Borderline variational problems for fractional Hardy-Schrödinger operators

by

Shaya Shakerian

B.Sc., University of Tehran, 2009
M.Sc., Sharif University of Technology, 2012

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

The Faculty of Graduate and Postdoctoral Studies

(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

July 2017

Abstract

In this thesis, we study properties of the fractional Hardy-Schrödinger operator $L_{\gamma,\alpha} := (-\Delta)^{\frac{\alpha}{2}} - \frac{\gamma}{|x|^{\alpha}}$ both on $\mathbb{R}^n$ and on its bounded domains. The following functional inequality is key to our variational approach.

$$C\left(\int_{\mathbb{R}^n} \frac{|u|^{2_\alpha^*(s)}}{|x|^s} dx\right)^{\frac{2}{2_\alpha^*(s)}} \leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{\alpha}} dx, \tag{0.1}$$

where $0 \leq s < \alpha < 2$, $n > \alpha$, $2_\alpha^*(s) := \frac{2(n-s)}{n-\alpha}$ and $\gamma < \gamma_H(\alpha)$, the latter being the best fractional Hardy constant on $\mathbb{R}^n$. We address questions regarding the attainability of the best constant $C > 0$ attached to this inequality. This allows us to establish the existence of non-trivial weak solutions for the following doubly critical problem on $\mathbb{R}^n$,

$$L_{\gamma,\alpha} u = |u|^{2_\alpha^* - 2} u + \frac{|u|^{2_\alpha^*(s)-2} u}{|x|^s} \quad \text{in } \mathbb{R}^n \qquad \text{where } 2_\alpha^* := 2_\alpha^*(0).$$

We then look for least-energy solutions of the following linearly perturbed non-linear boundary value problem on bounded subdomains of $\mathbb{R}^n$ containing the singularity 0:

$$(L_{\gamma,\alpha} - \lambda I) u = \frac{u^{2_\alpha^*(s)-1}}{|x|^s} \quad \text{on } \Omega. \tag{0.2}$$

We show that if γ is below a certain threshold $\gamma_{crit}(\alpha)$, then such solutions exist for all $0 < \lambda < \lambda_1(L_{\gamma,\alpha})$, the latter being the first eigenvalue of $L_{\gamma,\alpha}$. On the other hand, for $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$, we prove existence of such solutions only for those λ in $(0, \lambda_1(L_{\gamma,\alpha}))$ for which Ω has a positive *fractional Hardy-Schrödinger mass* $m_{\gamma,\lambda}^{\alpha}(\Omega)$. This latter notion is introduced by way of an invariant of the linear equation $(L_{\gamma,\alpha} - \lambda I) u = 0$ on Ω.
We then study the effect of non-linear perturbation $h(x)u^{q-1}$, where $h \in C^0(\overline{\Omega})$, $h \geq 0$ and $2 < q < 2_\alpha^*$. Our analysis shows that the existence of solutions is guaranteed by this perturbation whenever $0 \leq \gamma \leq \gamma_{crit}(\alpha)$, while for $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$, it depends on both the perturbation and the geometry of the domain.

Lay Summary

Many physical and probabilistic phenomena can be described by either deterministic or stochastic differential equations. In this thesis, we focus on a class of partial differential equation driven by *non-local operators*. These describe long range interactions between the objects under study, and not only by those interacting with their closest neighbours. These equations have deep connections to probability theory and geometry, and they appear in many physical applications such as elasticity, thin obstacle, phase transition, flames propagation, as well as mathematical finance.

Preface

Much of this dissertation is adapted from three of the author's research papers: [38], [39] and [66]. In particular, Chapter 3, which evolves around the proof of Theorem 3.1 and 3.2, form the main content of [39], *Borderline variational problems involving fractional Laplacians and critical singularities.* All of Chapter 4 are adapted from [38], *Mass and asymptotics associated to fractional Hardy-Schrödinger operators in critical regimes.* Chapter 5, where the proof of Theorem 5.1 is presented, is in accordance with [66], *Existence results for non-linearly perturbed Hardy-Schrödinger problems: Local and non-local cases.* The first manuscript [39] (joint work with Dr. Ghoussoub) was published in Advanced Non-linear Studies Journal. The second paper [38] (joint work with Dr. Ghoussoub, Dr. Robert and Dr. Zhao) has been submitted, and the third one [66] has been written and will be submitted for publication soon.

Table of Contents

Acknowledgements

I would like to express my deep gratitude and appreciation to my supervisor, Professor Nassif Ghoussoub, for his invaluable guidance, direction, encouragement but also constructive criticism during the course of this dissertation. I thank him for not only caring about my current and future academic career, but also for his genuine interest, concern, and support for my well-being outside my academic life.

I would also like to acknowledge the profound gratitude I have for Professor Frédéric Robert, from whom i also learned so much. His role as a mentoring collaborator, and his persistent call for rigour throughout the progress of our work was absolutely critical for my quest to become a researcher.

Finally, I would like to thank my committee members Professor Juncheng Wei and Professor Stephen Gustafson for their instructive comments, and especially Professor Jun-cheng Wei, with whom I have had several motivating and inspiring discussions.

Chapter 1

Introduction

The main focus of research in this thesis is the study of borderline variational problems and corresponding functional inequalities involving the fractional Hardy-Schrödinger operator. This thesis is based on three research papers [38, 39, 66] that can be found in Chapters 3, 4 and 5. Each of the chapters begins with a detailed introduction to the results it contains. In this introduction, I will give a general description of the problems I address in this thesis.

Equations driven by non-local operators appear in many areas of mathematics such as probability theory, fluid mechanics, and geometry, as well as other sciences such as physics, and economics. In analysis and PDE, these phenomena originate in potential theory, conformal geometry, and a wide class of physical systems modeled by α-stable Levy processes and related stochastic interfaces. Primary examples of non-local operators are those represented by the fractional Laplacians $(-\Delta)^{\frac{\alpha}{2}}$ for $0 < \alpha < 2$, which are clear generalizations of the well-studied Laplace operator ($\alpha = 2$). Over the last decade or so, many well-known properties of standard elliptic and parabolic equations have been extended to their non-local counterparts. Much of the progress can be summarized as follows:

• Representation and regularity results such as those in Silvestre [62], Caffarelli-Silvestre [13, 14], Ros-Oton and Serra [58, 59] and others.

• Variatonal formulations and methods, such as in Servadei [65], Bisci-Radulescu-Servadei [6], Servadei-Valdinoci [64] and others.

• Conformal Geometry and Yamabe type problems, such as in Chang-Gonzalez [16], Gonzalez-Qing [42] and others.

In the first part of this thesis, we study problems of existence of equations involving the fractional Hardy-Schrödinger operator on $\mathbb{R}^n$. In [39], N. Ghoussoub and I consider issues of existence of the variational solutions of the following borderline problem associated with the fractional Hardy-Schrödinger operator $L_{\gamma,\alpha} := (-\Delta)^{\frac{\alpha}{2}} - \frac{\gamma}{|x|^\alpha}$ on $\mathbb{R}^n$:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u - \gamma \frac{u}{|x|^\alpha} = \frac{u^{2^*_\alpha(s)-1}}{|x|^s} & \text{in } \mathbb{R}^n \\ u \geq 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $0 < \alpha < 2$, $0 \leq s < \alpha < n$, $2^*_\alpha(s) = \frac{2(n-s)}{n-\alpha}$, and $\gamma < \gamma_H(\alpha) = 2^\alpha \frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})}$ is the best fractional Hardy constant on $\mathbb{R}^n$. For any $\alpha \in (0, 2)$, the fractional space $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is defined as the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|_{H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |2\pi\xi|^\alpha |\mathcal{F}u(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx,$$

where $\mathcal{F}(u)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} u(x)dx$ denotes the Fourier transform of u. In order to study problem (1.1), we address the attainability of the best constant corresponding to the fractional Hardy-Sobolev inequality in $\mathbb{R}^n$, that is

$$\mu_{\gamma,s,\alpha}(\mathbb{R}^n) := \inf_{u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx}{\left(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}}}. \tag{1.2}$$

In the following, we check for which parameters γ and s, the best constant $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ is attained. Using the Caffarelli-Silvestre representation [12] and Ekeland's variational principle [25], we establish the following.

Theorem 1.1 (Ghoussoub-Shakerian [39]). *Suppose $0 < \alpha < 2$, $0 \leq s < \alpha < n$, and $\gamma < \gamma_H(\alpha) := 2^\alpha \frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})}$.*

1. *If either $\{s > 0\}$ or $\{s = 0$ and $\gamma \geq 0\}$, then $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ is attained.*
2. *If $s = 0$ and $\gamma < 0$, then there are no extremals for $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$.*
3. *If either $\{0 < \gamma < \gamma_H(\alpha)\}$ or $\{0 < s < \alpha$ and $\gamma = 0\}$, then any non-negative minimizer for $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ is positive, radially symmetric, radially decreasing, and approaches zero as $|x| \to \infty$.*

The results in Theorem 1.1 allow us to show the existence of non-trivial weak solutions for the following doubly critical problem on $\mathbb{R}^n$,

$$(-\Delta)^{\frac{\alpha}{2}} u - \gamma \frac{u}{|x|^\alpha} = |u|^{2^*_\alpha - 2} u + \frac{|u|^{2^*_\alpha(s)-2} u}{|x|^s} \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

where $2^*_\alpha := \frac{2n}{n-\alpha}$ is the critical α-fractional Sobolev exponent, and $0 \leq \gamma < \gamma_H(\alpha)$. We used the Caffarelli-Silvestre representation [12] and the Mountain Pass lemma [3] to prove the following result.

Theorem 1.2 (Ghoussoub-Shakerian [39])**.** *Let $0 < \alpha < 2$, $0 < s < \alpha < n$ and $0 \le \gamma < \gamma_H(\alpha)$. Then, there exists a non-trivial weak solution of (1.3).*

In the next step, we use the ground state representation introduced by Frank-Lieb-Seiringer [34], and Moser iteration, to establish the following asymptotic properties of the extremals of $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ at zero and infinity.

Theorem 1.3 (Ghoussoub-Robert-Shakerian-Zhao [38])**.** *Assume $0 \le s < \alpha < 2$, $n > \alpha$ and $0 \le \gamma < \gamma_H(\alpha)$. Then, any positive extremal $u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ for $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ satisfies $u \in C^1(\mathbb{R}^n \setminus \{0\})$ and*

$$\lim_{x\to 0} |x|^{\beta_-(\gamma)} u(x) = \lambda_0 \text{ and } \lim_{|x|\to\infty} |x|^{\beta_+(\gamma)} u(x) = \lambda_\infty, \tag{1.4}$$

where $\lambda_0, \lambda_\infty > 0$ and $\beta_-(\gamma)$ (resp., $\beta_+(\gamma)$) is the unique solution in $\left(0, \frac{n-\alpha}{2}\right)$ (resp., in $\left(\frac{n-\alpha}{2}, n-\alpha\right)$) of the equation

$$\Psi_{n,\alpha}(t) := 2^\alpha \frac{\Gamma(\frac{n-t}{2})\Gamma(\frac{\alpha+t}{2})}{\Gamma(\frac{n-t-\alpha}{2})\Gamma(\frac{t}{2})} = \gamma.$$

In particular, there exist $C_1, C_2 > 0$ such that

$$\frac{C_1}{|x|^{\beta_-(\gamma)} + |x|^{\beta_+(\gamma)}} \le u(x) \le \frac{C_2}{|x|^{\beta_-(\gamma)} + |x|^{\beta_+(\gamma)}} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

Remark 1.4. *We point out that the functions $u_1(x) = |x|^{-\beta_-(\gamma)}$ and $u_2(x) = |x|^{-\beta_+(\gamma)}$ are the fundamental solutions for the fractional Hardy-Schrödinger operator $L_{\gamma,\alpha} := (-\Delta)^{\frac{\alpha}{2}} - \frac{\gamma}{|x|^\alpha}$ on $\mathbb{R}^n$. Indeed, a straightforward computation yields (see Section 4.2)*

$$L_{\gamma,\alpha}|x|^{-\beta} = (\Psi_{n,\alpha}(\beta) - \gamma)|x|^{-\beta} = 0 \text{ on } \mathbb{R}^n \setminus \{0\} \quad \text{for } \beta \in \{\beta_-(\gamma), \beta_+(\gamma)\},$$

which implies that $\beta_+(\gamma)$ and $\beta_-(\gamma)$ satisfy $\Psi_{n,\alpha}(\beta) = \gamma$.

We then tackle the more challenging problems on bounded domains Ω of $\mathbb{R}^n$ with an interior singularity by considering the role of linear and non-linear perturbations on the existence of positive solutions. More precisely, if Ω is now a smooth bounded domain in $\mathbb{R}^n$ containing 0 in its interior, we then consider the fractional Sobolev space $H_0^{\frac{\alpha}{2}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{H_0^{\frac{\alpha}{2}}(\Omega)} = \frac{c_{n,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+\alpha}} dxdy \quad \text{where } C_{n,\alpha} = \frac{2^\alpha \Gamma\left(\frac{n+\alpha}{2}\right)}{\pi^{\frac{n}{2}} \left|\Gamma\left(-\frac{\alpha}{2}\right)\right|},$$

as well as the best constant in the fractional Hardy-Sobolev inequality on domain Ω, namely,

$$\mu_{\gamma,s,\alpha}(\Omega) := \inf_{u \in H_0^{\frac{\alpha}{2}}(\Omega)\setminus\{0\}} \frac{\frac{c_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy - \gamma\int_\Omega \frac{|u|^2}{|x|^\alpha}dx}{\left(\int_\Omega \frac{|u|^{2_\alpha^*(s)}}{|x|^s}dx\right)^{\frac{2}{2_\alpha^*(s)}}}. \tag{1.5}$$

As in the local case, one can show by translating, scaling and cutting off the extremals of $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ that $\mu_{\gamma,s,\alpha}(\Omega) = \mu_{\gamma,s,\alpha}(\mathbb{R}^n)$, which means that $\mu_{\gamma,s,\alpha}(\Omega)$ has no extremals if Ω is bounded. We therefore resort to a setting popularized by Brezis-Nirenberg [11] in the local case, where one dehomogenizes the problem by considering the following equation:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}u - \gamma\frac{u}{|x|^\alpha} = \frac{u^{2_\alpha^*(s)-1}}{|x|^s} + \lambda u & \text{in } \Omega \\ \qquad\qquad u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.6}$$

where $0 < \lambda < \lambda_1(L_{\gamma,\alpha})$ and $\lambda_1(L_{\gamma,\alpha})$ is the first eigenvalue of the operator $L_{\gamma,\alpha} = (-\Delta)^{\frac{\alpha}{2}} - \frac{\gamma}{|x|^\alpha}$ with the Dirichlet boundary condition. One then considers the quantity

$$\mu_{\gamma,s,\alpha,\lambda}(\Omega) = \inf_{u \in H_0^{\frac{\alpha}{2}}(\Omega)\setminus\{0\}} K_\Omega(u),$$

where

$$K_\Omega(u) := \frac{\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy - \gamma\int_\Omega \frac{u^2}{|x|^\alpha}dx - \lambda\int_\Omega u^2 dx}{\left(\int_\Omega \frac{u^{2_\alpha^*(s)}}{|x|^s}dx\right)^{\frac{2}{2_\alpha^*(s)}}},$$

and uses the fact that compactness is restored as long as $\mu_{\gamma,s,\alpha,\lambda}(\Omega) < \mu_{\gamma,s,\alpha}(\mathbb{R}^n)$; see [11, 37]. Janelli [45] showed that the behaviour of problem (1.6) - in the case when $\alpha = 2$ and $s = 0$ - is deeply influenced by the value of the parameter γ. Roughly speaking, when γ is sufficiently near to $\gamma_H(\alpha)$ then problem (1.6) becomes critical, in the sense of Pucci-Serrin [55]. Following [45], we prove that there exists a constant $0 < \gamma_{crit}(\alpha) < \gamma_H(\alpha)$ such that the operator $L_{\gamma,\alpha}$ becomes critical when $\gamma \in (\gamma_{crit}(\alpha), \gamma_H(\alpha))$.

We also show that the existence results in [64] can be extended to the case when $0 < s < \alpha$ and $0 < \gamma < \gamma_H(\alpha)$ as long as the operator $L_{\gamma,\alpha}$ is not critical.

The critical case is more delicate and the existence of solutions requires the domain Ω to satisfy a *positive mass condition,* defined as follows. Indeed, following the work of [36, 37] in the local setting, we define the fractional Hardy singular interior mass of a domain is the following way.

Theorem 1.5 (Ghoussoub-Robert-Shakerian-Zhao [38]). *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ $(n > \alpha)$ and consider, for $0 < \alpha < 2$, the boundary value problem*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} H - \left(\frac{\gamma}{|x|^\alpha} + a(x)\right) H = 0 & \text{in } \Omega \setminus \{0\} \\ H > 0 & \text{in } \Omega \setminus \{0\} \\ H = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.7}$$

where $a(x) \in C^{0,\tau}(\overline{\Omega})$ for some $\tau \in (0,1)$. Assuming the operator $(-\Delta)^{\frac{\alpha}{2}} - (\frac{\gamma}{|x|^\alpha} + a(x))$ is coercive, there exists then a threshold $-\infty < \gamma_{crit}(\alpha) < \gamma_H(\alpha)$ such that for any γ with $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$, there exists a unique solution to (1.7) (in the sense of Definition 4.6) $H : \Omega \to \mathbb{R}$, $H \not\equiv 0$, and a constant $c \in \mathbb{R}$ such that

$$H(x) = \frac{1}{|x|^{\beta_+(\gamma)}} + \frac{c}{|x|^{\beta_-(\gamma)}} + o\left(\frac{1}{|x|^{\beta_-(\gamma)}}\right) \qquad \text{as } x \to 0.$$

We define the fractional Hardy-singular internal mass *of Ω associated to the operator $L_{\gamma,\alpha}$ to be*

$$m^\alpha_{\gamma,a}(\Omega) := c \in \mathbb{R}.$$

We then establish the following results.

Theorem 1.6 (Ghoussoub-Robert-Shakerian-Zhao [38]). *Let Ω be a smooth bounded domain in $\mathbb{R}^n (n > \alpha)$ such that $0 \in \Omega$, and let $0 \le s < \alpha$, $0 \le \gamma < \gamma_H(\alpha)$, and $0 < \lambda < \lambda_1(L_{\gamma,\alpha})$. Then, there are extremals for $\mu_{\gamma,s,\alpha,\lambda}(\Omega)$ under one of the following two conditions:*

(1) $0 \le \gamma \le \gamma_{crit}(\alpha)$

(2) $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$ and $m^\alpha_{\gamma,\lambda}(\Omega)$ is positive.

One can then complete the picture as follows.

Linearly perturbed problem (1.6) with $0 \in \Omega$ and $0 < \lambda < \lambda(L_{\gamma,\alpha})$

Hardy term	Dimension	Singularity	Analytic. cond.	Ext.
$0 \le \gamma \le \gamma_{crit}(\alpha)$	$n \ge 2\alpha$	$s \ge 0$	$\lambda > 0$	Yes
$\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$	$n \ge 2\alpha$	$s \ge 0$	$m^\alpha_{\gamma,\lambda}(\Omega) > 0$	Yes
$0 \le \gamma < \gamma_H(\alpha)$	$\alpha < n < 2\alpha$	$s \ge 0$	$m^\alpha_{\gamma,\lambda}(\Omega) > 0$	Yes

In this direction, we address the following non-linearly perturbed problem associated with the operator $L_{\gamma,\alpha} - \lambda I$ on bounded domains Ω :

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u - \gamma \dfrac{u}{|x|^{\alpha}} - \lambda u = \dfrac{u^{2^*_\alpha(s)-1}}{|x|^s} + h(x)u^{q-1} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.8}$$

where $h \in C^0(\overline{\Omega})$, $h \geq 0$ and $q \in (2, 2^*_\alpha)$. We study the combined effect of the non-linear perturbation ($i.e.$, $h(x)u^{q-1}$) and the geometry of the domain (i.e., *the mass* introduced in Theorem 1.5) on the existence of a positive solution for (1.8).

Inspired by the work of Jaber [44], in a Riemannian context, we investigate the existence of solutions using the Mountain Pass Lemma of Ambrosetti- Rabinowitz [3] (see Lemma 3.6). It turns out that the existence of a solution for (1.8) depends only on the non-linear perturbation whenever the operator $L_{\gamma,\alpha}$ is non-critical (i.e., when $0 \leq \gamma \leq \gamma_{crit}(\alpha)$). On the other hand, in the critical case (i.e., when $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$), solvabiliy can either depend on the non-linear perturbation or the global geometry of the domain, or both. The difference between these regimes is determined by another threshold, but this time on the values of q in $(2, 2^*_\alpha)$.

We shall utilize ideas of [38] and [44] to prove the following.

Theorem 1.7 (Shakerian[66])**.** *Let Ω be a smooth bounded domain in $\mathbb{R}^n (n > \alpha)$ such that $0 \in \Omega$, and let $2^*_\alpha(s) := \frac{2(n-s)}{n-\alpha}$, $0 \leq s < \alpha$, $-\infty < \lambda < \lambda_1(L_{\gamma,\alpha})$, and $0 \leq \gamma < \gamma_H(\alpha)$. We also assume that $2 < q < 2^*_\alpha$, $h \in C^0(\overline{\Omega})$ and $h \geq 0$. Then, there exists a non-negative solution $u \in H_0^{\frac{\alpha}{2}}(\Omega)$ to (1.8) under one of the following conditions:*

(1) $0 \leq \gamma \leq \gamma_{crit}(\alpha)$ *and* $h(0) > 0$.

(2) $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$ *and* $\begin{cases} h(0) > 0 & \text{if } q > q_{crit} \\ c_1 h(0) + c_2 m^{\alpha}_{\gamma,\lambda}(\Omega) > 0 & \text{if } q = q_{crit} \\ m^{\alpha}_{\gamma,\lambda}(\Omega) > 0 & \text{if } q < q_{crit}. \end{cases}$

Here c_1, c_2 are two positive constant that can be computed explicitly (see Section 5.4), while $q_{crit} = 2^*_\alpha - 2\frac{\beta_+ - \beta_-}{n-\alpha} \in (2, 2^*_\alpha)$.

In order to summarize the results in Theorem 1.7, we set $\gamma_{crit} := \gamma_{crit}(\alpha)$, $\gamma_H := \gamma_H(\alpha)$ and $m^{\alpha}_{\gamma,\lambda} := m^{\alpha}_{\gamma,\lambda}(\Omega)$, and assume that $0 \leq s < \alpha$ and $2 < q < 2^*_\alpha$.

One can then complete the picture as follows.

Non-linearly perturbed problem (1.8): $n > \alpha$ and $\lambda < \lambda(L_{\gamma,\alpha})$

Hardy term	Dim.	q	λ	Analytic. cond	Ext.
$0 \le \gamma \le \gamma_{crit}$	$n \ge 2\alpha$	> 2	$> -\infty$	$h(0) > 0$	Yes
$\gamma_{crit} < \gamma < \gamma_H$ $0 \le \gamma < \gamma_H$	$n \ge 2\alpha$ $n < 2\alpha$	$> q_{crit}$	$> -\infty$	$h(0) > 0$	Yes
$\gamma_{crit} < \gamma < \gamma_H$ $0 \le \gamma < \gamma_H$	$n \ge 2\alpha$ $n < 2\alpha$	$= q_{crit}$	$> -\infty$	$c_1 h(0) + c_2 m^{\alpha}_{\gamma,\lambda} > 0$	Yes
$\gamma_{crit} < \gamma < \gamma_H$ $0 \le \gamma < \gamma_H$	$n \ge 2\alpha$ $n < 2\alpha$	$< q_{crit}$	> 0	$m^{\alpha}_{\gamma,\lambda} > 0$	Yes

Chapter 2

Preliminaries and a Description of The Functional Setting

In this chapter, we introduce the concepts and the classical results related to the non-local framework that will be the focus of this dissertation. Our main sources for this chapter are [6] and [20].

The basic operator is the so-called fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$. This chapter is devoted to introducing its various formulations, its associated fractional Sobolev spaces as well as the corresponding fractional Hardy-Sobolev inequalities. Since our main focus in this thesis is the study of fractional laplacians with Dirichlet boundary data via variational methods, we first introduce suitable function spaces required for the variational principles to apply. In Section 2.3, we introduce a weighted Sobolev space in higher dimension $\mathbb{R}^n \times (0,\infty)$, which will provide a suitable function space for our approach. Its construction is based on the α-extension formula that Caffarelli-Silvestre [12] associate to equations involving fractional laplacians. We later introduce functional inequalities involving the fractional Hardy-Schrödinger operator $L_{\gamma,\alpha}$, which play a crucial role in the non-local problems under study.

2.1 The fractional Laplacian operator

In this section, we start by presenting the different definitions for the fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$. First, consider the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ which consists of all rapidly decreasing functions in $C^\infty(\mathbb{R}^n)$, equipped with the following norm: If

$$\mathbb{Z}^n_+ = \{a = (a_1, a_2, ..., a_n) : a_i \geq 0 \text{ and } a_i \in \mathbb{Z} \text{ for } i = 1, 2, 3, ..., n\},$$

and

$$\partial^a := (\frac{\partial}{\partial x_1})^{a_1}(\frac{\partial}{\partial x_2})^{a_2}...(\frac{\partial}{\partial x_n})^{a_n},$$

then, the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of all functions $\psi \in C^\infty(\mathbb{R}^n)$ such that

$$\|\psi\|_{a,b} = \sup_{x\in\mathbb{R}^n} |x^a \partial^b \psi(x)|$$

is finite for every pair of multi-indices $a, b \in \mathbb{Z}_+^n$.

Remark 2.1. *An element of $\mathcal{S}(\mathbb{R}^n)$ is essentially a smooth function $\psi(x)$ such that $\psi(x)$ and all of its derivatives exist everywhere on $\mathbb{R}$ and decay faster than any inverse power of x, as $|x| \to \pm\infty$. In particular, $\mathcal{S}(\mathbb{R}^n)$ is a subspace of the function space $C^\infty(\mathbb{R}^n)$ of infinitely differentiable functions.*

The Fourier transform of a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ is then defined as

$$\mathcal{F}\psi(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\psi(x)dx. \tag{2.1}$$

Note that for every $\psi \in \mathcal{S}(\mathbb{R}^n)$, we have $\mathcal{F}\psi \in \mathcal{S}(\mathbb{R}^n)$. It is easy to verify that the Fourier transform and the inverse Fourier transform, given by

$$\mathcal{F}^{-1}\psi(x) := \int_{\mathbb{R}^n} e^{2\pi i\xi\cdot x}\psi(\xi)d\xi$$

are both continuous on $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. Moreover, we have

$$\mathcal{F}\mathcal{F}^{-1}\psi = \mathcal{F}^{-1}\mathcal{F}\psi = \psi,$$

which implies that each of $\mathcal{F}$ and $\mathcal{F}^{-1}$ is an isomorphism and a homomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$.

The dual space of $\mathcal{S}(\mathbb{R}^n)$, denoted by $\mathcal{S}'(\mathbb{R}^n)$, is the space of continuous linear functionals $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$. The elements of $\mathcal{S}'(\mathbb{R}^n)$ are called tempered distributions. The space $\mathcal{S}'(\mathbb{R}^n)$ is a linear space under the pointwise addition and scalar multiplication of functionals. If $T \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier transform of T can be defined as the tempered distribution given by

$$\langle \mathcal{F}T, \psi\rangle = \langle T, \mathcal{F}\psi\rangle \quad \text{for every } \psi \in \mathcal{S}(\mathbb{R}^n),$$

where $\langle ., .\rangle$ denotes the usual duality bracket between $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$. One can use the definition (2.1) to prove the Parseval-Plancherel formula (2.2), which will be crucial in what follows for proving the equivalence between the fractional spaces will be defined in the next section; see formula (2.6).

$$\|u\|^2_{L^2(\mathbb{R}^n)} = \|\mathcal{F}u\|^2_{L^2(\mathbb{R}^n)} \quad \text{for all } u \in L^2(\mathbb{R}^n). \tag{2.2}$$

For a detailed introduction to the classical theory of distribution and Fourier transform, we refer to the monograph [60] and to the recent book [18] for several applications to elliptic problems of linear and nonlinear functional analysis.

We now present various definitions for the fractional Laplacian operator on Schwartz space $\mathcal{S}(\mathbb{R}^n)$. In the first definition, we define the operator $(-\Delta)^{\frac{\alpha}{2}}$ as a singular integral.

Definition 2.2. *For any $\alpha \in (0,2)$, the fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$ is defined on the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ as*

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}}u &= C_{n,\alpha}P.V.\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}dxdy\\ &= C_{n,\alpha}\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\epsilon(x)}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}dxdy,\end{aligned}$$

where P.V. stands for the principal-value and

$$C_{n,\alpha} = \left(\int_{\mathbb{R}^n}\frac{1-\cos(\zeta_1)}{\zeta}d\zeta\right)^{-1}.$$

We refer the readers to [6, Section 1.3.1] and [20] for properties of the constant $C_{n,\alpha}$. It has been shown there that

$$C_{n,\alpha} = \left(\int_{\mathbb{R}^n}\frac{1-\cos(\zeta_1)}{\zeta}d\zeta\right)^{-1} = \frac{2^\alpha\Gamma(\frac{n+\alpha}{2})}{\pi^{\frac{n}{2}}|\Gamma(-\frac{\alpha}{2})|}.$$

The operator $(-\Delta)^{\frac{\alpha}{2}}$ can be also defined via the Fourier transform. It was shown in [20, Proposition 3.3] that the fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$ can be viewed as a pseudo-differential operator as follows.

Definition 2.3. *Let $\alpha \in (0,2)$. For any $u \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$(-\Delta)^{\frac{\alpha}{2}}u = \mathcal{F}^{-1}(|2\pi\xi|^\alpha(\mathcal{F}u)) \quad \forall\xi\in\mathbb{R}^n. \tag{2.3}$$

There is another way to define the fractional Laplace operator via the inverse Fourier transform (2.3). In fact, in the case $\alpha = 1$, the half-Laplacian acting on a function u in the whole space $\mathbb{R}^n$ can be computed as the normal derivative on the boundary of its harmonic extension to the upper half-space $\mathbb{R}^{n+1} := \mathbb{R}^n \times (0,+\infty)$. In other words, it is the Dirichlet to Neumann operator (see [12]). The operator $(-\Delta)^{\frac{\alpha}{2}}$ can be characterized in a similar way, by defining its α-harmonic extension to the upper half-space (See formula 2.5). Indeed, Caffarelli-Silvestre [12] proved the following:

Definition 2.4. *Let $\alpha \in (0,2)$. The fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$ on $\mathbb{R}^n$ can be expressed on the higher dimension $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ in the following way:*

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \frac{\partial w}{\partial \nu^\alpha} := -k_\alpha \lim_{y \to 0^+} y^{1-\alpha} \frac{\partial w}{\partial y}(x,y), \tag{2.4}$$

where $k_\alpha = \frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}$, and $w : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ is the α-harmonic extension of u, that solves

$$\begin{cases} \operatorname{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ w = u & \text{on } \mathbb{R}^n \times \{y=0\}. \end{cases} \tag{2.5}$$

See also [6, 20] and references therein for the basics on the fractional Laplacian.

2.2 The fractional Sobolev spaces

In this section, we introduce the classical fractional Sobolev space on $\mathbb{R}^n$ and its bounded subsets.

2.2.1 The classical fractional Sobolev space

For $\alpha \in (0,2)$, the classical fractional Sobolev space of order $\frac{\alpha}{2}$ is defined as

$$H^{\frac{\alpha}{2}}(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) \text{ such that } \int_{\mathbb{R}^n} (1 + |2\pi\xi|^\alpha)|\mathcal{F}u(\xi)|^2 d\xi < \infty \right\}$$

equipped with the norm

$$\|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \|\mathcal{F}u(\xi)\|_{L^2(\mathbb{R}^n)} + \||2\pi\xi|^{\frac{\alpha}{2}}\mathcal{F}u(\xi)\|_{L^2(\mathbb{R}^n)}.$$

One can use Parseval-Plancherel's formula (2.2) to rewrite this norm as

$$\|u\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} |2\pi\xi|^\alpha |\mathcal{F}(u)(\xi)|^2 d\xi.$$

The space $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is well defined and is a Hilbert space for every $\alpha > 0$. The following relation between the fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$ and the fractional Sobolev space $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ was proved in [34, Lemma 3.1]; see also [20, Proposition 3.4]:

$$\int_{\mathbb{R}^n} |2\pi\xi|^\alpha |\mathcal{F}u(\xi)|^2 d\xi = \frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}} dxdy, \tag{2.6}$$

for all $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$. Using this relation, one can obtain a new expression for the norm of the space $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$:

$$\|u\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy. \tag{2.7}$$

Consequently, the space $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ can be defined as the linear space of Lebesgue measurable functions u from $\mathbb{R}^n$ to $\mathbb{R}$ such that the norm defined in (2.7) is finite.

As in the classical case, any function in the fractional Sobolev space $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ can be approximated by a sequence of smooth functions with compact support. Indeed, for $0 < \alpha < 2$,

$$H^{\frac{\alpha}{2}}(\mathbb{R}^n) := \overline{C_0^\infty(\mathbb{R}^n)}^{\|\,.\,\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)}}.$$

In addition, the operator $(-\Delta)^{\frac{\alpha}{2}}$ can be extended by density from $\mathcal{S}(\mathbb{R}^n)$ to $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$. In this way, the associated scalar product can be formulated as follows

$$\begin{aligned}\langle u,v\rangle_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)} &:= \langle u,v\rangle + (u,v)\\ &= \frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}dxdy + \int_{\mathbb{R}^n} uvdx.\end{aligned}$$

Clearly, the norm $\|\,.\,\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)}$ defined in (2.7) induced by this scalar product. The following proposition states a crucial formulation which will be used freely in this thesis and it is a direct consequence of (2.6) coupled with Proposition 3.6 in [20].

Proposition 2.5. *Let $0 < \alpha < 2$ and $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$. Then,*

$$\int_{\mathbb{R}^n}|2\pi\xi|^\alpha|\mathcal{F}u(\xi)|^2 d\xi = \int_{\mathbb{R}^n}|(-\Delta)^{\frac{\alpha}{4}}u|^2dx = \frac{C_{n,\alpha}}{2}\int_{(\mathbb{R}^n)^2}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy.$$

Let now $\alpha \in (0,2)$ be fixed and Ω be an open-bounded subset of $\mathbb{R}^n$ with $(n > \alpha)$. We define $H^{\frac{\alpha}{2}}(\Omega)$ as the space of measurable functions u such that the following norm is finite

$$\|u\|^2_{H^{\frac{\alpha}{2}}(\Omega)} = \|u\|^2_{L^2(\Omega)} + \frac{C_{n,\alpha}}{2}\int_{\Omega}\int_{\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy. \tag{2.8}$$

We also define $\overline{H}_0^{\frac{\alpha}{2}}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\,.\,\|_{H^{\frac{\alpha}{2}}(\Omega)}$, that is

$$\overline{H}_0^{\frac{\alpha}{2}}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\,.\,\|_{H^{\frac{\alpha}{2}}(\Omega)}}.$$

One can find many properties of the above spaces in [20] and [6, Section 1].

Remark 2.6. *It is well known that if $u \in H^1(\Omega)$, which is the classical Sobolev space corresponding to the case $\alpha = 2$, then its null extension outside Ω, $\bar{u}$, is in $H^1(\mathbb{R}^n)$ and $\|u\|_{H^1(\Omega)} = \|\bar{u}\|_{H^1(\mathbb{R}^n)}$. This is, however, not true in general for functions in $H^{\frac{\alpha}{2}}(\Omega)$, for $0 < \alpha < 2$.*

In order to deal with problems on bounded domain, we shall need function spaces where null extensions are controlled. This will be done in the next section.

2.2.2 The variational fractional Sobolev space

We now consider the following Hilbert space on $\mathbb{R}^n$:

Definition 2.7. *For $0 < \alpha < 2$, we define the fractional Sobolev space $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ as the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm*

$$\|u\|^2_{H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx = \frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+\alpha}} dxdy.$$

Let $\mathcal{O} = \mathbb{R}^n \times \mathbb{R}^n \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$, where $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$, and define $\dot{H}^{\frac{\alpha}{2}}(\Omega)$ as the linear space of all Lebesgue measurable functions u from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function u in $\dot{H}^{\frac{\alpha}{2}}(\Omega)$ belongs to $L^2(\Omega)$, and

$$\text{the mapp } (x,y) \mapsto (u(x) - u(y))|x-y|^{\frac{1}{2}} \text{ is in } L^2(\mathcal{O}, dxdy).$$

The norm in $\dot{H}^{\frac{\alpha}{2}}(\Omega)$ is defined as follows:

$$\|u\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} = \|u\|^2_{L^2(\Omega)} + \frac{C_{n,\alpha}}{2} \int_{\mathcal{O}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+\alpha}} dxdy. \tag{2.9}$$

Remark 2.8. *Note that the norms in (2.8) and (2.9) are not the same because $\Omega \times \Omega$ is strictly contained in $\mathcal{O}$: this makes the classical fractional Sobolev space approach not sufficient for studying the nonlocal problem we consider in this thesis.*

We are now ready to introduce a suitable function space on a bounded domain Ω for the variational setting that will be needed later in this thesis. One can easily check that if $u \in \dot{H}^{\frac{\alpha}{2}}(\Omega)$, then the null extension of u outside Ω is in $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$. This allows us to define the desired space as

$$H_0^{\frac{\alpha}{2}}(\Omega) = \left\{ u \in \dot{H}^{\frac{\alpha}{2}}(\Omega) \text{ such that } u = 0 \text{ a.e. in } \mathcal{C}\Omega \right\}.$$

Observe that since $u = 0$ in $\mathcal{C}\Omega$, the integral in formula (2.9) can be extended to $\mathbb{R}^n$; that is for any $H_0^{\frac{\alpha}{2}}(\Omega)$, the norm can be written as:

$$\|u\|^2_{H_0^{\frac{\alpha}{2}}(\Omega)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy. \tag{2.10}$$

It has been shown in [6] that the norm on $H_0^{\frac{\alpha}{2}}(\Omega)$ defined by (2.10) is equivalent to the following norm – still denoted by $\|\ .\ \|_{H_0^{\frac{\alpha}{2}}(\Omega)}$:

$$\|u\|^2_{H_0^{\frac{\alpha}{2}}(\Omega)} = \frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy.$$

The density properties of smooth and compactly supported functions in space $H_0^{\frac{\alpha}{2}}(\Omega)$ is studied in [6], and it was proved there that the space $C_0^\infty(\Omega)$ is a dense subspace of $H_0^{\frac{\alpha}{2}}(\Omega)$; see [6, Theorem 2.6]. So, it follows from this density result that the space $H_0^{\frac{\alpha}{2}}(\Omega)$ eventually can be expressed in the following way:

Definition 2.9. *Let $0 < \alpha < 2$. We define the fractional Sobolev space $H_0^{\frac{\alpha}{2}}(\Omega)$ on the smooth bounded domain Ω as the completion of $C_0^\infty(\Omega)$ under the norm*

$$\|u\|^2_{H_0^{\frac{\alpha}{2}}(\Omega)} = \frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy. \tag{2.11}$$

2.3 The α-harmonic extension and weighted Sobolev space

In this subsection, we present an useful representation given in [5] and [9] for the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ as a trace class operator, as well as a corresponding representation for the space $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$.

