Geometry of Random Spaces

Geodesics and Susceptibility

by

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Abstract

This thesis investigates the geometry of random spaces.

Geodesics in Random Surfaces

The *Brownian map,* developed by Le Gall [98] and Miermont [108], is a random metric space arising as the scaling limit of random planar maps. Its construction involves Aldous' [7] *continuum random tree,* the canonical random real tree, and *Brownian motion,* an almost surely continuous but nowhere differentiable path. As a result, the Brownian map is a non-differentiable surface with a fractal geometry that is much closer to that of a real tree than a smooth surface.

A key feature, observed by Le Gall [97], is the *confluence of geodesics* phenomenon, which states that any two geodesics to a typical point coalesce before reaching the point. We show that, in fact, geodesics to anywhere near a typical point pass through a common *confluence point.* This leads to information about special points that had remained largely mysterious.

Our main result is the almost everywhere continuity and uniform stability of the *cut locus* of the Brownian map. We also classify *geodesic networks* that are dense and find the Hausdorff dimension of the set of pairs that are joined by each type of network.

Susceptibility of Random Graphs

Given a graph $G = (V, E)$ and an initial set I of *active* vertices in V, the *r-neighbour bootstrap percolation process,* attributed to Chalupa, Leath and Reich [50], is a cellular automaton that evolves by activating vertices with at least r active neighbours. If all vertices in V are activated eventually, we say that I is *contagious.* A graph with a small contagious set is called *susceptible.*

Bootstrap percolation has been analyzed on several deterministic graphs, such as grids, lattices and trees. More recent work studies the model on random graphs, such as the fundamental Erdős–Rényi [60] graph $\mathcal{G}_{n,p}$.

We identify thresholds for the susceptibility of $\mathcal{G}_{n,p}$, refining approximations by Feige, Krivelevich and Reichman [62]. Along the way, we obtain large deviation estimates that complement central limit theorems of Janson, Łuczak, Turova and Vallier [84]. We also study *graph bootstrap percolation,* a variation due to Bollobás [39]. Our main result identifies the sharp threshold for K_4*-percolation,* solving a problem of Balogh, Bollobás and Morris [24].

Lay Summary

We analyze two aspects of random spaces.

Geodesics in Random Surfaces

The *Brownian map* is a random fractal surface, identified by Le Gall and Miermont. We study *geodesics,* which are shortest paths between points. By strengthening an observation of Le Gall, that geodesics to a typical point in the Brownian map coalesce before reaching the point, we reveal several properties of its rich geometry. In particular, we analyze points from which geodesics leave in different directions but arrive at the same destination.

Susceptibility of Random Graphs

Bootstrap percolation is a cellular automaton, attributed to Chalupa et al., that models an evolving network. A network is called *susceptible* if a small part can influence the whole network. We identify thresholds for the susceptibility of a fundamental random network, called the *Erdős–Rényi graph,* in which all elements are equally likely to be directly connected. This refines approximations by Balogh et al. and Feige et al.

Preface

This thesis is based on articles [11, 12, 13, 91], as introduced in Part I.

Stability of geodesics in the Brownian map [13] is a joint work with Omer Angel (the author's advisor) and Grégory Miermont. This article is the topic of Part II and is to appear in the *Annals of Probability*.

Part III is based on a series of articles. *Thresholds for contagious sets in random graphs* [12] and *Minimal contagious sets in random graphs* [11] are joint works with Omer Angel. Article [12] is to appear in the *Annals of Applied Probability* and [11] is currently under review for publication. *Sharp threshold for K_4-percolation* [91] is an independent work of the author, currently under review for publication.

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Acknowledgments

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One of the most important moments during my graduate studies was the 2012 PIMS-Mprime Summer School in Probability. This event was held at UBC at the end of my M.Sc. studies, and led me to make the decision to continue on towards a Ph.D. I thank Omer Angel and Grégory Miermont for their inspiring lectures, and the summer school organizers, Omer Angel, Ed Perkins and Gordon Slade.

The year after this summer school, I visited Grégory for six weeks at ENS Lyon, while Omer was on sabbatical in Paris. I thank Omer, Grégory and UBC for support with travelling to Lyon, and Grégory for his kind hospitality. During this time, we began working on a project related to Grégory's summer school course that would eventually become a paper, appearing in Part II of this thesis. It has been a great pleasure to be involved in this area of research. I am truly grateful for the guidance and attention I received from Omer and Grégory during this visit, and in the years that followed.

In 2015 I had the opportunity to visit the Isaac Newton Institute for a special semester, titled Random Geometry. I thank Nathanaël Berestycki for hosting me and Killam Trusts and NSERC for their support. I benefited from many seminars, reading groups and the presence of leading researchers. I presented the results in Part II to experts in the field, and made considerable progress towards the work that appears in Part III of this thesis.

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To my parents

Part I

Introduction

Chapter 1

Brownian map

A *planar map (of the sphere)* is a discretization of the 2-dimensional sphere $\mathbb{S}^2$. More specifically, a planar map is a proper embedding (without crossing edges) of a finite, connected planar graph into $\mathbb{S}^2$, viewed up to orientation-preserving homeomorphisms. The *faces* of the map are the connected components of $\mathbb{S}^2$ minus its *vertices* and *edges,* inherited from its underlying graph structure. It is natural to ask what types of objects are obtained in this way, as the number of faces tends to infinity, that is, as the maps become increasingly large. Only very recently has the answer to this question been revealed.

A planar map can be treated as a finite metric space, endowed with the distance from the underlying graph. Since we view planar maps up to orientation-preserving homeomorphisms (that is, a planar map is a certain equivalence class), there is only a finite number of planar maps of size n (with n faces, say). A triangulation/quadrangulation is a map in which all faces are delimited by three/four edges.

The seminal work of Angel and Schramm (2003) [14] was the first to obtain a limiting object associated with random planar maps. In [14] the *infinite volume, local limit* (see Section 1.7.1) of uniformly random triangulations is identified, and named the *uniform infinite planar triangulation,* or UIPT. The UIPQ, corresponding to random quadrangulations, was later developed by Krikun [93]. Although these random infinite lattices can be embedded in $\mathbb{R}^2$, their geometry is very different from that of the Euclidean lattice $\mathbb{Z}^2$. For instance, as shown by Benjamini and Curien [30], simple random walk on the UIPQ is sub-diffusive, typically moving a distance of at most order $n^{1/3}$ (up to a logarithmic factor) after n steps (whereas on $\mathbb{Z}^d$, $d \geq 1$, the typical displacement is of order $n^{1/2}$). Very loosely speaking, these random lattices are "discrete fractals."

In a pioneering study, Chassaing and Schaeffer (2004) [51] showed that typical distances in a uniformly random quadrangulation of size n are of order $n^{1/4}$. With this result in mind, Schramm (2007) [121], in his ICM survey, posed the question of identifying the *scaling limit* of random planar maps (scaling the metric by $n^{-1/4}$) with respect to the *Hausdorff topology* (see Section 1.2.3). This program was recently completed in independent works by Le Gall (2013) [98] and Miermont (2013) [108]. The resulting limit is called the *Brownian map,* due to the fact that its construction involves *Brownian motion,* the canonical uniformly random path. Unlike the UIPQ, the Brownian map is a *finite volume* limit, which is homeomorphic to $\mathbb{S}^2$, as shown by Le Gall and Paulin [100] and Miermont [106]. That being said, like that of the UIPQ, its geometry is far from Euclidean. Indeed, as proved by Le Gall [96], the Brownian map is of Hausdorff dimension 4. It thus exhibits a random, fractal, spherical geometry. In fact, these objects are closely related. Roughly speaking, by scaling distances in the UIPQ by λ, and letting $\lambda \to 0$, an infinite volume (homeomorphic to $\mathbb{R}^2$) variant of the Brownian map, called the *Brownian plane,* is obtained (see Section 1.7.2).

Brownian motion is a universal object, in the sense that it is the scaling limit of many different types of random discrete paths. Similarly, the Brownian map is a universal object of interest. To quote Le Gall (2014) [95], in his ICM survey:

> Just as Brownian motion can be viewed as a purely random continuous curve, the Brownian map seems to be the right model for a purely random surface.

Although the Brownian map has been identified, and several of its fundamental properties are understood, much of its intricate structure has yet to be uncovered. As Le Gall [95] states,

> the Brownian map remains a mysterious object in many respects.

As continues to become increasingly clear, the Brownian map is an important new addition to the modern theory of probability. Notably, a very recent work (and the last in a long series of articles) by Sheffield and

Miller (2016) [109] establishes the equivalence of the Brownian map with the $\sqrt{8/3}$-*Liouville quantum gravity sphere,* another candidate for a canonical uniformly random spherical surface. This result is important, since prior to their work the two objects had their own separate advantages: the Brownian map with its natural *metric* structure, and the $\sqrt{8/3}$-Liouville quantum gravity sphere with its natural *conformal* structure. The results of [109] unify the theories. To quote Sheffield and Miller [109], it is now that case that

> any theorem about the Brownian map is henceforth also a theorem about the $\sqrt{8/3}$-Liouville quantum gravity sphere and vice-versa.

1.1 Our objective

In Part II of this thesis, we develop properties of *geodesics,* or shortest paths, in the Brownian map, and so also equivalently, in the $\sqrt{8/3}$-Liouville quantum gravity sphere. Such paths give insight into the geometry of these fundamental random spaces. See Section 1.8 below for our results.

1.2 Preliminaries

1.2.1 Planar maps

A *rooted planar map (of the sphere)* is an equivalence class of finite, connected, planar graphs with a distinguished oriented edge, embedded in $\mathbb{S}^2$, and viewed up to orientation-preserving homeomorphisms. The orientated edge determines the orientation of the entire map. The initial vertex of the oriented edge is called the *root* of the map. See Figure 1.1.

The *faces* of a planar map are the connected components of $\mathbb{S}^2$ minus the *edges* and *vertices* of the map, given by its underlying graph structure. Each face of a planar map is delimited by a finite number of edges, which we call its *degree*. In particular, a *q-angulation* is a planar map for which all faces are of degree q. We say *triangulation* and *quadrangulation* in the case of $q = 3$ and $q = 4$, respectively.

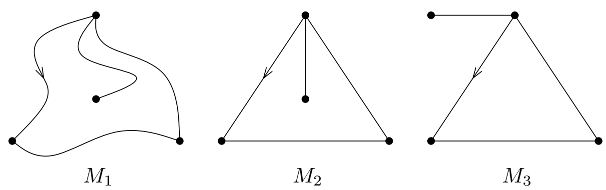

Figure 1.1: Three rooted planar maps: M_1 and M_2 are equivalent to each other, but not to M_3. The underlying graphs are all isomorphic.

1.2.2 CVS-bijection

Let (T, τ) be *rooted plane tree,* where $T = (V, E)$ and $\tau \in V$. We say that (T, τ) is *well-labelled* by a function $\ell : V \to \mathbb{N}$ if $\ell(\tau) = 1$ and $|\ell(a) - \ell(b)| \leq 1$, for all $(a, b) \in E$. We call (T, τ, ℓ) a well-labelled, rooted plane tree.

The *CVS-bijection,* due to Cori and Vauquelin [52] and Schaeffer [119], identifies the set of well-labelled, rooted plane trees with n edges with the set of rooted quadrangulations of the sphere with n faces. One direction of this bijection is as follows: Given (T, τ, ℓ), an extra (disconnected) vertex ρ is added and labelled 0. To begin, an oriented edge is drawn from ρ to τ. Then, proceeding along the *corners* (see Section 1.5.1) of T via the clockwise, *contour-ordered* path around T, an edge is drawn from a corner of T to the next corner with a smaller label, if such a corner exists. If there is no such corner (that is, in the case of a corner with label 1), an edge is instead drawn to ρ. Once the process is complete, and the edges in E are removed (while maintaining the vertices in V), a rooted quadrangulation (Q_T, ρ) on $V \cup \{\rho\}$ is obtained. The oriented edge from ρ to τ determines the orientation of Q_T. See Figure 1.2

We observe that the successive edges drawn by the CVS-bijection from a corner of a vertex $v \in V$ to ρ form a path in Q_T from v to ρ in such a way that the labels of the vertices visited by the path equal the graph distance (in Q_T) to ρ. Hence, for each $v \in V$ with k corners, the CVS-bijection specifies k distinguished geodesics from v to the root of the map ρ leaving from each of its corners.

We note that (more complicated) bijections for other classes of planar

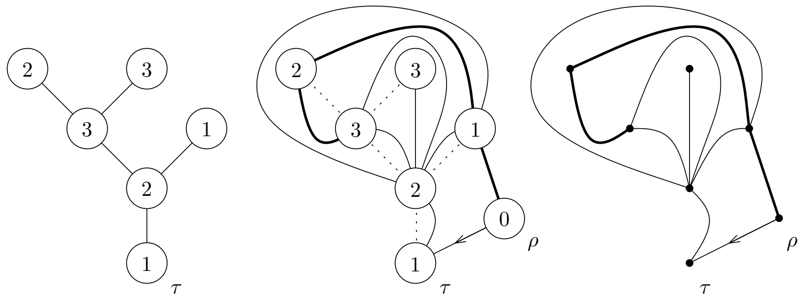

Figure 1.2: From left to right: A well-labelled plane tree (T, τ, ℓ), the CVS-bijection, and the corresponding rooted quadrangulation of the sphere (Q_T, ρ). One geodesic to ρ drawn by the CVS-bijection is highlighted.

maps are also available, see for instance the work of Bouttier, Di Francesco and Guitter [42].

In closing we mention that Tutte [130] (motivated by the four colour problem) developed enumerative methods for planar maps. In particular, by the *quadratic method* of [130], it follows that there are $\frac{2}{n+2}3^n\mathrm{Cat}_n$ quadrangulations of $\mathbb{S}^2$ with n faces, where Cat_n is the nth Catalan number. In light of the CVS-bijection, we see very clearly why this is the case: There are Cat_n trees with n edges, 2 ways to orient the resulting map (either the root edge is oriented from ρ to τ, or vice versa), and *almost* 3^n choices for the labels of the vertices in $V - \{\tau\}$. Since the labels cannot be less than $1 = \ell(\tau)$, there are in fact only $3^n/(n+2)$ ways to "well-label" (T, τ).

1.2.3 Gromov-Hausdorff metric

Let d_{H} denote the *Hausdorff distance* on non-empty, compact subsets of a metric space (S, θ). The *Gromov-Hausdorff metric* d_{GH} (see Edwards [59] and Gromov [73, 74]) on the set of all isometry classes of compact metric spaces $\mathbb{K}$ is obtained as follows: For two compact metric spaces $\mathbf{S}_1 = (S_1, \theta_1)$ and $\mathbf{S}_2 = (S_2, \theta_2)$ (representing isometry classes of $\mathbb{K}$), let $d_{\mathrm{GH}}(\mathbf{S}_1, \mathbf{S}_2)$ denote the infimum of $d_{\mathrm{H}}(\xi_1(S_1), \xi_2(S_2))$ over all isometric embeddings $\xi_i : S_i \to S$, $i = 1, 2$, into metric spaces (S, θ).

The space $(\mathbb{K}, d_{\mathrm{GH}})$ is a Polish space (see for instance Burago, Burago, and Ivanov [45]). A sequence of random, compact metric spaces $\mathbf{S}_n = (S_n, \theta_n)$ converges in distribution to a random, compact metric space $\mathbf{S} = (S, \theta)$ with respect to the *Gromov-Hausdorff topology* on $\mathbb{K}$ if almost surely $\mathbf{S}$ and the sequence $\mathbf{S}_n$ can be constructed so that $d_{\mathrm{GH}}(\mathbf{S}_n, \mathbf{S}) \to 0$ as $n \to \infty$.

1.2.4 Aldous' continuum random tree

Recall that a *real-tree*, or $\mathbb{R}$-*tree*, is a geodesic metric space in which each pair of points is connected by a unique simple path. Given a non-negative, continuous function h on $[0,1]$ satisfying $h(0) = 0 = h(1)$, an $\mathbb{R}$-tree (T_h, d_h) is obtained as follows: Put

$$d_h^*(s,t) = h(s) + h(t) - 2 \cdot \min_{s \wedge t \le u \le s \vee t} h(u), \quad s,t \in [0,1]$$

and let d_h denote the quotient distance induced by d_h^* on $T_h = [0,1]/\{d_h^* = 0\}$. We call h a *contour function* of the tree T_h. Note that two points $s < t$ are identified above if $h(s) = h(t)$ and $h(u) > h(s)$ for all $u \in (s,t)$. Informally, we think of constructing T_h by gluing underneath the graph of h, and then pressing it together from the sides, see Figure 1.3.

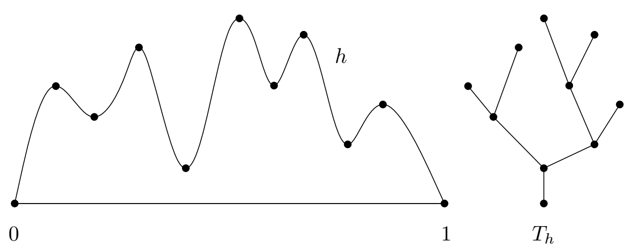

Figure 1.3: A tree T_h and its contour function h.

Aldous' [7] *continuum random tree*, or CRT, denoted $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$, is formed as above by taking h to be a normalized *Brownian excursion* $\mathbf{e} = \{\mathbf{e}_t : 0 \le t \le 1\}$. (Recall that $\mathbf{e}$ is Brownian motion conditioned to equal 0 at $t \in \{0,1\}$ and to be positive in $(0,1)$, see Revuz and Yor [117, Chapter XII].) Hence the relationship between the CRT and a Brownian excursion is analogous to that of a discrete tree and its contour function. We note that the CRT is the Gromov-Hausdorff scaling limit of uniformly random plane trees (see for instance Le Gall and Miermont [99, Theorem 3.9]). Moreover, it is the canonical uniformly random $\mathbb{R}$-tree, being also the limit of many other classes of trees.

1.3 Construction

The scaling limit of uniform planar maps has recently been identified.

Theorem 1.3.1 (Le Gall [98] and Miermont [108])**.** *Let Q_n be a uniformly random quadrangulation of $\mathbb{S}^2$. Let (M_n, d_n) denote the metric space obtained from Q_n by scaling its graph distance d_{Q_n} by $(8n/9)^{-1/4}$. Then, as $n \to \infty$, (M_n, d_n) converges in distribution with respect to the Gromov-Hausdorff topology to a random metric space (M, d), called the Brownian map.*

Recall (see Section 1.2.3) that this convergence means that almost surely we can construct (M, d) and the sequence (M_n, d_n) such that (M_n, d_n) converges to (M, d) in Gromov-Hausdorff distance. See Section 1.6 for a discussion on the proof of Theorem 1.3.1.

In [98] it is shown that, in fact, the same convergence in distribution holds (up an unimportant adjustment of the constant factor $(8/9)^{-1/4}$ in the scaling term) for uniformly random triangulations and $2k$-angulations, for all $k > 1$. Since then, the Brownian map has also been identified as the limit of several other types of maps, see for example [1, 2, 29, 36, 98]. In this sense, the Brownian map is a *universal* limiting object, in a similar way as Brownian motion is a universal limit of random paths and Aldous' CRT is a universal limit of random trees. Moreover, both of these fundamental objects play a leading role in its construction, which we now describe.

The general idea in the construction of the Brownian map is to extend (one direction of) the CVS-bijection to the CRT, in order to obtain a uniformly random, spherical metric space. Recall (see Section 1.2.2) that the CVS-bijection identifies random planar maps (specifically, quadrangulations) of the sphere with well-labelled plane trees. Thus, we require a method of "well-labelling" the CRT. Since the labels in a well-labelled tree increase or decrease by at most 1 along edges, the natural analogue for a label process in this continuum setting is Brownian motion. We proceed as follows.

The main ingredients are a normalized Brownian excursion $\mathbf{e} = \{\mathbf{e}_t : t \in [0,1]\}$, a random $\mathbb{R}$-tree $(\mathcal{T}_\mathbf{e}, d_\mathbf{e})$ indexed by $\mathbf{e}$, and a Brownian label process $Z = \{Z_a : a \in \mathcal{T}_\mathbf{e}\}$. More specifically, define $\mathcal{T}_\mathbf{e} = [0,1]/\{d_\mathbf{e} = 0\}$ as the quotient under the pseudo-distance

$$d_\mathbf{e}(s,t) = \mathbf{e}_s + \mathbf{e}_t - 2 \cdot \min_{s\wedge t \le u \le s\vee t} \mathbf{e}_u, \quad s,t \in [0,1]$$

and equip it with the quotient distance, again denoted by $d_\mathbf{e}$. The random metric space $(\mathcal{T}_\mathbf{e}, d_\mathbf{e})$ is Aldous' *continuum random tree*, or CRT (as discussed in Section 1.2.4). Let $p_\mathbf{e} : [0,1] \to \mathcal{T}_\mathbf{e}$ denote the canonical projection. Conditionally given $\mathbf{e}$, Z is a centred Gaussian process satisfying $\mathbf{E}[(Z_s - Z_t)^2] = d_\mathbf{e}(s,t)$ for all $s,t \in [0,1]$. The random process Z is the so-called *head of the Brownian snake* (see [99]). Note that Z is constant on each equivalence class $p_\mathbf{e}^{-1}(a)$, $a \in \mathcal{T}_\mathbf{e}$. In this sense, Z is Brownian motion indexed by the CRT.

Analogously to the definition of $d_\mathbf{e}$, we put

$$d_Z(s,t) = Z_s + Z_t - 2 \cdot \max\left\{ \inf_{u\in[s,t]} Z_u, \inf_{u\in[t,s]} Z_u \right\}, \quad s,t \in [0,1]$$

where we set $[s,t] = [0,t] \cup [s,1]$ in the case that $s > t$. Then, to obtain a pseudo-distance on $[0,1]$, we define

$$D^*(s,t) = \inf\left\{ \sum_{i=1}^{k} d_Z(s_i,t_i) : s_1 = s, t_k = t, d_\mathbf{e}(t_i, s_{i+1}) = 0 \right\}, \quad s,t \in [0,1].$$

Finally, we set $M = [0,1]/\{D^* = 0\}$ and endow it with the quotient distance induced by D^*, which we denote by d. An easy property (see [105, Section 4.3]) of the Brownian map is that $d_\mathbf{e}(s,t) = 0$ implies $D^*(s,t) = 0$, so that M can also be seen as a quotient of $\mathcal{T}_\mathbf{e}$, and we let $\Pi : \mathcal{T}_\mathbf{e} \to M$ denote the canonical projection, and put $\mathbf{p} = \Pi \circ p_\mathbf{e}$. Almost surely, the process Z attains a unique minimum on $[0,1]$, say at t_*. We set $\rho = \mathbf{p}(t_*)$. The random metric space $(M, d) = (M, d, \rho)$ is called the *Brownian map* and we call ρ its *root*.

