacts with a buyer (referred to as a retailer) using a generalized price-only contract. For some n exogenous, the supplier informs the retailer that n wholesale prices will be offered sequentially. For each wholesale price proposed, the retailer decides upon the quantity he will purchase at that price. Thus, indexing the offers forward, the supplier first offers a price w_1, and the retailer commits to buying q_1; then the supplier offers w_2, and the retailer commits to q_2, and so on so forth. The issue of commitment is not significant here. For example, we can simply assume that each transaction is completed and money and goods are exchanged before the next price is announced by the supplier. All decisions are made in anticipation of the demand. We will assume that both players are risk neutral. At the end of the n-th offer, trade occurs between the two players and the total amount of units that the retailer has purchased is denoted by $Q^n = q_1 + \cdots + q_n$. The retailer then uses Q^n to satisfy market demand.

We first assume that the demand faced by the retailer can be quite general. Let $R(Q)$ be the retailer's (expected) sales revenue given the total inventory level Q. For example, under the newsvendor setting, $R(Q) = p \int_0^Q \bar{F}(z)dz$, where p is the exogenous retail price and F is the CDF of demand distribution ($\bar{F} = 1 - F$). If demand is deterministic but price sensitive with inverse demand function $p = P(Q)$, we have $R(Q) = P(Q)Q$. We assume that $R(Q)$ is sufficiently smooth, as will be clear from the analysis. A stronger assumption can be $R(Q)$ is analytic, which requires that the function is infinitely differentiable and the power series converges to it. Further, we assume $R(Q)$ to be strictly concave. This holds for most economic settings we are interested in. For simplicity, we assume the supplier's marginal production cost $c = 0$. As long as costs are linear, this assumption is without loss of generality. We define π_R^n, π_S^n and π_T^n be the retailer, the supplier and the supply chain total profit, respectively, which are given by

$$\begin{aligned}
\pi_R^n &= R(q_1 + \cdots + q_n) - w_1 q_1 - w_2 q_2 - \cdots - w_n q_n \ , \\
\pi_S^n &= w_1 q_1 + w_2 q_2 + \cdots + w_n q_n \ , \\
\pi_T^n &= \pi_R^n + \pi_S^n = R(q_1 + \cdots + q_n) \ .
\end{aligned}$$

Denote by Q^{FB} the first-best order quantity which maximizes the total supply chain profit $\pi_T = R(Q)$. Since the problem is concave, Q^{FB} can be characterized by the first-order condition $R'(Q^{FB}) = 0$.

When the supplier and the retailer are independent self-interested rational players, we model the problem as a dynamic game with perfect information where the supplier and the retailer take actions sequentially with the supplier being the first mover. We are interested in the subgame perfect Nash equilibrium (SPNE). Due to the lack of commitment issue and the fact that in any epoch t, the only relevant information that both players use in computing their strategies is the length of the horizon and total order quantity traded up until t, relatively mild conditions are needed to guarantee the existence of a SPNE. Denote by w_t^* and q_t^* the equilibrium solution, $t = 1, \ldots, n$. We denote by x_t the pre-order inventory level at period t, so $x_1 = 0$ and $x_t = q_1 + \cdots + q_{t-1}$, $t = 2, \ldots, n$. We let $x_t^* = q_1^* + \cdots + q_{t-1}^*$ be the corresponding equilibrium pre-order inventory level at period t, and we let $Q^{n,*} = q_1^* + \cdots + q_n^*$ be the corresponding equilibrium total quantity purchased.

We use backward induction to find a SPNE. At the last offer, we first determine the retailer's optimal order quantity q_n^* for each possible wholesale price w_n and history of the previous offers. Then, given the retailer will follow his strategy, we compute what price w_n^* the supplier will offer for each possible history. Note that the pre-order inventory level x_n summarizes all useful information about the history of previous offers to determine the equilibrium solution. Hence, the equilibrium strategy can be denoted by $q_n^*(x_n, w_n)$ and $w_n^*(x_n)$. Now, one can show that there exists a mapping between w_n and q_n^* for any given x_n, so we equivalently denote the equilibrium strategy as $w_n^*(x_n, q_n)$ and $q_n^*(x_n)$. Therefore, solving the n-th offer problem allows us to naturally revert back to the $(n-1)$-th offer and compute the equilibrium strategy $w_{n-1}^*(x_{n-1}, q_{n-1})$ and $q_{n-1}^*(x_{n-1})$. We repeat this process until we solve the first offer problem $w_1^*(x_1, q_1)$ and $q_1^*(x_1)$. The following theorem provides a characterization of the SPNE for this problem.

Proposition 3.2.1. *Suppose the subgame perfect Nash equilibrium exists, the equilibrium solution at $t = n-k$ $(k \geq 0)$ satisfies the following condition:*

$$w^*_{n-k}(x_{n-k}, q_{n-k}) = (k+1)R'(Q^n)\prod_{j=1}^{k}(1+\frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}) - \sum_{j=1}^{k} w^*_{n-k+j}\frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}\prod_{m=1}^{j-1}(1+\frac{\partial q^*_{n-k+m}}{\partial x_{n-k+m}}) \quad (3.1)$$

$$q^*_{n-k}(x_{n-k}) = \begin{cases} \text{the solution of: } R'(Q^n)\prod_{j=1}^{k}(1+\frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}) + \frac{\partial w^*_{n-k}}{\partial q_{n-k}} q_{n-k} = 0 & \text{if } x_{n-k} < Q^{FB} \\ 0 & \text{if } x_{n-k} \geq Q^{FB} \end{cases} \quad (3.2)$$

where $Q^n = x_{n-k} + q_{n-k} + q^*_{n-k+1} + \cdots + q^*_n$.

Proposition 3.2.1 characterizes the equilibrium solution if it exists. Naturally, for general demand functions, condition (3.2) does not lead to closed form algebraic expressions. However, our interest is not to derive closed form expressions, but rather, primarily to analyze the properties of the equilibrium solution and the effect of multiple price offers on the supplier, the retailer and the whole supply chain.

Our first result is intuitive and shows that multiple price offers mitigate the double marginalization effect.

Theorem 3.2.2. *(a) The equilibrium total inventory $Q^{n,*}$ strictly increases in the number of offers n but will not exceed the first best inventory level Q^{FB}.*

(b) When the total number of price offers n increases, both the supplier's total profit and the retailer's total profit increase; moreover the supply chain's total profit strictly increases.

Intuitively, the opportunity to order multiple times will weaken the supplier's power and introduce a competition among the different offers. As a result, the supplier is forced to lower her wholesale prices and consequently, double marginalization is reduced. Furthermore, since the supply chain profit $\pi^{n,*}_T = R(Q^{n,*})$ is increasing in $Q^{n,*}$, Theorem 3.2.2(a) implies the total size of the pie, i.e., total supply chain profit, increases with the number of price offers. One reason for this is that multiple wholesale prices, in the case of stochastic demand, better matches the marginal return on risk for units of inventory that are purchased in anticipation of the demand. Furthermore, Theorem 3.2.2 claims that not only the whole supply chain profit increases in n but also neither the supplier nor the retailer will be worse off with more offers. As expected, the retailer takes advantage of the multiple chances for ordering and induces successively lower wholesale prices. On the other hand, by offering multiple wholesale prices, the supplier price-discriminates the retailer's orders. Therefore, the supplier can extract a higher profit and benefits herself. It should be noted that it is possible that either the supplier's profit or the retailer's profit "weakly increases", but the channel profit must strictly increase. We next show that full efficiency will be achieved in the limit. In other words, the total inventory level will get closer to the first-best when the number of price offers n increases. Thus, with the generalized price-only contract, the price of anarchy asymptotically vanishes.

Theorem 3.2.3. *The equilibrium total inventory level $Q^{n,*}$ approaches the first best order quantity Q^{FB} when n goes to infinity.*

So far, we have argued that under generalized price-only contract, the supplier's monopoly power will diminish and the markup she can charge will be reduced due to the retailer's previous orders. Nevertheless, the supplier is strategic and can anticipate this, so she would prefer to discourage the retailer from ordering large amounts in the early offers. As a result, we expect higher wholesale prices in earlier offers. This result is consistent with similar findings in other settings under dynamic contracting in supply chains, such as the one considered by Anand et al. [2].

Proposition 3.2.4. *Given a fixed n, on the equilibrium path, the wholesale prices strictly decrease, i.e. $w_1^* > w_2^* > \cdots > w_n^*$.*

As the equilibrium wholesale prices decrease in t, one may similarly expect the order quantities to correspondingly increase in t. Interestingly, we find out that the order quantities may not increase in t. In fact, the order quantities may exhibit fairly arbitrary patterns. Next, let us consider the supplier's per-period profit $w_{n-k}^* q_{n-k}^*$. Define α_{n-k} as follows

$$\alpha_{n-k} = 1 + \sum_{m=1}^{n} \prod_{j=m}^{k} (1 + \frac{\partial q_{n-k+j}^*}{\partial x_{n-k+j}}).$$

For instance, $\alpha_n = 1$ and $\alpha_{n-1} = 1 + (1 + \frac{\partial q_n^*}{\partial x_n}) = 2 + \frac{\partial q_n^*}{\partial x_n}$. One can directly verify from Proposition 3.2.1 that $w_{n-k}^* = R'(Q^n)\alpha_{n-k}$. The next result demonstrates that α_{n-k} plays an important role in comparing successive profits by the supplier.

Proposition 3.2.5. $w_{n-k+1}^* q_{n-k+1}^* (>, =, <) w_{n-k}^* q_{n-k}^*$ *if and only if* $\frac{\partial(\alpha_k/\alpha_{k+1})}{\partial x_k}(<, =, >)0$.

The proposition provides a necessary and sufficient condition for determining the pattern of the sequence of $w_{n-k}^* q_{n-k}^*$. For example, in the last two periods, $\partial(\alpha_{n-1}/\alpha_n)/\partial x_{n-1} = \partial^2 q_n^*/\partial x_n^2$. Therefore, whether $w_{n-1}^* q_{n-1}^*$ is greater than $w_n^* q_n^*$ depends on the convexity of q_n^* with respect to x_n. Let us consider the inverse demand function $p = a - bQ^\gamma$. One can show that if $\gamma = 1$, $\partial^2 q_n^*/\partial x_n^2 = 0$, so $w_{n-1}^* q_{n-1}^* = w_n^* q_n^*$; if $\gamma > 1$, $\partial^2 q_n^*/\partial x_n^2 > 0$, so $w_{n-1}^* q_{n-1}^* > w_n^* q_n^*$; if $\gamma < 1$, $\partial^2 q_n^*/\partial x_n^2 < 0$, so $w_{n-1}^* q_{n-1}^* < w_n^* q_n^*$. Thus, one can provide examples such that the supplier's per period profit decreases or increases or even stays the same over time for a fixed n.

Theorem 3.2.2 has shown that the size of the pie increases with the number of price offers. In addition, both the supplier's and the retailer's profits increase. A question of interest is whether the relative power of the two players changes with multiple offers. To illustrate this, note that when n =1, we have a simple static Stackelberg game. Under such game structure, the first mover (in our case the supplier) generally extracts a larger share of the overall profit. In fact, in newsvendor type settings, when demand is deterministic, linear and price sensitive, the supplier extracts two-thirds of the total profits. In what follows, we try to understand how the profit will be actually allocated between the two parties as n increases and whether the shares of the supplier and the retailer become more balanced. In particular, we investigate how multiple price offers will influence the ratio of the retailer's total profit over the supplier's total profit. One may suspect that this ratio will increase with the total number of price offers n. In other words, multiple price offers help to balance the channel power. In order to check this intuition, we will study three specific demand cases: exponential demand, uniform demand and linear (price-sensitive) demand.

Theorem 3.2.6. *In a n-stage generalized price-only contract:*

(1) Suppose demand is exponentially distributed with a parameter λ and market price p is exogenous, then the equilibrium strategies are

$$\begin{aligned} q_j^* &= \frac{1}{(n-j+1)\lambda} \quad j = 1, \dots, n \\ w_j^* &= (n-j+1)pe^{-\lambda \sum_{l=1}^{n} 1/l} \quad j = 1, \dots, n \,. \end{aligned}$$

(2) Suppose demand is uniformly distributed on $[0, M]$ and market price p is exogenous, then the equilibrium strategies are

$$\begin{aligned} q_1^* &= \frac{1}{2n} M \quad \text{and} \quad w_1^* = \beta_1 p \\ q_j^* &= \prod_{l=1}^{j-1} \frac{2(n-l)+1}{2(n-l)} q_1^* \quad j = 2, \dots, n \\ w_j^* &= \prod_{l=1}^{j-1} \frac{2(n-l)}{2(n-l)+1} w_1^* \quad j = 2, \dots, n \end{aligned}$$

where $\beta_n = \frac{1}{2}$ and $\beta_j = \frac{(2(n-j)+1)^2}{4(n-j)(n-j+1)} \beta_{j+1}$, $j = 1, \dots, n-1$.

(3) Suppose demand is deterministic with inverse demand function $p = a - bQ$, then the equilibrium strategies are

$$\begin{aligned} q_1^* &= \frac{a}{4bn} \quad \text{and} \quad w_1^* = \beta_1 a \\ q_j^* &= \prod_{l=1}^{j-1} \frac{2(n-l)+1}{2(n-l)} q_1^* \quad j = 1, \dots, n \\ w_j^* &= \prod_{l=1}^{j-1} \frac{2(n-l)}{2(n-l)+1} w_1^* \quad j = 1, \dots, n \end{aligned}$$

where $\beta_n = \frac{1}{2}$ and $\beta_j = \frac{(2(n-j)+1)^2}{4(n-j)(n-j+1)} \beta_{j+1}$, $j = 1, \dots, n-1$.

Thus, under exponential demand, uniform demand or linear (price-sensitive) demand, the problem is tractable and the analytical form of the equilibrium solution is clean and elegant. We find, remarkably, that in these three cases, for a fixed n, the supplier's per-period profit is identical, $w_t^* q_t^* = w_{t+1}^* q_{t+1}^*$, $t = 1, \ldots, n-1$. This result follows because the ratio, q_t^*/q_{t+1}^*, is always a constant which leads to a constant α_t, independent of q_t, implying that $\partial(\alpha_t/\alpha_{t+1})/\partial q_t = 0$. Thus, by Proposition 3.2.5, it follows that the supplier's profits per period are identical. Hence, note also that in these scenarios, since the equilibrium wholesale price decreases in t, the equilibrium order quantity increases in t.

Finally, let us investigate how the ratio of the retailer's total profit and the supplier's total profit changes with n for the three scenarios considered in Theorem 3.2.6. As is illustrated in Figure 3.3, the ratio, $\pi_R^{n,*}/\pi_S^{n,*}$, increases in n, but does not converge to 1. This can be verified by exploiting the expressions in Theorem 3.2.6. Therefore, based on these three examples, it appears that under generalized price-only contract, the supplier still has a somewhat decreasing first-mover advantage.

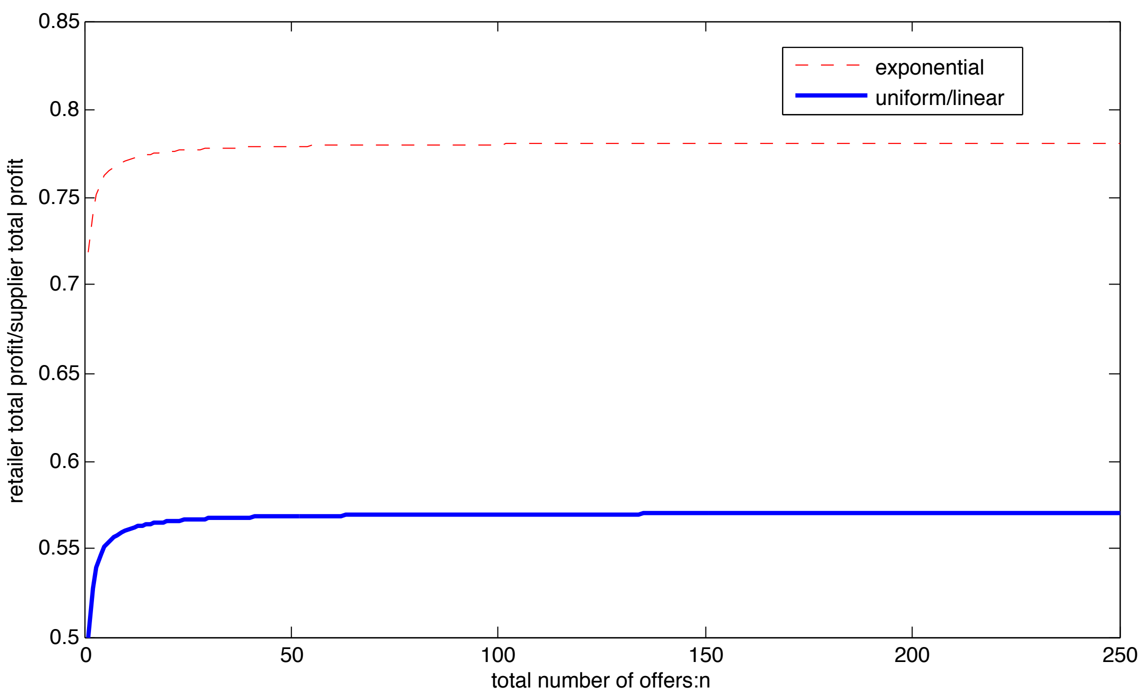


Figure 3.3: The ratio of the retailer's total profit over the supplier's total profit as the number of price offers n increases under (1) exponential demand (2) uniform demand or linear demand

3.3 Conclusion

In this paper, we study a generalization of the well-known wholesale price contract – the generalized price-only contract. We extend known results in the literature about this contract to more general demand settings and we derive some new interesting properties of the corresponding sub-game perfect equilibrium. We demonstrate that the introduction of additional trading opportunities benefits both players. Moreover, as the number of price offers n in the generalized price-only contract approaches infinity, the supply chain profit approaches the first best profit. We also show that for a given contract with a specified n, the wholesale prices monotonically decrease. We also reveal some curious properties of the generalized price-only contract, such as the stationarity of the supplier's per period profit in the three specific demand cases: exponential demand, uniform demand and linear (price-sensitive) demand, and we provide necessary and sufficient conditions for this to hold (see Proposition 3.2.5). Future research, for example, can attempt to derive theoretical characterizations of the balance of power, as the number of interactions increases, and bounds on the performance of this contract for a fixed n larger than one.

Chapter 4

Dynamic Short-term Supply Contracts under Private Inventory and Backorder Information

4.1 Introduction

We consider a two-echelon supply chain where a single supplier sells to a retailer in multiple periods. The sequence of events is as follows: In each period t, the supplier offers a one-period contract to the retailer. If the retailer rejects the contract, the relationship between the two parties is terminated and the game is over. If the retailer accepts the contract, he makes his order decision in anticipation of the random demand. The demand is then realized and the retailer collects his revenue (given an exogenous selling price r). Unmet demand is backordered, subject to a unit stock-out penalty b, and left-over inventory is carried over to the next period, subject to a unit holding cost h. In the next period $t+1$, the supplier designs a new contract, and the above events are repeated.

We make three fundamental assumptions in this paper: (1) Unmet demand in each period is backordered; (2) The retailer's inventory or backorder level at the beginning of each period is unobservable by the supplier; (3) A one-period contract is offered and executed in each period.

The first assumption differentiates our underlying inventory model from the lost-sales model. Backorder and lost-sales are the two standard dynamic inventory models. The lost-sales counterpart of our problem has been investigated by Zhang et al. [57], but the backorder setting has been left unattended until now. It will be shown in this chapter that the optimal solutions in these two settings are drastically different. We will come back to this assumption shortly. Note that the beginning inventory or backorder in period t is denoted by x_t, which can be positive (representing inventory) or negative (representing backorder). For convenience, we may refer to both inventory and backorder as (possibly negative) inventory when there is no confusion. In contrast, x_t is always non-negative in the lost-sales setting.

The second assumption characterizes our model as a dynamic adverse-selection model, or equivalently, a dynamic principal-agent model with hidden information. Asymmetric information is prevalent in operations management. As a core subject of operations management, inventory management has traditionally overlooked the transparency or accessibility of inventory information. In reality, retailers are reluctant to share sales and inventory information with their suppliers. For instance, according to a study of the Canadian logistics industry by Statistics Canada (2003) [11], only about 10% of Canadian retailers share inventory data over established platforms. There are a variety of reasons. One reason comes from strategic considerations by the retailer. Retailers may take advantage of this private information and underreport past sales to elicit discount from suppliers. Other reasons include the lack of trust between supply chain members and confidentiality restrictions of different parties.

It is well known that in the "selling to the newsvendor" model, simple contracts, such as wholesale price contract or buyback contract, is sub-optimal when the retailer has private information. Burnetas et al. [9] showed that in the static setting, the optimal contract entails concave quantity discount with the marginal unit payment decreasing in the order quantity. In general, the optimal contract can be very complex in a multi-period setting. Dynamic contracting is documented to be a challenging problem due to a host of technical and expositional difficulties. There is a large economics literature on dynamic contracting (e.g: [4] and [5]). The related literature in operations management is scant, with few published papers.

The third assumption delineates the type of contract under investigation. In the multi-period setting, there are at least two contracting modes that the supplier can choose from, short-term contracting and long-term contracting. Under long-term contracting, the principal (supplier) needs to commit to a contingency plan covering the entire decision horizon, while under short-term contracting, the principal only commits to a short-term (usually one-period) contract and a new contract is put in place once the previous one ends. The latter contracting mode is appropriate if the supplier lacks the credibility of carrying out a long-term contract or if she prefers the relative simplicity of managing one-period contracts. This paper will focus on short-term contracting with a new contract in each period. Indeed, there are a large number of real-world examples that the suppliers do not provide retailers with long-term price guarantees and replenishment schedules.

A key concept in the analysis of a short-term contracting problem is the belief of the private information. In our problem, knowing the demand distribution and the quantity ordered by the retailer in period $t-1$, the supplier can only tell the distribution of the retailer's inventory/backorder level at the beginning of period t (denoted by x_t), but not the exact value. The supplier's belief can be described by CDF $G_t(x_t)$ or PDF $g_t(x_t)$. One major source of difficulty in dynamic short-term contracting results from the complexity of the supplier's belief which is updated according to the contract and the retailer's response in every period following Bayes' rule.

To summarize, the three fundamental assumptions define our problem as a dynamic short-term adverse-selection problem on top of a backorder inventory system. It belongs to the most challenging type of contract design problems and fills a significant gap in the existing literature. The papers most closely related to our work are Zhang et al. [57] and Ilan and Xiao [27]. Zhang et al. [57] introduced the first dynamic contracting problem with private inventory information, which formalizes an important problem in supply chain management. They focus on dynamic short-term contracts in the lost-sales setting and show that the optimal contract takes the form of a batch-order contract under certain model assumptions. Ilan and Xiao [27] studied optimal long-term contracts in both lost-sales and backorder settings. They prove that in the backorder case, the optimal long-term contract consists of a menu of wholesale prices and upfront fees, whereas in the lost-sales case, the optimal long-term contract takes the same form with an additional option to lower the wholesale price (at a future time, after paying a fee). Our work fills the gap in this literature. The contracting takes place in every period, with inventory or backorder kept privately by the retailer. Our goal is to characterize the optimal short-term contract. We want to know if the optimal contract will still have a simple structure and be easy to implement. We are also interested in how short-term contracting plus the backorder assumption may lead to different managerial insights.

In this chapter, we will investigate three scenarios of the modeling horizon: (1) a single period; (2) two periods; and (3) an infinite horizon. For tractability of the model, we assume that the demand D_t is i.i.d. and follows an exponential distribution with rate λ. This assumption is supported by extant operations management literature (see Iglehart [26], Lau and Lau [32] and Nagarajan and Rajagophlan [41]). A summary of our main findings is as follows. (1) In the single-period setting, the optimal contract induces trade only when the retailer reports a negative inventory level, and the retailer obtains exactly his reservation profit. (2) In the two-period setting, the supplier's optimal contract in the first period involves at most two thresholds. The retailer obtains his reservation profit when the beginning inventory/backorder is below the lower threshold whereas he receives a positive information rent when it is above the upper threshold. In the middle range, the partnership will be terminated and the retailer will be excluded from future business with the supplier. (3) In the infinite-horizon setting, the optimal short-term contract is complicated and non-stationary in general. An interesting contract emerges. It induces a generalized base-stock policy, where the order-up-to level increases with and converges to the beginning inventory. This contract leaves zero information rent to the retailer (zero profit increment beyond his reservation profit) at any beginning inventory level. Although this contract is sub-optimal in general, it is intuitive, easy to implement, and provides a good heuristic for the optimal contract. We also conjecture that when the backorder cost is relatively low, the optimal contract induces a base-stock policy with an exclusion region for the beginning inventory.

The insights we obtain in the backorder case are substantially different from those in the lost-sales case. In our setting, the optimal short-term contract has a threshold structure, with possible exclusion in a middle range. The information rent under the optimal contract may be non-monotone in the retailer's beginning inventory. The supplier would sometimes prefer to deal with a retailer with high inventory, in contrast to the lost-sales case where the supplier always prefers a retailer with low inventory. In addition, the optimal contract may terminate the partnership when the retailer's inventory falls in a certain interval. The above observations accord with the troublesome phenomenon of "countervailing incentives" examined in the contract design literature (see e.g., Lewis and Sappington [34] and Jullien [29]). From this perspective, the backorder setting results in a more complex information structure than the lost-sales setting. Consequently, contrary to the common wisdom from inventory management, the dynamic short-term contracting problem is more involved under backorder than under lost sales.

The emergence of countervailing incentives in our problem is related to our assumptions about the backorder arrangement. To be consistent with the backorder inventory model, we assume that when ordering from the supplier, the retailer is required to take backorder from customers as needed. To avoid the arbitrage opportunity for the retailer to take backorder intentionally, we assume $b > (1-\delta)r$, i.e., the penalty for backorder is greater than the possible interest that can be earned. As a result, if the retailer abandons the relationship, he should not take backorder anymore, as it is unprofitable. The pressure of accommodating backorder devalues the relationship with the supplier and in turn increases the retailer's bargaining power. When the beginning inventory falls into a certain region, the supplier suffers even without leaving any information rent to the retailer and would rather let the retailer go.

The rest of the paper is organized as follows. Section 4.2 provides a brief literature review. In Section 4.3, we describe the general problem and establish the mathematical model. Then we analyze the single period case in Section 4.4, two-period case in Section 4.5 and infinite-horizon case in Section 4.6. Finally we conclude in Section 4.7. All proofs are deferred to the appendix.

4.2 Literature Review

Our work is related to the literature on dynamic adverse-selection problems. This topic is pioneered by economists and has recently found important applications in operations management. Myerson [40] introduced the famous revelation principle. He proves that any outcome that is implementable in equilibrium in an arbitrary mechanism can also be implemented in equilibrium via a direct revelation mechanism. The revelation principle serves as the starting point in analyzing adverse-selection problems, as well as other mechanism design problems. Dynamic adverse-selection problem is known to be a challenging problem and is much less understood than static adverse-selection problems due to a host of technical and expositional difficulties. Economists have made great efforts to characterize the optimal mechanism in certain specific settings. For instance, the majority of the literature focus on simplified cases where the hidden state is either constant or the state is sampled from time-independent distributions. See illustrative examples in Salanie [47] and Bolton and Dewatripont [8].

However, these models are too restrictive to capture the setting we are interested in. In this paper, we assume that the supplier cannot observe the retailer's inventory before ordering. The hidden state is the retailer's pre-order inventory level which will be affected by the state and action in the previous period. Thus, our problem belongs to a particular type of dynamic adverse-selection problems with an underlying Markov decision process. Battaglini [4] studied the optimal long-term contract between a monopolist and a buyer whose private preferences evolve as a two-state Markov process. Long-term contracting means that the manufacturer offers a contingency plan over the entire horizon and she has to commit to that plan at the beginning of the time horizon. It is considered a significant finding in Markovian adverse-selection problems, yet with a significant limitation about the two-state assumption. Zhang and Zenios [56] consider a general framework with more than two state, and develop a dynamic programming algorithm to derive optimal long-term contracts. To the best of our knowledge, dynamic short-term adverse-selection problems are even less studied in the literature, at least partly due to the very complex belief process mentioned in the introduction. The only noticeable work with an underlying Markov process is Zhang et al. [57]. Thus, the current work makes a theoretical contribution in deepening our understanding of this class of problems.

There is a growing number of papers that explore the contracting problem among self-interested firms with private information in a supply chain. For instance, Corbett and Groote [16], Ha [25] and Corbett et al. [17] examine the situation in which the supplier does not know the cost structure of the buyer. Other examples include Nazerzadeh and Perakis [42] on capacity information asymmetry and Cachon and Lariviere [10], Shin and Tunca [49] and Taylor and Xiao [54] on demand information asymmetry. All of these papers focus on static models. The paper by Burnetas et al. [9] studied a similar setting as our work, except that they only considered the single-period model. They show when a manufacturer sells to a newsvendor retailer with private demand information, the optimal contract takes the form of concave quantity discount.

Recently, several pioneering studies have explored multi-period contracting problem where private information arises over time and operational decisions need to be made dynamically based on available information. Ilan and Xiao [27] studied the optimal long-term contract when a manufacturer sells to a retailer over multiple periods with asymmetric demand and inventory information. They showed that in the backorder case, the optimal long-term contract consists of a menu of wholesale prices and associated upfront fees, whereas in the lost sales case, the optimal long-term contract takes the same form but has an additional option at a later time to lower the wholesale price after paying an option fee. In contrast, this paper will focus on short-term contracting, where the supplier only offers a one-period contract to the retailer in every period. The paper by Zhang et al. [57] is the closest to this paper. They examine the optimal short-term contract in the lost-sales case. They show that the optimal contracts take the form of batch-order contracts under certain cost regimes.

4.3 Problem Formulation

In this section, we formulate our problem and discuss some basic facts about the model. We will derive the optimal (short-term) contract in the single-period case, two-period case, and infinite-horizon case in subsequent sections. The sequence of events is as follows. In period t, the supplier offers a one-period contract to the retailer. If the retailer rejects the contract, the partnership is broken and the game is over. If the retailer accepts the contract, he makes his order decision in anticipation of the random demand. We assume zero lead time. Units are transferred to the retailer immediately and payments are received by the supplier. The demand is then realized and the retailer collects his sales revenue. After that, the retailer carries over any excess inventory or unmet backorder to the next period $t+1$. In period $t+1$, the supplier offers a new contract to the retailer, and the game is played dynamically. The total number of periods is T.

We introduce some notation. We write r for the unit selling price, c for the unit production cost, h for the unit holding cost, b for the backorder cost and $\delta \in (0,1)$ for the discount factor. Throughout this paper, we make the following assumption:

Assumption 4.3.1. *The demand D_t in every period is i.i.d. and follows an exponential distribution with rate λ.*

We assume that the supplier cannot observe the sales, i.e. the realized demand, at the retailer in any period. As a result, knowing the demand distribution and the quantity ordered by the retailer in period $t-1$, the supplier can barely tell the distribution of the retailer's pre-order inventory level x_t. We describe the supplier's belief about the pre-order inventory level by CDF $G_t(x_t)$ and PDF $g_t(x_t)$.

If the retailer orders q_t in period t, his post-order inventory level will be $y_t = x_t + q_t$. We define $v_t(y_t)$ as the retailer's one-period profit in period t before transferring any payment to the supplier. More specifically, $v_t(y_t)$ is equal to the retailer's expected revenue, minus the holding cost and backorder cost (holding cost and backorder cost are absent in the last period). We let $\Pi_{t+1}(y_t)$ be the supplier's expected profit-to-go from period $t+1$ onwards given the true y_t. Because x_t is only known by the retailer, the supplier may perceive the post-order inventory to be $\hat{y}_t$. The supplier's perception will affect her belief about x_{t+1} and thereby the optimal contract in period $t+1$. Therefore, we write $U_{t+1}(y_t|\hat{y}_t)$ as the retailer's expected profit-to-go from period $t+1$ onwards given his true post-order inventory y_t and the supplier's perception $\hat{y}_t$; and $\Psi_{t+1}(y_t) = \Pi_{t+1}(y_t) + U_{t+1}(y_t|y_t)$ as the channel's expected profit-to-go from period $t+1$ onwards if the supplier's perception is correct. Finally, we define $\underline{u}_t(x_t)$ as the retailer's reservation profit-to-go from period t onwards if the two parties have terminated their relationship at the beginning of period t.

As we study short-term contracting, a proper solution concept is the "Perfect Bayesian Equilibrium." (See [22] and [44].) In period t, the contract maximizes the supplier's expected profit-to-go given her belief about x_t; and assuming the optimal contracts are offered in all future periods, the retailer's response maximizes his expected profit-to-go given the contract; and the supplier's belief about x_{t+1} is derived according to the Bayes' rule.

According to the Revelation Principle (Myerson [40]), we can focus on direct revelation contracts. The supplier designs a menu contract $\{q_t(x_t), s_t(x_t)\}_{x_t \in (-\infty, \bar{x}_t]}$. We assume that the pre-order inventory x_t in period t is upper bounded by $\bar{x}_t$. For instance, x_t is the result of the previous period's sales, i.e. $x_t = \bar{x}_t - D_{t-1}$. For each possible x_t, the contract specifies a quantity plan $q_t(x_t)$ and payment plan $s_t(x_t)$. The supplier's goal is to maximize her expected profit-to-go, with respect to her belief G_t of the beginning inventory or backorder level x_t. We formulate the contracting problem using the principal-agent framework. The supplier solves the following problem:

$$\max_{s_t,q_t} \int_{-\infty}^{\bar{x}_t} \{s_t(x_t) - cq_t(x_t) + \delta\Pi_{t+1}(x_t + q_t(x_t))\} dG_t(x_t) \tag{4.1}$$

$$\begin{aligned} s.t. \quad & v_t(x_t + q_t(x_t)) - s_t(x_t) + \delta U_{t+1}(x_t + q_t(x_t)|x_t + q_t(x_t)) \\ & = \max_{\hat{x}_t} v_t(x_t + q_t(\hat{x}_t)) - s_t(\hat{x}_t) + \delta U_{t+1}(x_t + q_t(\hat{x}_t)|\hat{x}_t + q_t(\hat{x}_t)), \quad x_t, \hat{x}_t \in (-\infty, \bar{x}_t] \quad (4.2) \\ & v_t(x_t + q_t(x_t)) - s_t(x_t) + \delta U_{t+1}(x_t + q_t(x_t)|x_t + q_t(x_t)) \geq \underline{u}_t(x_t), \quad x_t \in (-\infty, \bar{x}_t] \quad (4.3) \end{aligned}$$

Constraint (4.2) is the "incentive compatibility" (IC) constraint. It prevents the retailer from lying. Reporting a different state $\hat{x}_t$ does not bring any benefit. Constraint (4.3) is the "individual rationality" (IR) constraint. It guarantees the retailer's participation. The profit from choosing $(q_t(x_t), s_t(x_t))$ is at least as good as his reservation profit-to-go. In each period $t = 1, \ldots, T$, the supplier needs to solve the above problem (4.1)-(4.3). In the following sections, we will investigate three situations (1) the single-period case: $T = 1$; (2) the two-period case: $T = 2$; and (3) the infinite-horizon case: $T = \infty$.

4.4 Single Period

First, we consider the single-period case which also includes the last period of a finite-horizon model. In this case, there is no backorder or inventory holding cost at the end. Instead, we assume that the retailer must refund any backorder or throw away any leftover units at the end of the period. Then, the retailer' revenue, given the post-order inventory y_1, is:

$$v_1(y_1) = \begin{cases} rE[\min(y_1, D)] = \frac{r}{\lambda}(1 - e^{-\lambda y_1}), & y_1 \geq 0 \\ ry_1, & y_1 < 0. \end{cases} \tag{4.4}$$

Recall that the post-order inventory $y_1 = x_1 + q_1$ is the sum of the pre-order inventory x_1 and the order quantity q_1. When $y_1 \geq 0$, the retailer collects all possible sales revenue and $v_1(y_1)$ coincides with the revenue function in the lost-sales case. When $y_1 < 0$, the retailer reimburse his consumers. As a result, the backorder assumption does not play a role here. The retailer's reservation profit, by quitting the business relationship, is:

$$\underline{u}_1(x_1) = v_1(x_1), \ x_1 \in \mathbb{R}. \tag{4.5}$$

We further assume that the initial inventory is the result of the previous period's sales, i.e., $x_1 = y_0 - D_0$ where y_0 is the period 0 post-order inventory level and D_0 is the period 0 demand. Without knowing the realized demand, the supplier is only able to form a belief about x_1

through CDF $G_1(x_1) = e^{-\lambda(y_0-x_1)}$ and PDF $g_1(x_1) = \lambda e^{-\lambda(y_0-x_1)}$.

As mentioned before, without loss of generality, we focus our attention on the direct revelation contracts. The supplier offers a menu contract $\{q_1(x_1), s_1(x_1)\}_{x_1\in(-\infty,y_0]}$. The retailer is asked to report his inventory or backorder x_1, and then the corresponding order $q_1(x_1)$ and payment $s_1(x_1)$ are executed. We define $u_1(x_1)$ as the net profit for the retailer, after transferring the payment to the supplier. Given the pre-order inventory x_1, $u_1(x_1) = v_1(x_1+q_1(x_1))-s_1(x_1)$. The optimal contract solves the following problem:

$$\max_{s_1,q_1} \int_{-\infty}^{y_0} \{s_1(x_1) - cq_1(x_1)\} dG_1(x_1) \tag{4.6}$$

$$s.t. \quad u_1(x_1) = \max_{\hat{x}_1\in(-\infty,y_0]} \{v_1(x_1 + q_1(\hat{x}_1)) - s_1(\hat{x}_1)\}, x_1 \in (-\infty, y_0] \tag{4.7}$$

$$u_1(x_1) \geq \underline{u}_1(x_1), x_1 \in (-\infty, y_0] \tag{4.8}$$

Given her belief $G_1(x_1)$ (or equivalently $g_1(x_1)$) of the retailer's pre-order inventory x_1, the supplier maximizes her expected profit subject to two constraints. As described, Constraint (4.7) is called the "incentive compatibility" (IC) constraint and (4.8) is called the "individual rationality" (IR) constraint. The IC constraint encourages the retailer to report his true state x_1. The IR constraint ensures that the retailer is willing to participate and accept the contract.