For a function $u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$, let $w = E_\alpha(u)$ be its α-harmonic extension to the upper half-space, $\mathbb{R}^{n+1}_+$, that is the solution of:

$$\begin{cases} \operatorname{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ w = u & \text{on } \mathbb{R}^n \times \{y = 0\}. \end{cases}$$

Recall that the extension function $w = E_\alpha(u)$ satisfies (2.4) and belongs to the Hilbert space $X^\alpha(\mathbb{R}^{n+1}_+)$ defined as the closure of $C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$ for the norm

$$\|w\|_{X^\alpha(\mathbb{R}^{n+1}_+)} := \left(k_\alpha\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla w|^2 dxdy\right)^{\frac{1}{2}},$$

where $k_\alpha = \frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}$ is a normalization constant chosen in such a way that the extension operator $E_\alpha(u) : H_0^{\frac{\alpha}{2}}(\mathbb{R}^n) \to X^\alpha(\mathbb{R}^{n+1}_+)$ is an isometry, that is, for any $u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$, we have

$$\|E_\alpha(u)\|_{X^\alpha(\mathbb{R}^{n+1}_+)} = \|u\|_{H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \|(-\Delta)^{\frac{\alpha}{4}} u\|_{L^2(\mathbb{R}^n)}. \tag{2.12}$$

Conversely, for a function $w \in X^\alpha(\mathbb{R}^{n+1}_+)$, we denote its trace on $\mathbb{R}^n \times \{y = 0\}$ as $\mathrm{Tr}(w) := w(.,0)$. This trace operator is also well defined and satisfies

$$\|w(.,0)\|_{H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)} \leq \|w\|_{X^\alpha(\mathbb{R}^{n+1}_+)}. \tag{2.13}$$

We shall frequently use the following useful fact: Since $\alpha \in (0,2)$, the weight $y^{1-\alpha}$ belongs to the Muckenhoupt class A_2; [56], which consists of all non-negative functions w on $\mathbb{R}^n$ satisfying for some constant C, the estimate

$$\sup_B\Big(\frac{1}{|B|}\int_B w dx\Big)\Big(\frac{1}{|B|}\int_B w^{-1} dx\Big) \leq C, \tag{2.14}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

If $\Omega \subset \mathbb{R}^{n+1}$ is an open domain, we denote by $L^2(\Omega, |y|^{1-\alpha})$ the space of all measurable functions on Ω such that $\|w\|^2_{L^2(\Omega,|y|^{1-\alpha})} = \int_\Omega |y|^{1-\alpha}|w|^2 dxdy < \infty$, and by $H^1(\Omega, |y|^{1-\alpha})$ the weighted Sobolev space

$$H^1(\Omega, |y|^{1-\alpha}) = \left\{ w \in L^2(\Omega, |y|^{1-\alpha}) : \nabla w \in L^2(\Omega, |y|^{1-\alpha}) \right\}.$$

It is remarkable that most of the properties of classical Sobolev spaces, including the embedding theorems have a weighted counterpart as long as the weight is in the Muckenhoupt class A_2; see [28] and [41]. Note that $H^1(\mathbb{R}^{n+1}_+, y^{1-\alpha})$ -up to a normalization factor- is also isometric to $X^\alpha(\mathbb{R}^{n+1}_+)$.

2.4 Fractional Hardy-Sobolev type inequalities

The starting point of the study of existence of weak solutions of borderline variational problems (1.1), (1.3) and (1.6) is based on the following fractional Hardy-Sobolev inequalities which guarantee that the associated functionals are well defined and sometimes bounded below on the right function spaces.

We start with the fractional Sobolev inequality [19], which asserts that for $n > \alpha$ and $0 < \alpha < 2$, there exists a constant $C(n,\alpha) > 0$ such that

$$\Big(\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx\Big)^{\frac{2}{2^*_\alpha}} \leq C(n,\alpha) \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx \quad \text{for all } u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n), \tag{2.15}$$

where $2^*_\alpha = \frac{2n}{n-\alpha}$.

Another important inequality is the fractional Hardy inequality (see [34] and [43]), which states that under the same conditions on n and α, we have

$$\gamma_H(\alpha) \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx \quad \text{for all } u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n), \tag{2.16}$$

where $\gamma_H(\alpha)$ is the best constant in the above inequality on $\mathbb{R}^n$, that is

$$\gamma_H(\alpha) := \inf \left\{ \frac{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx}{\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx};\ u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n) \setminus \{0\} \right\}. \tag{2.17}$$

It has also been shown there that $\gamma_H(\alpha) = 2^\alpha \frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})}$. Note that $\gamma_H(\alpha)$ converges to the best classical Hardy constant $\gamma_H(2) = \frac{(n-2)^2}{4}$ when $\alpha \to 2$. By interpolating these inequalities via Hölder's inequality, one gets the following fractional Hardy-Sobolev inequalities.

Lemma 2.10 (Fractional Hardy-Sobolev Inequalities)**.** *Assume that* $0 < \alpha < 2$ *and* $0 \leq s \leq \alpha < n$. *Then, there exists positive constant c such that*

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}} \leq c \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx \quad \textit{for all } u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n). \tag{2.18}$$

Moreover, if $\gamma < \gamma_H(\alpha) = 2^\alpha \frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})}$, *then*

$$C\left(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}} \leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx, \tag{2.19}$$

for all $u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ *and some positive constant C.*

Proof of Lemma 2.10. Note that for s = 0 (resp., s = α) the first inequality is just the fractional Sobolev (resp., the fractional Hardy) inequality. We therefore have to only consider the case where $0 < s < \alpha$ in which case $2^*_\alpha(s) > 2$. By applying Hölder's inequality, then the fractional Hardy and the fractional Sobolev inequalities, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx &= \int_{\mathbb{R}^n} \frac{|u|^{\frac{2s}{\alpha}}}{|x|^s} |u|^{2^*_\alpha(s)-\frac{2s}{\alpha}} dx \\
&\leq (\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx)^{\frac{s}{\alpha}} (\int_{\mathbb{R}^n} |u|^{(2^*_\alpha(s)-\frac{2s}{\alpha})\frac{\alpha}{\alpha-s}} dx)^{\frac{\alpha-s}{\alpha}} \\
&= (\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx)^{\frac{s}{\alpha}} (\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx)^{\frac{\alpha-s}{\alpha}} \\
&\leq C_1 (\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx)^{\frac{s}{\alpha}} C_2 (\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx)^{\frac{2^*_\alpha}{2}\cdot\frac{\alpha-s}{\alpha}} \\
&\leq c (\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx)^{\frac{n-s}{n-\alpha}} \\
&= c (\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx)^{\frac{2^*_\alpha(s)}{2}}.
\end{aligned}$$

From the definition of $\gamma_H(\alpha)$, it follows that for all $u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$,

$$\frac{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx}{(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx)^{\frac{2}{2^*_\alpha(s)}}} \geq (1 - \frac{\gamma}{\gamma_H(\alpha)}) \frac{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx}{(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx)^{\frac{2}{2^*_\alpha(s)}}}.$$

Hence (2.18) implies (2.19) whenever $\gamma < \gamma_H(\alpha)$. □

Remark 2.11. One can use (2.12) and (2.13) to rewrite inequalities (2.16), (2.18) and (2.19) as the following trace class inequalities:

$$\gamma_H(\alpha) \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} \, dx \leq \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)},$$

$$(\int_{\mathbb{R}^n} \frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s} \, dx)^{\frac{2}{2^*_\alpha(s)}} \leq c \, \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)},$$

and

$$C(\int_{\mathbb{R}^n} \frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s} \, dx)^{\frac{2}{2^*_\alpha(s)}} \leq \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx,$$

for all $w \in X^\alpha(\mathbb{R}^{n+1}_+)$ *and* $\gamma < \gamma_H(\alpha)$.

Chapter 3

Borderline Variational Problems Involving Fractional Laplacian and Critical Singularities

3.1 Introduction

In this chapter, we consider the problem of existence of nontrivial weak solutions to the following doubly critical problem on $\mathbb{R}^n$ involving the fractional Hardy-Schrödinger operator:

$$(-\Delta)^{\frac{\alpha}{2}} u - \gamma \frac{u}{|x|^\alpha} = |u|^{2^*_\alpha - 2} u + \frac{|u|^{2^*_\alpha(s) - 2} u}{|x|^s} \quad \text{in } \mathbb{R}^n, \tag{3.1}$$

where $0 \leq s < \alpha < 2$, $n > \alpha$, $2^*_\alpha := \frac{2n}{n-\alpha}$, $2^*_\alpha(s) := \frac{2(n-s)}{n-\alpha}$ and $\gamma \in \mathbb{R}$.

Problems involving two non-linearities have been studied in the case of local operators such as the Laplacian $-\Delta$, the p-Laplacian $-\Delta_p$ and the Biharmonic operator Δ^2 (See [8], [33], [47] and [68]). Problem (3.1) above is the non-local counterpart of the one studied by Filippucci-Pucci-Robert in [33], who treated the case of the p-Laplacian in an equation involving both the Sobolev and the Hardy-Sobolev critical exponents.

Questions of existence and non-existence of solutions for fractional elliptic equations with singular potentials were recently studied by several authors. All studies focus, however, on problems with only one critical exponent –mostly the non-linearity $u^{2^*_\alpha - 1}$– and to a lesser extent the critical Hardy-Sobolev singular term $\frac{u^{2^*_\alpha(s)-1}}{|x|^s}$ (see [19], [31], [70] and the references therein). These cases were also studied on smooth bounded domains (see for example [5], [7], [9], [29], [65] and the references therein). In general, the case of two critical exponents involve more subtleties and difficulties, even for local differential operators.

The variational approach that we adopt here, relies on the following fractional Hardy-Sobolev type inequality:

$$C\left(\int_{\mathbb{R}^n}\frac{|u|^{2_\alpha^*(s)}}{|x|^s}dx\right)^{\frac{2}{2_\alpha^*(s)}} \leq \int_{\mathbb{R}^n}|(-\Delta)^{\frac{\alpha}{4}}u|^2dx - \gamma\int_{\mathbb{R}^n}\frac{|u|^2}{|x|^\alpha}dx \quad \forall u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n), \tag{3.2}$$

where $\gamma < \gamma_H(\alpha) := 2^\alpha\frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})}$ is the best fractional Hardy constant on $\mathbb{R}^n$. Recall that the fractional space $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is defined as the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|^2_{H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n}|2\pi\xi|^\alpha|\mathcal{F}u(\xi)|^2d\xi = \int_{\mathbb{R}^n}|(-\Delta)^{\frac{\alpha}{4}}u|^2dx.$$

The best constant in the above fractional Hardy-Sobolev inequality is defined as:

$$\mu_{\gamma,s,\alpha}(\mathbb{R}^n) := \inf_{u\in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)\setminus\{0\}}\frac{\int_{\mathbb{R}^n}|(-\Delta)^{\frac{\alpha}{4}}u|^2dx - \gamma\int_{\mathbb{R}^n}\frac{|u|^2}{|x|^\alpha}dx}{\left(\int_{\mathbb{R}^n}\frac{|u|^{2_\alpha^*(s)}}{|x|^s}dx\right)^{\frac{2}{2_\alpha^*(s)}}}. \tag{3.3}$$

One step towards addressing Problem (3.1) consists of proving the existence of extremals for $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$, when $s \in [0,\alpha)$ and $\gamma \in (-\infty, \gamma_H(\alpha))$. Note that the Euler-Lagrange equation corresponding to the minimization problem for $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ is –up to a constant factor– the following:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}u - \gamma\frac{u}{|x|^\alpha} = \frac{u^{2_\alpha^*(s)-1}}{|x|^s} & \text{in } \mathbb{R}^n \\ u \geq 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{3.4}$$

When $\alpha = 2$, i.e., in the case of the standard Laplacian, the above minimization problem (3.3) has been extensively studied. See for example [15], [17], [33], [35], [36] and [40]. The non-local case has also been the subject of several studies, but in the absence of the Hardy term, i.e., when $\gamma = 0$. In [31], Fall, Minlend and Thiam proved the existence of extremals for $\mu_{0,s,\alpha}(\mathbb{R}^n)$ in the case $\alpha = 1$. Recently, J. Yang in [70] proved that there exists a positive, radially symmetric and non-increasing extremal for $\mu_{0,s,\alpha}(\mathbb{R}^n)$ when $\alpha \in (0,2)$. Asymptotic properties of the positive solutions were given by Y. Lei [48], Lu and Zhu [54], and Yang and Yu [71].

In section 3.2, we consider the remaining cases in the problem of deciding whether the best constant in the fractional Hardy-Sobolev inequality $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ is attained. We use Ekeland's variational principle to show the following.

Theorem 3.1. *Suppose* $0 < \alpha < 2$, $0 \leq s < \alpha < n$ *and* $\gamma < \gamma_H(\alpha) := 2^\alpha \frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})}$.

1. *If either* $\{s > 0\}$ *or* $\{s = 0$ *and* $\gamma \geq 0\}$, *then* $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ *is attained.*

2. *If* $s = 0$ *and* $\gamma < 0$, *then there are no extremals for* $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$.

3. *If either* $\{0 < \gamma < \gamma_H(\alpha)\}$ *or* $\{0 < s < \alpha$ *and* $\gamma = 0\}$, *then any non-negative minimizer for* $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ *is positive, radially symmetric, radially decreasing, and approaches zero as* $|x| \to \infty$.

In section 3.3, we consider problem (3.1) and use the Mountain Pass lemma to establish the following result.

Theorem 3.2. *Let* $0 < \alpha < 2$, $0 < s < \alpha < n$ *and* $0 \leq \gamma < \gamma_H(\alpha)$. *Then, there exists a non-trivial weak solution of (3.1).*

We say $u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is a weak solution of (3.1), if we have for all $\varphi \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{4}} u (-\Delta)^{\frac{\alpha}{4}} \varphi dx = \int_{\mathbb{R}^n} \gamma \frac{u}{|x|^\alpha} \varphi dx + \int_{\mathbb{R}^n} |u|^{2_\alpha^* - 2} u \varphi dx + \int_{\mathbb{R}^n} \frac{|u|^{2_\alpha^*(s)-2} u}{|x|^s} \varphi dx.$$

The standard strategy to construct weak solutions of (3.1) is to find critical points of the corresponding functional on $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$. However, (3.1) is invariant under the following conformal one parameter transformation group,

$$T_r : H_0^{\frac{\alpha}{2}}(\mathbb{R}^n) \to H_0^{\frac{\alpha}{2}}(\mathbb{R}^n); \qquad u(x) \to T_r[u](x) = r^{\frac{n-\alpha}{2}} u(rx) \quad r > 0, \tag{3.5}$$

which means that the convergence of Palais-Smale sequences is not a given. As it was argued in [33], there is an asymptotic competition between the energy carried by the two critical nonlinearities. Hence, the crucial step here is to balance the competition to avoid the domination of one term over another. Otherwise, there is vanishing of the weakest one, leading to a solution for the same equation but with only one critical nonlinearity. In order to deal with this issue, we choose a suitable minimax energy level, in such a way that after a careful analysis of the concentration phenomena, we could eliminate the possibility of a vanishing weak limit for these well chosen Palais-Smale sequences, while ensuring that none of the two nonlinearities dominate the other.

With representation (2.4), the non-local problem (3.1) can then be written as the following local problem:

$$\begin{cases} -\mathrm{div}\,(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \frac{\partial w}{\partial \nu^\alpha} = \gamma \frac{w(.,0)}{|x|^\alpha} + w(.,0)^{2^*_\alpha - 1} + \frac{w(.,0)^{2^*_\alpha(s)-1}}{|x|^s} & \text{on } \mathbb{R}^n, \end{cases} \tag{3.6}$$

for which $w \in X^\alpha(\mathbb{R}^{n+1}_+)$. Recall that the Hilbert space $X^\alpha(\mathbb{R}^{n+1}_+)$ defined as the closure of $C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$ for the norm

$$\|w\|_{X^\alpha(\mathbb{R}^{n+1}_+)} := \left(k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |\nabla w|^2 dxdy \right)^{\frac{1}{2}},$$

where $k_\alpha = \frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}$ is a normalization constant.

A function $w \in X^\alpha(\mathbb{R}^{n+1}_+)$ is said to be a weak solution to (3.6), if for all $\varphi \in X^\alpha(\mathbb{R}^{n+1}_+)$,

$$\begin{aligned} k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} \langle \nabla w, \nabla \varphi \rangle dxdy &= \gamma \int_{\mathbb{R}^n} \frac{w(x,0)}{|x|^\alpha} \varphi dx \\ &+ \int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha - 2} w(x,0) \varphi dx \\ &+ \int_{\mathbb{R}^n} \frac{|w(x,0)|^{2^*_\alpha(s)-2} w(x,0)}{|x|^s} \varphi dx. \end{aligned}$$

Note that for any weak solution w in $X^\alpha(\mathbb{R}^{n+1}_+)$ to (3.6), the function $u = w(.,0)$ defined in the sense of traces (see Section 2.3), is in $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ and is a weak solution to problem (3.1). The energy functional corresponding to (3.6) is

$$\begin{aligned} \Phi(w) = \frac{1}{2} \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \frac{\gamma}{2} \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx - \frac{1}{2^*_\alpha} \int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx \\ - \frac{1}{2^*_\alpha(s)} \int_{\mathbb{R}^n} \frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx. \end{aligned}$$

Hence, the associated trace of any critical point w of Φ in $X^\alpha(\mathbb{R}^{n+1}_+)$ is a weak solution for (3.1).

It follows from Remark 2.11 that there exist positive constants c, C such that the following fractional trace inequalities hold for all $w \in X^\alpha(\mathbb{R}^{n+1}_+)$ and $\gamma < \gamma_H(\alpha)$:

$$\gamma_H(\alpha)\int_{\mathbb{R}^n}\frac{|w(x,0)|^2}{|x|^\alpha}\,dx \le \ \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)}, \tag{3.7}$$

$$\left(\int_{\mathbb{R}^n}\frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^*_\alpha(s)}} \le c\ \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)}, \tag{3.8}$$

and

$$C\left(\int_{\mathbb{R}^n}\frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^*_\alpha(s)}} \le \ \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma\int_{\mathbb{R}^n}\frac{|w(x,0)|^2}{|x|^\alpha}dx. \tag{3.9}$$

The best constant $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ can then be written as:

$$S(n,\alpha,\gamma,s) = \inf_{w\in X^\alpha(\mathbb{R}^{n+1}_+)\setminus\{0\}} \frac{k_\alpha\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla w|^2 dxdy - \gamma\int_{\mathbb{R}^n}\frac{|w(x,0)|^2}{|x|^\alpha}dx}{\left(\int_{\mathbb{R}^n}\frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}}}.$$

We shall therefore investigate whether there exist extremal functions where this best constant is attained. Theorems 3.1 and 3.2 can therefore be stated in the following way:

Theorem 3.3. *Suppose $0<\alpha<2$, $0\le s<\alpha<n$ and $\gamma<\gamma_H(\alpha)$. We then have the following:*

1. *If $\{s>0\}$ or $\{s=0$ and $\gamma\ge 0\}$, then $S(n,\alpha,\gamma,s)$ is attained in $X^\alpha(\mathbb{R}^{n+1}_+)$.*
2. *If $s=0$ and $\gamma<0$, then there are no extremals for $S(n,\alpha,\gamma,s)$ in $X^\alpha(\mathbb{R}^{n+1}_+)$.*

Theorem 3.4. *Let $0<\alpha<2$, $0<s<\alpha<n$ and $0\le\gamma<\gamma_H(\alpha)$. Then, there exists a non-trivial weak solution to (3.6) in $X^\alpha(\mathbb{R}^{n+1}_+)$.*

3.2 Proof of Theorem 3.1

We shall minimize the functional

$$I_{\gamma,s}(w) = \frac{k_\alpha\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla w|^2 dxdy - \gamma\int_{\mathbb{R}^n}\frac{|w(x,0)|^2}{|x|^\alpha}dx}{\left(\int_{\mathbb{R}^n}\frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}}}$$

on the space $X^\alpha(\mathbb{R}^{n+1}_+)$. Whenever $S(n,\alpha,\gamma,s)$ is attained at some $w\in X^\alpha(\mathbb{R}^{n+1}_+)$, then it is clear that $u=\mathrm{Tr}(w):=w(.,0)$ will be a function in $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$, where $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ is attained.

Note first that inequality (3.7) asserts that $X^\alpha(\mathbb{R}^{n+1}_+)$ is embedded in the weighted space $L^2(\mathbb{R}^n, |x|^{-\alpha})$ and that this embeding is continuous. If $\gamma < \gamma_H(\alpha)$, it follows from (3.7) that

$$\|w\| := \left(k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |\nabla w|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx \right)^{\frac{1}{2}}$$

is well-defined on $X^\alpha(\mathbb{R}^{n+1}_+)$. Set $\gamma_+ = \max\{\gamma, 0\}$ and $\gamma_- = -\max\{\gamma, 0\}$. The following inequalities then hold for any $u \in X^\alpha(\mathbb{R}^{n+1}_+)$,

$$(1 - \frac{\gamma_+}{\gamma_H(\alpha)}) \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} \leq \|w\|^2 \leq (1 + \frac{\gamma_-}{\gamma_H(\alpha)}) \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)}. \tag{3.10}$$

Thus, $\| \, . \, \|$ is equivalent to the norm $\| \, . \, \|_{X^\alpha(\mathbb{R}^{n+1}_+)}$.

We start by considering the case when $s > 0$. Ekeland's variational principle [25] applied to the functional $I(w) := I_{\gamma,s}(w)$ yields the existence of a minimizing sequence $(w_k)_{k \in \mathbb{N}}$ for $S(n, \alpha, \gamma, s)$ such that as $k \to \infty$,

$$\int_{\mathbb{R}^n} \frac{|w_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = 1, \tag{3.11}$$

$$I(w_k) \longrightarrow S(n, \alpha, \gamma, s), \tag{3.12}$$

and

$$I'(w_k) \to 0 \text{ in } (X^\alpha(\mathbb{R}^{n+1}_+))', \tag{3.13}$$

where $(X^\alpha(\mathbb{R}^{n+1}_+))'$ denotes the dual of $X^\alpha(\mathbb{R}^{n+1}_+)$. Consider the functionals $J, K : X^\alpha(\mathbb{R}^{n+1}_+) \longrightarrow \mathbb{R}$ by

$$J(w) := \frac{1}{2}\|w\|^2 = \frac{k_\alpha}{2} \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |\nabla w|^2 dxdy - \frac{\gamma}{2} \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx,$$

and

$$K(w) := \frac{1}{2^*_\alpha(s)} \int_{\mathbb{R}^n} \frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx.$$

Straightforward computations yield that as $k \to \infty$,

$$J(w_k) \longrightarrow \frac{1}{2} S(n, \alpha, \gamma, s),$$

and

$$J'(w_k) - S(n, \alpha, \gamma, s) K'(w_k) \longrightarrow 0 \text{ in } (X^\alpha(\mathbb{R}^{n+1}_+))'. \tag{3.14}$$

Consider now the Levy concentration functions Q of $\frac{|w_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}$, defined as

$$Q(r) = \int_{B_r} \frac{|w_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \quad \text{for} \quad r > 0,$$

where B_r is the ball of radius r in $\mathbb{R}^n$. Since $\int_{\mathbb{R}^n} \frac{|w_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = 1$ for all $k \in \mathbb{N}$, then by continuity, and up to considering a subsequence, there exists $r_k > 0$ such that

$$Q(r_k) = \int_{B_{r_k}} \frac{|w_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = \frac{1}{2} \quad \text{for all } k \in \mathbb{N}.$$

Define the rescaled sequence $v_k(x,y) := r_k^{\frac{n-\alpha}{2}} w_k(r_k x, r_k y)$ for $k \in \mathbb{N}$ and $(x,y) \in \mathbb{R}^{n+1}_+$, in such a way that $(v_k)_{k\in\mathbb{N}}$ is also a minimizing sequence for $S(n,\alpha,\gamma,s)$. Indeed, it is easy to check that $v_k \in X^\alpha(\mathbb{R}^{n+1}_+)$ and that $\|w_k\|^2 = \|v_k\|^2$,

$$\lim_{k\to\infty} \left(k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |\nabla v_k|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|v_k(x,0)|^2}{|x|^\alpha} dx \right) = S(n,\alpha,\gamma,s) \tag{3.15}$$

and

$$\int_{\mathbb{R}^n} \frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = \int_{\mathbb{R}^n} \frac{|w_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = 1.$$

Moreover, we have that

$$\int_{B_1} \frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = \frac{1}{2} \quad \text{for all } k \in \mathbb{N}. \tag{3.16}$$

In addition, $\|v_k\|^2 = S(n,\alpha,\gamma,s) + o(1)$ as $k \to \infty$, so (3.10) yields that $(v_k)_{k\in\mathbb{N}}$ is bounded in $X^\alpha(\mathbb{R}^{n+1}_+)$. Therefore, without loss of generality, there exists a subsequence -still denoted v_k- such that

$$v_k \rightharpoonup v \text{ in } X^\alpha(\mathbb{R}^{n+1}_+) \ \text{ and } \ v_k(.,0) \to v(.,0) \text{ in } L^q_{loc}(\mathbb{R}^n) \text{ for all } 1 \le q < 2^*_\alpha. \tag{3.17}$$

We shall show that the weak limit of the minimizing sequence is not identically zero, that is $v \not\equiv 0$. Indeed, suppose $v \equiv 0$. It follows from (3.17) that

$$v_k \rightharpoonup 0 \text{ in } X^\alpha(\mathbb{R}^{n+1}_+) \ \text{ and } \ v_k(.,0) \to 0 \text{ in } L^q_{loc}(\mathbb{R}^n) \quad \text{for every } 1 \le q < 2^*_\alpha. \tag{3.18}$$

For $\delta > 0$, define $B_\delta^+ := \{(x,y) \in \mathbb{R}_+^{n+1} : |(x,y)| < \delta\}$, $B_\delta := \{x \in \mathbb{R}^n : |x| < \delta\}$ and let $\eta \in C_0^\infty(B_1^+)$ be a cut-off function such that $\eta \equiv 1$ in $B_{\frac{1}{2}}^+$ and $0 \le \eta \le 1$ in $\mathbb{R}_+^{n+1}$.

We use $\eta^2 v_k$ as test function in (3.14) to get that

$$\begin{aligned} k_\alpha \int_{\mathbb{R}_+^{n+1}} y^{1-\alpha} \nabla v_k . \nabla(\eta^2 v_k) dxdy - \gamma \int_{\mathbb{R}^n} \frac{v_k(x,0)(\eta^2 v_k(x,0))}{|x|^\alpha} dx \\ = S(n,\alpha,\gamma,s) \int_{\mathbb{R}^n} \frac{|v_k(x,0)|^{2_\alpha^*(s)-1}(\eta^2 v_k(x,0))}{|x|^s} dx + o(1). \end{aligned} \tag{3.19}$$

Simple computations yield $|\nabla(\eta v_k)|^2 = |v_k \nabla \eta|^2 + \nabla v_k . \nabla(\eta^2 v_k)$, so that we have

$$\begin{aligned} & k_\alpha \int_{\mathbb{R}_+^{n+1}} y^{1-\alpha} |\nabla(\eta v_k)|^2 dxdy - k_\alpha \int_{\mathbb{R}_+^{n+1}} y^{1-\alpha} \nabla v_k . \nabla(\eta^2 v_k) dxdy \\ & = k_\alpha \int_{\mathbb{R}_+^{n+1}} y^{1-\alpha} |v_k \nabla \eta|^2 dxdy \\ & = k_\alpha \int_E y^{1-\alpha} |\nabla \eta|^2 |v_k|^2 dxdy, \end{aligned}$$

where $E := \text{supp}(|\nabla \eta|)$. Since $\alpha \in (0,2)$, $y^{1-\alpha}$ is an A_2-weight, and since E is bounded, we have that the embedding $H^1(E, y^{1-\alpha}) \hookrightarrow L^2(E, y^{1-\alpha})$ is compact (See [5] and [41]). It follows from $(3.18)_1$ that

$$k_\alpha \int_E y^{1-\alpha} |\nabla \eta|^2 |v_k|^2 dxdy \to 0 \quad \text{ as } k \to \infty.$$

Therefore,

$$k_\alpha \int_{\mathbb{R}_+^{n+1}} y^{1-\alpha} |\nabla(\eta v_k)|^2 dxdy = k_\alpha \int_{\mathbb{R}_+^{n+1}} y^{1-\alpha} \nabla v_k . \nabla(\eta^2 w_k) dxdy + o(1).$$

By plugging the above estimate into (3.19), we get that

$$\begin{aligned} \|\eta v_k\|^2 &= k_\alpha \int_{\mathbb{R}_+^{n+1}} y^{1-\alpha} |\nabla(\eta v_k)|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^2}{|x|^\alpha} dx \\ &= S(n,\alpha,\gamma,s) \int_{\mathbb{R}^n} \frac{|v_k(x,0)|^{2_\alpha^*(s)-2}(|\eta v_k(x,0)|^2)}{|x|^s} dx + o(1) \\ &= S(n,\alpha,\gamma,s) \int_{B_1} \frac{|v_k(x,0)|^{2_\alpha^*(s)-2}(|\eta v_k(x,0)|^2)}{|x|^s} dx + o(1). \end{aligned} \tag{3.20}$$

Note that in the last equality we used the fact that $\mathrm{supp}(\eta(x,0)) \subset B_1$. We now apply Hölder's inequality with exponents $\frac{2^*_\alpha(s)}{2^*_\alpha(s)-2}$ and $\frac{2^*_\alpha(s)}{2}$ to get that

$$\int_{B_1} \frac{|v_k(x,0)|^{2^*_\alpha(s)-2}(|\eta v_k(x,0)|^2)}{|x|^s}dx = \int_{B_1} \frac{|v_k(x,0)|^{2^*_\alpha(s)-2}}{|x|^{s\frac{2^*_\alpha(s)-2}{2^*_\alpha(s)}}} \frac{|\eta v_k(x,0)|^2}{|x|^{s\frac{2}{2^*_\alpha(s)}}}dx$$
$$\leq \left(\int_{B_1} \frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2^*_\alpha(s)-2}{2^*_\alpha(s)}} \left(\int_{B_1} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}}.$$

It then follows from (3.16) that

$$\int_{B_1} \frac{|v_k(x,0)|^{2^*_\alpha(s)-2}(|\eta v_k(x,0)|^2)}{|x|^s}dx \leq \left(\frac{1}{2}\right)^{1-\frac{2}{2^*_\alpha(s)}} \left(\int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}}.$$

Plugging the above inequality into (3.20), we get that

$$\|\eta v_k\|^2 = k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta v_k)|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^2}{|x|^\alpha}dx$$
$$\leq \frac{S(n,\alpha,\gamma,s)}{2^{1-\frac{2}{2^*_\alpha(s)}}} \left(\int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}} + o(1).$$

On the other hand, it follows from the definition of $S(n,\alpha,\gamma,s)$ that

$$S(n,\alpha,\gamma,s) \left(\int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}} \leq \|\eta v_k\|^2$$
$$\leq \frac{S(n,\alpha,\gamma,s)}{2^{1-\frac{2}{2^*_\alpha(s)}}} \left(\int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}} + o(1).$$

Note that $\frac{S(n,\alpha,\gamma,s)}{2^{1-\frac{2}{2^*_\alpha(s)}}} < S(n,\alpha,\gamma,s)$ for $s \in (0,\alpha)$, which yields that

$$\int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx = o(1). \tag{3.21}$$

By straightforward computations and Hölder's inequality, we get that

$$\left(\int_{B_1} \frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{1}{2^*_\alpha(s)}} = \left(\int_{B_1} \frac{|\eta v_k(x,0) + (1-\eta)v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{1}{2^*_\alpha(s)}}$$
$$\leq \left(\int_{B_1} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{1}{2^*_\alpha(s)}} + \left(\int_{B_1} \frac{|(1-\eta)v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{1}{2^*_\alpha(s)}}$$
$$\leq \left(\int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{1}{2^*_\alpha(s)}} + C\left(\int_{B_1} |v_k(x,0)|^{2^*_\alpha(s)}dx\right)^{\frac{1}{2^*_\alpha(s)}}.$$

From $(3.18)_2$, and the fact that $2^*_\alpha(s) < 2^*_\alpha$, we obtain

$$\int_{B_1} |v_k(x,0)|^{2^*_\alpha(s)} dx \to 0 \quad \text{as } k \to \infty.$$

Therefore,

$$\left(\int_{B_1} \frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}} \leq \left(\int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}} + o(1). \tag{3.22}$$

It then follows from (3.21) and (3.22) that

$$o(1) = \int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = \int_{B_1} \frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx + o(1).$$

This contradicts (3.16) and therefore $v \not\equiv 0$.

We now conclude by proving that v_k converges weakly in $\mathbb{R}^{n+1}_+$ to v, and that

$$\int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = 1.$$

Indeed, for $k \in \mathbb{N}$, let $\theta_k = v_k - v$, and use the Brezis-Lieb Lemma (see [10] and [70]) to deduce that

$$1 = \int_{\mathbb{R}^n} \frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = \int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx + \int_{\mathbb{R}^n} \frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx + o(1),$$

which yields that both

$$\int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \text{ and } \int_{\mathbb{R}^n} \frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \text{ are in the interval } [0,1]. \tag{3.23}$$

The weak convergence $\theta_k \rightharpoonup 0$ in $X^\alpha(\mathbb{R}^{n+1}_+)$ implies that

$$\|v_k\|^2 = \|v + \theta_k\|^2 = \|v\|^2 + \|\theta_k\|^2 + o(1).$$

By using (3.14) and the definition of $S(n,\alpha,\gamma,s)$, we get that

$$\begin{aligned}
o(1) &= \|v_k\|^2 - S(n,\alpha,\gamma,s)\int_{\mathbb{R}^n}\frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx \\
&= \left(\|v\|^2 - S(n,\alpha,\gamma,s)\int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right) \\
&+ \left(\|\theta_k\|^2 - S(n,\alpha,\gamma,s)\int_{\mathbb{R}^n}\frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right) + o(1) \\
&\geq S(n,\alpha,\gamma,s)\left[\left(\int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}} - \int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right] \\
&+ S(n,\alpha,\gamma,s)\left[\left(\int_{\mathbb{R}^n}\frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}} - \int_{\mathbb{R}^n}\frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right] \\
&+ o(1).
\end{aligned}$$

Set now

$$A := \left(\int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}} - \int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx,$$

and

$$B := \left(\int_{\mathbb{R}^n}\frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}} - \int_{\mathbb{R}^n}\frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx.$$

Note that since $2^*_\alpha(s) > 2$, we have $a^{\frac{2}{2^*_\alpha(s)}} \geq a$ for every $a \in [0,1]$, and equality holds if and only if $a = 0$ or $a = 1$. It then follows from (3.23) that both A and B are non-negative. On the other hand, the last inequality implies that $A + B = o(1)$, which means that $A = 0$ and $B = o(1)$, that is

$$\int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx = \left(\int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}}.$$

Hence,

$$\text{either } \int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx = 0 \text{ or } \int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx = 1.$$

The fact that $v \not\equiv 0$ yields $\int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \neq 0$, and $\int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = 1$, which yields that

$$k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |\nabla v|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|v(x,0)|^2}{|x|^\alpha} dx = S(n, \alpha, \gamma, s).$$

Without loss of generality we may assume $v \geq 0$ (otherwise we take $|v|$ instead of v), and we then obtain a positive extremal for $S(n, \alpha, \gamma, s)$ in the case $s \in (0, \alpha)$.

Suppose now that $s = 0$ and $\gamma \geq 0$. By a result in [19], extremals exist for $S(n, \alpha, \gamma, s)$ whenever $s = 0$ and $\gamma = 0$. Hence, we only need to show that there exists an extremal for $S(n, \alpha, \gamma, 0)$ in the case $\gamma > 0$. First note that in this case, we have that

$$S(n, \alpha, \gamma, 0) < S(n, \alpha, 0, 0). \tag{3.24}$$

Indeed, if $w \in X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$ is an extremal for $S(n, \alpha, 0, 0)$, then by estimating the functional at w, and using the fact that $\gamma > 0$, we obtain

$$\begin{aligned} S(n, \alpha, \gamma, 0) &= \inf_{u \in X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}} \frac{\|u\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|u(x,0)|^2}{|x|^\alpha} dx}{\left(\int_{\mathbb{R}^n} |u(x,0)|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}}} \\ &\leq \frac{\|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx}{\left(\int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}}} \\ &< \frac{\|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)}}{\left(\int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}}} = S(n, \alpha, 0, 0). \end{aligned}$$

Now we show that $S(n, \alpha, \gamma, 0)$ is attained whenever $S(n, \alpha, \gamma, 0) < S(n, \alpha, 0, 0)$. Indeed, let $(w_k)_{k \in \mathbb{N}} \subset X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$ be a minimizing sequence for $S(n, \alpha, \gamma, 0)$. Up to multiplying by a positive constant, we assume that

$$\lim_{k \to \infty} \left(k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |\nabla w_k|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|w_k(x,0)|^2}{|x|^\alpha} dx \right) = S(n, \alpha, \gamma, 0) \tag{3.25}$$

and

$$\int_{\mathbb{R}^n} |w_k(x,0)|^{2^*_\alpha} dx = 1. \tag{3.26}$$

The sequence $\left(\|w_k\|_{X^\alpha(\mathbb{R}^{n+1}_+)}\right)_{k\in\mathbb{N}}$ is therefore bounded, and there exists a subsequence - still denoted w_k- such that $w_k \rightharpoonup w$ weakly in $X^\alpha(\mathbb{R}^{n+1}_+)$. The weak convergence implies that

$$\|w_k\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} = \|w_k - w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} + \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} + o(1),$$

and

$$\int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx = \int_{\mathbb{R}^n} \frac{|(w-w_k)(x,0)|^2}{|x|^\alpha} dx + \int_{\mathbb{R}^n} \frac{|w_k(x,0)|^2}{|x|^\alpha} dx + o(1).$$

The Brezis-Lieb Lemma ([10, Theorem 1]) and (3.26) yield that

$$\int_{\mathbb{R}^n} |(w_k - w)(x,0)|^{2^*_\alpha} dx \leq 1,$$

for large k, hence

$$\begin{aligned}
S(n,\alpha,\gamma,0) &= \|w_k\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|w_k(x,0)|^2}{|x|^\alpha} dx + o(1) \\
&\geq \|w_k - w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} + \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx + o(1) \\
&\geq S(n,\alpha,0,0)(\int_{\mathbb{R}^n} |(w_k - w)(x,0)|^{2^*_\alpha} dx)^{\frac{2}{2^*_\alpha}} \\
&+ S(n,\alpha,\gamma,0)(\int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx)^{\frac{2}{2^*_\alpha}} + o(1) \\
&\geq S(n,\alpha,0,0) \int_{\mathbb{R}^n} |(w_k - w)(x,0)|^{2^*_\alpha} dx \\
&+ S(n,\alpha,\gamma,0) \int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx + o(1).
\end{aligned}$$

Use the Brezis-Lieb Lemma again to get that

$$\begin{aligned}
S(n,\alpha,\gamma,0) &\geq (S(n,\alpha,0,0) - S(n,\alpha,\gamma,0)) \int_{\mathbb{R}^n} |(w_k - w)(x,0)|^{2^*_\alpha} dx \\
&+ S(n,\alpha,\gamma,0) \int_{\mathbb{R}^n} |w_k(x,0)|^{2^*_\alpha} dx + o(1) \\
&= (S(n,\alpha,0,0) - S(n,\alpha,\gamma,0)) \int_{\mathbb{R}^n} |(w_k - w)(x,0)|^{2^*_\alpha} dx \\
&+ S(n,\alpha,\gamma,0) + o(1).
\end{aligned}$$

Since $S(n,\alpha,\gamma,0) < S(n,\alpha,0,0)$, we get that $w_k(.,0) \to w(.,0)$ in $L^{2^*_\alpha}(\mathbb{R}^n)$, that is

$$\int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx = 1.$$

The lower semi-continuity of I then implies that w is a minimizer for $S(n,\alpha,\gamma,0)$. Note that $|w|$ is also an extremal in $X^\alpha(\mathbb{R}^{n+1}_+)$ for $S(n,\alpha,\gamma,0)$, therefore there exists a non-negative extremal for $S(n,\alpha,\gamma,s)$ in the case $\gamma > 0$ and $s = 0$, and this completes the proof of the case when $s = 0$ and $\gamma \geq 0$.