1.3.1 CVS-bijection, extended

Almost surely, for every pair of distinct points $s \neq t \in [0, 1]$, at most one of $d_{\mathbf{e}}(s, t) = 0$ or $d_Z(s, t) = 0$ holds, except in the particular case $\{s, t\} = \{0, 1\}$ where both identities hold simultaneously (see [100, Lemma 3.2]). Therefore only *leaves* (that is, *non-cut-points*) of $\mathcal{T}_{\mathbf{e}}$ are identified in the construction of the Brownian map, and this occurs if and only if they have the same label and along either the clockwise or counter-clockwise, contour-ordered path around $\mathcal{T}_{\mathbf{e}}$ between them, one only finds vertices of larger label. Thus, in the construction of the Brownian map, $(\mathcal{T}_{\mathbf{e}}, Z)$ is a continuum analogue for a well-labelled plane tree, and the quotient by $\{D^* = 0\}$ for the CVS-bijection (which recall identifies well-labelled plane trees with rooted planar maps, as discussed in Section 1.2.2).

1.4 Basic properties

1.4.1 Fractal, spherical geometry

Recall that an original motivation for developing this theory is to obtain a random spherical surface. Although a finite planar map is trivially homeomorphic to $\mathbb{S}^2$, this property is not a prior preserved in the Gromov-Hausdorff limit. Thus one of the most fundamental theorems regarding the Brownian map is as follows.

Theorem 1.4.1 (Le Gall and Paulin [100] and Miermont [106])**.** *Almost surely, the Brownian map (M, d) is homeomorphic to $\mathbb{S}^2$.*

The two proofs of this result take entirely different approaches. The original proof by Le Gall and Paulin [100] uses a general result of Moore [111] and works directly with the limiting object (M, d). On the other hand, Miermont [106] studies the discrete maps themselves, showing that large planar planar maps typically do not have cycles smaller than the scaling order $O(n^{1/4})$ that separate macroscopic (on the scaling order) areas of the map. In other words, the possible existence of "bottlenecks," capable of giving rise to non-spherical limits, is ruled out directly.

Recall that, for $d \geq 2$, the Hausdorff dimension of a path of Brownian motion in $\mathbb{R}^d$ is twice that of a smooth curve (see Kaufman [86]). Similarly, the Hausdorff dimension of (M, d) is twice that of $\mathbb{S}^2$.

Theorem 1.4.2 (Le Gall [96])**.** *Almost surely, the Hausdorff dimension of the Brownian map* (M, d) *is* 4*.*

Overall, we see by these theorems that the Brownian map has a spherical geometry, and an extremely singular metric space structure.

1.4.2 Volume and re-rooting invariance

Although the Brownian map is a rooted metric space, it is not so dependent on its root. The *volume measure* λ on M is defined as the push-forward of Lebesgue measure on $[0, 1]$ via $\mathbf{p}$. A fundamental result of Le Gall shows that the Brownian map is *invariant under re-rooting,* in the following sense.

Theorem 1.4.3 (Le Gall [97])**.** *Suppose that* U *is uniformly distributed over* $[0, 1]$ *and independent of* (M, d)*. Then* (M, d, ρ) *and* $(M, d, \mathbf{p}(U))$ *are equal in law.*

Therefore, to some degree, the root of the map is but an artifact of its construction. That being said, there is a dense set (of zero volume, but positive dimension) of special points that significantly contribute to its geometry. Indeed, investigating such points is a main focus in Part II of this thesis (see Section 1.8 for an overview).

1.5 Geodesics

A subset $\gamma \subset M$ is called a *geodesic segment* if (γ, d) is isometric to a compact interval. An isometry from such an interval to γ is a *geodesic* associated with the geodesic segment γ. We will however most often blur this distinction, referring to geodesic segments simply as geodesics. We note that the Brownian map, being the Gromov-Hausdorff limit of geodesic spaces, is almost surely a geodesic space (see for instance [45]).

Le Gall [97] obtained a complete description of the geodesics to the root ρ of the Brownian map, as discussed in the next section. This description implies several interesting properties of the Brownian map, and has played a critical role in its further analysis. Indeed, the work [97] predates the main Theorem 1.3.1, and is a key tool used in its proofs [98, 108] (see Section 1.6).

1.5.1 Simple geodesics

As discussed in Section 1.2.2, the CVS-bijection highlights geodesics from each corner of a well-labelled plane tree to the root of the resulting planar map. As it turns out, these are the only geodesics to the root that remain visible in the scaling limit.

A *corner* of a vertex v in a discrete plane tree T is a sector centred at v, delimited by the edges which precede and follow v along a contour-ordered path around T. Leaves of a tree have exactly one corner, and in general, the number of corners of v is equal to the number of connected components in $T - \{v\}$. Similarly, we may view the $\mathbb{R}$-tree $\mathcal{T}_{\mathbf{e}}$ as having corners, however in this continuum setting all sectors reduce to points. Hence, for the purpose of the following (slightly informal) discussion, let us think of each $t \in [0, 1]$ as corresponding to a *corner* of $\mathcal{T}_{\mathbf{e}}$ with label Z_t. Thus as $t \in [0, 1]$ varies from 0 to 1, we think of exploring the clockwise, contour-ordered path around $\mathcal{T}_{\mathbf{e}}$, encountering its corners labelled by Z_t along the way.

Recall (see Section 1.3) that $\rho = \mathbf{p}(t_*)$, such that Z_t attains its minimum at t_*. Put $Z_* = Z_{t_*}$. As it turns out, $d(\rho, \mathbf{p}(t)) = Z_t - Z_*$ for all $t \in [0, 1]$ (see [96]). In other words, up to a shift by the minimum label Z_*, the Brownian label of a point in $\mathcal{T}_{\mathbf{e}}$ is precisely the distance to ρ from the corresponding point in the Brownian map.

Simple geodesics to ρ are constructed as follows. For $t \in [0, 1]$ and $\ell \in [0, Z_t - Z_*]$, let $s_t(\ell)$ denote the point in $[0, 1]$ corresponding to the first corner with label $Z_t - \ell$ in the clockwise, contour-ordered path around $\mathcal{T}_{\mathbf{e}}$ beginning at the corner corresponding to t. For each such t, the image of the function $\Gamma_t : [0, Z_t - Z_*] \to M$ taking ℓ to $\mathbf{p}(s_t(\ell))$ is a geodesic segment from $\mathbf{p}(t)$ to ρ. In [97] it is shown that all geodesics to ρ are of this form.

Theorem 1.5.1 (Le Gall [97])**.** *Almost surely, all geodesics in (M, d) to the root ρ are simple geodesics Γ_t, $t \in [0, 1]$.*

This result has several important implications, as discussed in the sections that follow. In summary, the $\mathbb{R}$-tree structure of the Brownian map is revealed. We find that the space (M, d) is comprised of two topological $\mathbb{R}$-trees, $G(\rho)$ and $S(\rho)$: the former containing all points strictly inside geodesic segments to ρ, and the latter all points with multiple geodesics to ρ. These trees are dual to each other, in the sense that they are disjoint and both dense in (M, d). Loosely speaking, they are "intertwined." As it turns out, the Hausdorff dimensions of $G(\rho)$ and $S(\rho)$ are 1 and 2, and so, what remains of the Brownian map is a 4-dimensional set of points at their interface.

We remark here that, in brief, the main purpose of Part II of this thesis is to investigate the sets $G(x)$ and $S(x)$ for general points $x \in M$.

1.5.2 Tree of cut-points

Let $S(\rho)$ denote the set of points $y \in M$ with multiple geodesics to ρ. Note that the cut-points of $\mathcal{T}_\mathbf{e}$ (that is, points $a \in \mathcal{T}_\mathbf{e}$ such that $\mathcal{T}_\mathbf{e} - \{a\}$ has multiple connected components) are exactly the points in $\mathcal{T}_\mathbf{e}$ with multiple corners. Hence by Theorem 1.5.1 it follows that $S(\rho)$ is precisely the $\mathbb{R}$-tree $\mathcal{T}_\mathbf{e} = [0, 1]/\{d_\mathbf{e} = 0\}$ minus its leaves (that is, non-cut-points), projected into M. Informally, there is a geodesic to ρ leaving from each corner of the CRT (as is also the case for finite planar maps, see Section 1.2.2). Moreover, since the number of corners of a cut-point of $\mathcal{T}_\mathbf{e}$ is exactly the number of geodesics from the corresponding point in the map to ρ, we obtain the following result by standard properties of the CRT.

For $i \geq 1$, let $S_i(\rho)$ be the set of points $y \in M$ with exactly i geodesic segments to ρ.

Theorem 1.5.2 (Le Gall [97])**.** *Let ρ denote the root of the Brownian map. We have that* *(i)* $S(\rho) = S_2(\rho) \cup S_3(\rho)$*;*

(ii) $S_2(\rho)$ *is dense and has Hausdorff dimension* 2*;*

(iii) $S_3(\rho)$ *is dense and countable.*

Although the Brownian map is an extremely singular metric space, in this regard it nonetheless bears similarities with complete, analytic Riemannian surfaces homeomorphic to the sphere, for which the cut locus S of a point x is a tree and the number of "branches" emanating from a point in S is exactly the number of geodesics to x (see Poincaré [115] and Myers [113]). With this in mind, Le Gall [97] states that the set $S(\rho)$

> exactly corresponds to the cut locus of [the Brownian map] relative to the root.

Extending the notion of a cut locus to general points $x \in M$, however, turns out to be a more delicate matter, see Section 1.8.2.

1.5.3 Tree of geodesics

The *relative* interior of a geodesic segment γ between $x, y \in M$, is the set $\gamma - \{x, y\}$, that is, the segment minus its endpoints. Let $G(\rho)$ denote the set of points in the relative interior of a geodesic segment to ρ. We call $G(\rho)$ the *geodesic net* of ρ. Theorem 1.5.1 implies that $G(\rho)$ is precisely the $\mathbb{R}$-tree $\mathcal{T}_Z = [0,1]/\{d_Z = 0\}$ minus its leaves, projected into M. In this sense, there is a tree of geodesics to ρ in (M, d). As shown in [97], $G(\rho)$ is a relatively small subset of the map of Hausdorff dimension 1. Points in $S(\rho)$ correspond to leaves of $\mathcal{T}_Z$ (see [100, Lemma 3.2]), so the trees $S(\rho)$ and $G(\rho)$ are disjoint.

1.5.4 Confluence of geodesics

Perhaps the most striking consequence of Theorem 1.5.1 is that any two geodesics to the root coalesce before reaching the root. Le Gall [97] refers to this phenomenon as the *confluence of geodesics*. Let $B(x, \varepsilon)$ denote the ball of radius ε centred at $x \in M$. More specifically, almost surely we have that, for any $\varepsilon > 0$, there is some $\eta > 0$ such that all geodesic segments from points $y \in B(\rho,\varepsilon)^c$ to ρ coincide inside $B(\rho,\eta)$. In topological terminology (as pointed out in [105]), there is a unique *germ* of geodesics to ρ.

This observation follows by the continuity of Z_t and Theorem 1.5.1, as we now explain. In this discussion, for $s,t \in [0,1]$, let $[s,t]$ be the sub-interval of $[0,1]$ in the cyclic order, where $0 \equiv 1$. That is, $[s,t]$ denotes $[0,t] \cup [s,1]$ if $s > t$. Recall that $\rho = \mathbf{p}(t_*)$ and $Z_* = Z_{t_*}$ is the minimum of Z_t. Moreover $d(\rho,\mathbf{p}(t)) = Z_t - Z_*$ for all t (see Section 1.5.1). Therefore, for some $\xi > 0$, $Z_t - Z_* < \varepsilon$ for all $t \in [t_* - \xi, t_* + \xi]$. Hence $\mathbf{p}([t_* - \xi, t_* + \xi]) \subset B(\rho,\varepsilon)$. Let η be the minimum of $Z_t - Z_*$ on $[t_* + \xi, t_* - \xi]$. Note that $\eta > 0$. By the choice of η (and Theorem 1.5.1), all geodesics to ρ from points outside $B(\rho,\varepsilon)$ coincide inside $B(\rho,\eta)$. See Figure 1.4.

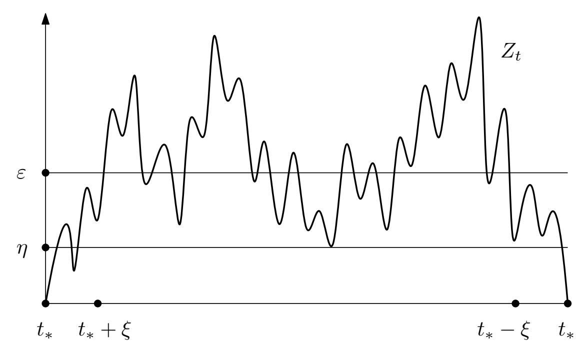

Figure 1.4: All geodesics to the root $\rho = \mathbf{p}(t_*)$ from points outside $B(\rho,\varepsilon)$ coincide inside $B(\rho,\eta)$, where η is the minimum of $Z_t - Z_*$ on $[t_* + \xi, t_* - \xi]$, and ξ is such that $Z_t - Z_* < \varepsilon$ on $[t_* - \xi, t_* + \xi]$.

Applying invariance under re-rooting (Theorem 1.4.3), we obtain the following result.

Theorem 1.5.3 (Le Gall [97, Corollary 7.7])**.** *Almost surely, for λ-almost every $x \in M$, the following holds. For any neighbourhood N of x, there is a sub-neighbourhood $N' \subset N$ so that all geodesics from x to points outside N coincide inside N'.*

Moreover, geodesics to the root of the map tend to coalesce quickly. For $t \in [0,1]$, let γ_t denote the image of the simple geodesic Γ_t from $\mathbf{p}(t)$ to the root of the map ρ (see Section 1.5.1). That is, γ_t is the geodesic segment associated with the geodesic Γ_t.

Lemma 1.5.4 (Miermont [108, Lemma 5])**.** *Almost surely, for all $s, t \in [0, 1]$, γ_s and γ_t coincide outside of $B(\mathbf{p}(s), d_Z(s, t))$.*

This result follows simply by noting that the distance from $\mathbf{p}(s)$ to the point at which γ_s and γ_t coalesce is no longer than the path from s to t in the tree $\mathcal{T}_Z$ (with equality holding if and only if $\gamma_t \subset \gamma_s$).

1.5.5 Regularity of geodesics

For $x, y \in M$, we call the set of points in some geodesic segment from x to y the *geodesic network* from x to y. In this section we note that by the observations in Sections 1.5.3 and 1.5.4, geodesic networks from ρ to points $y \in M$ have a specific topological structure.

We say that the ordered pair (x, y) is *regular* if any two distinct geodesic segments between x and y are disjoint inside, and coincide outside, a punctured ball centred at y of radius less than $d(x, y)$. Formally, if γ, γ' are geodesic segments between x and y, then for some $r \in (0, d(x, y))$, we have that $\gamma \cap \gamma' \cap B(y, r) = \{y\}$ and $\gamma - B(y, r) = \gamma' - B(y, r)$.

By the tree structure of $G(\rho)$ (see Section 1.5.3) and Theorems 1.4.3 and 1.5.3, we obtain the following result.

Proposition 1.5.5. *Almost surely, for λ-almost every $x \in M$, for all $y \in M$, (x, y) is regular.*

1.6 Uniqueness

The main obstacle to establishing the existence of the Brownian map (that is, the existence of a *unique* scaling limit) is to obtain more information about its geodesics, beyond the foundational results of Le Gall [97] (see Section 1.5.1).

A compactness argument of Le Gall [96] established scaling limits of planar maps along subsequences, however the question of uniqueness remained unresolved for some time. Some properties were known to hold regardless of what subsequence had been extracted, notably Le Gall's [97] description of geodesics to the root (Theorem 1.5.1) (and also, for instance, Theorems 1.4.1 and 1.4.2). This information, however, is not a priori sufficient to show that the limit exists.

The key to overcoming this difficulty is to relate a geodesic between a pair of typical points to geodesics to the root. Let γ be a geodesic segment between points selected uniformly according to λ. (Note that, by the confluence of geodesics phenomenon (Theorem 1.5.3), the root of the map is almost surely disjoint from γ.) In [98, 108] the set of points $z \in \gamma$ such that the relative interior of any geodesic from z to the root is disjoint from γ is shown to be small compared to γ. Roughly speaking, "most" points in "most" geodesics of the Brownian map are in a geodesic to the root. (See the discussion around equation (2) in [98] and [108, Section 2.3] for precise statements.) In this way, Le Gall [98] and Miermont [108] show that geodesics to the root do in fact provide enough information to characterize the Brownian map metric, leading to Theorem 1.3.1. See for example Miermont's Saint-Flour notes [105, Section 7] for a more detailed overview.

1.7 Related models

1.7.1 Local limits

As already mentioned, infinite volume, local limits of planar maps were developed prior to the results on scaling limits.

For a graph $G = (V, E)$, vertex $v \in V$ and $r \geq 0$, let $B_G(v, r)$ denote the ball of radius r in G, that is, the subgraph of G induced by the set of vertices whose graph distance to v is at most r. A sequence of rooted graphs (G_n, ρ_n) is said to *converge locally in distribution* to a rooted graph (G_∞, ρ), if for every $r \geq 0$ and graph H, the probability that $B_{G_n}(\rho_n, r) = H$ converges to the probability that $B_{G_\infty}(\rho, r) = H$ as $n \to \infty$.

Theorem 1.7.1 (Angel and Schramm [14])**.** *Let T_n be a uniformly random triangulation of $\mathbb{S}^2$ of size n. Then T_n converges locally in distribution to a* *random infinite graph T_∞, called the uniform infinite planar triangulation, or UIPT.*

The UIPQ, denoted by Q_∞, which arises as the local limit of uniformly random quadrangulations, was later developed by Krikun [93].

It is interesting to note that, just as the Brownian map is obtained via a continuum analogue of the CVS-bijection (see Sections 1.2.2 and 1.3), the UIPQ can also be constructed by an extension of this same bijection. This construction, as described below, is due to Curien, Ménard and Miermont [56]. In this case, the role of $\mathcal{T}_{\mathbf{e}}$ (the CRT) is replaced with that of $\mathcal{T}_\infty$, the critical Galton-Watson tree conditioned to survive (due to Kesten [87]).

Recall that $\mathcal{T}_\infty$ is obtained from a half-infinite line (called the *spine*), with edges between each $i \geq 0$ and $i+1$, by "grafting" critical (and so, almost surely finite) Galton-Watson trees on the left and right sides of each vertex $i \geq 0$. This tree can be "well-labelled" via a discrete version of the Brownian snake. Specifically, each edge in $\mathcal{T}_\infty$ is assigned an iid weight in $\{0, \pm 1\}$. We define the label ℓ_v of vertex v in $\mathcal{T}_\infty$ to be the sum of the edge weights along the (unique) path in $\mathcal{T}_\infty$ from 0 to v. Essentially, this labelling is (lazy) simple random walk indexed by $\mathcal{T}_\infty$. Extending the CVS-bijection to the object at hand, we draw an edge from each corner of a vertex v in $\mathcal{T}_\infty$ to the next corner in the clockwise, contour-ordered path around $\mathcal{T}_\infty$ with label $\ell_v - 1$. Since $\inf_{i \geq 0} \ell_i = -\infty$ (as the labels on the spine correspond to a standard (lazy) simple random walk), this procedure is well-defined. Moreover, in [56] it is shown that the object obtained in this way is equal in distribution to the UIPQ.

1.7.2 Brownian surfaces

An infinite volume version of the Brownian map, called the *Brownian plane* (P, D), has been introduced and studied by Curien and Le Gall [53]. The random metric space (P, D) is almost surely homeomorphic to the plane $\mathbb{R}^2$, and like the Brownian map, of Hausdorff dimension 4.

The spaces (M, d) and (P, D) have a similar local structure. Specifically, almost surely there are isometric neighbourhoods of the roots of (M, d) and

(P, D). The Brownian plane has an additional scale invariance property which makes it more amenable to analysis, see the works of Curien and Le Gall [54, 55]. We note that using these facts, properties of the Brownian plane can be deduced from those of the Brownian map.

Along the same lines as the construction of the Brownian map, the Brownian plane can be obtained through an extension of the CVS-bijection. In this setting, the role of the Brownian excursion $\mathbf{e}$ is replaced with that of a independent pair of three-dimensional Bessel process, indexed by $[0, \infty)$ and $(-\infty, 0]$. Furthermore, (P, D) can also be obtained as a *local* (non-compact) Gromov-Hausdorff scaling limit of the UIPQ (see Section 1.7.1), by scaling distances in the UIPQ by λ, and letting $\lambda \to 0$.

Bettinelli [33, 34, 35] has investigated *Brownian surfaces* of positive genus. In [33], the subsequential Gromov-Hausdorff convergence of uniform random bipartite quadrangulations of the g-torus $\mathbb{T}_g$ is established (also general orientable surfaces with a boundary are analyzed in [35]), and it is an ongoing work of Bettinelli and Miermont [37, 38] to confirm that a unique scaling limit exists. Some properties hold independently of which subsequence is extracted. For instance, any scaling limit of bipartite quadrangulations of $\mathbb{T}_g$ is homeomorphic to $\mathbb{T}_g$ (see [34]) and has Hausdorff dimension 4 (see [33]). Also, a confluence of geodesics is observed at typical points of the surface (see [35]). We also note that recently Baur, Miermont and Ray [28] have classified the scaling limits of uniform quadrangulations with a boundary.

1.8 Our results

In this section, we discuss some of the main results proved in Part II of this thesis. See Section 3.2 below for a complete overview.

1.8.1 Confluence points

We strengthen the confluence of geodesics phenomenon of Le Gall [97] (Theorem 1.5.3). We find that for any neighbourhood N of a typical point in the Brownian map, there is a *confluence point* x_0 between a sub-neighbourhood

$N' \subset N$ and the complement of N. See Figure 1.5.