By the envelop theorem, we obtain the local IC constraint from the global IC constraint (4.7):

$$u_1'(x_1) = \frac{\partial}{\partial x_1} \{v_1(x_1 + q_1(\hat{x}_1)) - s_1(\hat{x}_1)\}|_{\hat{x}_1=x_1} = v_1'(x_1 + q_1(x_1)) \tag{4.9}$$

One feature of the single-period problem is that the retailer does not need to pay any holding or backorder cost. In other words, it does not make a difference between the lost-sales case and backorder case when there is only one period. As a result, we are able to show similar results as [57]. We leave the proof and technical details to the Appendix C.

Lemma 4.4.1. *(1) A contract $\{q_1(x_1), s_1(x_1)\}$ satisfies the (global) IC constraint if and only if it satisfies the local IC constraint and $q_1(x_1)$ is weakly decreasing in x_1;*

(2) Under the optimal contract, the IR constraint must be binding at y_0 and redundant at $x_1 < y_0$ where $x_1 = y_0 - D_0$.

Lemma 4.4.1 suggests that we can replace the IC constraint (4.7) by the local IC constraint (4.9) and the monotonicity of $q_1(x_1)$. It is true as long as the problem satisfies the so-called single-crossing property $\frac{\partial^2 v_1(x_1+q_1)}{\partial x_1 \partial q_1} \leq 0$ (Topkis [55]). Indeed, the function $v_1(y_1)$ is concave and we have $v_1''(y_1) = \begin{cases} \lambda r e^{-\lambda y_1}, & y_1 > 0 \\ 0, & y_1 < 0 \end{cases}$. Moreover, we look at the information rent $u_1(x_1) - \underline{u}_1(x_1)$, which is interpreted as the extra profit to the retailer arising from his informational advantage. Its derivative is equal to $u_1'(x_1) - \underline{u}_1'(x_1) = v_1'(x_1+q_1(x_1)) - v_1'(x_1)$. As the function $v_1(y_1)$ is concave, it has decreasing differences and consequently the information rent decreases in x_1. In fact, for any $x_1 < \hat{x}_1$, we have

$$\begin{aligned} u_1(x_1) - \underline{u}_1(x_1) &= v_1(x_1 + q_1(x_1)) - s_1(x_1) - v_1(x_1) \\ &\geq v_1(x_1 + q_1(\hat{x}_1)) - s_1(\hat{x}_1) - v_1(x_1) \\ &= v_1(\hat{x}_1 + q_1(\hat{x}_1)) - s_1(\hat{x}_1) - v_1(\hat{x}_1) \\ &\quad +[v_1(x_1 + q_1(\hat{x}_1)) - v_1(\hat{x}_1 + q_1(\hat{x}_1)) + v_1(\hat{x}_1) - v_1(x_1)] \\ &\geq v_1(\hat{x}_1 + q_1(\hat{x}_1)) - s_1(\hat{x}_1) - v_1(\hat{x}_1) \\ &= u_1(\hat{x}_1) - \underline{u}_1(\hat{x}_1) \end{aligned}$$

Therefore, the retailer with a higher inventory is a "worse" type as he gets less information rent. In order to ensure that the retailer will not inflate his inventory level to elicit discounts, the supplier has to provide more incentives (i.e. larger rent) to those who have low inventory.

Next, we compute the so-called virtual surplus in the literature. We replace $s_1(x_1)$ with $v_1(x_1+q_1(x_1)) - u_1(x_1)$. Since the IR constraint is binding at y_0, we rewrite $u_1(x_1) = u_1(y_0) - \int_{x_1}^{y_0} u_1'(\xi)d\xi = \underline{u}_1(y_0) - \int_{x_1}^{y_0} v_1'(\xi + q_1(\xi))d\xi$. Thus, $s_1(x_1) = v_1(x_1+q_1(x_1)) - \underline{u}_1(y_0) + \int_{x_1}^{y_0} v_1'(\xi + q_1(\xi))d\xi$. Finally, the objective function (4.6) can be reformulated as follow:

$$\begin{aligned} & \int_{-\infty}^{y_0} \{s_1(x_1) - cq_1(x_1)\} dG_1(x_1) \\ = & \int_{-\infty}^{y_0} \{v_1(x_1 + q_1(x_1)) - \underline{u}_1(y_0) + \int_{x_1}^{y_0} v_1'(\xi + q_1(\xi))d\xi - cq_1(x_1)\} g_1(x_1)dx_1 \\ = & \int_{-\infty}^{y_0} \{v_1(x_1 + q_1(x_1)) - cq_1(x_1)\} g_1(x_1)dx_1 + \int_{-\infty}^{y_0}\int_{x_1}^{y_0} v_1'(\xi + q_1(\xi))d\xi g_1(x_1)dx_1 - \underline{u}_1(y_0) \\ = & \int_{-\infty}^{y_0} \{v_1(x_1 + q_1(x_1)) - cq_1(x_1)\} g_1(x_1)dx_1 + \int_{-\infty}^{y_0} v_1'(\xi + q_1(\xi))G(\xi)d\xi - \underline{u}_1(y_0) \\ = & \int_{-\infty}^{y_0} \{v_1(x_1 + q_1(x_1)) - cq_1(x_1) + v_1'(x_1 + q_1(x_1)\frac{G_1(x_1)}{g_1(x_1)}\} g_1(x_1)dx_1 - \underline{u}_1(y_0) \\ = & \int_{-\infty}^{y_0} J_1(q_1(x_1)|x_1) g_1(x_1)dx_1 - \underline{u}_1(y_0) \end{aligned} \tag{4.10}$$

We call $J_1(q_1|x_1) = v_1(x_1+q_1) - cq_1 + v_1'(x_1+q_1)\frac{G_1(x_1)}{g_1(x_1)}$ the "virtual surplus", which represents the redistributed profit for the supplier in relation to the order q_1 at x_1. The optimal quantity plan $q_1^*(x_1)$ maximizes the virtual surplus. Given x_1, the first-order derivative of $J_1(q_1|x_1)$ is

$$\begin{aligned} \frac{\partial J_1(q_1|x_1)}{\partial q_1} &= v_1'(x_1 + q_1) - c + v_1''(x_1 + q_1)\frac{G_1(x_1)}{g_1(x_1)} \\ &= \begin{cases} re^{-\lambda(x_1+q_1)} - c - \lambda re^{-\lambda(x_1+q_1)}\frac{G_1(x_1)}{g_1(x_1)}, & x_1 + q_1 > 0 \\ r - c, & x_1 + q_1 < 0 \end{cases} \\ &= \begin{cases} -c, & x_1 + q_1 > 0 \\ r - c, & x_1 + q_1 < 0 \end{cases} \end{aligned} \tag{4.11}$$

where the last equality holds because $\frac{G_1(x_1)}{g_1(x_1)} = 1/\lambda$. As we can see, $J_1(q_1|x_1)$ increases in $q_1 < 0$ but decreases in $q_1 > 0$. Therefore, the optimal quantity plan is $q^*(x_1) = \max\{0, -x_1\}$, i.e. the optimal order-up-to level is $y^*(x_1) = x_1 + q^*(x_1) = \max\{0, x_1\}$. Theorem 4.4.2 provides a full characterization of the optimal contract. The optimal contract entails a base-stock policy with base-stock level 0. Moreover, the payment is a linear function of the order quantity q_1 with marginal price r.

Theorem 4.4.2. *In the single period case, the optimal contract is:*

$$q_1^*(x_1) = \begin{cases} 0, & x_1 \geq 0 \\ -x_1, & x_1 < 0 \end{cases} \quad \textit{and} \quad s_1^*(x_1) = \begin{cases} 0, & x_1 \geq 0 \\ -rx_1, & x_1 < 0 \end{cases}. \tag{4.12}$$

Theorem 4.4.2 says that under the optimal contract, trade happens only when the retailer reports negative inventory. Moreover, the retailer gets exactly his reservation profit $u_1(x_1) = v_1(x_1) = \underline{u}_1(x_1)$. Under an exponential demand distribution, the distortion is so severe that it is optimal not paying any information rent to the retailer. The supplier is only willing to clear up the backorder and bring the inventory level up to 0. More interestingly, the optimal contract is independent of the supplier's belief G_1. In other words, no matter what belief the supplier has about the pre-order inventory x_1, she will always offer the contract proposed in Theorem 4.4.2. This property is uncommon in dynamic adverse-selection problems, and gives us hope that there might exist a simple optimal contract in more general cases.

4.5 Two Periods

We now look at the two-period case. We solve the problem by backward induction. Since we have already characterized the optimal contract in the last period, $t = 2$, in Theorem 4.4.2, we focus our attention on the first period, $t = 1$. Similarly as before, we assume that there exists a known y_0 and the pre-order inventory in period 1 is $x_1 = y_0 - D_0$. The supplier solves the following problem to find the optimal contract $\{q_1(x_1), s_1(x_1)\}_{x_1 \in (-\infty, y_0]}$:

$$\max_{s_1, q_1} \int_{-\infty}^{y_0} \{s_1(x_1) - cq_1(x_1) + \delta\Pi_2(x_1 + q_1(x_1))\} dG_1(x_1) \tag{4.13}$$

$$\begin{aligned} s.t. \quad & u_1(x_1) = v_1(x_1 + q_1(x_1)) - s_1(x_1) + \delta U_2(x_1 + q_1(x_1)) \\ & \qquad = \max_{\hat{x}_1} v_1(x_1 + q_1(\hat{x}_1)) - s_1(\hat{x}_1) + \delta U_2(x_1 + q_1(\hat{x}_1)), x_1 \in (-\infty, y_0] \quad (4.14) \\ & u_1(x_1) \geq \underline{u}_1(x_1), x_1 \in (-\infty, y_0] \quad (4.15) \end{aligned}$$

The retailer's expected profit in period 1 (before payment s_1) is given by the expected sales revenue minus the holding and backorder costs:

$$v_1(y_1) = rE[D] - bE[D - y_1]^+ - hE[y_1 - D])^+$$
$$= \begin{cases} \frac{r}{\lambda} - \frac{b}{\lambda}e^{-\lambda y_1} - hy_1 + \frac{h}{\lambda}(1 - e^{-\lambda y_1}), & y_1 \geq 0 \\ by_1 + (r - b)/\lambda, & y_1 < 0. \end{cases}$$

In addition, $\Pi_2(y_1)$ is the supplier's expected profit-to-go in period 2 given the post-order inventory y_1 in period 1 and $U_2(y_1)$ is the retailer's expected profit-to-go in period 2 given y_1. Notice that the optimal contract in period 2 is independent of the supplier's belief $G_2(x_2)$. Therefore, the supplier' perceived post-order inventory $\hat{y}_1$ is irrelevant and we can simplify the notation $U_2(y_1|\hat{y}_1)$ to $U_2(y_1)$. According to Theorem 4.4.2, for a given pre-order inventory x_2 in period 2, the retailer's profit in period 2 is $u_2(x_2) = \underline{u}_2(x_2) = \begin{cases} \frac{r}{\lambda}(1 - e^{-\lambda x_2}), & x_2 \geq 0 \\ rx_2, & x_2 < 0. \end{cases}$ and the supplier's profit in period 2 is $\pi_2(x_2) = s_2^*(x_2) - cq_2^*(x_2) = \begin{cases} 0, & x_2 \geq 0 \\ -(r - c)x_2, & x_2 < 0. \end{cases}$ As a result, given the post-order inventory y_1 in period 1, the retailer's expected profit-to-go in period 2 is:

$$U_2(y_1) = E[u_2(x_2)] = E[u_2(y_1 - D_1)] = \begin{cases} \frac{r}{\lambda} - \frac{2r}{\lambda}e^{-\lambda y_1} - ry_1 e^{-\lambda y_1}, & y_1 \geq 0 \\ ry_1 - \frac{r}{\lambda}, & y_1 < 0, \end{cases}$$

the supplier's expected profit-to-go in period 2 is

$$\Pi_2(y_1) = E[\pi_2(x_2)] = E[\pi_2(y_1 - D_1)] = \begin{cases} \frac{r-c}{\lambda}e^{-\lambda y_1} & y_1 \geq 0 \\ (r - c)(\frac{1}{\lambda} - y_1) & y_1 < 0 \end{cases},$$

and the expected profit-to-go for the whole channel is the sum of the two:

$$\Psi_2(y_1) = U_2(y_1) + \Pi_2(y_1) = \begin{cases} \frac{r}{\lambda} - \frac{r}{\lambda}e^{-\lambda y_1} - \frac{c}{\lambda}e^{-\lambda y_1} - ry_1 e^{-\lambda y_1}, & y_1 \geq 0 \\ cy_1 - \frac{c}{\lambda}, & y_1 < 0. \end{cases}$$

We need to be careful when defining the retailer's reservation profit-to-go in the backorder case. We assume that once the retailer decides to abandon the relationship, he will no longer take backorders. That is to say, if the retailer starts with a backorder, $x_1 < 0$, he will return the payment to customers and stop the business right away. If the retailer has positive initial inventory, $x_1 \geq 0$, he will keep satisfying demand until no inventory is left, but he will not take backorder at the end of period 1. This is consistent with the following assumption:

Assumption 4.5.1. *$b > (1 - \delta)r$.*

This assumption prevents the retailer from taking backorder intentionally. If $b \leq (1 - \delta)r$, the retailer has an arbitrage opportunity by carrying backorder all the time and refunding the customers at the end of the horizon. Therefore, it is more plausible to assume $b > (1-\delta)r$. Then, the retailer's reservation profit-to-go at the beginning of period 1 is equal to:

$$\begin{aligned}\underline{u}_1(x_1) &= \begin{cases} rE[\min(x_1, D)] - hE[x_1 - D]^+ + \delta \underline{U}_2(x_1), & x_1 \geq 0 \\ rx_1, & x_1 < 0 \end{cases} \\ &= \begin{cases} \frac{r}{\lambda}(1 - e^{-\lambda x_1}) - hx_1 + \frac{h}{\lambda}(1 - e^{-\lambda x_1}) + \delta[\frac{r}{\lambda} - \frac{r}{\lambda}e^{-\lambda x_1} - rx_1 e^{-\lambda x_1}], & x_1 \geq 0 \\ rx_1, & x_1 < 0, \end{cases}\end{aligned}$$

where $\underline{U}_2(x_1)$ is the retailer's reservation profit-to-go in period 2 given the beginning inventory x_1 in period 1. When $x_1 < 0$, we know $\underline{U}_2(x_1) = 0$. When $x_1 \geq 0$, we have

$$\underline{U}_2(x_1) = E[\underline{u}_2(x_1 - D_1)^+] = E[\underline{u}_2(x_2)^+] = \int_0^{x_1} \frac{r}{\lambda}(1 - e^{-\lambda x_2})\lambda e^{-\lambda(x_1 - x_2)} dx_2 = \frac{r}{\lambda} - \frac{r}{\lambda}e^{-\lambda x_1} - rx_1 e^{-\lambda x_1}$$

For convenience, we define $\mu_1(y_1) = v_1(y_1) + \delta U_2(y_1)$, more specifically,

$$\mu_1(y_1) = v_1(y_1) + \delta U_2(y_1) = \begin{cases} \frac{r+h}{\lambda} - \frac{b+h}{\lambda}e^{-\lambda y_1} - hy_1 + \delta[\frac{r}{\lambda} - \frac{2r}{\lambda}e^{-\lambda y_1} - ry_1 e^{-\lambda y_1}], & y_1 \geq 0 \\ \frac{r-b}{\lambda} + by_1 + \delta(ry_1 - \frac{r}{\lambda}), & y_1 < 0. \end{cases}$$

We interpret $\mu_1(y_1)$ as the pre-transfer profit-to-go function for the retailer, if he orders up to y_1, before transferring any payment to the supplier. Note that the retailer is assumed to take backorder at the end of period 1, if needed, as a requirement for doing business with the supplier. The IC constraint (4.14) becomes

$$u_1(x_1) = \mu_1(x_1 + q_1(x_1)) - s_1(x_1) = \max_{\hat{x}_1} \mu_1(x_1 + q_1(\hat{x}_1)) - s_1(\hat{x}_1), x_1 \in (-\infty, y_0] \tag{4.16}$$

We make a few observations about the pre-transfer profit-to-go function $\mu_1(y_1)$. First of all, we have $\mu_1''(y_1) = \begin{cases} -\lambda(b + h + \delta r\lambda y_1)e^{-\lambda y_1} & y_1 > 0 \\ 0 & y_1 < 0 \end{cases}$, and hence the period 1 problem still satisfies the single-crossing property. By a similar proof as Lemma 4.4.1, we can show that the optimal quantity plan in period 1, $q_1^*(x_1)$, will satisfy the following local IC constraint

$$u_1'(x_1) = \mu_1'(x_1 + q_1(x_1)) \tag{4.17}$$

and weakly decrease in $x_1 \in (-\infty, y_0]$.

Next, we compare $\mu_1(x_1)$ and $\underline{u}_1(x_1)$ to obtain insights for the optimal contract.

$$\mu_1(x_1) - \underline{u}_1(x_1) = \begin{cases} -\frac{1}{\lambda}e^{-\lambda x_1}[b - (1-\delta)r], & x_1 \geq 0 \\ -(\frac{1}{\lambda} - x_1)[b - (1-\delta)r], & x_1 < 0. \end{cases}$$

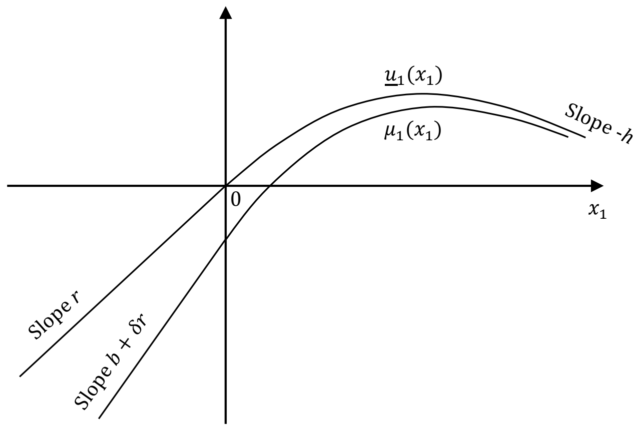


Figure 4.1: The retailer's reservation and pre-transfer profit-to-go functions in period 1.

Cleary, whether $\mu_1(x_1)$ is greater or smaller than $\underline{u}_1(x_1)$ depends on the term $b-(1-\delta)r$. By Assumption 4.5.1, we have $\mu_1(x_1)-\underline{u}_1(x_1)<0$ for all x_1. The two functions are illustrated in Figure 4.1. It implies that the supplier has to provide incentive in order to keep the retailer in the relationship. More importantly, the base-stock policy with base-stock level 0 proposed in Theorem 4.4.2 (i.e. $y_1(x_1)=\max\{0,x_1\}$, for all $x_1\in(-\infty,y_0]$) violates the IR constraint and is no longer a feasible quantity plan in period 1. In fact, under the quantity plan $y_1(x_1)=\max\{0,x_1\}$, we have

$$u_1'(x_1)-\underline{u}_1'(x_1)=\begin{cases}\mu_1'(x_1)-\underline{u}_1'(x_1), & x_1\geq 0\\ \mu_1'(0)-\underline{u}_1'(x_1), & x_1<0\end{cases}=\begin{cases}(b+\delta r-r)e^{-\lambda x_1} & x_1\geq 0\\ b+\delta r-r, & x_1<0\end{cases}.$$

We end up with $u_1'(x_1)>\underline{u}_1'(x_1)$ for all $x_1\in(-\infty,y_0]$. As a result, we claim that as long as there exists a point x_1^* at which $u_1(x_1^*)=\underline{u}_1(x_1^*)$, the IR constraint must be violated for all $x_1<x_1^*$. So the quantity plan $y_1(x_1)=\max\{0,x_1\}$ ($x_1\in(-\infty,y_0]$) is infeasible. We shall expect that the optimal contract in period 1 to be quite different from that in period 2.

In the single-period case, we have shown that the information rent decreases in the pre-order inventory level. However, when there are two periods to go, we fail to have a similar result. The information rent $u_1(x_1)-\underline{u}_1(x_1)$, the difference between the retailer's profit-to-go under the contract and his reservation profit-to-go, may not be monotone. In fact, the first-order derivative of the information rent is

$$\begin{aligned}u_1'(x_1)-\underline{u}_1'(x_1) &= \mu_1'(y_1(x_1))-\underline{u}_1'(x_1)\\ &= \begin{cases}(b+h+\delta r)e^{-\lambda y_1}+\delta r\lambda y_1 e^{-\lambda y_1}-(r+h)e^{-\lambda x_1}-\delta r\lambda x_1 e^{-\lambda x_1}, & y_1\geq x_1\geq 0\\ (b+h+\delta r)e^{-\lambda y_1}+\delta r\lambda y_1 e^{-\lambda y_1}-h-r, & y_1\geq 0>x_1\\ b-(1-\delta)r. & 0>y_1\geq x_1\end{cases}\end{aligned}$$

Whether the information rent is monotone will depend on the choice of y_1. That is to say, it

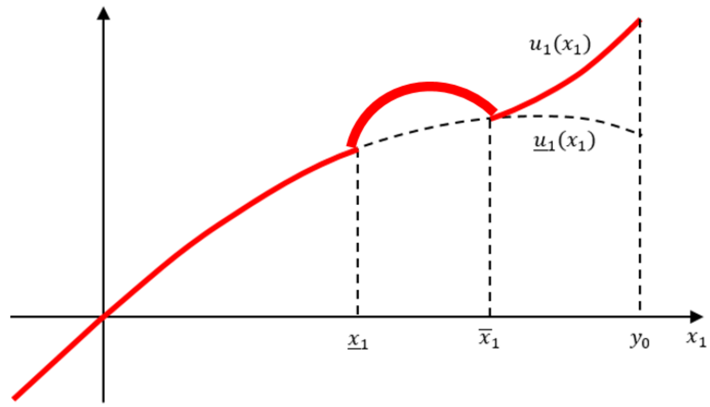


Figure 4.2: Illustration of "bump"

is unclear at which point the IR constraint will be binding. This is the main difference from the lost-sales case where the IR constraint must be binding at the highest inventory level and redundant at other points. In the backorder case, the IR constraint might be binding at a single point or multiple points. The good news is that we can exclude the possibility of "bump" in the retailer's profit-to-go function. Here "bump" means that the IR constraint is binding at two points but redundant at other points in between. If a "bump" occurs, the optimal contract would be too complicated to analyze. See Figure 4.2 for an illustration of "bump".

Proposition 4.5.2. *Under the optimal contract, there does not exist a "bump" in the retailer's profit-to-go function, i.e. there does not exist two points x_1^+ and x_1^- such that the IR constraint is binding at x_1^+ and x_1^- but redundant at $x_1 \in (x_1^-, x_1^+)$.*

Proposition 4.5.2 guarantees that the optimal contract will not lead to a "bump." Furthermore, we know for sure that there must exist at least one point x_1^* where the IR constraint is binding. If not, the supplier will be able to increase the payment uniformly across all possible x_1 to increase her profit. As a result, we conclude that at $x_1 < x_1^*$, the IR constraint is either always binding or never binding. Similar argument holds at $x_1 > x_1^*$.

Next, we derive the virtual surplus function $J_1(y_1|x_1)$. By a similar approach as the single-period case, we replace $s_1(x_1)$ with $\mu_1(x_1 + q_1(x_1)) - u_1(x_1) = \mu_1(y_1(x_1)) - u_1(x_1)$. At $x_1 < x_1^*$, the IR constraint is anchored at the right (or top). So we rewrite $u_1(x_1) = u_1(x_1^*) - \int_{x_1}^{x_1^*} u_1'(\xi)d\xi = \underline{u}_1(x_1^*) - \int_{x_1}^{x_1^*} \mu_1'(y_1(\xi))d\xi$. At $x_1 > x_1^*$, the IR constraint is anchored at the left (or bottom). So we rewrite $u_1(x_1) = u_1(x_1^*) + \int_{x_1^*}^{x_1} u_1'(\xi)d\xi = \underline{u}_1(x_1^*) + \int_{x_1^*}^{x_1} \mu_1'(y_1(\xi))d\xi$. Finally, we reformulate the objective function (4.13) as

$$\int_{-\infty}^{x_1^*} \{s_1(x_1) - cy_1(x_1) + cx_1 + \delta\Pi_2(y_1(x_1))\}g_1(x_1)dx_1 + \int_{x_1^*}^{y_0} \{s_1(x_1) - cy_1(x_1) + cx_1 + \delta\Pi_2(y_1(x_1))\}g_1(x_1)dx_1$$

where the first part is equal to

$$
\begin{aligned}
&\int_{-\infty}^{x_1^*} \{s_1(x_1) - cy_1(x_1) + cx_1 + \delta\Pi_2(y_1(x_1))\}g_1(x_1)dx_1 \\
= &\int_{-\infty}^{x_1^*} \{\mu_1(y_1(x_1)) - \underline{u}_1(x_1^*) + \int_{x_1}^{x_1^*} \mu_1'(y_1(\xi))d\xi - cy_1(x_1) + cx_1 + \delta\Pi_2(y_1(x_1))\}g_1(x_1)dx_1 \\
= &\int_{-\infty}^{x_1^*} \{\mu_1(y_1(x_1)) - cy_1(x_1) + cx_1 + \delta\Pi_2(y_1(x_1)) + \mu_1'(y_1(x_1))\frac{G_1(x_1)}{g(x_1)}\}g_1(x_1)dx_1 \\
&\qquad - \int_{-\infty}^{x_1^*} \underline{u}_1(x_1^*)g_1(x_1)dx_1, \qquad (4.18)
\end{aligned}
$$

and the second part is equal to

$$
\begin{aligned}
&\int_{x_1^*}^{y_0} \{s_1(x_1) - cy_1(x_1) + cx_1 + \delta\Pi_2(y_1(x_1))\}g_1(x_1)dx_1 \\
= &\int_{x_1^*}^{y_0} \{\mu_1(y_1(x_1)) - \underline{u}_1(x_1^*) - \int_{x_1^*}^{x_1} \mu_1'(y_1(\xi))d\xi - cy_1(x_1) + cx_1 + \delta\Pi_2(y_1(x_1))\}g_1(x_1)dx_1 \\
= &\int_{x_1^*}^{y_0} \{\mu_1(y_1(x_1)) - cy_1(x_1) + cx_1 + \delta\Pi_2(y_1(x_1)) - \mu_1'(y_1(x_1))\frac{1 - G_1(x_1)}{g(x_1)}\}g_1(x_1)dx_1 \\
&\qquad - \int_{x_1^*}^{y_0} \underline{u}_1(x_1^*)g_1(x_1)dx_1. \qquad (4.19)
\end{aligned}
$$

(4.18) and (4.19) lead to the following expressions for the virtual surplus anchored at the right end and the left end, respectively:

$$
J_1(y_1|x_1) = \begin{cases} cx_1 - cy_1 + \mu_1(y_1) + \delta\Pi_2(y_1) + \mu_1'(y_1)\frac{G_1(x_1)}{g(x_1)}, & x_1 < x_1^* \\ cx_1 - cy_1 + \mu_1(y_1) + \delta\Pi_2(y_1) - \mu_1'(y_1)\frac{1-G_1(x_1)}{g(x_1)}, & x_1 > x_1^* \end{cases} \qquad (4.20)
$$

The virtual surplus takes different forms for $x_1 > x_1^*$ and $x_1 < x_1^*$. It will be used later in finding the optimal contract. The optimal quantity plan $y^*(x)$ will maximize the virtual surplus subject to the IR constraint. In order to characterize the optimal contract in period 1, we first introduce two special quantity plans.

Definition 1. *Define $y_1^R(x_1)$ as the solution of $\underline{u}_1'(x_1) = \mu_1'(y_1)$, i.e., $y_1^R(x_1)$ solves*

$$
(b + h + \delta r + \delta r\lambda y_1^R)e^{-\lambda y_1^R} = (r + h + \delta r\lambda x_1)e^{-\lambda x_1} \qquad (4.21)
$$

for $x_1 \in [0, y_0]$, and $y_1^R(x_1) = y_1^R(0)$ for $x_1 \in (-\infty, 0)$.

Definition 2. *Define $y_1^L(x_1)$ as the solution of the first-order condition $\frac{d}{dy_1}J_1(y_1|x_1) = 0$, given the "virtual surplus" anchored at the left (or bottom)*

$$
J_1(y_1|x_1) = cx_1 - cy_1 + \mu_1(y_1) + \delta\Pi_2(y_1) - \mu_1'(y_1)\frac{1 - G_1(x_1)}{g(x_1)}, \qquad (4.22)
$$

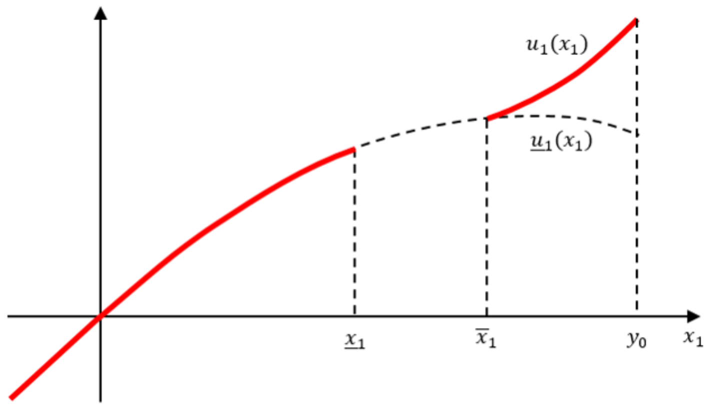


Figure 4.3: Retailer's profit-to-go in period 1 under the optimal contract

i.e., $y_1^L(x_1)$ *solves*

$$\delta c e^{-\lambda y_1^L} + e^{\lambda(y_0 - x_1)}(b + h + \delta r \lambda y_1^L)e^{-\lambda y_1^L} = c + h, \tag{4.23}$$

or $y_1^L(x_1) = x_1$ *if the above solution is less than* x_1*.*

If the supplier implements the quantity plan $y_1^R(x_1)$, the retailer will receive exactly his reservation profit-to-go, because $y_1^R(x_1)$ is such that $u_1'(x_1) = \mu_1'(y_1^R(x_1)) = \underline{u}_1'(x_1)$. The first equality follows from the local IC constraint. The quantity plan $y_1^L(x_1)$ maximizes the "virtual surplus" anchored at the left. By taking the derivative, we can easily check that $y_1^R(x_1)$ increases in x_1 while $y_1^L(x_1)$ decreases in x_1 whenever $y_1^L(x_1) > x_1$.

The full characterization of the optimal contract in period 1 is given by the following theorem.

Theorem 4.5.3. *The optimal contract in period 1 has at most two thresholds* $x_1^{**} \leq x_1^* \leq y_0$*:*

(a) At $x_1 \in (-\infty, x_1^{**}]$*, the retailer orders up to* $y_1^R(x_1)$ *and gets his reservation profit;*

(b) at $x_1 \in (x_1^{**}, x_1^*]$*, the retailer is excluded;*

(c) and at $x_1 \in (x_1^*, y_0]$*, the retailer orders up to* $y_1^L(x_1)$ *and receives a positive information rent.*

Theorem 4.5.3 indicates that the optimal contract in period 1 consists of three regions at most, as illustrated in Figure 4.3. In the first region $x_1 \in (-\infty, x_1^{**}]$, the retailer participates and gets exactly his reservation profit-to-go. To see that, we examine the virtual surplus anchored at the right $J_1(y_1|x_1) = cx_1 - cy_1 + \mu_1(y_1) + \delta\Pi_2(y_1) + \mu_1'(y_1)\frac{G_1(x_1)}{g(x_1)}$. Recall that $x_1 = y_0 - D_0$, therefore we have $G_1(x_1) = e^{-\lambda(y_0 - x_1)}$ and $g_1(x_1) = \lambda e^{-\lambda(y_0 - x_1)}$. The first-order derivative of

$J_1(y_1|x_1)$ is given by

$$\begin{aligned}\frac{dJ_1(y_1|x_1)}{dy_1} &= -c + \mu_1'(y_1) + \delta\Pi_2'(y_1) + \mu_1''(y_1)\frac{G_1(x_1)}{g(x_1)} \\ &= \begin{cases} -h - c + \delta c e^{-\lambda y_1} < 0, & y_1 > 0 \\ b + \delta c - c > 0, & y_1 < 0 \end{cases}\end{aligned}$$

The supplier wants to maximize the virtual surplus subject to the IR constraints. However, the function $J_1(y_1|x_1)$ increases in $y_1 < 0$ whereas it decreases in $y_1 > 0$. As a result, the base-stock policy with bast-stock level 0 (i.e. $y_1(x_1) = \max\{0, x_1\}$) maximizes the virtual surplus. However, we have discussed that the quantity plan $y_1(x_1) = \max\{0, x_1\}$ does not satisfy the IR constraint at all $x_1 \le x_1^*$. In fact, if the retailer quits, he does not need to incur the higher backorder cost, which gives him more bargaining power. The information rent corresponding to the base-stock policy is not large enough (in fact it yields negative profit) to ensure the retailer's participation. In other words, the supplier faces a trade-off between maximizing her profit and keeping the retailer in the relationship. Since $J_1(y_1|x_1)$ is decreasing in $y_1 > 0$, the optimal quantity plan will be $y_1^*(x_1) = y_1^R(x_1)$, which makes the IR constraint binding.

The second region $(x_1^{**}, x_1^*]$ suggests that the supplier may be able to improve her profit by excluding some types of retailers. Figure 4.4 shows the supplier's profit $\pi_1(x_1)$ for each possible pre-order inventory x_1 if she follows the quantity plan $y_1^R(x_1)$, i.e

$$\pi_1(x_1) = v_1(y_1^R(x_1)) + \delta U_2(y_1^R(x_1)) - u_1(x_1) - cy_1^R(x_1) + cx_1 + \delta\Pi_2(y_1^R(x_1)). \tag{4.24}$$

We observe that when x_1 is large, the supplier starts getting negative profit $\pi_1(x_1) < 0$. In this case, it is better to exclude the retailer, as even zero information rent would lead to negative profit for the supplier. In the backorder case, as the retailer has more bargaining power by threatening to quit, the optimal contract induces partial participation. The threshold x_1^{**} is determined by sending (4.24) to 0. We can prove that the threshold x_1^{**} can be uniquely determined if the costs are not too high.

Lemma 4.5.4. *Suppose $r + h + \delta r > e(h + c)$. There exists a unique x_1^{**} that satisfies $v_1(y_1^R(x_1)) + \delta U_2(y_1^R(x_1)) - u_1(x_1) - cy_1^R(x_1) + cx_1 + \delta\Pi_2(y_1^R(x_1)) = 0$.*

In fact, even when $r + h + \delta r < e(h + c)$, we observe numerically that the threshold x_1^{**} is unique. So we postulate that under all circumstances, there always exists a unique x_1^{**} that solves $v_1(y_1^R(x_1)) + \delta U_2(y_1^R(x_1)) - u_1(x_1) - cy_1^R(x_1) + cx_1 + \delta\Pi_2(y_1^R(x_1)) = 0$.

Finally, in the third region $x_1 \in (x_1^*, y_0]$, we investigate the virtual surplus anchored at the

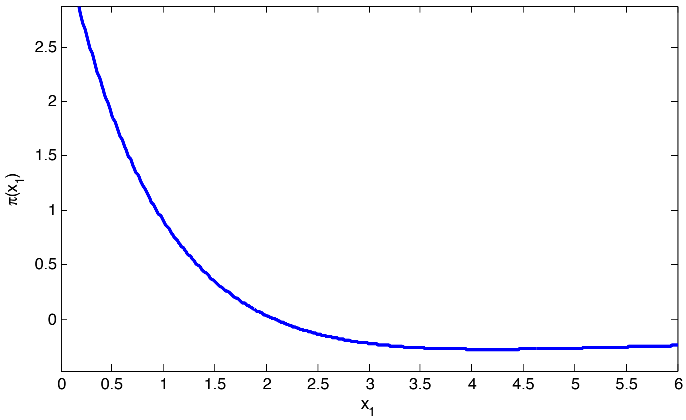


Figure 4.4: Supplier's profit-to-go in period 1 under $y_1^R(x_1)$. (Parameters: $r = 10$, $c = 5$, $b = 3$, $h = 3$, $\lambda = 1$ and $\delta = 0.9$.)

left $J_1(y_1|x_1) = cx_1 - cy_1 + \mu_1(y_1) + \delta\Pi_2(y_1) + \mu_1'(y_1)\frac{1-G_1(x_1)}{g(x_1)}$. Its derivative is given by

$$\begin{aligned}\frac{dJ_1(y_1|x_1)}{dy_1} &= c + \mu_1'(y_1) + \delta\Pi_2'(y_1) + \mu_1''(y_1)\frac{1-G_1(x_1)}{g(x_1)} \\ &= \begin{cases} -c - h + \delta c e^{-\lambda y_1} + e^{\lambda(y_0 - x_1)}(b + h + \delta r \lambda y_1)e^{-\lambda y_1}, & y_1 > 0 \\ b + \delta c - c, & y_1 < 0 \end{cases}\end{aligned}$$

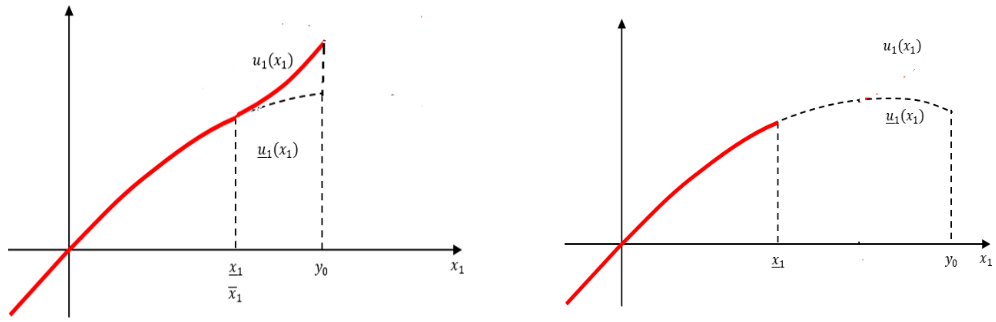
By definition, the quantity plan $y_1^L(x_1)$ maximizes the virtual surplus when $x_1 \in (x_1^*, y_0]$. In order to satisfy the IR constraint, the threshold x_1^* has a lower bound.