Now we consider the case when $\gamma < 0$.

Claim 3.5. If $\gamma \leq 0$, then $S(n,\alpha,\gamma,0) = S(n,\alpha,0,0)$, hence, there are no extremals for $S(n,\alpha,\gamma,0)$ whenever $\gamma < 0$.

Indeed, we first note that for $\gamma \leq 0$, we have $S(n,\alpha,\gamma,0) \geq S(n,\alpha,0,0)$. On the other hand, if we consider $w \in X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$ to be an extremal for $S(n,\alpha,0,0)$ and define for $\delta \in \mathbb{R}$, and $\bar{x} \in \mathbb{R}^n$, the function $w_\delta := w(x-\delta\bar{x}, y)$ for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}_+$, then by a change of variable, we get

$$\begin{aligned} S(n,\alpha,\gamma,0) \leq I_\delta : &= \frac{\|w_\delta\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|w_\delta(x,0)|^2}{|x|^\alpha} dx}{\left(\int_{\mathbb{R}^n} |w_\delta(x,0)|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}}} \\ &= \frac{\|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x+\delta\bar{x}|^\alpha} dx}{\left(\int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}}}, \end{aligned}$$

so that

$$S(n,\alpha,\gamma,0) \leq \lim_{\delta\to\infty} I_\delta = \frac{\|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)}}{\left(\int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}}} = S(n,\alpha,0,0).$$

Therefore, $S(n,\alpha,\gamma,0) = S(n,\alpha,0,0)$. Since there are extremals for $S(n,\alpha,0,0)$ (see [19]), there is none for $S(n,\alpha,\gamma,0)$ whenever $\gamma < 0$. This establishes (2) and completes the proof of Theorem 3.3.

Back to Theorem 3.1, since the non-negative α-harmonic function w is a minimizer for $S(n,\alpha,\gamma,s)$ in $X^\alpha(\mathbb{R}^{n+1}_+)\setminus\{0\}$, which exists from Theorem 3.3, then $u := \mathrm{Tr}(w) = w(.,0) \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n) \setminus \{0\}$ and by (2.12), u is a minimizer for $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ in $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n) \setminus \{0\}$. Therefore, (1) and (2) of Theorem 3.1 hold. For (3), let u^* be the Schwarz symmetrization of u. By the fractional Polya-Szegö inequality [57], we have

$$\|(-\Delta)^{\frac{\alpha}{2}} u^*\|^2_{L^2(\mathbb{R}^n)} \leq \|(-\Delta)^{\frac{\alpha}{2}} u\|^2_{L^2(\mathbb{R}^n)}.$$

Furthermore, it is clear (Theorem 3.4. of [51]) that

$$\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx \leq \int_{\mathbb{R}^n} \frac{|u^*|^2}{|x|^\alpha} dx \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx \leq \int_{\mathbb{R}^n} \frac{|u^*|^{2^*_\alpha(s)}}{|x|^s} dx.$$

Combining the above inequalities and the fact that $\gamma \geq 0$, we get that

$$\begin{aligned}
\mu_{\gamma,s,\alpha}(\mathbb{R}^n) &\leq \frac{\|(-\Delta)^{\frac{\alpha}{2}} u^*\|^2_{L^2(\mathbb{R}^n)} - \gamma \int_{\mathbb{R}^n} \frac{|u^*|^2}{|x|^\alpha} dx}{\left(\int_{\mathbb{R}^n} \frac{|u^*|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}}} \\
&\leq \frac{\|(-\Delta)^{\frac{\alpha}{2}} u\|^2_{L^2(\mathbb{R}^n)} - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx}{\left(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}}} = \mu_{\gamma,s,\alpha}(\mathbb{R}^n).
\end{aligned}$$

This implies that u^* is also a minimizer and achieves the infimum of $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$. Therefore the equality sign holds in all the inequalities above, that is

$$\gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx = \gamma \int_{\mathbb{R}^n} \frac{|u^*|^2}{|x|^\alpha} dx \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx = \int_{\mathbb{R}^n} \frac{|u^*|^{2^*_\alpha(s)}}{|x|^s} dx.$$

From Theorem 3.4. of [51], in the case of equality, it follows that $u = |u| = u^*$ if either $\gamma \neq 0$ or if $s \neq 0$. In particular, u is positive, radially symmetric and decreasing about origin. Hence u must approach a limit as $|x| \to \infty$, which must be zero.

3.3 Proof of Theorem 3.2

We shall now use the existence of extremals for the fractional Hardy-Sobolev type inequalities, established in Section 3.2, to prove that there exists a non-trivial weak solution for (3.6). The energy functional Ψ associated to (3.6) is defined as follows:

$$\Psi(w) = \frac{1}{2}\|w\|^2 - \frac{1}{2^*_\alpha} \int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx - \frac{1}{2^*_\alpha(s)} \int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx \qquad \text{for } w \in X^\alpha(\mathbb{R}^{n+1}_+), \tag{3.27}$$

where again $u := \mathrm{Tr}(w) = w(.,0)$. Fractional trace Hardy, Sobolev and Hardy-Sobolev inequalities yield that $\Psi \in C^1(X^\alpha(\mathbb{R}^{n+1}_+))$. Note that a weak solution to (3.6) is a non-trivial critical point of Ψ.

Throughout this section, we use the following notation for any sequence $(w_k)_{k\in\mathbb{N}} \in X^\alpha(\mathbb{R}^{n+1}_+)$:

$$u_k := \mathrm{Tr}(w_k) = w_k(.,0) \quad \text{for all } k \in \mathbb{N}.$$

We split the proof in three parts:

3.3.1 Existence of a suitable Palais-Smale sequence

We first verify that the energy functional Ψ satisfies the conditions of the Mountain Pass Lemma leading to a minimax energy level that is below a suitable threshold. The following is standard.

Lemma 3.6 (Ambrosetti and Rabinowitz [3]). *Let $(V, \| \|)$ be a Banach space and $\Psi : V \to \mathbb{R}$ a C^1-functional satisfying the following conditions:*
(a) $\Psi(0) = 0$,
(b) There exist $\rho, R > 0$ such that $\Psi(u) \geq \rho$ for all $u \in V$, with $\|u\| = R$,
(c) There exists $v_0 \in V$ such that $\limsup_{t\to\infty} \Psi(tv_0) < 0$.
Let $t_0 > 0$ be such that $\|t_0 v_0\| > R$ and $\Psi(t_0 v_0) < 0$, and define

$$c_{v_0}(\Psi) := \inf_{\sigma\in\Gamma} \sup_{t\in[0,1]} \Psi(\sigma(t)),$$

where

$$\Gamma := \{\sigma \in C([0,1], V) : \sigma(0) = 0 \text{ and } \sigma(1) = t_0 v_0\}.$$

Then, $c_{v_0}(\Psi) \geq \rho > 0$, and there exists a Palais-Smale sequence at level $c_{v_0}(\Psi)$, that is there exists a sequence $(w_k)_{k\in\mathbb{N}} \in V$ such that

$$\lim_{k\to\infty} \Psi(w_k) = c_{v_0}(\Psi) \quad \text{and} \quad \lim_{k\to\infty} \Psi'(w_k) = 0 \qquad \text{strongly in} \, V'.$$

Moreover, we have that $c_{v_0}(\Psi) \leq \sup_{t\geq 0} \Psi(tv_0)$.

We now prove the following.

Proposition 3.7. *Suppose $0 \leq \gamma < \gamma_H(\alpha)$ and $0 \leq s < \alpha$, and consider Ψ defined in (3.27) on the Banach space $X^\alpha(\mathbb{R}^{n+1}_+)$. Then, there exists $w \in X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$ such that $w \geq 0$ and $0 < c_w(\Psi) < c^\star$, where*

$$c^\star = \min\left\{ \frac{\alpha}{2n} S(n,\alpha,\gamma,0)^{\frac{n}{\alpha}}, \frac{\alpha - s}{2(n-s)} S(n,\alpha,\gamma,s)^{\frac{n-s}{\alpha-s}} \right\}, \tag{3.28}$$

and a Palais-Smale sequence $(w_k)_{k\in\mathbb{N}}$ in $X^\alpha(\mathbb{R}^{n+1}_+)$ at energy level $c_w(\Psi)$, that is,

$$\lim_{k\to\infty} \Psi(w_k) = c_w(\Psi) \text{ and } \lim_{k\to\infty} \Psi'(w_k) = 0 \qquad \text{strongly in } (X^\alpha(\mathbb{R}^{n+1}_+))'. \tag{3.29}$$

Proof of Proposition 3.7. In the sequel, we will use freely the following elementary identities involving $2^*_\alpha(s)$:

$$\tfrac{1}{2} - \tfrac{1}{2^*_\alpha} = \tfrac{\alpha}{2n}, \quad \tfrac{2^*_\alpha}{2^*_\alpha - 2} = \tfrac{n}{\alpha}, \quad \tfrac{1}{2} - \tfrac{1}{2^*_\alpha(s)} = \tfrac{\alpha - s}{2(n-s)} \quad \text{and} \quad \tfrac{2^*_\alpha(s)}{2^*_\alpha(s) - 2} = \tfrac{n-s}{\alpha - s}.$$

First, we note that the functional Ψ satisfies the hypotheses of Lemma 3.6, and that condition (c) is satisfied for any $w \in X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$. Indeed, it is standard to show that $\Psi \in C^1(X^\alpha(\mathbb{R}^{n+1}_+))$ and clearly $\Psi(0) = 0$, so that (a) of Lemma 3.6 is satisfied. For (b), note that by the definition of $S(n, \alpha, \gamma, s)$, we have that

$$S(n, \alpha, \gamma, 0)\Big(\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx\Big)^{\frac{2}{2^*_\alpha}} \leq \|w\|^2$$

and

$$S(n, \alpha, \gamma, s)\Big(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\Big)^{\frac{2}{2^*_\alpha(s)}} \leq \|w\|^2.$$

Hence,

$$\begin{aligned}
\Psi(w) &\geq \frac{1}{2}\|w\|^2 - \frac{1}{2^*_\alpha} S(n, \alpha, \gamma, 0)^{-\frac{2^*_\alpha}{2}} \|w\|^{2^*_\alpha} - \frac{1}{2^*_\alpha(s)} S(n, \alpha, \gamma, s)^{-\frac{2^*_\alpha(s)}{2}} \|w\|^{2^*_\alpha(s)} \\
&= \left(\frac{1}{2} - \frac{1}{2^*_\alpha} S(n, \alpha, \gamma, 0)^{-\frac{2^*_\alpha}{2}} \|w\|^{2^*_\alpha - 2} - \frac{1}{2^*_\alpha(s)} S(n, \alpha, \gamma, s)^{-\frac{2^*_\alpha(s)}{2}} \|w\|^{2^*_\alpha(s) - 2}\right) \|w\|^2.
\end{aligned} \tag{3.30}$$

Since $s \in [0, \alpha)$, we have that $2^*_\alpha - 2 > 0$ and $2^*_\alpha(s) - 2 > 0$. Thus, by (3.10), we can find $R > 0$ such that $\Psi(w) \geq \rho$ for all $w \in X^\alpha(\mathbb{R}^{n+1}_+)$ with $\|w\|_{X^\alpha(\mathbb{R}^{n+1}_+)} = R$. Regarding (c), note that

$$\Psi(tw) = \frac{t^2}{2}\|w\|^2 - \frac{t^{2^*_\alpha}}{2^*_\alpha} \int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx - \frac{t^{2^*_\alpha(s)}}{2^*_\alpha(s)} \int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx,$$

hence $\lim\limits_{t \to \infty} \Psi(tw) = -\infty$ for any $w \in X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$, which means that there exists $t_w > 0$ such that $\|t_w w\|_{X^\alpha(\mathbb{R}^{n+1}_+)} > R$ and $\Psi(tw) < 0$, for $t \geq t_w$.

Now we show that there exists $w \in X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$ such that $w \geq 0$ and

$$c_w(\Psi) < \frac{\alpha}{2n} S(n, \alpha, \gamma, 0)^{\frac{n}{\alpha}}. \tag{3.31}$$

From Theorem 3.3, we know that there exists a non-negative extremal w in $X^\alpha(\mathbb{R}^{n+1}_+)$ for $S(n, \alpha, \gamma, 0)$ whenever $\gamma \geq 0$. By the definition of t_w, and the fact that $c_w(\Psi) > 0$, we obtain

$$c_w(\Psi) \le \sup_{t\ge 0} \Psi(tw) \le \sup_{t\ge 0} f(t),$$

where

$$f(t) = \frac{t^2}{2}\|w\|^2 - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx \quad \forall t > 0.$$

Simple computations yield that $f(t)$ attains its maximum at the point $\tilde{t} = \left(\frac{\|w\|^2}{\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx}\right)^{\frac{1}{2^*_\alpha - 2}}$. It then follows that

$$\sup_{t\ge 0} f(t) = (\frac{1}{2} - \frac{1}{2^*_\alpha}) \left(\frac{\|w\|^2}{(\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx)^{\frac{2}{2^*_\alpha}}} \right)^{\frac{2^*_\alpha}{2^*_\alpha - 2}} = \frac{\alpha}{2n} \left(\frac{\|w\|^2}{(\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx)^{\frac{2}{2^*_\alpha}}} \right)^{\frac{n}{\alpha}}.$$

Since w is an extremal for $S(n, \alpha, \gamma, 0)$, we get that

$$c_w(\Psi) \le \sup_{t\ge 0} f(t) = \frac{\alpha}{2n} S(n, \alpha, \gamma, 0)^{\frac{n}{\alpha}}.$$

We now need to show that equality does not hold in (3.31). Indeed, otherwise we would have that $0 < c_w(\Psi) = \sup_{t\ge 0} \Psi(tw) = \sup_{t\ge 0} f(t)$. Consider $t_1 > 0$ (*resp.*, $t_2 > 0$) where $\sup_{t\ge 0} \Psi(tw)$ (*resp.*, $\sup_{t\ge 0} f(t)$) is attained. We get that

$$f(t_1) - \frac{t_1^{2^*_\alpha(s)}}{2^*_\alpha(s)} \int_{\mathbb{R}^n} \frac{|w(x, 0)|^{2^*_\alpha(s)}}{|x|^s} dx = f(t_2),$$

which means that $f(t_1) > f(t_2)$ since $t_1 > 0$. This contradicts the fact that t_2 is a maximum point of $f(t)$, hence the strict inequality in (3.31) holds.

To finish the proof of Proposition 3.7, we can assume without loss that

$$\frac{\alpha - s}{2(n-s)} S(n, \alpha, \gamma, s)^{\frac{n-s}{\alpha - s}} < \frac{\alpha}{2n} S(n, \alpha, \gamma, 0)^{\frac{n}{\alpha}}.$$

Let now w in $X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$ be a positive minimizer for $S(n, \alpha, \gamma, s)$, whose existence was established in Section 3.2, and set

$$\bar{f}(t) = \frac{t^2}{2}\|w\|^2 - \frac{t^{2^*_\alpha(s)}}{2^*_\alpha(s)} \int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx.$$

As above, we have

$$c_w(\Psi) \le \sup_{t\ge 0} f(t) = (\frac{1}{2} - \frac{1}{2^*_\alpha(s)}) \left(\frac{\|w\|^2}{(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx)^{\frac{2}{2^*_\alpha(s)}}} \right)^{\frac{2^*_\alpha(s)}{2^*_\alpha(s)-2}}$$
$$= \frac{\alpha - s}{2(n-s)} S(n,\alpha,\gamma,s)^{\frac{n-s}{\alpha-s}}.$$

Again, if equality holds, then $0 < c_w(\Psi) \le \sup_{t\ge 0} \Psi(tw) = \sup_{t\ge 0} \bar{f}(t)$, and if $t_1, t_2 > 0$ are points where the respective suprema are attained, then a contradiction is reached since

$$\bar{f}(t_1) - \frac{t_1^{2^*_\alpha}}{2^*_\alpha} \int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx = \bar{f}(t_2).$$

Therefore,

$$0 < c_w(\Psi) < c^\star = \min\left\{ \frac{\alpha}{2n} S(n,\alpha,\gamma,0)^{\frac{n}{\alpha}}, \frac{\alpha - s}{2(n-s)} S(n,\alpha,\gamma,s)^{\frac{n-s}{\alpha-s}} \right\}.$$

Finally, the existence of a Palais-Smale sequence at that level follows immediately from Lemma 3.6. □

3.3.2 Analysis of the Palais-Smale sequences

We now study the concentration properties of weakly null Palais-Smale sequences. For $\delta > 0$, we shall write $B_\delta^+ := \{(x,y) \in \mathbb{R}^{n+1}_+ : |(x,y)| < \delta\}$ and $B_\delta := \{x \in \mathbb{R}^n : |x| < \delta\}$.

Proposition 3.8. *Let $0 \le \gamma < \gamma_H(\alpha)$ and $0 < s < \alpha$. Assume that $(w_k)_{k\in\mathbb{N}}$ is a Palais-Smale sequence of Ψ at energy level $c \in (0, c^\star)$. If $w_k \rightharpoonup 0$ in $X^\alpha(\mathbb{R}^{n+1}_+)$ as $k \to \infty$, then there exists a positive constant $\epsilon_0 = \epsilon_0(n,\alpha,\gamma,c,s) > 0$ such that for every $\delta > 0$, one of the following holds:*

1. $\limsup_{k\to\infty} \int_{B_\delta} |u_k|^{2^*_\alpha} dx = \limsup_{k\to\infty} \int_{B_\delta} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx = 0;$

2. $\limsup_{k\to\infty} \int_{B_\delta} |u_k|^{2^*_\alpha} dx$ *and* $\limsup_{k\to\infty} \int_{B_\delta} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx \ge \epsilon_0.$

The proof of Proposition 3.8 requires the following two lemmas.

Lemma 3.9. *Let $(w_k)_{k\in\mathbb{N}}$ be a Palais-Smale sequence as in Proposition 3.8. If $w_k \rightharpoonup 0$ in $X^\alpha(\mathbb{R}^{n+1}_+)$, then for any $\omega \subset\subset \mathbb{R}^n \setminus \{0\}$ and any $D \subset\subset \mathbb{R}^{n+1}_+ \setminus \{0\}$, there exists a subsequence of $(w_k)_{k\in\mathbb{N}}$, still denoted by $(w_k)_{k\in\mathbb{N}}$, such that*

$$\lim_{k\to\infty} \int_\omega \frac{|u_k|^2}{|x|^\alpha} dx = \lim_{k\to\infty} \int_\omega \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx = 0 \tag{3.32}$$

and

$$\lim_{k\to\infty} \int_{D^*} |u_k|^{2^*_\alpha} dx = \lim_{k\to\infty} \int_D y^{1-\alpha} |\nabla w_k|^2 dxdy = 0, \tag{3.33}$$

where $u_k := w_k(.,0)$ for all $k \in \mathbb{N}$, and $D^* := \{(x,y) \in D : y = 0\} \subset\subset \mathbb{R}^n \setminus \{0\}$.

Proof of Lemma 3.9. Fix $\omega \subset\subset \mathbb{R}^n \setminus \{0\}$, and note that the following fractional Sobolev embedding is compact:

$$H_0^{\frac{\alpha}{2}}(\mathbb{R}^n) \hookrightarrow L^q(\omega) \quad \text{ for every } 1 \le q < 2^*_\alpha.$$

Using the trace inequality (2.13), and the assumption $w_k \rightharpoonup 0$ in $X^\alpha(\mathbb{R}^{n+1}_+)$, we get that

$$u_k \to 0 \quad \text{ strongly for every } 1 \le q < 2^*_\alpha.$$

On the other hand, the fact that $|x|^{-1}$ is bounded on $\omega \subset\subset \mathbb{R}^n \setminus \{0\}$ implies that there exist constants $C_1, C_2 > 0$ such that

$$0 \le \lim_{k\to\infty} \int_\omega \frac{|u_k|^2}{|x|^\alpha} dx \le C_1 \lim_{k\to\infty} \int_\omega |u_k|^2 dx$$

and

$$0 \le \lim_{k\to\infty} \int_\omega \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx \le C_2 \lim_{k\to\infty} \int_\omega |u_k|^{2^*_\alpha(s)} dx.$$

Since $s \in (0,\alpha)$, we have that $1 \le 2, 2^*_\alpha(s) < 2^*_\alpha$. Thus, (3.32) holds.

To show (3.33), we let $\eta \in C_0^\infty(\mathbb{R}^{n+1}_+)$ be a cut-off function such that $\eta_* := \eta(.,0) \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, $\eta \equiv 1$ in D and $0 \le \eta \le 1$ in $\mathbb{R}^{n+1}_+$. We first note that

$$k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |\nabla(\eta w_k)|^2 dxdy = k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |\eta \nabla w_k|^2 dxdy + o(1). \tag{3.34}$$

Indeed, apply the following elementary inequality for vectors X, Y in $\mathbb{R}^{n+1}$,

$$\big||X+Y|^2 - |X|^2\big| \le C(|X||Y| + |Y|^2),$$

with $X = y^{\frac{1-\alpha}{2}}\eta\nabla w_k$ and $Y = y^{\frac{1-\alpha}{2}} w_k\nabla\eta$, to get for all $k\in\mathbb{N}$, that

$$\left|y^{1-\alpha}|\nabla(\eta w_k)|^2 - y^{1-\alpha}|\eta\nabla w_k|^2\right| \le C\left(y^{1-\alpha}|\eta\nabla w_k||w_k\nabla\eta| + y^{1-\alpha}|w_k\nabla\eta|^2\right).$$

By Hölder's inequality, we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2 dxdy - \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\eta\nabla w_k|^2 dxdy\right| \\
&\le\ C_3\left[\|w_k\|_{X^\alpha(\mathbb{R}^{n+1}_+)}\ (\int_{\operatorname{supp}(\nabla\eta)} y^{1-\alpha}|w_k|^2 dxdy)^{\frac12} + \int_{\operatorname{supp}(\nabla\eta)} y^{1-\alpha}|w_k|^2 dxdy\right] \\
&\le C_4\left[(\int_{\operatorname{supp}(\nabla\eta)} y^{1-\alpha}|w_k|^2 dxdy)^{\frac12} + \int_{\operatorname{supp}(\nabla\eta)} y^{1-\alpha}|w_k|^2 dxdy\right].
\end{aligned} \tag{3.35}$$

Since the embedding $H^1(\operatorname{supp}(\nabla\eta), y^{1-\alpha}) \hookrightarrow L^2(\operatorname{supp}(\nabla\eta), y^{1-\alpha})$ is compact, and $w_k \rightharpoonup 0$ in $X^\alpha(\mathbb{R}^{n+1}_+)$, we get that

$$\int_{\operatorname{supp}(\nabla\eta)} y^{1-\alpha}|w_k|^2 dxdy = o(1),$$

which gives

$$\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2 dxdy = \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\eta\nabla w_k|^2 dxdy + o(1).$$

Thus, (3.34) holds.
Now recall that the sequence $(w_k)_{k\in\mathbb{N}}$ has the following property:

$$\lim_{k\to\infty}\Psi'(w_k) = 0 \quad \text{strongly in } (X^\alpha(\mathbb{R}^{n+1}_+))'. \tag{3.36}$$

Since $\eta^2 w_k \in X^\alpha(\mathbb{R}^{n+1}_+)$ for all $k\in\mathbb{N}$, we can use it as a test function in (3.36) to get that

$$\begin{aligned}
o(1) &= \langle\Psi'(w_k), \eta^2 w_k\rangle \\
&= k_\alpha\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}\langle\nabla w_k, \nabla(\eta^2 w_k)\rangle dxdy - \gamma\int_{\mathbb{R}^n}\frac{\eta_*^2|u_k|^2}{|x|^\alpha}dx \\
&\quad - \int_{\mathbb{R}^n}\eta_*^2|u_k|^{2^*_\alpha}dx - \int_{\mathbb{R}^n}\frac{\eta_*^2|u_k|^{2^*_\alpha(s)}}{|x|^s}dx.
\end{aligned}$$

Regarding the first term, we have

$$k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}\langle \nabla w_k, \nabla(\eta^2 w_k)\rangle dxdy = k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\eta\nabla w_k|^2 dxdy$$
$$+ k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} w_k \langle \nabla(\eta^2), \nabla w_k\rangle dxdy.$$

From Hölder's inequality, and the fact that $w_k \to 0$ in $L^2(\mathrm{supp}(|\nabla\eta|), y^{1-\alpha})$, it follows that as $k \to \infty$,

$$\begin{aligned}
&\left| k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}\langle \nabla w_k, \nabla(\eta^2 w_k)\rangle dxdy - k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\eta\nabla w_k|^2 dxdy \right| \\
&\quad = \left| k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} w_k \langle \nabla(\eta^2), \nabla w_k\rangle dxdy \right| \\
&\quad \le k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|w_k||\nabla(\eta^2)||\nabla w_k| dxdy \\
&\quad \le C \int_{\mathrm{supp}(|\nabla\eta|)} y^{1-\alpha}|w_k||\nabla w_k| dxdy \\
&\quad \le C\|w_k\|_{X^\alpha(\mathbb{R}^{n+1}_+)} \left(\int_{\mathrm{supp}(|\nabla\eta|)} y^{1-\alpha}|w_k|^2 dxdy \right)^{\frac{1}{2}} \\
&\quad = o(1).
\end{aligned}$$

Thus, we have proved that

$$k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}\langle \nabla w_k, \nabla(\eta^2 w_k)\rangle dxdy = k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\eta\nabla w_k|^2 dxdy + o(1).$$

Using the above estimate coupled with (3.34), we obtain

$$\begin{aligned}
o(1) &= \langle \Psi'(w_k), \eta^2 w_k\rangle \\
&= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\eta\nabla w_k|^2 dxdy - \gamma \int_K \frac{\eta_*^2|u_k|^2}{|x|^\alpha} dx \\
&\quad - \int_{\mathbb{R}^n} \eta_*^2 |u_k|^{2^*_\alpha} dx - \int_K \frac{\eta_*^2|u_k|^{2^*_\alpha(s)}}{|x|^s} dx + o(1) \qquad (3.37) \\
&= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2 dxdy - \int_{\mathbb{R}^n} \eta_*^2|u_k|^{2^*_\alpha} dx + o(1) \\
&\ge \|\eta w_k\|^2 - \int_{\mathbb{R}^n} \eta_*^2|u_k|^{2^*_\alpha} dx + o(1) \quad \text{as } k \to \infty,
\end{aligned}$$

where $K = \mathrm{supp}(\eta_*)$. Therefore,

$$\|\eta w_k\|^2 \leq \int_{\mathbb{R}^n} |\eta_* u_k|^2 |u_k|^{2^*_\alpha - 2} dx + o(1) \quad \text{ as } k \to \infty. \tag{3.38}$$

By Hölder's inequality, and using the definition of $S(n, \alpha, \gamma, 0)$, we then get that

$$\begin{aligned} \|\eta w_k\|^2 &\leq \left(\int_{\mathbb{R}^n} |\eta_* u_k|^{2^*_\alpha} dx \right)^{\frac{2}{2^*_\alpha}} \left(\int_{\mathbb{R}^n} |u_k|^{2^*_\alpha} dx \right)^{\frac{2^*_\alpha - 2}{2^*_\alpha}} + o(1) \\ &\leq S(n, \alpha, \gamma, 0)^{-1} \|\eta w_k\|^2 \left(\int_{\mathbb{R}^n} |u_k|^{2^*_\alpha} dx \right)^{\frac{2^*_\alpha - 2}{2^*_\alpha}} + o(1). \end{aligned} \tag{3.39}$$

Thus,

$$\left[1 - S(n, \alpha, \gamma, 0)^{-1} \left(\int_{\mathbb{R}^n} |u_k|^{2^*_\alpha} dx \right)^{\frac{2^*_\alpha - 2}{2^*_\alpha}} \right] \|\eta w_k\|^2 \leq o(1). \tag{3.40}$$

In addition, it follows from (3.29) that

$$\Psi(w_k) - \frac{1}{2} \langle \Psi'(w_k), w_k \rangle = c + o(1),$$

that is,

$$\left(\frac{1}{2} - \frac{1}{2^*_\alpha}\right) \int_{\mathbb{R}^n} |u_k|^{2^*_\alpha} dx + \left(\frac{1}{2} - \frac{1}{2^*_\alpha(s)}\right) \int_{\mathbb{R}^n} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx = c + o(1), \tag{3.41}$$

from which follows that

$$\int_{\mathbb{R}^n} |u_k|^{2^*_\alpha} dx \leq \frac{2n}{\alpha} c + o(1) \quad \text{ as } k \to \infty. \tag{3.42}$$

Plugging (3.42) into (3.40), we obtain that

$$\left[1 - S(n, \alpha, \gamma, 0)^{-1} \left(\frac{2n}{\alpha} c\right)^{\frac{\alpha}{n}} \right] \|\eta w_k\|^2 \leq o(1) \quad \text{ as } k \to \infty.$$

On the other hand, by the upper bound (3.28) on c, we have that

$$c < \frac{\alpha}{2n} S(n, \alpha, \gamma, 0)^{\frac{n}{\alpha}}.$$

This yields that $1 - S(n, \alpha, \gamma, 0)^{-1} (\frac{2n}{\alpha} c)^{\frac{\alpha}{n}} > 0$, and therefore,

$$\lim_{k \to \infty} \|\eta w_k\|^2 = 0.$$

Using (2.13) and (3.10), we obtain that

$$\lim_{k\to\infty} k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2 dxdy = 0.$$

It also follows from the definition of $S(n,\alpha,\gamma,0)$ that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |\eta_* u_k|^{2^*_\alpha} dx = 0.$$

Since $\eta\big|_D \equiv 1$ and $\eta_*\big|_{D^*} \equiv 1$, the last two equality yield (3.33). □

Lemma 3.10. *Let $(w_k)_{k\in\mathbb{N}}$ be Palais-Smale sequence as in Proposition 3.8. For any $\delta > 0$, set*

$$\begin{aligned} &\theta := \limsup_{k\to\infty} \int_{B_\delta} |u_k|^{2^*_\alpha} dx; \qquad \zeta := \limsup_{k\to\infty} \int_{B_\delta} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx \text{ and} \\ &\mu := \lim_{k\to\infty} \int_{B^+_\delta} y^{1-\alpha}|\nabla w_k|^2 dxdy - \gamma \int_{B_\delta} \frac{|u_k|^2}{|x|^\alpha} dx, \end{aligned} \tag{3.43}$$

where $u_k := Tr(w_k) = w_k(.,0)$ for all $k \in \mathbb{N}$. If $w_k \rightharpoonup 0$ in $X^\alpha(\mathbb{R}^{n+1}_+)$ as $k \to \infty$, then the following hold:

1. $\theta^{\frac{2}{2^*_\alpha}} \le S(n,\alpha,\gamma,0)^{-1}\mu$ *and* $\zeta^{\frac{2}{2^*_\alpha(s)}} \le S(n,\alpha,\gamma,s)^{-1}\mu$.
2. $\mu \le \theta + \zeta$.

Proof of Lemma 3.10. First note that it follows from Lemma 3.9 that θ, ζ and μ are well-defined and are independent of the choice of $\delta > 0$. Let now $\eta \in C^\infty_0(\mathbb{R}^{n+1}_+)$ be a cut-off function such that $\eta_* := \eta(.,0) \in C^\infty_0(\mathbb{R}^n \setminus \{0\})$, $\eta \equiv 1$ in B^+_δ, and $0 \le \eta \le 1$ in $\mathbb{R}^{n+1}_+$.

1. Since $\eta w_k \in X^\alpha(\mathbb{R}^{n+1}_+)$, we get from the definition of $S(n,\alpha,\gamma,s)$

$$\begin{aligned} &S(n,\alpha,\gamma,0)\Big(\int_{\mathbb{R}^n} |\eta_* u_k|^{2^*_\alpha} dx\Big)^{\frac{2}{2^*_\alpha}} \\ &\qquad \le k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta_* u_k|^2}{|x|^\alpha} dx. \end{aligned} \tag{3.44}$$

On the other hand, from the definition of η, it follows that

$$
\begin{aligned}
& k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta_* u_k|^2}{|x|^\alpha} dx \\
& = \int_{B_\delta^+} y^{1-\alpha}|\nabla w_k|^2 dxdy - \gamma \int_{B_\delta} \frac{|u_k|^2}{|x|^\alpha} dx \\
& + \int_{\mathrm{supp}(\eta)\setminus B_\delta^+} y^{1-\alpha}|\nabla w_k|^2 dxdy - \gamma \int_{\mathrm{supp}(\eta_*)\setminus B_\delta} \frac{|\eta_* u_k|^2}{|x|^\alpha} dx,
\end{aligned}
$$

and

$$
\Big(\int_{B_\delta} |u_k|^{2^*_\alpha} dx\Big)^{\frac{2}{2^*_\alpha}} \leq \Big(\int_{\mathbb{R}^n} |\eta_* u_k|^{2^*_\alpha} dx\Big)^{\frac{2}{2^*_\alpha}}.
$$

Note that $\mathrm{supp}(\eta) \setminus B_\delta^+ \subset\subset \mathbb{R}^{n+1}_+ \setminus \{0\}$ and $\mathrm{supp}(\eta_*) \setminus B_\delta \subset\subset \mathbb{R}^n \setminus \{0\}$. Therefore, taking the upper limits at both sides of (3.44), and using Lemma 3.9, we get that

$$
S(n,\alpha,\gamma,0)\Big(\int_{B_\delta} |u_k|^{2^*_\alpha} dx\Big)^{\frac{2}{2^*_\alpha}} \leq \int_{B_\delta^+} y^{1-\alpha}|\nabla w_k|^2 dxdy - \gamma \int_{B_\delta} \frac{|u_k|^2}{|x|^\alpha} dx + o(1),
$$

as $k \to \infty$, which gives

$$
\theta^{\frac{2}{2^*_\alpha}} \leq S(n,\alpha,\gamma,0)^{-1}\mu.
$$

Similarly, we can prove that

$$
\zeta^{\frac{2}{2^*_\alpha(s)}} \leq S(n,\alpha,\gamma,s)^{-1}\mu.
$$

2. Since $\eta^2 w_k \in X^\alpha(\mathbb{R}^{n+1}_+)$ and $\langle \Psi'(w_k), \eta^2 w_k \rangle = o(1)$ as $k \to \infty$, we have

$$
\begin{aligned}
o(1) &= \langle \Psi'(w_k), \eta^2 w_k \rangle \\
&= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} \langle \nabla w_k, \nabla(\eta^2 w_k) \rangle dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta_* u_k|^2}{|x|^\alpha} dx \\
&\quad - \int_{\mathbb{R}^n} \eta_*^2 |u_k|^{2^*_\alpha} dx - \int_{\mathbb{R}^n} \frac{\eta_*^2 |u_k|^{2^*_\alpha(s)}}{|x|^s} dx \\
&= \left(k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |\eta \nabla w_k|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta_* u_k|^2}{|x|^\alpha} dx \right) \\
&\quad - \int_{\mathbb{R}^n} \eta_*^2 |u_k|^{2^*_\alpha} dx - \int_{\mathbb{R}^n} \frac{\eta_*^2 |u_k|^{2^*_\alpha(s)}}{|x|^s} dx \\
&\quad + k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} w_k \langle \nabla(\eta^2), \nabla w_k \rangle dxdy.
\end{aligned}
\tag{3.45}
$$

By Hölder's inequality, and the fact that $w_k \to 0$ in $L^2(\mathrm{supp}(|\nabla \eta|), y^{1-\alpha})$, we obtain that

$$
\begin{aligned}
\left| k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} w_k \langle \nabla(\eta^2), \nabla w_k \rangle dxdy \right| &\le k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |w_k| |\nabla(\eta^2)| |\nabla w_k| dxdy \\
&\le C \int_{\mathrm{supp}(|\nabla \eta|)} y^{1-\alpha} |w_k| |\nabla w_k| dxdy \\
&\le C \|w_k\|_{X^\alpha(\mathbb{R}^{n+1}_+)} \|w_k\|_{L^2(\mathrm{supp}(|\nabla \eta|), y^{1-\alpha})} \\
&\le o(1) \quad \text{as } k \to \infty.
\end{aligned}
$$

Plugging the above estimate into (3.45), and using (2.13), we get that

$$
\begin{aligned}
o(1) &= \langle \Psi'(w_k), \eta^2 w_k \rangle \\
&= \left(k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |\nabla(\eta w_k)|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta_* u_k|^2}{|x|^\alpha} dx \right) \\
&\quad - \int_{\mathbb{R}^n} \eta_*^2 |u_k|^{2^*_\alpha} dx - \int_{\mathbb{R}^n} \frac{\eta_*^2 |u_k|^{2^*_\alpha(s)}}{|x|^s} dx + o(1) \\
&\ge \left(\int_{B_\delta^+} y^{1-\alpha} |\nabla w_k|^2 dxdy - \gamma \int_{B_\delta} \frac{|u_k|^2}{|x|^\alpha} dx \right) \\
&\quad - \left(\int_{B_\delta} |u_k|^{2^*_\alpha} dx \right) - \left(\int_{B_\delta} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx \right)
\end{aligned}
$$

$$-\int_{\mathrm{supp}(\eta_*)\setminus B_\delta}\left(\gamma\frac{|\eta_* u_k|^2}{|x|^\alpha}dx+\eta_*^2|u_k|^{2^*_\alpha}dx+\frac{\eta_*^2|u_k|^{2^*_\alpha(s)}}{|x|^s}\right)dx+o(1).$$

Noting that $\mathrm{supp}(\eta_*)\setminus B_\delta \subset\subset \mathbb{R}^n\setminus\{0\}$, and taking the upper limits on both sides, we get that $\mu\le\theta+\zeta$. □

Proof of Proposition 3.8. It follows from Lemma 3.10 that

$$\theta^{\frac{2}{2^*_\alpha}}\le S(n,\alpha,\gamma,0)^{-1}\mu\le S(n,\alpha,\gamma,0)^{-1}\theta+S(n,\alpha,\gamma,0)^{-1}\zeta,$$

which gives

$$\theta^{\frac{2}{2^*_\alpha}}(1-S(n,\alpha,\gamma,0)^{-1}\theta^{\frac{2^*_\alpha-2}{2^*_\alpha}})\le S(n,\alpha,\gamma,0)^{-1}\zeta. \tag{3.46}$$

On the other hand, by (3.41), we have

$$\theta\le\frac{2n}{\alpha}c.$$

Substituting the last inequality into (3.46), we get that

$$(1-S(n,\alpha,\gamma,0)^{-1}(\frac{2n}{\alpha}c)^{\frac{\alpha}{n}})\theta^{\frac{2}{2^*_\alpha}}\le S(n,\alpha,\gamma,0)^{-1}\zeta.$$

Recall that the upper bounded (3.28) on c implies that

$$1-S(n,\alpha,\gamma,0)^{-1}(\frac{2n}{\alpha}c)^{\frac{\alpha}{n}}>0.$$

Therefore, there exists $\delta_1=\delta_1(n,\alpha,\gamma,c)>0$ such that $\theta^{\frac{2}{2^*_\alpha}}\le\delta_1\zeta$. Similarly, there exists $\delta_2=\delta_2(n,\alpha,\gamma,c,s)>0$ such that $\zeta^{\frac{2}{2^*_\alpha(s)}}\le\delta_2\theta$. These two inequalities yield that there exists $\epsilon_0=\epsilon_0(n,\alpha,\gamma,c,s)>0$ such that

$$\text{either}\quad \theta=\zeta=0\quad\text{or}\quad\{\theta\ge\epsilon_0\text{ and }\zeta\ge\epsilon_0\}. \tag{3.47}$$

It follows from the definition of θ and ζ that

$$\begin{aligned}&\text{either}\ \limsup_{k\to\infty}\int_{B_\delta}|u_k|^{2^*_\alpha}dx=\limsup_{k\to\infty}\int_{B_\delta}\frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx=0;\\ &\text{or}\quad \limsup_{k\to\infty}\int_{B_\delta}|u_k|^{2^*_\alpha}dx\ge\epsilon_0\quad\text{and}\quad\limsup_{k\to\infty}\int_{B_\delta}\frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx\ge\epsilon_0.\end{aligned}$$

□

3.3.3 End of proof of Theorem 3.4

We shall first eliminate the possibility of a zero weak limit for the Palais-Smale sequence of Ψ, then we prove that the nontrivial weak limit is indeed a weak solution of Problem (3.6). In the sequel $(w_k)_{k\in\mathbb{N}}$ will denote the Palais-Smale sequence for Ψ obtained in Proposition 3.8.