Theorem 1.8.1 (Angel, Kolesnik and Miermont [13]). *Almost surely, for λ-almost every $x \in M$, the following holds. For any neighbourhood N of x, there is a sub-neighbourhood $N' \subset N$ and some $x_0 \in N - N'$ so that all geodesics between any points $x' \in N'$ and $y \in N^c$ pass through x_0.*

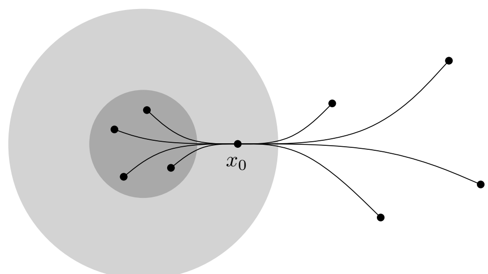

Figure 1.5: All geodesics from points in N' to points in the complement of $N \supset N'$ pass through a confluence point x_0.

Using this key result, we establish several properties of the Brownian map (see Section 3.2). In the remaining sections of this chapter, we discuss some of our more easily-stated results.

1.8.2 Cut loci

The cut locus of a point p in a Riemannian manifold, first examined by Poincaré [115], is the set of points $q \neq p$ which are endpoints of maximal (minimizing) geodesics from p. This collection of points is more subtle than just the set of points with multiple geodesics to p. In fact, it is generally the closure thereof (see Klingenberg [90, Theorem 2.1.14]).

In the Brownian map this equivalence breaks completely. Indeed, as shown in Chapter 3, almost all (in the sense of volume and Baire category) points are the end of a maximal geodesic, and every point is joined by multiple geodesics to a dense set of points. Moreover, whereas in the Brownian map there are points with multiple geodesics to the root which coalesce before reaching the root, in a Riemannian manifold any (minimizing) geodesic which is not the unique geodesic between its endpoints cannot be extended (see, for example, the "short-cut principle" discussed in Shiohama, Shioya and Tanaka [124, Remark 1.8.1]).

As it would seem that the Brownian map is about as far from Riemannian as possible, the following cautionary note by Berger [31] seems as appropriate as ever:

> The cut-locus is essentially a Riemannian notion if one expects a reasonable form of behavior. As soon as one goes to more general metric spaces things can become very, very wild.

That being said, we wish to extend this notion to the Brownian map since, to quote Berger [32] once again, it is

> interesting to contrast cut loci in Riemannian manifolds with those of a more general metric space.

We need only take care in order to define a suitable notion of cut locus for this highly singular metric space. We proceed as follows, defining the *(strong) cut locus* $C(x)$ of a point $x \in M$ to be the set of points $y \in M$ to which there are at least two geodesics from x that are disjoint in a neighbourhood of y. Thus, roughly speaking, $y \in C(x)$ if there are geodesics from x that approach y from different directions. We believe that this definition captures the essence of a cut locus as effectively as possible.

We show that the cut locus of the Brownian map is uniformly stable, in the following sense.

Theorem 1.8.2 (Angel, Kolesnik and Miermont [13])**.** *Almost surely, for all $x, y \in M$, $C(x)$ and $C(y)$ coincide outside a closed, nowhere dense set of zero λ-measure.*

Note that this result holds for *any* points $x, y \in M$, not only for typical points.

Moreover, for typical points $x \in M$, a small perturbation from x has only a small, local effect on the cut locus. In this sense, the cut locus of the Brownian map is continuous almost everywhere.

Theorem 1.8.3 (Angel, Kolesnik and Miermont [13])**.** *Almost surely, for λ-almost every $x \in M$, for any neighbourhood N of x, there is a sub-neighbourhood $N' \subset N$ so that $C(x') - N$ is the same for all $x' \in N'$.*

On the other hand, we define the *weak cut locus* $S(x)$ to be simply the set of points $y \in M$ with multiple geodesics to x. By Proposition 1.5.5, the two notions are typically one and the same.

Proposition 1.8.4. *Almost surely, for λ-almost every $x \in M$, $S(x) = C(x)$, that is, the weak and strong cut loci coincide.*

That being said, their general behaviour is markedly distinct. While the strong cut locus is uniformly stable, the weak cut locus behaves quite wildly, oscillating in dimension and volume, see Section 3.5.3.1.

1.8.3 Geodesic networks

By the results of Le Gall [97] discussed in Section 1.5, all geodesic networks to ρ are regular, and consist of at most three geodesics. We find that all except very few geodesic networks in the Brownian map are, in the following sense, a concatenation of two regular networks.

For $x, y \in M$ and $j, k \in \mathbb{N}$, we say that the ordered pair (x, y) induces a *normal (j,k)-network*, and write $(x, y) \in N(j, k)$, if for some z in the relative interior of all geodesic segments between x and y, (z, x) and (z, y) are regular (see Section 1.5.5) and z is connected to x and y by exactly j and k geodesic segments, respectively. See Figure 1.6.

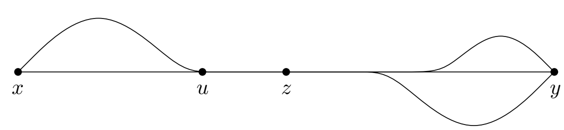

Figure 1.6: As depicted, $(x, y) \in N(2, 3)$. Note that (u, x) does not induce a normal (j, k)-network.

In particular, note if x, y are joined by exactly k geodesics and (x, y) is regular, then $(x, y) \in N(1, k)$. (Take z to be a point in the relative interior of the geodesic segment contained in all k segments from x to y.)

Not all networks are normal (j,k)-networks. For instance, if $(x,y) \in N(j,k)$ and $j > 1$, then there is a point u that is joined to x by two geodesics with disjoint relative interiors. See Figure 1.6. That being said, we find that most pairs induce normal (j,k)-networks. Moreover, for each $j,k \in \{1,2,3\}$, there are many normal (j,k)-networks in the map. Hence, in particular, we establish the existence of atypical networks comprised of more than three geodesics (and up to nine).

Theorem 1.8.5 (Angel, Kolesnik and Miermont [13]). *The following hold almost surely.*

(i) For any $j,k \in \{1,2,3\}$, $N(j,k)$ *is dense in* M^2.

(ii) $M^2 - \bigcup_{j,k\in\{1,2,3\}} N(j,k)$ *is nowhere dense in* M^2.

By Theorem 3.2.15, there are essentially only six types of geodesic networks that are dense in the Brownian map. See Figure 1.7.

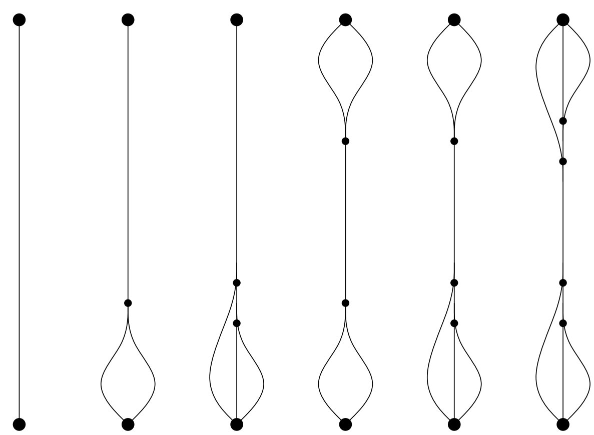

Figure 1.7: Classification of geodesic networks that are dense in the Brownian map (up to symmetries and homeomorphisms of the sphere).

Finally, we also obtain the Hausdorff dimension of the set of pairs joined by each type of normal network. For a subset $A \subset M$, let $\dim A$ denote its Hausdorff dimension (see Section 3.3.4).

Theorem 1.8.6 (Angel, Kolesnik and Miermont [13]). *Almost surely, we have that* $\dim N(j,k) = 2(6-j-k)$, *for all* $j,k \in \{1,2,3\}$. *Moreover,* $N(3,3)$ *is countable.*

In closing, we remark that it remains an interesting open problem to fully classify all types of geodesic networks in the Brownian map. Even showing that almost surely there are no $x, y \in M$ joined by *infinitely* many geodesics is open, although the upper bound of nine seems plausible. See also the intriguing possible existence of *ghost geodesics* in the Brownian map, as discussed in Section 3.2.4. Such geodesics (if they exist), behave more like Euclidean geodesics than typical geodesics in the Brownian map, in the sense that they do not coalesce with any other geodesics. We call them "ghosts," since in this way, they are undetected by all other geodesics.

Chapter 2

Bootstrap percolation

Let $G = (V, E)$ be a graph and r a positive integer. Given an initial set of *active* vertices $V_0 \subset V$, the *r-neighbour bootstrap percolation process* evolves by activating vertices with at least r active neighbours. Formally, let V_{t+1} be the union of V_t and the set of all vertices with at least r neighbours in V_t, that is,

$$V_{t+1} = V_t \cup \Big\{ v : |N(v) \cap V_t| \geq r \Big\},$$

where $N(v)$ is the set of neighbours of a vertex v. The sets V_t are increasing, and so converge to some set of eventually active vertices, denoted by $\langle V_0, G \rangle_r$. If $\langle V_0, G \rangle_r = V$, that is, all vertices in V are eventually activated, we say that G *percolates*.

Bootstrap percolation is most often attributed to Chalupa, Leath and Reich (1979) [50], who studied the model on the Bethe lattice (the infinite d-regular tree T_d). However, as noted in the survey paper by Alder and Lev [5], the idea was presented earlier by Pollak and Riess (1975) [116] (see also the private communication with Kopelman cited therein).

In fact, similar models had been considered even earlier. Since the status of any given vertex at any given time of the process depends only on the status of its neighbourhood, bootstrap percolation is an example of a *cellular automaton,* as developed by von Neumann (1966) [134], following Ulam (1950) [131]. Bollobás' (1968) [39] study of *weakly k-saturated graphs* leads to a variation called *graph bootstrap percolation,* see Section 2.5 below. Also of note is a model proposed by McCullogh and Pitts (1943) [104] (see also the modern review by Piccinini [114]) for neuronal interactions in the brain, which bears similarities with bootstrap percolation. In this model, the underlying graph is directed and its edges are assigned weights. A vertex become active/inactive if the weighted sum over the edges directed towards it from its active neighbours is larger/smaller than a certain threshold.

In any case, the term “bootstrap percolation” originates from [50], and the works [50, 116] are the first to present the idea from the perspective of statistical physics. The basic motivation is to study the effect of an impurity on a magnetic system. Since magnetism is a phenomenon that comes about through interaction, it is assumed in [50, 116] that a magnetic particle, which is no longer in direct contact with sufficiently many other magnetic particles, becomes non-magnetic, and so thereafter can be treated as an impurity itself. As a result, given the right initial conditions, the model exhibits an abrupt, first-order phase transition.

Since its introduction, bootstrap percolation has found many applications in mathematics, physics, and in other fields, including computer science and sociology, see for instance [4, 5, 9, 57, 58, 63, 64, 65, 68, 69, 70, 89, 112, 126, 133, 136, 137] and further references therein.

2.1 Our objective

The bootstrap percolation process is well-studied on several classes of deterministic graphs, such as grids, lattices, trees and hypercubes. More recently, there has been interest in studying the model on random graphs. The focus of Part III of this thesis is to analyze the model on the fundamental Erdős–Rényi [60] graph $\mathcal{G}_{n,p}$. Recall that $\mathcal{G}_{n,p}$ is the random subgraph of the complete graph K_n (the graph on $[n] = \{1, 2, \ldots, n\}$ containing all possible (undirected) edges $\{i, j\}$, where $i, j \in [n]$), obtained by including each possible edge independently with probability p. Our results are presented in Section 2.6 below.

2.2 Main questions and terminology

If $\langle V_0, G\rangle_r = V$, that is, all vertices in V are eventually active if V_0 is initially active, then we say that the set V_0 is *contagious* for G. Note that if G is finite then $\langle V_0, G\rangle_r = V_\tau$, where τ is the smallest t such that $V_t = V_{t+1}$.

The main questions of interest in the field revolve around the size of the set of eventually active vertices $\langle V_0, G\rangle_r$. In most works, the object of study is the probability that a *random* initial set V_0 is contagious. Usually V_0 is obtained either by initially activating each vertex in V independently with probability p, or else by selecting a random subset of V of a given size.

2.2.1 Critical thresholds

Suppose that each vertex of a graph G is initially active independently with probability p. *Thresholds* p_ε are defined as the infimum over p such that G percolates with probability at least ε. We put $p_c = p_{1/2}$, and refer to this quantity as the *critical probability,* or *critical threshold.* Sometimes we write $p_c(G, r)$ to explicitly denote the critical probability for r-bootstrap percolation on G.

The *ε-window,* or *scaling window,* is the interval $[p_\varepsilon, p_{1-\varepsilon}]$. Suppose that $G = G(n)$ is a sequence of graphs, obtained for instance by selecting G uniformly at random from a certain class of graphs. Then $p_\varepsilon = p_\varepsilon(n)$. If, for any $\varepsilon \in (0, 1/2)$, we have that $p_{1-\varepsilon} - p_\varepsilon = o(p_c)$ as $n \to \infty$, the percolation threshold p_c is called *sharp,* and *coarse* otherwise.

We note that since the event of percolation is monotone increasing in p, by a general principle of Bollobás and Thomason [41] a threshold p_c exists such that that if $p \gg p_c$ then G percolates with high probability and if $p \ll p_c$ then G does not percolate with high probability. Moreover, in some cases the existence of a sharp threshold follows by a general result of Friedgut [67, Theorem 1.4]. That being said, in this thesis our motivation is in actually *locating* certain sharp thresholds of interest (that is, we aim to identify a function $\theta = \theta(n)$, such that $p_c \sim \theta$ as $n \to \infty$).

2.2.2 Minimal contagious sets

Rather than studying random contagious sets, it also is natural is ask what types of contagious sets *exist* for a graph. This is a more difficult question to answer, since now interactions need to be considered. For instance, in order to apply the standard second moment method to show that a (random) graph $G = (V, E)$ has a contagious set of size q, estimates are required for the probability that, for sets $I \neq I' \subset V$ of size q and various values of $k > q$, we have that $|\langle I, G\rangle_r| \geq k$ and $|\langle I', G\rangle_r| \geq k$. As a result, there are comparatively less results in this direction, and indeed, this is a main focus of our thesis in Chapters 4 and 5.

Let $m(G, r)$ denote the size of a *minimal* contagious set for G. Note that $m(G, r) \geq r$. We call a graph *susceptible,* and say that it r*-percolates,* if it has a contagious set of the smallest possible size r. More generally, a graph with a contagious set of size q is called (q, r)*-susceptible,* or equivalently (q, r)*-percolating.* Critical thresholds are defined as is p_c in Section 2.2.1, and we use similar notation to denote them when it is clear from the context.

2.3 Brief survey

A well-known problem in the field is to show that any contagious set for 2-bootstrap percolation on the finite grid $[n]^2$ contains at least n vertices (see for instance [15] for a discussion). An elegant solution involves considering the length of the boundary of the set of active vertices, noting that this remains constant when an additional vertex is activated.

2.3.1 First results

Apart from the founding articles [50, 116] already mentioned, the first results in the literature concern bootstrap percolation on the infinite lattices $\mathbb{Z}^d$ and finite grids $[n]^d$. The latter situation is referred to as the *finite volume,* or *metastable,* regime. In all of the works discussed in this section, the process is started by declaring each vertex initially active independently with probability p. Recall that we let $p_c(G, r)$ denote the critical probability at which r-boostrap percolation occurs on G with probability at least $1/2$.

The first rigorous result is due to van Enter [133], who showed that $p_c(\mathbb{Z}^d, 2) = 0$ in all dimensions $d \geq 2$. More generally, Schonmann [120] proved that $p_c(\mathbb{Z}^d, r)$ is equal to 0 if $r \leq d$ and equal to 1 otherwise. In this sense, p_c is trivial on $\mathbb{Z}^d$. Another early result is that of Aizenman and

Lebowitz [6], which identifies the order of p_c for 2-bootstrap percolation in the metastable regime in all dimensions as $p_c([n]^d, 2) = \Theta(\log^{1-d} n)$. This result was generalized many years later by Cerf and Manzo [47], who proved that $p_c([n]^d, r) = \Theta(\log_{(r-1)}^{1-d} n)$, where $\log_{(r-1)}$ denotes the iterated logarithm, defined by $\log_{(1)} = \log$ and $\log_{(k+1)} = \log\log_{(k)}$.

A famous result of Holroyd [79], the first to identify a sharp threshold, shows that

$$p_c([n]^2, 2) = \frac{\pi^2}{18}\frac{1+o(1)}{\log n}.$$

Besides its precision, part of what makes this result exciting is that the constant $\pi^2/18 \approx 0.5483$ does not compare well at all with the numerical estimate 0.245 ± 0.015 reported by Adler, Stauffer and Aharony [3]. This discrepancy is partially explained by the refined bounds for p_c obtained by Gravner and Holroyd [71] and Gravner, Holroyd and Morris [72]. There are constants $c, C > 0$, so that for all large n,

$$\frac{c}{\log^{3/2} n} \le \frac{\pi^2}{18\log n} - p_c([n]^2, 2) \le \frac{C(\log\log n)^3}{\log^{3/2} n}.$$

Essentially, the issue seems to be that the lower order terms in the expansion for p_c are only of lower order for extremely large n, lying well outside computational range. In other words, $p_c \log n$ converges to $\pi^2/18$, but very slowly.

Finally, solving a long-standing problem in the field, Balogh, Bollobás, Duminil-Copin and Morris [20] identified the sharp threshold for all r in all dimensions. For any $2 \le r \le d$, we have that, as $n \to \infty$,

$$p_c([n]^d, r) = \left(\frac{\lambda(d,r) + o(1)}{\log_{(r-1)} n}\right)^{d-r+1}$$

where $\lambda(d, r)$ is an implicitly defined constant (without a simple closed form expression for most values of d, r).

2.3.2 More results

Beyond the celebrated results discussed in the previous Section 2.3.1, there lies is a vast literature. We close this section by only naming a few more results of interest. Extensive surveys can be found in the introductory sections of articles [20, 22, 24, 27, 84], for instance.

Generalizing the work of [50] on the infinite d-regular tree T_d, Balogh, Peres and Pete [25] calculate the critical probability for bootstrap percolation on a large class of trees and graphs, which includes in particular all trees of bounded degree. For instance, it is shown that

$$\lim_{\varepsilon \downarrow 0} p_\varepsilon(T_d, 2) = 1 - \frac{(d-2)^{2d-5}}{(d-1)^{d-2}(d-3)^{d-3}}.$$

Galton-Watson trees T were considered by Bollobás, Gunderson, Holmgren, Janson and Przykucki [40] (see also Gunderson and Przykucki [76]). If T has branching number b, then $p_c(T, r) = \Omega(e^{-b/(r-1)}/b)$.

Balogh, Bollobás and Morris [22] study *majority bootstrap percolation* on the hypercube $Q_n = [2]^n$, where a vertex is activated if at least half of its neighbours are active. The first two terms in the expansion for p_c are identified. For n even, we have that

$$p_c(Q_n, n/2) = \frac{1}{2}\left(1 - \sqrt{\frac{\log n}{n}}\right) + \Theta\left(\frac{\log\log n}{\sqrt{n \log n}}\right).$$

Bootstrap percolation has given insight into the Ising model at zero (or a very low) temperature, see the works by Cerf and Manzo [48, 49], Fontes, Schonmann and Sidoravicius [65] and Morris [112]. For instance, the *zero-temperature Glauber dynamics* on $\mathbb{Z}^d$ are studied in [112], under which at random times (determined by independent exponential clocks) vertices of $\mathbb{Z}^d$ update their status to agree with the majority of their neighbours (breaking ties at random). Initially each vertex in $\mathbb{Z}^d$ is assigned a positive spin with probability p and a negative spin with probability $1-p$, independently of all other vertices. The process is said to *fixate* if eventually all spins are positive. In this setting $p_c(\mathbb{Z}^d)$ is the infimum over p such that $\mathbb{Z}^d$ fixates with probability 1. A result of Arratia [16] (see also Schwartz [122] and Lootgieter [101]) implies that $p_c(\mathbb{Z}) = 1$. On the other hand, it is a long-standing conjecture that $p_c(\mathbb{Z}^d) = 1/2$ for all $d \geq 2$. Using ideas from bootstrap percolation, it is shown in [112] that $p_c(\mathbb{Z}^d) \to 1$ as $d \to \infty$.

2.4 Random graphs

More recently, bootstrap percolation has been studied on random graphs. Balogh and Pittel [27] investigate bootstrap percolation on $\mathcal{G}_{n,d}$, the uniformly random d-regular graph of size n. With high probability $\mathcal{G}_{n,d}$ and the infinite d-regular tree T_d have a similar local structure. In [27], it is verified that p_c for $\mathcal{G}_{n,d}$ coincides with that for T_d (which recall is computed in [25], see Section 2.3.2). Moreover, the width of the scaling window is analyzed.

Majority bootstrap percolation (see Section 2.3.2) on the Erdős–Rényi random graph $\mathcal{G}_{n,p}$ (discussed in more detail in the next Section 2.4.1) has been analyzed by Holmgren, Kettle and Juškevičius [78] (see also Kettle [88], Juškevičius [85] and Stefánsson and Vallier [125]). In this setting, for sufficiently small p, it turns out that p_c for $\mathcal{G}_{n,p}$ and the hypercube Q_n (as studied in [22], see Section 2.3.2) are comparable.

Turova and Vallier [129] analyzed a variation of $\mathcal{G}_{n,p}$, where in addition to its usual random edges, each vertex $i \in [n]$ is connected to vertex $i+1$ (mod n) by an edge with probability 1. This is a simplified version of a model proposed by Turova and Villa [128] for neuronal networks, where it seems that the strength of connections within such a network depend on a mixture of random effects and distances between neurons. These additional "local connections" tighten the scaling window and cause percolation to occur in some situations in which $\mathcal{G}_{n,p}$ is unlikely to percolate.

Bootstrap percolation on random graphs with given degrees has been studied by Amini [9], Amini and Fountoulakis [10] and Janson [83].

2.4.1 Bootstrap percolation on $\mathcal{G}_{n,p}$

The remainder of this section concerns bootstrap percolation on the Erdős–Rényi [60] random graph $\mathcal{G}_{n,p}$, which is our focus in Chapters 4 and 5. Recall that $\mathcal{G}_{n,p}$ is the random subgraph of the complete graph K_n, obtained by including each possible edge independently with probability p.