Lemma 4.5.5. *The threshold x_1^* satisfies $x_1^* \geq x_1^L$, where x_1^L is the unique solution of $y_1^R(x_1) = y_1^L(x_1)$. In addition,* $y_1^R(x_1) - y_1^L(x_1) = \begin{cases} < 0, & x_1 < x_1^L \\ > 0, & x_1 > x_1^L \end{cases}$.

The IR constraint requires $u_1(x_1) = u_1(x_1^*) + \int_{x_1^*}^{x_1} v_1'(y^L(\xi))d\xi \geq \underline{u}_1(x_1) = \underline{u}_1(x_1^*) + \int_{x_1^*}^{x_1} v_1'(y^R(\xi))d\xi$. Note that the IR constraint is binding at x_1^*. By rearranging the terms, we argue that x_1 should satisfy $\int_{x_1^*}^{x_1}[\mu_1'(y^L(\xi)) - \mu_1'(y^R(\xi))]d\xi \geq 0$. Now we let x_1 approach x_1^* and the IR constraint becomes $\mu_1'(y^L(x_1^*)) - \mu_1'(y^R(x_1^*)) \geq 0$, which implies $x_1^* \geq x_1^L$ due to the concavity of μ_1. Finally, the IR constraint will be automatically satisfied at $x_1 \in (x_1^*, y_0]$ because we have $\mu_1'(y^L(x_1)) - \mu_1'(y^R(x_1)) \geq 0$ for all $x_1 > x_1^*$. As a result, $y_1^L(x_1)$ is the optimal quantity plan in this case.

Moreover, the retailer starts getting positive information rent $u_1(x_1) - \underline{u}_1(x_1) = \int_{x_1^*}^{x_1}[\mu_1'(y^L(\xi)) - \mu_1'(y^R(\xi))]d\xi$. Surprisingly, the information rent is increasing in x_1 since $u_1'(x_1) - \underline{u}_1'(x_1) = \mu_1'(y^L(x_1)) - \mu_1'(y^R(x_1)) > 0$. The retailer with a higher inventory becomes a "better" type in the two-period case. This is opposite to the single-period case. As we see, the backorder case leads to totally different insights from the lost-sales case.

However, there may not be a closed-form expression for the threshold x_1^*. It will be found



(a) Special Case 1: no exclusion

(b) Special Case 2: no tail

Figure 4.5: Two special cases of the optimal contract in period 1.

numerically by maximizing the supplier's expected profit in the region $(x_1^*, y_0]$.

Depending on the parameters, there might exist two special cases as shown in Figure 4.5. The first case (Figure 4.5(a)) is $x_1^{**} = x_1^*$ where we do not have an exclusion region. The second one (Figure 4.5(b)) is $x_1^* = y_0$ where the positive information rent region (or the "tail") disappears.

4.6 Infinite Horizon

In Section 4.5, we have derived the optimal contract in the two-period case. We have seen that the optimal contract can be fairly complex and involve three regions. This manifests the complexity of the optimal contract in the general T-period case. There seems little hope to derive closed-form expressions for the optimal contracts. To eliminate the end-of-horizon effects, we consider the infinite horizon case in this section. We would like to investigate if there exists a stationary optimal contract with a simple structure.

Our analysis proceeds as follows. We first propose a simple stationary contract. Then we compute the supplier's profit-to-go functions under such a contract. Finally we show that when the model parameters lie in a certain region, the proposed contract is indeed the optimal short-term contract.

4.6.1 Retailer's Reservation Profit-to-go

We start with the retailer's reservation profit-to-go. Suppose the retailer decides to quit at the beginning of period t, we make the following assumptions. If the retailer has backorder on hand, he will return the payments to customers and leave the market right away. On the other hand, if the retailer has positive inventory, he will keep selling the goods but no longer take backorder (because it is unprofitable due to $b > (1 - \delta r)$). Once the retailer sells out all the inventory, he will leave the market. Therefore, the retailer's reservation profit in period t from inventory x_t (without ordering) is

$$\underline{v}_t(x_t) = \begin{cases} rE[\min(D, x_t)] - hE[x_t - D]^+ = \frac{r+h}{\lambda}(1 - e^{-\lambda x_t}) - hx_t, & x_t \geq 0 \\ rx_t, & x_t < 0. \end{cases}$$

We further define $\underline{U}_{t+1}(x_t)$ as the retailer's expected reservation profit-to-go (with no future orders or backorders) given the previous period's inventory x_t. Clearly, when $x_t \leq 0$, the retailer will reimburse his customers and leave the market in period t. Hence, we have $\underline{U}_{t+1}(x_t) = 0$. When $x_t > 0$, $\underline{U}_{t+1}(x_t)$ can be determined recursively as

$$\begin{aligned} \underline{U}_{t+1}(x_t) &= \int_0^{x_t} \{\underline{v}_{t+1}(x_{t+1}) + \delta \underline{U}_{t+2}(x_{t+1})\} dG_{t+1}(x_{t+1}) \\ &= \int_0^{x_t} \{\underline{v}_{t+1}(x_{t+1}) + \delta \underline{U}_{t+2}(x_{t+1})\} \lambda e^{-\lambda(x_t - x_{t+1})} dx_{t+1} \end{aligned} \tag{4.25}$$

Over the infinite horizon, both $\underline{U}_{t+1}$ and $\underline{U}_{t+2}$ can be replaced by a stationary function $\underline{U}$ (the time index is unnecessary). Thus, Equation (4.25) becomes

$$\underline{U}(x) = \int_0^x \{\underline{v}(z) + \delta \underline{U}(z)\} \lambda e^{-\lambda(x-z)} dz \tag{4.26}$$

where x denotes the beginning inventory of the "previous" period and z denotes the beginning inventory of the "current" period.

We multiply both sides of (4.26) by $e^{\lambda x}$ which yields $e^{\lambda x}\underline{U}(x) = \int_0^x \{\underline{v}(z) + \delta \underline{U}(z)\} \lambda e^{\lambda z} dz$. By a transformation $\tilde{\underline{U}}(x) = e^{\lambda x}\underline{U}(x)$, we get $\tilde{\underline{U}}(x) = \int_0^x \{\underline{v}(z)\lambda e^{\lambda z} + \delta\lambda\tilde{\underline{U}}(z)\} dz$. Finally, we take derivative on both sides and obtain the following ODE:

$$\tilde{\underline{U}}'(x) = \lambda e^{\lambda x}\underline{v}(x) + \delta\lambda\tilde{\underline{U}}(x) \tag{4.27}$$

Through straightforward algebra, the solution to (4.27) can be found as

$$\tilde{\underline{U}}(x) = \frac{r+h}{\delta\lambda} + \frac{r(1-\delta) + h[2 - \lambda x - \delta(1 - \lambda x)]}{\lambda(1-\delta)^2} e^{\lambda x} + e^{\delta\lambda x} M_{\underline{U}}$$

in which $M_{\underline{U}}$ is a constant to be determined from the boundary condition $\tilde{\underline{U}}(0) = e^{\lambda 0}\underline{U}(0) = 0$. So we have $M_{\underline{U}} = -\frac{h+r(1-\delta)}{\delta\lambda(1-\delta)^2}$.

In conclusion,

$$\begin{aligned} \underline{U}(x) &= \begin{cases} \frac{r+h}{\delta\lambda}e^{-\lambda x} + \frac{r(1-\delta)+h[2-\lambda x-\delta(1-\lambda x)]}{\lambda(1-\delta)^2} - \frac{h+r(1-\delta)}{\delta\lambda(1-\delta)^2}e^{-\lambda x(1-\delta)}, & x > 0 \\ 0, & x \leq 0 \end{cases} \\ &= \begin{cases} \frac{h(2-\delta)+r(1-\delta)}{\lambda(1-\delta)^2} - \frac{hx}{1-\delta} + \frac{r+h}{\delta\lambda}e^{-\lambda x} - \frac{h+r(1-\delta)}{\delta\lambda(1-\delta)^2}e^{-\lambda x(1-\delta)}, & x > 0 \\ 0, & x \leq 0 \end{cases} \end{aligned} \tag{4.28}$$

$$\begin{aligned}\underline{u}(x) &= \underline{v}(x) + \delta \underline{U}(x) \\ &= \begin{cases} \frac{r+h}{\lambda} - hx + \delta \frac{r(1-\delta)+h[2-\lambda x-\delta(1-\lambda x)]}{\lambda(1-\delta)^2} - \frac{h+r(1-\delta)}{\lambda(1-\delta)^2} e^{-\lambda x(1-\delta)}, & x > 0 \\ rx, & x \leq 0 \end{cases} \\ &= \begin{cases} \frac{h+r(1-\delta)}{\lambda(1-\delta)^2} - \frac{hx}{1-\delta} - \frac{h+r(1-\delta)}{\lambda(1-\delta)^2} e^{-\lambda x(1-\delta)}, & x > 0 \\ rx. & x \leq 0 \end{cases} \end{aligned} \tag{4.29}$$

4.6.2 The Zero-rent Plan $y^R(x)$

In the two-period case, we observe that when the pre-order inventory is relatively small (below x_1^{**}), the retailer should receive exactly his reservation profit-to-go under the optimal contract. The corresponding quantity plan does not depend on the supplier's belief and is easy to characterize. Motivated by the two-period problem, we compute the quantity plan $y^R(x)$ such that the retailer always gets his reservation profit-to-go. In other words, we propose a feasible contract with quantity plan $y^R(x)$. Following such a contract, the retailer receives his reservation profit-to-go $\underline{u}(x)$ in each period. One of our goals is to find conditions such that this contract is indeed the optimal short-term contract.

First, given his participation, the retailer's expected pre-transfer profit in period t from the post-order inventory y_t is equal to

$$v_t(y_t) = rE[D] - bE[D-y_t]^+ - hE[y_t - D]^+ = \begin{cases} \frac{r}{\lambda} - \frac{b}{\lambda} e^{-\lambda y_t} - hy_t + \frac{h}{\lambda}(1 - e^{-\lambda y_t}), & y_t \geq 0 \\ by_t + (r-b)/\lambda, & y_t < 0, \end{cases}$$

with first-order derivative $v_t'(y_t) = \begin{cases} (b+h)e^{-\lambda y_t} - h, & y_t \geq 0 \\ b, & y_t < 0 \end{cases}$ and second-order derivative $v_t''(y_t) = \begin{cases} -\lambda(b+h)e^{-\lambda y_t}, & y_t > 0 \\ 0, & y_t < 0 \end{cases}$.

We define $U(y)$ as the retailer's expected profit-to-go given the post-order inventory y of the "previous" period. By following the proposed quantity plan $y^R(x)$, the retailer always gets his reservation profit in the "current" period. It implies $U(y) = \int_{-\infty}^{y} \underline{u}(z)\lambda e^{-\lambda(y-z)}dz$. If $y \leq 0$, we obtain $U(y) = \int_{-\infty}^{y} rz\lambda e^{-\lambda(y-z)}dz = ry - \frac{r}{\lambda}$. If $y > 0$, we have

$$\begin{aligned}
U(y) &= \int_{-\infty}^{y} \underline{u}(z)\lambda e^{-\lambda(y-z)}dz \\
&= \int_{-\infty}^{0} \underline{u}(z)\lambda e^{-\lambda(y-z)}dz + \int_{0}^{y} \underline{u}(z)\lambda e^{-\lambda(y-z)}dz \\
&= \int_{-\infty}^{0} rz\lambda e^{-\lambda(y-z)}dz + \underline{U}(y) \\
&= -\frac{r}{\lambda}e^{-\lambda y} + \underline{U}(y) \\
&= \frac{h(2-\delta) + r(1-\delta)}{\lambda(1-\delta)^2} - \frac{hy}{1-\delta} + \frac{h + r(1-\delta)}{\delta\lambda}e^{-\lambda y} - \frac{h + r(1-\delta)}{\delta\lambda(1-\delta)^2}e^{-\lambda y(1-\delta)}
\end{aligned}$$

We compare the "no-order" profit-to-go $v(x)+\delta U(x)$ for the retailer with his reservation profit-to-go:

$$v(x) + \delta U(x) - \underline{u}(x) = \begin{cases} -(b + \delta r - r)e^{-\lambda x}/\lambda < 0, & x > 0 \\ (b + \delta r - r)(x - 1/\lambda) < 0, & x \leq 0. \end{cases}$$

As $b > (1-\delta)r$, the "no-order" profit-to-go $v(x) + \delta U(x)$ is smaller than $\underline{u}_1(x_1)$. Similar as the two-period case, the retailer is forced to accommodate backorders if he is doing business with the supplier. However, as $b > (1-\delta)r$, the retailer gains bargaining power by threatening to quit and avoid possible backorder penalty. As a result, the supplier needs to provide sufficient incentive to keep him in the relationship.

We further define $u(x)$ as the retailer's profit-to-go under the contract and the pre-order inventory x of the "current" period, i.e $u(x) = v(y(x)) + \delta U(y(x)) - s(x)$ in which $y(x)$ is the order-up-to level and $s(x)$ is the payment to the supplier. The local IC constraint indicates

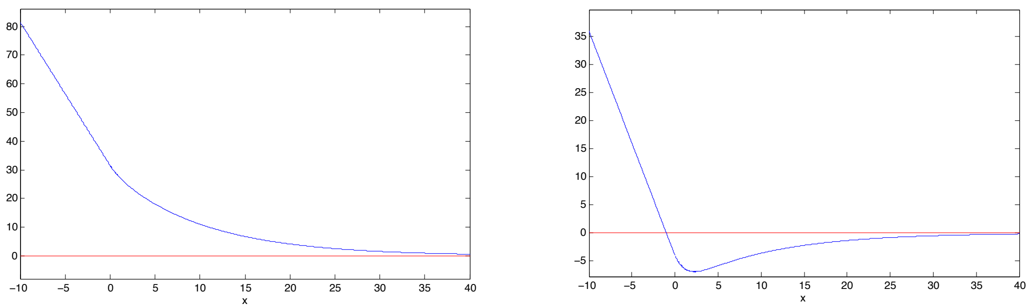
$$u'(x) = v'(y(x)) + \delta U'(y(x)) = \begin{cases} [b - r(1-\delta)]e^{-\lambda y} + \frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y(1-\delta)} - \frac{h}{1-\delta}, & y > 0 \\ b + \delta r. & y \leq 0 \end{cases}$$

By definition, the quantity plan $y^R(x)$ is such that the retailer receives his reservation profit-to-go. Thus it should satisfy $u'(x) = v'(y^R(x)) + \delta U'(y^R(x)) = \underline{u}'(x)$, i.e. $y^R(x)$ solves

$$[b - r(1-\delta)]e^{-\lambda y} + \frac{h + r(1-\delta)}{1-\delta}e^{-\lambda y(1-\delta)} - \frac{h}{1-\delta} = \begin{cases} \frac{h+r(1-\delta)}{1-\delta}e^{-\lambda x(1-\delta)} - \frac{h}{1-\delta}, & x > 0 \\ r, & x \leq 0 \end{cases} \quad (4.30)$$

Clearly, the solution $y^R(x)$ is positive for all x. Moreover, when $x \leq 0$, $y^R(x)$ is a constant, $y^R(0)$. When $x > 0$, $y^R(x) > x$ is uniquely determined. In fact, $y^R(x)$ has the following properties:

Lemma 4.6.1. *When $x > 0$,*



(a) Parameters: $r = 10$, $c = 5$, $b = 3$, $h = 3$, $\lambda = 1$ and $\delta = 0.9$ (b) Parameters: $r = 10$, $c = 6$, $b = 7$, $h = 5$, $\lambda = 1$ and $\delta = 0.9$

Figure 4.6: Supplier's profit-to-go under $y^R(x)$

(1) the quantity plan $y^R(x)$ is strictly increasing and convex in x.

(2) $\lim_{x\to\infty}[y^R(x) - x] = 0$.

In fact, when $x > 0$, $y^R(x)$ satisfies $[b-r(1-\delta)]e^{-\lambda y^R(x)} + \frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y^R(x)(1-\delta)} = \frac{h+r(1-\delta)}{1-\delta}e^{-\lambda x(1-\delta)}$. By taking derivative with respect to x on both sides of the equation, we will get Lemma 4.6.1(1). To show Lemma 4.6.1(2), we rewrite the equation to be $[b-r(1-\delta)]e^{-\lambda[y^R(x)-x]-\delta\lambda x} + \frac{h+r(1-\delta)}{1-\delta}e^{-\lambda[y^R(x)-x](1-\delta)} = \frac{h+r(1-\delta)}{1-\delta}$. Clearly, as $x \to \infty$, we must have $[y^R(x) - x] \to 0$. Otherwise, the limit of the left-hand side of the equation would not equal to the constant $\frac{h+r(1-\delta)}{1-\delta}$.

Finally we compute the supplier's profit-to-go under the quantity plan $y^R(x)$. We define $\pi_t(x)$ as the supplier's profit-to-go from period t onwards given pre-order inventory x in period t; $\Pi_{t+1}(y)$ as the supplier's profit-to-go from period $t+1$ onwards given post-order inventory y in period t. $\pi_t(x)$ and $\Pi_{t+1}(y)$ have the following relationships:

$$\begin{aligned}\pi_t(x) &= v_t(y(x)) + \delta U_{t+1}(y(x)) - u_t(x) - cy(x) + cx + \delta\Pi_{t+1}(y(x)) \\ \Pi_{t+1}(y) &= \int_{-\infty}^{y}\pi_{t+1}(z)dG(z) = \int_{-\infty}^{y}\pi_{t+1}(z)\lambda e^{-\lambda(y-z)}dz\end{aligned}$$

By assumption, the supplier implements the quantity plan $y^R(x)$ in each period. In this case, we apply the contraction mapping theorem to obtain the convergence of π_t and Π_t as the total number of periods T goes to infinity.

Lemma 4.6.2. *Suppose the supplier implements the quantity plan $y^R(x)$ in each period. As the total number of periods $T \to \infty$, the supplier's profit-to-go function $\pi = \lim_{T\to\infty}\pi_t$ exists and is unique.*

In the infinite-horizon case, the supplier's profit-to-go functions are stationary and satisfy:

$$\pi(x) = v(y^R(x)) + \delta U(y^R(x)) - \underline{u}(x) - cy^R(x) + cx + \delta\Pi(y^R(x)) \quad (4.31)$$

$$\Pi(y) = \int_{-\infty}^{y}\pi(z)dG(z) = \int_{-\infty}^{y}\pi(z)\lambda e^{-\lambda(y-z)}dz \quad (4.32)$$

Figures 4.6 demonstrates the supplier's profit-to-go $\pi(x)$ under the quantity plan $y^R(x)$. As we can see, $\pi(x)$ presents different structures given different cost parameters. In Figure 4.6(a), $\pi(x) \geq 0$ for all x. The supplier always gets positive profit by implementing $y^R(x)$. However, in Figure 4.6(b), $\pi(x)$ is first positive but later becomes negative. In this case, the supplier gets negative profit by implementing $y^R(x)$ for some types of retailers. The supplier is better off to exclude such types of retailers. Therefore, we conjecture that the optimal short-term contract in the backorder case may involve an exclusion region in certain parameter regimes.

4.6.3 The Optimal Contract

We explore if the optimal short-term contract is such that the retailer gets his reservation profit in each period. In order to show the optimality of the zero-rent contract, we use inductive argument. Suppose it holds from the "next" period onwards, we want to show the proposed contract with $y^R(x)$ is also optimal in the "current" period. By a similar approach as the two-period case, we obtain the virtual surplus anchored at the right end or left end:

$$J^R(y|x) = cx - cy + v(y) + \delta U(y) + \delta\Pi(y) + [v'(y) + \delta U'(y)]\frac{G(x)}{g(x)}, \quad (4.33)$$

$$J^L(y|x) = cx - cy + v(y) + \delta U(y) + \delta\Pi(y) - [v'(y) + \delta U'(y)]\frac{(1-G(x))}{g(x)}. \quad (4.34)$$

We first look at the virtual surplus anchored at the right $J^R(y|x)$. Its first-order derivative is:

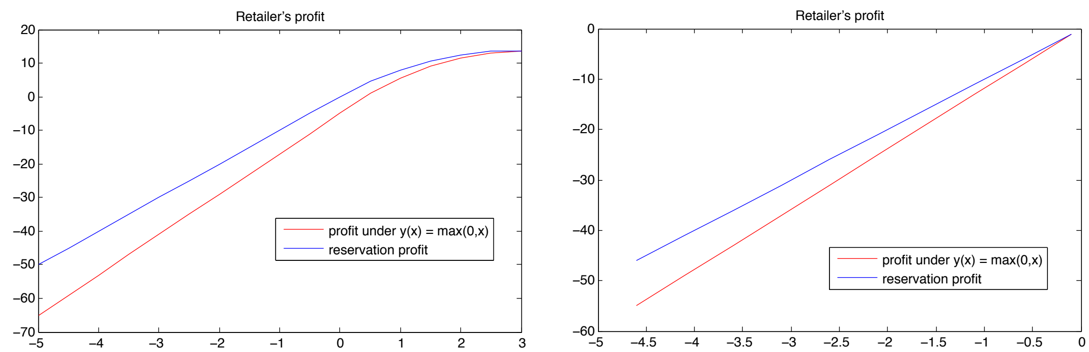
$$\frac{dJ^R(y|x)}{dy} = -c + v'(y) + \delta U'(y) + \delta\Pi'(y) + [v''(y) + \delta U''(y)]\frac{G(x)}{g(x)}.$$

We assume $x = y_0 - D$ for some known y_0. Therefore, $G(x) = e^{-\lambda(y_0-x)}$ and $g(x) = \lambda e^{-\lambda(y_0-x)}$. By straightforward algebra, we have that when $y < 0$, $\frac{dJ^R(y|x)}{dy} = -c + b + \delta r + \delta\Pi'(y)$, and when $y > 0$,

$$\begin{aligned}\frac{dJ^R(y|x)}{dy} &= -c + [b - r(1-\delta)]e^{-\lambda y} + \frac{h + r(1-\delta)}{1-\delta}e^{-\lambda y(1-\delta)} - \frac{h}{1-\delta} + \delta\Pi'(y) \\ &\quad -\lambda\{[b - r(1-\delta)]e^{-\lambda y} + [h + r(1-\delta)]e^{-\lambda y(1-\delta)}\}\frac{1}{\lambda} \\ &= -c - \frac{h}{1-\delta} + \delta\frac{h + r(1-\delta)}{1-\delta}e^{-\lambda y(1-\delta)} + \delta\Pi'(y).\end{aligned}$$

The following proposition gives us several properties of $J^R(y|x)$ which will play a role in characterizing the optimal contract.

Proposition 4.6.3. *Suppose that the supplier offers the zero-rent contract $y^R(x)$ from the "next" period onwards. For any given x, the virtual surplus anchored at the right $J^R(y|x)$ increases when $y < 0$ but decreases when $y > 0$. Moreover, $J^R(y|x)$ is concave at $y > 0$. In*



(a) If IR is binding at $x^* = 3$. (b) If IR is binding at $x^* = -0.1$.

Figure 4.7: Retailer's profit-to-go under quantity plan $y(x) = \max\{0, x\}$ and his reservation profit-to-go. (Parameters: $r = 10$, $c = 5$, $b = 3$, $h = 3$, $\lambda = 1$ and $\delta = 0.9$.)

other words, it satisfies

$$\frac{dJ^R(y|x)}{dy} = \begin{cases} < 0, & y > 0 \\ > 0, & y < 0 \end{cases} \quad \text{and} \quad \frac{d^2 J^R(y|x)}{dy^2} < 0 \text{ when } y > 0.$$

By the inductive assumption, from the "next" period onwards, the supplier offers the contract with quantity plan $y^R(x)$. So $\pi(z) = v(y^R(z)) + \delta U(y^R(z)) - \underline{u}(z) - cy^R(z) + cz + \delta\Pi(y^R(z))$. When $y \leq 0$, $y^R(z) = y^R(0)$ is a constant for all $z \leq y$. Therefore, $\pi(z) = v(y^R(0)) + \delta U(y^R(0)) - \underline{u}(z) - cy^R(0) + cz + \delta\Pi(y^R(0)) = \pi(0) - (r-c)z$. We can show that $\Pi(y) = \int_{-\infty}^{y} \pi(z)\lambda e^{-\lambda(y-z)}dz$ is also a linear function of y with slope $-(r-c)$. As a result, when $y < 0$, $\frac{dJ^R(y|x)}{dy} = -c + b + \delta r + \delta\Pi'(y) = -c + b + \delta r - \delta(r-c) = b + \delta c - c > 0$. When $y > 0$, we need to prove the result by induction. Please refer to the Appendix C for more details.

Proposition 4.6.3 indicates that the virtual surplus $J^R(y|x)$ is maximized at point 0 for any given x. Suppose the IR constraint is binding at some point x^*. For $x < x^*$, the quantity plan $y(x) = \max\{0, x\}$ maximizes $J^R(y|x)$. However, Figure 4.7 shows $y(x) = \max\{0, x\}$ is infeasible as the IR constraint is violated at $x < x^*$. We will soon prove that $y^R(x)$ is actually the optimal quantity plan when $x < x^*$.

Next we consider the case $x > x^*$. We look at the virtual surplus anchored at the left

$J^L(y|x)$. Its derivative is:

$$\begin{aligned}
\frac{dJ^L(y|x)}{dy} &= -c + v'(y) + \delta U'(y) + \delta\Pi'(y) + [v''(y) + \delta U''(y)]\frac{G(x)-1}{g(x)} \\
&= \begin{cases} -c - \frac{h}{1-\delta} + \delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y(1-\delta)} + \delta\Pi'(y) \\ \quad +\{[b - r(1-\delta)]e^{-\lambda y} + [h + r(1-\delta)]e^{-\lambda y(1-\delta)}\}e^{\lambda(y_0 - x)}, & y > 0 \\ -c + b + \delta r + \delta\Pi'(y), & y < 0 \end{cases} \\
&= \begin{cases} -c - \frac{h}{1-\delta} + \delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y(1-\delta)} + \delta\Pi'(y) \\ \quad +\{[b - r(1-\delta)]e^{-\lambda y} + [h + r(1-\delta)]e^{-\lambda y(1-\delta)}\}e^{\lambda(y_0 - x)}, & y > 0 \\ -c + b + \delta c, & y < 0 \end{cases}
\end{aligned}$$

The y that maximizes $J^L(y|x)$ is either the boundary point 0 or the solution(s) of the first-order condition $\frac{dJ^L(y|x)}{dy} = 0$. Thanks to Proposition 4.6.4 below, which guarantees there exists at most one point $y^L(x) > 0$ such that $\frac{dJ^L(y|x)}{dy} = 0$, we just need to compare between 0 and $y^L(x)$, and see which one leads to a larger $J^L(y|x)$.

Proposition 4.6.4. *Suppose the supplier offers the zero-rent contract $y^R(x)$ from the "next" period onwards. For any given x, the first-order condition $\frac{dJ^L(y|x)}{dy} = 0$ has at most one positive solution, denoted as $y^L(x)$. In addition, $y^L(x)$ is decreasing in x.*

In fact, the first-order condition can be written as

$$\frac{c + \frac{h}{1-\delta} - \delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y(1-\delta)} - \delta\Pi'(y)}{[b - r(1-\delta)]e^{-\lambda y} + [h + r(1-\delta)]e^{-\lambda y(1-\delta)}} = e^{\lambda(y_0 - x)}.$$

The right-hand side is independent of y whereas the left-hand side is decreasing in x, because the numerator is increasing in y by Proposition 4.6.3 yet the denominator is decreasing in y. Therefore, the solution $y^L(x)$ is unique (if exists) and is decreasing in x.

We have discussed the quantity plans that maximize $J^R(y|x)$ and $J^L(y|x)$ respectively. However, it is based on the fact of no "bump" in the retailer's profit-to-go function. If there exists a "bump", where the IR constraint is binding at two points but redundant in between, our previous analysis would not hold. In addition to maximizing the virtual surplus, we would have to take into account the boundary conditions at the two end points when characterizing the optimal quantity plan. As such, the existence of a "bump" would make the analysis more complicated. However, we are able to show if the supplier offers the proposed contract $y^R(x)$ from the "next" period onwards, the optimal contract in the "current" period will not lead to any "bump" in the retailer's profit-to-go function.

Theorem 4.6.5. *Suppose that the supplier implements the zero-rent plan $y^R(x)$ from the "next" period onwards. In the "current" period, the optimal contract will not create a "bump" in the retailer's profit-to-go function, i.e. there can not exist two points x^+ and x^- such that the IR*

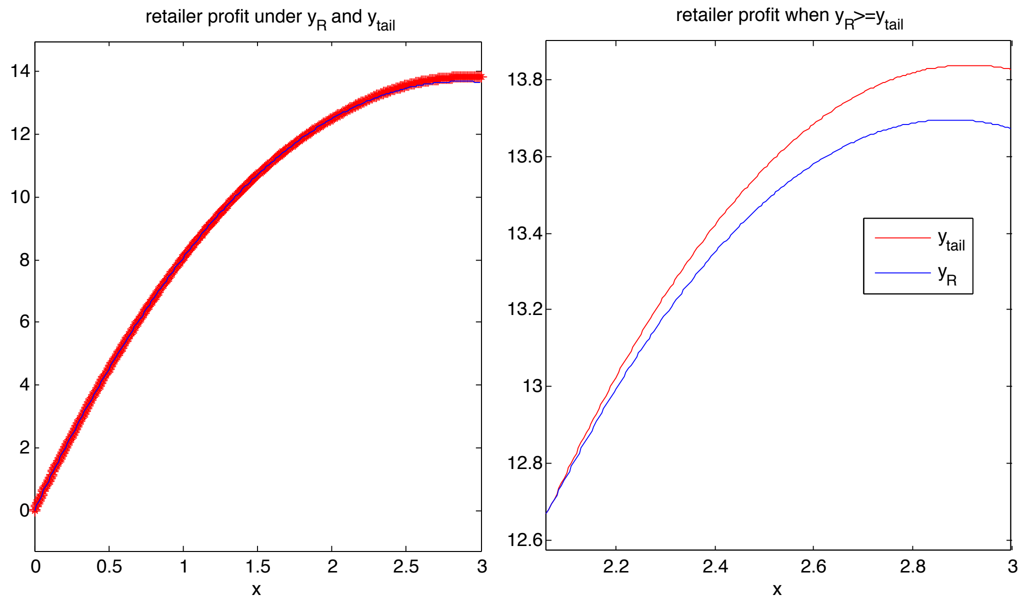


Figure 4.8: Retailer's profit-to-go under quantity plans $y^R(x)$ and $\max\{y^L(x), x\}$. (Parameters: $r = 10$, $c = 5$, $b = 3$, $h = 3$, $y_0 = 3$, $\lambda = 1$ and $\delta = 0.9$.)

constraint is binding at x^+ and x^- but redundant at $x \in (x^-, x^+)$.

Theorem 4.6.5 simplifies our analysis in finding the optimal short-term contract in the following sense. Under the optimal contract, there must exist a point x^* where the IR constraint is binding. Thanks to Theorem 4.6.5, we expect the optimal contract to have the following property: at $x < x^*$ (and at $x > x^*$), the IR constraint is either always binding (with the possibility of exclusion) or never binding. As a result, it suffices to only study the virtual surplus anchored at the right $J^R(y|x)$ and at the left $J^L(y|x)$.

In fact, when $x < x^*$, we look at $J^R(y|x)$ and we have known that $y = 0$ maximizes the virtual surplus. But the IR constraint cannot be satisfied given the quantity plan $y(x) = \max\{0, x\}$. It implies that the IR constraint will always be binding at $x < x^*$. In other words, it is optimal to implement $y^R(x)$ for $x < x^*$. However, when $x > x^*$, this may not be the case. We found examples where, instead of $y^R(x)$, it is better for the supplier to implement $\max\{y^L(x), x\}$. Interestingly, by doing so, both the supplier and the retailer are better off. See Figures 4.8 and 4.9.

We need some conditions on the anchor point x^* in order to satisfy the IR constraint when $x > x^*$. By similar analysis as the two-period case, we require $v'(y^L(x^*)) + \delta U'(y^L(x^*)) > \underline{u}'(x^*) = v'(y^R(x^*)) + \delta U'(y^R(x^*))$. The following lemma provides a lower bound for the threshold x^*, which is x^L.

Lemma 4.6.6. *The threshold x^* satisfies $x^* \geq x^L$, where x^L is the unique solution of $y^R(x) = y^L(x)$. In addition,* $y^R(x) - y^L(x) = \begin{cases} < 0, & x < x^L \\ > 0, & x > x^L \end{cases}$.

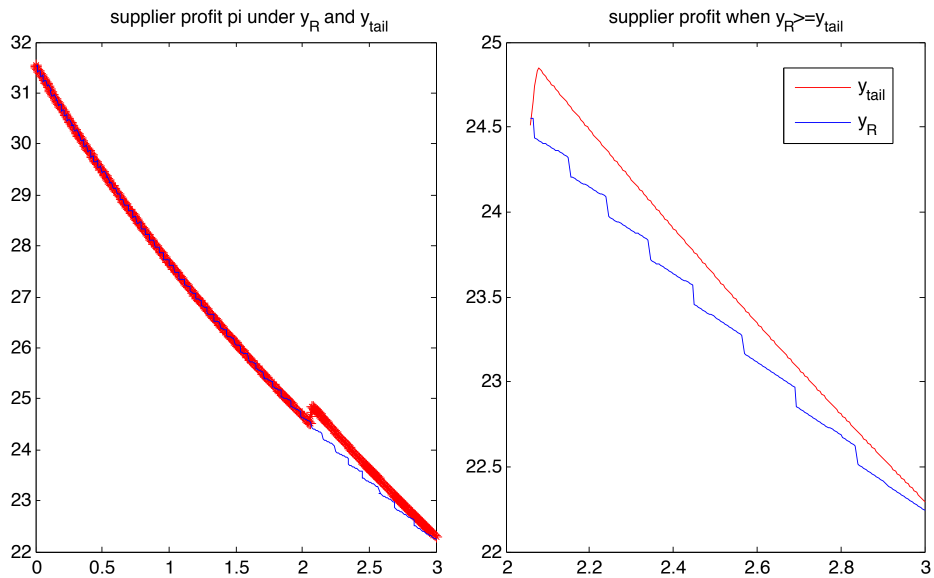


Figure 4.9: Supplier's profit-to-go under quantity plans $y^R(x)$ and $\max\{y^L(x), x\}$. (Parameters: $r = 10$, $c = 5$, $b = 3$, $h = 3$, $y_0 = 3$, $\lambda = 1$ and $\delta = 0.9$.)

Because $y^R(x)$ strictly increases in x and $y^L(x)$ strictly decreases in x when $y^L(x) > x$, the solution of $y^R(x) = y^L(x)$ must be unique, as shown in Figure 4.10. As we can see, the optimal short-term contract can be much more complex than the proposed zero-rent contract $y^R(x)$. In the "current" period, the optimal contract is such that when $x < x^*$, the retailer either orders up to $y^R(x)$ or is excluded; when $x > x^*$, the retailer orders up to $\max\{y^L(x), x\}$. The quantity plan $y^L(x)$ depends on the supplier's belief $G(x)$. Moreover, the optimal contract may result in a "pooling" region in which the order quantity is 0 and the supplier is unable to differentiate the retailer's true inventory level x. In this sense, we do not expect a stationary optimal (short-term) contract, as the supplier's belief evolves over time in a very complex pattern.

However, the proposed zero-rent contract $y^R(x)$, though not optimal, serves as a good heuristic. We show numerically that this contract is inferior than the optimal one by only a small percentage. But it is much simpler to understand and implement. Table 4.1 presents the gap between the zero-rent contract $y^R(x)$ and the optimal contract in the supplier's profit-to-go, which can be less than 2%. Figure 4.11 illustrates a sample trajectory of pre-order and post-order inventory levels under the zero-rent contract $y^R(x)$. We start with $y_0 = 0$. Demand unravels and the retailer has backorders $x_1 < 0$ at the beginning of period 1. According to the contract, the retailer orders up to $y^R(0)$. As long as the retailer holds backorder at the beginning of the periods, his order-up-to remains the same as $y^R(0)$. Once the retailer has positive pre-order inventory $x_t > 0$ in period 4, he orders up to a higher level $y^R(x_4) > y^R(0)$. In addition, when the pre-order inventory x is positive, the order-up-to level $y^R(x)$ is an strictly increasing function of x. As we can see, the zero-rent contract $y^R(x)$ leads to a generalized base-stock policy.