First we show that

$$\limsup_{k\to\infty}\int_{\mathbb{R}^n}|u_k|^{2^*_\alpha}dx > 0. \tag{3.48}$$

Indeed, otherwise $\lim\limits_{k\to\infty}\int_{\mathbb{R}^n}|u_k|^{2^*_\alpha}dx = 0$, which once combined with the fact that $\langle\Psi'(w_k), w_k\rangle \to 0$ yields

$$\|w_k\|^2 = \int_{\mathbb{R}^n}\frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx + o(1) \qquad \text{as } k\to\infty.$$

By combining this estimate with the definition of $S(n,\alpha,\gamma,s)$, we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}} &\le S(n,\alpha,\gamma,s)^{-1}\|w_k\|^2\\ &\le S(n,\alpha,\gamma,s)^{-1}\int_{\mathbb{R}^n}\frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx + o(1),\end{aligned}$$

which implies that

$$\left(\int_{\mathbb{R}^n}\frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}}\left[1 - S(n,\alpha,\gamma,s)^{-1}\Big(\int_{\mathbb{R}^n}\frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx\Big)^{\frac{2^*_\alpha(s)-2}{2^*_\alpha(s)}}\right] \le o(1).$$

It follows from (3.28) and (3.41) that as $k\to\infty$,

$$\int_{\mathbb{R}^n}\frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx = 2c\frac{n-s}{\alpha-s} + o(1)$$

and

$$(1 - S(n,\alpha,\gamma,s)^{-1}(2c\frac{n-s}{\alpha-s})^{\frac{\alpha-s}{n-s}}) > 0.$$

Hence,

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx = 0. \tag{3.49}$$

Using that $\lim_{k\to\infty} \int_{\mathbb{R}^n} |u_k|^{2^*_\alpha} dx = 0$, in conjunction with (3.49) and (3.41), we get that $c + o(1) = 0$, which contradicts the fact that $c > 0$. This completes the proof of (3.48).

Now, we show that for small enough $\epsilon > 0$, there exists another Palais-Smale sequence $(v_k)_{k\in\mathbb{N}}$ for Ψ satisfying the properties of Proposition 3.8, which is also bounded in $X^\alpha(\mathbb{R}^{n+1}_+)$ and satisfies

$$\int_{B_1} |v_k(x,0)|^{2^*_\alpha} dx = \epsilon \quad \text{for all } k \in \mathbb{N}. \tag{3.50}$$

For that, consider ϵ_0 as given in Proposition 3.8. Let $\beta = \limsup_{k\to\infty} \int_{\mathbb{R}^n} |u_k|^{2^*_\alpha} dx$, which is positive by (3.48). Set $\epsilon_1 := \min\{\beta, \frac{\epsilon_0}{2}\}$ and fix $\epsilon \in (0, \epsilon_1)$. Up to a subsequence, there exists by continuity a sequence of radii $(r_k)_{k\in\mathbb{N}}$ such that $\int_{B_{r_k}} |u_k|^{2^*_\alpha} dx = \epsilon$ for each $k \in \mathbb{N}$. Let now

$$v_k(x,y) := r_k^{\frac{n-\alpha}{2}} w_k(r_k x, r_k y) \quad \text{for } x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}_+.$$

It is clear that

$$\int_{B_1} |v_k(x,0)|^{2^*_\alpha} dx = \int_{B_{r_k}} |u_k|^{2^*_\alpha} dx = \epsilon \quad \text{for all } k \in \mathbb{N}. \tag{3.51}$$

It is easy to check that $(v_k)_{k\in\mathbb{N}}$ is also a Palais-Smale sequence for Ψ that satisfies the properties of Proposition 3.8.

We now show that $(v_k)_{k\in\mathbb{N}}$ is bounded in $X^\alpha(\mathbb{R}^{n+1}_+)$. Indeed, since $(v_k)_{k\in\mathbb{N}}$ is a Palais-Smale sequence, there exist positive constants $C_1, C_2 > 0$ such that

$$\begin{aligned} C_1 + C_2\|v_k\| &\geq \Psi(v_k) - \frac{1}{2^*_\alpha(s)}\langle \Psi'(v_k), v_k\rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*_\alpha(s)}\right)\|v_k\|^2 + \left(\frac{1}{2^*_\alpha} - \frac{1}{2^*_\alpha(s)}\right)\int_{\mathbb{R}^n} |v_k(x,0)|^{2^*_\alpha} dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*_\alpha(s)}\right)\|v_k\|^2. \end{aligned} \tag{3.52}$$

The last inequality holds since $2 < 2^*_\alpha(s) < 2^*_\alpha$. Combining (3.52) with (3.10), we obtain that $(v_k)_{k\in\mathbb{N}}$ is bounded in $X^\alpha(\mathbb{R}^{n+1}_+)$.

It follows that there exists a subsequence – still denoted by v_k – such that $v_k \rightharpoonup v$ in $X^\alpha(\mathbb{R}^{n+1}_+)$ as $k \to \infty$. We claim that v is a nontrivial weak solution of (3.6). Indeed, if $v \equiv 0$, then Proposition 3.8 yields that

$$\text{either} \quad \limsup_{k\to\infty} \int_{B_1} |v_k(x,0)|^{2^*_\alpha} dx = 0 \quad \text{or} \quad \limsup_{k\to\infty} \int_{B_1} |v_k(x,0)|^{2^*_\alpha} dx \geq \epsilon_0.$$

Since $\epsilon \in (0, \frac{\epsilon_0}{2})$, this is in contradiction with (3.51), thus, $v \not\equiv 0$.
To show that $v \in X^\alpha(\mathbb{R}^{n+1}_+)$ is a weak solution of (3.6), consider any $\varphi \in C_0^\infty(\mathbb{R}^{n+1}_+)$, and write

$$
\begin{aligned}
o(1) &= \langle \Psi'(v_k), \varphi \rangle \\
&= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} \langle \nabla v_k, \nabla \varphi \rangle dxdy - \gamma \int_{\mathbb{R}^n} \frac{v_k(x,0)\varphi}{|x|^\alpha} dx \\
&\quad - \int_{\mathbb{R}^n} |v_k(x,0)|^{2^*_\alpha - 2} v_k(x,0)\varphi dx - \int_{\mathbb{R}^n} \frac{|v_k(x,0)|^{2^*_\alpha(s)-2} v_k(x,0)\varphi}{|x|^s} dx.
\end{aligned}
\tag{3.53}
$$

Since $v_k \rightharpoonup v$ in $X^\alpha(\mathbb{R}^{n+1}_+)$ as $k \to \infty$, we have that

$$
\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} \langle \nabla v_k, \nabla \varphi \rangle dxdy \to \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} \langle \nabla v, \nabla \varphi \rangle dxdy, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^{n+1}_+).
$$

In addition, the boundedness of v_k in $X^\alpha(\mathbb{R}^{n+1}_+)$ yields that

$$
v_k(.,0), |v_k(.,0)|^{2^*_\alpha-2} v_k(.,0) \text{ and } |v_k(.,0)|^{2^*_\alpha(s)-2} v_k(.,0)
$$

are bounded in $L^2(\mathbb{R}^n, |x|^{-\alpha}), L^{\frac{2^*_\alpha}{2^*_\alpha - 1}}(\mathbb{R}^n)$ and $L^{\frac{2^*_\alpha(s)}{2^*_\alpha(s)-1}}(\mathbb{R}^n, |x|^{-s})$, respectively. Therefore, we have the following weak convergence:

$$
\begin{aligned}
& v_k(.,0) \rightharpoonup v(.,0) \quad \text{ in } L^2(\mathbb{R}^n, |x|^{-\alpha}) \\
& |v_k(.,0)|^{2^*_\alpha-2} v_k(.,0) \rightharpoonup |v(.,0)|^{2^*_\alpha-2} v(.,0) \quad \text{ in } L^{\frac{2^*_\alpha}{2^*_\alpha-1}}(\mathbb{R}^n) \\
& |v_k(.,0)|^{2^*_\alpha(s)-2} v_k(.,0) \rightharpoonup |v(.,0)|^{2^*_\alpha(s)-2} v(.,0) \quad \text{ in } L^{\frac{2^*_\alpha(s)}{2^*_\alpha(s)-1}}(\mathbb{R}^n, |x|^{-s}).
\end{aligned}
$$

Thus, taking limits as $k \to \infty$ in (3.53), we obtain that

$$
\begin{aligned}
0 &= \langle \Psi'(v), \varphi \rangle \\
&= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} \langle \nabla v, \nabla \varphi \rangle dxdy - \gamma \int_{\mathbb{R}^n} \frac{v(x,0)\varphi}{|x|^\alpha} dx \\
&\quad - \int_{\mathbb{R}^n} |v(x,0)|^{2^*_\alpha-2} v(x,0)\varphi dx - \int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)-2} v(x,0)\varphi}{|x|^s} dx.
\end{aligned}
$$

Hence v is a weak solution of (3.6).

Chapter 4

Mass and Asymptotics Associated to Fractional Hardy-Schrödinger Operators in Critical Regimes

4.1 Introduction

Throughout this chapter, we shall assume that

$$0 < \alpha < n \quad \text{and} \quad 0 \le \gamma < \gamma_H(\alpha) = 2^{\alpha} \frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})}, \tag{4.1}$$

which is the best fractional Hardy constant on $\mathbb{R}^n$ (see below). We may also sometimes use the following notations for $\beta_+(\gamma)$ and $\beta_-(\gamma)$ introduced in Remark 1.4:

$$\beta_+ := \beta_+(\gamma) \quad \text{and} \quad \beta_- := \beta_-(\gamma).$$

Our main focus will be on the case when $\alpha < 2$, that is when $(-\Delta)^{\frac{\alpha}{2}}$ is not a local operator. We shall study problems on bounded domains, but will start by recalling the properties of $(-\Delta)^{\frac{\alpha}{2}}$ on the whole of $\mathbb{R}^n$, where it can be defined on the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ (the space of rapidly decaying C^∞ functions on $\mathbb{R}^n$) via the Fourier transform,

$$(-\Delta)^{\frac{\alpha}{2}} u = \mathcal{F}^{-1}(|2\pi\xi|^{\alpha}\mathcal{F}(u)).$$

Here $\mathcal{F}(u)$ is the Fourier transform of u. For $\alpha \in (0,2)$, the fractional Sobolev space $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is defined as the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|^2_{H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |2\pi\xi|^{\alpha}|\mathcal{F}u(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx.$$

As we mentioned before in (2.17), the fractional Hardy inequality in $\mathbb{R}^n$ then states that

$$\gamma_H(\alpha) := \inf\left\{ \frac{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}}u|^2 dx}{\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx}; \, u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n) \setminus \{0\} \right\} = 2^\alpha \frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})},$$

which means that the fractional Hardy-Schrödinger operator $L_{\gamma,\alpha}$ is positive whenever (4.1) is satisfied. In this case, a Hardy-Sobolev type inequality holds for $L_{\gamma,\alpha}$. It states that if $0 \leq s < \alpha < n$, and $2^*_\alpha(s) = \frac{2(n-s)}{n-\alpha}$, then $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ is finite and strictly positive, where the latter is the best constant

$$\mu_{\gamma,s,\alpha}(\mathbb{R}^n) := \inf_{u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}}u|^2 dx - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx}{(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx)^{\frac{2}{2^*_\alpha(s)}}}. \tag{4.2}$$

Note that any minimizer for (4.2) leads –up to a constant– to a variational solution of the following borderline problem on $\mathbb{R}^n$,

$$\left\{ \begin{array}{rl} (-\Delta)^{\frac{\alpha}{2}}u - \gamma \frac{u}{|x|^\alpha} = \frac{u^{2^*_\alpha(s)-1}}{|x|^s} & \text{in } \mathbb{R}^n \\ u \geq 0 \, ; \, u \not\equiv 0 & \text{in } \mathbb{R}^n. \end{array} \right. \tag{4.3}$$

Indeed, a function $u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is said to be a weak solution to (4.3) if $u \geq 0$, $u \not\equiv 0$ and for any $\varphi \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$, we have

$$\frac{C_{n,\alpha}}{2} \int_{(\mathbb{R}^n)^2} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+\alpha}} \, dxdy = \int_{\mathbb{R}^n} (\gamma \frac{u}{|x|^\alpha} + \frac{u^{2^*_\alpha(s)-1}}{|x|^s}) \varphi \, dx.$$

Unlike the case of the Laplacian ($\alpha = 2$), no explicit formula is known for the best constant $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ nor for the extremals where it is achieved. We therefore try to describe their asymptotic profile whenever they exist. This was considered in Ghoussoub-Shakerian [39]; where Theorem 3.1 is proved.

Note that the cases when $\gamma = 0$ are by now well known. Indeed, it was stated in [19] that the infimum in $\mu_{0,0,\alpha}(\mathbb{R}^n)$ is attained. Actually, a function $\tilde{u} \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n) \setminus \{0\}$ is an extremal for $\mu_{0,0,\alpha}(\mathbb{R}^n)$ if and only if there exist $x_0 \in \mathbb{R}^n$, $k \in \mathbb{R} \setminus \{0\}$ and $r > 0$ and such that

$$\tilde{u}(x) = k\left(r^2 + |x-x_0|^2\right)^{-\frac{(n-\alpha)}{2}} \qquad \text{for all } x \in \mathbb{R}^n.$$

Asymptotic properties of the positive extremals of $\mu_{0,s,\alpha}(\mathbb{R}^n)$ (i.e., when $\gamma = 0$ and $0 < s < \alpha$) were given by Y. Lei [48], Lu-Zhu [54], and Yang-Yu

[71]. The latter proved that an extremal $\bar{u}(x)$ for $\mu_{0,s,\alpha}(\mathbb{R}^n)$ must have the following behaviour: There is $C > 0$ such that

$$C^{-1}\left(1+|x|^2\right)^{-\frac{(n-\alpha)}{2}} \leq \bar{u}(x) \leq C\left(1+|x|^2\right)^{-\frac{(n-\alpha)}{2}} \quad \text{for all } x \in \mathbb{R}^n. \tag{4.4}$$

Recently, Dipierro-Montoro-Peral-Sciunzi [21] found a similar control of the extremal for $\mu_{\gamma,0,\alpha}(\mathbb{R}^n)$ (i.e., when $0 < \gamma < \gamma_H(\alpha)$ and $s = 0$). Our first result is an improvement of their estimate since it gives the exact asymptotic behaviour of the extremal of $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ in the general case. For that, we consider the function

$$\Psi_{n,\alpha}(\beta) := 2^{\alpha}\frac{\Gamma(\frac{n-\beta}{2})\Gamma(\frac{\alpha+\beta}{2})}{\Gamma(\frac{n-\beta-\alpha}{2})\Gamma(\frac{\beta}{2})}. \tag{4.5}$$

Theorem 4.1. *Assume $0 \leq s < \alpha < 2$, $n > \alpha$ and $0 \leq \gamma < \gamma_H(\alpha)$. Then, any positive extremal $u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ for $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ satisfies $u \in C^1(\mathbb{R}^n \setminus \{0\})$ and*

$$\lim_{x\to 0}|x|^{\beta_-(\gamma)}u(x) = \lambda_0 \text{ and } \lim_{|x|\to\infty}|x|^{\beta_+(\gamma)}u(x) = \lambda_\infty, \tag{4.6}$$

where $\lambda_0, \lambda_\infty > 0$ and $\beta_-(\gamma)$ (resp., $\beta_+(\gamma)$) is the unique solution in $\left(0, \frac{n-\alpha}{2}\right)$ (resp., in $\left(\frac{n-\alpha}{2}, n-\alpha\right)$) of the equation $\Psi_{n,\alpha}(t) = \gamma$. In particular, there exist $C_1, C_2 > 0$ such that

$$\frac{C_1}{|x|^{\beta_-(\gamma)}+|x|^{\beta_+(\gamma)}} \leq u(x) \leq \frac{C_2}{|x|^{\beta_-(\gamma)}+|x|^{\beta_+(\gamma)}} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

Remark 4.2. Note that a direct consequence of Theorem 4.1 is (4.4) and the corresponding control by Dipierro-Montoro-Peral-Sciunzi [21].

Also note that if $\alpha = 2$, that is when the fractional Laplacian is the classical Laplacian, the best constant in the Hardy inequality is then $\gamma_H(2) = \frac{(n-2)^2}{4}$. The best constant associated with the Hardy-Sobolev inequality is

$$\mu_{\gamma,s,2}(\mathbb{R}^n) := \inf_{u\in D^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n}|\nabla u|^2dx - \gamma\int_{\mathbb{R}^n}\frac{|u|^2}{|x|^2}dx}{(\int_{\mathbb{R}^n}\frac{|u|^{2^\star(s)}}{|x|^s}dx)^{\frac{2}{2^\star(s)}}},$$

where $s \in [0,2)$, $2^*(s) := \frac{2(n-s)}{n-2}$, $0 \leq \gamma < \gamma_H(2) = \frac{(n-2)^2}{4}$ and $D^{1,2}(\mathbb{R}^n)$ is the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|u\|^2 = \int_{\mathbb{R}^n}|\nabla u|^2dx$.

The extremals for $\mu_{\gamma,s,2}(\mathbb{R}^n)$ are then explicit and are given by multiples of the functions $u_\epsilon(x) = \epsilon^{-\frac{n-2}{2}} U(\frac{x}{\epsilon})$ for $\epsilon > 0$, where

$$U(x) = \frac{1}{\left(|x|^{\frac{(2-s)\sigma_-(\gamma)}{n-2}} + |x|^{\frac{(2-s)\sigma_+(\gamma)}{n-2}}\right)^{\frac{n-2}{2-s}}} \quad \text{for } \mathbb{R}^n \setminus \{0\},$$

and

$$\sigma_\pm(\gamma) = \frac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} - \gamma}.$$

Note that the radial function $u(x) = |x|^{-\beta}$ is a solution of $L_{\gamma,2}(u) = 0$ on $\mathbb{R}^n \setminus \{0\}$ if and only if

$$\beta \in \{\sigma_-(\gamma), \sigma_+(\gamma)\}. \tag{4.7}$$

Back to the case $0 < \alpha < 2$, we now turn to when Ω is a smooth bounded domain in $\mathbb{R}^n$ with 0 in its interior. The best constant in the corresponding fractional Hardy-Sobolev inequality is then,

$$\mu_{\gamma,s,\alpha}(\Omega) := \inf_{u \in H_0^{\frac{\alpha}{2}}(\Omega)\setminus\{0\}} \frac{\frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}} dxdy - \gamma \int_\Omega \frac{|u|^2}{|x|^\alpha} dx}{\left(\int_\Omega \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}}},$$

where $H_0^{\frac{\alpha}{2}}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{H_0^{\frac{\alpha}{2}}(\Omega)} = \frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}} dxdy = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx.$$

In Proposition 4.15, we note that –just like the case when $\alpha = 2$– we have $\mu_{\gamma,s,\alpha}(\Omega) = \mu_{\gamma,s,\alpha}(\mathbb{R}^n)$, and therefore (4.3) restricted to Ω, with Dirichlet boundary condition has no extremal, unless Ω is essentially $\mathbb{R}^n$. We therefore resort to a setting popularized by Brezis-Nirenberg [11] by considering the following boundary value problem:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u - \gamma \dfrac{u}{|x|^\alpha} = \dfrac{u^{2^*_\alpha(s)-1}}{|x|^s} + \lambda u & \text{in } \Omega \\ \qquad\qquad u \geq 0 & \text{in } \Omega \\ \qquad\qquad u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{4.8}$$

where $0 < \lambda < \lambda_1(L_{\gamma,\alpha})$ and $\lambda_1(L_{\gamma,\alpha})$ is the first eigenvalue of the operator $L_{\gamma,\alpha} = (-\Delta)^{\frac{\alpha}{2}} - \frac{\gamma}{|x|^\alpha}$ with Dirichlet boundary condition, that is

$$\lambda_1 := \lambda_1(L_{\gamma,\alpha}) = \inf_{u \in H_0^{\frac{\alpha}{2}}(\Omega)\setminus\{0\}} \frac{\frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}} dxdy - \gamma \int_\Omega \frac{u^2}{|x|^\alpha}}{\int_\Omega u^2 dx}.$$

One then considers the quantity

$$\mu_{\gamma,s,\alpha,\lambda}(\Omega)=\inf_{u\in H_0^{\frac{\alpha}{2}}(\Omega)\setminus\{0\}}\frac{\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy-\gamma\int_\Omega\frac{u^2}{|x|^\alpha}dx-\lambda\int_\Omega u^2dx}{\left(\int_\Omega\frac{u^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}}},$$

and uses the fact that compactness is restored as long as $\mu_{\gamma,s,\alpha,\lambda}(\Omega)<\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$; see Proposition 4.10 and also [11] for more details. This type of condition is now classical in borderline variational problems; see Aubin [4] and Brezis-Nirenberg [11].

When $\alpha=2$, i.e., in the case of the standard Laplacian, the minimization problem $\mu_{\gamma,s,\alpha,\lambda}(\Omega)$ has been extensively studied, see for example Lieb [50], Chern-Lin [17], Ghoussoub-Moradifam [35] and Ghoussoub-Robert [36]. The non-local case has also been the subject of several studies, but in the absence of the Hardy term, i.e., when $\gamma=0$. In [65], Servadei proved the existence of extremals for $\mu_{0,0,\alpha,\lambda}(\mathbb{R}^n)$, and completed the study of problem (4.8) which has been initiated by Servadei-Valdinoci [63, 64]. Recently, it has been shown by Yang-Yu [71] that there exists a positive extremal for $\mu_{0,s,\alpha,\lambda}(\mathbb{R}^n)$ when $s\in[0,2)$. In this chapter, we consider the remaining cases.

In the spirit of Jannelli [45], who dealt with the Laplacian case, we observe that problem (4.8) is deeply influenced by the value of the parameter γ. Roughly speaking, if γ is sufficiently small then $\mu_{\gamma,s,\alpha,\lambda}(\Omega)$ is attained for any $0<\lambda<\lambda_1$. This is essentially what was obtained by Servadei-Valdinoci [64] when $s=\gamma=0$ and $n\geq 2\alpha$ via local arguments. This is, however not the case, when γ is closer to $\gamma_H(\alpha)$, which amounts to dealing with low dimensions: see for instance Servadei-Valdinoci [63]. In this context of low dimension, the local arguments generally fail, and it is necessary to use *global* arguments via the introduction of a notion of mass in the spirit of Schoen [61]. In the present case, and as in the work of Ghoussoub-Robert [37], we define a notion of mass for the operator $L_{\gamma,\alpha}-\lambda I$, which again turns out to be critical for this non-local case. The mass is defined via the following key result.

Theorem 4.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ $(n>\alpha)$ and consider, for $0<\alpha<2$, the boundary value problem*

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}H-\left(\frac{\gamma}{|x|^\alpha}+a(x)\right)H=0 & \text{in } \Omega\setminus\{0\}\\ H>0 & \text{in } \Omega\setminus\{0\}\\ H=0 & \text{in } \mathbb{R}^n\setminus\Omega,\end{cases}\tag{4.9}$$

where $a(x)\in C^{0,\tau}(\overline{\Omega})$ for some $\tau\in(0,1)$. Assuming the operator $(-\Delta)^{\frac{\alpha}{2}}-(\frac{\gamma}{|x|^\alpha}+a(x))$ coercive, there exists then a threshold $-\infty<\gamma_{crit}(\alpha)<\gamma_H(\alpha)$

such that for any γ with $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$, there exists a unique solution to (4.9) (in the sense of Definition 4.6) $H : \Omega \to \mathbb{R}$, $H \not\equiv 0$, and a constant $c \in \mathbb{R}$ such that

$$H(x) = \frac{1}{|x|^{\beta_+(\gamma)}} + \frac{c}{|x|^{\beta_-(\gamma)}} + o\left(\frac{1}{|x|^{\beta_-(\gamma)}}\right) \qquad \text{as } x \to 0.$$

We define the fractional Hardy-singular internal mass *of Ω associated to the operator $L_{\gamma,\alpha}$ to be*

$$m^{\alpha}_{\gamma,a}(\Omega) := c \in \mathbb{R}.$$

We then prove the following existence result, which complements those in [65] and [71] to the case when $\gamma > 0$.

Theorem 4.4. *Let Ω be a smooth bounded domain in $\mathbb{R}^n (n > \alpha)$ such that $0 \in \Omega$, and let $0 \le s < \alpha$, $0 \le \gamma < \gamma_H(\alpha)$. Then, there exist extremals for $\mu_{\gamma,s,\alpha,\lambda}(\Omega)$ under one of the following two conditions:*

1. $0 \le \gamma \le \gamma_{crit}(\alpha)$ *and* $0 < \lambda < \lambda_1(L_{\gamma,\alpha})$,
2. $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$, $0 < \lambda < \lambda_1(L_{\gamma,\alpha})$ *and* $m^{\alpha}_{\gamma,\lambda}(\Omega) > 0$.

The idea of studying how critical behavior occurs while varying a parameter γ on which an operator $L_{\gamma,\alpha}$ continuously depends goes back to [45], who considered the classical Hardy-Schrödinger operator $L_{\gamma,2} := -\Delta - \frac{\gamma}{|x|^2}$, and showed the existence of extremals for any $\lambda > 0$ provided $0 \le \gamma \le \frac{(n-2)^2}{4} - 1$. In this case, $\gamma_{crit}(2) = \frac{(n-2)^2}{4} - 1$. The definition of the mass and the counterpart of Theorem 4.4 for the operator $L_{\gamma,2}$ was established by Ghoussoub-Robert [37]. The complete picture can be described as follows.

Hardy term	Dimension	Singularity	Analytic. cond.	Ext.
$0 \le \gamma \le \gamma_{crit}(\alpha)$	$n \ge 2\alpha$	$s \ge 0$	$\lambda > 0$	Yes
$\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$	$n \ge 2\alpha$	$s \ge 0$	$m^{\alpha}_{\gamma,\lambda}(\Omega) > 0$	Yes
$0 \le \gamma < \gamma_H(\alpha)$	$\alpha < n < 2\alpha$	$s \ge 0$	$m^{\alpha}_{\gamma,\lambda}(\Omega) > 0$	Yes

Even though the constructions and the methods are heavily inspired by the work of Ghoussoub-Robert [37] on the Laplacian case, the fact that the operator is nonlocal here induces several fundamental difficulties that had to be overcome. First, the construction of the mass in the local case uses a precise classification of singularities for solutions of corresponding elliptic equations, that follows from the comparison principle stating that behavior in a domain is governed by the behavior on its boundary. In the nonlocal case, this fails since one needs to consider the whole complement of the domain, and not only its boundary. We were able to bypass this difficulty by using sharp regularity results available for the fractional Laplacian. Another difficulty we had to face came from the test-functions estimates in the presence of the mass. In the classical local case, one estimates the associated functional on a singular test-function, counting on the mass to appear after suitable integrations by parts. In the nonlocal context, this strategy fails. We overcome this difficulty by looking at the integral on the boundary of a domain as a limit of integrals on the domain after multiplying by a cut-off functions whose support converge to the boundary. This process is well-defined in the nonlocal context and proves to be efficient in tackling the estimates involving the mass.

4.2 The fractional Hardy-Schrödinger operator $L_{\gamma,\alpha}$ on $\mathbb{R}^n$

In this section, we study the local behavior of solutions of the fractional Hardy-Schrödinger operator $L_{\gamma,\alpha} := (-\Delta)^{\frac{\alpha}{2}} - \frac{\gamma}{|x|^\alpha}$ on $\mathbb{R}^n$. The most basic solutions for $L_{\gamma,\alpha}u = 0$ on $\mathbb{R}^n$ are of the form $u(x) = |x|^{-\beta}$, and a straightforward computation yields (see [34])

$$(-\Delta)^{\frac{\alpha}{2}}|x|^{-\beta} = \Psi_{n,\alpha}(\beta)|x|^{-\beta-\alpha} \text{ in the sense of } \mathcal{S}'(\mathbb{R}^n) \text{ when } 0 < \beta < n-\alpha,$$

where

$$\Psi_{n,\alpha}(\beta) := 2^\alpha \frac{\Gamma(\frac{n-\beta}{2})\Gamma(\frac{\alpha+\beta}{2})}{\Gamma(\frac{n-\beta-\alpha}{2})\Gamma(\frac{\beta}{2})}. \tag{4.10}$$

Recall that the best constant in the fractional Hardy inequality

$$\gamma_H(\alpha) := \mu_{0,\alpha,\alpha}(\mathbb{R}^n) = \inf\left\{ \frac{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}}u|^2 dx}{\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx};\ u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n) \setminus \{0\} \right\}$$

is never achieved (see Fall [29]), is equal to $\Psi_{n,\alpha}(\frac{n-\alpha}{2}) = 2^\alpha \frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})}$ (see Herbst and Yafaev [43, 69]), and it converges to the best classical Hardy constant $\gamma_H(2) = \frac{(n-2)^2}{4}$ whenever $\alpha \to 2$.

We summarize some properties of the function $\beta \mapsto \Psi_{n,\alpha}(\beta)$ which will be used freely in this section. They are essentially consequences from known properties of Gamma function Γ.

Proposition 4.5 (Frank-Lieb-Seiringer [34]). *The following properties hold:*

1. $\Psi_{n,\alpha}(\beta) > 0$ *for all* $\beta \in (0, n-\alpha)$.
2. *The graph of* $\Psi_{n,\alpha}$ *in* $(0, n-\alpha)$ *is symmetric with respect to* $\frac{n-\alpha}{2}$, *that is,*
$$\Psi_{n,\alpha}(\beta) = \Psi_{n,\alpha}(n-\alpha-\beta) \text{ for all } \beta \in (0, n-\alpha).$$
3. $\Psi_{n,\alpha}$ *is strictly increasing in* $(0, \frac{n-\alpha}{2})$, *and strictly decreasing in* $(\frac{n-\alpha}{2}, n-\alpha)$.
4. $\Psi_{n,\alpha}\left(\dfrac{n-\alpha}{2}\right) = \gamma_H(\alpha)$.
5. $\lim\limits_{\beta \searrow 0} \Psi_{n,\alpha}(\beta) = \lim\limits_{\beta \nearrow n-\alpha} \Psi_{n,\alpha}(\beta) = 0$.
6. *For any* $\gamma \in (0, \gamma_H(\alpha))$, *there exists a unique* $\beta_-(\gamma) \in (0, \frac{n-\alpha}{2})$ *such that* $\Psi_{n,\alpha}(\beta_-(\gamma)) = \gamma$.
7. *For any* $0 < \beta \leq n-\alpha$, *we have that*
$$(-\Delta)^{\frac{\alpha}{2}}|x|^{-\beta} = \Psi_{n,\alpha}(\beta)|x|^{-\alpha-\beta} + c_{n,\alpha}\mathbf{1}_{\{\beta=n-\alpha\}}\delta_0 \text{ in } \mathcal{S}'(\mathbb{R}^n), \quad (4.11)$$
where we define $\Psi_{n,\alpha}(n-\alpha) = 0$ *and* $c_{n,\alpha} > 0$ *is a constant.*

In particular, for $0 < \beta < n-\alpha$,
$$\left((-\Delta)^{\frac{\alpha}{2}} - \frac{\gamma}{|x|^{\alpha}}\right)|x|^{-\beta} = 0 \text{ in } \mathcal{S}'(\mathbb{R}^n) \text{ if and only if } \beta \in \{\beta_+(\gamma), \beta_-(\gamma)\},$$
where $0 < \beta_-(\gamma) < \frac{n-\alpha}{2}$ is as in Proposition 4.5 and $\beta_+(\gamma) := n-\alpha-\beta_-(\gamma) \in \left(\frac{n-\alpha}{2}, n-\alpha\right)$. In particular, it follows from Proposition 4.5 that $\beta_-(\gamma), \beta_+(\gamma)$ are the only solutions to $\Psi_{n,\alpha}(\beta) = \gamma$ in $(0, n-\alpha)$. Since $0 < \beta_-(\gamma) < \frac{n-\alpha}{2} < \beta_+(\gamma) < n-\alpha$, we get that $x \mapsto |x|^{-\beta_-(\gamma)}$ is locally in $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$. It is the"small" or variational solution, while $x \mapsto |x|^{-\beta_+(\gamma)}$ is the"large" or singular solution. We extend $\beta_-(\gamma), \beta_+(\gamma)$ to the whole interval $[0, \gamma_H(\alpha)]$ by defining
$$\beta_-(0) := 0, \quad \beta_+(0) := n-\alpha, \quad \text{and} \quad \beta_-(\gamma_H(\alpha)) = \beta_+(\gamma_H(\alpha)) = \frac{n-\alpha}{2}, \quad (4.12)$$
which is consistant with Proposition 4.5.

We now proceed to define a critical threshold $\gamma_{crit}(\alpha)$ as follows. Assuming first that $n > 2\alpha$, then $\frac{n-\alpha}{2} < \frac{n}{2} < n - \alpha$ and therefore, by Proposition 4.5, there exists $\bar{\gamma}(\alpha) \in (0, \gamma_H(\alpha))$ such that

$$\begin{cases} \dfrac{n}{2} < \beta_+(\gamma) < n - \alpha & \text{if } \gamma \in (0, \bar{\gamma}(\alpha)) \\ \beta_+(\gamma) = \dfrac{n}{2} & \text{if } \gamma = \bar{\gamma}(\alpha) \\ \dfrac{n-\alpha}{2} < \beta_+(\gamma) < \dfrac{n}{2} & \text{if } \gamma \in (\bar{\gamma}(\alpha), \gamma_H(\alpha)). \end{cases}$$

We then set

$$\gamma_{crit}(\alpha) := \begin{cases} \bar{\gamma}(\alpha) & \text{if } \; n > 2\alpha \\ 0 & \text{if } n = 2\alpha \\ -1 & \text{if } n < 2\alpha. \end{cases} \tag{4.13}$$

One can easily check that for $\gamma \in [0, \gamma_H(\alpha))$, we have that

$$\gamma \in (\gamma_{crit}(\alpha), \gamma_H(\alpha)) \quad \Leftrightarrow \quad \beta_+(\gamma) < \frac{n}{2} \quad \Leftrightarrow \; x \mapsto |x|^{-\beta_+(\gamma)} \in L^2_{loc}(\mathbb{R}^n).$$

We now introduce the following terminology in defining a notion of solution on a punctured domain.

Definition 4.6. *Let Ω be a smooth domain (not necessarily bounded) of $\mathbb{R}^n$, $n > 1$. Let f be a function in $L^1_{loc}(\Omega \setminus \{0\})$. We say that $u : \Omega \to \mathbb{R}$ is a solution to*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = f & \text{in } \Omega \setminus \{0\} \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

provided

1. *For any $\eta \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, we have that $\eta u \in H_0^{\frac{\alpha}{2}}(\Omega)$;*
2. $\int_\Omega \frac{|u(x)|}{1+|x|^{n+\alpha}} \, dx < \infty$;
3. *For any $\varphi \in C_c^\infty(\Omega \setminus \{0\})$, we have that*

$$\frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+\alpha}} \, dxdy = \int_{\mathbb{R}^n} f(x)\varphi(x) \, dx.$$

Note that the third condition is consistent thanks to the two preceding it. If Ω is bounded, the second hypothesis rewrites as $u \in L^1(\Omega)$.

4.3 Profile of solutions

Throughout this chapter, we shall frequently use the following fact:

Proposition 4.7. *A measurable function $u : \mathbb{R}^n \to \mathbb{R}$ belongs to $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n} |u|^{2^\star_\alpha(0)}\, dx < +\infty$ and $\int_{(\mathbb{R}^n)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}\, dxdy < +\infty$.*

The proof consists of approximating u by a compactly supported function satisfying the same properties. Then, by convoluting with a smooth mollifier, this approximation is achieved by a smooth compactly supported function. The rest is classical and the details are left to the reader.

To prove Theorem 4.1, we shall use a similar argument as in Dipierro-Montoro-Peral-Sciunzi [21]. The main idea is to transform problem (4.3) into a different nonlocal problem in a weighted fractional space by using a representation introduced in Frank-Lieb-Seiringer [34].