2.4.1.1 Random contagious sets

Bootstrap percolation on $\mathcal{G}_{n,p}$ was first studied by Vallier [132] (see also the related works of Ball and Britton [17, 18] and Scalia-Tomba [118]). The work of [132] was expanded upon by Janson, Łuczak, Turova and Vallier [84]. Among many other detailed results, the following is proved.

Theorem 2.4.1 (Janson et al. [84, Theorem 3.1])**.** *Fix $r \geq 2$. Suppose that $\vartheta = \vartheta(n)$ satisfies $1 \ll \vartheta \ll n$. Put $\ell_r = \ell_r(\vartheta) = \frac{r}{r-1}\vartheta$ and*

$$\alpha_r = (r-1)!\left(\frac{r-1}{r}\right)^{2(r-1)}, \qquad p = p(n,\vartheta) = \left(\frac{\alpha_r}{n\vartheta^{r-1}}\right)^{1/r}.$$

Suppose that $I = I(n) \subset [n]$ is independent of $\mathcal{G}_{n,p}$ and such that $|I|/\ell_r \to \varepsilon$, as $n \to \infty$. If $\varepsilon \in [0,1)$ then with high probability $|\langle I, \mathcal{G}_{n,p}\rangle_r| < \frac{r}{r-1}|I|$. If $\varepsilon > 1$ then with high probability $|\langle I, \mathcal{G}_{n,p}\rangle_r| = n(1-o(1))$, that is, all except possibly very few vertices are eventually activated.

In this sense, ℓ_r is the critical size for a *random* set (selected independently of $\mathcal{G}_{n,p}$) to be contagious for $\mathcal{G}_{n,p}$. (By symmetry, the probability that such a set I is contagious is the same as for the set of vertices labelled 1 through $|I|$, or for any other given set of size $|I|$ that is independent of $\mathcal{G}_{n,p}$.) A heuristic for the criticality of ℓ_r is given in Section 2.4.2.2. Theorem 2.4.1 and a related central limit theorem are discussed in greater detail in Section 2.4.2.1 below.

2.4.1.2 Small contagious sets

More recently, and in contrast with the work of [84] discussed in the previous Section 2.4.1.1, Feige, Krivelevich and Reichman [62] study small contagious sets in $\mathcal{G}_{n,p}$, in a range of p. Although it is very unlikely for a random set (selected independently of $\mathcal{G}_{n,p}$) of size $\ell < \ell_r$ to be contagious, there typically *exist* contagious sets in $\mathcal{G}_{n,p}$ that are much smaller than ℓ_r.

Recall that $m(G,r)$ denotes the size of a minimal contagious set for G.

Theorem 2.4.2 (Feige et al. [62, Theorem 1.1])**.** *Fix $r \geq 2$. Suppose that $\vartheta = \vartheta(n)$ satisfies*

$$\frac{\log^2 n}{\log\log n} \ll \vartheta \ll n.$$

Let

$$\alpha_r = (r-1)!\left(\frac{r-1}{r}\right)^{2(r-1)}, \qquad p = p(n,\vartheta) = \left(\frac{\alpha_r}{n\vartheta^{r-1}}\right)^{1/r}.$$

Then, with high probability,

$$c_r \leq \frac{m(\mathcal{G}_{n,p},r)}{\psi(n,\vartheta)} \leq C_r$$

where

$$\psi(n,\vartheta) = \frac{\vartheta}{\log(n/\vartheta)},$$

$c_r < r$, and $c_r \to 2$ and $C_r = \Omega(r^{r-2})$, as $r \to \infty$.

Note that $d = np$ in [62] corresponds to $(\alpha_r(n/\vartheta)^{(r-1)})^{1/r}$ in this context. The lower bound holds in fact for all ϑ. (Although this is not stated in [62, Theorem 1.1], it follows from the proof, see [62, Corollaries 2.1 and 4.1].)

The inequality $c_r < r$ (which is relevant to our results in Section 2.6.2) is not shown in [62], so we briefly explain it here: In [62, Lemma 4.2 and Corollary 4.1], it is observed that a graph of size k with a contagious set of size ℓ has at least $r(k-\ell)$ edges. By this observation, it follows easily that with high probability

$$m(\mathcal{G}_{n,p},r) \geq \xi\frac{r-1}{r}\frac{n}{d^{r/(r-1)}\log d},$$

provided that $\xi^{r-1}e^{r+2}/(2r)^r < 1$. Since $(r-1)! > e((r-1)/e)^{r-1}$, this leads to the bound $m(\mathcal{G}_{n,p}, r) \geq c\psi(n, \vartheta)$, where

$$c < 2\left(\frac{r}{r-1}\right)^3 \left(\frac{2r}{e^4}\right)^{1/(r-1)} < r,$$

for all $r \geq 2$.

Recall that a graph is susceptible if it contains a contagious set of the smallest possible size r. Another result of [62] identifies the order of the threshold for p above which $\mathcal{G}_{n,p}$ is likely to be susceptible.

Theorem 2.4.3 (Feige et al. [62, Theorem 1.2])**.** *Let $r \geq 2$. Let $p_c(n, r)$ denote the critical threshold for the susceptibility of $\mathcal{G}_{n,p}$. As $n \to \infty$, we have that $p_c(n, r) = \Theta((n \log^{r-1} n)^{-1/r})$.*

In particular, we note that $p_c(n, 2) = \Theta(1/\sqrt{n \log n})$, an observation relevant to Section 2.5 below.

2.4.2 Binomial chains

In this section, we discuss the *binomial chain* construction used in [84] (and in Chapter 5 below) to analyze the spread of activation from an initially active set I in $\mathcal{G}_{n,p}$. This representation of the bootstrap percolation dynamics is due to Scalia-Tomba [118] (see also Sellke [123]). We refer to [84, Section 2] for a detailed description, and here only present the properties relevant to this thesis. The main idea is to reveal the graph one vertex at a time. As a vertex is revealed, we mark its neighbours. Once a vertex has been marked r times, we know it will be activated, and add it to the list of active vertices.

Formally, sets $A(t)$ and $U(t)$ of *active* and *used* vertices at time $t \geq 0$ are defined as follows: Let $A(0) = I$ and $U(0) = \emptyset$. For $t > 0$, choose some unused, active vertex $v_t \in A(t-1) - U(t-1)$, and give each neighbour of v_t a *mark*. Then let $A(t)$ be the union of $A(t-1)$ and the set of all vertices in $\mathcal{G}_{n,p}$ with at least r marks, and put $U(t) = U(t-1) \cup \{v_t\}$. The process terminates at time $t = \tau$, where $\tau = \min\{t \geq 0 : A(t) = U(t)\}$, that is, when all active vertices have been used. It is easy to see that $A(\tau) = \langle I, \mathcal{G}_{n,p}\rangle_r$.

Let $S(t) = |A(t)| - |I|$. By exploring the edges of $\mathcal{G}_{n,p}$ one step at a time, revealing the edges from v_t only at time t, the random variables $S(t)$

can be constructed in such a way that $S(t) \sim \mathrm{Bin}(n - |I|, \pi(t))$, where $\pi(t) = \mathbf{P}(\mathrm{Bin}(t,p) \geq r)$, see [84, Section 2]. Moreover, for $s < t$, we have that $S(t) - S(s) \sim \mathrm{Bin}(n - |I|, \pi(t) - \pi(s))$. Finally, it is shown that $|\langle I, \mathcal{G}_{n,p}\rangle_r| \geq k$ if and only if $\tau \geq k$ if and only if $S(t) + |I| > t$ for all $t < k$. Thus to determine the size of the eventually active set $\langle I, \mathcal{G}_{n,p}\rangle_r$, it suffices to analyze the process $S(t)$.

2.4.2.1 Activation by small sets

Making use of the binomial chain construction described in the previous Section 2.4.2, many results are developed in [84]. In this section we discuss two results which are relevant to the topic of Chapter 5 of this thesis (introduced in Section 2.6.2 below).

The following quantities play an important role in [84]. We denote

$$k_r = k_r(\vartheta) = \left(\frac{r}{r-1}\right)^2 \vartheta, \qquad \ell_r = \ell_r(\vartheta) = \frac{r-1}{r} k_r.$$

For $\varepsilon \in [0,1]$, we define $\delta_\varepsilon \in [0, \varepsilon]$ implicitly by

$$\frac{\delta_\varepsilon^r}{r} = \delta_\varepsilon - \varepsilon_r, \qquad \varepsilon_r = \frac{r-1}{r}\varepsilon.$$

(We note that $\ell_r, k_r, \delta_\varepsilon$ correspond to $a_c, t_c, \varphi(\varepsilon)$ in [84].)

As shown by Theorem 2.4.1 above, if

$$p = p(n,\vartheta) = \left(\frac{\alpha_r}{n\vartheta^{r-1}}\right)^{1/r} = \left(\frac{(r-1)!}{nk_r^{r-1}}\right)^{1/r} \tag{2.4.1}$$

then ℓ_r is the critical size for a random set to be contagious (see Section 2.4.2.2 for a heuristic explanation for this fact).

More precisely, the following results are proved in [84].

Theorem 2.4.4 ([84, Theorem 3.1]). *Fix $r \geq 2$. Let p be as in (2.4.1), where $\vartheta = \vartheta(n)$ satisfies $1 \ll \vartheta \ll n$. Suppose that $I = I(n) \subset [n]$ is independent of $\mathcal{G}_{n,p}$ and such that $|I|/\ell_r \to \varepsilon$, as $n \to \infty$. If $\varepsilon \in [0,1)$, then with high probability $|\langle I, \mathcal{G}_{n,p}\rangle_r| = (\delta_\varepsilon + o(1))k_r$. On the other hand, if $\varepsilon > 1$, then with* *high probability* $|\langle I, \mathcal{G}_{n,p}\rangle_r| = n(1-o(1))$.

(If $np \gg \log n + (r-1)\log\log n$, then, in fact, with high probability I is contagious, that is $|\langle I, \mathcal{G}_{n,p}\rangle_r| = n$, see [84, Theorem 3.1](iii).) Moreover, the following central limit theorem is established. Recall that a sequence of random variables X_n is *asymptotically normal* with mean μ_n and variance σ_n^2 if $(X_n - \mu_n)/\sigma_n$ converges in distribution to a standard normal.

Theorem 2.4.5 ([84, Theorem 3.8(i)])**.** *Fix $r \geq 2$. Let p be as in* (2.4.1)*, where $\vartheta = \vartheta(n)$ satisfies $1 \ll \vartheta \ll n$. Suppose that $I = I(n) \subset [n]$ is independent of $\mathcal{G}_{n,p}$ and such that $|I|/\ell_r \to \varepsilon \in (0,1)$, as $n \to \infty$. Then $|\langle I, \mathcal{G}_{n,p}\rangle_r|$ is asymptotically normal with mean $\mu \sim \delta_\varepsilon k_r$ and variance $\sigma^2 = \delta'_\varepsilon k_r$, where $\delta'_\varepsilon = \delta_\varepsilon^r (1-\delta_\varepsilon^{r-1})^{-2}/r$.*

(See (3.13) and (3.22) in [84] for the definition of μ.) In particular, note that the mean and variance of $|\langle I, \mathcal{G}_{n,p}\rangle_r|$ are of the same order as k_r.

2.4.2.2 Criticality of ℓ_r

Let p, ℓ_r, k_r be as defined in the previous Section 2.4.2.1. In [84, Section 6] a heuristic is provided for the criticality of ℓ_r, which we recount here. By the law of large numbers, with high probability $S(t) \approx \mathbf{E}S(t)$. A calculation shows that if $|I| > \ell_r$ then $|I| + \mathbf{E}S(t) \geq t$ for all $t < n - o(n)$, whereas if $|I| < \ell_r$ then already for $t = k_r$ we get $|I| + \mathbf{E}S(k_r) < k_r$.

In particular, for $t \leq k_r$, since $\vartheta \ll n$ we have that

$$pt \leq pk_r = O((\vartheta/n)^{1/r}) \ll 1.$$

It follows that $\pi(t) \sim (tp)^r/r!$. We therefore have for $t = xk_r$ that

$$\mathbf{E}S(xk_r) = (n - |I|)\pi(t) \sim \frac{x^r}{r} k_r \cdot \frac{k_r^{r-1} np^r}{(r-1)!} = \frac{x^r}{r} k_r.$$

If $|I| < \ell_r$, then for $x = 1$ we have

$$|I| + \mathbf{E}S(k_r) < \ell_r + k_r/r = k_r.$$

2.5 Graph bootstrap percolation

In this section, we discuss a variation of bootstrap percolation due to Bollobás [39], which is the topic of Chapter 6.

Fix a graph H. Following [39], *H-bootstrap percolation* is a cellular automaton that adds edges to a graph $G = (V, E)$ by iteratively completing all copies of H missing a single edge. Formally, given a graph $G_0 = G$, let G_{i+1} be G_i together with every edge whose addition creates a subgraph that is isomorphic to H. For a finite graph G, this procedure terminates once $G_{\tau+1} = G_\tau$, for some $\tau = \tau(G)$. We denote the resulting graph G_τ by $\langle G\rangle_H$. If $\langle G\rangle_H$ is the complete graph on V, the graph G is said to *H-percolate,* or equivalently, that G is *H-percolating.*

Balogh, Bollobás and Morris [24] study H-bootstrap percolation in the case that $G = \mathcal{G}_{n,p}$ and $H = K_k$. The case $k = 4$ is the minimal case of interest. Indeed, *all* graphs K_2-percolate, and a graph K_3-percolates if and only if it is connected. Thus the case K_3 follows by a classical result of Erdős and Rényi [60]. If $p = (\log n + \varepsilon)/n$ then $\mathcal{G}_{n,p}$ is K_3-percolating with probability $\exp(-e^{-\varepsilon})(1 + o(1))$, as $n \to \infty$.

One may define the critical thresholds for H-bootstrap percolation by

$$p_c(n, H) = \inf \{p > 0 : \mathbf{P}(\langle \mathcal{G}_{n,p}\rangle_H = K_n) \geq 1/2\}.$$

It is expected that this property has a sharp threshold for $H = K_k$ for all k, in the sense that for some $p_c = p_c(k)$ we have that $\mathcal{G}_{n,p}$ is K_k-percolating with high probability for $p > (1 + \delta)p_c$ and with probability tending to 0 for $p = (1 - \delta)p_c$. Some bounds for $p_c(n, K_k)$ are obtained in [24]. A main result of [24] identifies the order of the threshold for K_4-percolation.

Theorem 2.5.1 (Balogh et al. [24, Theorem 2])**.** *Let $p_c(n, K_4)$ denote the critical threshold for K_4-bootstrap percolation on $\mathcal{G}_{n,p}$. As $n \to \infty$, we have that $p_c(n, K_4) = \Theta(1/\sqrt{n \log n})$.*

Note that the order of $p_c(n, K_4)$ coincides with that for the susceptibility of $\mathcal{G}_{n,p}$ in the case that $r = 2$, see Theorem 2.4.3. This connection is discussed further in Section 2.6.3 below. For larger k, the order of p_c is known only up to a poly-logarithmic factor, see [24, Theorem 1].

2.5.1 Clique processes

In [24], the *clique process* is introduced as a way of analyzing K_4-percolation on graphs. This process plays a key role in Chapter 6.

Definition 2.5.2. We say that three graphs $G_i = (V_i, E_i)$ *form a triangle* if there are distinct vertices x, y, z such that $x \in V_1 \cap V_2$, $y \in V_1 \cap V_3$ and $z \in V_2 \cap V_3$.

In [24], the following observation is made.

Lemma 2.5.3. *Suppose that $G_i = (V_i, E_i)$ are K_4-percolating.*
(i) If $|V_1 \cap V_2| > 1$ then $G_1 \cup G_2$ is K_4-percolating.
(ii) If the G_i form a triangle then $G_1 \cup G_2 \cup G_3$ is K_4-percolating.

By these observations, the K_4-percolation dynamics are classified in [24] as follows.

Definition 2.5.4. A *clique process* for a graph G is a sequence $(\mathcal{S}_t)_{t=1}^{\tau}$ of collections of subgraphs of G with the following properties:

(i) $\mathcal{S}_0 = E(G)$ is the edge set of G.
(ii) For each $t < \tau$, $\mathcal{S}_{t+1}$ is constructed from $\mathcal{S}_t$ by either (a) merging two subgraphs $G_1, G_2 \in \mathcal{S}_t$ with at least two common vertices, or (b) merging three subgraphs $G_1, G_2, G_3 \in \mathcal{S}_t$ that form a triangle.
(iii) $\mathcal{S}_\tau$ is such that no further operations as in (ii) are possible.

Lemma 2.5.5. *Let G be a finite graph and $(\mathcal{S}_t)_{t=1}^{\tau}$ a clique process for G. For all $t \le \tau$, $\mathcal{S}_t$ is a collection of edge-disjoint, K_4-percolating subgraphs of G. Furthermore, $\langle G \rangle_{K_4}$ is the edge-disjoint, triangle-free union of the cliques $\langle H \rangle$, $H \in \mathcal{S}_\tau$. Hence G is K_4-percolating if and only if $\mathcal{S}_\tau = \{G\}$. In particular, if two clique processes for G terminate at $\mathcal{S}_\tau$ and $\mathcal{S}'_{\tau'}$, then necessarily $\mathcal{S}_\tau = \mathcal{S}'_{\tau'}$.*

The existence of such a concise description of the dynamics is the reason why the results of [24] are stronger for K_4-percolation than for K_k-percolation, $k > 4$. Indeed, the bounds for $p_c(n, K_k)$ obtained in [24] for the cases $k > 4$ hold for $p_c(n, H)$ for all graphs H in a certain class that in particular contains K_k. As it stands now, the order of $p_c(n, K_4)$ is unknown for $k > 4$.

2.6 Our results

Finally, we introduce our main contributions to the study of bootstrap percolation on random graphs, in relation to the results discussed above. The following results are proved in Part III.

2.6.1 Susceptibility

The susceptibility of $\mathcal{G}_{n,p}$ is analyzed in Chapter 4.

Theorem 2.6.1 (Angel and Kolesnik [12]). *Fix $r \geq 2$ and $\alpha > 0$. Let*

$$\alpha_r = (r-1)!\left(\frac{r-1}{r}\right)^{2(r-1)}, \qquad p = \theta_r(\alpha, n) = \left(\frac{\alpha}{n\log^{r-1} n}\right)^{1/r}.$$

If $\alpha > \alpha_r$ then with high probability $\mathcal{G}_{n,p}$ is susceptible. If $\alpha < \alpha_r$ then with high probability $\mathcal{G}_{n,p}$ has no contagious set of size r.

As a result, we identity the sharp thresholds for the susceptibility of $\mathcal{G}_{n,p}$ as $p_c(n, r) \sim \theta_r(\alpha_r, n)$, improving the estimates given by Feige, Krivelevich and Reichman [62] in Theorem 2.4.3.

We identify p_c using the standard first and second moment methods. That being said, due to the fact that contagious sets are highly correlated, establishing the upper bound for p_c involves a fairly involved application of the second moment method. Roughly speaking, we restrict to a sub-process of the r-bootstrap percolation process that evolves without forming triangles. As it turns out, triangle-free percolating subgraphs of $\mathcal{G}_{n,p}$ are much less correlated, and by using Mantel's [102] theorem, their approximate independence is readily established. It then remains to show that the threshold for this sub-process coincides with p_c up to smaller order terms. See Section 4.2.4 below for a more detailed outline of the proof.

It is interesting to compare this result with Theorem 2.4.1. We find that if $p = \theta_r(\alpha, n)$, for some $\alpha = (1+\delta)\alpha_r$, then with high probability $\mathcal{G}_{n,p}$ has a contagious set of size r, however a random set (selected independently of $\mathcal{G}_{n,p}$) is likely to be contagious only if it is of size (roughly) at least $\frac{r}{r-1}\log n$.

Moreover, for sub-critical p, we obtain the following information about the influence of sets of size r.

Theorem 2.6.2 (Angel and Kolesnik [12])**.** *Fix $r \geq 2$. Let $p = \theta_r(\alpha, n)$, for some $\alpha \in (0, \alpha_r)$. With high probability the maximum of $|\langle I, \mathcal{G}_{n,p}\rangle_r|$ over sets $I \subset [n]$ of size r is equal to $(\beta_* + o(1))\log n$, where $\beta_*(\alpha) \in (0, (\frac{r}{r-1})^2)$ satisfies*

$$r + \beta \log\left(\frac{\alpha\beta^{r-1}}{(r-1)!}\right) - \frac{\alpha\beta^r}{r!} - \beta(r-2) = 0.$$

In other words, for any $\delta > 0$, with high probability there exist sets I of size r that activate more than $(1-\delta)\beta_* \log n$ vertices, however none that activate more than $(1+\delta)\beta_* \log n$.

2.6.2 Minimal contagious sets

In Chapter 5, we study minimal contagious sets in $\mathcal{G}_{n,p}$. Recall that $m(G, r)$ denotes the size of minimal contagious sets for a graph G. We obtain the following improved bounds for $m(\mathcal{G}_{n,p}, r)$, for all $r \geq 2$.

Theorem 2.6.3 (Angel and Kolesnik [11])**.** *Fix $r \geq 2$. Suppose that $\vartheta = \vartheta(n)$ satisfies $1 \ll \vartheta \ll n$. Let*

$$\alpha_r = (r-1)!\left(\frac{r-1}{r}\right)^{2(r-1)}, \qquad p = p(n, \vartheta) = \left(\frac{\alpha_r}{n\vartheta^{r-1}}\right)^{1/r}.$$

Then, with high probability,

$$m(\mathcal{G}_{n,p}, r) \geq r\psi(1 + o(1))$$

where

$$\psi = \psi(n, \vartheta) = \frac{\vartheta}{\log(n/\vartheta)}$$

and $o(1)$ *depends only on* n*.*

This result improves the lower bounds of Feige, Krivelevich and Reichman [62] in Theorem 2.4.2, noting that $c_r < r$ for all $r \geq 2$. To give some intuition for this significant improvement, recall (as discussed below Theorem 2.4.2) that the bound $m(\mathcal{G}_{n,p}, r) \geq c_r \psi$ in Theorem 2.4.2 is proved simply by noting that a graph of size k with a contagious set of size ℓ has at least $r(k - \ell)$ edges. On the other hand, in Chapter 5 we in a sense track the full trajectory of activation in percolating graphs, rather than using only a rough estimate for graphs arrived at by such trajectories. Using (discrete) variational calculus, we identify the optimal trajectory from a set of size ℓ in $\mathcal{G}_{n,p}$ to an eventually active set of k vertices. This leads to refined bounds for the structure of percolating subgraphs of $\mathcal{G}_{n,p}$ with unusually small contagious sets, and so an improved bound for $m(\mathcal{G}_{n,p}, r)$.