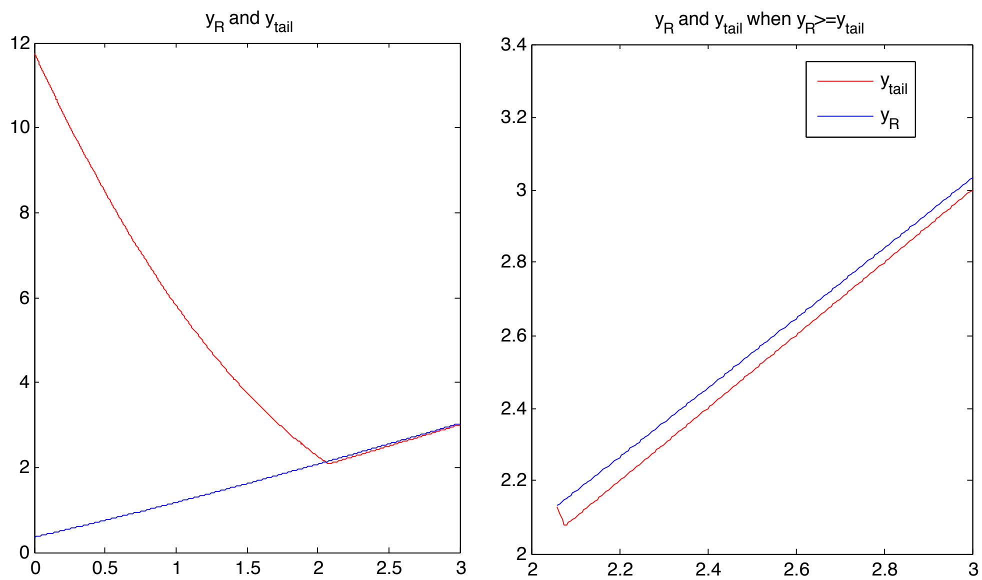


Figure 4.10: Two quantity plans $y^R(x)$ and $\max\{y^L(x), x\}$. (Parameters: $r = 10$, $c = 5$, $b = 3$, $h = 3$, $y_0 = 3$, $\lambda = 1$ and $\delta = 0.9$.)

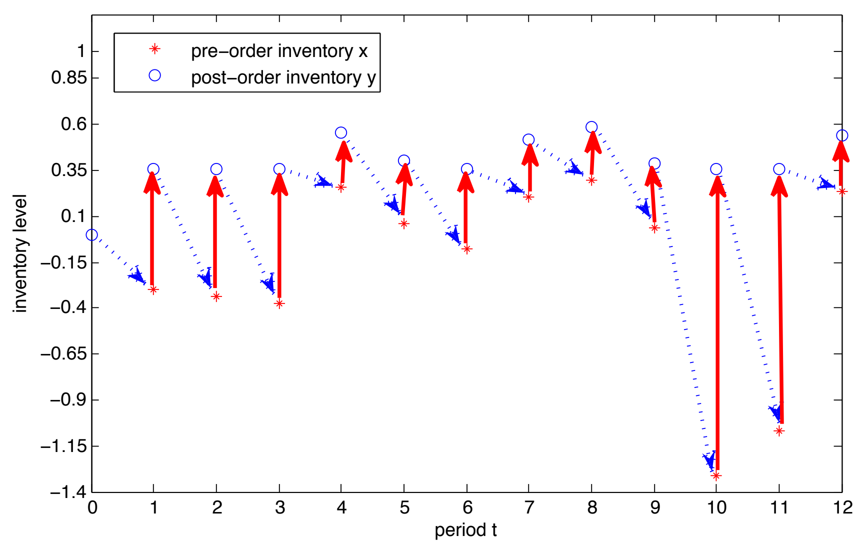


Figure 4.11: Sample inventory trajectory under the quantity plan $y^R(x)$. (Parameters: $r = 10$, $c = 5$, $b = 3$, $h = 3$, $y_0 = 0$, $\lambda = 1$ and $\delta = 0.9$.)

x	0	0.5	1	1.5	2	2.2	2.4	2.6	2.8	3
$\pi^R(x)$	31.51	29.44	27.73	26.07	24.63	24.15	23.64	23.11	22.68	22.24
$\pi^*(x)$	31.51	29.44	27.73	26.07	24.63	24.48	23.9	23.35	22.81	22.29
$(\pi^*(x) - \pi^R(x))/\pi^*(x)$	0	0	0	0	0	1.38%	1.11%	0.99%	0.56%	0.25%

Table 4.1: Supplier's profit-to-go under the zero-rent contract $y^R(x)$ and the optimal contract. (Parameters: $r = 10$, $c = 5$, $b = 3$, $h = 3$, $y_0 = 3$, $\lambda = 1$ and $\delta = 0.9$.)

x	0	0.5	1	1.5	2	2.2	2.4	2.6	2.8	3
$\pi^R(x)$	-3.99	-5.36	-6.215	-6.6775	-6.8764	-6.91	-6.89	-6.88	-6.83	-6.79
$\pi^L(x)$	-3.99	-5.36	-6.215	-6.6775	-6.8764	-5.93	-6.11	-6.26	-6.38	-6.47

Table 4.2: Supplier's profit-to-go under $y^R(x)$ and $\max\{y^L(x), x\}$. (Parameters: $r = 10$, $c = 6$, $b = 7$, $h = 5$, $y_0 = 3$, $\lambda = 1$ and $\delta = 0.9$.)

However, we find that when the cost parameters are large enough, the supplier gets negative profit-to-go no matter she implements $y^R(x)$ or $\max\{y^L(x), x\}$ (see Table 4.2). The supplier is able to improve her profit-to-go by terminating the relationship with such retailer. In this case, we can actually show that the optimal short-term contract is stationary and takes a simple form. The optimal contract consists of a threshold x^{**} (maybe different from the x^* described above) and a base-stock policy with a positive order-up-to level $y^R(0)$. When $x \leq x^{**}$, the retailer orders up to $y^R(0)$. Yet when $x > x^{**}$, the supplier terminates the relationship with the retailer. In other words, the optimal short-term contract in the backorder case induces partial participation. When the beginning inventory is high, it would be too expensive for the supplier to encourage the retailer's participation, and the supplier would rather exclude the retailer. So far this result is only shown numerically. We state it as a conjecture.

Conjecture 1. *Suppose the cost parameters (c, b, h and r) are such that (i) $v(y^R(0)) + \delta U(y^R(0)) - cy^R(0) + \delta\frac{r-c}{\lambda}e^{-\lambda y^R(0)} < 0$ and (ii) $y^R(0) < x^L$. The optimal short-term contract is stationary and takes the following form: there exists a threshold $x^{**} \leq 0$; at $x \leq x^{**}$, the retailer participates and orders up to $y^R(0)$; at $x > x^{**}$, the retailer is excluded. The threshold x^{**} solves $(r-c)x^{**} = v(y^R(0)) + \delta U(y^R(0)) - cy^R(0) + \delta\frac{r-c}{\lambda}e^{-\lambda(y^R(0)-x^{**})}$.*

Figure 4.12 demonstrates a sample trajectory of pre-order and post-order inventory levels under the contract in Conjecture 1. The corresponding threshold is $x^{**} = -0.009$. We still start with $y_0 = 0$. Demand is realized and the retailer has pre-order inventory x_1 which is below the threshold. Then the retailer orders up to $y^R(0)$. As long as the retailer's pre-order inventory level is smaller than the threshold $x^{**} = -0.009$, the order-up-to level is always $y^R(0)$. Yet at the beginning of period 5, the retailer holds positive inventory $x_5 > 0$, the two parties terminates their relationship and the game ends.

As we can see, the optimal short-term contract in the backorder case can be drastically different from the lost-sales case. Our results yield valuable insights to practitioners. In order to improve customer satisfaction, the supplier may require the retailer to take backorder instead

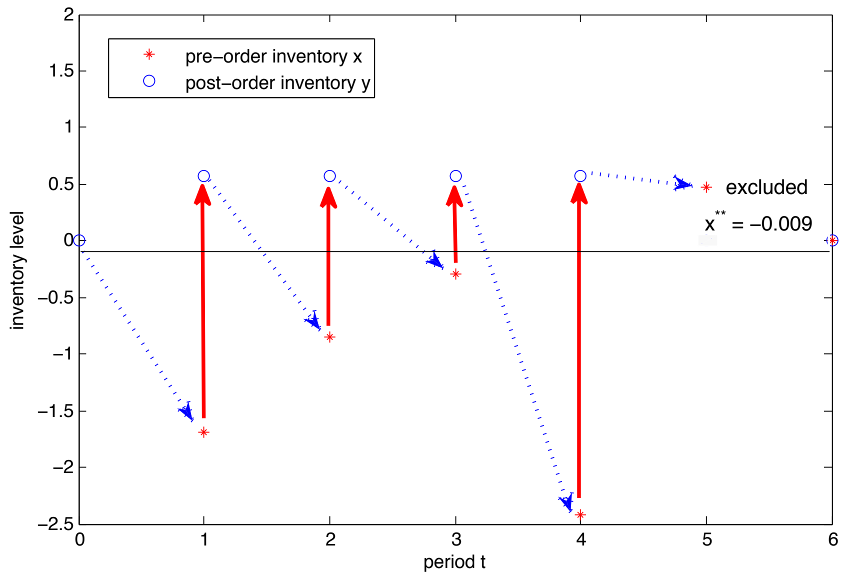


Figure 4.12: Sample inventory trajectory under the quantity plan in Conjecture 1. (Parameters: $r = 10$, $c = 6$, $b = 7$, $h = 5$, $y_0 = 0$, $\lambda = 1$ and $\delta = 0.9$.)

of simply losing excess demand. This has a huge impact on the contract design. If the cost parameters lie in a certain region, the supplier's optimal short-term contract has a simple structure. It entails a base-stock policy and an exclusion region. However, in other cases, the optimal short-term contract may be complex and hard to implement. As a result, the supplier may search for some simple contract that has a good, though not optimal, performance, such as the zero-rent contract. Alternatively, the supplier may consider switching to long-term contracting.

4.7 Conclusion

We study a dynamic adverse-selection model in which a supplier sells to a retailer with private inventory and backorder information. Our work fills a significant gap and tackles an open problem in the dynamic contracting literature and the supply chain management literature.

First, we show that in the single-period case, the supplier's optimal contract consists of a base-stock policy with base-stock level 0. With backorder (negative inventory) on hand, the retailer should order up to 0 whereas with positive inventory, the retailer should order nothing. He will receive zero information rent in any case. Similar results do not hold in the multi-period setting. In the two-period case, we demonstrate that the optimal contract in the first period can be fairly complex. It has a threshold structure with possibly two thresholds. More interestingly, the retailer starts getting positive information rent when his inventory is high enough. It is drastically different from the lost-sales case in which a higher inventory level makes the retailer a "worse" type. In the backorder case, it may be a good thing to have high inventory. Moreover, the optimal contract may entail an exclusion region. When the retailer's beginning inventory falls into that region, the supplier will terminate the relationship with him, because it is too costly for the supplier to ensure the retailer's participation.

Finally, we analyze the optimal short-term contract in the infinite-horizon case. We show that a stationary optimal contract may not exist in general. However, in certain cost regimes, the optimal short-term contract may have a simple threshold structure. If the retailer's beginning inventory is below the threshold, he orders up to a positive base-stock level, and he obtains his reservation profit. In this case, the contract looks similar as the single-period case. However, if the retailer's beginning inventory is beyond the threshold, he will be excluded, i.e. the supplier terminates the business relationship with the retailer.

In the paper, we make a few assumptions to improve the tractability of the analysis. For instance, we only consider the exponential demand. In the future, we will further explore the problem under more general demand types, such as Erlang distribution. Notice that Erlang distribution with shape parameter 1 is exponential distribution, so we may extend the results to cases with shape parameters other than 1. Although there may not be a closed-form solution under general demand, we will pursue interesting structural properties. For instance, under more general demand assumptions, can we still have a threshold structure? If so, how will the threshold change when the number of no-ordering periods is larger?

Chapter 5

Conclusion

The first essay explores the use of incentivized action in mobile games. To our best knowledge, our work provides the first analytical model to study incented actions. We provide sufficient conditions for the optimality of a threshold strategy of offering incented actions to low-engaged players and then removing them to encourage real-money purchases once a player is sufficiently engaged. We also explore the settings where the optimality of the threshold policy breaks down. Moreover, we provide managerial insights and assist game publishers in targeting which types of games can take most advantage of delivering incented actions. The results and modeling approach will be useful to researchers as well as practitioners.

In the future, we plan to investigate the setting where transition probabilities are unknown and therefore some statistical learning algorithm would be required. We are also interested in the situation where engagement is difficult to define or measure and a partially observed Markov decision process (POMDP) model would be required. Also in the age of big data, with the increasing availability of player-level data, we would like to develop data-driven approaches to establish appropriate player behavior models, estimate game parameters, and derive insights on the impact of certain policies. Furthermore, for games hosted on mobile platforms, the platform holder is able to make interventions into the practice of incented actions. In fact, the platform holder and the game publisher have misaligned incentives. Typically, the revenue derived from incented actions is not processed through the platform whereas in-app purchases are. We would like to investigate the incentive misalignment problem between the platform and game publisher, possibly as a dynamic contracting problem.

The second essay studies a simple but new dynamic contract that generalizes the well-known wholesale price-only contract and is related to well-known ideas such as double marginalization, contract structure and commitment issues. We show that the generalized price-only contract benefits both players. Moreover, the inefficiency approaches 0 as the number of price offers n approaches infinity. We also demonstrate that for a given contract with a specific n, the wholesale prices monotonically decrease. However, somewhat surprisingly, for a fixed n, the order quantities within the n periods may not be monotone. We provide necessary and sufficient conditions for the stationarity of the supplier's per period profit. As one of the future research directions, we are interested in characterizing the bound on the performance of generalized price-only contract for a fixed n. Another interesting direction is to investigate the performance of other contracts when subjected to the same dynamics as in this paper.

The third essay contributes to the dynamic contracting literature. We analyze a dynamic adverse-selection problem where a supplier sells to a retailer with private inventory or backlog information. Our work fills the gap in the literature and focuses on dynamic short-term contracts. We demonstrate that the information rent (profit yielded to the retailer) under the optimal contract may be non-monotone in the retailer's inventory (or backlog) level. In the lost-sales case, Zhang et al. [57] shows that the retailer with a higher inventory is a "worse" type because he gets less information rent. However, in the backlogging case, the information rent will sometimes increase in the retailer's inventory (or backlog) level. Hence, the retailer with a higher inventory can be a "better" type. More interestingly, we find that the supplier may be better off by excluding some types of retailers. Specially, under exponentially distributed demand, if the cost parameters fall into a certain regime, the optimal short-term contract entails a base-stock order policy and a exclusion region. It is drastically different from the lost-sales setting and yields new insights to academics and practitioners. In the future, we will further explore the problem under more general demand types. Although there may not be closed-form solution under general demand, we will pursue interesting structural properties. For instance, under more general demand assumptions, can we still have a threshold structure? If so, how will the threshold change when the number of no-ordering periods is larger?

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Appendix A

Proofs of Results in Chapter 2

Derivation of expected total value in (2.8)

Given a policy y, the induced stochastic process underlying our problem is an *absorbing Markov chain* (for a discussion on absorbing Markov chains see Chapter III.4 of Taylor and Karlin [53]). An absorbing Markov chain is one where every state can reach (with nonzero probability) an absorbing state. In our setting the absorbing state is the quit state -1 and (A3.3) assures that the quit state is reachable from every engagement level.

The absorbing Markov chain structure allows for clean formulas for the total expected reward. Policy y induces a Markov chain transition matrix

$$P^y := \begin{bmatrix} S^y & s^y \\ 0 & 1 \end{bmatrix} \tag{A.1}$$

where S^y is an $n+1$ by $n+1$ matrix with entries corresponding to the transition probabilities between engagement levels, given the policy y (see Example 9 below for an illustration). The vector s^y has entries corresponding to the quitting probabilities of the engagement levels, and the bottom right corner "1" indicates the absorbing nature of the quitting state -1.

Associated with policy y and the transition matrix P^y is a *fundamental matrix*

$$M^y := \sum_{k=0}^{\infty} S^k = (I_{n+1} - S)^{-1} \tag{A.2}$$

where I_{n+1} is the $n+1$ by $n+1$ identity matrix. The fundamental matrix is a key ingredient for analyzing absorbing Markov chains. Its entries have the following useful interpretation: the (e, e')th entry $n^y_{e,e'}$ of M^y is the expected number of visits to engagement level e' starting in engagement level e before being absorbed in the quit state. Using the entries of the fundamental matrix we can write a closed-form formula for the total expected revenue of policy y:

$$W^y(e) = \sum_{e' \in E} n^y_{e,e'} r(e', y(e')). \tag{A.3}$$

An advantage of (A.3) over (2.7) is that the former is a finite sum over the number of engagement levels and does not explicitly involve the time index t. However, this formula can be simplified further. Observe that $n_{e,e'} = 0$ for $e' < e$ since engagement can only increase over time (provided the player does not quit). Hence we can write:

$$W^y(e) = \sum_{e' \geq e} n^y_{e,e'} r(e', y(e')).$$

Example 9. *In this example we derive the formulas given in Example 1 using the fundamental matrix. For policy y^1 the matrix S^1 introduced in* (A.1) *is*

$$S^1 = \begin{bmatrix} q_M(0)(1-\tau_M) & q_M(0)\tau_M \\ 0 & q_M(1) \end{bmatrix}$$

where the entries come from the transition probabilities in (2.5)*. The fundamental matrix is*

$$M^1 = \begin{bmatrix} \frac{1}{1-q_M(0)(1-\tau_M)} & \frac{q_M(0)\tau_M}{1-q_M(0)(1-\tau_M)(1-q_M(1))} \\ 0 & \frac{1}{1-q_M(1)} \end{bmatrix}$$

and the total expected rewards (2.8) *for in each of the two starting engagement levels are:*

$$W^1(0) = \tfrac{1}{1-q_M(0)(1-\tau_M)} q_M(0)\mu_M + \tfrac{q_M(0)\tau_M}{(1-q_M(0)(1-\tau_M))(1-q_M(1))} q_M(1)\mu_M$$

and

$$W^1(1) = \tfrac{q_M(1)\mu_M}{q_Q(1)}$$

respectively.

Proof of Proposition 2.4.1

The following is an important lemma to understand the nature of the fundamental matrix in our setting:

Lemma A.0.1. *The matrix Q is upper bidiagonal and its component is denoted by $k_{i,j}$,*

$$S = \begin{bmatrix} k_{1,1} & k_{1,2} & 0 & \dots & 0 & 0 \\ 0 & k_{2,2} & k_{2,3} & 0 & \dots & 0 \\ & & \dots & \dots & & \\ 0 & 0 & \dots & 0 & k_{N-1,N-1} & k_{N-1,N} \\ 0 & 0 & 0 & 0 & \dots & k_{N,N} \end{bmatrix}$$

the corresponding fundamental matrix $(I-S)^{-1}$ is upper triangular and its (i,j)-th entry is $\frac{1}{(1-k_{i,i})}$ if $i = j$ and $\frac{\prod_{v=i}^{j-1} k_{v,v+1}}{\prod_{v=i}^{j}(1-k_{v,v})}$ if $i < j$, i.e.

$$(I-S)^{-1} = \begin{bmatrix} \frac{1}{(1-k_{1,1})} & \frac{k_{1,2}}{(1-k_{1,1})(1-k_{2,2})} & \dots & \frac{\prod_{j=1}^{N-2} k_{j,j+1}}{\prod_{j=1}^{N-1}(1-k_{j,j})} & \frac{\prod_{j=1}^{N-1} k_{j,j+1}}{\prod_{j=1}^{N}(1-k_{j,j})} \\ 0 & \frac{1}{(1-k_{2,2})} & \frac{k_{2,3}}{(1-k_{2,2})(1-k_{3,3})} & \dots & \frac{\prod_{j=2}^{N-1} k_{j,j+1}}{\prod_{j=2}^{N}(1-k_{j,j})} \\ & & \dots & & \\ 0 & \dots & 0 & \frac{1}{(1-k_{N-1,N-1})} & \frac{k_{N-1,N}}{(1-k_{N-1,N-1})(1-k_{N,N})} \\ 0 & 0 & 0 & \dots & \frac{1}{(1-k_{N,N})} \end{bmatrix}$$

Proof. We prove the lemma by showing $(I-S)\times(I-S)^{-1} = I$ where $(I-S)^{-1}$ is proposed above. Denote R as the production of $(I-S)$ and $(I-S)^{-1}$. The (i,j)-th entry of R results from the multiplication of the i-th row of $(I-S)$ and the j-th column of $(I-S)^{-1}$.

The i-th row of $(I-S)$ is $(0,\dots,0,\underbrace{1-k_{i,i}}_{i},\underbrace{-k_{i,i+1}}_{i+1},0,\dots,0)$. For the j-th column of $(I-S)^{-1}$, we consider three possible cases:

1) If $j<i$, the j-th column of $(I-S)^{-1}$ is $(\frac{\prod_{v=1}^{j-1} k_{v,v+1}}{\prod_{v=1}^{j}(1-k_{v,v})},\dots,\underbrace{0}_{i},\underbrace{0}_{i+1},0,\dots,0)^T$. Clearly the (i,j)-th entry of R is 0.

2) If $j=i$, the j-th column of $(I-S)^{-1}$ is $(\frac{\prod_{v=1}^{j-1} k_{v,v+1}}{\prod_{v=1}^{j}(1-k_{v,v})},\dots,\underbrace{\frac{1}{(1-k_{i,i})}}_{i},\underbrace{0}_{i+1},0,\dots,0)^T$. So the (i,i)-th entry of R is 1.

3) If $j>i$, the j-th column of $(I-S)^{-1}$ is $(\frac{\prod_{v=1}^{j-1} k_{v,v+1}}{\prod_{v=1}^{j}(1-k_{v,v})},\dots,\underbrace{\frac{\prod_{v=i}^{j-1} k_{v,v+1}}{\prod_{v=i}^{j}(1-k_{v,v})}}_{i},\underbrace{\frac{\prod_{v=i+1}^{j-1} k_{v,v+1}}{\prod_{v=i+1}^{j}(1-k_{v,v})}}_{i+1},0,\dots,0)^T$. By simply algebra, we obtain the (i,j)-th entry of R is 0.

In conclusion, the (i,j)-th entry of R is 1 if $i=j$ and is 0 otherwise. This implies that R is an identity matrix.

In our model, $k_{i,j}$ indicates the transition probability from state i to state j. Suppose the policy is y^1, we have $k_{\bar{e},\bar{e}} = q_M(\bar{e})(1-\tau_M)$ and $k_{\bar{e},\bar{e}+1} = q_M(\bar{e})\tau_M$. Suppose the policy is y^2, we have $k_{\bar{e},\bar{e}} = p_M(\bar{e})(1-\tau_M) + p_I(\bar{e})(1-\tau_I)$ and $k_{\bar{e},\bar{e}+1} = p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I$. By definition, the expected number of visits $n^y_{\bar{e},e}$ is the $(\bar{e},e)$-th entry of the fundamental matrix M^y. According to Lemma A.0.1, we have $n^1_{\bar{e},e} = \frac{q_M(\bar{e})\tau_M}{1-q_M(\bar{e})(1-\tau_M)} \times \frac{\prod_{j=\bar{e}+1}^{e-1} k_{j,j+1}}{\prod_{j=\bar{e}+1}^{e}(1-k_{j,j})}$ and $n^2_{\bar{e},e} = \frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_M}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} \times \frac{\prod_{j=\bar{e}+1}^{e-1} k_{j,j+1}}{\prod_{j=\bar{e}+1}^{e}(1-k_{j,j})}$. (We assume $\prod_{j=\bar{e}+1}^{\bar{e}} k_{j,j+1} = 1$). Because we only make local change of the policy at engagement level $\bar{e}$, $n^1_{\bar{e},e}$ and $n^2_{\bar{e},e}$ share the same term

$\frac{\prod_{j=\bar{e}+1}^{e-1} k_{j,j+1}}{\prod_{j=\bar{e}+1}^{e}(1-k_{j,j})}$ where $k_{j,j}$ and $k_{j,j+1}$ depend on the policy $y^1(j)$ for $j > \bar{e}$. In fact,

$$k_{j,j} = \begin{cases} q_M(j)(1-\tau_M) & \text{if } y^*(j) = 0 \text{ and } j < N \\ p_M(j)(1-\tau_M) + p_I(j)(1-\tau_I) & \text{if } y^*(j) = 1 \text{ and } j < N \\ q_M(j) & \text{if } y^*(j) = 0 \text{ and } j = N \\ p_M(j) + p_I(j) & \text{if } y^*(j) = 1 \text{ and } j = N \end{cases}$$

$$k_{j,j+1} = \begin{cases} q_M(j)\tau_M & \text{if } y^*(j) = 0 \text{ and } j < N \\ p_M(j)\tau_M + p_I(j)\tau_I & \text{if } y^*(j) = 0 \text{ and } j < N \end{cases}$$

Moreover, we find out that $n_{\bar{e}+1,e}^{y^1(\bar{e}+1)} = \frac{\prod_{j=\bar{e}+1}^{e-1} k_{j,j+1}}{\prod_{j=\bar{e}+1}^{e}(1-k_{j,j})}$. Hence, we rewrite $n_{\bar{e},e}^1 = \frac{q_M(\bar{e})\tau_M}{1-q_M(\bar{e})(1-\tau_M)} n_{\bar{e}+1,e}^{y^1(\bar{e}+1)}$ and $n_{\bar{e},e}^2 = \frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_M}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} n_{\bar{e}+1,e}^{y^1(\bar{e}+1)}$ for all $e > \bar{e}$. Finally, the progression effect is equivalent to the following:

$$\begin{aligned} \Delta_n(e|\bar{e}) &= n_{\bar{e},e}^2 - n_{\bar{e},e}^1 \\ &= \tfrac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_M}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} n_{\bar{e}+1,e}^{y^1(\bar{e}+1)} - \tfrac{q_M(\bar{e})\tau_M}{1-q_M(\bar{e})(1-\tau_M)} n_{\bar{e}+1,e}^{y^1(\bar{e}+1)} \\ &= [\tfrac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_M}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} - \tfrac{q_M(\bar{e})\tau_M}{1-q_M(\bar{e})(1-\tau_M)}] n_{\bar{e}+1,e}^{y^1(\bar{e}+1)} \\ &= \tfrac{p_I(\bar{e})\{\tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[(1-\tau_I)\tau_M - (1-\tau_M)\tau_I]\}}{[1-q_M(\bar{e})(1-\tau_M)][1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]} n_{\bar{e}+1,e}^{y^1(\bar{e}+1)} \end{aligned} \tag{A.4}$$

Since the denominator of (A.4) is positive and $n_{\bar{e}+1,e}^{y^1(\bar{e}+1)}$ is positive, the sign of $\Delta_n(e|\bar{e})$ is completely determined by the term $\tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[(1-\tau_I)\tau_M - (1-\tau_M)\tau_I]$ for all $e > \bar{e}$. It is only affected by $\bar{e}$ but not e. It means that the progression effect is uniform in sign with respect to e. □

Proof of Proposition 2.4.2

By definition, $n_{\bar{e},\bar{e}}^y$ is the $(\bar{e},\bar{e})$-th entry of the fundamental matrix N^y. According to Lemma A.0.1 if the policy is y^1, $k_{\bar{e},\bar{e}}^1 = q_M(\bar{e})(1-\tau_M)$ and thereby $n_{\bar{e},\bar{e}}^1 = \frac{1}{1-q_M(\bar{e})(1-\tau_M)}$. If the policy is y^2, $k_{\bar{e},\bar{e}}^2 = p_M(\bar{e})(1-\tau_M) + p_I(\bar{e})(1-\tau_I)$ and consequently $n_{\bar{e},\bar{e}}^2 = \frac{1}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)}$.

Therefore, the retention effect is equal to

$$\begin{aligned} \Delta_n(\bar{e}|\bar{e}) &= n_{\bar{e},\bar{e}}^2 - n_{\bar{e},\bar{e}}^1 \\ &= \tfrac{1}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} - \tfrac{1}{1-q_M(\bar{e})(1-\tau_M)} \\ &= \tfrac{[1-q_M(\bar{e})(1-\tau_M)]-[1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]}{[1-q_M(\bar{e})(1-\tau_M)][1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]} \\ &= \tfrac{p_I(\bar{e})[(1-\tau_I)-\alpha(\bar{e})(1-\tau_M)]}{[1-q_M(\bar{e})(1-\tau_M)][1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]} \end{aligned}$$

where the last equality comes from the fact $q_M(\bar{e}) = p_M(\bar{e}) + \alpha(\bar{e})p_I(\bar{e})$. The sign of $\Delta_n(\bar{e}|\bar{e})$ completely depends on $(1-\tau_I) - \alpha(\bar{e})(1-\tau_M)$. Under Assumptions 2.3.1–2.3.4, we have

$(1-\tau_I) \geq (1-\tau_M) \geq \alpha(\bar{e})(1-\tau_M)$. Hence the retention effect is always nonnegative, i.e $\Delta_n(\bar{e}|\bar{e}) \geq 0$ for all $\bar{e}$. □

Proof of Theorem 2.5.2

In order to prove the Theorem, we first introduce the following lemma.

Lemma A.0.2. *For any* $e = 1, \ldots, N$, $W(e, y=0) \geq \frac{q_M(e)\mu_M}{1-q_M(e)}$ *and* $W(e, y=1) \geq \frac{p_M(e)\mu_M + p_I(e)\mu_I}{1-p_M(e)-p_I(e)}$

Proof of Lemma A.0.2: The proof is by induction. Clearly, at the highest engagement level $e = N$, we have $W(N, y=0) = \frac{q_M(N)\mu_M}{1-q_M(N)}$ and $W(N, y=1) = \frac{p_M(N)\mu_M + p_I(N)\mu_I}{1-p_M(N)-p_I(N)}$. Now suppose it holds for level $j \geq e+1$, we would like to show that the result still holds for level e.

$$
\begin{aligned}
& W(e, y=0) - \frac{q_M(e)\mu_M}{1-q_M(e)} \\
= \;& \frac{q_M(e)\mu_M}{1-q_M(e)(1-\tau_M)} + \frac{q_M(e)\tau_M}{1-q_M(e)(1-\tau_M)} W(e+1) - \frac{q_M(e)\mu_M}{1-q_M(e)} \\
= \;& \frac{[q_M(e)\mu_M + q_M(e)\tau_M W(e+1)][1-q_M(e)] - q_M(e)\mu_M[1-q_M(e)(1-\tau_M)]}{[1-q_M(e)(1-\tau_M)][1-q_M(e)]} \\
= \;& \frac{q_M(e)\tau_M[W(e+1)(1-q_M(e)) - q_M(e)\mu_M]}{[1-q_M(e)(1-\tau_M)][1-q_M(e)]}
\end{aligned}
$$

By the inductive assumption, $W(e+1) \geq W(e+1, y=0) \geq \frac{q_M(e+1)\mu_M}{1-q_M(e+1)} \geq \frac{q_M(e)\mu_M}{1-q_M(e)}$, we finally obtain $W(e, y=0) - \frac{q_M(e)\mu_M}{1-q_M(e)} \geq 0$. Similarly,

$$
\begin{aligned}
& W(e, y=1) - \frac{p_M(e)\mu_M + p_I(e)\mu_I}{1-p_M(e)-p_I(e)} \\
= \;& \frac{p_M(e)\mu_M + p_I(e)\mu_I}{1-p_M(e)(1-\tau_M) - p_I(e)(1-\tau_I)} + \frac{[p_M(e)\tau_M + p_I\tau_I]W(e+1)}{1-p_M(e)(1-\tau_M) - p_I(e)(1-\tau_I)} - \frac{p_M(e)\mu_M + p_I(e)\mu_I}{1-p_M(e)-p_I(e)} \\
= \;& \frac{(p_M(e)\tau_M + p_I\tau_I)[W(e+1)(1-p_M(e)-p_I(e)) - p_M(e)\mu_M - p_I(e)\mu_I]}{[1-p_M(e)(1-\tau_M) - p_I(e)(1-\tau_I)][1-p_M(e)-p_I(e)]}
\end{aligned}
$$

By the inductive assumption, we have $W(e+1) \geq W(e+1, y=1) \geq \frac{p_M(e+1)\mu_M + p_I(e+1)\mu_I}{1-p_M(e+1)-p_I(e+1)} \geq \frac{p_M(e)\mu_M + p_I(e)\mu_I}{1-p_M(e)-p_I(e)}$. Therefore, $W(e, y=1) - \frac{p_M(e)\mu_M + p_I(e)\mu_I}{1-p_M(e)-p_I(e)} \geq 0$. This completes the proof for Lemma A.0.2. □

We return to the proof of Theorem 2.5.2, to show the optimal value function $W(e)$ is increasing in e. It suffices to show $W(e, y=1) \leq W(e+1)$ and $W(e, y=0) \leq W(e+1)$ for any $e < N$, in that we will have $W(e) = \max\{W(e, y=1), W(e, y=0)\} \leq W(e+1)$. First, we compare $W(e, y = 1)$ and $W(e + 1)$.

$$
\begin{aligned}
& W(e, y = 0) - W(e + 1) \\
= \quad & \frac{q_M(e)\mu_M}{1 - q_M(e)(1 - \tau_M)} + \frac{q_M(e)\tau_M}{1 - q_M(e)(1 - \tau_M)} W(e + 1) - W(e + 1) \\
= \quad & \frac{q_M(e)\mu_M}{1 - q_M(e)(1 - \tau_M)} - \frac{1 - q_M(e)}{1 - q_M(e)(1 - \tau_M)} W(e + 1) \\
= \quad & \frac{1 - q_M(e)}{1 - q_M(e)(1 - \tau_M)} [\frac{q_M(e)\mu_M}{1 - q_M(e)} - W(e + 1)] \leq 0 \qquad \text{(A.5)}
\end{aligned}
$$

The inequality (A.5) holds because $\frac{q_M(e)\mu_M}{1-q_M(e)} \leq \frac{q_M(e+1)\mu_M}{1-q_M(e+1)} \leq W(e + 1, y = 0) \leq W(e + 1)$. Similarly, we compare $W(e, y = 1)$ and $W(e + 1)$.

$$
\begin{aligned}
& W(e, y = 1) - W(e + 1) \\
= \quad & \frac{p_M(e)\mu_M + p_I(e)\mu_I}{1 - p_M(e)(1 - \tau_M) - p_I(e)(1 - \tau_I)} + \frac{p_M(e)\tau_M + p_I(e)\tau_I}{1 - p_M(e)(1 - \tau_M) - p_I(e)(1 - \tau_I)} W(e + 1) - W(e + 1) \\
= \quad & \frac{p_M(e)\mu_M + p_I(e)\mu_I}{1 - p_M(e)(1 - \tau_M) - p_I(e)(1 - \tau_I)} - \frac{1 - p_M(e) - p_I(e)}{1 - p_M(e)(1 - \tau_M) - p_I(e)(1 - \tau_I)} W(e + 1) \\
= \quad & \frac{1 - p_M(e) - p_I(e)}{1 - p_M(e)(1 - \tau_M) - p_I(e)(1 - \tau_I)} [\frac{p_M(e)\mu_M + p_I(e)\mu_I}{1 - p_M(e) - p_I(e)} - W(e + 1)] \leq 0 \qquad \text{(A.6)}
\end{aligned}
$$

where the inequality (A.6) holds since $\frac{p_M(e)\mu_M+p_I(e)\mu_I}{1-p_M(e)-p_I(e)} \leq \frac{p_M(e+1)\mu_M+p_I(e+1)\mu_I}{1-p_M(e+1)-p_I(e+1)} \leq W(e + 1, y = 1) \leq W(e + 1)$.

Finally, since $W(e) = max\{W(e, y = 1), W(e, y = 0)\}$ and we have shown that both $W(e, y = 1)$ and $W(e, y = 0)$ are no greater than $W(e + 1)$, we conclude that $W(e) \leq W(e + 1)$. □

Proof of Proposition 2.5.1

We denote $W^2(\bar{e}) - W^1(\bar{e}) = C(\bar{e}) + F(\bar{e})$, where $C(\bar{e})$ represents the "current" benefits of offering incented actions and $F(\bar{e})$ represents the "future" benefits. In order to prove the optimality of the myopic policy, we first take a close look at $C(\bar{e})$ and $F(\bar{e})$. By definition,

$$
\begin{aligned}
C(\bar{e}) &= \frac{p_M(\bar{e})\mu_M+p_I(\bar{e})\mu_I}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} - \frac{q_M(\bar{e})\mu_M}{1-q_M(\bar{e})(1-\tau_M)} \\
&= \frac{\{[p_M(\bar{e})\mu_M+p_I(\bar{e})\mu_I][1-q_M(\bar{e})(1-\tau_M)]-q_M(\bar{e})\mu_M[1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]\}}{[1-q_M(\bar{e})(1-\tau_M)][1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]} \\
&= \frac{p_I(\bar{e})}{[1-q_M(\bar{e})(1-\tau_M)][1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]} \{\mu_I - \alpha(\bar{e})\mu_M + q_M(\bar{e})[(1 - \tau_I)\mu_M - (1 - \tau_M)\mu_I]\} \\
F(\bar{e}) &= \{\frac{p_M(\bar{e})\tau_M+p_I(\bar{e})\tau_I}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} - \frac{q_M(\bar{e})\tau_M}{1-q_M(\bar{e})(1-\tau_M)}\}\{\sum_{e'>\bar{e}} n^{y^1}_{\bar{e}+1,e'} r(e', y(e'))\} \\
&= \frac{\{[p_M(\bar{e})\tau_M+p_I(\bar{e})\tau_I][1-q_M(\bar{e})(1-\tau_M)]-q_M(\bar{e})\tau_M[1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]\}}{[1-q_M(\bar{e})(1-\tau_M)][1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]} \{\sum_{e'>\bar{e}} n^{y^1}_{\bar{e}+1,e'} r(e', y(e'))\} \\
&= \frac{p_I(\bar{e})\{\sum_{e'>\bar{e}} n^{y^1(\bar{e}+1)}_{\bar{e}+1,e'} r(e',y(e'))\}}{[1-q_M(\bar{e})(1-\tau_M)][1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]} \{\tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[(1 - \tau_I)\tau_M - (1 - \tau_M)\tau_I]\}
\end{aligned}
$$

We further define

$$\delta_1(\bar{e}) = \mu_I - \alpha(\bar{e})\mu_M + q_M(\bar{e})[(1-\tau_I)\mu_M - (1-\tau_M)\mu_I]$$
$$\delta_2(\bar{e}) = \tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[(1-\tau_I)\tau_M - (1-\tau_M)\tau_I]$$

Clearly, the sign of $C(\bar{e})$ is determined by $\delta_1(\bar{e})$ and the sign of $F(\bar{e})$ is determined by $\delta_2(\bar{e})$. Now suppose $\tau_I/\tau_M = \mu_I/\mu_M$, it leads to $\delta_1(\bar{e})/\mu_M = \delta_2(\bar{e})/\tau_M$. Therefore, $\delta_1(\bar{e})$ and $\delta_2(\bar{e})$ must have the same sign. It implies that whenever $\delta_1(\bar{e})$ is positive, we must have $\delta_2(\bar{e})$ positive and thereby $W^2(\bar{e}) - W^1(\bar{e})$ positive. Similarly, whenever $\delta_1(\bar{e})$ is negative, we must have $\delta_2(\bar{e})$ negative and thereby $W^2(\bar{e}) - W^1(\bar{e})$ negative.