Lemma 4.8 (Ground State Representation [34]; Formula (4.3))**.** *Assume $0 < \alpha < 2$, $n > \alpha$, $0 < \beta < \frac{n-\alpha}{2}$. For $u \in C_c^\infty(\mathbb{R}^n \backslash \{0\})$, we let $v_\beta(x) = |x|^\beta u(x)$ in $\mathbb{R}^n \backslash \{0\}$. Then,*

$$\frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}\, dxdy = \Psi_{n,\alpha}(\beta) \int_{\mathbb{R}^n} \frac{u^2(x)}{|x|^\alpha}\, dx \\ + \frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v_\beta(x)-v_\beta(y)|^2}{|x-y|^{n+\alpha}}\, \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta}.$$

Let now $u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ be a positive weak solution to (4.3). Then by (4.4) and Remark 4.4 in [21], we have

$$\frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}\, dxdy = \gamma \int_{\mathbb{R}^n} \frac{u^2(x)}{|x|^\alpha}\, dx + \int_{\mathbb{R}^n} \frac{u^{2^*_\alpha(s)}(x)}{|x|^s}\, dx.$$

Set $v(x) = |x|^{\beta_-(\gamma)} u(x)$ on $\mathbb{R}^n \backslash \{0\}$. It follows from Lemma 4.8 and the definition of $\beta_-(\gamma)$ that

$$\begin{aligned}
&\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^2}{|x-y|^{n+\alpha}}\,\frac{dx}{|x|^{\beta_-(\gamma)}}\frac{dy}{|y|^{\beta_-(\gamma)}}\\
&\quad=\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}\,dxdy-\Psi_{n,\alpha}(\beta_-(\gamma))\int_{\mathbb{R}^n}\frac{u^2(x)}{|x|^\alpha}\,dx\\
&\quad=\gamma\int_{\mathbb{R}^n}\frac{u^2(x)}{|x|^\alpha}\,dx+\int_{\mathbb{R}^n}\frac{u^{2^*_\alpha(s)}(x)}{|x|^s}\,dx-\Psi_{n,\alpha}(\beta_-(\gamma))\int_{\mathbb{R}^n}\frac{u^2(x)}{|x|^\alpha}\,dx\\
&\quad=\int_{\mathbb{R}^n}\frac{u^{2^*_\alpha(s)}(x)}{|x|^s}\,dx\\
&\quad=\int_{\mathbb{R}^n}\frac{v^{2^*_\alpha(s)}(x)}{|x|^{s+\beta_-(\gamma)2^*_\alpha(s)}}dx.
\end{aligned}$$

For $0<\beta<\frac{n-\alpha}{2}$, define the space $H_0^{\frac{\alpha}{2},\beta}(\mathbb{R}^n)$ as the completion of $C_c^\infty(\mathbb{R}^n\setminus\{0\})$ with respect to the norm

$$\|\phi\|_{H_0^{\frac{\alpha}{2},\beta}(\mathbb{R}^n)}:=\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+\alpha}}\,\frac{dx}{|x|^\beta}\frac{dy}{|y|^\beta}\right)^{\frac{1}{2}}.$$

Many of the properties of the space $H_0^{\frac{\alpha}{2},\beta}(\mathbb{R}^n)$ were established in [22]. By Lemma 4.8, Remark 4.4 in [21] and [1], we have that $v\in H_0^{\frac{\alpha}{2},\beta}(\mathbb{R}^n)$. Now, we introduce the operator $(-\Delta_\beta)^{\frac{\alpha}{2}}$, whose action on a function w is given via the following duality: For $\phi\in H_0^{\frac{\alpha}{2},\beta}(\mathbb{R}^n)$,

$$\langle(-\Delta_\beta)^{\frac{\alpha}{2}}w,\phi\rangle=\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(w(x)-w(y))(\phi(x)-\phi(y))}{|x-y|^{n+\alpha}}\,\frac{dx}{|x|^\beta}\frac{dy}{|y|^\beta}.$$

This means that v is a weak solution to

$$(-\Delta_{\beta_-(\gamma)})^{\frac{\alpha}{2}}v=\frac{v^{2^*_\alpha(s)-1}}{|x|^{s+\beta_-(\gamma)2^*_\alpha(s)}}\quad\text{in }\mathbb{R}^n,\tag{4.14}$$

in the sense that for any $\phi\in H^{\frac{\alpha}{2},\beta_-}(\mathbb{R}^n)$, we have that

$$\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{n+\alpha}}\,\frac{dx}{|x|^{\beta_-}}\frac{dy}{|y|^{\beta_-}}=\int_{\mathbb{R}^n}\frac{v^{2^*_\alpha(s)-1}}{|x|^{s+\beta_-2^*_\alpha(s)}}\phi\,dx.$$

The following proposition gives a regularity result and a Harnack inequality for weak solutions of (4.14).

Proposition 4.9. *Assume* $0 < s < \alpha < 2$, $n > \alpha$ *and* $0 < \beta < \frac{n-\alpha}{2}$, *and let* $v \in H_0^{\frac{\alpha}{2},\beta}(\mathbb{R}^n)$ *be a non-negative, non-zero weak solution to the problem*

$$(-\Delta_\beta)^{\frac{\alpha}{2}} v = \frac{v^{2_\alpha^*(s)-1}}{|x|^{s+\beta 2_\alpha^*(s)}} \quad \text{in } \mathbb{R}^n.$$

Then, $v \in L^\infty(\mathbb{R}^n)$ *and there exist constants* $R > 0$ *and* $C > 0$ *such that* $C \le v(x)$ *in* $B_R(0)$.

Proof. The statement that $v(x) \ge C$ in $B_R(0)$ is essentially the Harnack inequality for super-harmonic functions associated to the nonlocal operator $(-\Delta_\beta)^{\frac{\alpha}{2}}$, which is just Theorem 3.4 in Abdellaoui-Medina-Peral-Primo [2]. See also the proof of Lemma 3.10 in [2] and also [21]. We now show that $v \in L^\infty(\mathbb{R}^n)$ by using a similar argument as in [21]. For any $p \ge 1$ and $T > 0$, define the function

$$\phi_{p,T}(t) = \begin{cases} t^p & \text{if } 0 \le t \le T \\ pT^{p-1}(t-T) + T^p & \text{if } t > T. \end{cases}$$

It is easy to check that the function $\phi_{p,T}(t)$ has the following properties:

- $\phi_{p,T}(t)$ is convex and Lipschitz in $[0, \infty)$.
- $\phi_{p,T}(t) \le t^p$ for all $t \ge 0$.
- $t\phi'_{p,T}(t) \le 2p\phi_{p,T}(t)$ for all $t \ge 0$, since

$$t\phi'_{p,T}(t) = \begin{cases} p\phi_{p,T}(t) & \text{if } 0 < t < T \\ pT^{p-1}t & \text{if } t > T. \end{cases}$$

- If $T_2 > T_1 > 0$, then $\phi_{p,T_1}(t) \le \phi_{p,T_2}(t)$ for all $t \ge 0$.

Since $\phi_{p,T}(t)$ is convex and Lipschitz, then as noted in [49],

$$(-\Delta_\beta)^{\frac{\alpha}{2}} \phi_{p,T}(v) \le \phi'_{p,T}(v)(-\Delta_\beta)^{\frac{\alpha}{2}} v \quad \text{in } \mathbb{R}^n. \tag{4.15}$$

Since $\phi_{p,T}(t)$ is Lipschitz and $\phi_{p,T}(0) = 0$, then $\phi_{p,T}(v) \in H_0^{\frac{\alpha}{2},\beta}(\mathbb{R}^n)$. By the weighted fractional Hardy-Sobolev inequality, the ground state representation formula, Lemma 4.8, and (4.2), we get that there exists some constant $C_0 > 0$ which only depends on n, α, s and β such that

$$\left[\int_{\mathbb{R}^n} \frac{|\phi_{p,T}(v)|^{2_\alpha^*(s)}}{|x|^{s+\beta 2_\alpha^*(s)}} dx\right]^{\frac{2}{2_\alpha^*(s)}} \le \frac{C_0}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi_{p,T}(v(x)) - \phi_{p,T}(v(y))|^2}{|x-y|^{n+\alpha}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta}. \tag{4.16}$$

Since $\phi_{p,T}(t) \geq 0$ for all $t \geq 0$, we get from (4.15) that

$$\begin{aligned}
\int_{(\mathbb{R}^n)^2} \frac{|\phi_{p,T}(v(x)) - \phi_{p,T}(v(y))|^2}{|x-y|^{n+\alpha}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} &= \int_{\mathbb{R}^n} \phi_{p,T}(v)(-\Delta_\beta)^{\frac{\alpha}{2}} \phi_{p,T}(v) dx \\
&\leq \int_{\mathbb{R}^n} \phi_{p,T}(v)\phi'_{p,T}(v)(-\Delta_\beta)^{\frac{\alpha}{2}} v dx \\
&= \int_{\mathbb{R}^n} \phi_{p,T}(v)\phi'_{p,T}(v) \frac{v^{2^*_\alpha(s)-1}}{|x|^{s+\beta 2^*_\alpha(s)}} dx \\
&\leq 2p \int_{\mathbb{R}^n} |\phi_{p,T}(v)|^2 \frac{v^{2^*_\alpha(s)-2}}{|x|^{s+\beta 2^*_\alpha(s)}} dx.
\end{aligned}$$

Note that the last inequality holds, since $t\phi'_{p,T}(t) \leq 2p\phi(t)$ for all $t \geq 0$. By (4.16), we have

$$\left[\int_{\mathbb{R}^n} \frac{|\phi_{p,T}(v)|^{2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}} \, dx\right]^{\frac{2}{2^*_\alpha(s)}} \leq pC_0 \int_{\mathbb{R}^n} |\phi_{p,T}(v)|^2 \frac{v^{2^*_\alpha(s)-2}}{|x|^{s+\beta 2^*_\alpha(s)}} \, dx. \quad (4.17)$$

Letting $p_1 = \dfrac{2^*_\alpha(s)}{2}$, then

$$\left[\int_{\mathbb{R}^n} \frac{|\phi_{p_1,T}(v)|^{2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}} \, dx\right]^{\frac{2}{2^*_\alpha(s)}} \leq p_1 C_0 \int_{\mathbb{R}^n} |\phi_{p_1,T}(v)|^2 \frac{v^{2^*_\alpha(s)-2}}{|x|^{s+\beta 2^*_\alpha(s)}} \, dx. \quad (4.18)$$

For $m > 0$, a simple computation and Hölder's inequality yield that

$$\begin{aligned}
& p_1 C_0 \int_{\mathbb{R}^n} |\phi_{p_1,T}(v)|^2 \frac{v^{2^*_\alpha(s)-2}}{|x|^{s+\beta 2^*_\alpha(s)}} dx \\
&= p_1 C_0 \int_{v(x) \leq m} |\phi_{p_1,T}(v)|^2 \frac{v^{2^*_\alpha(s)-2}}{|x|^{s+\beta 2^*_\alpha(s)}} \, dx \\
&\quad + p_1 C_0 \int_{v(x) > m} |\phi_{p_1,T}(v)|^2 \frac{v^{2^*_\alpha(s)-2}}{|x|^{s+\beta 2^*_\alpha(s)}} dx \\
&\leq p_1 C_0 m^{2^*_\alpha(s)-2} \int_{v(x) \leq m} \frac{|\phi_{p_1,T}(v)|^2}{|x|^{s+\beta 2^*_\alpha(s)}} dx \\
&\quad + p_1 C_0 \int_{v(x) > m} \frac{|\phi_{p_1,T}(v)|^2}{|x|^{\frac{2(s+\beta 2^*_\alpha(s))}{2^*_\alpha(s)}}} \frac{v^{2^*_\alpha(s)-2}}{|x|^{s+\beta 2^*_\alpha(s) - \frac{2(s+\beta 2^*_\alpha(s))}{2^*_\alpha(s)}}} dx
\end{aligned}$$

$$
\begin{aligned}
&\leq p_1 C_0 m^{2^*_\alpha(s)-2} \int_{\mathbb{R}^n} \frac{|\phi_{p_1,T}(v)|^2}{|x|^{s+\beta 2^*_\alpha(s)}} dx \\
&\quad + p_1 C_0 \left[\int_{v(x)>m} \frac{|\phi_{p_1,T}(v)|^{2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}} dx\right]^{\frac{2}{2^*_\alpha(s)}} \left[\int_{v(x)>m} \frac{v^{2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}\ dx\right]^{\frac{\alpha-s}{n-s}} \\
&\leq p_1 C_0 m^{2^*_\alpha(s)-2} \int_{\mathbb{R}^n} \frac{|\phi_{p_1,T}(v)|^2}{|x|^{s+\beta 2^*_\alpha(s)}}\ dx \\
&\quad + p_1 C_0 \left[\int_{\mathbb{R}^n} \frac{|\phi_{p_1,T}(v)|^{2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}\ dx\right]^{\frac{2}{2^*_\alpha(s)}} \left[\int_{v(x)>m} \frac{v^{2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}\ dx\right]^{\frac{\alpha-s}{n-s}}.
\end{aligned}
$$

Recall that $v \in H_0^{\frac{\alpha}{2},\beta}(\mathbb{R}^n)$, hence $\int_{\mathbb{R}^n} \frac{v^{2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}\ dx < \infty$. Thus, we can take a large $M_0 \gg 1$ and fix it in such a way that

$$
p_1 C_0 \left[\int_{v(x)>M_0} \frac{v^{2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}\ dx\right]^{\frac{\alpha-s}{n-s}} \leq \frac{1}{2}.
$$

Since $\phi_{p_1,T}(t) \leq t^{p_1}$ for all $t \geq 0$, then by (4.18) and the fact that $p_1 = \dfrac{2^*_\alpha(s)}{2}$, we get

$$
\begin{aligned}
\left[\int_{\mathbb{R}^n} \frac{|\phi_{p_1,T}(v)|^{2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}\ dx\right]^{\frac{2}{2^*_\alpha(s)}} &\leq 2p_1 C_0 M_0^{2^*_\alpha(s)-2} \int_{\mathbb{R}^n} \frac{|\phi_{p_1,T}(v)|^2}{|x|^{s+\beta 2^*_\alpha(s)}}\ dx \\
&\leq 2p_1 C_0 M_0^{2^*_\alpha(s)-2} \int_{\mathbb{R}^n} \frac{|v|^{2p_1}}{|x|^{s+\beta 2^*_\alpha(s)}}\ dx \\
&= 2p_1 C_0 M_0^{2^*_\alpha(s)-2} \int_{\mathbb{R}^n} \frac{v^{2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}\ dx. \quad (4.19)
\end{aligned}
$$

Let $C_1 = 2C_0 M_0^{2^*_\alpha(s)-2}$. By taking $T \to \infty$ in (4.19) and applying Fatou's lemma, we get that

$$
\left[\int_{\mathbb{R}^n} \frac{v^{p_1 2^*_\alpha(s)}}{|x|^{s+\beta 2_\alpha*(s)}}\ dx\right]^{\frac{2}{2_\alpha*(s)}} \leq p_1 C_1 \int_{\mathbb{R}^n} \frac{v^{2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}\ dx < \infty.
$$

Define now recursively the sequence $\{p_k\}_{k=2}^{\infty}$ as follows:

$$
2p_{k+1} + 2^*_\alpha(s) - 2 = p_k 2^*_\alpha(s) \quad \text{for all } k \geq 1. \tag{4.20}
$$

Using (4.17) and (4.20), we have

$$\begin{aligned}\left[\int_{\mathbb{R}^n}\frac{|\phi_{p_{k+1},T}(v)|^{2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}dx\right]^{\frac{2}{2^*_\alpha(s)}} &\leq p_{k+1}C_0\int_{\mathbb{R}^n}|\phi_{p_{k+1},T}(v)|^2\frac{v^{2^*_\alpha(s)-2}}{|x|^{s+\beta 2^*_\alpha(s)}}dx\\ &\leq p_{k+1}C_0\int_{\mathbb{R}^n}v^{2p_{k+1}}\frac{v^{2^*_\alpha(s)-2}}{|x|^{s+\beta 2^*_\alpha(s)}}dx\\ &= C_0p_{k+1}\int_{\mathbb{R}^n}\frac{v^{p_k 2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}dx. \qquad (4.21)\end{aligned}$$

We also have used the fact that $\phi_{p_{k+1},T}(t)\leq t^{p_{k+1}}$ for all $t\geq 0$. By taking $T\to\infty$ in (4.21) and applying Fatou's lemma, we get that

$$\left[\int_{\mathbb{R}^n}\frac{v^{p_{k+1}2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}dx\right]^{\frac{2}{2^*_\alpha(s)}}\leq C_0p_{k+1}\int_{\mathbb{R}^n}\frac{v^{p_k 2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}dx \quad \text{for all } k\geq 1.$$

Hence, by (4.20), we obtain that

$$\begin{aligned}&\left[\int_{\mathbb{R}^n}\frac{v^{p_{k+1}2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}\;dx\right]^{\frac{1}{2^*_\alpha(s)(p_{k+1}-1)}}\\ &\qquad\leq (C_0p_{k+1})^{\frac{1}{2(p_{k+1}-1)}}\left[\int_{\mathbb{R}^n}\frac{v^{p_k 2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}\;dx\right]^{\frac{1}{2(p_{k+1}-1)}}\\ &\qquad= (C_0p_{k+1})^{\frac{1}{2(p_{k+1}-1)}}\left[\int_{\mathbb{R}^n}\frac{v^{p_k 2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}\;dx\right]^{\frac{1}{2^*_\alpha(s)(p_k-1)}}.\end{aligned}$$

For $k\geq 1$, set

$$I_k:=\left[\int_{\mathbb{R}^n}\frac{v^{p_k 2^*_\alpha(s)}}{|x|^{s+\beta 2^*_\alpha(s)}}\;dx\right]^{\frac{1}{2^*_\alpha(s)(p_k-1)}} \quad \text{and } D_k=(C_0p_{k+1})^{\frac{1}{2(p_{k+1}-1)}}.$$

We have $I_{k+1}\leq D_kI_k$ for all $k\geq 1$, and

$$\ln I_{k+1} \;\leq\; \ln D_k+\ln I_k\leq\sum_{j=1}^k\ln D_j+\ln I_1\leq\sum_{j=1}^k\frac{\ln C_0+\ln p_{j+1}}{2(p_{j+1}-1)}+\ln I_1.$$

It follows from (4.20) that $p_{k+1}=p_1^k(p_1-1)+1$ for all $k\geq 0$. This coupled with the fact that $p_1 > 1$ yield

$$\begin{aligned}\ln I_{k+1} &\leq \sum_{j=1}^{k} \frac{\ln C_0}{2p_1^j(p_1-1)} + \sum_{j=1}^{k} \frac{\ln[p_1^j(p_1-1)+1]}{2p_1^j(p_1-1)} + \ln I_1 \\ &\leq \sum_{j=1}^{k} \frac{\ln C_0}{2p_1^j(p_1-1)} + \sum_{j=1}^{k} \frac{\ln p_1^{j+1}}{2p_1^j(p_1-1)} + \ln I_1 < C_2 < \infty.\end{aligned}$$

For any fix $R \geq 1$, we then have

$$\left[\int_{|x|\leq R} \frac{v^{p_k 2_\alpha^*(s)}}{|x|^{s+\beta 2_\alpha^*(s)}}\, dx\right]^{\frac{1}{2_\alpha^*(s)(p_k-1)}} \leq I_k \leq e^{C_2} =: C_3 \quad \text{for all } k \geq 1.$$

Since $s + \beta 2_\alpha^*(s) > 0$, we then get

$$\left[\int_{|x|\leq R} v^{p_k 2_\alpha^*(s)}\, dx\right]^{\frac{1}{2_\alpha^*(s)p_k}} \leq C_3 R^{\frac{s+\beta 2_\alpha^*(s)}{2_\alpha^*(s)p_k}} \quad \text{for all } k \geq 1.$$

Since $\lim\limits_{k\to\infty} p_k = \infty$, we have

$$\|v\|_{L^\infty(B_R(0))} = \lim_{k\to\infty} \left[\int_{|x|\leq R} v^{p_k 2_\alpha^*(s)}\, dx\right]^{\frac{1}{2_\alpha^*(s)p_k}} \leq C_3,$$

and finally, that $\|v\|_{L^\infty(\mathbb{R}^n)} \leq C_3$. □

Proof of Theorem 4.1. Let $v(x) = |x|^{\beta_-(\gamma)}u(x)$ in $\mathbb{R}^n\backslash\{0\}$, by the discussion before at the beginning of section 3, we know that $v \in H_0^{\frac{\alpha}{2},\beta}(\mathbb{R}^n)$ is a positive weak solution to (4.14). We deduce from Proposition 4.9 that for all $R > 0$, there exist some constant $C > 1$ such that $C^{-1} \leq v(x) \leq C$ in $B_R(0)$. Since $v(x) = |x|^{\beta_-(\gamma)}u(x)$ in $\mathbb{R}^n\backslash\{0\}$, then

$$\frac{C^{-1}}{|x|^{\beta_-(\gamma)}} \leq u(x) \leq \frac{C}{|x|^{\beta_-(\gamma)}} \text{ in } B_R(0)\backslash\{0\}. \tag{4.22}$$

In order to prove the asymptotic behavior at zero, it is enough to show that $\lim\limits_{x\to 0} |x|^{\beta_-(\gamma)}u(x)$ exists. To that end, we proceed as follows:

Claim 1: $u \in C^1(\mathbb{R}^n \setminus \{0\})$.

This is consequence of regularity theory and we only sketch the proof. First we define $f_0(x) := \gamma |x|^{-\alpha} u + u^{2^*_\alpha(s)-1}|x|^{-s}$, so that for any $\omega \subset\subset \mathbb{R}^n \setminus \{0\}$, we have that $(-\Delta)^{\alpha/2} u = f_0$ in ω in the sense that $u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ and

$$\frac{C_{n,\alpha}}{2} \int_{(\mathbb{R}^n)^2} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+\alpha}} dxdy = \int_\omega f_0 \varphi \, dx \text{ for all } \varphi \in C_c^\infty(\omega).$$

It follows from (4.22) that $f_0 \in L^\infty(\omega)$. Since $u \geq 0$ and $f_0 \in L^\infty(\omega)$, it follows from Remark 2.5 (see also Theorem 2.1) in Jin-Li-Xiong [46] that there exists $\tau > 0$ such that $u \in C^{0,\tau}_{loc}(\mathbb{R}^n \setminus \{0\})$. Then, using recursively Theorem 2.1 in Jin-Li-Xiong [46], we get that $u \in C^1(\mathbb{R}^n \setminus \{0\})$. This proves the claim.

Claim 2: There exists $C > 0$ such that $|x|^{\beta_-(\gamma)+1}|\nabla u(x)| \leq C$ for all $x \in B_1(0) \setminus \{0\}$.

If not, then there exists a sequence $(x_i)_{i\in\mathbb{N}} \in B_1(0) \setminus \{0\}$ such that

$$\lim_{i\to+\infty} |x_i|^{\beta_-(\gamma)+1}|\nabla u(x_i)| = +\infty.$$

For simplicity, we write $\beta_- := \beta_-(\gamma)$. It follows from from Claim 1 that

$$\lim_{i\to+\infty} x_i = 0.$$

We define $r_i := |x_i|$ and we set

$$u_i(x) := r_i^{\beta_-} u(r_i x) \text{ for all } x \in \mathbb{R}^n \setminus \{0\}.$$

It is easy to see that $u_i \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$, $u_i \geq 0$ for all $i \in \mathbb{N}$ and $(-\Delta)^{\alpha/2} u_i = f_i$ in $\omega \subset\subset \mathbb{R}^n \setminus \{0\}$ where $f_i(x) := \gamma |x|^{-\alpha} u_i + r_i^{(2^*_\alpha(s)-2)(\frac{n-\alpha}{2}-\beta_-)} u_i^{2^*_\alpha(s)-1}|x|^{-s}$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Using the apriori bound of Remark 2.5 (see also Theorem 2.1) in Jin-Li-Xiong [46], we get that there exists $\tau > 0$ such that for any $R > 1$, there exists $C(R) > 0$ such that $\|u_i\|_{C^{0,\tau}(B_R(0)-B_{R^{-1}}(0))} \leq C(R)$ for all $i \in \mathbb{N}$. Using recursively Theorem 2.1 of [46] as in Step 1, we get that for any $\omega \subset\subset \mathbb{R}^n \setminus \{0\}$, there exists $C(\omega) > 0$ such that $\|u_i\|_{C^1(\omega)} \leq C(\omega)$. Taking ω large enough and estimating $|\nabla u_i(\frac{x_i}{|x_i|})|$, we get a contradiction, which proves Claim 2.

Set now $h(x) := \frac{u^{2^*_\alpha(s)-2}}{|x|^s}$, so that $(-\Delta)^{\alpha/2} u - \frac{\gamma}{|x|^\alpha} u = h(x) u$ in $\mathbb{R}^n$. It follows from Claims 1 and 2, that $h \in C^1(\mathbb{R}^n \setminus \{0\})$, and for some $C > 0$,

$$|h(x)| + |x| \cdot |\nabla h(x)| \leq C|x|^{\theta-\alpha} \text{ for all } x \in B_1(0) \setminus \{0\},$$

where $\theta := (2^*_\alpha(s) - 2)(\frac{n-\alpha}{2} - \beta_-) > 0$. It then follows from Lemma 4.14 below that there exists $\lambda_0 > 0$ such that

$$\lim_{x\to 0} |x|^{\beta_-} u(x) = \lambda_0 > 0.$$

In order to deal with the behavior at infinity, let w be the fractional Kelvin transform of u, that is,

$$w(x) = |x|^{\alpha-n} u(x^*) := |x|^{\alpha-n} u\left(\frac{x}{|x|^2}\right) \text{ in } \mathbb{R}^n \setminus \{0\}.$$

By Lemma 2.2 and Corollary 2.3 in [32], we have that $w \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$. A simple calculation gives us that w is also a positive weak solution to (4.3). Indeed, we have

$$(-\Delta)^{\frac{\alpha}{2}} w(x) = \frac{1}{|x|^{n+\alpha}} \left((-\Delta)^{\frac{\alpha}{2}} u\right)\left(\frac{x}{|x|^2}\right) = \gamma \frac{w(x)}{|x|^\alpha} + \frac{w^{2^*_\alpha(s)-1}(x)}{|x|^s}.$$

Arguing as in the first part of the proof, we get that there exists $\lambda_\infty > 0$ such that

$$\lim_{x\to 0} |x|^{\beta_-(\gamma)} w(x) = \lambda_\infty > 0.$$

Coming back to u, this implies that

$$\lim_{|x|\to\infty} |x|^{\beta_+(\gamma)} u(x) = \lambda_\infty > 0.$$

This ends the proof of Theorem 4.1. □

4.4 Analytic conditions for the existence of extremals

Let $a \in C^{0,\tau}(\overline{\Omega})$ for some $\tau \in (0,1)$, and define the functional $J_a^\Omega : H_0^{\frac{\alpha}{2}}(\Omega) \longrightarrow \mathbb{R}$ by

$$J_a^\Omega(u) := \frac{\frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}} dxdy - \gamma \int_\Omega \frac{u^2}{|x|^\alpha} dx - \int_\Omega a u^2 dx}{\left(\int_\Omega \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}}},$$

in such a way that

$$\mu_{\gamma,s,\alpha,a}(\Omega) := \inf\left\{J_a^\Omega(u) : u \in H_0^{\frac{\alpha}{2}}(\Omega) \setminus \{0\}\right\}.$$

We now prove the following proposition, which gives analytic conditions for the existence of extremals for $\mu_{\gamma,s,\alpha,a}(\Omega)$.

Proposition 4.10. *Let Ω be a bounded domain in $\mathbb{R}^n$ $(n > \alpha)$ such that $0 \in \Omega$, and assume that $0 \le \gamma < \gamma_H(\alpha)$ and $0 \le s \le \alpha$.*

1. *If $\mu_{\gamma,s,\alpha,a}(\Omega) < \mu_{\gamma,s,\alpha}(\mathbb{R}^n)$, then there are extremals for $\mu_{\gamma,s,\alpha,a}(\Omega)$ in $H_0^{\frac{\alpha}{2}}(\Omega)$.*

2. *If $a(x)$ is a constant λ, with $0 < \lambda < \lambda_1(L_{\gamma,\alpha})$ and if $s < \alpha$, then $\mu_{\gamma,s,\alpha,a}(\Omega) > 0$.*

Proof. Let $(u_k)_{k\in\mathbb{N}} \subset H_0^{\frac{\alpha}{2}}(\Omega) \setminus \{0\}$ be a minimizing sequence for $\mu_{\gamma,s,\alpha,a}(\Omega)$, that is,

$$J_a^{\Omega}(u_k) = \mu_{\gamma,s,\alpha,a}(\Omega) + o(1) \quad \text{as } k \to \infty.$$

Up to multiplying by a constant, we may assume that

$$\int_\Omega \frac{|u_k|^{2_\alpha^*(s)}}{|x|^s}\, dx = 1 \tag{4.23}$$

$$\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^2}{|x-y|^{n+\alpha}}dxdy - \int_\Omega \left(\frac{\gamma}{|x|^\alpha} + a\right) u_k^2 dx = \mu_{\gamma,s,\alpha,\lambda}(\Omega) + o(1) \tag{4.24}$$

as $k \to +\infty$. By (4.23), we have $\displaystyle\int_\Omega u_k^2 dx \le C < \infty$ for all k. Since $0 \le \gamma < \gamma_H(\alpha)$, the fractional Hardy inequality combined with (4.24) yields that $\|u_k\|_{H_0^{\frac{\alpha}{2}}(\Omega)} \le C$ for all k. It then follows that there exists $u \in H_0^{\frac{\alpha}{2}}(\Omega)$ such that, up to a subsequence, such that (u_k) goes to u weakly in $H_0^{\frac{\alpha}{2}}(\Omega)$ and strongly in $L^2(\Omega)$ as $k \to \infty$.

We first show that $\int_\Omega \frac{|u|^{2_\alpha^*(s)}}{|x|^s}dx = 1$. Define $\theta_k = u_k - u$ for all $k \in \mathbb{N}$. It follows from the boundedness in $H_0^{\frac{\alpha}{2}}(\Omega)$ that, up to a subsequence, we have that $\theta_k \rightharpoonup 0$ weakly in $H_0^{\frac{\alpha}{2}}(\Omega)$, strongly in $L^2(\Omega)$ as $k \to \infty$, and $\theta_k(x) \to 0$ for a.e. $x \in \Omega$ as $k \to +\infty$. Hence, by the Brezis-Lieb lemma (see [10] and [70]), we get that

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^2}{|x-y|^{n+\alpha}}dxdy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\theta_k(x)-\theta_k(y)|^2}{|x-y|^{n+\alpha}}dxdy \\ &\quad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy + o(1),\end{aligned}$$

$$1 = \int_\Omega \frac{|u_k|^{2_\alpha^*(s)}}{|x|^s}dx = \int_\Omega \frac{|\theta_k|^{2_\alpha^*(s)}}{|x|^s}dx + \int_\Omega \frac{|u|^{2_\alpha^*(s)}}{|x|^s}dx + o(1),$$

$$\int_\Omega \frac{u_k^2}{|x|^\alpha}dx = \int_\Omega \frac{\theta_k^2}{|x|^\alpha}dx + \int_\Omega \frac{u^2}{|x|^\alpha}dx + o(1),$$

and

$$\int_\Omega u_k^2 dx = \int_\Omega u^2 dx + o(1), \qquad \text{as } k \to \infty.$$

Thus, we have

$$\begin{aligned}\mu_{\gamma,s,\alpha,a}(\Omega) = &\left[\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy - \int_\Omega\left(\frac{\gamma}{|x|^\alpha}+a\right)u^2dx\right]\\ &+\left[\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\theta_k(x)-\theta_k(y)|^2}{|x-y|^{n+\alpha}}dxdy - \gamma\int_\Omega\frac{\theta_k^2}{|x|^\alpha}dx\right]+o(1)\end{aligned} \tag{4.25}$$

as $k \to +\infty$. The definition of $\mu_{\gamma,s,\alpha,a}(\Omega)$ and $H_0^{\frac{\alpha}{2}}(\Omega) \subset H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ yield

$$\begin{aligned}\mu_{\gamma,s,\alpha,a}(\Omega)\left(\int_\Omega\frac{|u|^{2_\alpha^*(s)}}{|x|^s}dx\right)^{\frac{2}{2_\alpha^*(s)}} \le &\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy\\ &-\int_\Omega\left(\frac{\gamma}{|x|^\alpha}+a\right)u^2dx,\end{aligned}$$

and

$$\begin{aligned}\mu_{\gamma,s,\alpha}(\mathbb{R}^n)\left(\int_\Omega\frac{|\theta_k|^{2_\alpha^*(s)}}{|x|^s}dx\right)^{\frac{2}{2_\alpha^*(s)}} \le &\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\theta_k(x)-\theta_k(y)|^2}{|x-y|^{n+\alpha}}dxdy\\ &-\gamma\int_\Omega\frac{\theta_k^2}{|x|^\alpha}dx.\end{aligned} \tag{4.26}$$

Summing these two inequalities and using (4.23) and (4.25), and passing to the limit $k \to \infty$, we obtain

$$\mu_{\gamma,s,\alpha}(\mathbb{R}^n)\left(1-\int_\Omega\frac{|u|^{2_\alpha^*(s)}}{|x|^s}dx\right)^{\frac{2}{2_\alpha^*(s)}} \le \mu_{\gamma,s,\alpha,a}(\Omega)\left(1-\left(\int_\Omega\frac{|u|^{2_\alpha^*(s)}}{|x|^s}dx\right)^{\frac{2}{2_\alpha^*(s)}}\right).$$

Finally, the fact that $\mu_{\gamma,s,\alpha,a}(\Omega) < \mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ implies that $\int_\Omega \frac{|u|^{2_\alpha^*(s)}}{|x|^s}dx = 1$. It remains to show that u is an extremal for $\mu_{\gamma,s,\alpha,a}(\Omega)$. For that, note that since $\int_\Omega \frac{|u|^{2_\alpha^*(s)}}{|x|^s}dx = 1$, the definition of $\mu_{\gamma,s,\alpha,a}(\Omega)$ yields that

$$\mu_{\gamma,s,\alpha,a}(\Omega) \le \frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy - \int_\Omega\left(\frac{\gamma}{|x|^\alpha}+a\right)u^2dx.$$

The second term in the right-hand-side of (4.25) is nonnegative due to (4.26). Therefore, we get that

$$\mu_{\gamma,s,\alpha,a}(\Omega) = \frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy - \int_\Omega\left(\frac{\gamma}{|x|^\alpha}+a\right)u^2dx.$$

This proves the first claim of the Proposition.

Now assume that $\lambda \in (0, \lambda_1(L_{\gamma,\alpha}))$ and $0 \leq \gamma < \gamma_H(\alpha)$, then for all $u \in H_0^{\frac{\alpha}{2}}(\Omega)\setminus\{0\}$,

$$\begin{aligned}
J_\lambda^\Omega(u) &= \frac{\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy - \int_\Omega\left(\frac{\gamma}{|x|^\alpha}+\lambda\right)u^2dx}{\left(\int_\Omega\frac{|u|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}}}\\
&\geq \left(1-\frac{\lambda}{\lambda_1(L_{\gamma,\alpha})}\right)\frac{\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}dxdy - \gamma\int_\Omega\frac{u^2}{|x|^\alpha}dx}{\left(\int_\Omega\frac{|u|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}}}\\
&\geq \left(1-\frac{\lambda}{\lambda_1(L_{\gamma,\alpha})}\right)\left(1-\frac{\gamma}{\gamma_H(\alpha)}\right)\mu_{0,s,\alpha,0}(\Omega)\\
&= \left(1-\frac{\lambda}{\lambda_1(L_{\gamma,\alpha})}\right)\left(1-\frac{\gamma}{\gamma_H(\alpha)}\right)\mu_{0,s,\alpha,0}(\mathbb{R}^n) > 0.
\end{aligned}$$

Therefore, $\mu_{\gamma,s,\alpha,\lambda}(\Omega) > 0$. □

4.5 The fractional Hardy singular interior mass of a domain in the critical case

In this section, we define the fractional Hardy singular interior mass of a domain by proving Theorem 4.3. We shall need the following five lemmae.

Lemma 4.11. *Assume $0 < \beta \leq n-\alpha$, and let $\eta \in C_c^\infty(\Omega)$ be a cut-off function such that $0 \leq \eta(x) \leq 1$ in Ω, and $\eta(x) \equiv 1$ in $B_\delta(0)$, for some $\delta > 0$ small. Then $x \mapsto \eta(x)|x|^{-\beta} \in H_0^{\frac{\alpha}{2}}(\Omega)$ and there exists $f_\beta \in L^\infty_{loc}(\mathbb{R}^n)$ with $f_\beta(x) \geq 0$ on $B_\delta(0)$ and $f_\beta \in C^1(B_\delta(0))$ such that*

$$(-\Delta)^{\frac{\alpha}{2}}(\eta|x|^{-\beta}) = \Phi_{n,\alpha}(\beta)|x|^{-\alpha}\eta|x|^{-\beta} + f_\beta \quad \text{in } \mathcal{D}'(\Omega\setminus\{0\}), \tag{4.27}$$

in the sense that, if $v_\beta(x) := \eta(x)|x|^{-\beta}$, then for all $\varphi \in C_c^\infty(\Omega \setminus \{0\})$,

$$\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(v_\beta(x)-v_\beta(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+\alpha}}dxdy = \Phi_{n,\alpha}(\beta)\int_\Omega \frac{v_\beta \varphi}{|x|^\alpha}dx + \int_\Omega f_\beta \varphi(x)\, dx.$$

Moreover, if $\beta < \frac{n-\alpha}{2}$, then $v_\beta \in H_0^{\frac{\alpha}{2}}(\Omega)$ and equality (4.27) holds in the classical sense of $H_0^{\frac{\alpha}{2}}(\Omega)$.

Proof. When $\beta < \frac{n-\alpha}{2}$, it follows from Proposition 4.7 that $x \mapsto \eta(x)|x|^{-\beta} \in H_0^{\frac{\alpha}{2}}(\Omega)$. In the general case, for $\varphi \in C_c^\infty(\Omega \setminus \{0\})$, straightforward computations yield

$$\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(v_\beta(x)-v_\beta(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+\alpha}}dxdy = \langle(-\Delta)^{\frac{\alpha}{2}}|x|^{-\beta}, \eta\varphi\rangle + \int_\Omega f_\beta \varphi\, dx,$$

where

$$f_\beta(x) := C_{n,\alpha}\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}\frac{\eta(x)-\eta(y)}{|x-y|^{n+\alpha}}\cdot\frac{1}{|y|^\beta}\,dy \quad \text{for all } x \in \mathbb{R}^n.$$

Note that $f_\beta \in L^\infty_{loc}(\mathbb{R}^n)$, and for $x \in B_\delta(0)$, we have that

$$f_\beta(x) := C_{n,\alpha}\int_{\mathbb{R}^n}\frac{1-\eta(y)}{|x-y|^{n+\alpha}}\cdot\frac{1}{|y|^\beta}\,dy \geq 0,$$

yielding that $f_\beta \in C^1(B_\delta(0))$. Since $\varphi \equiv 0$ around 0, the lemma is a consequence of (4.11). $\square$

Lemma 4.12 (A comparison principle via coercivity)**.** *Suppose Ω be a bounded smooth domain in $\mathbb{R}^n$, $0<\alpha<2$, $\gamma < \gamma_H(\alpha)$ and $a(x) \in C^{0,\tau}(\overline{\Omega})$ for some $\tau \in (0,1)$. Assume that the operator $(-\Delta)^{\frac{\alpha}{2}} - (\frac{\gamma}{|x|^\alpha} + a(x))$ is coercive. Let u be a function in $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ that satisfies*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}u - \left(\dfrac{\gamma}{|x|^\alpha} + a(x)\right)u & \geq 0 \quad \text{in } \Omega \\ \qquad\qquad\qquad\qquad\qquad\quad u & \geq 0 \quad \text{on } \partial\Omega, \end{cases}$$

in the sense that $u \geq 0$ in $\mathbb{R}^n \setminus \Omega$ and

$$\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}dxdy - \int_\Omega\left(\frac{\gamma}{|x|^\alpha}+a(x)\right)uvdx \geq 0,$$

for all $v \in H_0^{\frac{\alpha}{2}}(\Omega)$ with $v \geq 0$ a.e. in Ω. Then, $u \geq 0$ in Ω.