Moreover, since $c_r \to 2$, our bound is larger by a factor of roughly $r/2$ for large r. Hence the improvement of our bound increases with r. This is due to the fact that the crude bound of $r(k - \ell)$ for the number of edges in a graph of size k with a contagious set of size ℓ is an increasingly inaccurate estimate for the combinatorics of such graphs as $r \to \infty$.

Hence, in particular, we find that $m(\mathcal{G}_{n,p}, r)/\psi(n, \vartheta)$ grows at least linearly in r. It seems plausible that this is the truth, and that moreover, our bound is asymptotically sharp. In any case, as it stands now, a substantial gap remains between our linear lower bound and the super-exponential upper bound in Theorem 2.4.2. This upper bound has the advantage of being proved by a procedure that with high probability locates a contagious set in polynomial time. That being said, this set is possibly much larger than a *minimal* contagious set, especially for large r. In future work, we hope to (1) identify $m(\mathcal{G}_{n,p}, r)$ up to a factor of $1 + o(1)$ and (2) efficiently locate contagious sets that are as close as possible to minimal.

As a consequence, we obtain lower bounds for the critical threshold $p_c(n, r, q)$ for the (q, r)-susceptibility of $\mathcal{G}_{n,p}$ (see Section 2.2.2).

Corollary 2.6.4 (Angel and Kolesnik [11])**.** *Fix $r \geq 2$. Suppose that $r \leq q = q(n) \ll n/\log n$. As $n \to \infty$,*

$$p_c(n, r, q) \geq \left(\frac{\alpha_{r,q}}{n \log^{r-1} n}\right)^{1/r} (1 + o(1)),$$

where $\alpha_{r,q} = \alpha_r (r/q)^{r-1}$.

We note that the results in Section 2.6.1 confirm that this bound is sharp in the special case $q = r$.

These results follows by large deviation estimates for the number of vertices eventually activated by a set that is smaller than the critical amount ℓ_r, as defined in Theorem 2.4.1 (and discussed further in Sections 2.4.2.1 and 2.4.2.2 above).

We let $P(\ell, k)$ denote the probability that for a given set $I \subset [n]$ (independent of $\mathcal{G}_{n,p}$), with $|I| = \ell$, we have that $|\langle I, \mathcal{G}_{n,p}\rangle_r| \geq k$. Recall $k_r, \ell_r, \delta_\varepsilon, \varepsilon_r$ as defined in Section 2.4.2.1. The following is the key result of Chapter 5.

Theorem 2.6.5 (Angel and Kolesnik [11])**.** *Fix $r \geq 2$. Let p be as in* (2.4.1)*, where $\vartheta = \vartheta(n)$ satisfies $1 \ll \vartheta \ll n$. Let $\varepsilon \in [0, 1)$ and $\delta \in [\delta_\varepsilon, 1]$. Suppose that $\ell/\ell_r \to \varepsilon$ and $k/k_r \to \delta$, as $n \to \infty$. Then, as $n \to \infty$, we have that $P(\ell, k) = \exp[\xi k_r (1 + o(1))]$, where $\xi = \xi(\varepsilon, \delta)$ is equal to*

$$-\frac{\delta^r}{r} + \begin{cases} (\delta - \varepsilon_r)\log(e r^{-1}\delta^r/(\delta - \varepsilon_r)), & \delta \in [\delta_\varepsilon, \varepsilon); \\ (\varepsilon/r)\log\left(e\varepsilon^{r-1}\right) - (r-2)(\delta - \varepsilon) + (r-1)\log(\delta^\delta/\varepsilon^\varepsilon), & \delta \in [\varepsilon, 1], \end{cases}$$

and $o(1)$ depends only on n.

We note that $t = k_r$ is the point at which the binomial chain $S(t)$ becomes super-critical (see Section 2.4.2), so we have that $P(\varepsilon\ell_r, \delta k_r) = e^{o(k_r)} P(\varepsilon\ell_r, k_r)$ for $\delta > 1$.

These estimates complement the central limit theorems (Theorem 2.4.5) of Janson, Łuczak, Turova and Vallier [84]. Indeed, since the mean and variance of $|\langle I, \mathcal{G}_{n,p}\rangle_r|$ are of the same order (see Theorem 2.4.5), the event that $|\langle I, \mathcal{G}_{n,p}\rangle_r| \geq \delta k_r$, for some $\delta \in (\delta_\varepsilon, 1]$, represents a large deviation from the typical behaviour. Hence $-(\frac{r}{r-1})^2\xi$ is the large deviations rate function corresponding to the events of interest $\{|\langle I, \mathcal{G}_{n,p}\rangle_r| \geq k\}$.

2.6.3 K_4-percolation

In Chapter 6, we study K_4-bootstrap percolation on $\mathcal{G}_{n,p}$. We identify the sharp threshold as $p_c(n, K_4) \sim 1/\sqrt{3n\log n}$, improving the estimates of Balogh, Bollobás and Morris [24] in Theorem 2.5.1, thereby solving Problem 2 stated in [24].

Theorem 2.6.6 (Kolesnik [91]). *Let $p = \sqrt{\alpha/(n\log n)}$. If $\alpha > 1/3$ then $\mathcal{G}_{n,p}$ is K_4-percolating with high probability. If $\alpha < 1/3$ then with high probability $\mathcal{G}_{n,p}$ does not K_4-percolate.*

We note that the super-critical case $\alpha > 1/3$ follows by the results in Section 2.6.1 (joint work with Angel [12]) in the case of $r = 2$, as explained below the statement of the next theorem. It thus remains to study the sub-critical case $\alpha < 1/3$.

In the sub-critical case, we also identify the size of the largest K_4-percolating subgraphs of $\mathcal{G}_{n,p}$.

Theorem 2.6.7 (Kolesnik [91]). *Let $p = \sqrt{\alpha/(n\log n)}$, for some $\alpha \in (0, 1/3)$. With high probability the largest clique in $\langle\mathcal{G}_{n,p}\rangle_{K_4}$ has size $(\beta_* + o(1))\log n$, where $\beta_*(\alpha) \in (0, 3)$ satisfies $3/2 + \beta\log(\alpha\beta) - \alpha\beta^2/2 = 0$.*

By the results discussed in Section 2.6.1 (joint work with Angel [12]), it follows that with high probability $\langle\mathcal{G}_{n,p}\rangle_{K_4}$ has cliques of size at least $(\beta_* + o(1))\log n$. Our contribution is to show that these are typically the largest cliques.

The main ingredients in the proof are the large deviation estimates discussed in Section 2.6.2 (joint work with Angel [11]) and a connection with the susceptibility of $\mathcal{G}_{n,p}$, in the case of $r = 2$ (as observed in [12]). Indeed, since $p_c(n, K_4)$ and $p_c(n, 2)$ are on the same order (see Theorems 2.4.3 and 2.5.1), it is natural to ask how the two processes are related. We note that if a graph $G = (V, E)$ has a contagious pair $\{u, v\} \subset V$ that is joined by an edge $(u, v) \in E$, then G is K_4-percolating (see Section 4.2.2). In this case, we call G a *seed graph* and (u, v) a *seed edge*. The above theorems show that, although a graph can K_4-percolate in a variety of ways (see Section 2.5.1), up to smaller order terms $p_c(n, K_4)$ coincides with the threshold for the event that $\mathcal{G}_{n,p}$ has a seed edge.

It is a general phenomenon of $\mathcal{G}_{n,p}$ that often the threshold for a property of interest coincides with that of a more fundamental event. Moreover, even stronger results hold in some cases. For example, one of the first results on $\mathcal{G}_{n,p}$, in the original paper [60], shows that with high probability $\mathcal{G}_{n,p}$ is connected (equivalently, K_3-percolating) if and only if it has no isolated vertices. Komlós and Szemerédi [92] showed that with high probability $\mathcal{G}_{n,p}$ is Hamiltonian if and only if its minimum degree is at least 2.

In closing, we mention that it seems possible that K_4-percolation is more complicated than K_3-percolation. Perhaps, for p in the scaling window (see Section 2.2.1), the probability that $\mathcal{G}_{n,p}$ has a seed edge converges to a constant in $(0, 1)$, and with non-vanishing probability $\mathcal{G}_{n,p}$ is K_4-percolating due instead to a small K_4-percolating subgraph C of size $O(1)$ that plays the role of a seed edge (i.e., is K_4-percolating and causes $\mathcal{G}_{n,p}$ to K_4-percolate by successively adding doubly connected vertices).

Part II

Geodesics in Random Surfaces

Chapter 3

Stability of Geodesics in the Brownian Map

3.1 Overview

The Brownian map is a random geodesic metric space arising as the scaling limit of random planar maps. We strengthen the so-called *confluence of geodesics* phenomenon observed at the root of the map, and with this, reveal several properties of its rich geodesic structure.

Our main result is the continuity of the cut locus at typical points. A small shift from such a point results in a small, local modification to the cut locus. Moreover, the cut locus is uniformly stable, in the sense that any two cut loci coincide outside a closed, nowhere dense set of zero measure.

We obtain similar stability results for the set of points inside geodesics to a fixed point. Furthermore, we show that the set of points inside geodesics of the map is of first Baire category. Hence, most points in the Brownian map are endpoints.

Finally, we classify the types of geodesic networks which are dense. For each $k \in \{1, 2, 3, 4, 6, 9\}$, there is a dense set of pairs of points which are joined by networks of exactly k geodesics and of a specific topological form. We find the Hausdorff dimension of the set of pairs joined by each type of network. All other geodesic networks are nowhere dense.*

*This chapter is joint work with Omer Angel and Grégory Miermont [13], to appear in the *Annals of Probabilty*.

3.2 Background and main results

A universal scaling limit of random planar maps has recently been identified by Le Gall [98] (triangulations and $2k$-angulations, $k > 1$) and Miermont [108] (quadrangulations) as a random geodesic metric space called the *Brownian map* (M, d). In this chapter, we establish properties of the Brownian map which are a step towards a complete understanding of its geodesic structure.

The works of Cori and Vauquelin [52] and Schaeffer [119] describe a bijection from well-labelled plane trees to rooted planar maps. The Brownian map is obtained as a quotient of Aldous' [7, 8] *continuum random tree*, or CRT, by assigning Brownian labels to the CRT and then identifying some of its non-cut-points, or *leaves*, according to a continuum analogue of the *CVS-bijection* (see Section 3.3.1). The resulting object is homeomorphic to the sphere $\mathbb{S}^2$ (Le Gall and Paulin [100] and Miermont [106]) and of Hausdorff dimension 4 (Le Gall [96]) and is thus in a sense a random, fractal, spherical surface.

Le Gall [97] classifies the geodesics to the root, which is a certain distinguished point of the Brownian map (see Section 3.3.1), in terms of the label process on the CRT (see Section 3.3.2). Moreover, the Brownian map is shown to be invariant in distribution under uniform re-rooting from the volume measure λ on M (see Section 3.3.1). Hence, geodesics to typical points exhibit a similar structure as those to the root. It thus remains to investigate geodesics from special points of the Brownian map.

3.2.1 Geodesic nets

A striking consequence of Le Gall's description of geodesics to the root is that any two such geodesics are bound to meet and then coalesce before reaching the root, a phenomenon referred to as the *confluence of geodesics* (see Section 3.3.3). In fact, the set of points in the relative interior of a geodesic to the root is a small subset which is homeomorphic to an $\mathbb{R}$-tree and of Hausdorff dimension 1 (see [97]).

Definition 3.2.1. We call a subset $\gamma \subset M$ a *geodesic segment* if (γ, d) is isometric to a compact interval. The *extremities* of the geodesic segment are the images, say x and y, of the extremities of the source interval, and we say that γ is a geodesic segment between x and y (or from x to y if we insist on distinguishing one orientation of γ).

We will often denote a particular geodesic segment between $x, y \in M$ as $[x, y]$, and denote its relative interior by $(x, y) = [x, y] - \{x, y\}$. (Since there might be more than one such geodesic segment, we will be careful in lifting any ambiguity that might arise from this notation.) We define $[x, y)$ and $(x, y]$ similarly.

Definition 3.2.2. For $x \in M$, the *geodesic net* of x, denoted $G(x)$, is the set of points $y \in M$ that are contained in the relative interior of a geodesic segment to x.

Although geodesics to the root of the Brownian map are understood, the structure of geodesics to general points remains largely mysterious. Indeed, the main obstacle in establishing the existence of the Brownian map is to relate a geodesic between a pair of typical points to geodesics to the root. A compactness argument of Le Gall [96] yields scaling limits of planar maps along subsequences, however the question of uniqueness remained unresolved for some time. Finally, making use of Le Gall's description of geodesics to the root, Le Gall [98] and Miermont [108] show that distances to the root provide enough information to characterize the Brownian map metric. Let γ be a geodesic between points selected uniformly according to λ. (By the confluence of geodesics phenomenon, the root of the map is almost surely disjoint from γ.) In [98, 108] the set of points $z \in \gamma$ such that the relative interior of any geodesic from z to the root is disjoint from γ is shown to be small compared to γ. Hence, roughly speaking, "most" points in "most" geodesics of the Brownian map are in a geodesic to the root. (See the discussion around equation (2) in [98] and [108, Section 2.3] for precise statements.)

In this chapter, we show that for *any* two points $x, y \in M$, points which are in a geodesic to x but not in a geodesic to y are exceptional. Hence, to a considerable extent, the geodesic structure of the Brownian map is similar as viewed from any point of the map, providing further evidence that it is, to quote Le Gall [95], "very regular in its irregularity."

Theorem 3.2.3. *Almost surely, for all $x, y \in M$, $G(x)$ and $G(y)$ coincide outside a closed, nowhere dense set of zero λ-measure.*

Furthermore, for most points $x \in M$, the effect of small perturbations of x on $G(x)$ is localized.

Theorem 3.2.4. *Almost surely, the function $x \mapsto G(x)$ is continuous almost everywhere in the following sense.*

For λ-almost every $x \in M$, for any neighbourhood N of x, there is a sub-neighbourhood $N' \subset N$ so that $G(x') - N$ is the same for all $x' \in N'$.

The *uniform infinite planar triangulation*, or UIPT, introduced by Angel and Schramm [14], is a random lattice which arises as the *local limit* of random triangulations of the sphere. The case of quadrangulations, giving rise to the UIPQ, is due to Krikun [93]. We remark that Theorem 3.2.4 is in a sense a continuum analogue to a result of Krikun [94] (see also Curien, Ménard, and Miermont [56]) which shows that the "Schaeffer's tree" of the UIPQ only changes locally after relocating its root.

Next, we find that the union of all geodesic nets is relatively small.

Definition 3.2.5. Let $F = \bigcup_{x \in M} G(x)$ denote the set of points in the relative interior of a geodesic in (M, d). We refer to F as the *geodesic framework* and $E = F^c$ as the *endpoints of the Brownian map*.

Theorem 3.2.6. *Almost surely, the geodesic framework of the Brownian map, $F \subset M$, is of first Baire category.*

Hence, the endpoints of the Brownian map, $E \subset M$, is a residual subset. This property of the Brownian map is reminiscent of a result of Zamfirescu [138], which states that for most convex surfaces — that is, for all surfaces in a residual subset of the Baire space of convex surfaces in $\mathbb{R}^n$ endowed with the Hausdorff metric — the endpoints form a residual set.

3.2.2 Cut loci

Recall that the cut locus of a point p in a Riemannian manifold, first examined by Poincaré [115], is the set of points $q \neq p$ which are endpoints of maximal (minimizing) geodesics from p. This collection of points is more subtle than merely the set of points with multiple geodesics to p, and in fact, is generally the closure thereof (see Klingenberg [90, Theorem 2.1.14]).

In the Brownian map this equivalence breaks completely. Indeed, almost all (in the sense of volume, by the confluence of geodesics phenomenon and invariance under re-rooting) and most (in the sense of Baire category, by Theorem 3.2.6) points are the end of a maximal geodesic, and every point is joined by multiple geodesics to a dense set of points (see the note after the proof of Proposition 3.5.2). Moreover, whereas in the Brownian map there are points with multiple geodesics to the root which coalesce before reaching the root, in a Riemannian manifold any (minimizing) geodesic which is not the unique geodesic between its endpoints cannot be extended (see, for example, the "short-cut principle" discussed in Shiohama, Shioya and Tanaka [124, Remark 1.8.1]).

We introduce the following notions of cut locus for the Brownian map.

Definition 3.2.7. For $x \in M$, the *weak cut locus* of x, denoted $S(x)$, is the set of points $y \in M$ with multiple geodesics to x. The *strong cut locus* of x, denoted $C(x)$, is the set of points $y \in M$ to which there are at least two geodesics from x that are disjoint in a neighbourhood of y.

We will see that for most points x, it holds that $S(x) = C(x)$ (Proposition 3.5.3). However, in some sense, $C(x)$ is better-behaved than $S(x)$ for the remaining exceptional points, and we will argue in Section 3.5.3 below that $C(x)$ is more effective at capturing the essence of a cut-locus for the metric space (M, d).

The construction of the Brownian map as a quotient of the CRT gives a natural mapping from the CRT to the map. Let ρ denote the root of the map. Cut-points of the CRT correspond to a dense subset $S(\rho) \subset M$ of Hausdorff dimension 2 (see [97]). Le Gall's description of geodesics reveals that $S(\rho)$ is almost surely exactly the set of points with multiple geodesics to ρ (see Section 3.3.2). More specifically, for any $y \in M$, the number of connected components of $S(\rho) - \{y\}$ is precisely the number of geodesics from y to ρ. This is similar to the case of a complete, analytic Riemannian surface homeomorphic to the sphere (see Poincaré [115] and Myers [113]) where the cut locus S of a point x is a tree and the number of "branches" emanating from a point in S is exactly the number of geodesics to x.

Since the strong cut locus of the root of the Brownian map corresponds to the CRT minus its leaves — that is, almost surely $S(\rho) = C(\rho)$, where ρ is the root (see Section 3.3.2) — it is a fundamental subset of the map.

We obtain analogues of Theorems 3.2.3 and 3.2.4 for the strong cut locus.

Theorem 3.2.8. *Almost surely, for all $x, y \in M$, $C(x)$ and $C(y)$ coincide outside a closed, nowhere dense set of zero λ-measure.*

Theorem 3.2.9. *Almost surely, the function $x \mapsto C(x)$ is continuous almost everywhere in the following sense.*

For λ-almost every $x \in M$, for any neighbourhood N of x, there is a sub-neighbourhood $N' \subset N$ so that $C(x') - N$ is the same for all $x' \in N'$.

Theorem 3.2.9 brings to mind the results of Buchner [44] and Wall [135], which show that the cut locus of a fixed point in a compact manifold is continuously stable under perturbations of the metric on an open, dense subset of its Riemannian metrics (endowed with the Whitney topology).

As for the geodesic nets in Theorem 3.2.6, we show that the union of all strong cut loci is a small subset of the map.

Theorem 3.2.10. *Almost surely, $\bigcup_{x \in M} C(x)$ is of first Baire category.*

We remark that Gruber [75] (see also Zamfirescu [139]) shows that for most (in the sense of Baire category) convex surfaces X, for any point $x \in X$, the set of points with multiple geodesics to x is of first Baire category. Since for typical points $x \in M$, $C(x)$ is exactly the set of points with multiple geodesics to x (that is, $C(x) = S(x)$, see Proposition 3.5.3), Theorem 3.2.10 shows that this property holds almost surely for almost every point of the Brownian map. That being said, there is a dense set of atypical points D such that every $x \in D$ is connected to *all* points outside a small neighbourhood of x by multiple geodesics (see Proposition 3.5.2).

3.2.3 Geodesic networks

Next, we investigate the structure of geodesic segments between pairs of points in the Brownian map.

Definition 3.2.11. For $x, y \in M$, the *geodesic network* between x and y, denoted by $G(x, y)$, is the set of points in some geodesic segment between x and y.

Geodesic networks with one endpoint being the root of the map (or a typical point by invariance under re-rooting) are well understood. As discussed in Section 3.2.2, for any $y \in M$, the number of connected components in $S(\rho) - \{y\}$ gives the number of geodesics from y to ρ. Hence, by properties of the CRT, almost surely there is a dense set with Hausdorff dimension 2 of points with exactly two geodesics to the root; a dense, countable set of points with exactly three geodesics to the root; and no points connected to the root by more than three geodesics. By invariance under re-rooting, it follows that the set of pairs that are joined by multiple geodesics is a zero-volume subset of $(M^2, \lambda \otimes \lambda)$ (see also Miermont [107]). Hence the vast majority of networks in the Brownian map consist of a single geodesic segment. Furthermore, by Le Gall's description of geodesics to the root and invariance under re-rooting, geodesic segments from a typical point of the Brownian map have a specific topological structure.

For $x \in M$, let $B(x, \varepsilon)$ denote the open ball of radius ε centred at x.

Definition 3.2.12. We say that the ordered pair of distinct points (x, y) is *regular* if any two distinct geodesic segments between x and y are disjoint inside, and coincide outside, a punctured ball centred at y of radius less than $d(x, y)$. Formally, if γ and γ' are geodesic segments between x and y, then there exists $r \in (0, d(x, y))$ such that $\gamma \cap \gamma' \cap B(y, r) = \{y\}$ and $\gamma - B(y, r) = \gamma' - B(y, r)$.

For typical points x, all pairs (x, y) are regular (see Section 3.3.2).

We note that this notion is not symmetric, that is, (x, y) being regular does not imply that (y, x) is regular. In fact, observe that (x, y) and (y, x) are regular if and only if there is a unique geodesic from x to y.

A key property is the following.

Lemma 3.2.13. *If (x, y) is regular and γ is a geodesic segment between x and y, then for any point z in the relative interior of γ, the segment $[x, z] \subset \gamma$ is the unique geodesic segment between x and z. Hence, any points $z \neq z'$ in the relative interior of γ are joined by a unique geodesic.*

Consequently, any geodesic segment γ' to x that intersects the relative interior of γ at some point z coalesces with γ from that point on, that is, $\gamma \cap B(x, d(x, z)) = \gamma' \cap B(x, d(x, z))$.