Notice that the previous analysis does not rely on the policy for higher engagement level $e > \bar{e}$. Even if we fix $y^1(e)$ to be the optimal action which is solved by backward induction for $e > \bar{e}$, we still have that the "current" benefit $C(\bar{e})$ and the "future" benefit $F(\bar{e})$ share the same sign. Therefore, the optimal action at engagement level $\bar{e}$ can be determined by whether $C(\bar{e})$ is positive or negative. Equivalently speaking, the myopically-optimal policy will be the optimal policy. □

Proof of Proposition 2.5.4

By definition, the revenue effect is

$$\Delta_r(\bar{e}) = p_M(\bar{e})\mu_M + p_I(\bar{e})\mu_I - q_M(\bar{e})\mu_M = p_I(\bar{e})[\mu_I - \alpha(\bar{e})\mu_M]$$

where the last equality comes from the fact $q_M(\bar{e}) = p_M(\bar{e}) + \alpha(\bar{e})p_I(\bar{e})$.

First of all, as $p_I(\bar{e})$ is non-negative, the sign of $\Delta_r(\bar{e})$ is determined by $\mu_I - \alpha(\bar{e})\mu_M$ which is decreasing in $\bar{e}$. Thus if $\Delta_r(\bar{e}) < 0$ for some $\bar{e}$, we will also have $\Delta_r(e') < 0$ for all $e' \geq \bar{e}$.

Moreover, both $p_I(\bar{e})$ and $\mu_I - \alpha(\bar{e})\mu_M$ are nonincreasing in $\bar{e}$, so we conclude that $\Delta_r(\bar{e})$ is nonincreasing in $\bar{e}$ whenever $\mu_I - \alpha(\bar{e})\mu_M > 0$, i.e. whenever $\Delta_r(\bar{e}) > 0$. □

Proof of Proposition 2.5.6

(a) Recall the expression for $\Delta_n(e|\bar{e})$.

$$\Delta_n(e|\bar{e}) = \frac{p_I(\bar{e})n_{\bar{e}+1,e}^{y^1(\bar{e}+1)}}{[1-q_M(\bar{e})(1-\tau_M)][1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]}\{\tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[(1-\tau_I)\tau_M - (1-\tau_M)\tau_I]\}$$

The sign of $\Delta_n(e|\bar{e})$ is completely decided by $\delta_2(\bar{e}) = \tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[(1-\tau_I)\tau_M - (1-\tau_M)\tau_I] = \tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[\tau_M - \tau_I]$.

Assumption 2.5.5 indicates that $-\alpha(\bar{e})\tau_M + q_M(\bar{e})\tau_M$ will decrease in $\bar{e}$. Besides, $-q_M(\bar{e})\tau_I$ will also decrease in $\bar{e}$. Therefore, $\delta_2(\bar{e})$ will be a decreasing function of $\bar{e}$. It implies $\Delta_n(e|\bar{e})$ satisfies the following property: if $\Delta_n(e|\bar{e}) > 0$ for some $\bar{e}$, then $\Delta_n(e|e') > 0$ for all $e' \leq \bar{e}$; and if $\Delta_n(e|\bar{e}) < 0$ for some $\bar{e}$, then $\Delta_n(e|e') < 0$ for all $e' \geq \bar{e}$

(b) From the previous analysis, we have already seen that the sign of $C(\bar{e})$ only depends on the term $\delta_1(\bar{e}) = \mu_I - \alpha(\bar{e})\mu_M + q_M(\bar{e})[(1-\tau_I)\mu_M - (1-\tau_M)\mu_I]$ and the sign of $F(\bar{e})$ only depends on the term $\delta_2(\bar{e}) = \tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[\tau_M - \tau_I]$. From (a), we have already shown that $\delta_2(\bar{e})$ will decrease in $\bar{e}$. Therefore, if $F(\bar{e}) > 0$ for some $\bar{e}$, then $F(\bar{e}) > 0$ for all $e' \leq \bar{e}$; if $F(\bar{e}) < 0$ for some $\bar{e}$, then $F(\bar{e}) < 0$ for all $e' \geq \bar{e}$.

Similarly, Assumption 2.5.5 ensures that $-\alpha(\bar{e})\mu_M + q_M(\bar{e})(1-\tau_I)\mu_M$ will also decrease in $\bar{e}$, in that $\alpha(\bar{e}+1)\mu_M - q_M(\bar{e}+1)(1-\tau_I)\mu_M - \alpha(\bar{e})\mu_M + q_M(\bar{e})(1-\tau_I)\mu_M = [\alpha(\bar{e}+1) - \alpha(\bar{e})]\mu_M - [q_M(\bar{e}+1) - q_M(\bar{e})]\mu_M(1-\tau_I) \geq \mu_M[(\alpha(\bar{e}+1) - \alpha(\bar{e})) - (q_M(\bar{e}+1) - q_M(\bar{e}))] \geq 0$. In addition, $-q_M(\bar{e})(1-\tau_M)\mu_I$ will decrease in $\bar{e}$ as well. As a result, $\delta_1(\bar{e})$ will also be an decreasing function of $\bar{e}$. Hence $C(\bar{e})$ satisfies the following property: if $C(\bar{e}) > 0$ for some $\bar{e}$, then $C(\bar{e}) > 0$ for all $e' \leq \bar{e}$; if $C(\bar{e}) < 0$ for some $\bar{e}$, then $C(\bar{e}) < 0$ for all $e' \geq \bar{e}$. □

Proof of Theorem 2.5.8

In order to prove the optimal policy is a threshold policy, it suffices to show if there exists some $\bar{e}$ such that $W(\bar{e}, y=1) - W(\bar{e}, y=0) > 0$, then we must have $W(e, y=1) - W(e, y=0) > 0$ for all $e < \bar{e}$.

First of all, Assumption 2.5.3 guarantees that it is optimal not to offer incented action at the highest engagement level. Because $W(N, y=1) = \frac{p_M(N)\mu_M + p_I(N)\mu_I}{p_Q(N)} < \frac{p_M(N)\mu_M + p_I(N)\mu_M}{p_Q(N)} = \frac{q_M(N)\mu_M}{q_Q(N)} = W(N, y=0)$. Suppose that $W(\bar{e}, y=1) - W(\bar{e}, y=0) > 0$ for some $\bar{e} < N$ where

$$W(\bar{e}, y=1) - W(\bar{e}, y=0) = \frac{p_I(\bar{e})}{[1-q_M(\bar{e})(1-\tau_M)][1-p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)]}\{\delta_1(\bar{e}) + \delta_2(\bar{e})\sum_{e>\bar{e}} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)} r(e, y(e))\}$$

so it is equivalent to assume $\delta_1(\bar{e}) + \delta_2(\bar{e})\sum_{e>\bar{e}} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)} r(e, y(e)) > 0$. It implies that at least one of $\delta_1(\bar{e})$ and $\delta_2(\bar{e})$ has to be positive.

We would like to show $W(\bar{e}-1, y=1) - W(\bar{e}-1, y=0) > 0$. It suffices to show $\delta_1(\bar{e}-1) + \delta_2(\bar{e}-1)\sum_{e>\bar{e}-1} n_{\bar{e},e}^{y^*(\bar{e})} r(e, y(e)) > 0$. Now we consider three possible scenarios:

(1.1) Suppose $\delta_1(\bar{e}) > 0$ and $\delta_2(\bar{e}) > 0$. We have already proven that $\delta_1(\bar{e})$ and $\delta_2(\bar{e})$ are decreasing functions in $\bar{e}$ under Assumption 2.5.5. Therefore, $\delta_1(\bar{e}-1) > \delta_1(\bar{e}) > 0$ and $\delta_2(\bar{e}-1) > \delta_2(\bar{e}) > 0$. Clearly, we have $W(\bar{e}-1, y=1) - W(\bar{e}-1, y=0) > 0$ in this case.

(1.2) Suppose $\delta_1(\bar{e}) < 0$ but $\delta_2(\bar{e}) > 0$ (which may happen only if $\mu_I/\mu_M < \tau_I/\tau_M$). We first prove under Assumption 2.5.7, $\delta_2(\bar{e}-1)\sum_{e>\bar{e}-1} n_{\bar{e},e}^{y^*(\bar{e})} r(e, y(e)) > \delta_2(\bar{e})\sum_{e>\bar{e}} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)} r(e, y(e))$. Notice that

$$\delta_2(\bar{e}-1)\sum_{e>\bar{e}-1} n_{\bar{e},e}^{y^*(\bar{e})} r(e, y(e)) = \delta_2(\bar{e}-1)\{n_{\bar{e},\bar{e}}^{y^*(\bar{e})} r(\bar{e}, y^*(\bar{e})) + n_{\bar{e},\bar{e}+1}^{y^*(\bar{e})} r(\bar{e}+1, y^*(\bar{e}+1)) + \cdots + n_{\bar{e},N}^{y^*(\bar{e})} r(N, y^*(N))\}$$

$$\delta_2(\bar{e})\sum_{e>\bar{e}} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)} r(e, y(e)) = \delta_2(\bar{e})\{n_{\bar{e}+1,\bar{e}+1}^{y^*(\bar{e}+1)} r(\bar{e}+1, y^*(\bar{e}+1)) + \cdots + n_{\bar{e}+1,N}^{y^*(\bar{e}+1)} r(N, y^*(N))\}$$

For $e > \bar{e}$, we have such relationship $n_{\bar{e},e}^1 = \frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I}{1-p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)}$ since $y^*(\bar{e}) = 1$.

As a result,

$$\begin{aligned}
&\delta_2(\bar{e}-1)\sum_{e>\bar{e}-1} n_{\bar{e},e}^{y^*(\bar{e})} r(e,y(e)) \\
&= \delta_2(\bar{e}-1) n_{\bar{e},\bar{e}}^{y^*(\bar{e})} r(\bar{e},y^*(\bar{e})) + \delta_2(\bar{e}-1)\tfrac{p_M(\bar{e})\tau_M+p_I(\bar{e})\tau_I}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} \sum_{e>\bar{e}} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)} r(e,y(e))
\end{aligned}$$

Next we are going to show $\delta_2(\bar{e}-1)\frac{p_M(\bar{e})\tau_M+p_I(\bar{e})\tau_I}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} > \delta_2(\bar{e})$ under Assumption 2.5.7. Since $\delta_2(\bar{e}-1) > \delta_2(\bar{e}) > 0$, we can compare $\frac{\delta_2(\bar{e})}{\delta_2(\bar{e}-1)}$ with $\frac{p_M(\bar{e})\tau_M+p_I(\bar{e})\tau_I}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)}$. For the ratio $\frac{\delta_2(\bar{e})}{\delta_2(\bar{e}-1)}$, we have

$$\begin{aligned}
\tfrac{\delta_2(\bar{e})}{\delta_2(\bar{e}-1)} &= \tfrac{\tau_I-\alpha(\bar{e})\tau_M+q_M(\bar{e})[\tau_M-\tau_I]}{\tau_I-\alpha(\bar{e}-1)\tau_M+q_M(\bar{e}-1)[\tau_M-\tau_I]} = \tfrac{\tau_I/\tau_M-\alpha(\bar{e})+q_M(\bar{e})[1-\tau_I/\tau_M]}{\tau_I/\tau_M-\alpha(\bar{e}-1)+q_M(\bar{e}-1)[1-\tau_I\tau_M]} \\
&= \tfrac{1-q_M(\bar{e})}{1-q_M(\bar{e}-1)} + \tfrac{[\alpha(\bar{e}-1)-q_M(\bar{e}-1)]\frac{1-q_M(\bar{e})}{1-q_M(\bar{e}-1)}-[\alpha(\bar{e})-q_M(\bar{e})]}{\tau_I/\tau_M-\alpha(\bar{e}-1)+q_M(\bar{e}-1)[1-\tau_I/\tau_M]} \le \tfrac{1-\alpha(\bar{e})}{1-\alpha(\bar{e}-1)} \qquad \text{(A.7)}
\end{aligned}$$

Because $[\alpha(\bar{e}-1)-q_M(\bar{e}-1)]\frac{1-q_M(\bar{e})}{1-q_M(\bar{e}-1)} - [\alpha(\bar{e})-q_M(\bar{e})] < 0$ and $1-q_M(\bar{e}-1) > 0$, the ratio $\frac{\delta_2(\bar{e})}{\delta_2(\bar{e}-1)}$ will increase in τ_I/τ_M and reach maximum when $\tau_I/\tau_M = 1$. We achieve (A.7).

Finally, Assumption 2.5.7 claims that $\frac{1-\alpha(\bar{e})}{1-\alpha(\bar{e}-1)} \le \frac{p_M(\bar{e})\tau_M+p_I(\bar{e})\tau_I}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)}$. Hence, we end up with $\delta_2(\bar{e}-1)\frac{p_M(\bar{e})\tau_M+p_I(\bar{e})\tau_I}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} > \delta_2(\bar{e})$. We further have

$$\begin{aligned}
&\delta_2(\bar{e}-1)\sum_{e>\bar{e}-1} n_{\bar{e},e}^{y^*(\bar{e})} r(e,y(e)) \\
&= \delta_2(\bar{e}-1) n_{\bar{e},\bar{e}}^{2} r(\bar{e},y^*(\bar{e})) + \delta_2(\bar{e}-1)\tfrac{p_M(\bar{e})\tau_M+p_I(\bar{e})\tau_I}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} \sum_{e>\bar{e}} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)} r(e,y(e)) \\
&> \delta_2(\bar{e})\sum_{e>\bar{e}} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)} r(e,y(e))
\end{aligned}$$

Finally we conclude

$$\delta_1(\bar{e}-1) + \delta_2(\bar{e}-1)\sum_{e>\bar{e}-1} n_{\bar{e},e}^{y^*(\bar{e})} r(e,y(e)) > \delta_1(\bar{e}) + \delta_2(\bar{e})\sum_{e>\bar{e}} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)} r(e,y(e)) > 0$$

equivalently, we have $W(\bar{e}-1,y=1) - W(\bar{e}-1,y=0) > 0$ in this case.

(1.3) Suppose $\delta_1(\bar{e}) > 0$ but $\delta_2(\bar{e}) < 0$ (which may happen only if $\mu_I/\mu_M > \tau_I/\tau_M$). If $\delta_2(\bar{e}-1) \ge 0$, we easily get $\delta_1(\bar{e}-1) + \delta_2(\bar{e}-1)W(\bar{e}) > \delta_1(\bar{e}) + \delta_2(\bar{e}-1)W(\bar{e}) > 0$. Else if $0 > \delta_2(\bar{e}-1) \ge \delta_2(\bar{e})$, Theorem 2.5.2 indicates that $0 \le W(\bar{e}) \le W(\bar{e}+1)$, therefore we have $\delta_2(\bar{e}-1)W(\bar{e}) > \delta_2(\bar{e})W(\bar{e}) > \delta_2(\bar{e})W(\bar{e}+1)$. Finally, $\delta_1(\bar{e}-1) + \delta_2(\bar{e}-1)\sum_{e>\bar{e}-1} n_{\bar{e},e}^{y^*(\bar{e})} r(e,y(e)) = \delta_1(\bar{e}-1) + \delta_2(\bar{e}-1)W(\bar{e}) > \delta_1(\bar{e}) + \delta_2(\bar{e})W(\bar{e}+1) = \delta_1(\bar{e}) + \delta_2(\bar{e})\sum_{e>\bar{e}} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)} r(e,y(e)) > 0$. Therefore, $W(\bar{e}-1,y=1) - W(\bar{e}-1,y=0) > 0$.

In conclusion, we have shown that once $W(\bar{e},y=1) - W(\bar{e},y=0) > 0$ for some $\bar{e}$, we must also have $W(\bar{e}-1,y=1) - W(\bar{e}-1,y=0) > 0$. As a result, the optimal policy should be a forward threshold policy. □

Example of a non-threshold policy when Assumption 2.5.5 is violated

Consider the following two engagement level example. Assume $\mu_M = 1$, $\mu_I = 0.27$, $\tau_M = 0.99$, $\tau_I = 0.25$. At level 0, $p_M(0) = 0.23$, $p_I(0) = 0.54$, $\alpha(0) = 0.75$ and thereby $q_M(0) = 0.635$. At level 1, $p_M(1) = 0.34$, $p_I(1) = 0.52$, $\alpha(1) = 0.81$ and thereby $q_M(1) = 0.7612$. At level 2, $p_M(2) = 0.42$, $p_I(2) = 0.45$, $\alpha(2) = 1$ and thereby $q_M(1) = 0.87$.

The optimal policy is $y^* = (0, 1, 0)$. We use backward induction. At the highest level 2, we have $y^*(2) = 0$ and $W(2) = 0.87/0.13 = 6.692$. At level 1,

$$
\begin{aligned}
W(1, y=1) &= \tfrac{p_M(1)\mu_M + p_I(1)\mu_I}{1-p_M(1)(1-\tau_M)-p_I(1)(1-\tau_I)} + \tfrac{p_M(1)\tau_M + p_I(1)\tau_I}{(1-p_M(1)(1-\tau_M)-p_I(1)(1-\tau_I))} \frac{q_M(2)\mu_M}{q_Q(2)} \\
&= \tfrac{0.4804}{0.6066} + \tfrac{0.4666}{0.6066}(6.692) = 0.7920 + 0.7692(6.692) = 5.9395 \\
W(1, y=0) &= \tfrac{q_M(1)\mu_M}{1-q_M(1)(1-\tau_M)} + \tfrac{q_M(1)\tau_M}{1-q_M(1)(1-\tau_M)} \frac{q_M(2)\mu_M}{q_Q(2)} \\
&= \tfrac{0.7612}{0.9924} + \tfrac{0.7536}{0.9924}(6.692) = 0.7670 + 0.7594(6.692) = 5.849
\end{aligned}
$$

therefore $y^*(1) = 1$ and $W(1) = W(1, y = 1) = 5.9395$. Moreover, $C(1) = 0.7920 - 0.7670 = 0.025$ and $F(1) = (0.7692 - 0.7594)(6.692) = 0.0098(6.692) = 0.066$. Finally, we look at level 0.

$$
\begin{aligned}
W(0, y=1) &= \tfrac{p_M(0)\mu_M + p_I(0)\mu_I}{1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I)} + \tfrac{p_M(0)\tau_M + p_I(0)\tau_I}{(1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I))} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.3758}{0.5927} + \tfrac{0.3627}{0.5927}(5.9395) = 0.6340 + 0.6119(5.9395) = 4.2684 \\
W(0, y=0) &= \tfrac{q_M(0)\mu_M}{1-q_M(0)(1-\tau_M)} + \tfrac{q_M(0)\tau_M}{1-q_M(0)(1-\tau_M)} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.635}{0.9937} + \tfrac{0.6287}{0.9937}(5.9395) = 0.6391 + 0.6327(5.9395) = 4.3969
\end{aligned}
$$

as we can see $y^*(0) = 0$ and $W(0) = W(0, y = 0) = 4.3969$. Besides, $C(0) = 0.6340 - 0.6391 = -0.0051$ and $F(0) = (0.6119 - 0.6327)(5.9395) = -0.0208(5.9395) = -0.1232$. The optimal policy is not a threshold policy.

In fact, Assumption 2.5.5 is violated because $\alpha(1) - \alpha(0) = 0.81 - 0.75 = 0.06$ while $q_M(1) - q_M(0) = 0.7612 - 0.635 = 0.1262$. Assumption 2.5.7 is satisfied since $1 - \alpha(1) = 1 - 0.81 = 0.19$ and $(1-\alpha(0))\frac{p_M(1)\tau_M + p_I(1)\tau_I}{(1-p_M(1)(1-\tau_M)-p_I(1)(1-\tau_I))} = (1 - 0.75)\frac{0.4666}{0.6066} = 0.1923$.

Example of a non-threshold policy when Assumption 2.5.7 is violated

Consider the following two engagement level example. Assume $\mu_M = 1$, $\mu_I = 0.2$, $\tau_M = 0.91$, $\tau_I = 0.47$. At level 0, $p_M(0) = 0.03$, $p_I(0) = 0.51$, $\alpha(0) = 0.59$ and thereby $q_M(0) = 0.3309$. At level 1, $p_M(1) = 0.05$, $p_I(1) = 0.5$, $\alpha(1) = 0.62$ and thereby $q_M(1) = 0.36$. At level 2, $p_M(2) = 0.34$, $p_I(2) = 0.45$, $\alpha(2) = 1$ and thereby $q_M(1) = 0.79$.

The optimal policy is $y^* = (0, 1, 0)$. We use backward induction. At the highest level 2, we have $y^*(2) = 0$ and $W(2) = 0.79/0.21 = 3.7619$. At level 1,

$$\begin{aligned}
W(1, y=1) &= \tfrac{p_M(1)\mu_M + p_I(1)\mu_I}{1-p_M(1)(1-\tau_M)-p_I(1)(1-\tau_I)} + \tfrac{p_M(1)\tau_M + p_I(1)\tau_I}{(1-p_M(1)(1-\tau_M)-p_I(1)(1-\tau_I))} \frac{q_M(2)\mu_M}{q_Q(2)} \\
&= \tfrac{0.15}{0.7305} + \tfrac{0.2805}{0.7305}(3.7619) = 0.2053 + 0.3840(3.7619) = 1.6498 \\
W(1, y=0) &= \tfrac{q_M(1)\mu_M}{1-q_M(1)(1-\tau_M)} + \tfrac{q_M(1)\tau_M}{1-q_M(1)(1-\tau_M)} \frac{q_M(2)\mu_M}{q_Q(2)} \\
&= \tfrac{0.36}{0.9676} + \tfrac{0.3276}{0.9676}(3.7619) = 0.3721 + 0.3386(3.7619) = 1.6459
\end{aligned}$$

therefore $y^*(1) = 1$ and $W(1) = W(1, y=1) = 1.6498$. Moreover, $C(1) = 0.2053 - 0.3721 = -0.1668$ and $F(1) = (0.3840 - 0.3386)(3.7619) = 0.0454(3.7619) = 0.1708$. Finally, we look at level 0.

$$\begin{aligned}
W(0, y=1) &= \tfrac{p_M(0)\mu_M + p_I(0)\mu_I}{1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I)} + \tfrac{p_M(0)\tau_M + p_I(0)\tau_I}{(1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I))} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.1320}{0.7270} + \tfrac{0.2670}{0.7270}(1.6498) = 0.1816 + 0.3673(1.6498) = 0.7876 \\
W(0, y=0) &= \tfrac{q_M(0)\mu_M}{1-q_M(0)(1-\tau_M)} + \tfrac{q_M(0)\tau_M}{1-q_M(0)(1-\tau_M)} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.3309}{0.9702} + \tfrac{0.3011}{0.9702}(1.6498) = 0.3411 + 0.3104(1.6498) = 0.8532
\end{aligned}$$

as we can see $y^*(0) = 0$ and $W(0) = W(0, y=0) = 0.8532$. Besides, $C(0) = 0.1816 - 0.3411 = -0.1595$ and $F(0) = (0.3673 - 0.3104)(1.6498) = 0.0569(1.6498) = 0.0939$. The optimal policy is not a threshold policy.

In fact, Assumption 2.5.5 is satisfied because $\alpha(1) - \alpha(0) = 0.62 - 0.59 = 0.03$ while $q_M(1) - q_M(0) = 0.3600 - 0.3309 = 0.0291$. But Assumption 2.5.7 is violated since $(1 - \alpha(1)) = 1 - 0.62 = 0.38$ and $(1-\alpha(0))\tfrac{p_M(1)\tau_M + p_I(1)\tau_I}{(1-p_M(1)(1-\tau_M)-p_I(1)(1-\tau_I))} = (1 - 0.59)\tfrac{0.2805}{0.7305} = 0.1574$.

Proof of Proposition 2.6.1

We will restrict ourselves only to threshold policies. According to the backward induction, in order to prove the optimal threshold is non-decreasing in μ_I, we only need to show $W^2(e) - W^1(e)$ is non-decreasing in μ_I given that $y^1(e') = 0$ for $e' > e$. Because the optimal threshold e^* is solved by $W^2(e^*) - W^1(e^*) > 0$ where $y^1(e') = 0$ for $e' > e^*$ and $W^2(e^*+1) - W^1(e^*+1) \leq 0$ where $y^1(e') = 0$ for $e' > e^* + 1$.

We have already characterized the explicit expression for $W^2(e) - W^1(e)$ which is

$$W^2(e) - W^1(e) = \begin{cases} \tfrac{p_I(e)}{[1-q_M(e)(1-\tau_M)][1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)]}\{\delta_1(e) + \delta_2(e)\sum_{e'>e} n_{e+1,e'}^{y^1(e+1)} r(e', y(e'))\}, & e < N \\ \tfrac{p_I(N)}{p_Q(N)q_Q(N)}\{\mu_I - \alpha(N)\mu_M + q_M(N)(\mu_M - \mu_I)\}, & e = N \end{cases}$$

$$\begin{aligned}
\text{where } \delta_1(e) &= [\mu_I - \alpha(e)\mu_M + q_M(e)(1-\tau_I)\mu_M - q_M(e)(1-\tau_M)\mu_I] \\
\delta_2(e) &= [\tau_I - \alpha(e)\tau_M + q_M(e)(\tau_M - \tau_I)]
\end{aligned}$$

Clearly, $W^2(N) - W^1(N)$ will increase in μ_I. For any $e < N$, $\delta_1(e)$ will increase in μ_I while $\delta_2(e)$ will keep constant. In addition, both $n^{y^1(e+1)}_{e+1,e'}$ and $r(e', y(e'))$ will remain the same since we fix $y^1(e') = 0$ unchanged for all $e' > e$. Hence, $W^2(e) - W^1(e)$ will increase in μ_I. Let $\hat{e}^*$ be the largest engagement level such that $W^2(e) - W^1(e) > 0$. By definition, $\hat{e}^*$ is actually the new optimal threshold under a larger μ_I. Since originally $W^2(e^*) - W^1(e^*) > 0$ and the difference $W^2(e) - W^1(e)$ is increasing in μ_I, we conclude that $\hat{e}^* \geq e^*$. The optimal threshold must be non-decreasing in μ_I. □

Proof of Proposition 2.6.2

Similarly as Proposition 2.6.1, we will still restrict ourselves to threshold policies. It suffices to show $W^2(e) - W^1(e)$ is non-decreasing in τ_I given that $y^1(e') = 0$ for $e' > e$.

Obviously, $W^2(N) - W^1(N)$ does not depend on τ_I. For $e < N$, since $y^1(e') = 0$ for $e' > e$, we have $r(e', y(e'))$ and $n^{y^1(e+1)}_{e+1,e'}$ unrelated with τ_I. Therefore, $W^1(e) = \frac{q_M(e)\mu_M}{1-q_M(e)(1-\tau_M)} + \frac{q_M(e)\tau_M}{1-q_M(e)(1-\tau_M)} \sum_{e'>e} n^{y^1(e+1)}_{e+1,e'} r(e', y(e'))$ will not be affected by τ_I. Now we would like to show $W^2(e) = \frac{p_M(e)\mu_M + p_I(e)\mu_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)} + \frac{p_M(e)\tau_M + p_I(e)\tau_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)} \sum_{e'>e} n^{y^1(e+1)}_{e+1,e'} r(e', y(e'))$ will increase in τ_I. In fact, if τ_I increases by $\epsilon > 0$, we have

$$
\begin{aligned}
&[\tfrac{p_M(e)\mu_M + p_I(e)\mu_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I-\epsilon)} + \tfrac{p_M(e)\tau_M + p_I(e)(\tau_I+\epsilon)}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I-\epsilon)} \sum_{e'>e} n^{y^1(e+1)}_{e+1,e'} r(e', y(e'))] \\
&-[\tfrac{p_M(e)\mu_M + p_I(e)\mu_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)} + \tfrac{p_M(e)\tau_M + p_I(e)\tau_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)} \sum_{e'>e} n^{y^1(e+1)}_{e+1,e'} r(e', y(e'))] \\
&= \tfrac{\epsilon p_I(e) q_Q(e)}{[1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I-\epsilon)][1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)]} \{\sum_{e'>e} n^{y^1(e+1)}_{e+1,e'} r(e', y(e')) - \tfrac{p_M(e)\mu_M + p_I(e)\mu_I}{p_Q(e)}\} \\
&= \tfrac{\epsilon p_I(e) q_Q(e)}{[1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I-\epsilon)][1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)]} \{W^{y^1}(e+1) - \tfrac{p_M(e)\mu_M + p_I(e)\mu_I}{p_Q(e)}\} > 0
\end{aligned}
\tag{A.8}
$$

where $W^{y^1}(e+1)$ is the revenue at engagement level $e+1$ if Publisher follows the policy $y^1(e') = 0$ for all $e' > e$. The inequality (A.8) holds because of Lemma A.0.2. As a result, $W^2(e)$ will increase in τ_I and consequently $W^2(e) - W^1(e)$ will increase in τ_I for all $e < N$. This implies that the optimal threshold must be non-decreasing in τ_I. □

Details of Example 6

Consider the following two engagement level example. Assume $\mu_M = 1$, $\mu_I = 0.05$, $\tau_M = 0.8$, $\tau_I = 0.2$. At level 0, $p_M(0) = 0.3$, $p_I(0) = 0.5$, $\alpha(0) = 0.7$ and thereby $q_M(0) = 0.65$. At level 1, $p_M(1) = 0.5$, $p_I(1) = 0.4$, $\alpha(1) = 1$ and thereby $q_M(1) = 0.9$.

Because $\alpha(1) = 1$, we easily get $y^*(1) = 0$ and $W(1) = \frac{q_M(1)\mu_M}{1-q_M(1)} = 9$. Now we solve for level

0. At level 0,

$$
\begin{aligned}
W(0, y=1) &= \tfrac{p_M(0)\mu_M + p_I(0)\mu_I}{1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I)} + \tfrac{p_M(0)\tau_M + p_I(0)\tau_I}{(1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I))} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.325}{0.54} + \tfrac{0.34}{0.54}(9) = 0.6019 + 0.6296(9) = 6.2683 \\
W(0, y=0) &= \tfrac{q_M(0)\mu_M}{1-q_M(0)(1-\tau_M)} + \tfrac{q_M(0)\tau_M}{1-q_M(0)(1-\tau_M)} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.65}{0.87} + \tfrac{0.52}{0.87}(9) = 0.7471 + 0.5977(9) = 6.1264
\end{aligned}
$$

hence $y^*(0) = 1$ and $W(0) = \max\{W(0, y=1), W(0, y=0)\} = 6.2683$.

Now we change the parameters as follows: $\mu_M = 1$, $\mu_I = 0.05$, $\tau_M = 0.8$, $\tau_I = 0.25$. At level 0, $p_M(0) = 0.1$, $p_I(0) = 0.7$, $\alpha(0) = 0.7$ and thereby $q_M(0) = 0.59$. At level 1, $p_M(1) = 0.3$, $p_I(1) = 0.6$, $\alpha(1) = 1$ and thereby $q_M(1) = 0.9$.

We still have $y^*(1) = 0$ and $W(1) = \frac{q_M(1)\mu_M}{1-q_M(1)} = 9$. But at level 0,

$$
\begin{aligned}
W(0, y=1) &= \tfrac{p_M(0)\mu_M + p_I(0)\mu_I}{1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I)} + \tfrac{p_M(0)\tau_M + p_I(0)\tau_I}{(1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I))} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.135}{0.425} + \tfrac{0.225}{0.425}(9) = 0.3176 + 0.5294(9) = 5.0823 \\
W(0, y=0) &= \tfrac{q_M(0)\mu_M}{1-q_M(0)(1-\tau_M)} + \tfrac{q_M(0)\tau_M}{1-q_M(0)(1-\tau_M)} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.59}{0.882} + \tfrac{0.472}{0.882}(9) = 0.6689 + 0.5351(9) = 5.4852
\end{aligned}
$$

hence $y^*(0) = 0$ and $W(0) = \max\{W(0, y=1), W(0, y=0)\} = 5.4852$.

Suppose the incented action becomes so attractive (e.g. Candy Crush), it will increase τ_I. At the same time it will attract from Monetization to Incented action. In other words, it will increase p_I and decrease p_M. From the above example, we observe that the optimal threshold may decrease.

Details of Example 8

Consider the following two engagement level example. Assume $\mu_M = 1$, $\mu_I = 0.0001$, $\tau_M = 0.01$, $\tau_I = 0.009$. At level 0, $p_M(0) = 0.05$, $p_I(0) = 0.68$. At level 1, $p_M(1) = 0.3$, $p_I(1) = 0.65$. Besides, we set α step size be 0.6, i.e. $\alpha(1) = \alpha(0) + 0.6$.

We start with $\alpha(0) = 0.25$ and $\alpha(1) = 0.85$. Therefore, $q_M(0) = 0.05 + 0.68(0.25) = 0.22$ and $q_M(1) = 0.3 + 0.65(0.85) = 0.8525$. We solve the optimal policy by backward induction. At level 1,

$$
\begin{aligned}
W(1, y=1) &= \tfrac{p_M(1)\mu_M + p_I(1)\mu_I}{1-p_M(1)-p_I(1)} = \tfrac{0.300065}{0.05} = 0.4616 \\
W(1, y=0) &= \tfrac{q_M(1)}{1-q_M(1)} = \tfrac{0.8525}{0.1475} = 5.7797
\end{aligned}
$$

therefore, $y^*(1) = 0$ and $W(1) = 5.7797$. Now we solve for level 0. At level 0,

$$\begin{aligned}
W(0, y=1) &= \tfrac{p_M(0)\mu_M + p_I(0)\mu_I}{1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I)} + \tfrac{p_M(0)\tau_M + p_I(0)\tau_I}{(1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I))} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.050068}{0.27662} + \tfrac{0.00662}{0.27662}(5.7797) = 0.18099 + 0.02393(5.7797) = 0.3193 \\
W(0, y=0) &= \tfrac{q_M(0)\mu_M}{1-q_M(0)(1-\tau_M)} + \tfrac{q_M(0)\tau_M}{1-q_M(0)(1-\tau_M)} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.22}{0.7822} + \tfrac{0.0022}{0.7822}(5.7797) = 0.28125 + 0.00281(5.7797) = 0.2975
\end{aligned}$$

hence $y^*(0) = 1$ and $W(0) = \max\{W(0, y=1), W(0, y=0)\} = 0.3193$. In conclusion, the optimal policy is $y^* = (0, 0)$ under this case.

Next we increase $\alpha(0)$ by 0.1 but keep all the other parameters unchanged, i.e. $\alpha(0) = 0.35$ and $\alpha(1) = 0.95$. Correspondingly, $q_M(0) = 0.05 + 0.68(0.35) = 0.288$ and $q_M(1) = 0.3 + 0.65(0.95) = 0.9175$. Under this case, at level 1,

$$\begin{aligned}
W(1, y=1) &= \tfrac{p_M(1)\mu_M + p_I(1)\mu_I}{1-p_M(1)-p_I(1)} = \tfrac{0.300065}{0.05} = 0.4616 \\
W(1, y=0) &= \tfrac{q_M(1)}{1-q_M(1)} = \tfrac{0.9175}{0.0825} = 11.1212
\end{aligned}$$

therefore, $y^*(1) = 0$ and $W(1) = 11.1212$. Now we solve for level 0. At level 0,

$$\begin{aligned}
W(0, y=1) &= \tfrac{p_M(0)\mu_M + p_I(0)\mu_I}{1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I)} + \tfrac{p_M(0)\tau_M + p_I(0)\tau_I}{(1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I))} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.050068}{0.27662} + \tfrac{0.00662}{0.27662}(11.1212) = 0.18099 + 0.02393(11.1212) = 0.4471 \\
W(0, y=0) &= \tfrac{q_M(0)\mu_M}{1-q_M(0)(1-\tau_M)} + \tfrac{q_M(0)\tau_M}{1-q_M(0)(1-\tau_M)} \frac{q_M(1)\mu_M}{q_Q(1)} \\
&= \tfrac{0.288}{0.71488} + \tfrac{0.00288}{0.71488}(11.1212) = 0.40286 + 0.00403(11.1212) = 0.4477
\end{aligned}$$

hence $y^*(0) = 0$ and $W(0) = \max\{W(0, y=1), W(0, y=0)\} = 0.4477$. In conclusion, the optimal policy is $y^* = (0, 0)$ under this case. The optimal threshold decreases as $\alpha(0)$ increases.