Proof. Let $u_-(x) = -\min(u(x), 0)$ be the negative part of u. It follows from Proposition 4.7 that $u_- \in H_0^{\frac{\alpha}{2}}(\Omega)$. We can therefore use it as a test function to get

$$\begin{aligned}\langle Lu, u_- \rangle := \frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(u_-(x) - u_-(y))}{|x-y|^{n+\alpha}} dxdy \\ - \gamma \int_\Omega \frac{uu_-}{|x|^\alpha} dx - \int_\Omega a(x) uu_- dx \geq 0.\end{aligned}$$

Let

$$\Omega^+ := \{x : u(x) \geq 0\} \quad \text{and} \quad \Omega^- := \{x : u(x) < 0\}.$$

Straightforward computations yield

$$\begin{aligned}0 \leq -\langle Lu_-, u_- \rangle - \frac{C_{n,\alpha}}{2} \int_{\Omega^-} \int_{\Omega^+} \frac{(u(x) - u(y))u_-(y)}{|x-y|^{n+\alpha}} dxdy \\ + \frac{C_{n,\alpha}}{2} \int_{\Omega^+} \int_{\Omega^-} \frac{(u(x) - u(y))u_-(x)}{|x-y|^{n+\alpha}} dxdy,\end{aligned}$$

which yields via coercivity

$$c\|u_-\|^2_{H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)} \leq \langle Lu_-, u_- \rangle \leq 0.$$

Thus, $u_- \equiv 0$, and therefore, $u \geq 0$ on Ω. □

Lemma 4.13. *Assume that $u \in H_0^{\frac{\alpha}{2}}(\Omega)$ is a weak solution of*

$$(-\Delta)^{\frac{\alpha}{2}} u - \left(\frac{\gamma + O(|x|^\tau)}{|x|^\alpha} \right) u = 0 \quad \text{in } H_0^{\frac{\alpha}{2}}(\Omega),$$

for some $\tau > 0$. If $u \not\equiv 0$ and $u \geq 0$, then there exists a constant $C > 0$ such that

$$C^{-1} \leq |x|^{\beta_-(\gamma)} u(x) \leq C \quad \textit{for } x \to 0, \ x \in \Omega.$$

Proof. We use the weak Harnack inequality to prove the lower bound. Indeed, using Theorem 3.4 and Lemma 3.10 in [2], we get that there exists $C_1 > 0$ such that for $\delta_1 > 0$ small enough,

$$u(x) \geq C_1 |x|^{-\beta_-(\gamma)} \text{ in } B_{\delta_1}.$$

The other inequality goes as in the iterative scheme used to prove Proposition 4.9. □

Lemma 4.14 (See Fall-Felli [30]). *Consider an open subset $\omega \subset \Omega$ with $0 \in \omega$, and a function $h \in C^1(\omega)$ such that for some $\tau > 0$,*

$$|h(x)| + |x| \cdot |\nabla h(x)| \leq C|x|^{\tau-\alpha} \text{ for all } x \in \omega \setminus \{0\}.$$

Let $u \in H_0^{\frac{\alpha}{2}}(\Omega)$ be a weak solution of

$$(-\Delta)^{\frac{\alpha}{2}} u - \frac{\gamma}{|x|^\alpha} u = h(x)u \quad \text{ in } \omega \subset \Omega,$$

in the sense that for all $\varphi \in C_c^\infty(\omega)$,

$$\frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+\alpha}} dxdy - \gamma \int_\Omega \frac{u\varphi}{|x|^\alpha} dx = \int_\Omega h(x)u\varphi \, dx.$$

Assume further that there exists $C > 0$ such that

$$C^{-1} \leq |x|^{\beta_-(\gamma)} u(x) \leq C \quad \text{for } x \to 0, \ x \in \Omega.$$

Then, there exists $l > 0$ such that

$$\lim_{x \to 0} |x|^{\beta_-(\gamma)} u(x) = l.$$

Proof. This result is an extension of Theorem 1.1 proved by Fall-Felli [30], who showed that under these conditions, one has

$$\lim_{\tau \to 0} |\tau x|^{\frac{n-\alpha}{2} - \sqrt{\left(\frac{n-\alpha}{2}\right)^2 + \mu}} u(\tau x) = \psi\left(0, \frac{x}{|x|}\right) \text{ in } C^1_{loc}(B_1(0)) \setminus \{0\} \quad (4.28)$$

where $\mu \in \mathbb{R}$ and $\psi : \mathbb{S}^{n+1}_+ := \{\theta \in \mathbb{S}^{n+1} \ \theta_1 > 0\} \to \mathbb{R}$ are respectively an eigenvalue and an eigenfunctions for the problem

$$\begin{cases} -\mathrm{div}(\theta_1^{1-\alpha} \nabla \psi) = \mu \theta_1^{1-\alpha} \psi & \text{in } \mathbb{S}^{n+1}_+ \\ -\lim_{\theta_1 \to 0} {\theta_1}^{1-\alpha} \partial_\nu \psi(\theta_1, \theta') = \gamma k_\alpha \psi(0, \theta') & \text{for } \theta' \in \partial \mathbb{S}^{n+1}_+, \end{cases} \quad (4.29)$$

where k_α is a positive constant defined in Section 2.3. We refer to [30] for the explicit definition of this eigenvalue problem, in particular the relevant spaces used via the Caffarelli-Silvestre classical representation [12]. It then follows from the pointwise control (4.22) that

$$\beta_-(\gamma) := \frac{n - \alpha}{2} - \sqrt{\left(\frac{n - \alpha}{2}\right)^2 + \mu},$$

and by Proposition 2.3 in Fall-Felli [30], that μ is the first eigenvalue of the eigenvalue problem (4.29). Then, using classical arguments, we get that the corresponding eigenspace is one-dimensional and is spanned by any positive eigenfunction of (4.29) (no matter the value of μ, it must necessarily be the first eigenvalue).

We are left with proving that $\psi(0, \frac{x}{|x|})$ is independant of x. In view of the remarks above, this amonts to prove the existence of a positive eigenfunction that is constant on the boundary.

We now exhibit such an eigenfunction by following the argument in Proposition 2.3 in [30]. First, use ([29], Lemma 3.1) to obtain $\Gamma \in C^0([0,+\infty) \times \mathbb{R}^n) \cap C^2((0,+\infty) \times \mathbb{R}^n)$ such that

$$\begin{cases} -\mathrm{div}(t^{1-\alpha}\nabla\Gamma) = 0 & \text{in } (0,+\infty) \times \mathbb{R}^n \\ -\lim_{t\to 0} t^{1-\alpha}\partial_\nu \Gamma(t,x) = k_\alpha \frac{\gamma}{|x|^\alpha} & \text{for } x \in \mathbb{R}^n = \partial((0,+\infty) \times \mathbb{R}^n) \\ \Gamma(0,x) = |x|^{-\beta_-(\gamma)} & \text{for } x \in \mathbb{R}^n = \partial((0,+\infty) \times \mathbb{R}^n). \end{cases} \tag{4.30}$$

Moreover, Γ is in the relevant function space, $\Gamma > 0$ and satisfies

$$\Gamma(z) = |z|^{-\beta_-(\gamma)}\Gamma\left(\frac{z}{|z|}\right) \text{ for all } z \in (0,+\infty) \times \mathbb{R}^n$$

where $|z| = \sqrt{t^2 + |x|^2}$ if $z = (t,x)$. In particular, we have that $\Gamma(z) = |z|^{-\beta_-(\gamma)}\psi_0(\theta)$ for $\theta := \frac{z}{|z|}$ and some $\psi_0 \in C^0(\overline{\mathbb{S}^{n+1}_+}) \cap C^2(\mathbb{S}^{n+1}_+)$. Following [30], we get that ψ_0 is an eigenfunction for the problem (4.29). Moreover, $\psi_0 > 0$. Therefore, ψ_0 corresponds to the first eigenvalue and spans the corresponding eigenspace. Finally, we remark that for $\theta \in \partial\mathbb{S}^{n+1}_+$, we have that

$$\psi_0(0,\theta) = \Gamma(0,\theta) = |\theta|^{-\beta_-(\gamma)} = 1.$$

Since the eigenspace is one-dimensional, there exists $l \in \mathbb{R}$ such that $\psi = l \cdot \psi_0$. Therefore $\psi(0, \frac{x}{|x|}) = l$ for all $x \in B_1(0) \setminus \{0\} \subset \mathbb{R}^n$. It then follows from (4.28) that

$$\lim_{x\to 0} |x|^{\beta_-(\gamma)}u(x) = l > 0,$$

which complete the proof of Lemma 4.14. $\square$

Proof of Theorem 4.3. We first prove the existence of a solution. For $\delta > 0$ small enough, let $\eta \in C_c^\infty(\Omega)$ be a cut-off function as in Lemma 4.11 such that $\eta(x) \equiv 1$ in $B_\delta(0)$. Set $\beta := \beta_+(\gamma) \le n - \alpha$ in (4.27) and define

$$\begin{aligned} f(x) : &= -\left((-\Delta)^{\frac{\alpha}{2}} - \left(\frac{\gamma}{|x|^\alpha} + a(x) \right) \right) (\eta |x|^{-\beta_+(\gamma)}) \\ &= -f_{\beta_+(\gamma)} + \frac{a\eta}{|x|^{\beta_+(\gamma)}} \qquad \text{in } \Omega \setminus \{0\}, \end{aligned}$$

in the distribution sense. In particular, $f \in C^1(B_\delta(0) \setminus \{0\})$ and there exists a positive constant $C > 0$ such that

$$|f(x)| + |x| \cdot |\nabla f(x)| \le C|x|^{-\beta_+(\gamma)} \text{ for } x \neq 0 \text{ close to } 0. \tag{4.31}$$

In the sequel, we write $\beta_+ := \beta_+(\gamma)$ and $\beta_- := \beta_-(\gamma)$. Note that the assumption $\gamma > \gamma_{crit}(\alpha)$ implies that $\beta_+ < \frac{n}{2} < \frac{n+\alpha}{2}$. Thus, using (4.31) and the fact that $\beta_+ < \frac{n+\alpha}{2}$, we get that $f \in L^{\frac{2n}{n+\alpha}}(\Omega)$. Since $L^{\frac{2n}{n+\alpha}}(\Omega) = \left(L^{\frac{2n}{n-\alpha}}(\Omega)\right)' \subset \left(H_0^{\frac{\alpha}{2}}(\Omega)\right)'$, there exists $g \in H_0^{\frac{\alpha}{2}}(\Omega)$ such that

$$\left((-\Delta)^{\frac{\alpha}{2}} - \left(\frac{\gamma}{|x|^\alpha} + a(x) \right) \right) g = f \text{ weakly in } H_0^{\frac{\alpha}{2}}(\Omega).$$

Set

$$H(x) := \frac{\eta(x)}{|x|^{\beta_+}} + g(x) \text{ for all } x \in \overline{\Omega} \setminus \{0\}. \tag{4.32}$$

Thanks to (4.27), $H : \Omega \to \mathbb{R}$ is a solution to

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} H - \left(\frac{\gamma}{|x|^\alpha} + a(x) \right) H = 0 & \text{in } \Omega \setminus \{0\} \\ \qquad\qquad\qquad\qquad\quad H = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{4.33}$$

in the sense of Definition 4.6. The idea is to now write f as the difference of two positive C^1 functions. The decomposition $f = |f| - 2f_-$ does not work here since the resulting functions are not necessarily C^1. To smooth out the functions $x \mapsto |x|$ and $x \mapsto x_-$, we consider

$$\varphi_1(x) := \sqrt{1 + x^2} \text{ and } \varphi_2(x) := \varphi_1(x) - x \text{ for all } x \in \mathbb{R}.$$

It is clear that $\varphi_1, \varphi_2 \in C^1(\mathbb{R})$ and there exists $C > 0$ such that

$$0 \le \varphi_i(x) \le C(1 + |x|)\,,\ |\varphi_i'(x)| \le C \text{ and } x = \varphi_1(x) - \varphi_2(x), \tag{4.34}$$

for all $x \in \mathbb{R}$ and $i = 1, 2$. Define $f_i := \varphi_i \circ f$ for $i = 1, 2$. In particular, $f = f_1 - f_2$. Let $g_1, g_2 \in H_0^{\frac{\alpha}{2}}(\Omega)$ be solutions to

$$(-\Delta)^{\frac{\alpha}{2}} g_i - \left(\frac{\gamma}{|x|^\alpha} + a(x) \right) g_i = f_i \text{ weakly in } H_0^{\frac{\alpha}{2}}(\Omega) \tag{4.35}$$

for $i = 1, 2$. Since $f_1, f_2 \geq 0$, Lemma 4.12 yields $g_1, g_2 \geq 0$. Also

$$\left((-\Delta)^{\frac{\alpha}{2}} - \left(\frac{\gamma}{|x|^\alpha} + a(x)\right)\right)(g - (g_1 - g_2)) = f - (f_1 - f_2) = 0.$$

It follows from coercivity that $g = g_1 - g_2$. Assuming $g_1 \not\equiv 0$, it follows from Lemma 3.10 in [2] that there exists $K' > 0$ such that $g_1(x) \geq K'|x|^{-\beta_-}$ in $B_\delta(0) \setminus \{0\}$.

Since $g_1 \in H_0^{\frac{\alpha}{2}}(\Omega)$, it follows from (4.35) and Theorem 2.1 of Jin-Li-Xiong [46] that $g_1 \in C^{0,\tau}_{loc}(\Omega \setminus \{0\})$ for some $\tau > 0$. Arguing as in the proof of Theorem 4.1, we get that $g_1 \in C^1(\Omega \setminus \{0\})$. Setting

$$h(x) := \frac{f_1(x)}{g_1(x)} \text{ for } x \text{ close to } 0,$$

we have that $h \in C^1(B_\delta(0))$. Now use (4.31) and (4.34) to get that

$$\begin{aligned} f_1(x) &\leq C(1 + |f(x)|) \leq C|x|^{-\beta_+} = C|x|^{-\beta_-}|x|^{\alpha-(\beta_+-\beta_-)}|x|^{-\alpha} \\ &\leq K_1|x|^{-\alpha+(\alpha-(\beta_+-\beta_-))}g_1(x). \end{aligned}$$

Using the fact that $\gamma > \gamma_{crit}(\alpha)$ if and only if $\alpha - (\beta_+ - \beta_-) > 0$, we get that $|h(x)| \leq C|x|^{\tau-\alpha}$ for $x \to 0$ where $\tau := \alpha - (\beta_+ - \beta_-) > 0$. Therefore, we have that

$$(-\Delta)^{\frac{\alpha}{2}} g_1 - \frac{\gamma + O(|x|^{(\alpha-(\beta_+-\beta_-))})}{|x|^\alpha} g_1 = 0 \text{ weakly in } H_0^{\frac{\alpha}{2}}(\Omega),$$

with $g_1 \geq 0$ and $g_1 \not\equiv 0$. It then follows from Lemma 4.13 that there exists $c > 0$ such that $c^{-1} \leq |x|^{\beta_-} g_1(x) \leq c$ for $x \in \Omega$, $x \neq 0$ close to 0. Arguing as in Claim 2 in the proof of Theorem 4.1, we get that there exists $C > 0$ such that

$$c^{-1} \leq |x|^{\beta_-} g_1(x) \leq c \text{ and } |x|^{\beta_-+1}|\nabla g_1(x)| \leq C \text{ for all } x \in B_\delta(0). \tag{4.36}$$

We now deal with the differential of h. With the controls (4.31), (4.34) and (4.36), we get that

$$|x| \cdot |\nabla h(x)| \leq C|x|^{\tau-\alpha} \text{ for } x \in B_{\delta/2}(0) \setminus \{0\}.$$

Now, writing $(-\Delta)^{\frac{\alpha}{2}} g_1 - \frac{\gamma}{|x|^\alpha} g_1 = h(x) g_1$ in Ω and using Lemma 4.14, we get that $|x|^{-\beta_-} g_1(x)$ has a finite limit as $x \to 0$. Note that this is also clearly the case if $g_1 \equiv 0$. The same holds for g_2. Therefore, there exists a constant $c \in \mathbb{R}$ such that $|x|^{-\beta_-} g(x) \to c$ as $x \to 0$. In other words,

$$H(x) = \frac{1}{|x|^{\beta_+}} + \frac{c}{|x|^{\beta_-}} + o\left(\frac{1}{|x|^{\beta_-}}\right) \qquad \text{as } x \to 0,$$

and there exists $C > 0$ such that $|g(x)| \leq C|x|^{-\beta_-}$ for all $x \in \Omega$.

We now prove that $H > 0$ in $\Omega \setminus \{0\}$. Indeed, from the above asymptotic expansion we have that $H(x) > 0$ for $x \to 0$, $x \neq 0$. Since $\chi H \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ for all $\chi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, it follows from Proposition 4.7 that $H_- \in H_0^{\frac{\alpha}{2}}(\Omega \setminus B_\epsilon(0))$ for some $\epsilon > 0$ small. We then test (4.33) against H_-, and arguing as in the proof of Lemma 4.12 we get that $H_- \equiv 0$, and then $H \geq 0$. Since $H \not\equiv 0$, $H \in C^1(\Omega \setminus \{0\})$, it follows from the Harnack inequality (see Lemma 3.10 in [2]) that $H > 0$ in $\Omega \setminus \{0\}$. This proves the existence of a solution u to Problem (4.9) with the relevant asymptotic behavior.

We now deal with uniqueness. Assume that there exists another solution u' satisfying the hypothesis of Theorem 4.3. We define $\bar{u} := u - u'$. Then $\bar{u} : \Omega \to \mathbb{R}$ is a solution to

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} \bar{u} - \left(\frac{\gamma}{|x|^\alpha} + a(x)\right) \bar{u} = 0 & \text{in } \Omega \setminus \{0\} \\ \bar{u} = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

in the sense of Definition 4.6. Since $|\bar{u}(x)| \leq C|x|^{-\beta_-}$ for all $x \in \Omega$ where $C > 0$ is some uniform constant, then by using Proposition 4.7 one concludes that $\bar{u} \in H_0^{\frac{\alpha}{2}}(\Omega)$ is a weak solution to

$$(-\Delta)^{\frac{\alpha}{2}} \bar{u} - \left(\frac{\gamma}{|x|^\alpha} + a(x)\right) \bar{u} = 0 \text{ in } \Omega,$$

that is, for all $\varphi \in H_0^{\frac{\alpha}{2}}(\Omega)$,

$$\frac{C_{n,\alpha}}{2} \int_{(\mathbb{R}^n)^2} \frac{(\bar{u}(x) - \bar{u}(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+\alpha}} dxdy - \int_{\mathbb{R}^n} \left(\frac{\gamma}{|x|^\alpha} + a(x)\right) \bar{u}\varphi \, dx = 0.$$

Taking $\varphi := \bar{u}$ and using the coercivity, we get that $\bar{u} \equiv 0$, and then $u \equiv u'$, which yields the uniqueness. □

4.6 Existence of extremals

This section is devoted to prove the main result, which is Theorem 4.4. By choosing a suitable test function, we estimate the functional $J_a^\Omega(u)$, and we show that the condition $\mu_{\gamma,s,\alpha,a}(\Omega) < \mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ holds under suitable conditions on the dimension or on the mass of the domain. Recall that Proposition 4.10 implies that it is this strict inequality that guarantees the existence of extremals for $\mu_{\gamma,s,\alpha,a}(\Omega)$.

We fix $a \in C^{0,\tau}(\overline{\Omega})$, $\tau \in (0,1)$ and $\eta \in C_c^\infty(\Omega)$ such that

$$\eta \equiv 1 \text{ in } B_\delta(0) \text{ and } \eta \equiv 0 \text{ in } \mathbb{R}^n \setminus B_{2\delta}(0) \text{ with } B_{4\delta}(0) \subset \Omega. \tag{4.37}$$

Let $U \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ be an extremal for $\mu_{\gamma,s,\alpha,0}(\mathbb{R}^n)$. It follows from Theorem 4.1 that, up to multipliying by a nonzero constant, U satisfies for some $\kappa > 0$,

$$(-\Delta)^{\frac{\alpha}{2}} U - \frac{\gamma}{|x|^\alpha} U = \kappa \frac{U^{2_\alpha^*(s)-1}}{|x|^s} \text{ weakly in } H_0^{\frac{\alpha}{2}}(\mathbb{R}^n). \tag{4.38}$$

Moreover, $U \in C^1(\mathbb{R}^n \setminus \{0\})$, $U > 0$ and

$$\lim_{|x|\to\infty} |x|^{\beta_+} U(x) = 1. \tag{4.39}$$

Set

$$J_a^\Omega(u) := \frac{\frac{C_{n,\alpha}}{2} \int_{(\mathbb{R}^n)^2} \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\, dxdy - \int_\Omega \left(\frac{\gamma}{|x|^\alpha} + a\right) u^2\, dx}{\left(\int_\Omega \frac{|u|^{2_\alpha^*(s)}}{|x|^s}\, dx\right)^{\frac{2}{2_\alpha^*(s)}}} = \frac{A(u)}{B(u)^{\frac{2}{2_\alpha^*(s)}}}, \tag{4.40}$$

where

$$A(u) := \langle u, u\rangle - \int_\Omega a u^2\, dx \quad \text{ and } \quad B(u) := \int_\Omega \frac{|u|^{2_\alpha^*(s)}}{|x|^s}\, dx \tag{4.41}$$

with

$$\begin{aligned} \langle u, v\rangle \quad := \quad & \frac{C_{n,\alpha}}{2} \int_{(\mathbb{R}^n)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\, dxdy \\ & - \int_{\mathbb{R}^n} \frac{\gamma}{|x|^\alpha} uv\, dx \text{ for } u, v \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n). \end{aligned} \tag{4.42}$$

Consider

$$u_\epsilon(x) := \epsilon^{-\frac{n-\alpha}{2}} U(\epsilon^{-1} x) \text{ for } x \in \mathbb{R}^n \setminus \{0\}.$$

It follows from Proposition 4.7 that $\eta u_\epsilon \in H_0^{\frac{\alpha}{2}}(\Omega)$.

4.6.1 General estimates for ηu_ϵ

We define the following bilinear form B_η on $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ as follows: For any $\varphi, \psi \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$,

$$\begin{aligned} B_\eta(\varphi,\psi) &:= \langle \eta\varphi, \psi\rangle - \langle \varphi, \eta\psi\rangle \\ &= \frac{C_{n,\alpha}}{2}\int_{(\mathbb{R}^n)^2} \frac{\eta(x)-\eta(y)}{|x-y|^{n+\alpha}}\left(\varphi(y)\psi(x)-\varphi(x)\psi(y)\right)\,dxdy. \end{aligned} \tag{4.43}$$

This expression makes sense since $\eta \equiv 1$ around 0 and $\eta \equiv 0$ around ∞. Note that

$$\langle \eta u_\epsilon, \eta u_\epsilon\rangle \quad = \quad \langle u_\epsilon, \eta^2 u_\epsilon\rangle + \epsilon^{\beta_+-\beta_-} B_\eta\left(\frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}, \frac{\eta u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}\right). \tag{4.44}$$

It follows from (4.38) and the definition of u_ϵ that

$$\langle u_\epsilon, \varphi\rangle = \kappa\int_{\mathbb{R}^n} \frac{u_\epsilon^{2_\alpha^*(s)-1}}{|x|^s}\varphi\,dx \text{ for all } \varphi \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n).$$

By a change of variable, we get as $\epsilon \to 0$,

$$\begin{aligned} \langle u_\epsilon, \eta^2 u_\epsilon\rangle &= \kappa\int_{\mathbb{R}^n} \frac{\eta^2 u_\epsilon^{2_\alpha^*(s)}}{|x|^s}\,dx \\ &= \kappa\int_{\mathbb{R}^n} \frac{u_\epsilon^{2_\alpha^*(s)}}{|x|^s}\,dx + O\left(\int_{\mathbb{R}^n\setminus B_\delta(0)} \frac{u_\epsilon^{2_\alpha^*(s)}}{|x|^s}\,dx\right) \\ &= \kappa\int_{\mathbb{R}^n} \frac{U^{2_\alpha^*(s)}}{|x|^s}\,dx + O\left(\int_{\mathbb{R}^n\setminus B_{\epsilon^{-1}\delta}(0)} \frac{U^{2_\alpha^*(s)}}{|x|^s}\,dx\right). \end{aligned}$$

With (4.39), we get that

$$\begin{aligned} \langle u_\epsilon, \eta^2 u_\epsilon\rangle &= \kappa\int_{\mathbb{R}^n} \frac{U^{2_\alpha^*(s)}}{|x|^s}\,dx + O\left(\epsilon^{\frac{2_\alpha^*(s)}{2}(\beta_+-\beta_-)}\right) \\ &= \kappa\int_{\mathbb{R}^n} \frac{U^{2_\alpha^*(s)}}{|x|^s}\,dx + o\left(\epsilon^{\beta_+-\beta_-}\right). \end{aligned} \tag{4.45}$$

We now deal with the second term of (4.44). First note that

$$B_\eta\left(\frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}, \frac{\eta u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}\right) = \frac{C_{n,\alpha}}{2}\int_{(\mathbb{R}^n)^2} \frac{(\eta(x)-\eta(y))^2}{|x-y|^{n+\alpha}} \frac{u_\epsilon(x)}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}\cdot\frac{u_\epsilon(y)}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}\,dxdy.$$

It follows from (4.39) and the pointwise control of Theorem 4.1 that there exists $C>0$ such that for any $x\in\mathbb{R}^n\setminus\{0\}$, we have that

$$\lim_{\epsilon\to 0}\frac{u_\epsilon(x)}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}=S(x):=\frac{1}{|x|^{\beta_+}}\text{ and }\left|\frac{u_\epsilon(x)}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}\right|\le\frac{C}{|x|^{\beta_+}}. \tag{4.46}$$

Since $\eta(x)=1$ for all $x\in B_\delta(0)$ and $\beta_+(\gamma)<n$, Lebesgue's convergence theorem yields

$$\begin{aligned}\lim_{\epsilon\to 0}B_\eta\left(\frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}},\frac{\eta u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}\right)&=\frac{C_{n,\alpha}}{2}\int_{(\mathbb{R}^n)^2}\frac{(\eta(x)-\eta(y))^2}{|x-y|^{n+\alpha}}S(x)S(y)\,dxdy\\&=B_\eta(S,\eta S).\end{aligned} \tag{4.47}$$

By plugging together (4.44), (4.45) and (4.47), we get as $\epsilon\to 0$,

$$\langle\eta u_\epsilon,\eta u_\epsilon\rangle=\kappa\int_{\mathbb{R}^n}\frac{U^{2^*_\alpha(s)}}{|x|^s}\,dx+B_\eta(S,\eta S)\epsilon^{\beta_+-\beta_-}+o\left(\epsilon^{\beta_+-\beta_-}\right). \tag{4.48}$$

Arguing as in the proof of (4.45), we obtain as $\epsilon\to 0$,

$$\int_{\mathbb{R}^n}\frac{(\eta u_\epsilon)^{2^*_\alpha(s)}}{|x|^s}\,dx=\int_{\mathbb{R}^n}\frac{U^{2^*_\alpha(s)}}{|x|^s}\,dx+o(\epsilon^{\beta_+-\beta_-}). \tag{4.49}$$

As an immediate consequence, we get

Proposition 4.15. *Suppose that* $0\le s<\alpha<n$, $0<\alpha<2$ *and* $0\le\gamma<\gamma_H(\alpha)$. *Then,*

$$\mu_{\gamma,s,\alpha,0}(\Omega)=\mu_{\gamma,s,\alpha}(\mathbb{R}^n).$$

Proof. It follows from the definition of $\mu_{\gamma,s,\alpha}(\Omega)$ that $\mu_{\gamma,s,\alpha,0}(\Omega)\ge\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$. We now show the reverse inequity. Using the estimates (4.48) and (4.49) above, we have as $\epsilon\to 0$,

$$\begin{aligned}J_0^\Omega(U_\epsilon)&=\frac{\kappa\int_{\mathbb{R}^n}\frac{U^{2^*_\alpha(s)}}{|x|^s}\,dx}{\left(\int_{\mathbb{R}^n}\frac{U^{2^*_\alpha(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^*_\alpha(s)}}}+O(\epsilon^{\beta_+-\beta_-})\\&=J_0^{\mathbb{R}^n}(U)+O(\epsilon^{\beta_+-\beta_-})=\mu_{\gamma,s,\alpha}(\mathbb{R}^n)+O(\epsilon^{\beta_+-\beta_-}).\end{aligned}$$

Letting $\epsilon\to 0$ yields $\mu_{\gamma,s,\alpha,0}(\Omega)\le\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ from which follows that

$$\mu_{\gamma,s,\alpha,0}(\Omega)=\mu_{\gamma,s,\alpha}(\mathbb{R}^n).$$

□

4.6.2 The test functions for the non-critical case

We now estimate $J(\eta u_\epsilon)$ when $0 \le \gamma \le \gamma_{crit}(\alpha)$, that is in the case when $\beta_- \ge \frac{n}{2}$. Note that since $\beta_- + \beta_+ = n - \alpha$, we have that

$$\beta_+ - \beta_- > \alpha \text{ when } \gamma < \gamma_{crit}(\alpha) \quad \text{and} \quad \beta_+ - \beta_- = \alpha \text{ if } \gamma = \gamma_{crit}(\alpha). \tag{4.50}$$

We start with the following:

Proposition 4.16. *Let $0 \le s < \alpha < 2$, $0 \le \gamma \le \gamma_{crit}(\alpha)$ and $n \ge 2\alpha$. Then, as $\epsilon \to 0$,*

$$\int_\Omega a(\eta u_\epsilon)^2 dx = \begin{cases} \epsilon^\alpha \left(\int_{\mathbb{R}^n} U^2 dx\right) a(0) + o(\epsilon^\alpha) & \text{if } 0 \le \gamma < \gamma_{crit}(\alpha) \\ \omega_{n-1} a(0) \epsilon^\alpha \ln(\epsilon^{-1}) + o(\epsilon^\alpha \ln \epsilon) & \text{if } \gamma = \gamma_{crit}(\alpha). \end{cases}$$

Proof of Proposition 4.16. We write

$$\begin{aligned} \int_\Omega a(\eta u_\epsilon)^2 dx &= \int_{B_\delta} a u_\epsilon^2 dx + \int_{B_{2\delta} \setminus B_\delta} a(\eta u_\epsilon)^2 dx \\ &= \epsilon^\alpha \int_{B_{\epsilon^{-1}\delta}} a(\epsilon x) U^2 dx + O(\epsilon^{\beta_+ - \beta_-}). \end{aligned}$$

Assume that $\gamma < \gamma_{crit}(\alpha)$. Since $\beta_+ > \frac{n}{2}$ and $U \in C^1(\mathbb{R}^n \setminus \{0\})$ satisfies (4.6), we get that $U \in L^2(\mathbb{R}^n)$ and therefore, Lebesgue's convergence theorem and the assumption $\beta_+(\gamma) - \beta_-(\gamma) > \alpha$ yield

$$\int_\Omega a(\eta u_\epsilon)^2\, dx = \epsilon^\alpha \left(\int_{\mathbb{R}^n} U^2 dx\right) a(0) + o(\epsilon^\alpha) \quad \text{ as } \epsilon \to 0.$$

If now $\gamma = \gamma_{crit}(\alpha)$, then $\lim_{|x| \to \infty} |x|^{\frac{n}{2}} U(x) = 1$ and $\beta_+ - \beta_- = \alpha$. Therefore

$$\int_\Omega (\eta u_\epsilon)^2 dx = \omega_{n-1} a(0) \epsilon^\alpha \ln(\epsilon^{-1}) + o(\epsilon^\alpha \ln \epsilon) \quad \text{ as } \epsilon \to 0.$$

This proves Proposition 4.16. □

Plugging together (4.48), (4.49) and Proposition 4.16 then yields, as $\epsilon \to 0$,

$$\begin{aligned} J_a^\Omega(\eta u_\epsilon) &= \frac{\kappa \int_{\mathbb{R}^n} \frac{U^{2^*_\alpha(s)}}{|x|^s}\, dx}{\left(\int_{\mathbb{R}^n} \frac{U^{2^*_\alpha(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^*_\alpha(s)}}} - a(0) \frac{\int_{\mathbb{R}^n} U^2\, dx}{\left(\int_{\mathbb{R}^n} \frac{U^{2^*_\alpha(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^*_\alpha(s)}}} \epsilon^\alpha + o(\epsilon^\alpha) \\ &= J_0^{\mathbb{R}^n}(U) - a(0) \frac{\int_{\mathbb{R}^n} U^2\, dx}{\left(\int_{\mathbb{R}^n} \frac{U^{2^*_\alpha(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^*_\alpha(s)}}} \epsilon^\alpha + o(\epsilon^\alpha), \end{aligned} \tag{4.51}$$

when $\gamma < \gamma_{crit}(\alpha)$, and

$$J_a^\Omega(\eta u_\epsilon) = J_0^{\mathbb{R}^n}(U) - a(0)\frac{\omega_{n-1}}{\left(\int_{\mathbb{R}^n} \frac{U^{2_\alpha^*(s)}}{|x|^s}\,dx\right)^{\frac{2}{2_\alpha^*(s)}}}\epsilon^\alpha \ln\frac{1}{\epsilon} + o(\epsilon^\alpha \ln\frac{1}{\epsilon}), \quad (4.52)$$

when $\gamma = \gamma_{crit}(\alpha)$.

4.6.3 The test function for the critical case

Here, we assume that $\gamma > \gamma_{crit}(\alpha)$. It follows from Theorem 4.3 that there exists $H : \Omega \setminus \{0\} \to \mathbb{R}$ such that

$$\begin{cases} H \in C^1(\Omega\setminus\{0\})\ ,\ \xi H \in H_0^{\frac{\alpha}{2}}(\Omega) & \text{for all } \xi \in C_c^\infty(\mathbb{R}^n\setminus\{0\}),\\ (-\Delta)^{\frac{\alpha}{2}}H - \left(\frac{\gamma}{|x|^\alpha}+a\right)H = 0 & \text{weakly in } \Omega\setminus\{0\}\\ H>0 & \text{in } \Omega\setminus\{0\}\\ H=0 & \text{in } \partial\Omega\\ \text{and } \lim_{x\to 0}|x|^{\beta_+}H(x)=1. & \end{cases}$$

Here the solution is in the sense of Definition 4.6. In other words, the second identity means that for any $\varphi \in C_c^\infty(\Omega\setminus\{0\})$, we have that

$$\frac{C_{n,\alpha}}{2}\int_{(\mathbb{R}^n)^2}\frac{(H(x)-H(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+\alpha}}\,dxdy - \int_{\mathbb{R}^n}\left(\frac{\gamma}{|x|^\alpha}+a\right)H\varphi\,dx = 0. \quad (4.53)$$

Note that this latest identity makes sense since $H \in L^1(\Omega)$ (since $\beta_+ < n$). Let now η be as in (4.37). Following the construction of the singular function H in (4.32), there exists $g \in H_0^{\frac{\alpha}{2}}(\Omega)$ such that

$$H(x) := \frac{\eta(x)}{|x|^{\beta_+}} + g(x) \text{ for } x \in \Omega\setminus\{0\},$$

where

$$(-\Delta)^{\frac{\alpha}{2}}g - \left(\frac{\gamma}{|x|^\alpha}+a\right)g = f, \quad (4.54)$$

with $f \in L^\infty(\Omega)$ and $f \in C^1(B_\delta(0)\setminus\{0\})$. It follows from (4.31) that there exists $c > 0$ such that

$$|f(x)| \le \frac{C}{|x|^{\beta_+}} \text{ for } x \in \Omega\setminus\{0\} \text{ and } |\nabla f(x)| \le \frac{C}{|x|^{\beta_++1}} \text{ for all } x \in B_{\delta/2}(0)\setminus\{0\}. \quad (4.55)$$

We also have that

$$g(x)=\frac{m^{\alpha}_{\gamma,a}(\Omega)}{|x|^{\beta_-}}+o\left(\frac{1}{|x|^{\beta_-}}\right)\text{ as }x\to 0,\text{ and }|g(x)|\leq C|x|^{-\beta_-}\text{ for all }x\in\Omega. \tag{4.56}$$

Define the test function as

$$T_\epsilon(x)=\eta u_\epsilon(x)+\epsilon^{\frac{\beta_+-\beta_-}{2}}g(x)\quad\text{ for all }x\in\overline{\Omega}\setminus\{0\},$$

where

$$u_\epsilon(x):=\epsilon^{-\frac{n-\alpha}{2}}U(\epsilon^{-1}x)\text{ for }x\in\mathbb{R}^n\setminus\{0\},$$

and $U\in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is such that $U>0$, $U\in C^1(\mathbb{R}^n\setminus\{0\})$ and satisfies (4.38) above for some $\kappa>0$ and also (4.39). It is easy to see that $T_\epsilon\in H_0^{\frac{\alpha}{2}}(\Omega)$ for all $\epsilon>0$.

This subsection is devoted to computing the expansion of $J_a^\Omega(T_\epsilon)$ where J_a^Ω is defined in (4.40), (4.41) and (4.42). For simplicity, we set $S(x):=\frac{1}{|x|^{\beta_+}}$ for $x\in\mathbb{R}^n\setminus\{0\}$. In particular, it follows from (4.11) that we have that

$$(-\Delta)^{\frac{\alpha}{2}}S-\frac{\gamma}{|x|^\alpha}S=0\text{ weakly in }\mathbb{R}^n\setminus\{0\}, \tag{4.57}$$

in the sense that $\langle S,\varphi\rangle=0$ for all $\varphi\in C_c^\infty(\mathbb{R}^n\setminus\{0\})$. First note that

$$\lim_{\epsilon\to 0}\frac{T_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}=H\text{ in }L^\infty_{loc}(\overline{\Omega}\setminus\{0\}).$$

Therefore, since $|\epsilon^{-\frac{\beta_+-\beta_-}{2}}T_\epsilon(x)|\leq C|x|^{-\beta_+}$ for $x\in\Omega\setminus\{0\}$ with $2\beta_+<n$, Lebesgue's theorem yields as $\epsilon\to 0$,

$$\int_\Omega aT_\epsilon^2\,dx=\epsilon^{\beta_+-\beta_-}\int_\Omega aH^2\,dx+o\left(\epsilon^{\beta_+-\beta_-}\right),$$

Since $T_\epsilon=\eta u_\epsilon+\epsilon^{\frac{\beta_+-\beta_-}{2}}g$, we have that

$$\begin{aligned}A(T_\epsilon)&=\langle T_\epsilon,T_\epsilon\rangle-\epsilon^{\beta_+-\beta_-}\int_\Omega aH^2\,dx+o\left(\epsilon^{\beta_+-\beta_-}\right)\\&=\langle\eta u_\epsilon,\eta u_\epsilon\rangle+2\epsilon^{\frac{\beta_+-\beta_-}{2}}\langle\eta u_\epsilon,g\rangle+\epsilon^{\beta_+-\beta_-}\langle g,g\rangle\\&-\epsilon^{\beta_+-\beta_-}\int_\Omega aH^2\,dx+o\left(\epsilon^{\beta_+-\beta_-}\right)\end{aligned}$$

We are now going to estimate these terms separately. First, Formula (4.43) and (4.48) yield, as $\epsilon \to 0$

$$A(T_\epsilon) = \kappa \int_{\mathbb{R}^n} \frac{U^{2^*_\alpha(s)}}{|x|^s}\, dx + 2\epsilon^{\frac{\beta_+-\beta_-}{2}} \langle u_\epsilon, \eta g\rangle + \epsilon^{\beta_+-\beta_-} M_\epsilon + o\left(\epsilon^{\beta_+-\beta_-}\right), \tag{4.58}$$

where

$$M_\epsilon := B_\eta\left(S, \eta S\right) + 2B_\eta\left(\frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}, g\right) + \langle g, g\rangle - \int_\Omega aH^2\, dx.$$

As to the second term of (4.58), we have

$$\langle u_\epsilon, \eta g\rangle = \kappa \int_{\mathbb{R}^n} \frac{u_\epsilon^{2^*_\alpha(s)-1}\eta g}{|x|^s}\, dx.$$

We set $\theta_\epsilon := \int_{\mathbb{R}^n} \frac{u_\epsilon^{2^*_\alpha(s)-1}\eta g}{|x|^s}\, dx$. It is easy to check that, since $\eta g \in H_0^{\frac{\alpha}{2}}(\Omega)$, $(u_\epsilon)_\epsilon$ is bounded in $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ and goes to 0 weakly as $\epsilon \to 0$, we have that

$$\lim_{\epsilon\to 0} \theta_\epsilon = 0. \tag{4.59}$$

Therefore we can rewrite (4.58) as

$$A(T_\epsilon) = \kappa \int_{\mathbb{R}^n} \frac{U^{2^*_\alpha(s)}}{|x|^s}\, dx + 2\kappa\epsilon^{\frac{\beta_+-\beta_-}{2}} \theta_\epsilon + \epsilon^{\beta_+-\beta_-} M_\epsilon + o\left(\epsilon^{\beta_+-\beta_-}\right) \tag{4.60}$$

as $\epsilon \to 0$.