Proof. Let (x, y) be regular and let γ be a geodesic segment between x and y. Assume that there are two distinct geodesic segments γ_1, γ_2 between z and x, where z is some point in the relative interior of γ. By adding the sub-segment $[y, z] \subset \gamma$ to γ_1 and γ_2, we obtain two distinct geodesic segments between y and x that coincide in the non-empty neighbourhood $B(y, d(y, z))$ of y, contradicting the definition of regularity for (x, y). This gives the first part of the statement, and the second part is a straightforward consequence. ■

We find that all except very few geodesic networks in the Brownian map are, in the following sense, a concatenation of two regular networks.

Definition 3.2.14. For $(x, y) \in M^2$ and $j, k \in \mathbb{N}$, we say that (x, y) induces a *normal* (j, k)*-network*, and write $(x, y) \in N(j, k)$, if for some z in the relative interior of all geodesic segments between x and y, (z, x) and (z, y) are regular and z is connected to x and y by exactly j and k geodesic segments, respectively.

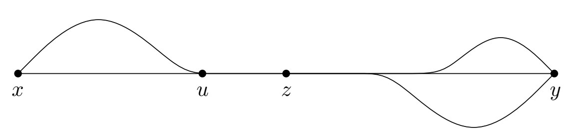

Figure 3.1: As depicted, $(x, y) \in N(2, 3)$. Note that (u, x) does not induce a normal (j, k)-network.

In particular, note if x, y are joined by exactly k geodesics and (x, y) is regular, then $(x, y) \in N(1, k)$. (Take z to be a point in the relative interior of the geodesic segment contained in all k segments from x to y.)

Not all networks are normal (j, k)-networks. For instance, if $(x, y) \in N(j, k)$ and $j > 1$, then there is a point $u \in G(x, y)$ so that u is joined to x by two geodesics with disjoint relative interiors. See Figure 3.1. That being said, most pairs induce normal (j, k)-networks. Moreover, for each $j, k \in \{1, 2, 3\}$, there are many normal (j, k)-networks in the map. Hence, in particular, we establish the existence of atypical networks comprised of more than three geodesics (and up to nine).

Theorem 3.2.15. *The following hold almost surely.*

(i) For any $j, k \in \{1, 2, 3\}$, $N(j, k)$ is dense in M^2.

(ii) $M^2 - \bigcup_{j,k\in\{1,2,3\}} N(j, k)$ is nowhere dense in M^2.

By Theorem 3.2.15, there are essentially only six types of geodesic networks which are dense in the Brownian map. See Figure 3.2.

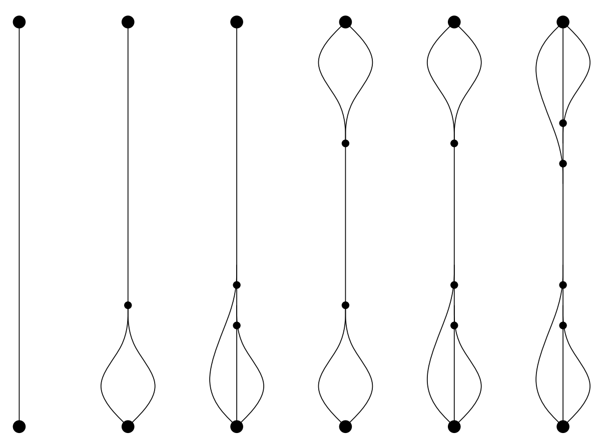

Figure 3.2: Theorem 3.2.15: Classification of networks which are dense in the Brownian map (up to symmetries and homeomorphisms of the sphere).

Since the geodesic net of the root, or a typical point by invariance under re-rooting, is a binary tree — which follows by the uniqueness of local minima of the label process Z, see [100, Lemma 3.1], and since $G(\rho)$ is the tree $[0,1]/\{d_Z = 0\}$, see Section 3.3.2 — it can be shown using ideas in the proof of Theorem 3.2.16 below that the pairs of small dots near the large dots in the 3rd, 5th and 6th networks in Figure 3.2 are indeed distinct points. (That is, Theorem 3.2.15 would still hold if we were to further require that normal networks have this additional property.) For instance, in Figure 3.7 below, note that all geodesic segments from y to y' are sub-segments of geodesics from y to the typical point z_n, and hence do not coalesce at the same point. We omit further discussion on this small detail.

It remains an interesting open problem to fully classify the types of geodesic networks in the Brownian map.

Additionally, we obtain the dimension of the sets $N(j,k)$, $j,k \le 3$.

For a set $A \subset M$, let $\dim A$ and $\dim_{\mathrm{P}} A$ denote its Hausdorff and packing dimensions, respectively (see Section 3.3.4).

Theorem 3.2.16. *Almost surely, we have* $\dim N(j,k) = \dim_{\mathrm{P}} N(j,k) = 2(6-j-k)$*, for all* $j,k \in \{1,2,3\}$*. Moreover,* $N(3,3)$ *is countable.*

We remark that since $N(j,k)$, for any $j,k \in \{1,2,3\}$, is dense in M^2 (by Theorem 3.2.15) its Minkowski dimension is that of M^2, which by Proposition 3.3.5 below is almost surely equal to 8.

Definition 3.2.17. For each $k \in \mathbb{N}$, let $P(k) \subset M^2$ denote the set of pairs of points that are connected by exactly k geodesics.

Theorems 3.2.15 and 3.2.16 imply the following results.

Corollary 3.2.18. *Put* $K = \{1,2,3,4,6,9\}$*. The following hold almost surely.*

(i) For each $k \in K$*,* $P(k)$ *is dense in* M^2*.*

(ii) $M^2 - \bigcup_{k\in K} P(k)$ *is nowhere dense in* M^2*.*

Corollary 3.2.19. *Almost surely, we have that* $\dim P(2) \ge 6$, $\dim P(3) \ge 4$, $\dim P(4) \ge 4$ *and* $\dim P(6) \ge 2$*.*

We expect the lower bounds in Corollary 3.2.19 to give the correct Hausdorff dimensions of the sets $P(k)$, $k \in K - \{1,9\}$. As discussed in

Section 3.2.2, $P(1)$ is of full volume, and hence $\dim P(1) = 8$. We suspect that $P(9)$ is countable. It would be of interest to determine if the set $P(k)$ is non-empty for some $k \notin K$, and whether there is any $k \notin K$ for which it has positive dimension. We hope to address these issues in future work.

3.2.4 Confluence points

Our key tool is a strengthening of the confluence of geodesics phenomenon of Le Gall [97] (see Section 3.3.3). We find that for any neighbourhood N of a typical point in the Brownian map, there is a *confluence point* x_0 between a sub-neighbourhood $N' \subset N$ and the complement of N. See Figure 3.3.

Proposition 3.2.20. *Almost surely, for λ-almost every $x \in M$, the following holds. For any neighbourhood N of x, there is a sub-neighbourhood $N' \subset N$ and some $x_0 \in N - N'$ so that all geodesics between any points $x' \in N'$ and $y \in N^c$ pass through x_0.*

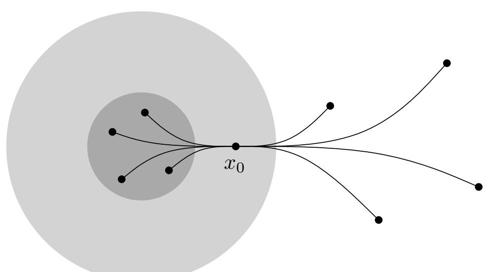

Figure 3.3: Proposition 3.2.20: All geodesics from points in N' to points in the complement of $N \supset N'$ pass through a confluence point x_0.

Definition 3.2.21. We say that a sequence of geodesic segments γ_n converges to a geodesic segment γ, and write $\gamma_n \to \gamma$, if γ_n converges to γ with respect to the Hausdorff topology.

Since (M, d) is almost surely homeomorphic to $\mathbb{S}^2$, and hence almost surely compact, the following lemma is a straightforward consequence of the Arzelà-Ascoli Theorem (see, for example, Bridson and Haefliger [43, Corollary 3.11]).

Lemma 3.2.22. *Almost surely, the set of geodesic segments in* (M, d) *is compact (with respect to the Hausdorff topology).*

Our key result, Proposition 3.2.20, is related to the fact that many sequences of geodesic segments in the Brownian map converge in a stronger sense.

Definition 3.2.23. We say that a sequence of geodesic segments $[x_n, y_n]$ *converges strongly* to $[x, y]$, and write $[x_n, y_n] \rightrightarrows [x, y]$, if $x_n \to x$, $y_n \to y$, and for any geodesic segment $[x', y'] \subset (x, y)$ (excluding the endpoints) we have that $[x', y'] \subset [x_n, y_n]$ for all sufficiently large n.

Strong convergence is stronger than convergence in the Hausdorff topology. Indeed, if x', y' are ε away from x, y along $[x, y]$, then for large n $[x', y'] \subset [x_n, y_n]$. Moreover, since $d(x_n, x') \leq d(x_n, x) + \varepsilon$ for all such n, $[x_n, x']$ is eventually contained in $B(x, 2\varepsilon)$. Similarly, $[y', y_n]$ is eventually contained in $B(y, 2\varepsilon)$. In the Euclidean plane, or generic smooth manifolds, strong convergence does not occur. In contrast, in the Brownian map it is the norm, as we shall see below. In light of this we also make the following definition.

Definition 3.2.24. A geodesic segment γ is called a *stable geodesic* if whenever $[x_n, y_n] \to \gamma$ we also have $[x_n, y_n] \rightrightarrows \gamma$. Otherwise, γ is called a *ghost geodesic*.

Proposition 3.2.25. *Almost surely, for* λ*-almost every* $x \in M$*, for all* $y \in M$*, all sub-segments of all geodesic segments* $[x, y]$ *are stable.*

Proposition 3.2.20 follows by combining Proposition 3.2.25 with the confluence of geodesics phenomenon and the fact that (M, d) is almost surely compact (see Section 3.4).

In closing, we remark that it would be interesting to know if Proposition 3.2.25 holds for all $x \in M$, that is, are all geodesics in M stable, or are there any ghost geodesics? Ghost geodesics have various properties, and in particular they intersect every other geodesic in at most one point. It would be quite surprising if such geodesics exist, and we hope to rule them out in future work. We thus expect an analogue of Proposition 3.2.20 to hold for all $x \in M$. If so, then as a consequence, we would obtain the following result.

Conjecture 3.2.26. *Almost surely, the geodesic framework of the Brownian map, $F \subset M$, is of Hausdorff dimension 1.*

In this way, we suspect that although the Brownian map is a complicated object of Hausdorff dimension 4, it has a relatively simple geodesic framework which is of first Baire category (Theorem 3.2.6) and Hausdorff dimension 1.

3.3 Preliminaries

In this section, we briefly recount the construction of the Brownian map and what is known regarding its geodesics.

3.3.1 The Brownian map

Fix $q \in \{3\} \cup 2(\mathbb{N}+1)$ and set c_q equal to $6^{1/4}$ if $q = 3$ or $(9/q(q-2))^{1/4}$ if $q > 3$. Let M_n denote a uniform q-angulation of the sphere (see Le Gall and Miermont [99]) with n faces, and d_n the graph distance on M_n scaled by $c_q n^{-1/4}$. The works of Le Gall [98] and Miermont [108] (for $q = 4$) show that in the Gromov-Hausdorff topology on isometry classes of compact metric spaces (see Burago, Burago and Ivanov [45]), (M_n, d_n) converges in distribution to a random metric space called the *Brownian map* (M, d).

The Brownian map has also been identified as the scaling limit of several other types of maps, see [1, 2, 29, 36, 98].

The construction of the Brownian map involves a normalized Brownian excursion $\mathbf{e} = \{\mathbf{e}_t : t \in [0,1]\}$, a random $\mathbb{R}$-tree $(\mathcal{T}_\mathbf{e}, d_\mathbf{e})$ indexed by $\mathbf{e}$, and a Brownian label process $Z = \{Z_a : a \in \mathcal{T}_\mathbf{e}\}$. More specifically, define $\mathcal{T}_\mathbf{e} = [0,1]/\{d_\mathbf{e} = 0\}$ as the quotient under the pseudo-distance

$$d_\mathbf{e}(s,t) = \mathbf{e}_s + \mathbf{e}_t - 2 \cdot \min_{s \wedge t \le u \le s \vee t} \mathbf{e}_u, \quad s,t \in [0,1]$$

and equip it with the quotient distance, again denoted by $d_\mathbf{e}$. The random metric space $(\mathcal{T}_\mathbf{e}, d_\mathbf{e})$ is Aldous' *continuum random tree*, or CRT. We let $p_\mathbf{e} : [0,1] \to \mathcal{T}_\mathbf{e}$ denote the canonical projection. Conditionally given $\mathbf{e}$, Z is a centred Gaussian process satisfying $\mathbf{E}[(Z_s - Z_t)^2] = d_\mathbf{e}(s,t)$ for all $s,t \in [0,1]$.

The random process Z is the so-called *head of the Brownian snake* (see [99]). Note that Z is constant on each equivalence class $p_{\mathbf{e}}^{-1}(a)$, $a \in \mathcal{T}_{\mathbf{e}}$. In this sense, Z is Brownian motion indexed by the CRT.

Analogously to the definition of $d_{\mathbf{e}}$, we put

$$d_Z(s,t) = Z_s + Z_t - 2 \cdot \max\left\{ \inf_{u \in [s,t]} Z_u, \inf_{u \in [t,s]} Z_u \right\}, \quad s,t \in [0,1]$$

where we set $[s,t] = [0,t] \cup [s,1]$ in the case that $s > t$. Then, to obtain a pseudo-distance on $[0,1]$, we define

$$D^*(s,t) = \inf\left\{ \sum_{i=1}^{k} d_Z(s_i,t_i) : s_1 = s, t_k = t, d_{\mathbf{e}}(t_i, s_{i+1}) = 0 \right\}, \quad s,t \in [0,1].$$

Finally, we set $M = [0,1]/\{D^* = 0\}$ and endow it with the quotient distance induced by D^*, which we denote by d. An easy property (see [105, Section 4.3]) of the Brownian map is that $d_{\mathbf{e}}(s,t) = 0$ implies $D^*(s,t) = 0$, so that M can also be seen as a quotient of $\mathcal{T}_{\mathbf{e}}$, and we let $\Pi : \mathcal{T}_{\mathbf{e}} \to M$ denote the canonical projection, and put $\mathbf{p} = \Pi \circ p_{\mathbf{e}}$. Almost surely, the process Z attains a unique minimum on $[0,1]$, say at t_*. We set $\rho = \mathbf{p}(t_*)$. The random metric space $(M,d) = (M,d,\rho)$ is called the *Brownian map* and we call ρ its *root*. Being the Gromov-Hausdorff limit of geodesic spaces, (M,d) is almost surely a geodesic space (see [45]).

Almost surely, for every pair of distinct points $s \neq t \in [0,1]$, at most one of $d_{\mathbf{e}}(s,t) = 0$ or $d_Z(s,t) = 0$ holds, except in the particular case $\{s,t\} = \{0,1\}$ where both identities hold simultaneously (see [100, Lemma 3.2]). Hence, only leaves (that is, non-cut-points) of $\mathcal{T}_{\mathbf{e}}$ are identified in the construction of the Brownian map; and this occurs if and only if they have the same label and along either the clockwise or counter-clockwise, contour-ordered path around $\mathcal{T}_{\mathbf{e}}$ between them, one only finds vertices of larger label. Thus, as mentioned at the beginning of Section 3.2, in the construction of the Brownian map, $(\mathcal{T}_{\mathbf{e}}, Z)$ is a continuum analogue for a well-labelled plane tree, and the quotient by $\{D^* = 0\}$ for the CVS-bijection (which, as discussed in Section 3.2, identifies well-labelled plane trees with rooted planar maps).

See Section 3.3.2 for more details.

Lastly, we note that although the Brownian map is a rooted metric space, it is not so dependent on its root. The volume measure λ on M is defined as the push-forward of Lebesgue measure on $[0,1]$ via $\mathbf{p}$. Le Gall [97] shows that the Brownian map is *invariant under re-rooting* in the sense that if U is uniformly distributed over $[0,1]$ and independent of (M,d), then (M,d,ρ) and $(M,d,\mathbf{p}(U))$ are equal in law. Hence, to some extent, the root of the map is but an artifact of its construction.

3.3.2 Simple geodesics

Recall that a *corner* of a vertex v in a discrete plane tree T is a sector centred at v and delimited by edges which precede and follow v along a contour-ordered path around T. Leaves of a tree have exactly one corner, and in general, the number of corners of v is equal to the number of connected components in $T - \{v\}$. Similarly, we may view the $\mathbb{R}$-tree $\mathcal{T}_{\mathbf{e}}$ as having corners, however in this continuum setting all sectors reduce to points. Hence, for the purpose of the following (informal) discussion, let us think of each $t \in [0,1]$ as corresponding to a *corner* of $\mathcal{T}_{\mathbf{e}}$ with label Z_t.

Put $Z_* = Z_{t_*}$. As it turns out, $d(\rho, \mathbf{p}(t)) = Z_t - Z_*$ for all $t \in [0,1]$ (see [96]). In other words, up to a shift by the minimum label Z_*, the Brownian label of a point in $\mathcal{T}_{\mathbf{e}}$ is precisely the distance to ρ from the corresponding point in the Brownian map.

All geodesics to ρ are *simple geodesics*, constructed as follows. For $t \in [0,1]$ and $\ell \in [0, Z_t - Z_*]$, let $s_t(\ell)$ denote the point in $[0,1]$ corresponding to the first corner with label $Z_t - \ell$ in the clockwise, contour-ordered path around $\mathcal{T}_{\mathbf{e}}$ beginning at the corner corresponding to t. For each such t, the image of the function $\Gamma_t : [0, Z_t - Z_*] \to M$ taking ℓ to $\mathbf{p}(s_t(\ell))$ is a geodesic segment from $\mathbf{p}(t)$ to ρ. Moreover, the main result of [97] shows that all geodesics to ρ are of this form. Hence, the geodesic net of the root, $G(\rho)$, is precisely the set of cut-points of the $\mathbb{R}$-tree $\mathcal{T}_Z = [0,1]/\{d_Z = 0\}$ projected into M.

These results mirror the fact that from each corner of a labelled, discrete plane tree, the CVS-bijection draws geodesics to the root of the resulting map in such a way that the label of a vertex visited by any such geodesic equals the distance to the root. See [95, 97] for further details.

Moreover, since the cut-points of $\mathcal{T}_{\mathbf{e}}$ are its vertices with multiple corners, we see that the set $S(\rho)$ (discussed in Section 3.2.2) of points with multiple geodesics to ρ is exactly the set of cut-points of the $\mathbb{R}$-tree $\mathcal{T}_{\mathbf{e}} = [0,1]/\{d_{\mathbf{e}} = 0\}$ projected into M.

Furthermore, since points in $S(\rho)$ correspond to leaves of $\mathcal{T}_Z$ (see [100, Lemma 3.2]), geodesics to the root of the map (or a typical point, by invariance under re-rooting) have a particular topological structure, as discussed in Section 3.2.3. We state this here for the record.

Proposition 3.3.1. *Almost surely, for λ-almost every x, for all $y \in M$, (x, y) is regular.*

Hence, as mentioned in Section 3.2.2, we have that $S(\rho) = C(\rho)$. That is, all points with multiple geodesics to the root are in the strong cut locus of the root.

3.3.3 Confluence at the root

As discussed in Section 3.2.1, a *confluence of geodesics* is observed at the root of the Brownian map. Combining this with invariance under re-rooting, the following result is obtained.

Lemma 3.3.2 (Le Gall [97, Corollary 7.7])**.** *Almost surely, for λ-almost every $x \in M$, the following holds. For every $\varepsilon > 0$ there is an $\eta \in (0, \varepsilon)$ so that if $y, y' \in B(x, \varepsilon)^c$, then any pair of geodesics from x to y and y' coincide inside of $B(x, \eta)$.*

Moreover, geodesics to the root of the map tend to coalesce quickly.

For $t \in [0, 1]$, let γ_t denote the image of the simple geodesic Γ_t from $\mathbf{p}(t)$ to the root of the map ρ (see Section 3.3.2).

Lemma 3.3.3 (Miermont [108, Lemma 5])**.** *Almost surely, for all $s, t \in [0, 1]$, γ_s and γ_t coincide outside of $B(\mathbf{p}(s), d_Z(s, t))$.*

We require the following lemma.

Lemma 3.3.4. *Almost surely, for λ-almost every $x \in M$, the following holds. For any $y \in M$ and neighbourhood N of y, there exists a sub-neighbourhood $N' \subset N$ so that if $y' \in N'$, then any geodesic from x to y' coincides with a geodesic from x to y outside of N.*

Proof. Let ρ denote the root of the map. Let $y \in M$ and a neighbourhood N of y be given. Select $\varepsilon > 0$ so that $B(y, \varepsilon) \subset N$. Let N_ε denote the set of points $y' \in M$ with the property that for all $t' \in [0, 1]$ for which $\mathbf{p}(t') = y'$, there exists some $t \in [0, 1]$ so that $\mathbf{p}(t) = y$ and $d_Z(t, t') < \varepsilon$. As discussed in Section 3.3.2, Le Gall [97] shows that all geodesics to ρ are simple geodesics. Hence, by Lemma 3.3.3, any geodesic from ρ to a point $y' \in N_\varepsilon$ coincides with some geodesic from ρ to y outside of N.

We claim that N_ε is a neighbourhood of y. To see this, note that if $\mathbf{p}(t_n) = y_n \to y$ in (M, d), then there is a subsequence t_{n_k} so that for some $t_y \in [0, 1]$, we have that $t_{n_k} \to t_y$ as $k \to \infty$. Hence $d_Z(t_y, t_{n_k}) < \varepsilon$ for all large k, and since $\mathbf{p}$ is continuous (see [97]), $\mathbf{p}(t_y) = y$. Therefore, for any $y_n \to y$ in (M, d), $y_n \notin N_\varepsilon$ for at most finitely many n, giving the claim.