Now we further increase $\alpha(0)$ by 0.05 but still keep all the other parameters unchanged, i.e. $\alpha(0) = 0.4$ and $\alpha(1) = 1$. Therefor $q_M(0) = 0.05 + 0.68(0.4) = 0.322$ and $q_M(1) = 0.3 + 0.65 = 0.95$. Under the new parameters, at level 1,

$$\begin{aligned}
W(1, y=1) &= \tfrac{p_M(1)\mu_M + p_I(1)\mu_I}{1-p_M(1)-p_I(1)} = \tfrac{0.300065}{0.05} = 0.4616 \\
W(1, y=0) &= \tfrac{q_M(1)}{1-q_M(1)} = \tfrac{0.95}{0.05} = 19
\end{aligned}$$

so $y^*(1) = 0$ and $W(1) = 19$. At level 0,

$$\begin{aligned} W(0, y=1) &= \tfrac{p_M(0)\mu_M + p_I(0)\mu_I}{1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I)} + \tfrac{p_M(0)\tau_M + p_I(0)\tau_I}{(1-p_M(0)(1-\tau_M)-p_I(0)(1-\tau_I))} \frac{q_M(1)\mu_M}{q_Q(1)} \\ &= \tfrac{0.050068}{0.27662} + \tfrac{0.00662}{0.27662}(19) = 0.18099 + 0.02393(19) = 0.63566 \\ W(0, y=0) &= \tfrac{q_M(0)\mu_M}{1-q_M(0)(1-\tau_M)} + \tfrac{q_M(0)\tau_M}{1-q_M(0)(1-\tau_M)} \frac{q_M(1)\mu_M}{q_Q(1)} \\ &= \tfrac{0.322}{0.68122} + \tfrac{0.00322}{0.68122}(19) = 0.47268 + 0.00488(19) = 0.5654 \end{aligned}$$

therefore we get $y^*(0) = 1$ and $W(0) = \max\{W(0, y=1), W(0, y=0)\} = 0.63566$. In conclusion, the optimal policy is $y^* = (1, 0)$. The optimal threshold increases as $\alpha(0)$ increases.

To summarize, the optimal threshold may increase or decrease with $\alpha(0)$ and it is possible to have U-shape.

Appendix B

Proofs of Results in Chapter 3

We first introduce a lemma.

Lemma B.0.3. *At each $t = 1, \ldots, n$, the solution of* (3.2) *satisfies:* $1 + \frac{\partial q_t^*}{\partial x_t} > 0$, *i.e. the post-order inventory* $x_t + q_t^*(x_t)$ *strictly increases in* x_t.

Proof of Lemma B.0.3: Suppose $q_{n-k}^*(x_{n-k})$ satisfies (3.2), i.e.

$$R'(Q^n)\prod_{j=1}^{k}(1 + \frac{\partial q_{n-k+j}^*}{\partial x_{n-k+j}}) + \frac{\partial w_{n-k}^*(x_{n-k}, q_{n-k}^*)}{\partial q_{n-k}} q_{n-k}^* = 0 \ .$$

We apply the Implicit Function Theorem and take derivative with respect to x_{n-k} on both sides which yields

$$\frac{d}{dx_{n-k}}[R'(Q^n)\prod_{j=1}^{k}(1 + \frac{\partial q_{n-k+j}^*}{\partial x_{n-k+j}}) + \frac{\partial w_{n-k}^*}{\partial q_{n-k}} q_{n-k}]|_{q_{n-k}=q_{n-k}^*}(1 + \frac{\partial q_{n-k}^*}{\partial x_{n-k}}) + \frac{\partial w_{n-k}^*(x_{n-k}, q_{n-k}^*)}{\partial q_{n-k}} \frac{\partial q_{n-k}^*}{\partial x_{n-k}} = 0 \quad \text{(B.1)}$$

$q_{n-k}^*(x_{n-k})$ satisfies the second-order condition, i.e., at $q_{n-k} = q_{n-k}^*(x_{n-k})$

$$\frac{d}{dx_{n-k}}[R'(Q^n)\prod_{j=1}^{k}(1 + \frac{\partial q_{n-k+j}^*}{\partial x_{n-k+j}}) + \frac{\partial w_{n-k}^*}{\partial q_{n-k}} q_{n-k}] \leq 0 \ . \quad \text{(B.2)}$$

Suppose the derivative (B.2) is equal to 0, then from (B.1),

$$\frac{\partial w_{n-k}^*(x_{n-k}, q_{n-k}^*)}{\partial q_{n-k}} \frac{\partial q_{n-k}^*}{\partial x_{n-k}} = 0 \Longrightarrow \frac{\partial q_{n-k}^*}{\partial x_{n-k}} = 0 \Longrightarrow 1 + \frac{\partial q_{n-k}^*}{\partial x_{n-k}} = 1 > 0 \ .$$

Suppose the derivative (B.2) is less than 0, then from (B.1),

$$1 + \frac{\partial q_{n-k}^*}{\partial x_{n-k}} = \frac{\frac{\partial w_{n-k}^*(x_{n-k}, q_{n-k}^*)}{q_{n-k}}}{\frac{d}{dx_{n-k}}[R'(Q^n)\prod_{j=1}^{k}(1 + \frac{\partial q_{n-k+j}^*}{\partial x_{n-k+j}}) + \frac{\partial w_{n-k}^*}{\partial q_{n-k}} q_{n-k}]|_{q_{n-k}=q_{n-k}^*(x_{n-k})}} > 0 \ ,$$

where the inequality holds because $w_{n-k}^*(x_{n-k}, q_{n-k}^*)$ must decrease in q_{n-k}^*. Otherwise, if $\frac{\partial w_{n-k}^*(x_{n-k}, q_{n-k}^*)}{\partial q_{n-k}} \geq 0$, (3.2) is positive for all $q_{n-k} \geq 0$, and the equilibrium solution at $t = n - k$ does not exist.

In conclusion, we have shown $1 + \frac{\partial q_t^*}{\partial x_t} > 0$, and the proof is complete. □

Proof of Proposition 3.2.1

The proof is by induction.

At the last offer $t = n$, given pre-order inventory level x_n and wholesale price w_n, the retailer decides upon his optimal order quantity to maximize his profit which is $\pi_R^n = R(x_n + q_n) - w_1 q_1 - \cdots - w_n q_n$. The problem is concave in q_n and the first-order derivative is $d\pi_R^n/dq_n = R'(x_n + q_n) - w_n$. If $w_n \geq R'(x_n)$, then $\partial\pi_R^n/\partial q_n = R'(x_n + q_n) - w_n < R'(x_n) - w_n \leq 0$ for all $q_n > 0$. Thus the retailer should order 0; else if $w_n < R'(x_n)$, then the retailer will order a positive quantity, $q_n^*(x_n, w_n)$, which satisfies the first-order condition $w_n = R'(x_n + q_n)$.

As for the supplier, she anticipates the retailer's order for any wholesale price and chooses w_n to maximize $\pi_S^n = w_1 q_1 + \cdots + w_{n-1} q_{n-1} + w_n q_n^*(x_n, w_n)$. Trivially, if $w_n > R'(x_n)$, the supplier will earn nothing since the retailer will not order. We focus on the case $w_n \leq R'(x_n)$. Since R' is strictly decreasing, there exists a one-to-one map between w_n ($w_n \leq R'(x_n)$) and $q_n^*(x_n, w_n)$ ($q_n^* \geq 0$). We can focus on an equivalent problem where the supplier faces the inverse demand curve $w_n^*(x_n, q_n) = R'(x_n + q_n)$ and has to decide upon an optimal quantity q_n to maximize her profit $\pi_S^n = w_1 q_1 + \cdots + w_{n-1} q_{n-1} + w_n^*(x_n, q_n) q_n$, where the first-order derivative is $\partial\pi_S^n/\partial q_n = R'(x_n + q_n) + R''(x_n + q_n) q_n$. The optimal q_n^* has to be either the boundary solution 0, or the solution of first-order condition. If $x_n \geq Q^{FB}$, $R'(x_n + q_n) + R''(x_n + q_n) q_n \leq R'(x_n + q_n) < 0$ for all $q_n > 0$, hence we have $q_n^* = 0$; else if $x_n < Q^{FB}$, $\partial\pi_S^n/\partial q_n|_{q_n=0} = R'(x_n) > 0$, so the optimal solution q_n^* cannot be the boundary solution, and thereby must be a solution of the first order condition $R'(x_n + q_n) + R''(x_n + q_n) q_n = R'(Q^n) + \frac{\partial w_n^*}{\partial q_n} q_n = 0$. The Theorem holds for the last offer.

Suppose the equilibrium solution $\{w_{n-k+j}^*(x_{n-k+j}, q_{n-k+j}), q_{n-k+j}^*(x_{n-k+j})\}$ satisfy (3.1) and (3.2) for $j = 1, \ldots, k$. For $t = n - k$, the retailer's profit is

$$\pi_R^n = R(x_{n-k} + q_{n-k} + q_{n-k+1}^* + \cdots + q_n^*) - w_1 q_1 - \cdots - w_{n-k} q_{n-k} - w_{n-k+1}^* q_{n-k+1}^* - \cdots - w_n^* q_n^* \ ,$$

and its corresponding first-order derivative is

$$\begin{aligned} \frac{\partial \pi_R^n}{\partial q_{n-k}} &= R'(Q^n)\frac{dQ^n}{dq_{n-k}} - w_{n-k} - \frac{d}{dq_{n-k}}(w_{n-k+1}^* q_{n-k+1}^* + \cdots + w_n^* q_n^*) \qquad \text{(B.3)} \\ &= R'(Q^n)(1 + \sum_{j=1}^{k} \frac{dq_{n-k+j}^*}{dq_{n-k}}) - w_{n-k} - \sum_{j=1}^{n}(q_{n-k+j}^* \frac{dw_{n-k+j}^*}{dq_{n-k}} + w_{n-k+j}^* \frac{dq_{n-k+j}^*}{dq_{n-k}}) \ . \end{aligned}$$

Next, we closely investigate the terms $\frac{dq_{n-k+j}^*}{dq_{n-k}}$ and $\frac{dw_{n-k+j}^*}{dq_{n-k}}$. Note that both q_{n-k+j}^* and w_{n-k+j}^* are functions of x_{n-k+j} which is equal to $x_{n-k} + q_{n-k} + q_{n-k+1}^* + \cdots + q_{n-k+j-1}^*$. Furthermore, for all $1 \leq m \leq j-1$, q^*_{n-k+m} also depends on q_{n-k}. Therefore, we have

$$\begin{aligned}
\frac{dq^*_{n-k+j}(x_{n-k+j})}{dq_{n-k}} &= \frac{dq^*_{n-k+j}(x_{n-k} + q_{n-k} + \sum_{m=1}^{j-1} q^*_{n-k+m})}{dq_{n-k}} \\
&= \frac{\partial q^*_{n-k+j}}{\partial x^*_{n-k+j}}\{1 + \sum_{m=1}^{j-1} \frac{dq^*_{n-k+m}}{dq_{n-k}}\} \\
&= \frac{\partial q^*_{n-k+j}}{\partial x^*_{n-k+j}}\{1 + \frac{\partial q^*_{n-k+1}}{\partial x^*_{n-k+1}} + \frac{\partial q^*_{n-k+2}}{\partial x^*_{n-k+2}}(1 + \frac{\partial q^*_{n-k+1}}{\partial x^*_{n-k+1}}) + \ldots \\
&\quad + \frac{\partial q^*_{n-k+j-1}}{\partial x^*_{n-k+j-1}}(1 + \frac{\partial q^*_{n-k+1}}{\partial x^*_{n-k+1}})(1 + \frac{\partial q^*_{n-k+2}}{\partial x^*_{n-k+2}}) \ldots (1 + \frac{\partial q^*_{n-k+j-2}}{\partial x^*_{n-k+j-2}})\} \\
&= \frac{\partial q^*_{n-k+j}}{\partial x^*_{n-k+j}} \prod_{m=1}^{j-1} (1 + \frac{\partial q^*_{n-k+m}}{\partial x_{n-k+m}}) \ ,
\end{aligned}$$

and

$$\begin{aligned}
\frac{dw^*_{n-k+j}(x_{n-k+j}, q^*_{n-k+j}(x_{n-k+j}))}{dq_{n-k}} &= \frac{\partial w^*_{n-k+j}}{\partial x^*_{n-k+j}}(\frac{dx_{n-k+j}}{dq_{n-k}} + \frac{dq^*_{n-k+j}(x_{n-k+j})}{dq_{n-k}}) \\
&= \frac{\partial w^*_{n-k+j}}{\partial x^*_{n-k+j}}\{\prod_{m=1}^{j-1}(1 + \frac{\partial q^*_{n-k+m}}{\partial x^*_{n-k+m}}) + \frac{\partial q^*_{n-k+j}}{\partial x^*_{n-k+j}} \prod_{m=1}^{j-1}(1 + \frac{\partial q^*_{n-k+m}}{\partial x^*_{n-k+m}})\} \\
&= \frac{\partial w^*_{n-k+j}}{\partial x^*_{n-k+j}} \prod_{m=1}^{j} (1 + \frac{\partial q^*_{n-k+m}}{\partial x_{n-k+m}}) \ .
\end{aligned}$$

In addition, we simplify $\frac{dQ^n}{dq_{n-k}}$ as follows:

$$\begin{aligned}
\frac{dQ^n}{dq_{n-k}} &= \frac{d(x_{n-k} + q_{n-k} + q^*_{n-k+1} + \cdots + q^*_n)}{dq_{n-k}} = 1 + \sum_{j=1}^{k} \frac{dq^*_{n-k+j}(x_{n-k+j})}{dq_{n-k}} \\
&= 1 + \sum_{j=1}^{k} \frac{\partial q^*_{n-k+j}}{\partial x^*_{n-k+j}} \prod_{m=1}^{j-1} (1 + \frac{\partial q^*_{n-k+m}}{\partial x_{n-k+m}}) \\
&= 1 + \frac{\partial q^*_{n-k+1}}{\partial x^*_{n-k+1}} + \frac{\partial q^*_{n-k+2}}{\partial x^*_{n-k+2}}(1 + \frac{\partial q^*_{n-k+1}}{\partial x^*_{n-k+1}}) + \cdots + \frac{\partial q^*_n}{\partial x^*_n} \prod_{m=1}^{n-1} (1 + \frac{\partial q^*_{n-k+m}}{\partial x_{n-k+m}}) \\
&= \prod_{j=1}^{k} (1 + \frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}) \ .
\end{aligned}$$

By the inductive assumption, we have

$$\begin{aligned} q^*_{n-k+j}\frac{dw^*_{n-k+j}}{dq_{n-k}} &= q^*_{n-k+j}\frac{\partial w^*_{n-k+j}}{\partial x^*_{n-k+j}}\prod_{m=1}^{j}(1+\frac{\partial q^*_{n-k+m}}{\partial q^*_{n-k+m-1}}) \\ &= -R'(Q^n)\prod_{m=1}^{k}(1+\frac{\partial q^*_{n-k+j+m}}{\partial x_{n-k+j+m}})\prod_{m=1}^{j}(1+\frac{\partial q^*_{n-k+m}}{\partial x_{n-k+m}}) \\ &= -R'(Q^n)\prod_{m=1}^{k}(1+\frac{\partial q^*_{n-k+m}}{\partial x_{n-k+m}}) \ . \end{aligned}$$

Finally, by the first-order condition, we end up with the inverse demand function

$$\begin{aligned} w^*_{n-k}(x_{n-k},q_{n-k}) &= R'(Q^n)\frac{dQ^n}{dq_{n-k}} - \sum_{j=1}^{n}(q^*_{n-k+j}\frac{dw^*_{n-k+j}}{dq_{n-k}} + w^*_{n-k+j}\frac{dq^*_{n-k+j}}{dq_{n-k}}) \\ &= R'(Q^n)\prod_{j=1}^{k}(1+\frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}) + \sum_{j=1}^{k}R'(Q^n)\prod_{m=1}^{k}(1+\frac{\partial q^*_{n-k+m}}{\partial x_{n-k+m}}) \\ &\quad -\sum_{j=1}^{k}w^*_{n-k+j}\frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}\prod_{m=1}^{j-1}(1+\frac{\partial q^*_{n-k+m}}{\partial x_{n-k+m}}) \\ &= (k+1)R'(Q^n)\prod_{j=1}^{k}(1+\frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}) - \sum_{j=1}^{k}w^*_{n-k+j}\frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}\prod_{m=1}^{j-1}(1+\frac{\partial q^*_{n-k+m}}{\partial x_{n-k+m}}) \ . \end{aligned}$$

Given the inverse demand function $w^*_{n-k}(x_{n-k},q_{n-k})$, the supplier chooses q_{n-k} to maximize her profit which is

$$\pi^n_S = w_1q_1 + \cdots + w^*_{n-k}q_{n-k} + w^*_{n-k+1}q^*_{n-k+1} + \cdots + w^*_nq^*_n$$

and the corresponding first-order derivative is

$$\begin{aligned} \frac{\partial\pi^n_S}{\partial q_{n-k}} &= w^*_{n-k} + \frac{\partial w^*_{n-k}}{\partial q_{n-k}}q_{n-k} + \frac{d}{dq_{n-k}}(w^*_{n-k+1}q^*_{n-k+1} + \cdots + w^*_nq^*_n) \\ &= R'(Q^n)\prod_{j=1}^{k}(1+\frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}) + \frac{\partial w^*_{n-k}}{\partial q_{n-k}}q_{n-k} \ , \end{aligned} \tag{B.4}$$

where the equality holds because w^*_{n-k} satisfies (B.3).

Now if $x_{n-k} \geq Q^{FB}$, we have $\frac{\partial\pi^n_S}{\partial q_{n-k}} = R'(Q^n)\prod_{j=1}^{k}(1+\frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}})+\frac{\partial w^*_{n-k}}{\partial q_{n-k}}q_{n-k} \leq R'(Q^n)\prod_{j=1}^{k}(1+\frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}) < 0$ for all $q_{n-k} > 0$. So the optimal solution is the boundary solution $q^*_{n-k} = 0$. If $x_{n-k} < Q^{FB}$, then $\frac{\partial\pi^n_S}{\partial q_{n-k}}|_{q_{n-k}=0} = R'(Q^n)\prod_{j=1}^{k}(1+\frac{\partial q^*_{n-k+j}}{\partial q_{n-k+j-1}}) > 0$ (by Lemma B.0.3), so the optimal solution must satisfy the first-order condition rather than the boundary solution, and the proof is completed. □

Proof of Theorem 3.2.2

(a) Consider $Q^n = q_1 + q_2^* + \cdots + q_n^*$ as a function of q_1. Lemma B.0.3 implies that Q^n strictly increases in q_1 because

$$\frac{\partial Q^n}{\partial q_1} = \prod_{j=1}^{n-1}(1 + \frac{\partial q_{j+1}^*}{\partial x_{j+1}}) > 0 \ .$$

Notice that the equilibrium total inventory $Q^{n-1,*}$ under the $(n-1)$-offer case is equivalent to the total inventory $Q^n_{q_1=0}$ under the n-offer case by forcing $q_1 = 0$. The equilibrium total inventory $Q^{n,*}$ under the n-offer case has $q_1 = q_1^* > 0$. Therefore, we conclude that $Q^{n,*} = Q^n_{q_1=q_1^*} > Q^n_{q_1=0} = Q^{n-1,*}$, and we have shown that $Q^{n-1,*} < Q^{n,*}$.

Finally, we would like to show that the equilibrium total inventory $Q^{n,*} \leq Q^{FB}$. We prove it by contradiction. Suppose $Q^{n,*} > Q^{FB}$, then there exists a certain $\hat{t}$, $1 \leq \hat{t} \leq n-1$, such that $x_{\hat{t}}^* \leq Q^{FB}$ but $x_{\hat{t}+1}^* > Q^{FB}$. The retailer's total profit is $\pi_R^{n,*} = R(Q^{n,*}) - w_1^* q_1^* - w_2^* q_2^* - \cdots - w_n^* q_n^*$.

We construct a new strategy for the retailer where $\hat{q}_j$ is the same as q_j^* except that (i) $\hat{q}_j(x_j) = 0$ whenever $x_j > Q^{FB}$ and (ii) at $j = \hat{t}$, we decrease the order quantity from $q_{\hat{t}}^*(x_{\hat{t}})$ to $\hat{q}_{\hat{t}}(x_{\hat{t}}) = Q^{FB} - x_{\hat{t}}$. Under this strategy, the retailer's order quantities will be $\{q_1^*, \ldots, q_{\hat{t}-1}^*, \hat{q}_{\hat{t}}, 0, \ldots, 0\}$ and the retailer's total profit will be $\hat{\pi}_R^n = R(Q^{FB}) - w_1^* q_1^* - w_2^* q_2^* - \cdots - w_{\hat{t}}^* \hat{q}_{\hat{t}}$. Thus,

$$\hat{\pi}_R^n - \pi_R^{n,*} = R(Q^{FB}) - R(Q^{n,*}) + w_{\hat{t}}^* q_{\hat{t}}^* + \cdots + w_n^* q_n^* - w_{\hat{t}}^* \hat{q}_{\hat{t}} \ .$$

Since the function $R(Q)$ strictly decreases when $Q \geq Q^{FB}$, we have $R(Q^{FB}) - R(Q^{n,*}) > 0$ as $Q^{n,*} > Q^{FB}$. Further, the supplier's wholesale price w_t^* must be non-negative and $q_{\hat{t}}^* > \hat{q}_{\hat{t}}$. Therefore, $w_{\hat{t}}^* q_{\hat{t}}^* + \cdots + w_n^* q_n^* - w_{\hat{t}}^* \hat{q}_{\hat{t}} > 0$. We found a strictly profitable unilateral deviation for the retailer which is a contradiction. In conclusion, we must have $Q^{n,*} \leq Q^{FB}$. □

We prove (b) by contradiction. Suppose the retailer's total profit, $\pi_R^{n,*}$, or the supplier's total profit, $\pi_S^{n,*}$, is not monotonically increasing. In other words, there exists some $\hat{n} > 1$, such that $\pi_R^{\hat{n},*} < \pi_R^{\hat{n}-1,*}$ and/or $\pi_S^{\hat{n},*} < \pi_S^{\hat{n}-1,*}$.

As we can see, the pre-order inventory, x_t, at each offer provides all necessary information to determine the equilibrium strategy. Therefore, in the $\hat{n}$-offer case, if the retailer (mistakenly) orders zero at the first offer, then the remaining game will have the same equilibrium outcome as the game with $\hat{n}-1$ offers.

Now suppose the retailer's equilibrium total profits satisfy $\pi_R^{\hat{n},*} < \pi_R^{\hat{n}-1,*}$. Under the $\hat{n}$-offer game, the retailer is able to unilaterally deviate to a strategy where he orders 0 in the first period but returns to his equilibrium strategy in later periods. By doing so, the retailer will get total profit $\pi_R^{\hat{n}-1,*}$. So there exists a strictly profitable unilateral deviation for the retailer, which is a contradiction. Therefore, we must have $\pi_R^{\hat{n},*} \geq \pi_R^{\hat{n}-1,*}$.

Similarly, suppose the supplier's equilibrium total profit satisfies $\pi_S^{\hat{n},*} < \pi_S^{\hat{n}-1,*}$. Under the

$\hat{n}$-offer game, the supplier can unilaterally deviate to a strategy where she uses the equilibrium strategy in periods $t \geq 2$ but in the first period she sets a wholesale price so high that the retailer's best response is to order 0. In particular, if the supplier proposes a wholesale price to be $\max R(Q)$, the retailer will definitely order nothing. Otherwise the retailer will surely get negative profit if he orders, since his profit is not greater than $R(Q) - w_1 q_1 < R(Q) - \max R(Q) q_1 < 0$. As a result, by applying this new strategy, the supplier can achieve a total profit $\pi_S^{\hat{n}-1,*}$. The supplier has a strictly profitable unilateral deviation, which is a contradiction. Therefore, we must have $\pi_S^{\hat{n},*} \geq \pi_S^{\hat{n}-1,*}$.

In conclusion, we proved that $\pi_R^{n,*} \geq \pi_R^{n-1,*}$ and $\pi_S^{n,*} \geq \pi_S^{n-1,*}$ for all $n \geq 2$. Furthermore, the supply chain profit $\pi^{n,*} = R(Q^{n,*})$ will strictly increase in $Q^{n,*}$ as $Q^{n,*} \leq Q^{FB}$. Therefore, Theorem 3.2.2(a) implies $\pi_T^{n,*} \neq \pi_T^{n-1,*}$. We conclude that $\pi_T^{n,*} > \pi_T^{n-1,*}$, i.e., the supply chain total profit is strictly increasing. □

Remark: An implicit assumption that was made is that $\max R(Q)$ is finite. However, we merely need the existence of a price w, high enough, for which the retailer will not order. Taking $\max R(Q)$ is only an example.

Proof of Theorem 3.2.3

By Theorem 3.2.2, the equilibrium total inventory level $Q^{n,*}$ strictly increases in n. Moreover, Theorem 3.2.2 guarantees that $Q^{n,*}$ is bounded from above by Q^{FB}. Therefore, the limit of $Q^{n,*}$ exists as n goes to infinity. We denote it as Q^*, i.e $\lim_{n\to+\infty} Q^{n,*} = Q^*$. By a similar proof as Theorem 3.2.2, we can show, on the equilibrium path, x_n^* strictly increases in n and is bounded from above by Q^{FB}, so its limit also exists, denoted as $\lim_{n\to+\infty} x_n^* = x^*$. Consequently, $\lim_{n\to+\infty} q_n^*(x_n^*) = \lim_{n\to+\infty} Q^{n,*} - x_n^* = Q^* - x^*$, i.e. the limit of $q_n^*(x_n^*)$ also exists. Finally, since the last period wholesale price is $w_n^* = R'(Q^{n,*})$, it will strictly decreases and bounded from below by 0. So the limit of $w_n^*(x_n^*)$ exists and is denoted as $\lim_{n\to+\infty} w_n^*(x_n^*) = w^*$

We now prove the theorem by contradiction. Suppose that $Q^* < Q^{FB}$. As n goes to infinity, the optimality condition (3.1) for the last period becomes $w^* = R'(Q^*) > 0$. Because the equilibrium supply chain profit $\pi_T^{n,*}$ is strictly increasing in n and bounded from above by the first-best supply chain profit π^{FB}, $\pi_T^{n,*}$ will converge to a constant when n goes to infinity, which also implies that

$$\lim_{n\to+\infty} \frac{d\pi_T^n}{dx_n} = 0 \ .$$

However, since $\pi_T^n = R(x_n + q_n)$ and $w_n = R'(x_n + q_n)$, we have

$$\frac{d\pi_T^n(x_n)}{dx_n} = R'(x_n + q_n^*)[1 + \frac{dq_n^*(x_n)}{dx_n}] = w_n^*(x_n)[1 + \frac{dq_n^*(x_n)}{dx_n}] \ .$$

Therefore

$$\begin{aligned}\lim_{n\to+\infty}\frac{d\pi_T^n(x_n)}{dx_n} &= \lim_{n\to+\infty} w_n^*(x_n)[1+\frac{dq_n^*(x_n)}{dx_n}] \\ &= w^*\lim_{n\to+\infty}[1+\frac{dq_n^*(x_n)}{dx_n}] \\ &= w^*\lim_{n\to+\infty}\frac{R''(x_n+q_n^*(x_n))}{2R''(x_n+q_n^*(x_n))+R^{(3)}(x_n+q_n^*(x_n))q_n^*(x_n)} ,\end{aligned}$$

The last equality follows from (B.1). Specifically, by Proposition 3.2.1, $q_n^*(x_n)$ is the solution of $R'(Q^n)+\frac{\partial w_n^*}{\partial q_n}q_n = R'(Q^n)+R''(Q^n)q_n = 0$. We apply the Implicit Function Theorem,

$$R''(Q^n)(1+\frac{dq_n^*(x_n)}{dx_n})+R^{(3)}(Q^n)(1+\frac{dq_n^*(x_n)}{dx_n})q_n^*(x_n)+R''(Q^n)\frac{dq_n^*(x_n)}{dx_n}=0 ,$$

which leads to

$$1+\frac{dq_n^*(x_n)}{dx_n}=\frac{R''(x_n+q_n^*(x_n))}{2R''(x_n+q_n^*(x_n))+R^{(3)}(x_n+q_n^*(x_n))q_n^*(x_n)} .$$

As we assume $w^*\neq 0$, in order to achieve $\lim_{n\to+\infty}\frac{d\pi_T^n}{dx_n}=0$, we should have

$$\lim_{n\to+\infty}\frac{R''(x_n+q_n(x_n))}{2R''(x_n+q_n(x_n))+R^{(3)}(x_n+q_n(x_n))q(x_n)}=\frac{R''(x^*+q^*)}{2R''(x^*+q^*)+R^{(3)}(x^*+q^*)q^*}=0 .$$

However, if the equality holds, $R''(x^*+q^*)=0$, the optimality condition (3.2) for the last period becomes $R'(x^*+q^*)+R''(x^*+q^*)q^* = R'(x^*+q^*)=0$, implying that $x^*+q^*=Q^{FB}$. But $R''(Q^{FB})<0$ and as a result, $\lim_{n\to+\infty}\frac{d\pi_T^n}{dx_n}\neq 0$, which leads to a contradiction. In conclusion, we must have $\lim_{n\to+\infty}Q^{n,*}=Q^{FB}$. □

Proof of Proposition 3.2.4

By (3.1), we have

$$\begin{aligned}w_{n-k}^* &= (k+1)R'(Q^n)\prod_{j=1}^{k}(1+\frac{\partial q_{n-k+j}^*}{\partial x_{n-k+j}})-\sum_{j=1}^{k}w_{n-k+j}^*\frac{\partial q_{n-k+j}^*}{\partial x_{n-k+j}}\prod_{m=1}^{j-1}(1+\frac{\partial q_{n-k+m}^*}{\partial x_{n-k+m}}) \\ w_{n-k+1}^* &= kR'(Q^n)\prod_{j=1}^{k-1}(1+\frac{\partial q_{n-k+1+j}^*}{\partial x_{n-k+1+j}})-\sum_{j=1}^{k-1}w_{n-k+1+j}^*\frac{\partial q_{n-k+1+j}^*}{\partial x_{n-k+1+j}}\prod_{m=1}^{j-1}(1+\frac{\partial q_{n-k+1+m}^*}{\partial x_{n-k+1+m}}) .\end{aligned}$$

Therefore,

$$
\begin{aligned}
& w^*_{n-k} - w^*_{n-k+1} \\
= \; & (1 + \frac{\partial q^*_{n-k+1}}{\partial x_{n-k+1}})[(k+1)R'(Q^n)\prod_{j=2}^{k}(1 + \frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}) - w^*_{n-k+1} - \sum_{j=2}^{k} w^*_{n-k+j}\frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}\prod_{m=1}^{j-1}(1 + \frac{\partial q^*_{n-k+m}}{\partial x_{n-k+m}})] \\
= \; & (1 + \frac{\partial q^*_{n-k+1}}{\partial x_{n-k+1}})[(k+1)R'(Q^n)\prod_{j=2}^{k}(1 + \frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}) - kR'(Q^n)\prod_{j=2}^{k}(1 + \frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}})] \\
= \; & R'(Q^n)\prod_{j=1}^{k}(1 + \frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}) \geq 0 \; , \qquad \text{(B.5)}
\end{aligned}
$$

where the inequality holds by Lemma B.0.3. □

Proof of Proposition 3.2.5

Since (B.5) provides a recursive equation for the equilibrium wholesale price, we have

$$
\begin{aligned}
w^*_{n-k} &= R'(Q^n)\prod_{j=1}^{k}(1 + \frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}}) + R'(Q^n)\prod_{j=1}^{k-1}(1 + \frac{\partial q^*_{n-k+1+j}}{\partial x_{n-k+1+j}}) + \cdots + R'(Q^n)(1 + \frac{\partial q^*_n}{\partial x_n}) + R'(Q^n) \\
&= R'(Q^n)\{1 + \sum_{m=1}^{n}\prod_{j=m}^{k}(1 + \frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}})\} \\
&= R'(Q^n)\alpha_{n-k} \; .
\end{aligned}
$$

According to Proposition 3.2.1, we have

$$
\begin{aligned}
\frac{w^*_{n-k}}{w^*_{n-k+1}} &= \frac{\alpha_{n-k}}{\alpha_{n-k+1}} \\
\frac{q^*_{n-k+1}}{q^*_{n-k}} &= \frac{-R'(Q^n)\prod_{j=1}^{k-1}(1 + \frac{\partial q^*_{n-k+1+j}}{\partial x_{n-k+1+j}})/\frac{\partial w^*_{n-k+1}}{\partial q_{n-k+1}}}{-R'(Q^n)\prod_{j=1}^{k}(1 + \frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}})/\frac{\partial w^*_{n-k}}{\partial q_{n-k}}} = \frac{\frac{\partial w^*_{n-k}}{\partial q_{n-k}}}{(1 + \frac{q^*_{n-k+1}}{x_{n-k+1}})\frac{\partial w^*_{n-k+1}}{\partial q_{n-k+1}}} \; .
\end{aligned}
$$

As $w^*_{n-k} = \frac{\alpha_{n-k}}{\alpha_{n-k+1}} w^*_{n-k+1}$, we further obtain

$$
\begin{aligned}
\frac{q^*_{n-k+1}}{q^*_{n-k}} &= \frac{\frac{\partial \frac{\alpha_{n-k}}{\alpha_{n-k+1}} w^*_{n-k+1}}{\partial q_{n-k}}}{(1+\frac{q^*_{n-k+1}}{x_{n-k+1}})\frac{\partial w^*_{n-k+1}}{\partial q_{n-k+1}}} \\
&= \frac{\alpha_{n-k}}{\alpha_{n-k+1}} \frac{\frac{\partial w^*_{n-k+1}}{\partial q_{n-k}}}{(1+\frac{q^*_{n-k+1}}{x_{n-k+1}})\frac{\partial w^*_{n-k+1}}{\partial q_{n-k+1}}} + \frac{\partial(\frac{\alpha_{n-k}}{\alpha_{n-k+1}})}{\partial q_{n-k}} \frac{w^*_{n-k+1}}{(1+\frac{q^*_{n-k+1}}{x_{n-k+1}})\frac{\partial w^*_{n-k+1}}{\partial q_{n-k+1}}} \\
&= \frac{\alpha_{n-k}}{\alpha_{n-k+1}} + \frac{\partial(\frac{\alpha_{n-k}}{\alpha_{n-k+1}})}{\partial q_{n-k}} \frac{w^*_{n-k+1}}{(1+\frac{q^*_{n-k+1}}{x_{n-k+1}})\frac{\partial w^*_{n-k+1}}{\partial q_{n-k+1}}} \\
&= \frac{w^*_{n-k}}{w^*_{n-k+1}} + \frac{\partial(\frac{\alpha_{n-k}}{\alpha_{n-k+1}})}{\partial q_{n-k}} \frac{w^*_{n-k+1}}{(1+\frac{q^*_{n-k+1}}{x_{n-k+1}})\frac{\partial w^*_{n-k+1}}{\partial q_{n-k+1}}} \, .
\end{aligned}
$$

As a result, $w^*_{n-k+1}q^*_{n-k+1} - w^*_{n-k}q^*_{n-k}$, or, equivalently, whether $\frac{q^*_{n-k+1}}{q^*_{n-k}} - \frac{w^*_{n-k}}{w^*_{n-k+1}}$ is positive or negative completely depends on the term $\frac{\partial(\frac{\alpha_{n-k}}{\alpha_{n-k+1}})}{\partial q_{n-k}} \frac{w^*_{n-k+1}}{(1+\frac{q^*_{n-k+1}}{x_{n-k+1}})\frac{\partial w^*_{n-k+1}}{\partial q_{n-k+1}}}$. Since $w^*_{n-k+1} \geq 0$, $(1+\frac{q^*_{n-k+1}}{x_{n-k+1}}) > 0$ and $\frac{\partial w^*_{n-k+1}}{\partial q_{n-k+1}} < 0$, we conclude that $\frac{q^*_{n-k+1}}{q^*_{n-k}} - \frac{w^*_{n-k}}{w^*_{n-k+1}}(>,=,<)0$ if and only if $\frac{\partial(\frac{\alpha_{n-k}}{\alpha_{n-k+1}})}{\partial q_{n-k}}(<,=,>)0$. □

Proof of Theorem 3.2.6

(1) Exponential demand: We will show that at each offer $n-k$, the equilibrium strategy is as follows:

$$
\begin{aligned}
w^*_{n-k}(x_{n-k}, q_{n-k}) &= (k+1)pe^{-\lambda Q^n} && \text{(B.6)} \\
q^*_{n-k}(x_{n-k}) &= \frac{1}{\lambda(k+1)} \, . && \text{(B.7)}
\end{aligned}
$$

Under exponential demand, the revenue function is $R(Q) = p\int_0^Q e^{-\lambda\xi}d\xi$, so $R'(Q) = pe^{-\lambda Q}$ and $R''(Q) = p\lambda e^{-\lambda Q}$.

According to Proposition 3.2.1, at the last offer, $w^*_n(x_n, q_n) = R'(Q^n) = pe^{-\lambda Q^n}$ and q^*_n is the solution of $pe^{-\lambda Q^n} - p\lambda e^{-\lambda Q^n} q_n = 0$. Therefore, $q^*_n(x_n) = 1/\lambda$. Suppose these results hold for offers $n-k+j$, with $j = 1, \ldots, k$, and consider offer $n-k$. By the inductive assumption, q^*_{n-k+j} is a constant for all $j \geq 1$, hence $\frac{\partial q^*_{n-k+j}}{\partial x_{n-k+j}} = 0$. Proposition 3.2.1 leads to

$$
\begin{aligned}
&w^*_{n-k}(x_{n-k}, q_{n-k}) = (k+1)pe^{-\lambda Q^n} \\
&q^*_{n-k} \text{ solves } pe^{-\lambda Q^n} - p(k+1)\lambda e^{-\lambda Q^n} q_{n-k} = 0 \, ,
\end{aligned}
$$

hence $q^*_{n-k} = \frac{1}{\lambda(k+1)}$ and we have proven (B.6) and (B.7).