We now estimate M_ϵ. First, we write

$$B_\eta\left(\frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}, g\right) = \frac{C_{n,\alpha}}{2} \int_{(\mathbb{R}^n)^2} F_\epsilon(x,y)\, dxdy,$$

where

$$F_\epsilon(x,y) := \frac{\eta(x)-\eta(y)}{|x-y|^{n+\alpha}} \left(\frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}(y)g(x) - \frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}(x)g(y)\right).$$

Remembering that $\eta \equiv 1$ in $B_\delta(0)$ and $\eta \equiv 0$ in $B_{2\delta}(0)^c$ and using (4.56), we get that

$$\left|F_\epsilon(x,y)\mathbb{1}_{|x|<\delta/2}\right| \le C\mathbb{1}_{|x|<\delta/2}\mathbb{1}_{|y|>\delta}|x|^{-\beta_+}|y|^{-(n+\alpha+\beta_-)} \in L^1((\mathbb{R}^n)^2).$$

Similarly, we have a bound on F_ϵ on $\{|x| > 3\delta\}$. By symmetry, this yields also a bound on $\{|y| < \delta/2\} \cup \{|y| > 3\delta\}$. We are then left with getting a bound on $\mathcal{A} := \left[B_{3\delta}(0) \setminus B_{\delta/2}(0)\right]^2$.

For $(x, y) \in \mathcal{A}$, we have that

$$\begin{aligned}
|F_\epsilon(x,y)| &\leq C\frac{|x-y|}{|x-y|^{n+\alpha}} \cdot \left(\frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}(y) - \frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}(x)\right) g(x) \\
&+ C\frac{|x-y|}{|x-y|^{n+\alpha}} \cdot \left(\frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}(x)(g(x)-g(y))\right) \\
&\leq C|x-y|^{1-\alpha-n}\left(\left|\frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}(y) - \frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}(x)\right| + |g(x)-g(y)|\right).
\end{aligned}$$

As noticed in the proof of Theorem 4.3, it follows from elliptic theory that $g \in C^1(\Omega \setminus \{0\})$. Therefore, there exists $C > 0$ such that $|g(x) - g(y)| \leq C|x - y|$ for all $(x, y) \in \mathcal{A}$.

Setting $\tilde{u}_\epsilon := \epsilon^{-\frac{\beta_+-\beta_-}{2}} u_\epsilon$, it follows from (4.38) that

$$(-\Delta)^{\frac{\alpha}{2}}\tilde{u}_\epsilon - \frac{\gamma}{|x|^\alpha}\tilde{u}_\epsilon = \kappa\epsilon^{\frac{2^*_\alpha(s)-2}{2}(\beta_+-\beta_-)}\frac{\tilde{u}_\epsilon^{2^*_\alpha(s)-1}}{|x|^s} \text{ weakly in } H_0^{\frac{\alpha}{2}}(\mathbb{R}^n).$$

It then follows from (4.46) and arguments similar to the Proof of Theorem 4.1 (see Remark 2.5 and Theorem 2.1 of Jian-Li-Xiong [46]) that $(\tilde{u}_\epsilon)_\epsilon$ is bounded in $C^1_{loc}(\mathbb{R}^n \setminus \{0\})$. Therefore, there exists $C > 0$ such that $|\tilde{u}_\epsilon(x) - \tilde{u}_\epsilon(y)| \leq C|x - y|$ for all $(x, y) \in \mathcal{A}$. Then, we get

$$|F_\epsilon(x,y)| \leq C|x-y|^{2-\alpha-n} \in L^1(\mathcal{A}).$$

Therefore, (F_ϵ) is uniformly dominated on $(\mathbb{R}^n)^2$. Noting that $\frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}(x) \to S(x)$ as $\epsilon \to 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, Lebesgue's theorem yields

$$\lim_{\epsilon\to 0} B_\eta\left(\frac{u_\epsilon}{\epsilon^{\frac{\beta_+-\beta_-}{2}}}, g\right) = B_\eta(S, g)\,. \tag{4.61}$$

Here again, note that $B_\eta(S, g)$ makes sense. Therefore, we get that $M_\epsilon = M + o(1)$ as $\epsilon \to 0$ where

$$M := B_\eta(S, \eta S) + 2B_\eta(S, g) + \langle g, g\rangle - \int_\Omega aH^2\,dx. \tag{4.62}$$

We now estimate $B(T_\epsilon)$. Note first that since $p>2$, there exists $C(p)>0$ such that

$$\left||x+y|^p-|x|^p-p|x|^{p-2}xy\right|\leq C(p)\left(|x|^{p-2}y^2+|y|^p\right) \text{ for all } x,y\in\mathbb{R}.$$

We therefore get that

$$\begin{aligned}
B(T_\epsilon) &= \int_{\mathbb{R}^n}\frac{\left|\eta u_\epsilon+\epsilon^{\frac{\beta_+-\beta_-}{2}}g\right|^{2^*_\alpha(s)}}{|x|^s}\,dx\\
&= \int_{\mathbb{R}^n}\frac{(\eta u_\epsilon)^{2^*_\alpha(s)}}{|x|^s}\,dx+2^*_\alpha(s)\epsilon^{\frac{\beta_+-\beta_-}{2}}\int_{\mathbb{R}^n}\frac{u_\epsilon^{2^*_\alpha(s)-1}\eta^{2^*_\alpha(s)-1}g}{|x|^s}\,dx\\
&+O\left(\epsilon^{\beta_+-\beta_-}\int_{\mathbb{R}^n}\frac{u_\epsilon^{2^*_\alpha(s)-2}\eta^{2^*_\alpha(s)-2}g^2}{|x|^s}\,dx+\epsilon^{\frac{2^*_\alpha(s)}{2}(\beta_+-\beta_-)}\int_{\mathbb{R}^n}\frac{|g|^{2^*_\alpha(s)}}{|x|^s}\,dx\right).
\end{aligned}$$

Since $\eta\equiv 1$ around 0, we get that

$$\begin{aligned}
\int_{\mathbb{R}^n}\frac{u_\epsilon^{2^*_\alpha(s)-1}\eta^{2^*_\alpha(s)-1}g}{|x|^s}\,dx &= \int_{\mathbb{R}^n}\frac{u_\epsilon^{2^*_\alpha(s)-1}\eta g}{|x|^s}\,dx+O\left(\int_{\Omega\setminus B_\delta(0)}\frac{u_\epsilon^{2^*_\alpha(s)-1}g}{|x|^s}\,dx\right)\\
&=\theta_\epsilon+o\left(\epsilon^{\frac{\beta_+-\beta_-}{2}}\right),
\end{aligned}$$

as $\epsilon\to 0$. Therefore, in view of (4.49), we deduce that

$$B(T_\epsilon)=\int_{\mathbb{R}^n}\frac{U^{2^*_\alpha(s)}}{|x|^s}\,dx+2^*_\alpha(s)\epsilon^{\frac{\beta_+-\beta_-}{2}}\theta_\epsilon+o\left(\epsilon^{\beta_+-\beta_-}\right), \tag{4.63}$$

as $\epsilon\to 0$. Plugging (4.60), (4.59) and (4.63) into (4.40), we get that

$$\begin{aligned}
J_a^\Omega(T_\epsilon) &= \frac{\kappa\int_{\mathbb{R}^n}\frac{U^{2^*_\alpha(s)}}{|x|^s}\,dx}{\left(\int_{\mathbb{R}^n}\frac{U^{2^*_\alpha(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^*_\alpha(s)}}}\left(1+\frac{M}{\kappa\int_{\mathbb{R}^n}\frac{U^{2^*_\alpha(s)}}{|x|^s}\,dx}\epsilon^{\beta_+-\beta_-}+o\left(\epsilon^{\beta_+-\beta_-}\right)\right)\\
&= J_0^{\mathbb{R}^n}(U)\left(1+\frac{M}{\kappa\int_{\mathbb{R}^n}\frac{U^{2^*_\alpha(s)}}{|x|^s}\,dx}\epsilon^{\beta_+-\beta_-}+o\left(\epsilon^{\beta_+-\beta_-}\right)\right)
\end{aligned} \tag{4.64}$$

as $\epsilon\to 0$, where M is defined in (4.62) and $J_0^{\mathbb{R}^n}$ is as in (4.40).

We now express M in term of the mass. Note that in the classical (pointwise) setting, an integration by parts yield that $B_\eta(\varphi,\psi)$ defined in (4.43) is an integral on the boundary of a domain. Hence, the mass appears by simply integrating by part independently the singular function H. The central remark we make here is that the integral on the boundary on a domain (defined in the local setting) can be seen as the limit of an integral on the domain via multiplication by a cut-off function with support converging to the boundary –which happened to be defined in the nonlocal setting. Therefore, despite the nonlocal aspect of our problem, we shall be able to apply the same strategy as in the local setting.

We shall be performing the following computations in the same order as the ones above made to get $A(T_\epsilon)$. The constant M will therefore appear naturally in the two settings.

Let $\chi \in C^\infty(\mathbb{R}^n)$ such that $\chi \equiv 0$ in $B_1(0)$ and $\chi \equiv 1$ in $\mathbb{R}^n \setminus B_2(0)$. For $k \in \mathbb{N} \setminus \{0\}$, define $\chi_k(x) := \chi(kx)$ for $x \in \mathbb{R}^n$, so that

$$\chi_k(x) = 0 \text{ for } |x| < \frac{1}{k} \text{ and } \chi_k(x) = 1 \text{ for } |x| > \frac{2}{k}.$$

In particular, $(\chi_k)_k$ is bounded in $L^\infty(\mathbb{R}^n)$ and $\chi_k(x) \to 1$ as $k \to +\infty$ for a.e. $x \in \mathbb{R}^n$. Since $\chi_k H \in H_0^{\frac{\alpha}{2}}(\Omega)$, then by the very definition of H (see (4.53)), we have that

$$\begin{aligned}
0 &= \langle H, \chi_k H\rangle - \int_{\mathbb{R}^n} aH\chi_k H\,dx \\
&= \langle \eta S + g, \chi_k \eta S + \chi_k g\rangle - \int_{\mathbb{R}^n} \chi_k a H^2\,dx \\
&= \langle \eta S, \chi_k \eta S\rangle + \langle \eta S, \chi_k g\rangle + \langle \chi_k \eta S, g\rangle + \langle g, \chi_k g\rangle - \int_{\mathbb{R}^n} \chi_k a H^2\,dx \\
&= \langle S, \chi_k \eta^2 S\rangle + B_\eta(S, \chi_k \eta S) + \langle S, \eta \chi_k g\rangle + B_\eta(S, \chi_k g) + \langle S, \chi_k \eta g\rangle \\
&\quad + B_{\chi_k \eta}(S, g) + \langle g, \chi_k g\rangle - \int_{\mathbb{R}^n} \chi_k a H^2\,dx.
\end{aligned}$$

Since $aH^2 \in L^1(\Omega)$ (this is a consequence of $2\beta_+ < n$) and S is a solution to (4.57), we get that

$$0 = B_\eta(S, \chi_k \eta S) + B_\eta(S, \chi_k g) + B_{\chi_k \eta}(S, g) + \langle g, \chi_k g\rangle - \int_{\mathbb{R}^n} aH^2\,dx + o(1),$$

as $k \to +\infty$. We now estimate these terms separately.
Our first claim is that

$$\lim_{k \to +\infty} \langle \chi_k g, g\rangle = \langle g, g\rangle. \tag{4.65}$$

Indeed,

$$\begin{aligned}\|\chi_k g - g\|^2_{H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)} &= \frac{C_{n,\alpha}}{2}\int_{(\mathbb{R}^n)^2} \frac{|(1-\chi_k)(x)g(x)-(1-\chi_k)(y)g(y)|^2}{|x-y|^{n+\alpha}}\,dxdy\\ &\leq C_{n,\alpha}\int_{(\mathbb{R}^n)^2}|1-\chi_k(x)|^2\frac{|g(x)-g(y)|^2}{|x-y|^{n+\alpha}}\,dxdy\\ &+ C_{n,\alpha}\int_{(\mathbb{R}^n)^2} g(y)^2\frac{|\chi_k(x)-\chi_k(y)|^2}{|x-y|^{n+\alpha}}\,dxdy.\end{aligned}$$

The first integral goes to 0 as $k\to+\infty$ with Lebesgue's convergence theorme since $g\in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$. For the second term, we use the change of variable $X = kx$, $Y = ky$ and the control of $g(x)$ by $|x|^{-\beta_-}$. This proves that $(\chi_k g)\to g$ in $H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ as $k\to+\infty$. The claim follows and (4.65) is proved.

We now write

$$B_\eta(S,\chi_k\eta S) \quad = \quad \frac{C_{n,\alpha}}{2}\left(\int_{(\mathbb{R}^n)^2}\chi_k(x)\tilde{F}(x,y)\,dxdy + \int_{(\mathbb{R}^n)^2} G_k(x,y)\,dxdy\right),$$

where

$$\tilde{F}(x,y) := \frac{\eta(x)-\eta(y)}{|x-y|^{n+\alpha}}\left(S(y)(\eta S)(x) - S(x)(\eta S)(y)\right),$$

and

$$G_k(x,y) := \frac{(\eta(x)-\eta(y))(\chi_k(x)-\chi_k(y))(\eta S)(y)S(x)}{|x-y|^{n+\alpha}}.$$

As in the proof of (4.61) and (4.47), $\tilde{F}\in L^1((\mathbb{R}^n)^2)$ and Lebesgue's convergence theorem yields

$$\lim_{k\to+\infty}\frac{C_{n,\alpha}}{2}\int_{(\mathbb{R}^n)^2}\chi_k(x)\tilde{F}(x,y)\,dxdy = B_\eta(S,\eta S).$$

Arguing as in the proof of (4.61), we get the existence of $G\in L^1((\mathbb{R}^n)^2)$ such that $|G_k(x,y)|\leq G(x,y)$ for all $(x,y)\in(\mathbb{R}^n)^2$ such that $|x|<\delta/2$ or $|x|>3\delta$. By symmetry, a similar control also holds for $(x,y)\in(\mathbb{R}^n)^2$ such that $|y|<\delta/2$ or $|y|>3\delta$. Moreover, for $\delta>0$ small enough, we have that $G_k(x,y)=0$ for $(x,y)\in(\mathbb{R}^n)^2$ such that $|x|>\delta/2$ and $|y|>\delta/2$ (this is due to the definition of χ_k). Therefore, since $\lim_{k\to+\infty}(\chi_k(x)-\chi_k(y))=0$ for a.e. $(x,y)\in(\mathbb{R}^n)^2$, Lebesgue's convergence theorem yields $\int_{(\mathbb{R}^n)^2}G_k(x,y)\,dxdy\to 0$ as $k\to+\infty$. We can then conclude that

$$\lim_{k\to+\infty}B_\eta(S,\chi_k\eta S) = B_\eta(S,\eta S).$$

Similar arguments yield

$$\lim_{k\to+\infty} B_\eta(S,\chi_k g) = B_\eta(S,g).$$

Therefore, we get that

$$0 = B_\eta(S,\eta S) + B_\eta(S,g) + B_{\chi_k\eta}(S,g) + \langle g,g\rangle - \int_{\mathbb{R}^n} aH^2\,dx + o(1),$$

as $k\to+\infty$. We also have that

$$\begin{aligned} B_{\chi_k\eta}(S,g) &= \frac{C_{n,\alpha}}{2}\int_{(\mathbb{R}^n)^2}\chi_k(x)\frac{\eta(x)-\eta(y)}{|x-y|^{n+\alpha}}\left(S(y)g(x)-S(x)g(y)\right)\,dxdy \\ &+ \frac{C_{n,\alpha}}{2}\int_{(\mathbb{R}^n)^2}\eta(y)\frac{\chi_k(x)-\chi_k(y)}{|x-y|^{n+\alpha}}\left(S(y)g(x)-S(x)g(y)\right)\,dxdy. \end{aligned}$$

As above, the first integral of the right-hand-side goes to $B_\eta(S,g)$ as $k\to+\infty$. We now deal with the second integral. Using that $\beta_++\beta_-=n-\alpha$, the change of variables $X=kx$ and $Y=ky$ yield

$$\int_{(\mathbb{R}^n)^2}\eta(y)\frac{\chi_k(x)-\chi_k(y)}{|x-y|^{n+\alpha}}\left(S(y)g(x)-S(x)g(y)\right)dxdy = \int_{(\mathbb{R}^n)^2}F_k(X,Y)dXdY,$$

where

$$F_k(X,Y) := \eta\left(\frac{Y}{k}\right)\frac{\chi(X)-\chi(Y)}{|X-Y|^{n+\alpha}}\left(\frac{1}{|Y|^{\beta_+}}g\left(\frac{X}{k}\right)k^{-\beta_-} - \frac{1}{|X|^{\beta_+}}g\left(\frac{Y}{k}\right)k^{-\beta_-}\right).$$

Note that there exists $C>0$ such that $|g(x)|\le C|x|^{-\beta_-}$ for all $x\in\Omega\setminus\{0\}$. Since $\chi(X)=0$ for $|X|<1$ and $\chi(X)=1$ for $|X|>2$, arguing as in the proof of (4.61), we get that $|F_k(X,Y)|$ is uniformly bounded from above by a function in $L^1((\mathbb{R}^n)^2)$ for $(X,Y)\in(\mathbb{R}^n)^2$ such that $X\notin B_3(0)\setminus B_{1/2}(0)$ or $Y\notin B_3(0)\setminus B_{1/2}(0)$.

There exists $C>0$ such that $|\eta(X)-\eta(Y)|\le C|X-Y|$ for all $(X,Y)\in[B_3(0)\setminus B_{1/2}(0)]^2$. Therefore, for such (X,Y), we have that

$$\begin{aligned} |F_k(X,Y)| &\le C|X-Y|^{1-\alpha-n}\left|\left(\frac{1}{|Y|^{\beta_+}}-\frac{1}{|X|^{\beta_+}}\right)g\left(\frac{X}{k}\right)k^{-\beta_-}\right| \\ &\quad +C|X-Y|^{1-\alpha-n}\frac{1}{|X|^{\beta_+}}\left|g\left(\frac{X}{k}\right)k^{-\beta_-}-g\left(\frac{Y}{k}\right)k^{-\beta_-}\right| \\ &\le C|X-Y|^{2-\alpha-n} \\ &\quad +C|X-Y|^{1-\alpha-n}\left|g\left(\frac{X}{k}\right)k^{-\beta_-}-g\left(\frac{Y}{k}\right)k^{-\beta_-}\right|. \end{aligned}$$

Define $g_k(X) := g\left(\frac{X}{k}\right) k^{-\beta_-}$ for $X \in k\Omega$. It follows from (4.54) and (4.55) that

$$(-\Delta)^{\frac{\alpha}{2}} g_k - \left(\frac{\gamma}{|X|^\alpha} + k^{-\alpha} a(k^{-1}X)\right) g_k = f_k \text{ weakly in } H_0^{\alpha/2}(k\Omega),$$

where

$$f_k(X) := k^{-\beta_- -\alpha} f(k^{-1}X) \text{ so that } |f_k(X)| \leq C k^{-(\alpha-(\beta_+-\beta_-))}|X|^{-\beta_+},$$

for all $X \in k\Omega$. Here again, elliptic regularity yields that (g_k) is bounded in $C^1_{loc}(\mathbb{R}^n \setminus \{0\})$. Therefore, there exists $C > 0$ such that

$$|g_k(X) - g_k(Y)| \leq C|X - Y|,$$

for all $(X, Y) \in [B_3(0) \setminus B_{1/2}(0)]^2$. Therefore, we get that

$$|F_k(X, Y)| \leq C|X - Y|^{2-\alpha-n} \text{ for all } (X, Y) \in [B_3(0) \setminus B_{1/2}(0)]^2.$$

Therefore, since $\alpha < 2$, (F_k) is also dominated on this domain, and then on $(\mathbb{R}^n)^2$. Finally, it follows from the definition (4.56) of the mass that

$$\lim_{k\to+\infty} F_k(X, Y) = m^\alpha_{\gamma,a}(\Omega)\frac{\chi(X) - \chi(Y)}{|X - Y|^{n+\alpha}}\left(\frac{1}{|Y|^{\beta_+}|X|^{\beta_-}} - \frac{1}{|X|^{\beta_+}|Y|^{\beta_-}}\right),$$

for a.e. $(X, Y) \in (\mathbb{R}^n)^2$. Therefore, Lebesgue's convergence theorem yields

$$0 = B_\eta(S, \eta S) + 2B_\eta(S, g) + K \cdot m^\alpha_{\gamma,a}(\Omega) + \langle g, g\rangle - \int_{\mathbb{R}^n} aH^2\, dx,$$

where

$$K := \frac{C_{n,\alpha}}{2}\int_{(\mathbb{R}^n)^2} \frac{\chi(X) - \chi(Y)}{|X - Y|^{n+\alpha}}\left(\frac{1}{|Y|^{\beta_+}|X|^{\beta_-}} - \frac{1}{|X|^{\beta_+}|Y|^{\beta_-}}\right) dXdY.$$

Without loss of generality, we can assume that χ is radially symetrical and nondecreasing. Therefore, we get that $K > 0$. With (4.62), we then get that

$$M = -K \cdot m^\alpha_{\gamma,a}(\Omega) \text{ with } K > 0.$$

Plugging this identity in (4.64) yields

$$J^\Omega_a(T_\epsilon) = J^{\mathbb{R}^n}_0(U)\left(1 - \frac{K}{\kappa\int_{\mathbb{R}^n}\frac{U^{2^*_\alpha(s)}}{|x|^s}\,dx}\cdot m^\alpha_{\gamma,a}(\Omega)\epsilon^{\beta_+-\beta_-} + o\left(\epsilon^{\beta_+-\beta_-}\right)\right). \tag{4.66}$$

4.6.4 Proof of Theorem 4.4

Theorem 4.4 is now a direct consequence of (4.51), (4.52), (4.66) and Proposition 4.10.

Chapter 5

Existence Results for Non-linearly Perturbed Fractional Hardy-Schrödinger Problems

5.1 Introduction

Throughout this chapter, we shall use the following notations:

$$\beta_+ := \beta_+(\gamma),\ \beta_- := \beta_-(\gamma) \text{ and } m^{\alpha}_{\gamma,\lambda} := m^{\alpha}_{\gamma,\lambda}(\Omega).$$

Recall that the mass $m^{\alpha}_{\gamma,\lambda}(\Omega)$ is defined in Theorem 1.5, and parameters $\beta_+(\gamma)$ and $\beta_-(\gamma)$ are introduced in Section 4.2; see also Remark 1.4.

We consider the following perturbed problem associated with the operator $L_{\gamma,\alpha}$ on bounded domains $\Omega \subset \mathbb{R}^n$ with $0 \in \Omega$:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u - \gamma \dfrac{u}{|x|^{\alpha}} - \lambda u = \dfrac{u^{2^*_\alpha(s)-1}}{|x|^s} + hu^{q-1} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{5.1}$$

where $0 \leq s < \alpha < 2$, $n > \alpha$, $2^*_\alpha(s) := \frac{2(n-s)}{n-\alpha}$, and $\lambda, \gamma \in \mathbb{R}$. We also assume that $h \in C^0(\overline{\Omega})$, $h \geq 0$, and $q \in (2, 2^*_\alpha)$ with $2^*_\alpha := 2^*_\alpha(0)$.

When ($h \equiv 0$), Problem (5.1) has been studied in both the local and non-local setting. See [17, 36, 37] and [38] and the references therein. In [44], Jaber considered the local version of the problem in the Riemannian context but in the absence of the Hardy term ($i.e., \gamma = 0$). A similar problem, with the second order operator replaced by the fourth order Paneitz operator, was studied by Esposito-Robert [26] (see also Djadli- Hebey-Ledoux [23]).

By using ideas from [38] and [44], we now investigate the role of the linear perturbation (*i.e.*, λu), the non-linear perturbation (*i.e.*, hu^{q-1}), as well as the geometry of the domain (*the mass* $m^{\alpha}_{\gamma,\lambda}$) on the existence of a positive solution of (5.1).

As in Jaber [44], our main tool here to investigate the existence of solutions is the Mountain Pass Lemma of Ambrosetti- Rabinowitz [3] (see Lemma 3.6). We shall use the extremal of $\mu_{\gamma,s,\alpha}(\mathbb{R}^n)$ (defined in (1.2)) and its profile at zero and infinity to build appropriate test-functions for the functional under study.

Our analysis shows that the existence of a solution for problem (5.1) depends only on the non-linear perturbation when the operator $L_{\gamma,\alpha}$ is non-critical ($i.e., 0 \leq \gamma \leq \gamma_{crit}(\alpha)$). The critical case ($i.e., \gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$) is more complicated and depends on other conditions involving both the perturbation and the global geometry of the domain. More precisely, when $0 \leq \gamma \leq \gamma_{crit}(\alpha)$, the competition is between the linear and non-linear perturbations, and since $q > 2$, the non-linear term dominates. In the critical case, this competition is more challenging as it is between *the mass* and the non-linear perturbation. In this situation, there exists a threshold $q_{crit} \in (2, 2^*_\alpha)$, where the dominant factor switches from the non-linear perturbation to the mass. The transition at 2^*_α is most interesting. We shall establish the following result.

Theorem 5.1 (Shakerian [66]). *Let Ω be a smooth bounded domain in $\mathbb{R}^n (n > \alpha)$ such that $0 \in \Omega$, and let $2^*_\alpha(s) := \frac{2(n-s)}{n-\alpha}$, $0 \leq s < \alpha$, $-\infty < \lambda < \lambda_1(L_{\gamma,\alpha})$, and $0 \leq \gamma < \gamma_H(\alpha)$. We consider $2 < q < 2^*_\alpha$, $h \in C^0(\overline{\Omega})$ and $h \geq 0$. Then, there exists a positive solution $u \in H_0^{\frac{\alpha}{2}}(\Omega)$ to (5.1) under one of the following conditions:*

(1) $0 \leq \gamma \leq \gamma_{crit}(\alpha)$ *and* $h(0) > 0$.

(2) $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$ *and* $\begin{cases} h(0) > 0 & \text{if } q > q_{crit} \\ c_1 h(0) + c_2 m^{\alpha}_{\gamma,\lambda} > 0 & \text{if } q = q_{crit} \\ m^{\alpha}_{\gamma,\lambda} > 0 & \text{if } q < q_{crit}, \end{cases}$

where c_1, c_2 are two positive constant and can be computed explicitly (see Section 5.4), and $q_{crit} = 2^*_\alpha - 2\frac{\beta_+ - \beta_-}{n-\alpha} \in (2, 2^*_\alpha)$.

Remark 5.2. When the operator $L_{\gamma,\alpha}$ is non-critical, our condition depends only on the non-linear perturbation h ($i.e., h(0) > 0$), and not on the positivity of λ, which was the case in the non-perturbed case.

For $\gamma < \gamma_H(\alpha)$, the same argument as in the beginning of Section 3.2 coupled with the fractional Hardy inequality yield that

$$|||u||| = \left(\frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}} \, dxdy - \gamma \int_{\Omega} \frac{u^2}{|x|^\alpha} \, dx \right)^{\frac{1}{2}}$$

is a well defined norm on $H_0^{\frac{\alpha}{2}}(\Omega)$, and is equivalent to the norm $\|u\|_{H_0^{\frac{\alpha}{2}}(\Omega)}$.
We shall consider the following functional $\Phi : H_0^{\frac{\alpha}{2}}(\Omega) \to \mathbb{R}$ whose critical points are solutions for (5.1): For $u \in H_0^{\frac{\alpha}{2}}(\Omega)$, let

$$\Phi(u) = \frac{1}{2}|||u|||^2 - \frac{\lambda}{2} \int_{\Omega} u^2 dx - \frac{1}{2_\alpha^*(s)} \int_{\Omega} \frac{u_+^{2_\alpha^*(s)}}{|x|^s} dx - \frac{1}{q} \int_{\Omega} h u_+^q dx,$$

where $u_+ = \max(0, u)$ is the non-negative part of u. Note that any critical point of the functional $\Phi(u)$ is essentially a variational solution of (5.1). Indeed, we have for any $v \in H_0^{\frac{\alpha}{2}}(\Omega)$,

$$\begin{aligned}\langle \Phi'(u), v \rangle = & \frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}} \, dxdy \\ & - \int_{\mathbb{R}^n} (\gamma \frac{u}{|x|^\alpha} + \lambda u + \frac{u_+^{2_\alpha^*(s)-1}}{|x|^s} + h u_+^{q-1}) v \, dx.\end{aligned}$$

5.2 The Palais-Smale condition below a critical threshold

In this section, we prove the following

Proposition 5.3. *If $c < \frac{\alpha-s}{2(n-s)} \mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-s}{\alpha-s}}$, then every Palais-Smale sequence $(u_k)_{k\in\mathbb{N}}$ for Φ at level c has a convergent subsequence in $H_0^{\frac{\alpha}{2}}(\Omega)$.*

Proof of Proposition 5.3. Assume $c < \frac{\alpha-s}{2(n-s)} \mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-s}{\alpha-s}}$ and let$(u_k)_{k\in\mathbb{N}} \in H_0^{\frac{\alpha}{2}}(\Omega)$ be a Palais-Smale sequence for Φ at level c, that is $\Phi(u_k) \to c$ and

$$\Phi'(u_k) \to 0 \quad \text{in } (H_0^{\frac{\alpha}{2}}(\Omega))', \tag{5.2}$$

where $(H_0^{\frac{\alpha}{2}}(\Omega))'$ denotes the dual of $H_0^{\frac{\alpha}{2}}(\Omega)$.
We first prove that $(u_k)_{k\in\mathbb{N}}$ is bounded in $H_0^{\frac{\alpha}{2}}(\Omega)$. One can use $u_k \in H_0^{\frac{\alpha}{2}}(\Omega)$

as a test function in (5.2) to get that

$$|||u_k|||^2-\lambda\int_\Omega u_k^2dx=\int_\Omega\frac{(u_k)_+^{2^*_\alpha(s)}}{|x|^s}dx+\int_\Omega h(u_k)_+^q dx+o(|||u_k|||)\quad\text{as } k\to\infty. \tag{5.3}$$

On the other hand, from the definition of Φ, we deduce that

$$|||u_k|||^2-\lambda\int_\Omega u_k^2dx=2\Phi(u_k)+\frac{2}{2^*_\alpha(s)}\int_\Omega\frac{(u_k)_+^{2^*_\alpha(s)}}{|x|^s}+\frac{2}{q}\int_\Omega h(u_k)_+^q dx. \tag{5.4}$$

It follows from the last two identities that as $k\to\infty$,

$$2\Phi_q(u_k)=\left(1-\frac{2}{2^*_\alpha(s)}\right)\int_\Omega\frac{(u_k)_+^{2^*_\alpha(s)}}{|x|^s}dx+\left(1-\frac{2}{q}\right)\int_\Omega h(u_k)_+^q dx+o(|||u_k|||). \tag{5.5}$$

This coupled with the Palais-Smale condition $\Phi(u_k)\to c$, and the fact that $h\ge 0$ yield

$$\begin{aligned}2c&=\left(1-\frac{2}{2^*_\alpha(s)}\right)\int_\Omega\frac{(u_k)_+^{2^*_\alpha(s)}}{|x|^s}dx+\left(1-\frac{2}{q}\right)\int_\Omega h(u_k)_+^q dx+o(1)\\&\ge\left(1-\frac{2}{2^*_\alpha(s)}\right)\int_\Omega\frac{(u_k)_+^{2^*_\alpha(s)}}{|x|^s}dx+o(1)\quad\text{as } k\to\infty.\end{aligned}$$

Thus,

$$\left(1-\frac{2}{q}\right)\int_\Omega h(u_k)_+^q dx=O(1)\quad\text{as } k\to\infty.$$

We finally obtain

$$|||u_k|||^2-\lambda\int_\Omega u_k^2dx\le O(1)+o(|||u_k|||)\quad\text{as } k\to\infty.$$

Using that $\lambda<\lambda_1(L_{\gamma,\alpha})$ and $\gamma<\gamma_H(\alpha)$, we get that

$$\begin{aligned}0&<\left(1-\frac{\gamma}{\gamma_H(\alpha)}\right)\left(1-\frac{\lambda}{\lambda_1(L_{\gamma,\alpha})}\right)\|u_k\|^2_{H_0^{\frac{\alpha}{2}}(\Omega)}\\&\le\left(1-\frac{\lambda}{\lambda_1(L_{\gamma,\alpha})}\right)|||u_k|||^2\\&\le|||u_k|||^2-\lambda\int_\Omega u_k^2dx\le O(1)+o(|||u_k|||)\quad\text{as } k\to\infty.\end{aligned}$$

We then deduce that $(u_k)_{k\in\mathbb{N}}$ is bounded in $H_0^{\frac{\alpha}{2}}(\Omega)$, which implies that there exists $u \in H_0^{\frac{\alpha}{2}}(\Omega)$ such that, up to a subsequence,

$$\begin{aligned}
&(1) \quad u_k \rightharpoonup u \text{ weakly in } H_0^{\frac{\alpha}{2}}(\Omega).\\
&(2) \quad u_k \to u \text{ strongly in } L^{p_1}(\Omega) \ \text{ for all } p_1 \in [2, 2_\alpha^*).\\
&(3) \quad u_k \to u \text{ strongly in } L^{p_2}(\Omega, |x|^{-s}dx) \ \text{ for all } p_2 \in [2, 2_\alpha^*(s)).
\end{aligned} \tag{5.6}$$

We now claim that, up to a subsequence, we have

$$|||u_k - u|||^2 = \int_\Omega \frac{(u_k - u)_+^{2_\alpha^*(s)}}{|x|^s} dx + o(1) \quad \text{ as } k \to \infty, \tag{5.7}$$

and

$$\frac{\alpha - s}{2(n-s)} |||u_k - u|||^2 \leq c + o(1) \quad \text{ as } k \to \infty. \tag{5.8}$$

Indeed, straightforward computations yield

$$\begin{aligned}
o(1) &= \langle \Phi'(u_k) - \Phi'(u), u_k - u \rangle \\
&= |||u_k - u|||^2 - \lambda \int_\Omega (u_k - u)^2 \\
&\quad - \int_\Omega (u_k - u) \frac{\left((u_k)_+^{2_\alpha^*(s)-1} - u_+^{2_\alpha^*(s)-1} \right)}{|x|^s} dx \\
&\quad + \int_\Omega h(u_k - u) \left[(u_k)_+^{q-1} - u_+^{q-1} \right] dx \quad \text{ as } k \to \infty.
\end{aligned} \tag{5.9}$$

We first write

$$\begin{aligned}
\int_\Omega (u_k - u) \frac{\left((u_k)_+^{2_\alpha^*(s)-1} - u_+^{2_\alpha^*(s)-1} \right)}{|x|^s} dx &= \int_\Omega \frac{(u_k)_+^{2_\alpha^*(s)}}{|x|^s} dx - \int_\Omega u_k \frac{u_+^{2_\alpha^*(s)-1}}{|x|^s} dx \\
&\quad - \int_\Omega u \frac{(u_k)_+^{2_\alpha^*(s)-1}}{|x|^s} dx + \int_\Omega \frac{u_+^{2_\alpha^*(s)}}{|x|^s} dx.
\end{aligned}$$

It now follows from integral theory that

$$\lim_{k\to\infty} \int_\Omega u_k \frac{u_+^{2_\alpha^*(s)-1}}{|x|^s} dx = \int_\Omega \frac{u_+^{2_\alpha^*(s)}}{|x|^s} dx = \lim_{k\to\infty} \int_\Omega u \frac{(u_k)_+^{2_\alpha^*(s)-1}}{|x|^s} dx.$$

So, we get that

$$\int_\Omega (u_k - u) \frac{\left((u_k)_+^{2_\alpha^*(s)-1} - u_+^{2_\alpha^*(s)-1} \right)}{|x|^s} dx = \int_\Omega \frac{(u_k)_+^{2_\alpha^*(s)}}{|x|^s} dx - \int_\Omega \frac{u_+^{2_\alpha^*(s)}}{|x|^s} dx.$$

In order to deal with the right hand side of the last identity, we use the following basic inequality:

$$\left|(u_k)_+^{2_\alpha^*(s)} - u_+^{2_\alpha^*(s)} - (u_k-u)_+^{2_\alpha^*(s)}\right| \leq c\left(u_+^{2_\alpha^*(s)-1}|u_k-u|+(u_k-u)_+^{2_\alpha^*(s)-1}|u|\right),$$

for some constant $c>0$. We multiply both sides of the above inequality by $|x|^{-s}$ and take integral over Ω, and then use (5.6) to get that

$$\lim_{k\to\infty}\int_\Omega \frac{(u_k)_+^{2_\alpha^*(s)} - (u_k-u)_+^{2_\alpha^*(s)}}{|x|^s}dx = \int_\Omega \frac{u_+^{2_\alpha^*(s)}}{|x|^s}dx.$$

We therefore have

$$\int_\Omega (u_k-u)\frac{\left((u_k)_+^{2_\alpha^*(s)-1} - u_+^{2_\alpha^*(s)-1}\right)}{|x|^s}dx = \int_\Omega \frac{(u_k-u)_+^{2_\alpha^*(s)}}{|x|^s}dx+o(1) \text{ as } k\to\infty.$$

In addition, the embeddings (5.6) yield that

$$\int_\Omega (u_k-u)^2 = o(1) \quad \text{ as } k\to\infty,$$

and

$$\int_\Omega h(u_k-u)\left[(u_k)_+^{q-1} - u_+^{q-1}\right]dx = \int_\Omega (u_k-u)_+^q dx + o(1) = o(1) \text{ as } k\to\infty.$$

Plugging back the last three estimates into (5.9) gives (5.7). On the other hand, since u is a weak solution of (5.1) then $\Phi(u)\geq 0$, and since $\Phi(u_k)\to c$ as $k\to\infty$, it follows that $\frac{\alpha-s}{2(n-s)}\|u_k-u\|^2\leq c+o(1)$. This proves the claim. We now show that

$$\lim_{k\to\infty} u_k = u \text{ in } H_0^{\frac{\alpha}{2}}(\Omega). \tag{5.10}$$

Indeed, test the inequality (1.5) on u_k-u, and use (5.6) and Proposition 4.15 to obtain that

$$\int_\Omega \frac{(u_k-u)_+^{2_\alpha^*(s)}}{|x|^s}dx \leq \mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{-\frac{2_\alpha^*(s)}{2}}|||u_k-u|||^{2_\alpha^*(s)}+o(1). \tag{5.11}$$

Combining this with (5.7), we get

$$|||u_k-u|||^2\left(1-\mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{-\frac{2_\alpha^*(s)}{2}}|||u_k-u|||^{2_\alpha^*(s)-2}+o(1)\right)\leq o(1).$$

It then follows from the last inequality and (5.8) that

$$\left(1-\mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{-\frac{2^*_\alpha(s)}{2}}\Big(\frac{2(n-s)}{\alpha-s}c\Big)^{\frac{2^*_\alpha(s)-2}{2}}+o(1)\right)|||u_k-u|||^2\le o(1).$$

Note that the assumption $c<\frac{\alpha-s}{2(n-s)}\mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-s}{\alpha-s}}$ implies that

$$\Big(\frac{2(n-s)}{\alpha-s}c\Big)^{\frac{2^*_\alpha(s)-2}{2}}<\mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{2^*_\alpha(s)}{2}},$$

and therefore

$$\left(1-\mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{-\frac{2^*_\alpha(s)}{2}}\Big(\frac{2(n-s)}{\alpha-s}c\Big)^{\frac{2^*_\alpha(s)-2}{2}}\right)>0.$$

Thus, $|||u_k-u|||\to 0$ as $k\to\infty$, and this proves (5.10).
Finally, we have that $\Phi(u)=c$, since the functional is continuous on $H_0^{\frac{\alpha}{2}}(\Omega)$.
□

5.3 Mountain pass geometry and existence of a Palais-Smale sequence

Proposition 5.4. *For every $w\in H_0^{\frac{\alpha}{2}}(\Omega)\setminus\{0\}$ with $w\ge 0$, there exists an energy level c, with*

$$0<c\le\sup_{t\ge 0}\Phi(tw),\tag{5.12}$$

and a Palais-Smale sequence $(u_k)_k$ for Φ at level c, that is

$$\Phi(u_k)\to c\quad\textit{and}\quad\Phi'(u_k)\to 0\ \textit{in}\ (H_0^{\frac{\alpha}{2}}(\Omega))'.$$

Proof. We show that the functional Φ satisfies the hypotheses of the mountain pass lemma 3.6. It is standard to show that $\Phi\in C^1(H_0^{\frac{\alpha}{2}}(\Omega))$ and clearly $\Phi(0)=0$, so that (a) of Lemma 3.6 is satisfied.