Hence the lemma follows by invariance under re-rooting. ■

We remark that the size of N' in Lemma 3.3.4 depends strongly on x and y. For instance, for a fixed $\varepsilon > 0$ and convergent sequences of typical points x_n (that is, points satisfying the statement of Lemma 3.3.4) and general points y_n, for each n let $\eta_n > 0$ be such that the statement of the lemma holds for the pair x_n, y_n with $N_n = B(y_n, \varepsilon)$ and $N'_n = B(y_n, \eta_n)$. It is quite possible that $\eta_n \to 0$ as $n \to \infty$.

3.3.4 Dimensions

Finally, we collect some facts about the dimension of various subsets of the Brownian map. These statements are easily derived from established results, but are not explicitly stated in the literature.

For a metric space $X \subset M$, let $\dim X$ denote its Hausdorff dimension, $\dim_{\mathrm{P}} X$ its packing dimension, and $\underline{\mathrm{Dim}}\, X$ (resp. $\overline{\mathrm{Dim}}\, X$) its lower (resp.

upper) Minkowski dimension. If the lower and upper Minkowski dimensions coincide, we denote their common value by $\operatorname{Dim} X$. We note that for any metric space X we have

$$\dim X \le \underline{\operatorname{Dim}}\, X \le \overline{\operatorname{Dim}}\, X \qquad \text{and} \qquad \dim X \le \dim_{\mathrm{P}} X \le \overline{\operatorname{Dim}}\, X.$$

See Mattila [103], for instance, for detailed definitions and other properties of these dimensions. (For example, $\dim A = \inf\{t : \mathcal{H}^t(A) = 0\}$, where $\mathcal{H}^t$ is the Hausdorff measure, defined by $\mathcal{H}^t(A) = \lim_{\delta \downarrow 0} \mathcal{H}^t_\delta(A)$, where $\mathcal{H}^t_\delta(A)$ is the infimum over sums $\sum_i \delta_i^t$ such that there is a countable cover of A by sets A_i with diameters $\delta_i \le \delta$.)

We require the following result, which is implicit in Le Gall's [96] proof that $\dim M = 4$. For completeness, we include a proof via the uniform volume estimates of balls in the Brownian map.

Proposition 3.3.5. *Almost surely, for any non-empty, open subset $U \subset M$, we have that $\lambda(U) > 0$ (hence λ has full support) and $\dim U = \dim_{\mathrm{P}} U = \operatorname{Dim} U = 4$.*

Proof. Let a non-empty, open subset $U \subset M$ be given. Fix some arbitrary $\eta > 0$.

By [108, Lemma 15], there is a $c \in (0, \infty)$ and $\varepsilon_0 > 0$ so that for all $\varepsilon \in (0, \varepsilon_0)$ and $x \in M$, we have that $\lambda(B(x, \varepsilon)) \ge c\varepsilon^{4+\eta}$. In particular, $\lambda(U) > 0$. For $\varepsilon > 0$, let $N(\varepsilon)$ denote the number of balls of radius ε required to cover M. By a standard argument, it follows that there exists a $c' \in (0, \infty)$ so that for all $\varepsilon \in (0, 2\varepsilon_0)$ we have $N(\varepsilon) \le c'\varepsilon^{-(4+\eta)}$. It follows directly that $\overline{\operatorname{Dim}}\, M \le 4 + \eta$, and the same bound holds for $U \subset M$.

On the other hand, by [108, Lemma 14] (a consequence of [96, Corollary 6.2]), there is a $C \in (0, \infty)$ so that for all $\varepsilon > 0$ and $x \in M$, we have that $\lambda(B(x, \varepsilon)) \le C\varepsilon^{4-\eta}$. In particular, for all $\varepsilon > 0$ and $x \in U$ we have $\lambda(B(x, \varepsilon) \cap U) \le C\varepsilon^{4-\eta}$. It follows that $\dim U \ge 4 - \eta$ (see, for example, Falconer [61, Exercise 1.8]).

Since $\eta > 0$ is arbitrary, the general dimension inequalities imply the claim. ∎

Definition 3.3.6. For $x \in M$, and $k \geq 1$ or $k = \infty$, let $S_k(x)$ denote the set of points $y \in M$ with exactly k geodesics to x.

We believe that $S_\infty(x)$ is empty for all x. In fact, it is plausible that all $S_k(x)$ are empty for all $k > k_0$ (perhaps even $k_0 = 9$).

In particular, the weak cut locus $S(x)$, as defined in Section 3.2.2, is equal to $S_\infty(x) \cup \bigcup_{k \geq 2} S_k(x)$. As discussed in Section 3.2.3, by Le Gall's description of geodesics to the root, properties of the CRT, and invariance under re-rooting, we have the following result.

Proposition 3.3.7. *Almost surely, for λ-almost every $x \in M$*

(i) $S(x) = S_2(x) \cup S_3(x)$*;*

(ii) $S_2(x)$ *is dense, and has Hausdorff dimension 2 (and measure 0);*

(iii) $S_3(x)$ *is dense and countable.*

We observe that the proof in [97, Proposition 3.3] that $S(\rho)$ is almost surely of Hausdorff dimension 2 gives additional information.

Proposition 3.3.8. *Almost surely, for λ-almost every $x \in M$, for any non-empty, open set $U \subset M$ and each $k \in \{1, 2, 3\}$, we have that*

$$\dim(S_k(x) \cap U) = \dim_{\mathrm{P}}(S_k(x) \cap U) = 2(3 - k).$$

Proof. By invariance under re-rooting, it suffices to prove the claim holds almost surely when $x = \rho$ is the root of the map.

Let a non-empty, open subset $U \subset M$ be given.

Let $S = S(x)$ and $S_i = S_i(x)$ for $i = 1, 2, 3$. By Proposition 3.3.7(i), $S = S_2 \cup S_3$ and $M - \{x\} = S_1 \cup S$.

First, we note that by Proposition 3.3.7(iii), $S_3 \cap U$ is countable, and so has Hausdorff and packing dimension 0.

From [97] we have that S is the image of the cut-points (or skeleton) of the CRT, $\mathrm{Sk} \subset \mathcal{T}_{\mathbf{e}}$, under the projection $\Pi : \mathcal{T}_{\mathbf{e}} \to M$. Moreover, Π is Hölder continuous with exponent $1/2 - \varepsilon$ for any $\varepsilon > 0$, and restricted to Sk, Π is a homeomorphism from Sk onto S.

Note that Sk is of packing dimension 1, being the countable union of sets which are isometric to line segments (recall that the packing dimension of a countable union of sets is the supremum of the dimension of the sets). Hence, by the Hölder continuity of Π, it follows that $\dim_{\mathrm{P}} S \leq 2$ (see, for instance, [103, Exercise 6, p. 108]) and so in particular, we find that $\dim_{\mathrm{P}}(S \cap U) \leq 2$.

On the other hand, by the density of S in M and since Π is a homeomorphism from Sk to S, we see that there is a geodesic segment in Sk that is projected to a path in $S \cap U$. In the proof of [97, Proposition 3.3] it is shown that the Hausdorff dimension of any such path is at least 2. Hence $\dim(S \cap U) \geq 2$.

Altogether, by the general dimension inequality $\dim A \leq \dim_{\mathrm{P}} A$, we find that $S \cap U$ has Hausdorff and packing dimension 2.

Therefore, since $S_3 \cap U$ has Hausdorff and packing dimension 0 and $S = S_2 \cup S_3$, it follows that $S_2 \cap U$ has Hausdorff and packing dimension 2. Moreover, since by Proposition 3.3.5, U has Hausdorff and packing dimension 4 and $M - \{x\} = S_1 \cup S$, we find that $S_1 \cap U$ has Hausdorff and packing dimension 4. ■

In closing, we note that Propositions 3.3.7 and 3.3.8 imply the following result.

Proposition 3.3.9. *Almost surely, for λ-almost every $x \in M$, $S(x)$ is dense, $\dim S(x) = \dim_{\mathrm{P}} S(x) = 2$, and $\lambda(S(x)) = 0$.*

3.4 Confluence near the root

We show that a confluence of geodesics is observed near the root of the Brownian map, strengthening the results discussed in Section 3.3.3. Specifically, we establish the following result.

Lemma 3.4.1. *Almost surely, for λ-almost every $x \in M$, the following holds. For any $y \in M$ and neighbourhoods N_x of x and N_y of y, there are sub-neighbourhoods N'_x and N'_y so that if $x' \in N'_x$ and $y' \in N'_y$, then any geodesic segment from x' to y' coincides with some geodesic segment from x to y outside of $N_x \cup N_y$.*

We note that Lemma 3.4.1 strengthens Lemma 3.3.4 in that it allows for perturbations of both endpoints of a geodesic.

Once Lemma 3.4.1 is established, our key result is a straightforward consequence of Lemma 3.3.2 and the fact that the Brownian map is almost surely compact.

Proof of Proposition 3.2.20. By invariance under re-rooting, it suffices to prove the claim when $x = \rho$ is the root of the map. Let an (open) neighbourhood N of x be given. By Lemma 3.3.2, there is a point $x_0 \in N - \{x\}$ which is contained in all geodesic segments between x and points $y \in N^c$. Hence, by Lemma 3.4.1, for each $y \in N^c$ there is an $\eta_y > 0$ so that x_0 is contained in all geodesic segments between points $x' \in B(x, \eta_y)$ and $y' \in B(y, \eta_y)$. Since N^c is compact, it can be covered by finitely many balls $B(y, \eta_y)$, say with $y \in Y$. Put $N' = B(x, \min_{y \in Y} \eta_y)$. If $y_0 \in N^c$, then $y_0 \in B(y, \eta_y)$ for some $y \in Y$, and thus all geodesics from points $x' \in N' \subset B(x, \eta_y)$ to y_0 pass through x_0. ■

The rest of this section contains the proof of Lemma 3.4.1. By invariance under re-rooting, we may and will assume that x is in fact the root of the Brownian map. In rough terms, we must rule out the existence of a sequence of geodesic segments $[x_n, y_n]$ converging to a geodesic segment $[x, y]$, but not converging strongly in the sense given in Section 3.2.4.

For the remainder of this section we fix a realization of the Brownian map exhibiting the almost sure properties of the random metric space (M, d) that will be required below, notably the fact that M is homeomorphic to the 2-dimensional sphere. Slightly abusing notation, let us refer to this realization as (M, d). We also fix a point $y \neq x \in M$ and a geodesic segment $\gamma = [x, y]$ between x and y.

We utilize a dense subset $T \subset M$ of points, which we refer to as *typical points,* containing the root x, and such that

(i) the claims of Proposition 3.3.1 and Lemma 3.3.4 hold for all $u \in T$;

(ii) for each $u, v \in T$, there is a unique geodesic from u to v.

Such a set exists almost surely. For example, the set of equivalence classes containing rational points almost surely works. We may assume that T exists for the particular realization of (M, d) we have selected. It is in fact possible to choose T to have full λ-measure, but for now, we only need it to be dense in M.

In what follows, we will at times shift our attention to the homeomorphic image of a neighbourhood of γ in which our arguments are more transparent. Whenever doing so, we will appeal only to topological properties of the map. We let d_{E} be the Euclidean distance on $\mathbb{C}$, and for $w \in \mathbb{C}$ and $r > 0$, we let $B_{\mathrm{E}}(w, r)$ be the open Euclidean ball centered at w with radius r.

Fix a homeomorphism τ from M to $\hat{\mathbb{C}}$. The image of γ under τ is a simple arc in $\hat{\mathbb{C}}$. Let ϕ be a homeomorphism from this arc onto the unit interval $I = [0, 1] \subset \mathbb{R} \subset \mathbb{C}$, with $\phi(\tau(x)) = 0$ and thus $\phi(\tau(y)) = 1$. By a variation of the Jordan-Schönflies Theorem (see Mohar and Thomassen [110, Theorem 2.2.6]), ϕ can be extended to a homeomorphism from $\hat{\mathbb{C}}$ onto $\hat{\mathbb{C}}$. Hence $\phi \circ \tau|_\gamma$ can be extended to a homeomorphism from M to $\hat{\mathbb{C}}$ sending γ onto I. We fix such a homeomorphism, and denote it by ψ.

Since M is homeomorphic to $\hat{\mathbb{C}}$, once the geodesic γ is fixed we can think of the Brownian map as just $\hat{\mathbb{C}}$ with a random metric (for which $[0, 1]$ is a geodesic). The reader may well do this, and then ψ becomes the identity. We do not take this route, since that would require showing that ψ can be constructed in a measurable way, which we prefer to avoid.

Definition 3.4.2. Let $\mathbb{H}_+ = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$ (resp. $\mathbb{H}_- = \{w \in \mathbb{C} : \operatorname{Im} w < 0\}$) denote the open upper (resp. lower) half-plane of $\mathbb{C}$. We refer to $L = \psi^{-1}(\mathbb{H}_+)$ (resp. $R = \psi^{-1}(\mathbb{H}_-)$) as the *left* (resp. *right*) *side of* γ.

Lemma 3.4.3. *Let $u, v \in \gamma$. For all $\delta > 0$, there are typical points $u_\ell \in B(u, \delta) \cap L \cap T$ and $v_\ell \in B(v, \delta) \cap L \cap T$ so that $[u_\ell, v_\ell] - \gamma$ is contained in $(B(u, \delta) \cup B(v, \delta)) \cap L$. (See Figure 3.4.) An analogous statement holds replacing L with R.*

Proof. Let $\delta > 0$ and $u, v \in \gamma$ be given. We only discuss the argument for the left side of γ, since the two cases are symmetrical. Moreover, we may assume that u, v, x, y are all distinct. Indeed, suppose the lemma holds with distinct u, v, x, y. If we shift u, v along γ by at most $\eta > 0$ and apply the

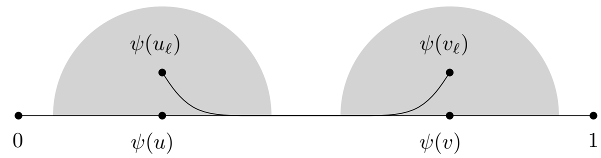

Figure 3.4: Lemma 3.4.3: $[u_\ell, v_\ell] - \gamma$ is contained in $(B(u,\delta) \cup B(v,\delta)) \cap L$ (as viewed through the homeomorphism ψ).

lemma with $\delta' = \delta - \eta$, the resulting u_ℓ, v_ℓ will satisfy the requirements of the lemma for u, v and δ. Without loss of generality, we further assume x, u, v, y appear on γ in that order.

We may and will assume that $\delta < d(u,x) \wedge d(v,y)$. In particular, $B(u,\delta)$ and $B(v,\delta)$ do not contain the extremities x, y of γ. Let $\delta' > 0$ be small enough so that $B_{\mathrm{E}}(\psi(v), \delta') \subset \psi(B(v,\delta))$. Note that the Euclidean ball $B_{\mathrm{E}}(\psi(v), \delta')$ does not contain $0, 1 \in \mathbb{C}$, and so $N = \psi^{-1}(B_{\mathrm{E}}(\psi(v), \delta'))$ does not intersect the extremities x, y of γ.

Let us apply Lemma 3.3.4 to the points x, v (using the fact that x is typical) and the neighbourhood $N = \psi^{-1}(B_{\mathrm{E}}(\psi(v), \delta'))$ of v defined above. According to this lemma, there exists a neighbourhood $N' \subset N$ of v such that any geodesic segment γ' between a point $v' \in N'$ and x coincides with some geodesic between v and x outside N. Since $x, y \notin N$, γ' must first encounter γ (if we see γ' as parameterized from v' to x) at a point w in the relative interior of γ. Since (x, y) is regular, we apply Lemma 3.2.13 to conclude that γ and γ' coincide between w and x and are disjoint elsewhere.

If we further assume that $v' \in N' \cap L$ is in the left side of γ, then we claim that the sub-arc $[v', w) \subset \gamma'$ is contained in L. Indeed, $\psi([v', w))$ is contained in the Euclidean ball $B_{\mathrm{E}}(\psi(v), \delta')$, starts in $\mathbb{H}_+$, and is disjoint of I, and so, it is contained in the upper half of the ball.

Since T is dense in M, we can take some typical $v_\ell \in N' \cap L \cap T$. For this choice, the geodesic segment $[x, v_\ell]$ is unique, and $[x, v_\ell] - \gamma$ is included in $B(v,\delta) \cap L$.

Assume also $\delta < \frac{1}{2} d(u, v)$. By a similar argument, in which v_ℓ assumes the role of x (which is a valid assumption since $v_\ell \in T$), for any u' close enough to u, any geodesic $[u', v_\ell]$ coalesces with $[x, v_\ell]$ within $B(u, \delta)$. Taking such a $u' = u_\ell$ in $T \cap L$, we get that $[v_\ell, u_\ell] - [v_\ell, x] \subset B(u, \delta) \cap L$, and hence $[u_\ell, v_\ell] - \gamma \subset (B(u, \delta) \cup B(v, \delta)) \cap L$, as required. ■

In the next lemma, recall the two notions of convergence (standard and strong) of geodesic segments given in Section 3.2.4.

Lemma 3.4.4. *Suppose that $[x', y'] \subset \gamma$ and $[x_n, y_n] \to [x', y']$ as $n \to \infty$. Then we have the strong convergence $[x_n, y_n] \rightrightarrows [x', y']$.*

The proof is somewhat involved. The general idea of the proof is to use Lemma 3.4.3 to obtain geodesic segments $\gamma_\ell = [u_\ell, v_\ell]$ and $\gamma_r = [u_r, v_r]$ between typical points in the left and right sides of γ, whose intersection $\gamma_\ell \cap \gamma_r$ contains a large segment from γ. Since γ_ℓ and γ_r are the unique geodesics between their (typical) endpoints, we deduce that γ_n contains $\gamma_\ell \cap \gamma_r$ for all large n. See Figure 3.5.

Proof. Let $\gamma_n = [x_n, y_n]$ and $\gamma' = [x', y']$, such that $\gamma_n \to \gamma'$, as in the lemma be given.

Let $\varepsilon > 0$ and put $\gamma'_\varepsilon = \gamma' - (B(x', \varepsilon) \cup B(y', \varepsilon))$. We show that γ_n contains γ'_ε for all large n. Since $\gamma_n \to \gamma'$ (and hence $x_n \to x'$ and $y_n \to y'$) this implies that $\gamma_n \rightrightarrows \gamma'$, as required.

We may assume that $\varepsilon < 2^{-1} d(x', y')$. Let u (resp. v) denote the point in γ' at distance $\varepsilon/2$ from x' (resp. y'). By Lemma 3.4.3, there are points $u_\ell \in B(u, \varepsilon/4) \cap L \cap T$ and $v_\ell \in B(v, \varepsilon/4) \cap L \cap T$ such that $[u_\ell, v_\ell] - \gamma$ is contained in $(B(u, \varepsilon/4) \cup B(v, \varepsilon/4)) \cap L$. We also let u_r, v_r be defined similarly, replacing L by R everywhere. Note that the geodesic segments $[u_\ell, v_\ell]$ and $[u_r, v_r]$ are unique since the extremities are all in T. Moreover, by our choice of ε, u, v, the segments $[u_\ell, v_\ell]$ and $[u_r, v_r]$ intersect γ and are disjoint from $\{x', y'\}$. Put

$$\delta = \frac{1}{2} \min\{d(u_\ell, \gamma), d(v_\ell, \gamma), d(u_r, \gamma), d(v_r, \gamma)\}$$

and note that $\delta > 0$. Let $[\gamma]_\delta = \{z \in M : d(z, \gamma) < \delta\}$ be the δ-neighbourhood of γ in M.

For $\eta > 0$, let us write $V_\eta = \{w \in \mathbb{C} : d_{\mathrm{E}}(w, I) < \eta\}$ for the η-neighbourhood of I in $\mathbb{C}$. Let $\eta_1 > 0$ be such that $V_{\eta_1} \subset \psi([\gamma]_\delta)$. Such an η_1 exists since, otherwise, we could find a sequence (z_n) of points in M such that $d(z_n, \gamma) \geq \delta$ but $d_{\mathrm{E}}(\psi(z_n), I) \to 0$ as $n \to \infty$, a clear contradiction since $\psi(\gamma) = I$ and (z_n) has convergent subsequences.

Note that $\psi(u_\ell), \psi(v_\ell), \psi(u_r), \psi(v_r) \notin V_{\eta_1}$ by the definition of δ. Put $I_\ell = \psi([u_\ell, v_\ell])$, and fix $\eta_2 > 0$ such that

$$\eta_2 < d_{\mathrm{E}}(\psi(x'), I_\ell) \wedge d_{\mathrm{E}}(\psi(y'), I_\ell)\,,$$

which is possible since $[u_\ell, v_\ell]$ does not intersect $\{x', y'\}$. Finally, we let $\eta_\ell = \eta_1 \wedge \eta_2$, and similarly define η_r, and set $\eta = \eta_\ell \wedge \eta_r$.

Consider I_ℓ as a parametrized simple path from $\psi(u_\ell)$ to $\psi(v_\ell)$. This path contains a single segment of I, since the geodesic $[u_\ell, v_\ell]$ is unique. Let u''_ℓ, v''_ℓ be defined by $I_\ell \cap I = [\psi(u''_\ell), \psi(v''_\ell)]$, with u''_ℓ the endpoint closer to x. Let the last point at which I_ℓ enters (the closure of) V_η before hitting I be $\psi(u'_\ell)$. Let the first point it exits V_η after separating from I be $\psi(v'_\ell)$. See Figure 3.5. Let H_ℓ denote the connected component of $V_\eta - \psi([u'_\ell, v'_\ell])$ that is contained in $\mathbb{H}_+$. Replacing u_ℓ, v_ℓ with u_r, v_r in the arguments above, we obtain u''_r, v''_r, H_r. Note that our choice of η implies that $\psi(x')$ and $\psi(y')$ are farther than η away (with respect to d_{E}) from H_ℓ, H_r.

Since $\gamma_n \to \gamma'$, we have that for every n large enough, $\psi(\gamma_n) \subset V_\eta$, $\psi(x_n) \in B_{\mathrm{E}}(\psi(x'), \eta)$, and $\psi(y_n) \in B_{\mathrm{E}}(\psi(y'), \eta)$. By our choice of η, for such an n, the extremities $\psi(x_n), \psi(y_n)$ of $\psi(\gamma_n)$ do not belong to $H_\ell \cup H_r$.