Finally, on the equilibrium path, $q^*_j = \frac{1}{\lambda(n-j+1)}$, so $Q^{n,*} = \sum_{l=1}^n 1/l$ and $w^*_j = (n-j+1)pe^{-\lambda\sum_{l=1}^n 1/l}$. □

(2) Uniform demand: Instead of directly using Proposition 3.2.1, we prove the result by induction. We will show that at each offer j, given any initial inventory x_j, the equilibrium strategy satisfies the following equation:

$$\begin{aligned} q^*_j(x_j) &= \frac{M}{2(n-j+1)} - \frac{x_j}{2(n-j+1)} && \text{(B.8)} \\ w^*_j(x_j, q^*_j(x_j)) &= \beta_j[1-\frac{x_j}{M}]p \ . && \text{(B.9)} \end{aligned}$$

Under uniform demand, the revenue function is $R(Q) = p(Q - Q^2/2M)$. So $R'(Q) = p(1-Q/M)$ and $R''(Q) = -p/M$. According to Proposition 3.2.1, at the last offer, $w^*_n(x_n, q_n) = R'(Q^n) = p(1-Q^n/M)$ and q^*_n is the solution of $p(1-Q^n/M) - pq_n/M = 0$. Hence, $q^*_n = \frac{M}{2} - \frac{x_n}{2}$. For simplicity we denote $w^*_n(x_n, q^*_n(x_n))$ as w^*_n, and note that $w^*_n = \frac{1}{2}[1-\frac{x_n}{M}]p$.

Suppose the result holds for offer $j \geq k+1$, and consider the offer k. For $j \geq k+1$, q^*_j and w^*_j satisfy the following recursive equations:

$$\begin{aligned} q^*_{j+1} &= \frac{M}{2(n-j)} - \frac{x_{j+1}}{2(n-j)} = \frac{M}{2(n-j)} - \frac{x_j + q^*_j}{2(n-j)} = \frac{M}{2(n-j)} - \frac{x_j + \frac{M}{2(n-j+1)} - \frac{x_j}{2(n-j+1)}}{2(n-j)} \\ &= \frac{2(n-j)+1}{2(n-j)}[\frac{M}{2(n-j+1)} - \frac{x_j}{2(n-j+1)}] = \frac{2(n-j)+1}{2(n-j)}q^*_j \ , \\ w^*_{j+1} &= \beta_{j+1}[1-\frac{x_{j+1}}{M}]p = \beta_{j+1}[1-\frac{x_j+q^*_j}{M}]p = \beta_{j+1}[1-\frac{x_j + \frac{M}{2(n-j+1)} - \frac{x_j}{2(n-j+1)}}{M}]p \\ &= \frac{2(n-j)+1}{2(n-j+1)}\beta_{j+1}[1-\frac{x_j}{M}]p = \frac{2(n-j)}{2(n-j)+1}\beta_j[1-\frac{x_j}{M}]p = \frac{2(n-j)}{2(n-j)+1}w^*_j \ . \end{aligned}$$

Moreover, the total inventory level Q^n can be expressed as

$$\begin{aligned} Q^n &= x_n + q^*_n = \frac{1}{2}(M+x_n) = \frac{1}{2}(M + x_{n-1} + q^*_{n-1}) = \frac{1}{2}(M + x_{n-1} + \frac{M - x_{n-1}}{4}) \\ &= M[\frac{1}{2} + \frac{1}{2*4}] + \frac{1}{2}*\frac{3}{4}x_{n-1} = M[\frac{1}{2} + \frac{1}{2*4}] + \frac{1}{2}*\frac{3}{4}(x_{n-2} + q^*_{n-2}) \\ &= \ldots \\ &= M\sum_{i=k+1}^{n}\frac{1}{2(n-i+1)}\prod_{l=i+1}^{n}\frac{2(n-l)+1}{2(n-l+1)} + \prod_{l=k+1}^{n}\frac{2(n-l)+1}{2(n-l+1)}x_{k+1} \\ &= A_{k+1} * M + B_{k+1} * (x_k + q_k) \ , \end{aligned}$$

where $A_{k+1} = \sum_{i=k+1}^n \frac{1}{2(n-i+1)}\prod_{l=i+1}^n \frac{2(n-l)+1}{2(n-l+1)}$ and $B_{k+1} = \prod_{l=k+1}^n \frac{2(n-l)+1}{2(n-l+1)}$.

We next compute the equilibrium for period k. The retailer's total profit is equal to

$$\begin{aligned}
\pi_R^n &= p(Q^n - (Q^n)^2/2M) - w_1 q_1 - \cdots - w_k q_k - w_{k+1}^* q_{k+1}^* - \cdots - w_n^* q_n^* \\
&= p[A_{k+1} * M + B_{k+1} * (x_k + q_k) - \frac{(A_{k+1} * M + B_{k+1} * (x_k + q_k))^2}{2M}] \\
&\quad - w_1 q_1 - \cdots - w_k q_k - (n-k) w_{k+1}^* q_{k+1}^* \\
&= p(A_{k+1} * M + B_{k+1} * (x_k + q_k) - \frac{(A_{k+1} * M + B_{k+1} * (x_k + q_k))^2}{2M}) \\
&\quad - w_1 q_1 - \cdots - w_k q_k - \beta_{k+1} p \frac{(M - (x_k + q_k))^2}{2M} \ ,
\end{aligned}$$

where the third equality follows the observation that $w_{k+1}^* q_{k+1}^* = w_{k+2}^* q_{k+2}^* = \cdots = w_n^* q_n^* = \beta_{k+1} p \frac{(M-(x_k+q_k))^2}{2(n-k)M}$. Hence, the first-order condition is

$$\frac{\partial \pi_R^n}{\partial q_k} = p(1 - \frac{A_{k+1} * M + B_{k+1} * (x_k + q_k)}{M}) B_{k+1} - w_k + \beta_{k+1} p \frac{M - (x_k + q_k)}{M} = 0 \ .$$

Therefore, the supplier's inverse demand function is

$$w_k^*(x_k, q_k) = p(1 - \frac{A_{k+1} * M + B_{k+1} * (x_k + q_k)}{M}) B_{k+1} + \beta_{k+1} p \frac{M - (x_k + q_k)}{M} \ .$$

The supplier needs to determine an order quantity q_k to maximize her total profit, given by

$$\begin{aligned}
\pi_S^n &= w_1 q_1 + \cdots + w_k^* q_k + w_{k+1}^* q_{k+1}^* + \cdots + w_n^* q_n^* \\
&= w_1 q_1 + \cdots + w_{k-1} q_{k-1} + [p(1 - \frac{A_{k+1} * M + B_{k+1} * (x_k + q_k)}{M}) B_{k+1} + \beta_{k+1} p \frac{M - (x_k + q_k)}{M}] q_k \\
&\quad + \beta_{k+1} p \frac{(M - (x_k + q_k))^2}{2M} \ .
\end{aligned}$$

The first-order condition is

$$\frac{\partial \pi_S^k}{\partial q_k} = p(1 - \frac{A_{k+1} * M + B_{k+1} * (x_k + q_k)}{M}) B_{k+1} - [\frac{p B_{k+1}^2}{M} + \frac{r \beta_{k+1}}{M}] q_k = 0 \ .$$

As a result,

$$q_k^*(x_k) = \frac{M(1 - A_{k+1}) B_{k+1}}{2 B_{k+1}^2 + \beta_{k+1}} - \frac{B_{k+1}^2 x_k}{2 B_{k+1}^2 + \beta_{k+1}} \ .$$

However, we have $\beta_{k+1} = 2(n-k) B_{k+1}^2$ and $A_{k+1} + B_{k+1} = 1$, which finally leads to

$$q_k^*(x_k) \quad = \quad \frac{M}{2(n-k+1)} - \frac{x_k}{2(n-k+1)} \ .$$

The corresponding wholesale price is

$$\begin{aligned} w_k^* = w_k^*(x_k, q_k^*(x_k)) &= p(1 - \frac{A_{k+1} * M + B_{k+1} * (x_k + q_k^*)}{M})B_{k+1} + \beta_{k+1} p \frac{M - (x_k + q_k^*)}{M} \\ &= [\frac{pB_{k+1}^2}{M} + \frac{p\beta_{k+1}}{M}]q_k^* + \beta_{k+1} p \frac{M - (x_k + q_k^*)}{M} \\ &= \frac{pB_{k+1}^2}{M}(\frac{M}{2(n-k+1)} - \frac{x_k}{2(n-k+1)}) + \beta_{k+1} p \frac{M - x_k}{M} \\ &= p(\frac{B_{k+1}^2}{2(n-k+1)} + \beta_{k+1})[1 - \frac{x_k}{M}] \\ &= p\frac{(2(n-k+1))^2}{2(n-k+1)2(n-k)}\beta_{k+1}[1 - \frac{x_k}{M}] \\ &= p\beta_k[1 - \frac{x_k}{M}] \ , \end{aligned}$$

and the proof is complete. □

(3) Linear demand: Under linear (price-sensitive) demand, the revenue function is $R(Q) = (a - bQ)Q$. Note that if we re-write $a = p$ and $b = p/2M$, the revenue function becomes $R(Q) = r(Q - Q^2/2M)$, the same as in the uniform demand case. Therefore, by parameter transformation, the problem under linear demand is equivalent to the one under uniform demand. □

Appendix C

Proofs of Results in Chapter 4

Proof of Lemma 4.4.1

The proof is almost the same as Zhang et al. [57] with little modification.

(1) $\Rightarrow$ direction: Suppose $\{q_1(x_1), s_1(x_1)\}$ satisfies the global IC constraint. By the envelop theorem, as analyzed in the main text, we can show $q_1(x_1)$ satisfies the local IC constraint.

Because $\frac{\partial^2 v_1(x_1+q_1)}{\partial x_1 \partial q_1} = v_1''(x_1+q_1) = \begin{cases} -\lambda r e^{-\lambda(x_1+q_1)} & x_1+q_1>0 \\ 0 & x_1+q_1<0 \end{cases}$. The function $v_1(x_1+q_1)-s_1(x_1)$ has decreasing differences and thereby $q_1(x_1)$ must be weakly decreasing in x_1. (See Topkis [55]) $\square$

$\Leftarrow$ direction: Suppose $q_1(x_1)$ satisfies the local IC constraint and is weakly decreasing in x_1. Without loss of generality, we only consider the case $x_1 > \hat{x}_1$. The case $x_1 < \hat{x}_1$ is similar.

$$
\begin{aligned}
& u_1(x_1) - [v_1(x_1+q_1(\hat{x}_1)) - s_1(\hat{x}_1)] \\
= \;& [u_1(\hat{x}_1) + \int_{\hat{x}_1}^{x_1} v_1'(\xi+q_1(\xi))d\xi] - [u_1(\hat{x}_1) + v_1(x_1+q_1(\hat{x}_1)) - v_1(\hat{x}_1+q_1(\hat{x}_1))] \\
= \;& [u_1(\hat{x}_1) + \int_{\hat{x}_1}^{x_1} v_1'(\xi+q_1(\xi))d\xi] - [u_1(\hat{x}_1) + \int_{\hat{x}_1}^{x_1} v_1'(\xi+q_1(\hat{x}_1))d\xi] \\
= \;& \int_{\hat{x}_1}^{x_1} [v_1'(\xi+q_1(\xi)) - v_1'(\xi+q_1(\hat{x}_1))]d\xi \geq 0
\end{aligned}
$$

The last inequality holds because $q_1(\xi) \leq q_1(\hat{x}_1)$ and v_1' is a decreasing function. So the global IC constraint holds. $\square$

(2) The IC constraint says $v_1(x_1+q_1(x_1))-s_1(x_1)-v_1(x_1) \geq v_1(x_1+q_1(y_0))-s_1(y_0)-v_1(x_1)$. In addition, since $v_1'' \leq 0$, i.e. v_1 has decreasing differences, we have $v_1(x_1+q_1(y_0)) - v_1(x_1) \geq v_1(y_0+q_1(y_0))-v_1(y_0)$. Combining the two inequalities, we get $v_1(x_1+q_1(x_1))-s_1(x_1)-v_1(x_1) \geq v_1(y_0+q_1(y_0))-s_1(y_0)-v_1(y_0) \geq 0$, for all $x_1 < y_0$. The supplier wants to maximize her expected profit, therefore she should let $v_1(y_0+q_1(y_0)) - s_1(y_0) - v_1(y_0) = 0$ at optimum. In conclusion, the IR constraint must be binding at y_0 and redundant at $x_1 < y_0$. $\square$

Proof of Theorem 4.4.2

Because the IR constraint must be binding at y_0, we replace $s_1(x_1)$ with $v_1(x_1+q_1(x_1)) - u_1(x_1) = v_1(x_1+q_1(x_1)) - v_1(y_0) + \int_{x_1}^{y_0} v_1'(z+q_1(z))dz$ and rewrite the supplier's objective function as

$$\int_{-\infty}^{y_0} [s_1(x_1) - cq_1(x_1)]dG_1(x_1) = \int_{-\infty}^{y_0} J_1(q_1(x_1)|x_1)g_1(x_1)dx_1 - \underline{u}_1(y_0)$$

The detailed analysis is in the main text. And $J_1(q_1|x_1) = v_1(x_1+q_1) - cq_1 + v_1'(x_1+q_1)\frac{G_1(x_1)}{g_1(x_1)}$ is the so-called virtual surplus. The optimal quantity plan $q_1^*(x_1)$ maximizes $J_1(q_1|x_1)$. We look at the derivative of $J_1(q_1|x_1)$ with respect to q_1.

$$\begin{aligned}\frac{\partial J_1(q_1|x_1)}{\partial q_1} &= v_1'(x_1+q_1) - c + v_1''(x_1+q_1)\frac{G_1(x_1)}{g_1(x_1)} \\ &= \begin{cases} re^{-\lambda(x_1+q_1)} - c - \lambda re^{-\lambda(x_1+q_1)}\frac{G_1(x_1)}{g_1(x_1)} & x_1+q_1 > 0 \\ r - c & x_1+q_1 < 0 \end{cases}\end{aligned}$$

Since we assume $x_1 = y_0 - D_0$ for some constant y_0, we have CDF $G_1(x_1) = e^{-\lambda(y_0-x_1)}$ and PDF $g_1(x_1) = \lambda e^{-\lambda(y_0-x_1)}$. Therefore, $G_1(x_1)/g_1(x_1) = \lambda^{-1}$. When $x_1+q_1 > 0$, $\partial J_1(q_1|x_1)/\partial q_1 = -c < 0$. But when $x_1+q_1 < 0$, $\partial J_1(q_1|x_1)/\partial q_1 = r - c > 0$. It indicates that $J_1(q_1|x_1)$ increases when $x_1+q_1 < 0$ but decreases when $x_1+q_1 > 0$. As a result, the optimal quantity should be $q_1^*(x_1) = \max\{0, -x_1\}$, or equivalently $y_1^*(x_1) = \max\{0, x_1\}$. The corresponding payment is solved by

$$\begin{aligned} s_1^*(x_1) &= v_1(x_1+q_1^*(x_1)) - u_1^*(x_1) \\ &= v_1(x_1+q_1^*(x_1)) - v_1(y_0) + \int_{x_1}^{y_0} v_1'(\xi+q_1^*(\xi))d\xi \\ &= \begin{cases} v_1(0) - v_1(y_0) + \int_{x_1}^{0} v_1'(0)d\xi + \int_0^{y_0} v_1'(\xi)d\xi, & x_1 < 0 \\ v_1(x_1) - v_1(y_0) + \int_{x_1}^{y_0} v_1'(\xi)d\xi, & x_1 \geq 0 \end{cases} \\ &= \begin{cases} -rx_1, & x_1 < 0 \\ 0, & x_1 \geq 0. \end{cases} \end{aligned}$$

□

Proof of Proposition 4.5.2

We prove the result by contradiction. Suppose under the optimal contract, there exists "bump" at $[x_1^-, x_1^+]$ in the retailer's profit-to-go function. Now we focus on the interval $[x_1^-, x_1^+]$ and we solve a subproblem $\mathcal{P}[x_1^-, x_1^+]$, where we keep the IC constraint unchanged but we let the IR constraint binding at the two endpoints x_1^- and x_1^+ and we ignore the IR constraint at points in between:

$$(\mathcal{P}[x_1^-, x_1^+]) \qquad \max_{s_1(x_1), y_1(x_1)} \int_{x_1^-}^{x_1^+} \{s_1(x_1) - cy_1(x_1) + cx_1 + \delta\Pi_2(y_1(x_1))\} dG_1(x_1)$$

$$s.t. \qquad \mu_1(x_1 + q_1(x_1)) - s_1(x_1) \geq \mu_1(x_1 + q_1(\hat{x}_1)) - s_1(\hat{x}_1), x_1, \hat{x}_1 \in [x_1^-, x_1^+]$$

$$\mu_1(y_1(x_1^-)) - s_1(x_1^-) = \underline{u}_1(x_1^-) \text{ and } \mu_1(y_1(x_1^+)) - s_1(x_1^+) = \underline{u}_1(x_1^+)$$

Suppose the hypothesis holds, the optimal solution of $\mathcal{P}[x_1^-, x_1^+]$ will automatically satisfy the IR constraint at $x_1 \in (x_1^-, x_1^+)$.

Next, we replace the global IC constraints by the local IC constraints $u_1'(x_1) = \mu_1'(y_1(x_1))$ and the monotonicity condition $y_1'(x_1) \leq 1$. Then we rewrite $u_1(x_1) = u_1(x_1^-) + \int_{x_1^-}^{x_1} \mu_1'(y_1(\xi))d\xi = \underline{u}_1(x_1^-) + \int_{x_1^-}^{x_1} \mu_1'(y_1(\xi))d\xi$ and replace $s_1(x_1)$ with $\mu_1(x_1 + q_1(x_1)) - u_1(x_1)$ in the objective function. By doing so, we obtain the virtual surplus anchoring at the bottom endpoint x_1^-:

$$J_1(y_1|x_1) = cx_1 - cy_1 + \mu_1(y_1) + \delta\Pi_2(y_1) - \mu_1'(y_1)\tfrac{G_1(x_1^+) - G_1(x_1)}{g_1(x_1)} \tag{C.1}$$

As a result, the subproblem $\mathcal{P}[x_1^-, x_1^+]$ can be re-formulated as follows:

$$(\mathcal{P}[x_1^-, x_1^+]) \qquad \max_{y_1(x_1)} \int_{x_1^-}^{x_1^+} J(y_1(x_1)|x_1) g_1(x_1) dx_1$$

$$s.t. \begin{cases} u_1'(x_1) = \mu_1'(y_1(x_1)), \quad x_1 \in [x_1^-, x_1^+] \\ u_1(x_1^-) = \underline{u}_1(x_1^-) \text{ and } u_1(x_1^+) = \underline{u}_1(x_1^+) \\ y_1(x_1) \geq x_1 \text{ and } y_1'(x_1) \leq 1 \quad x_1 \in [x_1^-, x_1^+] \end{cases}$$

Following the standard notations in the optimal control literature, we define $x_1 \to t$ as the time; $u_1(x_1) \to x(t)$ as the state variable; $y_1(x_1) \to u(t)$ as the control variable; $\mu_1'(y_1) \to g(t, x, u)$ as the state transition function; $J(y_1(x_1)|x_1)g_1(x_1) \to f(t, x, u)$ as the objective function; $x_1^- \to t_0$ and $x_1^+ \to t_f$ as the initial and final time; $\underline{u}_1(x_1^-) \to x_0$ and $\underline{u}_1(x_1^+) \to x_f$ as the initial and final state. Therefore, the subproblem $\mathcal{P}[x_1^-, x_1^+]$ is translated into an optimal control problem with two fixed endpoints and two constraints on the control variable $y_1'(x_1) \leq 1$ and $y_1(x_1) \geq x_1$. The corresponding Hamiltonian should be $H(y_1|x_1, \eta) = J(y_1|x_1)g_1(x_1) + \eta(x_1)\mu_1'(y_1)$ and the Lagrangian should be $\mathcal{L}(y_1|x_1, \eta, \rho) = H(y_1|x_1, \eta) + \rho_1(y_1 - x_1) + \rho_2(y_1'(x_1) - 1)$.

The maximum principle requires that the optimal control $y_1^*(x_1)$ and the optimal state variable $u_1^*(x_1)$ should satisfy the following conditions:

(1) Feasibility: $\dot{u}_1^*(x_1) = \mu_1'(y_1^*(x_1))$, $u_1^*(x_1^-) = \underline{u}_1(x_1^-)$ and $u_1^*(x_1^+) = \underline{u}_1(x_1^+)$; $y_1^*(x_1) \geq x_1$ and $\dot{y}_1^*(x_1) \leq 1$.

(2) Adjoint equation for η: η should satisfy $\begin{cases} \eta'(x_1) = \frac{\partial \mathcal{L}}{\partial u_1} = 0 \\ \eta(x_1^+) = \beta \text{ for some constant } \beta \end{cases}$. Clearly, η

is a constant, i.e. $\eta(x_1) = \beta$ for all $x_1 \in [x_1^-, x_1^+]$.

(3) Condition for Lagrange multiplier: ρ are such that $\frac{\partial \mathcal{L}}{\partial y_1}|_{y_1=y_1^*} = 0$; $\rho \geq 0$ and satisfy the complementary slackness condition $\rho_1(y_1^*(x_1) - x_1) = 0$, $\rho_2(\dot{y}_1^*(x_1) - 1)) = 0$.

(4) Hamiltonian maximization condition: $y_1^*(x_1)$ maximizes the Hamiltonian $H(y_1^*(x_1)|x_1, \eta) \geq H(y_1|x_1, \eta)$ for all y_1.

The analysis proceeds as follows. We will first characterize the optimal control $y_1^*(x_1)$. Then we will show under the optimal control, the IR constraint at the two endpoints can not be binding simultaneously. This will lead to a contradiction and thereby we will conclude there is no "bump".

We start with characterizing the optimal control $y_1^*(x_1)$. First of all, we look at the first-order derivative of the Hamiltonian:

$$\begin{aligned}\frac{\partial H}{\partial y_1} &= \frac{\partial J_1(y_1|x_1)}{\partial y_1} g_1(x_1) + \beta \mu_1''(y_1) \\ &= \begin{cases} g_1(x_1)\{b + \delta c - c\}, & y_1 < 0 \\ g_1(x_1)\{-h - c + \delta c e^{-\lambda y_1} + (b + h + \delta r \lambda y_1) e^{-\lambda y_1} e^{\lambda(y_0 - x_1)} (G_1(x_1^+) - \beta)\}. & y_1 > 0 \end{cases}\end{aligned} \tag{C.2}$$

Clearly, when $y_1 < 0$, we have $\frac{\partial H}{\partial y_1} > 0$ because $b > c(1-\delta)$. The function H increases in y_1 when y_1 is negative. However, when $y_1 > 0$, the sign of $\frac{\partial H}{\partial y_1}$ is determined by the function

$$\varphi(y_1|x_1, \beta) = -h - c + \delta c e^{-\lambda y_1} + (b + h + \delta r \lambda y_1) e^{-\lambda y_1} e^{\lambda(y_0 - x_1)} (G_1(x_1^+) - \beta) \tag{C.3}$$

In other words, the first order condition $\frac{\partial H}{\partial y_1} = 0$ is equivalent to $\varphi(y_1|x_1, \beta) = 0$.

One observation is that β must be smaller than $G_1(x_1^+)$ under the optimal control. If not, $\beta \geq G_1(x_1^+)$, we have $\varphi(y_1|x_1, \beta) \leq -h - c + \delta c e^{-\lambda y_1} \leq -h - c + \delta c < 0$. Therefore, the function H decreases in y_1 when y_1 is positive. Correspondingly, the optimal control will be $y^*(x_1) = \max\{0, x_1\}$. Notice that $y^*(x_1) < y_1^R(x_1)$ for all x_1. We obtain:

$$\begin{aligned} u_1^*(x_1^+) - \underline{u}_1(x_1^+) &= [u_1^*(x_1^-) + \int_{x_1^-}^{x_1^+} \mu_1'(y_1^*(z))dz] - [\underline{u}_1(x_1^-) + \int_{x_1^-}^{x_1^+} \mu_1'(y_1^R(z))dz] \\ &= \int_{x_1^-}^{x_1^+} (\mu_1'(y_1^*(z)) - \mu_1'(y_1^R(z))dz < 0 \end{aligned}$$

where the equality holds because $u_1^*(x_1^-) = \underline{u}_1(x_1^-)$ and the inequality holds because $y^*(x_1) < y_1^R(x_1)$. Hence, the constraint $u_1^*(x_1^+) = \underline{u}_1(x_1^+)$ can not be satisfied, which is a contradiction. From now on, we only consider the case $\beta < G_1(x_1^+)$.

Another observation is that there does not exist any constant β such that $y_1^*(x_1) = y^R(x_1)$ for all $x_1 \in [x_1^-, x_1^+]$. This can be easily checked by comparing the definition of y_1^R and the first order condition $\frac{\partial H}{\partial y_1} = 0$.

Now we would like to show that the optimal control $y_1^*(x_1)$ will be a decreasing function of x_1. We examine the structure of $\varphi(y_1|x_1,\beta)$. First of all, φ, as a function of y_1, will decrease in x_1, i.e. $\varphi(y_1|x_1,\beta) > \varphi(y_1|\tilde{x}_1,\beta)$ for all y_1 if $x_1 < \tilde{x}_1$. It is because

$$\frac{\partial\varphi(y_1|x_1,\beta)}{\partial x_1} = -\lambda(b+h+\delta r\lambda y_1)e^{-\lambda y_1}e^{\lambda(y_0-x_1)}(G_1(x_1^+)-\beta) < 0 \tag{C.4}$$

where the inequality holds in that $\beta < G_1(x_1)$.

Secondly, we have

$$\frac{\partial\varphi(y_1|x_1,\beta)}{\partial y_1} = \lambda e^{-\lambda y_1}\{-\delta c + [\delta r - (b+h+\delta r\lambda y_1)]e^{\lambda(y_0-x_1)}(G_1(x_1^+)-\beta)\}. \tag{C.5}$$

Given x_1 and β, there are two possible scenarios: either (i) $\frac{\partial\varphi(y_1|x_1,\beta)}{\partial y_1}$ is always negative or (ii) $\frac{\partial\varphi(y_1|x_1,\beta)}{\partial y_1}$ is first positive and then becomes negative. It implies that $\varphi(y_1|x_1,\beta)=0$ has most two solutions. More preciously, if $\varphi(y_1|x_1,\beta)=0$ has no solution, the optimal control $y_1^*(x_1) = \max_{\{} 0, x_1\}$; if $\varphi(y_1|x_1,\beta)=0$ has unique solution, the optimal control $y_1^*(x_1)$ is the maximum of this solution and x_1; and if $\varphi(y_1|x_1,\beta)=0$ has two solutions, we pick the larger one. When x_1 is negative, we compare it with 0 and choose the one that leads to a bigger H as the optimal control. When x_1 is positive, the $y_1^*(x_1)$ is the maximum of the solution and x_1. See Figure C.1 for different scenarios. However, no matter how many zero points $\varphi(y_1|x_1,\beta)$ has, we argue that as long as $y_1^*(x_1)$ is solved by the first-order condition, i.e. $\varphi(y_1|x_1,\beta)=0$, $y_1^*(x_1)$ must be decreasing in x_1. This is illustrated by Figure C.2.

Now we want to show under the optimal control $y_1^*(x_1)$, the IR constraint will be violated at some $x_1 \in (x_1^-, x_1^+)$. We need to introduce the following lemma:

Lemma C.0.4. *(a) Suppose at some point $0 \le \hat{x}_1 < y_0$, the optimal control is such that $y^*(\hat{x}_1) = \hat{x}_1$. We must also have $y^*(x_1) = x_1$ for $\hat{x}_1 < x_1 \le y_0$.*

(b) Suppose at some point $\hat{x}_1 < 0$, the optimal control is such that $y^*(\hat{x}_1) = 0$. We must also have $y^*(x_1) = 0$ for $\hat{x}_1 < x_1 \le 0$.

Proof of Lemma C.0.4:

(a) According to the feasibility condition, we must satisfy $y_1^*(x_1) \ge x_1$ and $\dot{y}_1^*(x_1) \le 1$. However, $\dot{y}_1^*(x_1) \le 1$ implies that the order quantity $q_1^*(x_1)$ is weakly decreasing in x_1. As a result, once $y_1^*(\hat{x}_1) = \hat{x}_1$, i.e. $q_1^*(\hat{x}_1) = 0$, we will have $q_1^*(x_1) = 0$ for all $x_1 > \hat{x}_1$. So $y_1^*(x_1) = x_1$ for all $x_1 > \hat{x}_1$. □

(b) We know $y_1^*(\hat{x}_1) = 0$. It can happen only when has $\varphi(y_1|x_1,\beta)=0$ has 0 solution or 2 solutions. We need to discuss these two possible cases.

If $\varphi(y_1|x_1,\beta)=0$ has 0 solution, it means $\varphi(y_1|\hat{x}_1,\beta) < 0$ for all $y_1 \ge 0$. We have already shown φ, as a function of y_1, will decrease in x_1, therefore $\varphi(y_1|x_1,\beta) < \varphi(y_1|\hat{x}_1,\beta) < 0$ for all $x_1 \ge \hat{x}_1$ and $y_1 \ge 0$. So we conclude that the optimal control will be $y_1^*(x_1) = 0$ for $\hat{x}_1 < x_1 \le 0$.

If $\varphi(y_1|x_1,\beta)=0$ has 2 solutions, as mentioned earlier, we pick the larger solution denoted as $y_1^{large-FOC}(x_1)$ and compare it with 0. Then we select the one which gives us a higher H as the

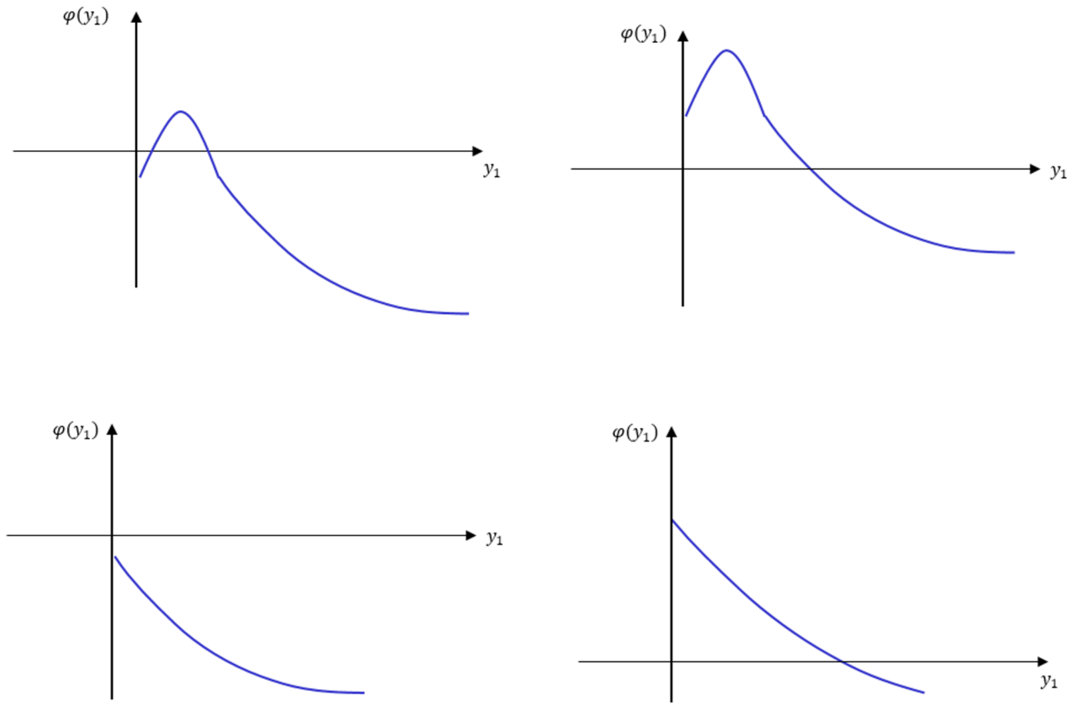

Figure C.1: Illustration of function $\varphi(y_1|x_1, \beta)$ under different x_1 and β.

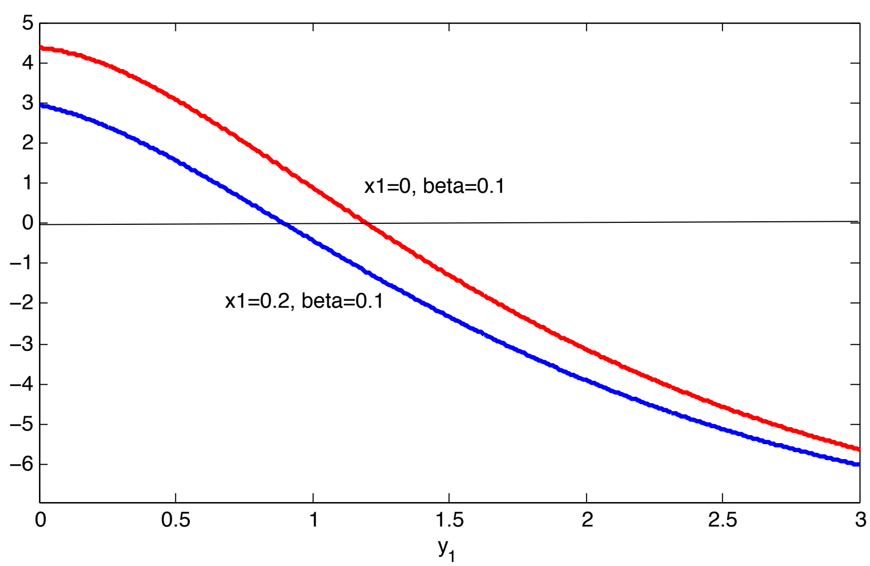

Figure C.2: Function $\varphi(y_1|x_1, \beta)$ when choosing two different x_1. Parameters: $r = 10$, $c = 5$, $b = 2$, $h = 3$, $y_0 = 3$, $\beta = 0.1$, $\lambda = 1$ and $\delta = 0.9$.

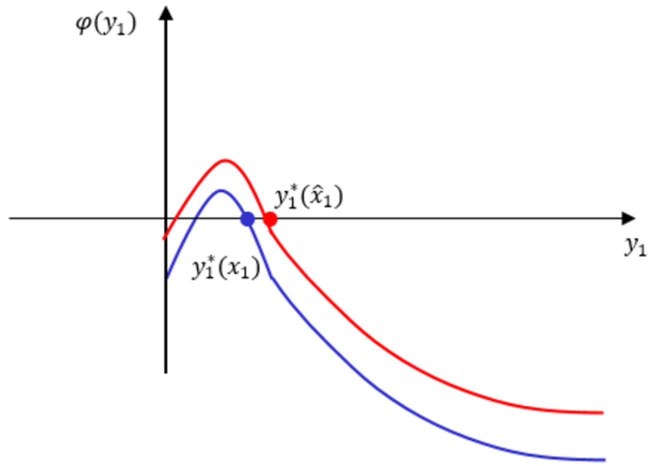

Figure C.3: Function $\varphi(y_1|x_1,\beta)=0$ has 2 solutions but the optimal control is $y_1^*(x_1)=0$.

optimal control. That is to say, $y_1^*(\hat{x}_1)=0$ implies that $H(y_1^{large-FOC}(\hat{x}_1)|\hat{x}_1,\beta)\le H(0|\hat{x}_1,\beta)$. Now we compare $H(y_1^{large-FOC}(x_1)|x_1,\beta)$ and $H(0|x_1,\beta)$ for $\hat{x}_1<x_1\le 0$. Notice that

$$H(y_1^{large-FOC}(\hat{x}_1)|\hat{x}_1,\beta) \quad = H(0|\hat{x}_1,\beta)+\int_0^{y_1^{large-FOC}(\hat{x}_1)} g_1(\hat{x}_1)\varphi(z|\hat{x}_1,\beta)dz,$$

and $H(y_1^{large-FOC}(\hat{x}_1)|\hat{x}_1,\beta)\le H(0|\hat{x}_1,\beta)$ implies $\int_0^{y_1^{large-FOC}(\hat{x}_1)}\varphi(z|\hat{x}_1,\beta)dz\le 0$. Moreover, we have $\varphi(z|\hat{x}_1,\beta)>\varphi(z|x_1,\beta)$ for all z when $\hat{x}_1<x_1$. Therefore,

$$\int_0^{y_1^{large-FOC}(x_1)}\varphi(z|x_1,\beta)dz<\int_0^{y_1^{large-FOC}(x_1)}\varphi(z|\hat{x}_1,\beta)dz$$

Also, we have claimed that $y_1^{large-FOC}(\hat{x}_1)>y_1^{large-FOC}(x_1)$ if both of them are solved by first-order condition. In addition, the function $\varphi(z|\hat{x}_1,\beta)$ is positive when $z\in(y_1^{large-FOC}(x_1),y_1^{large-FOC}(\hat{x}_1))$. Please refer to Figure C.3. As a result, we obtain

$$\int_0^{y_1^{large-FOC}(x_1)}\varphi(z|\hat{x}_1,\beta)dz<\int_0^{y_1^{large-FOC}(\hat{x}_1)}\varphi(z|\hat{x}_1,\beta)dz$$

Finally,

$$\int_0^{y_1^{large-FOC}(x_1)}\varphi(z|x_1,\beta)dz<\int_0^{y_1^{large-FOC}(x_1)}\varphi(z|\hat{x}_1,\beta)dz<\int_0^{y_1^{large-FOC}(\hat{x}_1)}\varphi(z|\hat{x}_1,\beta)dz\le 0$$

which implies

$$H(y_1^{large-FOC}(x_1)|x_1,\beta)=H(0|x_1,\beta)+\int_0^{y_1^{large-FOC}(x_1)}g_1(x_1)\varphi(z|x_1,\beta)dz<H(0|x_1,\beta).$$

Therefore, it is optimal to have $y_1^*(x_1) = 0$ for $0 \geq x_1 > \hat{x}_1$. □

Next, we show that the IR constraint cannot be satisfied at the two end points simultaneously. One necessary condition for the IR constraint satisfied at $x_1 \in [x_1^-, x_1^+]$ is $y_1^*(x_1^-) \leq y_1^R(x_1^-)$ and $y_1^*(x_1^+) \geq y_1^R(x_1^+)$. We consider three possible cases:

Case 1): Suppose $x_1^- < x_1^+ \leq 0$. In this case, $y_1^R(x_1) = y_1^R(0)$ is a constant for all $x_1 \in [x_1^-, x_1^+]$.