For (b), we show that 0 is a strict local minimum. Indeed, by the definition of $\lambda_1(L_{\gamma,\alpha})$ and $\mu_{\gamma,s,\alpha}(\Omega)$, we have that

$$\lambda_1(L_{\gamma,\alpha})\int_\Omega|w|^2dx\le|||w|||^2\quad\text{and}\quad\mu_{\gamma,s,\alpha}(\mathbb{R}^n)\Big(\int_\Omega\frac{|w|^{2^*_\alpha(s)}}{|x|^s}dx\Big)^{\frac{2}{2^*_\alpha(s)}}\le|||w|||^2.$$

In addition, it follows from $(5.6)_2$ that there exists a positive constant $S > 0$ such that

$$S(\int_\Omega h|w|^q dx)^{\frac{2}{q}} \leq |||w|||^2.$$

Hence,

$$\begin{aligned}\Phi(w) &\geq \frac{1}{2}|||w|||^2 - \frac{1}{2}\frac{\lambda}{\lambda_1(L_{\gamma,\alpha})}|||w|||^2 - \frac{1}{q}S^{-\frac{q}{2}}|||w|||^q \\ &\qquad - \frac{1}{2^*_\alpha(s)}\mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{-\frac{2^*_\alpha(s)}{2}}|||w|||^{2^*_\alpha(s)} \\ &= |||w|||^2\Bigg(\frac{1}{2}(1-\frac{\lambda}{\lambda_1(L_{\gamma,\alpha})}) - \frac{1}{q}S^{-\frac{q}{2}}|||w|||^{q-2} \\ &\qquad - \frac{1}{2^*_\alpha(s)}\mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{-\frac{2^*_\alpha(s)}{2}}|||w|||^{2^*_\alpha(s)-2}\Bigg).\end{aligned}$$

Since $\lambda < \lambda_1(L_{\gamma,\alpha})$, $q \in (2, 2^*_\alpha)$ and $s \in [0, \alpha)$, we have that $1 - \frac{\lambda}{\lambda_1(L_{\gamma,\alpha})} > 0$, $q - 2 > 0$ and $2^*_\alpha(s) - 2 > 0$, respectively. Thus, we can find $R > 0$ such that $\Phi(w) \geq \rho$ for all $w \in H_0^{\frac{\alpha}{2}}(\Omega)$ with $\|w\|_{H_0^{\frac{\alpha}{2}}(\Omega)} = R$.
Regarding (c), we have

$$\Phi(tw) = \frac{t^2}{2}|||w|||^2 - \frac{t^2\lambda}{2}\int_\Omega w^2 dx - \frac{t^{2^*_\alpha(s)}}{2^*_\alpha(s)}\int_\Omega \frac{w_+^{2^*_\alpha(s)}}{|x|^s}dx - \frac{t^q}{q}\int_\Omega h w_+^q dx,$$

hence $\lim\limits_{t\to\infty}\Phi(tw) = -\infty$ for any $w \in H_0^{\frac{\alpha}{2}}(\Omega)\setminus\{0\}$ with $w_+ \not\equiv 0$, which means that there exists $t_w > 0$ such that $\|t_w w\|_{H_0^{\frac{\alpha}{2}}(\Omega)} > R$ and $\Phi(tw) < 0$, for $t \geq t_w$. In other words,

$$0 < \rho \leq \inf\{\Phi(w); \|w\|_{H_0^{\frac{\alpha}{2}}(\Omega)} = R\} \leq c = \inf_{\gamma\in\mathcal{F}}\sup_{t\in[0,1]}\Phi(\gamma(t)) \leq \sup_{t\geq 0}\Phi(tw),$$

where $\mathcal{F}$ is the class of all path $\gamma \in C([0,1]; H_0^{\frac{\alpha}{2}}(\Omega))$ with $\gamma(0) = 0$ and $\gamma(1) = t_w w$.
The rest follows from the Ambrosetti-Rabinowitz lemma.

□

5.4 Proof of Theorem 5.1

This section is devoted to prove Theorem 5.1. Since Φ satisfies the Palais-Smale condition only up to level $\frac{\alpha-s}{2(n-s)}\mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-s}{\alpha-s}}$, we need to check which conditions on γ, q, h and the mass of the domain Ω guarantee that there exists a $w \in H_0^{\frac{\alpha}{2}}(\Omega)$ such that

$$\sup_{t\geq 0} \Phi(tw) < \frac{\alpha - s}{2(n-s)} \mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-s}{\alpha-s}}.$$

We shall use the test functions ηu_ϵ(resp., T_ϵ) constructed in Section 4.6 for the case when the operator $L_{\gamma,\alpha}$ is non-critical (resp., critical) to obtain the general condition of existence.

One can summarize the definition and properties of the test-functions as follows (see also Section 4.6):

$$v_\epsilon = \begin{cases} U_\epsilon := \eta u_\epsilon & \text{if } 0 \leq \gamma \leq \gamma_{crit}(\alpha) \\ T_\epsilon := U_\epsilon + \epsilon^{\frac{\beta_+ - \beta_-}{2}} g(x) & \text{if } \gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha), \end{cases} \tag{5.13}$$

where the cut-off function $\eta \in C_0^\infty(\Omega)$ verifies (4.37) and

$$u_\epsilon(x) := \epsilon^{-\frac{n-\alpha}{2}} U(\epsilon^{-1}x) \text{ for } x \in \mathbb{R}^n \setminus \{0\},$$

and $U \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is such that $U > 0$, $U \in C^1(\mathbb{R}^n \setminus \{0\})$ and satisfies (4.38) for some $\kappa > 0$ and also (4.39). Also, the function g satisfies

$$g(x) = \frac{m^\alpha_{\gamma,\lambda}}{|x|^{\beta_-}} + o\left(\frac{1}{|x|^{\beta_-}}\right) \text{ as } x \to 0, \text{ and } |g(x)| \leq C|x|^{-\beta_-} \text{ for all } x \in \Omega.$$

We refer the readers to Subsection 4.6.3 for the definition and properties of $g(x)$ in detail.

We now prove the following proposition which plays a crucial role in the proof of Theorem 5.1.

Proposition 5.5. *There exists $\tau_l > 0$ for $l = 1,..,5$ such that*

1) If $0 \leq \gamma \leq \gamma_{crit}(\alpha)$, then

$$\sup_{t\geq 0} \Phi(tv_\epsilon) = \Upsilon - \tau_1 h(0) \epsilon^{n-q\frac{n-\alpha}{2}} + o(\epsilon^{n-q\frac{n-\alpha}{2}}); \tag{5.14}$$

2) If $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$, then

$$\sup_{t\geq 0} \Phi(tv_\epsilon) = \Upsilon + \begin{cases} -\tau_2 h(0)\epsilon^{n-q\frac{n-\alpha}{2}} + o(\epsilon^{n-q\frac{n-\alpha}{2}}) & \text{if } q > q_{crit} \\ -(\tau_3 h(0) + \tau_4 m^\alpha_{\gamma,\lambda})\epsilon^{\beta_+-\beta_-} + o(\epsilon^{\beta_+-\beta_-}) & \text{if } q = q_{crit} \\ -\tau_5 m^\alpha_{\gamma,\lambda}\epsilon^{\beta_+-\beta_-} + o(\epsilon^{\beta_+-\beta_-}) & \text{if } q < q_{crit}, \end{cases} \tag{5.15}$$

where $\Upsilon := \frac{\alpha-s}{2(n-s)}\mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-s}{\alpha-s}}$ and $q_{crit} := 2^*_\alpha - 2\frac{\beta_+-\beta_-}{n-\alpha} \in (2, 2^*_\alpha)$.

Proof of Proposition 5.5. We expand $\Phi(tv_\epsilon)$ in the following way:

$$\Phi(tv_\epsilon) = \frac{t^2}{2} I_\epsilon - \frac{t^{2^*_\alpha(s)}}{2^*_\alpha(s)} J_\epsilon - \frac{t^q}{q} K_\epsilon \quad \text{ as } \epsilon \to 0,$$

where

$$I_\epsilon := |||v_\epsilon|||^2 - \lambda \|v_\epsilon\|^2_{L^2(\Omega)}, \; J_\epsilon := \int_\Omega \frac{|v_\epsilon|^{2^*_\alpha(s)}}{|x|^s} dx \text{ and } K_\epsilon := \int_\Omega h|v_\epsilon|^q dx.$$

Here is a summary of the estimates obtained in Section 4.6 which will be used freely in this section:

Let $E_U := \int_{\mathbb{R}^n} \frac{U^{2^*_\alpha(s)}}{|x|^s} dx$. There exist positive constant c_1, c_2, c_3 such that

$$I_\epsilon = \begin{cases} \kappa E_U - c_1 \lambda \epsilon^\alpha + o(\epsilon^\alpha) & \text{if } 0 \leq \gamma < \gamma_{crit}(\alpha) \\ \kappa E_U - c_2 \lambda \epsilon^\alpha \ln \epsilon^{-1} + o(\epsilon^\alpha \ln \epsilon^{-1}) & \text{if } \gamma = \gamma_{crit}(\alpha) \\ \kappa E_U - c_3 m^\alpha_{\gamma,\lambda} \epsilon^{\beta_+ - \beta_-} + 2\kappa \epsilon^{\frac{\beta_+ - \beta_-}{2}} \theta_\epsilon + o(\epsilon^{\beta_+ - \beta_-}) & \text{if } \gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha), \end{cases} \tag{5.16}$$

as $\epsilon \to 0$, and

$$J_\epsilon = \begin{cases} E_U + o(\epsilon^{\beta_+ - \beta_-}) & \text{if } 0 \leq \gamma \leq \gamma_{crit}(\alpha) \\ E_U + 2^*_\alpha(s) \epsilon^{\frac{\beta_+ - \beta_-}{2}} \theta_\epsilon + o(\epsilon^{\beta_+ - \beta_-}) & \text{if } \gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha), \end{cases} \tag{5.17}$$

as $\epsilon \to 0$. Here, $\theta_\epsilon := \int_{\mathbb{R}^n} \frac{u_\epsilon^{2^*_\alpha(s)-1} \eta g}{|x|^s} dx$ and we have $\lim_{\epsilon \to 0} \theta_\epsilon = 0$; see (4.59). We are then left with estimating K_ϵ.

Estimate for K_ϵ : We will consider two following cases.

Case 1: $0 \leq \gamma \leq \gamma_{crit}(\alpha)$. We split K_ϵ into two integrals as follows

$$K_\epsilon = \int_\Omega h|U_\epsilon|^q dx = \int_{B_\delta} h|U_\epsilon|^q dx + \int_{\Omega \setminus B_\delta} h|U_\epsilon|^q dx.$$

We start by estimating the first term:

$$\begin{aligned}
\int_{B_\delta} h|U_\epsilon|^q dx &= \epsilon^{q\frac{\alpha-n}{2}} \int_{B_\delta} h(x)|U(\frac{x}{\epsilon})|^q dx \\
&= \epsilon^{n-q\frac{n-\alpha}{2}} \int_{B_{\frac{\delta}{\epsilon}}} h(\epsilon X)|U(X)|^q dX \\
&= \epsilon^{n-q\frac{n-\alpha}{2}} h(0) \int_{\mathbb{R}^n} |U(X)|^q dX \\
&\quad + \epsilon^{n-q\frac{n-\alpha}{2}} \int_{\mathbb{R}^n \setminus B_{\frac{\delta}{\epsilon}}} h(\epsilon X)|U(X)|^q dX.
\end{aligned}$$

Note that we used the change of variable $x = \epsilon X$. From the asymptotic (4.39) and the fact that $q > 2$, it then follows that

$$\begin{aligned}
\int_{B_\delta} h|U_\epsilon|^q dx &= \epsilon^{n-q\frac{n-\alpha}{2}} h(0) \int_{\mathbb{R}^n} |U(X)|^q dX + O(\epsilon^{q\frac{\beta_+-\beta_-}{2}}) \\
&= \epsilon^{n-q\frac{n-\alpha}{2}} h(0) \int_{\mathbb{R}^n} |U(X)|^q dX + o(\epsilon^{\beta_+-\beta_-}).
\end{aligned}$$

Following the same argument that we treat the second integral in the last term yields that

$$\int_{\Omega\setminus B_\delta} h|U_\epsilon|^q dx = O(\epsilon^{q\frac{\beta_+-\beta_-}{2}}) = o(\epsilon^{\beta_+-\beta_-}).$$

Therefore,

$$\int_{\Omega} h|U_\epsilon|^q dx = \epsilon^{n-q\frac{n-\alpha}{2}} h(0) \left[\int_{\mathbb{R}^n} |U(X)|^q dX\right] + o(\epsilon^{\beta_+-\beta_-}). \tag{5.18}$$

It now follows from (4.50) that

$$\beta_+ - \beta_- \geq \alpha \quad \text{if } 0 \leq \gamma \leq \gamma_{crit}(\alpha).$$

On the other hand, the condition $2 < q < 2^*_\alpha$ implies that

$$0 < n - q\frac{n-\alpha}{2} < \alpha.$$

Combining the last two inequalities, we then get that

$$n - q\frac{n-\alpha}{2} < \beta_+ - \beta_-,$$

and therefore

$$K_\epsilon = \epsilon^{n-q\frac{n-\alpha}{2}} h(0) \left[\int_{\mathbb{R}^n} |U(X)|^q dX\right] + o(\epsilon^{n-q\frac{n-\alpha}{2}}), \tag{5.19}$$

when $0 \le \gamma \le \gamma_{crit}(\alpha)$.

Case 2: $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$. In order to estimate K_ϵ in the critical case, we need the following inequality: For $q > 2$, there exists $C = C(q) > 0$ such that

$$||X+Y|^q - |X|^q| - qXY|X|^{q-2}| \le C(|X|^{q-2}Y^2 + |Y|^q) \quad \text{for all } X, Y \in \mathbb{R}.$$

We write

$$K_\epsilon = \int_\Omega h|T_\epsilon|^q dx = \int_{B_\delta} h|U_\epsilon + \epsilon^{\frac{\beta_+ - \beta_-}{2}} g(x)|^q dx + O(\epsilon^{q\frac{\beta_+ - \beta_-}{2}}),$$

where the last term came from the fact that

$$\int_{\Omega \setminus B_\delta} h|T_\epsilon|^q dx = O(\epsilon^{q\frac{\beta_+ - \beta_-}{2}}) = o(\epsilon^{\beta_+ - \beta_-}).$$

Let now $X = U_\epsilon$ and $Y = \epsilon^{\frac{\beta_+ - \beta_-}{2}} g(x)$ in the above inequality. Taking integral from both sides then leads us to

$$\int_{B_\delta} h|U_\epsilon + \epsilon^{\frac{\beta_+ - \beta_-}{2}} g(x)|^q dx = \int_{B_\delta} h|U_\epsilon|^q dx + q\epsilon^{\frac{\beta_+ - \beta_-}{2}} \int_{B_\delta} hg|U_\epsilon|^{q-1} dx + R_\epsilon,$$

where

$$\begin{aligned} R_\epsilon &= O\left(\epsilon^{\beta_+ - \beta_-} \int_{B_\delta} h|U_\epsilon|^{q-2} g^2 dx + \epsilon^{q\frac{\beta_+ - \beta_-}{2}} \int_{B_\delta} hg^q dx\right) \\ &= O(\epsilon^{q\frac{\beta_+ - \beta_-}{2}}) = o(\epsilon^{\beta_+ - \beta_-}). \end{aligned} \tag{5.20}$$

Regarding the second term, we have

$$\begin{aligned} q\epsilon^{\frac{\beta_+ - \beta_-}{2}} \int_{B_\delta} hg|U_\epsilon|^{q-1} dx &= O(\epsilon^{(\beta_+ - \beta_-) + (n - q\frac{n-\alpha}{2})}) + O(\epsilon^{q\frac{\beta_+ - \beta_-}{2}}) \\ &= o(\epsilon^{\beta_+ - \beta_-}) \qquad \text{for all } q \in (2, 2^*_\alpha). \end{aligned} \tag{5.21}$$

Combining (5.18), (5.20) and (5.21), we get that there exist a constant $C > 0$ such that

$$K_\epsilon = Ch(0)\epsilon^{n-q\frac{n-\alpha}{2}} + o(\epsilon^{n-q\frac{n-\alpha}{2}}) + o(\epsilon^{\beta_+ - \beta_-}) \quad \text{if } \gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha).$$

We point out that the situation in the critical case is more delicate, and unlike the non-critical case, we have that both

$$\beta_+ - \beta_- \text{ and } n - q\frac{n-\alpha}{2} \text{ are in the interval } (0, \alpha).$$

Therefore, there is a competition between the terms $\epsilon^{\beta_+ - \beta_-}$ and $\epsilon^{n - q\frac{n-\alpha}{2}}$. In order to find the threshold, namely q_{crit}, we equate the exponents of the ϵ terms, and solve the equation for q to get that

$$q_{crit} = 2^*_\alpha - 2\frac{\beta_+ - \beta_-}{n - \alpha}.$$

One should note that $q_{crit} \in (2, 2^*_\alpha)$, since $\alpha > \beta_+ - \beta_- > 0$ in the critical case. This implies that

$$\begin{cases} o(\epsilon^{n-q\frac{n-\alpha}{2}}) + o(\epsilon^{\beta_+ - \beta_-}) = o(\epsilon^{n-q\frac{n-\alpha}{2}}) & \text{if } q > q_{crit} \\ o(\epsilon^{n-q\frac{n-\alpha}{2}}) + o(\epsilon^{\beta_+ - \beta_-}) = o(\epsilon^{n-q\frac{n-\alpha}{2}}) = o(\epsilon^{\beta_+ - \beta_-}) & \text{if } q = q_{crit} \\ o(\epsilon^{n-q\frac{n-\alpha}{2}}) + o(\epsilon^{\beta_+ - \beta_-}) = o(\epsilon^{\beta_+ - \beta_-}) & \text{if } q < q_{crit}. \end{cases}$$

We finally obtain

$$K_\epsilon = \begin{cases} c_4 h(0)\epsilon^{n-q\frac{n-\alpha}{2}} + o(\epsilon^{n-q\frac{n-\alpha}{2}}) & \text{if } q > q_{crit} \\ c_5 h(0)\epsilon^{\beta_+ - \beta_-} + o(\epsilon^{\beta_+ - \beta_-}) & \text{if } q = q_{crit} \\ o(\epsilon^{\beta_+ - \beta_-}) & \text{if } q < q_{crit}. \end{cases} \tag{5.22}$$

for some $c_4, c_5 > 0$, and as long as $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$.

We now define

$$I_0 := \lim_{\epsilon \to 0} I_\epsilon = \kappa \int_{\mathbb{R}^n} \frac{U^{2^*_\alpha(s)}}{|x|^s} dx \quad \text{and} \quad J_0 := \lim_{\epsilon \to 0} J_\epsilon = \int_{\mathbb{R}^n} \frac{U^{2^*_\alpha(s)}}{|x|^s} dx,$$

and it is easy to check that

$$\lim_{\epsilon \to 0} K_\epsilon = 0 \quad \text{ for all cases.}$$

In the next step, we claim that, up to a subsequence of $(u_\epsilon)_{\epsilon > 0}$, there exists $T_0 := T_0(n, s, \alpha) > 0$ such that

$$\sup_{t \geq 0} \Phi(tv_\epsilon) = d(\Psi(v_\epsilon))^{\frac{2^*_\alpha(s)}{2^*_\alpha(s) - 2}} - \frac{T_0^q}{q} K_\epsilon + o(K_\epsilon), \tag{5.23}$$

where

$$\Psi(v_\epsilon) = \frac{|||v_\epsilon|||^2 - \lambda \|v_\epsilon\|^2_{L^2(\Omega)}}{\left(\int_\Omega \frac{|v_\epsilon|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}}} = \frac{I_\epsilon}{J_\epsilon^{\frac{2}{2^*_\alpha(s)}}}.$$

The proof of this claim goes exactly as **Step II** in [44, Proposition 3]. We omit it here.
Let us now compute $\Psi(v_\epsilon)$. It follows from (4.51), (4.52) and (4.66) that there exist positive constants c_6, c_7, c_8 such that

$$\Psi(v_\epsilon) = \mu_{\gamma,s,\alpha}(\mathbb{R}^n)\Big(1 + \Theta_{\epsilon,\gamma}\Big) \tag{5.24}$$

where

$$\Theta_{\epsilon,\gamma} = \begin{cases} -c_6\lambda\epsilon^\alpha + o(\epsilon^\alpha) & \text{if } 0 \leq \gamma < \gamma_{crit}(\alpha) \\ -c_7\lambda\epsilon^\alpha \ln\epsilon^{-1} + o(\epsilon^\alpha \ln\epsilon^{-1}) & \text{if } \gamma = \gamma_{crit}(\alpha) \\ -c_8 m^\alpha_{\gamma,\lambda}\epsilon^{\beta_+-\beta_-} + o(\epsilon^{\beta_+-\beta_-}) & \text{if } \gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha). \end{cases}$$

We are now going to estimate $\sup\limits_{t\geq 0}\Phi(tv_\epsilon)$. This will be done again by considering two cases:

Case 1: $0 \leq \gamma \leq \gamma_{crit}(\alpha)$. In this case, plugging (5.19) and (5.24) into (5.23) implies that there exist constants $c_9, c_{10}, c_{11}, c_{12} > 0$ such that

$$\begin{aligned}\sup_{t\geq 0}\Phi(tv_\epsilon) &= \frac{\alpha - s}{2(n-s)}\mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-s}{\alpha-s}} \\ &\quad - c_9\lambda\epsilon^\alpha - c_{10}h(0)\epsilon^{n-q\frac{n-\alpha}{2}} + o(\epsilon^\alpha) + o(\epsilon^{n-q\frac{n-\alpha}{2}}),\end{aligned}$$

when $0 \leq \gamma < \gamma_{crit}(\alpha)$, and

$$\begin{aligned}\sup_{t\geq 0}\Phi(tv_\epsilon) &= \frac{\alpha - s}{2(n-s)}\mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-s}{\alpha-s}} \\ &\quad - c_{11}\lambda\epsilon^\alpha\ln(\epsilon^{-1}) - c_{12}h(0)\epsilon^{n-q\frac{n-\alpha}{2}} + o(\epsilon^\alpha\ln(\epsilon^{-1})) + o(\epsilon^{n-q\frac{n-\alpha}{2}}),\end{aligned}$$

when $\gamma = \gamma_{crit}(\alpha)$.
Recall that $\alpha > n - q\frac{n-\alpha}{2}$, since $q > 2$. This implies that

$$o(\epsilon^\alpha) + o(\epsilon^{n-q\frac{n-\alpha}{2}}) = o(\epsilon^{n-q\frac{n-\alpha}{2}}).$$

Thus, there exist a positive constant τ_1 such that, for every $0 \leq \gamma \leq \gamma_{crit}(\alpha)$, we have

$$\sup_{t\geq 0}\Phi(tv_\epsilon) = \Upsilon - \tau_1 h(0)\epsilon^{n-q\frac{n-\alpha}{2}} + o(\epsilon^{n-q\frac{n-\alpha}{2}}),$$

where $\Upsilon := \frac{\alpha-s}{2(n-s)}\mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-s}{\alpha-s}}$.

Case 2: $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$. The critical case needs a careful analysis as a new phenomena happens in this situation. We shall show that there is a competition between the geometry of the domain, *the mass* (i.e., $\epsilon^{\beta_+-\beta_-}m^{\alpha}_{\gamma,\lambda}$), and the non-linear perturbation (i.e., $\epsilon^{n-q\frac{n-\alpha}{2}}h(0)$). Indeed, it follows from plugging (5.22) and (5.24) into (5.23) that there exist constants $c_{12}, c_{13}, c_{14} > 0$ such that

$$\sup_{t\geq 0}\Phi(tv_\epsilon) = \Upsilon - c_{12}m^{\alpha}_{\gamma,\lambda}\epsilon^{\beta_+-\beta_-} + o(\epsilon^{\beta_+-\beta_-})$$
$$+\begin{cases} -c_{13}h(0)\epsilon^{n-q\frac{n-\alpha}{2}} + o(\epsilon^{n-q\frac{n-\alpha}{2}}) & \text{if } q > q_{crit} \\ -c_{14}h(0)\epsilon^{\beta_+-\beta_-} + o(\epsilon^{\beta_+-\beta_-}) & \text{if } q = q_{crit} \\ o(\epsilon^{\beta_+-\beta_-}) & \text{if } q < q_{crit}. \end{cases}$$

Following our analysis in the critical case of estimating K_ϵ, one can then summarize the competition results as follows.

Competitive Terms	$q > q_{crit}$	$q = q_{crit}$	$q < q_{crit}$.
$\epsilon^{n-q\frac{n-\alpha}{2}}h(0)$	Dominate	Equally Dominate	$\times$
$\epsilon^{\beta_+-\beta_-}m^{\alpha}_{\gamma,\lambda}$	$\times$	Equally Dominate	Dominate

Therefore, we finally deduce that there exists $\tau_l > 0$ for $l = 2, .., 5$ such that

$$\sup_{t\geq 0}\Phi(tv_\epsilon) = \Upsilon + \begin{cases} -\tau_2 h(0)\epsilon^{n-q\frac{n-\alpha}{2}} + o(\epsilon^{n-q\frac{n-\alpha}{2}}) & \text{if } q > q_{crit} \\ -(\tau_3 h(0) + \tau_4 m^{\alpha}_{\gamma,\lambda})\epsilon^{\beta_+-\beta_-} + o(\epsilon^{\beta_+-\beta_-}) & \text{if } q = q_{crit} \\ -\tau_5 m^{\alpha}_{\gamma,\lambda}\epsilon^{\beta_+-\beta_-} + o(\epsilon^{\beta_+-\beta_-}) & \text{if } q < q_{crit}, \end{cases}$$

where $\Upsilon := \frac{\alpha-s}{2(n-s)}\mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-s}{\alpha-s}}$.

□

Remark 5.6. *We point out that the value q_{crit} corresponds to the value $q = 4$ obtained in [44, Proposition 3]. Indeed, when $\alpha = 2$, $\gamma = 0$ and $n = 3$, our problem turns to the perturbed Hardy-Sobolev equation considered by Jaber [44] in the Riemannian setting. We then have that $\beta_+(0) = n - \alpha = 1$, $\beta_-(0) = 0$, and therefore*

$$q_{crit} = 2^*_\alpha - 2\frac{\beta_+(0) - \beta_-(0)}{n-\alpha} = 6 - 2 = 4.$$

Bibliography

[1] B. Abdellaoui, R. Bentiffour, *Caffarelli-Kohn-Nirenberg type inequalities of fractional order and applications*, preprint.

[2] B. Abdellaoui, M. Medina, I. Peral and A. Primo, *The effect of the Hardy potential in some Caldern-Zygmund properties for the fractional Laplacian*, Journal of Differential Equations 260.11 (2016): 8160-8206.

[3] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973), 349 - 381.

[4] T. Aubin, *Le probleme de Yamabe concernant la courbure scalaire*, Differential Geometry Pescola 1985. Springer Berlin Heidelberg, 1986. 66-72.

[5] B. Barrios, E. Colorado, A. De Pablo, and U. S'anchez, *On some critical problems for the fractional Laplacian operator,* J. Diff. Eqns 252 (2012), 6133-6162.

[6] G. M. Bisci, V. D. Radulescu and R. Servadei, *Variational methods for nonlocal fractional problems*, Vol. 162. Cambridge University Press, 2016.

[7] B. Barrios, M. Medina, and I. Peral, *Some remarks on the solvability of non-local elliptic problems with the Hardy potential*, Communications in Contemporary Mathematics 16, 04 (2014).

[8] M. Bhakta, *A note on semilinear elliptic equation with biharmonic operator and multiple critical nonlinearities*, arXiv preprint arXiv:1405.0162 (2014).

[9] C. Brandle, E. Colorado, A. De Pablo, and U. S'anchez, *A concave-convex elliptic problem involving the fractional Laplacian,* Proc. Roy. Soc. Edinburgh Sect. A, 143 (2013), 39-71.

[10] H. Brezis and E. Lieb, *A relation between point convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. 88 (1983), 486-490. MR 84c:28003

[11] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving criticalSobolev exponents*, Comm. Pure Appl. Math., 36, (1983), 437-477.

[12] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. in Part. Diff. Equa. 32 (2007), 1245-1260.

[13] L. Cafarelli and L. Silvestre, *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math. 62 (2009), 597-638.

[14] L. Cafarelli and L. Silvestre, *Regularity results for nonlocal equations by approximation*, Arch. Rat. Mech. Anal. 200 (2011), 59-88.

[15] F. Catrina and Z. Q. Wang, *On the Caffarelli-Kohn-Nirenberg inequalities sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Communications on Pure and Applied Mathematics 54.2 (2001): 229-258.

[16] A. Chang and M. Gonzalez, *Fractional Laplacian in conformal geometry*, Advances in Mathematics 226.2 (2011): 1410-1432.

[17] J. L. Chern and C. S. Lin, *Minimizers of Caffarelli-Kohn-Nirenberg inequalities with the singularity on the boundary*, Arch. Ration. Mech. Anal. 197 (2010), no. 2, 401-432.

[18] Ph. G. Ciarlet, *Linear and nonlinear functional analysis with applications*, Society for Industrial and Applied Mathematics, Philadelphia, 2013.

[19] A. Cotsiolis and N.K. Tavoularis, *Best constants for Sobolev inequalities for higher order fractional derivatives* J. Math. Anal. Appl., 295 (2004), pp. 225-236.

[20] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhikers guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012), no. 5, 521-573.

[21] S. Dipierro, L. Montoro, I. Peral and B. Sciunzi, *Qualitative properties of positive solutions to nonlocal critical problems involving the Hardy-Leray potential*, arXiv preprint arXiv:1506.07317 (2015).

[22] S. Dipierro and E. Valdinoci, *A density property for fractional weighted Sobolev spaces*, Atti Acca. Na. Lincei Rend. Lincei Math. Appl., availabe at http://arxiv.org/pdf/1501.04918.pdf.

[23] Z. Djadli, E. Hebey, and M. Ledoux, *Paneitz-type operators and applications*, Duke Mathematical Journal 104.1 (2000): 129-170.

[24] O. Druet, *Elliptic equations with critical Sobolev exponents in dimension 3*, Ann. Inst. H. Poincar e Anal. Non Lin eaire 19 (2002), no. 2, 125142.

[25] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. 47 (1974), 324-353.

[26] P. Esposito, and F. Robert, *Mountain pass critical points for Paneitz-Branson operators*, Calculus of Variations and Partial Differential Equations 15.4 (2002): 493-517.

[27] J. F. Escobar, *Sharp constant in a Sobolev trace inequality*, Indiana Math. J., 37 : 687-698, 1988.

[28] E.B. Fabes, C.E. Kenig, and R.P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations 7 (1982) 77-116.

[29] M. M. Fall, *Semilinear elliptic equations for the fractional Laplacian with Hardy potential* (2011); arXiv:1109.5530v4.

[30] M. M. Fall and V. Felli, *Unique continuation property and local asymptotics of solutions to fractional elliptic equations*, Communications in Partial Differential Equations 39.2 (2014): 354-397.

[31] M. M. Fall, I. A. Minlend, and E.A. Thiam, *The role of the mean curvature in a Hardy-Sobolev trace inequality*, arXiv preprint arXiv:1411.4414 (2014).

[32] M. M. Fall and T. Weth, *Nonexistence results for a class of fractional elliptic boundary value problems*, J. Funct. Anal. 263 (2012), 22052227.

[33] R. Filippucci, P. Pucci, and F. Robert, *On a p-Laplace equation with multiple critical nonlinearities*, J. Math. Pures Appl. (9) 91 (2009), no. 2, 156-177.

[34] R. L. Frank, E. H. Lieb, and R. Seiringer, *Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators*, J. Amer. Math. Soc. 21 (4) (2008) 925-950.

[35] N. Ghoussoub and A. Moradifam, *Functional inequalities: New perspectives and new applications*, Mathematical Surveys and Monographs, vol. 187, American Mathematical Society, Providence, RI, 2013.

[36] N. Ghoussoub and F. Robert, *On the Hardy-Schrödinger operator with a boundary singularity*, arXiv preprint arXiv:1410.1913 (2014).

[37] N. Ghoussoub and F. Robert, *The Hardy-Schrödinger operator with interior singularity: The remaining cases*, Submitted (2016). arXiv:1612.08355, Hal Id : hal-01422509.

[38] N. Ghoussoub, F. Robert, S. Shakerian and M. Zhao, *Mass and asymptotics associated to fractional Hardy-Schrödinger operators in critical regimes*, April 2017, https://arxiv.org/pdf/1704.08658.pdf.

[39] N. Ghoussoub and S. Shakerian, *Borderline variational problems involving fractional Laplacians and critical singularities*, Advanced Nonlinear Studies, 15 (2015) 527-555.

[40] N. Ghoussoub and C. Yuan, *Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents*, Trans. Amer. Math. Soc. 12 (2000), 5703-5743.

[41] V. Gol'dshtein and A. Ukhlov, *Weighted Sobolev spaces and embedding theorems*, Trans. Amer. Math. Soc., 361(2009), pp. 3829-3850.

[42] M. d. M. Gonzalez, J. Qing. *Fractional conformal Laplacians and fractional Yamabe problems*, Analysis and PDE 6.7 (2013), 15351576.

[43] I. W. Herbst, *Spectral theory of the operator* $(p^2 + m^2)^{1/2} - Ze^2/r$, Comm. Math. Phys. 53 (1977), no. 3, 285-294.

[44] H. Jaber, *Mountain pass solutions for perturbed Hardy-Sobolev equations on compact manifolds*, Analysis (Berlin) 36 (2016), no. 4, 287296.

[45] E. Jannelli, *The role played by space dimension in elliptic critical problems*, Journal of Differential Equations 156.2 (1999): 407-426.

[46] T. Jin, Y. Li and J. Xiong, *On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions*, arXiv preprint arXiv:1111.1332 (2011).

[47] D. Kang and G. Li, *On the elliptic problems involving multi-singular inverse square potentials and multi-critical Sobolev-Hardy exponents*, Nonlinear Anal., 66 (2007), 1806-1816.

[48] Y. Lei, *Asymptotic properties of positive solutions of the Hardy-Sobolev type equations*, Journal of Differential Equations 254 (2013), 1774-1799

[49] T. Leonori, I. Peral, A. Primo and F Soria, *Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations*, Discrete Contin. Dyn. Syst 35 (12), 6031-6068, 2015

[50] E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. Math. 118 : 349-374, 1983.

[51] E. H. Lieb and M. Loss, Analysis, Volume 14 of Graduate Studies in Mathematics, AMS, 1997.

[52] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case, part 1 and 2*, Revista Mathematica Iberoamericana, 1 : 145-201, 45-121, 1985.

[53] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, part 1 and 2*, Ann. I. H. P., Anal. Nonlin., 1: 109-145, 223-283, 1984.

[54] G. Lu and J. Zhu, *Symmetry and regularity of extremals of an integral equation related to the Hardy-Sobolev inequality*, Calc. Var. Partial Differential Equations 42 (2011), 563-577.

[55] P. Pucci, J. Serrin, *Critical exponents andcritical dimensions for polyharmonic operators*, J.Math.Pures Appl. 69, 55-83 (1990)

[56] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207-226.

[57] Y. J. Park, *Fractional Polya-Szegö inequality*, J. Chungcheong Math. Soc. 24 (2011), no. 2, 267-271.

[58] X. Ros-Oton, J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. 101 (2014), 275-302.

[59] X. Ros-Oton, J. Serra, *Boundary regularity for fully nonlinear integro-differential equations*, preprint arXiv (2014).

[60] S. Salsa, *Partial differential equations in action: From modelling to theory*, Springer-Verlag Italia, Milano, 2008.

[61] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. 20 (1984), no. 2, 479495.

[62] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Communications on pure and applied mathematics 60.1 (2007): 67-112.

[63] R. Servadei and E. Valdinoci, *A Brezis-Nirenberg result for non-local critical equations in low dimension*, Commun. Pure Appl. Anal., 12, no. 6, 2445-2464 (2013).

[64] R. Servadei and E. Valdinoci, *The Brezis-Nirenberg result for the fractional Laplacian*, Transactions of the American Mathematical Society 367.1 (2015): 67-102.

[65] R. Servadei, *A critical fractional Laplace equation in the resonant case*, Topol. Methods Nonlinear Anal 43.1 (2014): 251-267.

[66] S. Shakerian, *Existence results for non-linearly perturbed Hardy-Schrödinger problems: Local and non-local cases*, Submitted (2017).

[67] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. 110 : 353-372, 1976.

[68] B. Xuan and J. Wang, *Existence of a nontrivial weak solution to quasilinear elliptic equations with singular weights and multiple critical exponents*, Nonlinear Analysis: Theory, Methods and Applications 72.9 (2010): 3649-3658.

[69] D. Yafaev, *Sharp constants in the Hardy-Rellich inequalities*, Journal of Functional Analysis, 168:121144, 1999.

[70] J. Yang, *Fractional Sobolev-Hardy inequality in* $\mathbb{R}^N$, Nonlinear Analysis: Theory, Methods and Applications (2014).

[71] J. Yang and X. Yu, *Fractional Hardy-Sobolev elliptic problems*, arXiv preprint arXiv:1503.0021 6 (2015).