If $y_1^*(x_1^-)$ happens to be 0, Lemma C.0.4 implies that $y_1^*(x_1^+) = 0 < y_1^R(x_1^+)$. This violates the necessary condition.

If $y_1^*(x_1^-)$ is the (larger) solution of the first-order condition, we have $0 < y_1^*(x_1^-) \leq y_1^R(x_1^-)$. However, at the endpoint x_1^+, we may get $y_1^*(x_1^+) = 0$ or $y_1^*(x_1^+) > 0$. If $y_1^*(x_1^+) = 0$, it already violates the necessary condition since $0 < y_1^R(x_1^+)$. If $y_1^*(x_1^+) > 0$, it implies that the optimal control is solved by the first-order condition. Therefore, we have $y_1^*(x_1^+) < y_1^*(x_1^-) \leq y_1^R(x_1^-) = y_1^R(x_1^+)$, which also violates the necessary condition.

In conclusion, the IR constraints must be violated at some point in (x_1^-, x_1^+) in Case 1).

Case 2) Suppose $0 \leq x_1^- < x_1^+$. In this case, $y_1^R(x_1)$ will increase in $x_1 \in [x_1^-, x_1^+]$ because $\frac{dy_1^R}{dx_1} = \frac{u_1''(x_1)}{\mu_1''(x_1)} > 0$.

If $y_1^*(x_1^-) = x_1^-$, according to Lemma C.0.4, we have $y_1^*(x_1^+) = x_1^+ < y_1^R(x_1^+)$, which violates the necessary condition.

If $x_1^- < y_1^*(x_1^-) \leq y_1^R(x_1^-)$, we either have $y_1^*(x_1^+) = x_1^+$ or $y_1^*(x_1^+) > x_1^+$. If $y_1^*(x_1^+) = x_1^+$, it violates the necessary condition since $x_1^+ < y_1^R(x_1^+)$. If $y_1^*(x_1^+) > x_1^+$, it also violates the necessary condition in that $y_1^*(x_1^+) < y_1^*(x_1^-) \leq y_1^R(x_1^-) < y_1^R(x_1^+)$. The first inequality holds because $y_1^*(x_1)$ is the (larger) solution of the first-order condition and is decreasing in x_1.

In conclusion, the IR constraints must be violated at some point in (x_1^-, x_1^+) in Case 2).

Case 3) Suppose $x_1^- < 0 < x_1^+$, we apply a similar approach as the previous two cases.

If $y_1^*(x_1^-) = 0$, we also have $y_1^*(0) = 0$. Therefore, $y_1^*(x_1^+) = x_1^+ < y_1^R(x_1^+)$, which violates the necessary condition.

If $0 < y_1^*(x_1^-) \leq y_1^R(x_1^-)$, we either have $y_1^*(x_1^+) = x_1^+$ or $y_1^*(x_1^+) > x_1^+$. If $y_1^*(x_1^+) = x_1^+$, it violates the necessary condition. If $y_1^*(x_1^+) > x_1^+$, it also violates the necessary condition in that $y_1^*(x_1^+) < y_1^*(0) < y_1^*(x_1^-) \leq y_1^R(x_1^-) < y_1^R(x_1^+)$.

In conclusion, the IR constraints must be violated at some point in (x_1^-, x_1^+) in Case 3).

In summary, we conclude that the optimal contract does not have "bump" in the retailer's profit-to-go function. □

Proof of Lemma 4.5.4

$$\pi_1(x_1) = \mu_1(y_1^R(x_1)) - cy_1^R(x_1) + cx_1 + \delta\Pi_2(y_1^R(x_1)) - \underline{u}_1(x_1) \tag{C.6}$$

$$= \begin{cases} \frac{r+h}{\lambda} - \frac{b+h}{\lambda}e^{-\lambda y_1^R(0)} - hy_1^R(0) - cy_1^R(0) + cx_1 - rx_1 \\ \quad +\delta[\frac{r}{\lambda} - \frac{r+c}{\lambda}e^{-\lambda y_1^R(0)} - ry_1^R(0)e^{-\lambda y_1^R(0)}], & x_1 \le 0 \\ \frac{r+h}{\lambda} - \frac{b+h}{\lambda}e^{-\lambda y_1^R(x_1)} - hy_1^R(x_1) - cy_1^R(x_1) + cx_1 \\ \quad +\delta[\frac{r}{\lambda} - \frac{r+c}{\lambda}e^{-\lambda y_1^R(x_1)} - ry_1^R(x_1)e^{-\lambda y_1^R(x_1)}] \\ \quad -\{\frac{r}{\lambda}(1-e^{-\lambda x_1}) - hx_1 + \frac{h}{\lambda}(1-e^{-\lambda x_1}) + \delta[\frac{r}{\lambda} - \frac{r}{\lambda}e^{-\lambda x_1} - rx_1e^{-\lambda x_1}]\}. & x_1 > 0 \end{cases} \tag{C.7}$$

Recall the definition of $y_1^R(x_1)$:

$$(b+h+\delta r+\delta r\lambda y_1^R)e^{-\lambda y_1^R} = \begin{cases} (r+h+\delta r\lambda x_1)e^{-\lambda x_1}, & x_1 > 0 \\ r+h, & x_1 \le 0 \end{cases}$$

We can further simplify the supplier's profit as follows:

$$\pi_1(x_1) = \begin{cases} -(h+c)y_1^R(0) - (r-c)x_1 + \delta\frac{r-ce^{-\lambda y_1^R(0)}}{\lambda} & x_1 \le 0 \\ -(h+c)(y_1^R(x_1) - x_1) + \delta\frac{re^{-\lambda x_1} - ce^{-\lambda y_1^R(x_1)}}{\lambda} & x_1 > 0 \end{cases} \tag{C.8}$$

When $x_1 < 0$, $\pi_1(x_1)$ is a linear function x_1 with slope $-(r-c)$. Thus, when $x_1 \to -\infty$, we have $\pi_1(x_1) \to +\infty$. When $x_1 > 0$, $\pi_1(x_1)$ may not be monotone. But we can see that $\pi_1(x_1)$ approaches 0 from the negative side as $x_1 \to +\infty$. Therefore, $\pi_1(x_1)$ has at least one root. Now we look at its first-order derivative when $x_1 > 0$.

$$\begin{aligned} \pi_1'(x_1) &= \mu_1'(y_1^R(x_1))\frac{dy_1^R}{dx_1} - c\frac{dy_1^R}{dx_1} + c - \delta(r-c)e^{-\lambda y_1^R(x_1)}\frac{dy_1^R}{dx_1} - \underline{u}_1'(x_1) \\ &= [(r+h+\delta r\lambda x_1)e^{-\lambda x_1} - h - c](\frac{dy_1^R}{dx_1} - 1) - \delta(r-c)e^{-\lambda y_1^R(x_1)}\frac{dy_1^R}{dx_1} \\ &= (h+c)(1 - \frac{dy_1^R}{dx_1}) - \delta re^{-\lambda x_1} + \delta ce^{-\lambda y_1^R(x_1)}\frac{dy_1^R}{dx_1} \end{aligned} \tag{C.9}$$

Without loss of generality, we let $\lambda = 1$. We compare (C.8) and (C.9) when $x_1 > 0$:

$$\begin{aligned} \pi_1(x_1) &= -(h+c)q_1^R(x_1) + \delta e^{-x_1}[r - ce^{-q_1^R(x_1)}] \\ \pi_1'(x_1) &= -(h+c)\frac{dq_1^R(x_1)}{dx_1} - \delta e^{-x_1}[r - ce^{-q_1^R(x_1)}] + \delta ce^{-x_1-q_1^R(x_1)}\frac{dq_1^R(x_1)}{dx_1} \end{aligned}$$

Therefore, $\pi_1(x_1) + \pi_1'(x_1) = -(h+c)[q_1^R(x_1) + \frac{dq_1^R(x_1)}{dx_1}] + \delta ce^{-x_1-q_1^R(x_1)}\frac{dq_1^R(x_1)}{dx_1}$. Since $q_1^R(x_1)$ is decreasing in x_1, we have $\frac{dq_1^R(x_1)}{dx_1} < 0$. Next we examine the term $q_1^R(x_1) + \frac{dq_1^R(x_1)}{dx_1}$. Recall that

$q_1^R(x_1) = y_1^R(x_1) - x_1$, so we have

$$\frac{dq_1^R(x_1)}{dx_1} = \frac{dy_1^R(x_1)}{dx_1} - 1 = -\frac{\delta r(e^{q_1^R(x_1)}-1)}{b+h+\delta r y_1^R(x_1)} = -\frac{\delta r(e^{q_1^R(x_1)}-1)}{(r+h+\delta r x_1)e^{q_1^R(x_1)}-\delta r}$$

When $x_1 \geq 2 - (r+h)/(\delta r)$, we further have

$$-\frac{dq_1^R(x_1)}{dx_1} = \frac{\delta r(e^{q_1^R(x_1)}-1)}{(r+h+\delta r x_1)e^{q_1^R(x_1)}-\delta r} \leq \frac{\delta r(e^{q_1^R(x_1)}-1)}{(r+h+\delta r x_1-\delta r)e^{q_1^R(x_1)}} = \frac{\delta r}{(r+h+\delta r x_1-\delta r)}(1-e^{-q_1^R(x_1)}) \leq (1-e^{-q_1^R(x_1)}) \leq q_1^R(x_1)$$

As a result, we get $\pi_1(x_1) + \pi_1'(x_1) = (h+c)(-\frac{dq_1^R(x_1)}{dx_1} - q_1^R(x_1)) + \delta c e^{-x_1-q_1^R(x_1)}\frac{dq_1^R(x_1)}{dx_1} \leq 0$. That is to say, π_1 and π_1' cannot be positive simultaneously when $x_1 \geq 2-(r+h)/(\delta r)$.

When $x_1 < 2-(r+h)/(\delta r) < 1$, we rewrite the expression of $\pi_1'(x_1)$ as

$$\pi_1'(x_1) = [\underline{u}_1'(x_1) - c](\frac{dy_1^R}{dx_1} - 1) - \delta(r-c)e^{-\lambda y_1^R(x_1)}\frac{dy_1^R}{dx_1} \tag{C.10}$$

We assume $(r+h+\delta r)e^{-1} - (h+c) > 0$. Therefore, we have $\underline{u}_1'(x_1) \geq \underline{u}_1'(1) = (r+h+\delta r)e^{-1} - (h+c) \geq 0$. Because $\frac{dy_1^R}{dx_1}$, we conclude from (C.10) that $\pi_1'(x_1) < 0$ in this case.

Combining the two cases, when $x_1 \geq 2-(r+h)/(\delta r)$, we know that π_1 and π_1' cannot be positive at the same time. When $x_1 < 2-(r+h)/(\delta r)$, we know $\pi_1'(x_1) < 0$ and thereby $\pi_1(x_1)$ is decreasing in x_1. Now we claim that $\pi_1(x_1)$ has only one root. We prove by contradiction. Suppose π_1 has at least two roots. One of them must happen when $\pi_1(x_1)$ is crossing the x-axis from negative to positive. In other words, there exists some x_1 such that $\pi_1(x_1) > 0$ and $\pi_1'(x_1) > 0$. However, we have shown it will not happen. So we obtain a contradiction. In conclusion, $\pi_1(x_1)$ has only one root. □

Proof of Lemma 4.5.5

$y_1^R(x_1)$ strictly increases in x_1 and $y_1^R(x_1) > x_1$. Yet, $y_1^L(x_1)$ strictly decreases in x_1 when $y_1^L(x_1) > x_1$. Therefore, the solution of $y_1^R(x_1) = y_1^L(x_1)$ must be unique. □

Proof of Lemma 4.6.1

By definition, when $x > 0$, $y^R(x)$ satisfies

$$[b-r(1-\delta)]e^{-\lambda y^R(x)} + \frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y^R(x)(1-\delta)} = \frac{h+r(1-\delta)}{1-\delta}e^{-\lambda x(1-\delta)}$$

We take derivative with respect to x on both sides which leads to

$$\{[b-r(1-\delta)]e^{-\lambda y^R(x)} + [h+r(1-\delta)]e^{-\lambda y^R(x)(1-\delta)}\}\frac{dy^R}{dx} = [h+r(1-\delta)]e^{-\lambda x(1-\delta)}$$

Therefore,

$$\frac{dy^R}{dx} = \frac{[h + r(1-\delta)]e^{-\lambda x(1-\delta)}}{[b - r(1-\delta)]e^{-\lambda y^R(x)} + [h + r(1-\delta)]e^{-\lambda y^R(x)(1-\delta)}} > 0. \tag{C.11}$$

i.e. $y^R(x)$ is strictly increasing in x.

Next, we consider $\frac{d^2y^R}{dx^2}$ which satisfies

$$\begin{aligned} & \frac{d^2y^R}{dx^2}\{[b - r(1-\delta)]e^{-\lambda y^R(x)} + [h + r(1-\delta)]e^{-\lambda y^R(x)(1-\delta)}\}/\lambda \qquad \text{(C.12)} \\ = \;& \{[b - r(1-\delta)]e^{-\lambda y^R(x)} + (1-\delta)[h + r(1-\delta)]e^{-\lambda y^R(x)(1-\delta)}\}(\frac{dy^R}{dx})^2 - (1-\delta)[h + r(1-\delta)]e^{-\lambda x(1-\delta)} \end{aligned}$$

Clearly, whether $\frac{d^2y^R}{dx^2}$ is positive depends on the right-hand side of Equation (C.12). For the sake of presentation, we shorten the notation by letting $K_1 = [b - r(1-\delta)]e^{-\lambda y^R(x)}$, $K_2 = [h+r(1-\delta)]e^{-\lambda y^R(x)(1-\delta)}$ and $K_3 = [h+r(1-\delta)]e^{-\lambda x(1-\delta)}$. We have $\frac{dy^R}{dx} = \frac{K_3}{K_1+K_2}$ from (C.11). Moreover, by the definition of $y^R(x)$, we have $K_1 + \frac{K_2}{1-\delta} = \frac{K_3}{1-\delta}$. The right-hand side of Equation (C.12) can be simplified as

$$\begin{aligned} & [K_1 + (1-\delta)K_2](\frac{K_3}{K_1+K_2})^2 - (1-\delta)K_3 \\ = \;& \frac{K_3}{(K_1+K_2)^2}\{[K_1 + (1-\delta)K_2]K_3 - (1-\delta)(K_1+K_2)^2\} \\ = \;& \frac{K_3}{(K_1+K_2)^2}\{[K_1 + (1-\delta)K_2][(1-\delta)K_1 + K_2] - (1-\delta)(K_1+K_2)^2\} \\ = \;& \delta^2\frac{K_3}{(K_1+K_2)^2}K_1K_2 > 0 \end{aligned}$$

As a result, we conclude $\frac{d^2y^R}{dx^2} > 0$, i.e $y^R(x)$ is convex in x. □

Proof of Lemma 4.6.2

Suppose the supplier implements $y^R(x)$ in each period. Notice that $y^R(x)$ is stationary and does not depend on the supplier's belief. In addition, we have seen as $T \to \infty$, the retailer's profit-to-go $U(y)$ exists and is unique. We can compute π iteratively by the following equation:

$$\pi_n(z) = v(y^R(z)) + \delta U(y^R(z)) - \underline{u}(z) - cy^R(z) + cz + \delta\Pi_{n-1}(y^R(z)) \tag{C.13}$$

$$\Pi_{n-1}(y(z)) = \int_{-\infty}^{y(z)} \pi_{n-1}(\xi)\lambda e^{-\lambda(y(z)-\xi)}d\xi \tag{C.14}$$

We use the Contraction Mapping Theorem. In other words, for any two functions π_{n-1} and π'_{n-1} which satisfy $||\pi_{n-1} - \pi'_{n-1}|| := \max_z |\pi_{n-1}(z) - \pi'_{n-1}(z)| < \epsilon$, we want to prove $||\pi_n - \pi'_n|| := \max_z |\pi_n(z) - \pi'_n(z)| < \delta\epsilon$.

In fact, for any z,

$$\begin{aligned}
|\pi_n(z) - \pi_n'(z)| &= \delta|\Pi_{n-1}(z) - \Pi_{n-1}'(z)| \\
&= \delta|\int_{-\infty}^{y^R(z)} [\pi_{n-1}(\xi) - \pi_{n-1}'(\xi)]\lambda e^{-\lambda(y^R(z)-\xi)} d\xi| \\
&\leq \delta \int_{-\infty}^{y^R(z)} |\pi_{n-1}(\xi) - \pi_{n-1}'(\xi)|\lambda e^{-\lambda(y^R(z)-\xi)} d\xi \\
&< \delta \int_{-\infty}^{y^R(z)} \epsilon\lambda e^{-\lambda(y^R(z)-\xi)} d\xi = \delta\epsilon
\end{aligned}$$

therefore, $||\pi_n - \pi_n'|| := \max_z |\pi_n(z) - \pi_n'(z)| < \delta\epsilon$. Therefore, the iteration (C.13)-(C.14) is indeed a contraction mapping. By the Contraction Mapping Theorem, the sequence of π_n converges and its limit $\lim_{n\to\infty} \pi_n = \pi$ exists and is unique. In addition, $\Pi(y) = \lim_{n\to\infty} \Pi_n(y) = \lim_{n\to\infty} \int_{-\infty}^{y} \pi_{n-1}(\xi)\lambda e^{-\lambda(y(z)-\xi)} d\xi = \int_{-\infty}^{y} \pi(\xi)\lambda e^{-\lambda(y(z)-\xi)} d\xi$ also exists and is unique. □

Proof of Proposition 4.6.3

We first consider the case $z \leq 0$. Note that $\Pi(z) = \int_{-\infty}^{z} \pi(\xi)\lambda e^{-\lambda(z-\xi)} d\xi$. We assume the supplier implements the quantity plan y^R from the "next" period onwards, i.e. $\pi(\xi) = v(y^R(\xi)) + \delta U(y^R(\xi)) - \underline{u}(\xi) - cy^R(\xi) + c\xi + \delta\Pi(y^R(\xi))$. We have shown $y^R(\xi) = y^R(0)$ is a constant whenever $\xi \leq 0$. Therefore, $\pi(\xi) = v(y^R(0)) + \delta U(y^R(0)) - \underline{u}(\xi) - cy^R(0) + c\xi + \delta\Pi(y^R(0)) = \pi(0) - (r-c)\xi$. As a result, we have

$$\begin{aligned}
\Pi(z) &= \int_{-\infty}^{z} \pi(\xi)\lambda e^{-\lambda(z-\xi)} d\xi \\
&= \int_{-\infty}^{z} [\pi(0) - (r-c)\xi]\lambda e^{-\lambda(z-\xi)} d\xi \\
&= \pi(0) - (r-c)(z - \frac{1}{\lambda})
\end{aligned}$$

When $z \leq 0$, $\Pi(z)$ is a linear function of z with slope $-(r-c)$. Finally,

$$\frac{dJ^R(z|x)}{dz} = -c + b + \delta r + \delta\Pi'(z) = -c + b + \delta r - \delta(r-c) = b + \delta c - c > 0,$$

so $J^R(z|x)$ increases when $z < 0$. □

Next we examine the case $z > 0$. Note $\Pi(z) = \int_{-\infty}^{z} \pi(\xi)\lambda e^{-\lambda(z-\xi)} d\xi$. We take derivative on both sides with respect to z and we get

$$\begin{aligned}
\Pi'(z) &= \lambda\pi(z) - \lambda\int_{-\infty}^{z}\pi(\xi)\lambda e^{-\lambda(z-\xi)}d\xi \\
&= \lambda[\pi(z) - \Pi(z)] \\
&= \lambda\int_{-\infty}^{z}[\pi(z) - \pi(\xi)]\lambda e^{-\lambda(z-\xi)}d\xi \\
&= \lambda\int_{-\infty}^{z}[\int_{\xi}^{z}\pi'(\eta)d\eta]\lambda e^{-\lambda(z-\xi)}d\xi \\
&= \int_{-\infty}^{z}\pi'(\xi)\lambda e^{-\lambda(z-\xi)}d\xi,
\end{aligned}$$

where the last equality holds by changing the order of integration. We use the equation $\Pi'(z) = \int_{-\infty}^{z}\pi'(\xi)\lambda e^{-\lambda(z-\xi)}d\xi$ and prove the result by induction. Suppose $-c-\frac{h}{1-\delta}+\delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y(1-\delta)}+\delta\Pi'_{t+1}(y) < 0$. We want to show that similar inequality holds for $\Pi'_t(y)$. In fact,

$$\begin{aligned}
\Pi'_t(z) &= \int_{-\infty}^{z}\pi'_t(\xi)\lambda e^{-\lambda(z-\xi)}d\xi \\
&= \int_{-\infty}^{z}\{[v'(y^R(\xi) + \delta U'(y^R(\xi)) - c + \delta\Pi'_{t+1}(y^R(\xi))]\frac{dy^R(\xi)}{d\xi} + c - \underline{u}'(\xi)\}\lambda e^{-\lambda(z-\xi)}d\xi \\
&= \int_{-\infty}^{0} -(r-c)\lambda e^{-\lambda(z-\xi)}d\xi \\
&\quad + \int_{0}^{z}\{[v'(y^R(\xi) + \delta U'(y^R(\xi)) - c + \delta\Pi'_{t+1}(y^R(\xi))]\frac{dy^R(\xi)}{d\xi} + c - \underline{u}'(\xi)\}\lambda e^{-\lambda(z-\xi)}d\xi \\
&= -(r-c)e^{-\lambda z} + \int_{0}^{z}\{[v'(y^R(\xi) + \delta U'(y^R(\xi)) - c + \delta\Pi'_{t+1}(y^R(\xi))]\frac{dy^R(\xi)}{d\xi} + c - \underline{u}'(\xi)\}\lambda e^{-\lambda(z-\xi)}d\xi
\end{aligned}$$

We replace $\frac{dy^R(\xi)}{d\xi}$ by $\frac{[h+r(1-\delta)]e^{-\lambda x(1-\delta)}}{[b-r(1-\delta)]e^{-\lambda y^R(x)}+[h+r(1-\delta)]e^{-\lambda y^R(x)(1-\delta)}}$ in the integrand which leads to

$$\begin{aligned}
& [v'(y^R(\xi) + \delta U'(y^R(\xi)) - c + \delta\Pi'_{t+1}(y^R(\xi))]\frac{dy^R(\xi)}{d\xi} + c - \underline{u}'(\xi) \\
= \; & \{[b - r(1-\delta)]e^{-\lambda y^R(\xi)} + \frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y^R(\xi)(1-\delta)} - \frac{h}{1-\delta} - c + \delta\Pi'_{t+1}(y^R(\xi))\} \\
& \times\frac{[h+r(1-\delta)]e^{-\lambda\xi(1-\delta)}}{[b-r(1-\delta)]e^{-\lambda y^R(\xi)} + [h+r(1-\delta)]e^{-\lambda y^R(\xi)(1-\delta)}} + c - \frac{h+r(1-\delta)}{1-\delta}e^{-\xi(1-\delta)} + \frac{h}{1-\delta} \\
= \; & c + \frac{h}{1-\delta} - \delta\frac{h+r(1-\delta)}{1-\delta}e^{-\xi(1-\delta)} + \{\delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y^R(\xi)(1-\delta)} - \frac{h}{1-\delta} - c + \delta\Pi'_{t+1}(y^R(\xi))\}\frac{dy^R(\xi)}{d\xi}
\end{aligned}$$

For the first part,

$$\int_{0}^{z}\{c + \frac{h}{1-\delta} - \delta\frac{h+r(1-\delta)}{1-\delta}e^{-\xi(1-\delta)}\}\lambda e^{-\lambda(z-\xi)}d\xi = (c + \frac{h}{1-\delta})(1 - e^{-\lambda z}) - \frac{h+r(1-\delta)}{1-\delta}(e^{-\lambda z(1-\delta)} - e^{-\lambda z})$$

For the second part, by the inductive assumption, for any ξ, $\delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y^R(\xi)(1-\delta)} - \frac{h}{1-\delta} - c +$

$\delta\Pi'_{t+1}(y^R(\xi)) < 0$. However, Lemma 4.6.1 says $\frac{dy^R(\xi)}{d\xi} > 0$. Therefore, the whole integrand is negative. As a result,

$$\int_0^z \{\delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y^R(\xi)(1-\delta)} - \frac{h}{1-\delta} - c + \delta\Pi'_{t+1}(y^R(\xi))\}\frac{dy^R(\xi)}{d\xi}\lambda e^{-\lambda(z-\xi)}d\xi < 0$$

Finally,

$$\begin{aligned}
\Pi'_t(z) &= \int_{-\infty}^z \pi'_t(\xi)\lambda e^{-\lambda(z-\xi)}d\xi \\
&= -(r-c)e^{-\lambda z} + (c+\frac{h}{1-\delta})(1-e^{-\lambda z}) - \frac{h+r(1-\delta)}{1-\delta}(e^{-\lambda z(1-\delta)} - e^{-\lambda z}) \\
&\quad + \int_0^z \{\delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y^R(\xi)(1-\delta)} - \frac{h}{1-\delta} - c + \delta\Pi'_{t+1}(y^R(\xi))\}\frac{dy^R(\xi)}{d\xi}\lambda e^{-\lambda(z-\xi)}d\xi < 0 \\
&\leq (r-c)e^{-\lambda z} + (c+\frac{h}{1-\delta})(1-e^{-\lambda z}) - \frac{h+r(1-\delta)}{1-\delta}(e^{-\lambda z(1-\delta)} - e^{-\lambda z}) \\
&= c + \frac{h}{1-\delta} - \frac{h+r(1-\delta)}{1-\delta}e^{-\lambda z(1-\delta)}
\end{aligned}$$

By rearranging the terms, we have

$$\begin{aligned}
\frac{dJ^R(z|x)}{dz} &= -c - \frac{h}{1-\delta} + \delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda z(1-\delta)} + \delta\Pi'_t(z) \\
&\leq -c - \frac{h}{1-\delta} + \delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y(1-\delta)} + \delta[c + \frac{h}{1-\delta} - \frac{h+r(1-\delta)}{1-\delta}e^{-\lambda z(1-\delta)}] \\
&= -(1-\delta)(c+\frac{h}{1-\delta}) < 0
\end{aligned}$$

i.e. $\frac{dJ^R(z|x)}{dz} < 0$ when $z > 0$. □

In the following, we want to show the virtual surplus $J^R(z|x)$ is concave. For the sake of analysis, we define $\varphi(z) = c + \frac{h}{1-\delta} - \delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda z(1-\delta)} - \delta\Pi'(z)$. In fact, $\varphi(z) = -\frac{dJ^R(z|x)}{dz}$. Equivalently we want to prove $\varphi(z)$ increases in z. We prove the result by induction. Suppose $\varphi_{t+1}(z)$ is an increasing function in z.

From the previous analysis, we have seen

$$\begin{aligned}
\varphi_t(z) &= c + \frac{h}{1-\delta} - \delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda z(1-\delta)} - \delta\Pi'_t(z) \\
&= c + \frac{h}{1-\delta} - \delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda z(1-\delta)} \\
&\quad -\delta\{c + \frac{h}{1-\delta} - \frac{h+r(1-\delta)}{1-\delta}e^{-\lambda z(1-\delta)} \\
&\quad + \int_0^z \{\delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y^R(\xi)(1-\delta)} - \frac{h}{1-\delta} - c + \delta\Pi'_{t+1}(y^R(\xi))\}\frac{dy^R(\xi)}{d\xi}\lambda e^{-\lambda(z-\xi)}d\xi\} \\
&= (1-\delta)(c+\frac{h}{1-\delta}) + \delta\int_0^z \varphi_{t+1}(y^R(\xi))\frac{dy^R(\xi)}{d\xi}\lambda e^{-\lambda(z-\xi)}d\xi
\end{aligned}$$

We consider its derivative

$$\begin{aligned}\varphi_t'(z) &= \lambda\delta\{\varphi_{t+1}(y^R(z))\frac{dy^R(z)}{d\xi} - \int_0^z \varphi_{t+1}(y^R(\xi))\frac{dy^R(\xi)}{d\xi}\lambda e^{-\lambda(z-\xi)}d\xi\} \\ &= \lambda\delta\{\varphi_{t+1}(y^R(z))\frac{dy^R(z)}{d\xi}e^{-\lambda z} + \int_0^z [\varphi_{t+1}(y^R(z))\frac{dy^R(z)}{d\xi} - \varphi_{t+1}(y^R(\xi))\frac{dy^R(\xi)}{d\xi}]\lambda e^{-\lambda(z-\xi)}d\xi\}\end{aligned}$$

From the previous analysis, we have $\varphi_{t+1}(y^R(z)) > 0$ and $\frac{dy^R(z)}{d\xi} > 0$. So $\varphi_{t+1}(y^R(z))\frac{dy^R(z)}{d\xi}e^{-\lambda z} > 0$. In addition, by the inductive hypothesis, φ_{t+1} is an increasing function. For any $\xi < z$, $y^R(\xi) < y^R(z)$, thereby $\varphi_{t+1}(y^R(z)) > \varphi_{t+1}(y^R(\xi))$. Moreover, $\frac{dy^R(z)}{d\xi} > \frac{dy^R(\xi)}{d\xi}$ because $y^R(z)$ is convex. As a result, $\int_0^z[\varphi_{t+1}(y^R(z))\frac{dy^R(z)}{d\xi} - \varphi_{t+1}(y^R(\xi))\frac{dy^R(\xi)}{d\xi}]\lambda e^{-\lambda(z-\xi)}d\xi > 0$. Combining these two terms, we conclude $\varphi_t'(z) > 0$. Therefore, $\frac{d^2J^R(z|x)}{dz^2} < 0$, i.e. $J^R(z|x)$ is concave. □

Proof of Proposition 4.6.4

We examine the first-order condition $\frac{dJ^L(y|x)}{dy}0$. When $y \le 0$, we have seen in the main text $\frac{dJ^L(y|x)}{dy} = -c + b + \delta c > 0$. We now consider the case $y > 0$.

$$-c - \frac{h}{1-\delta} + \delta\frac{h + r(1-\delta)}{1-\delta}e^{-\lambda y(1-\delta)} + \delta\Pi'(y) + \{[b - r(1-\delta)e^{-\lambda y} + [h + r(1-\delta)]e^{-\lambda y(1-\delta)}\}e^{\lambda(y_0-x)} = 0$$

By moving the terms, we end up with the following equation

$$\frac{c + \frac{h}{1-\delta} - \delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y(1-\delta)} - \delta\Pi'(y)}{[b - r(1-\delta)e^{-\lambda y} + [h + r(1-\delta)]e^{-\lambda y(1-\delta)}} = e^{\lambda(y_0-x)} \tag{C.15}$$

Proposition 4.6.3 ensures that the numeration $c+\frac{h}{1-\delta} - \delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y(1-\delta)} - \delta\Pi'(y)$ is increasing in y. And the denominator $[b-r(1-\delta)e^{-\lambda y}+[h+r(1-\delta)]e^{-\lambda y(1-\delta)}$ is decreasing in y. Therefore, the fraction as a whole is a increasing function of y. As a result, for any fixed x, the solution of (C.15) is unique. What is more, the right-hand side of (C.15) is equal to $e^{\lambda(y_0-x)}$ decreasing in x. The solution $y^L(x)$ will also decrease in x. □

Proof of Theorem 4.6.5

We apply a similar proof as the two-period case. We prove the theorem by contradiction. Suppose under the optimal contract, there exists "bump" at $[x^-, x^+]$ in the retailer's profit-to-go function. We focus our attention on the sub-interval and construct the subproblem as an optimal control problem.

$$(\mathcal{P}[x^-,x^+]) \qquad \max_{y(x)} \int_{x^-}^{x^+} J(y(x)|x)g(x)dx$$
$$s.t. \begin{cases} u'(x) = v'(y(x)) + \delta U'(y(x)), \quad x \in [x^-,x^+] \\ u(x^-) = \underline{u}(x^-) \text{ and } u(x^+) = \underline{u}(x^+) \\ y(x) \geq x \text{ and } y'(x) \leq 1 \quad x \in [x^-,x^+] \end{cases}$$

where $J(y|x)$ is the virtual surplus anchoring at the bottom endpoint x^-.

$$J(y|x) = cx - cy + v(y) + \delta U(y) + \delta\Pi(y) - [v'(y) + \delta U'(y)]\tfrac{G_1(x_1^+)-G_1(x_1)}{g_1(x_1)}$$

Similar as before, the subproblem $\mathcal{P}[x^-,x^+]$ can be translated into an optimal control problem with two fixed endpoints and two constraints on the control variable $y'(x) \leq 1$ and $y(x) \geq x$. The corresponding Hamiltonian should be $H(y|x,\eta) = J(y|x)g(x) + \eta(x)[v'(y) + \delta U'(y)]$ and the Lagrangian should be $\mathcal{L}(y|x,\eta,\rho) = H(y|x,\eta) + \rho_1(y-x) + \rho_2(y'(x)-1)$. Furthermore, the adjoint parameter η should satisfy $\begin{cases} \eta'(x) = \frac{\partial \mathcal{L}}{\partial u} = 0 \\ \eta(x^+) = \beta \text{ for some constant } \beta \end{cases}$. So η should be a constant, i.e. $\eta(x) = \beta$ for all $x \in [x^-,x^+]$.

We first investigate the first-order derivative of the Hamiltonian:

$$\begin{aligned} \frac{\partial H}{\partial y}(y|x) &= \begin{cases} g(x)\{b + \delta c - c\} > 0, & y < 0 \\ g(x)\{-c - \frac{h}{1-\delta} + \delta\frac{h+r(1-\delta)}{1-\delta}e^{-\lambda y(1-\delta)} + \delta\Pi'(y) & \\ +e^{\lambda(y_0-x)}(G(x^+)-\beta)\{[b-r(1-\delta)]e^{-\lambda y} + [h+r(1-\delta)]e^{-\lambda y(1-\delta)}\}\}, & y > 0 \end{cases} \\ &= \begin{cases} g(x)\{b + \delta c - c\} > 0, & y < 0 \\ g(x)\{-\varphi(y) + e^{\lambda(y_0-x)}(G(x^+)-\beta)\{[b-r(1-\delta)]e^{-\lambda y} + [h+r(1-\delta)]e^{-\lambda y(1-\delta)}\}\}, & y > 0 \end{cases} \end{aligned}$$

As a result, the corresponding first-order condition should be

$$\frac{\varphi(y)}{[b-r(1-\delta)]e^{-\lambda y} + [h+r(1-\delta)]e^{-\lambda y(1-\delta)}} = e^{\lambda(y_0-x)}(G(x^+)-\beta) \tag{C.16}$$

Proposition 4.6.3 guarantees that the left-hand side of (C.16) is increasing in y whereas the right-hand side of (C.16) is decreasing in x. Therefore, (C.16) has at most one solution. Besides, the solution (if exists) must strictly decrease in x. In conclusion, when $x \leq 0$, the optimal control $y^*(x)$ is either 0 or the solution of (C.16). When $x > 0$, the optimal control $y^*(x)$ is the maximum between x and the solution of (C.16).

Suppose for all $x \in [x^-,x^+]$, the optimal control $y^*(x)$ is the solution of (C.16), we clearly have $y^*(x^-) > y^*(x^+)$. Now we need to take care about the other two cases $y^*(x) = 0$ and $y^*(x) = x$. Notice that $y'^*(x) \leq 1$, i.e. the order quantity $q^*(x) = y^*(x) - x$ decreases in x.

Once there is a point $\hat{x} < x^+$ such that $y^*(\hat{x}) = \hat{x}$, we must have $y^*(x) = x$ for all $x > \hat{x}$. On the other hand, recall that $\frac{\partial H}{\partial y}(y|x)$, as a function of y, is decreasing in x. We can easily argue that if there is a point $\hat{x} < x^+$ such that $y^*(\hat{x}) = 0$, we must have $y^*(x) = 0$ for all $\hat{x} < x < 0$.

No matter what cases it is, we always end up with $y^*(x^-) \leq y^*(x^+)$. However, one necessary condition for the existence of "bump" is $y^*(x^-) \leq y^R(x^-) < y^R(x^+) \leq y^*(x^+)$. The necessary condition cannot be satisfied, and thereby the IR constraint will be violated at some point between $[x^-, x^+]$. We obtain a contradiction. So there cannot exist "bump" in the retailer's profit-to-go function under the optimal contract. □

Proof of Lemma 4.6.6

As the two-period case, because $y^R(x) > x$ strictly increases in x and $y^L(x)$ strictly decreases in x. The solution of $y^R(x) = y^L(x)$ must be unique. □