We claim that, for all such n, $\psi(\gamma_n) \cap H_\ell = \emptyset$. Indeed, if $\psi(\gamma_n)$ were to intersect H_ℓ, then by the Jordan Curve Theorem it would intersect $\psi([u'_\ell, v'_\ell])$ at two points $\psi(u_0), \psi(v_0)$ such that the segment $\psi((u_0, v_0)) \subset \psi(\gamma_n)$ is contained in H_ℓ. Since $H_\ell \cap \psi([u'_\ell, v'_\ell]) = \emptyset$, it would then follow that there are distinct geodesics between $u_0, v_0 \in [u_\ell, u_r]$, contradicting the uniqueness $[u_\ell, u_r]$. Similarly, for all such n, $\psi(\gamma_n) \cap H_r = \emptyset$.

Let $[u'', v''] = [u''_\ell, v''_\ell] \cap [u''_r, v''_r]$, with u'' the endpoint closer to x. Recalling (from the third paragraph of the proof) that $d(x', u) = \varepsilon/2$, $d(y', v) = \varepsilon/2$, $u_\ell \in B(u, \varepsilon/4)$, $v_\ell \in B(v, \varepsilon/4)$, and $[u_\ell, v_\ell] - \gamma = [u_\ell, u''_\ell) \cup (v''_\ell, v_\ell]$ is contained

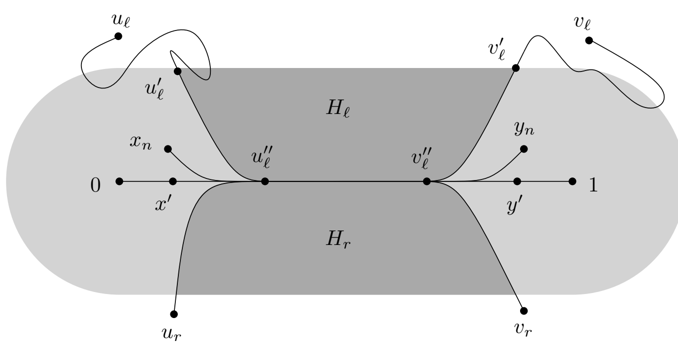

Figure 3.5: Lemma 3.4.4: Given $[x', y'] \subset \gamma$ we find a geodesic $\gamma_\ell = [u_\ell, v_\ell]$ which intersects γ in $[u''_\ell, v''_\ell]$, which is almost all of $[x', y']$, and similarly $[u_r, v_r]$. These are used to define the sets V_η (shaded), and subsets H_ℓ and H_r (dark gray). For large n, the geodesics γ_n are included in V_η and cannot enter $H_\ell \cup H_r$, leading to strong convergence. The points $u, v, u'_r, u''_r, v'_r, v''_r, u'', v''$ are not shown. For clarity, we omitted $\psi(\cdot)$ from all points (besides $\psi(x) = 0$ and $\psi(y) = 1$) named in the figure.

in $B(u, \varepsilon/4) \cup B(v, \varepsilon/4)$, it follows that $d(u''_\ell, x'), d(v''_\ell, y') < \varepsilon$. Similarly, since $u_r \in B(u, \varepsilon/4)$, $v_r \in B(v, \varepsilon/4)$, and $[u_r, v_r] - \gamma = [u_r, u''_r) \cup (v''_r, v_r]$ is contained in $B(u, \varepsilon/4) \cup B(v, \varepsilon/4)$, we have that $d(u''_r, x'), d(v''_r, y') < \varepsilon$. Hence $d(u'', x'), d(v'', y') < \varepsilon$, and so $\gamma'_\varepsilon \subset [u'', v'']$.

To conclude recall that, for all large n, we have that $\psi(\gamma_n) \subset V_\eta$, $\psi(x_n) \in B_{\mathrm{E}}(\psi(x'), \eta)$, $\psi(y_n) \in B_{\mathrm{E}}(\psi(y'), \eta)$, and $\psi(\gamma_n) \cap (H_\ell \cup H_r) = \emptyset$. By the Jordan Curve Theorem, it moreover follows that $[u'', v''] \subset \gamma_n$, and hence $\gamma'_\varepsilon \subset \gamma_n$, completing the proof. ∎

Proof of Proposition 3.2.25. Since $\gamma = [x, y]$ is a general geodesic segment from the root of the map, we obtain Proposition 3.2.25 immediately by Lemma 3.4.4 and invariance under re-rooting. ∎

With Proposition 3.2.25 at hand, Lemma 3.4.1 follows easily.

Proof of Lemma 3.4.1. By invariance under re-rooting, we may restrict to the case that x is the root of M. Let $y \in M$ and neighbourhoods N_x of x and N_y of y be given. Almost surely, there are at most 3 geodesics from x to y, which we call γ_i, for $i = 1, \ldots, k$ with $k \leq 3$. Suppose that $[x_n, y_n]$ is a sequence of geodesic segments with $x_n \to x$ and $y_n \to y$ in (M, d). If $[x_{n_k}, y_{n_k}]$ is a convergent subsequence of $[x_n, y_n]$, then by Lemma 3.2.22, $[x_{n_k}, y_{n_k}]$ converges to some γ_i. By Proposition 3.2.25, it follows that $[x_{n_k}, y_{n_k}] - (N_x \cup N_y)$ is contained in γ_i for all large k. We conclude that for any sequence $[x_n, y_n]$ as above, for all sufficiently large n we have that $[x_n, y_n] - (N_x \cup N_y)$ is contained in some geodesic segment from x to y. Hence sub-neighbourhoods N'_x and N'_y as in the lemma exist. ■

3.5 Proof of main results

In this section, we use Proposition 3.2.20 to establish our main results.

3.5.1 Typical points

To simplify the proofs below, we make use of a set of *typical points* $T \subset M$ (we slightly abuse notation by keeping the same notation as in Section 3.4). The set T will satisfy the following.

(i) $\lambda(T^c) = 0$;

(ii) Proposition 3.2.25 (and weaker results such as Proposition 3.2.20 and Lemmas 3.3.2, 3.3.4 and 3.4.1) holds for all $x \in T$;

(iii) Proposition 3.3.1 holds for all $x \in T$;

(iv) Proposition 3.3.7 holds for all $x \in T$;

(v) Proposition 3.3.8 holds for all $x \in T$;

(vi) For each $x, y \in T$, there is a unique geodesic from x to y.

To be precise, when we say above that a proposition holds for all $x \in T$, we mean that the property in the proposition, known to hold for λ-almost every point, holds for every point of T.

The almost sure existence of a set T satisfying (i)–(v) follows by invariance under re-rooting (and results cited or proved thus far). We note that property (vi) follows by (iii), since as mentioned in Section 3.2.3, if (x, y) and (y, x) are regular then there is a unique geodesic from x to y.

Hence, in the sections which follow, to show that various properties hold almost surely for λ-almost every $x \in M$, it suffices to confirm that they hold for points in T.

3.5.2 Geodesic nets

Theorems 3.2.3 and 3.2.4 follow by Proposition 3.2.20.

Proof of Theorem 3.2.3. Let $x, y \in M$ and $u \in T - \{x, y\}$ be given. Proposition 3.2.20 provides an (open) neighbourhood U_u of u and a point u_0 outside U_u so that all geodesics from any $v \in U_u$ to either x or y pass through u_0. In particular any geodesic $[v, x]$, with $v \in U_u$, can be written as $[v, u_0] \cup [u_0, x]$. By the choice of u_0, replacing the second segment by some $[u_0, y]$ gives a geodesic from v to y. The same holds with x, y reversed. Consequently, $G(x) \cap U_u = G(y) \cap U_u$.

Thus $G(x)$ and $G(y)$ coincide in $\bigcup_{T-\{x,y\}} U_u$. Since T is dense and has full measure, the theorem follows. ■

Proof of Theorem 3.2.4. Let $x \in T$ and a neighbourhood N of x be given. Select $\varepsilon > 0$ so that $B(x, 2\varepsilon) \subset N$. Let $N' \subset B(x, \varepsilon)$ and $x_0 \in B(x, \varepsilon) - N'$ be as in Proposition 3.2.20. By the choice of x_0, for any $y_0 \in N^c$ and $x' \in N'$, observe that $y_0 \in G(x')$ if and only if there is some $y \in B(x, \varepsilon)^c$ and geodesic $[x_0, y]$ so that $y_0 \in [x_0, y)$. This condition is independent of x'. Hence all $G(x')$, $x' \in N'$, coincide on N^c. ■

In support of our conjecture in Section 3.2.4, we show that the union of most geodesic nets is of Hausdorff dimension 1.

Proposition 3.5.1. *Almost surely, there is a subset $\Lambda \subset M$ of full volume, $\lambda(\Lambda^c) = 0$, satisfying* $\dim \bigcup_{x \in \Lambda} G(x) = 1$.

Proof. We prove the claim with $\Lambda = T$, which has full measure.

By property (ii) of points in T, there is a confluence of geodesics to all points $x \in T$ (that is, the statement of Lemma 3.3.2 holds). As discussed in Section 3.2.1, we thus have that $\dim G(x) = 1$ for all $x \in T$.

Let $\varepsilon > 0$ be given. For each $x \in T$, put $G_\varepsilon(x) = G(x) - B(x, \varepsilon)$. By Theorem 3.2.4, for each $x \in T$ there is an $\eta_x \in (0, \varepsilon)$ such that $G_{2\varepsilon}(x') \subset$

$G_\varepsilon(x)$ for all $x' \in B(x, \eta_x)$. Since (M, d) is a separable metric space and hence strongly Lindelöf (that is, all open subspaces of (M, d) are Lindelöf) there is a countable subset $T_\varepsilon \subset T$ such that $\bigcup_{x \in T_\varepsilon} B(x, \eta_x)$ is equal to $\bigcup_{x \in T} B(x, \eta_x)$, and in particular, contains T. Hence, by the choice of T_ε, $\bigcup_{x \in T} G_{2\varepsilon}(x)$ is contained in $\bigcup_{x \in T_\varepsilon} G_\varepsilon(x)$, a countable union of 1-dimensional sets, and so is 1-dimensional.

Taking a countable union over $\varepsilon = 1/n$, we see that $\dim \bigcup_{x \in T} G(x) = 1$, which yields the claim. ■

3.5.3 Cut loci

As discussed in Section 3.2.2, Le Gall's study of geodesics reveals a correspondence between cut-points of the CRT and points with multiple geodesics to the root of the Brownian map. Hence, Le Gall [97] states that $S(\rho)$ "exactly corresponds to the cut locus of [the Brownian map] relative to the root."

3.5.3.1 Weak cut loci

The main way in which the weak cut locus is badly behaved is that there is a dense set of points for which the weak cut locus has positive volume and full dimension (whereas typically it is much smaller, see Proposition 3.3.9).

Proposition 3.5.2. *Almost surely, for λ-almost every $x \in M$, for any neighbourhood N of x, there is a set D with $\dim D = 2$, dense in some neighbourhood $N' \subset N$ of x, such that $N^c \subset S(x')$ for all $x' \in D$.*

Proof. Let $x \in T$ and a neighbourhood N of x be given. Let $N' \subset N$ and $x_0 \in N - N'$ be as in Proposition 3.2.20. Fix some $u \in N^c \cap T$, and put $D = N' \cap S(u)$ so that by properties (iv),(v) of points in T, we have that D is dense in N' and satisfies $\dim D = 2$. By property (vi) of points in T, there is a unique geodesic from u to x. Since this geodesic passes through x_0, it follows that there is a unique geodesic from u to x_0. Hence, by the choice of D and x_0, we see that there are multiple geodesics from each point $x' \in D$ to x_0. We conclude, by the choice of x_0, that $N^c \subset S(x')$, for all $x' \in D$. ■

Since the weak cut locus relation is symmetric — that is, $y \in S(x)$ if and only if $x \in S(y)$ — we note that it follows immediately by Proposition 3.5.2 that almost surely, for *all* $x \in M$, $S(x)$ is dense in M (as mentioned in Section 3.2.2) and $\dim S(x) \geq 2$.

By the proof of Proposition 3.5.2, we find that $S(x)$ does not effectively capture the essence of a cut locus of a general point $x \in M$. Therein, observe that although all points $y \in N^c$ are in $S(x')$, $x' \in D$, this is due to the structure of the map near x' (namely the multiple geodesics to the confluence point x_0) and does not reflect on the map near y. For this reason, we also define a strong cut locus for the Brownian map, see Section 3.2.2.

3.5.3.2 Strong cut loci

By Le Gall's description of geodesics to the root and invariance under re-rooting, and in particular Proposition 3.3.1, we immediately obtain the following:

Proposition 3.5.3. *Almost surely, for λ-almost every $x \in M$, $S(x) = C(x)$, that is, the weak and strong cut loci coincide.*

We remark that the strong cut locus relation, unlike the weak cut locus, is not symmetric in x and y, that is, $y \in C(x)$ does not imply that $x \in C(y)$. See Figure 3.6.

Figure 3.6: Asymmetry of the strong cut locus relation: For a regular pair (x, y) joined by two geodesics, we have $y \in C(x)$, however $x \notin C(y)$, since all geodesics from y to x coincide near x.

Although more in tune with the singular geometry of the Brownian map, not all properties of cut loci in smooth manifolds apply for the Brownian map. For instance, $C(x)$ is much smaller than the *closure* of all points with multiple geodesics to x (as is the case with the cut locus of a smooth surface, see Klingenberg [90, Theorem 2.1.14]) since the set of such points is dense in M (as noted after the proof of Proposition 3.5.2). Moreover, it is not necessarily the case that all points $y \in C(x)$ are endpoints relative to x (that is, extremities y of a geodesic $[x, y]$ which cannot be extended to a geodesic $[x, y'] \supset [x, y]$ for any $y' \neq y$; in other words, $y \notin G(x)$). For instance, if γ, γ' are distinct geodesics from the root of the map ρ to some point x, with a common initial segment $[\rho, y] = \gamma \cap \gamma'$, then note that y is in $C(x)$ (by Proposition 3.3.1), however not an endpoint relative to x, being in the relative interior of γ.

Despite such differences, we propose that the set $C(x)$ is a more interesting notion of cut locus in our setting than $S(x)$ or, say, the set of *all* endpoints relative to x (that is, $G(x)^c - \{x\}$), which by Theorem 3.2.6 is a residual subset of the map.

As stated in Section 3.2.2, analogues of Theorems 3.2.3 and 3.2.4 hold for the strong cut locus. The proofs are very similar to those of Theorems 3.2.3 and 3.2.4.

Proof of Theorem 3.2.8. Let $x, y \in M$ and $u \in T - \{x, y\}$ be given. Proposition 3.2.20 provides an (open) neighbourhood U_u of u and a point u_0 outside U_u so that all geodesics from any $v \in U_u$ to either x or y pass through u_0. In particular any geodesic $[v, u_0]$ can be extended to each of x, y.

Since $v \in C(x)$ is determined by the structure of geodesics $[v, x]$ near v, a point $v \in U_u$ is in $C(x)$ if and only if $v \in C(y)$. Thus $C(x)$ and $C(y)$ agree in $\bigcup_{u \in T - \{x,y\}} U_u$. The result follows, since T is dense and has full measure. ■

Proof of Theorem 3.2.9. Let $x \in T$ and a neighbourhood N of x be given. Let $N' \subset N$ and $x_0 \in N - N'$ be as in Proposition 3.2.20. For any $x' \in N'$ and $y \in N^c$, $y \in C(x')$ if and only if there are multiple geodesics from x_0 to y which are distinct near y. Since this condition is independent of x', we conclude that all $C(x')$, $x' \in N'$, coincide on N^c. ■

Analogously to Proposition 3.5.1, we find that the union over most strong cut loci is of Hausdorff dimension 2.

Proposition 3.5.4. *Almost surely, there is a subset* $\Lambda \subset M$ *of full volume,* $\lambda(\Lambda^c) = 0$*, satisfying* $\dim \bigcup_{x \in \Lambda} C(x) = 2$.

Proof. The proposition follows by the proof of Proposition 3.5.1, but replacing its use of Theorem 3.2.4 with that of Theorem 3.2.9, and noting, by property (iv) of points in T, that $\dim C(x) = 2$ for all $x \in T$. We omit the details. ■

It would be interesting to know if almost surely $\bigcup_{x \in M} C(x)$ is of Hausdorff dimension 2.

3.5.4 Geodesic stars

A *geodesic star* is a formation of geodesic segments which share a common endpoint and are otherwise pairwise disjoint. Geodesic stars play a important role in [108]. While every point is the centre of a geodesic star with a single ray, almost every point is not the centre of a star with any more rays.

Definition 3.5.5. For $\varepsilon > 0$, let $Z(\varepsilon)$ denote the set of points $x \in M$ such that for some $y, y' \in B(x, \varepsilon)^c$ and geodesic segments $[x, y]$ and $[x, y']$, we have that $(x, y] \cap (x, y'] = \emptyset$. We call a point in $Z(\varepsilon)$ the *centre of a geodesic* ε*-star with two rays.*

Note that any point in the interior of a geodesic is in $Z(\varepsilon)$ for some $\varepsilon > 0$, but the converse need not hold.

Proposition 3.5.6. *Almost surely, for any* $\varepsilon > 0$, $Z(\varepsilon)$ *is nowhere dense in* M.

Proof. Let $\varepsilon > 0$ and $x \in T$ be given. Put $N = B(x, \varepsilon/2)$. Let $N' \subset N$ and $x_0 \in N - N'$ be as in Proposition 3.2.20. Since $N \subset B(x', \varepsilon)$ for all $x' \in N'$, x_0 is contained in all geodesic segments of length ε from points $x' \in N'$. Hence $Z(\varepsilon) \cap N' = \emptyset$. The result thus follows by the density of T. ■

Proof of Theorems 3.2.6 and 3.2.10. Note that if a point is either in the relative interior of a geodesic or in the strong cut locus of a point, then it is the centre of a geodesic ε-star with two rays, for some $\varepsilon > 0$. Therefore $\bigcup_{x \in M} G(x)$ and $\bigcup_{x \in M} C(x)$ are contained in $\bigcup_{n \geq 1} Z(n^{-1})$, a set of first Baire category by Proposition 3.5.6. The theorems follow. ■

3.5.5 Geodesic networks

In this section, we classify the types of geodesic networks which are dense in the Brownian map and calculate the dimension of the set of pairs with each type of network.

Proof of Theorem 3.2.15. Let $u \neq v \in T$ be given. By property (vi) of points in T, there is a unique geodesic $[u, v]$. Put $\varepsilon = \frac{1}{3}d(u, v)$. By property (ii) of points in T, we have by Lemma 3.4.1 that there is an $\eta > 0$ so that if $U = B(u, \eta)$ and $V = B(v, \eta)$, then for any $u' \in U$ and $v' \in V$, any geodesic segment $[u', v']$ coincides with $[u, v]$ outside of $B(u, \varepsilon) \cup B(v, \varepsilon)$.

Let z denote the midpoint of $[u, v]$. By the choice of η and since $u \in T$, we have by properties (iii),(iv) for points in T that for all $v' \in V$, the pair (z, v') is regular and joined by at most three geodesics. Hence we split $V = V_1 \cup V_2 \cup V_3$, where V_k consists of $v' \in V$ for which $(z, v') \in N(1, k)$. Similarly, we decompose $U = U_1 \cup U_2 \cup U_3$ according to the number of geodesics between z and $u' \in U$. Since $u, v \in T$, we see by property (iv) of points in T that all U_j, V_k are dense in U, V.

Finally, by the choice of η, observe that $U_j \times V_k \subset N(j, k)$, for all $j, k \in \{1, 2, 3\}$. Hence, parts (i),(ii) of the theorem follow by the density of T. ■

For the proof of Theorem 3.2.16, we require the following result concerning the dimension of cartesian products in arbitrary metric spaces.

Lemma 3.5.7 (Howroyd [81, 82])**.** *For any metric spaces X, Y we have that*

(i) $(\dim X) + (\dim Y) \le \dim(X \times Y)$*;*

(ii) $\dim_{\mathrm{P}}(X \times Y) \le (\dim_{\mathrm{P}} X) + (\dim_{\mathrm{P}} Y)$*,*

where the metric on $X \times Y$ is the L^1 metric on the product.

Proof of Theorem 3.2.16. Let $u \neq v \in T$ and U_j, V_k, $j, k \in \{1, 2, 3\}$, be as in the proof of Theorem 3.2.15. Since $u, v \in T$, we have by properties (iv),(v) of points in T that for all $j, k \in \{1, 2, 3\}$, $\dim U_j = \dim_{\mathrm{P}} U_j = 2(3 - j)$, $\dim V_k = \dim_{\mathrm{P}} V_k = 2(3 - k)$, and moreover, the sets U_3, V_3 are countable.

Recall that in the proof of Theorem 3.2.15, it is shown that for all $j, k \in \{1, 2, 3\}$, $U_j \times V_k \subset N(j, k)$. We thus obtain the lower bounds $\dim N(j, k) \ge$

$2(6-j-k)$ by Lemma 3.5.7(i). In particular, since $\dim A \leq \dim_{\mathrm{P}} A$, we obtain $8 \leq \dim N(1,1) \leq \dim_{\mathrm{P}} N(1,1) \leq \dim_{\mathrm{P}} M^2 \leq 8$, where the last inequality follows by Proposition 3.3.5 and Lemma 3.5.7(ii). Hence, we find that $\dim N(1,1) = \dim_{\mathrm{P}} N(1,1) = 8$.

It remains to give an upper bound on the dimensions of $N(j,k)$ when j,k are not both 1, in which case the complement of the geodesic network $G(x,y)$ is disconnected. By symmetry, we assume $j \neq 1$, so that there are multiple geodesics leaving x. Let $[x',y']$ be the closure of the intersection of all relative interiors (x,y) of geodesics from x to y. (Since $j \neq 1$, it follows that $x \neq x'$. If $k=1$ then $y=y'$.)

Fix a countable, dense subset $T_0 \subset T$. Take some $x_0 \in T_0$ in a component U_x of $G(x,y)^c$ whose closure contains x but not $[x',y']$. (See Figure 3.7.) By the Jordan Curve Theorem and the choice of $[x',y']$, for any geodesic $[x_0,y]$ we have that $[x_0,y]-U_x$ is contained in some geodesic from x to y, and in particular, contains $[x',y']$. Since x_0 is typical, by property (ii) of points in T, we have that all sub-segments of all geodesics $[x_0,y]$ are stable. Let z denote the midpoint of $[x',y']$. Note that, in particular, $[x',z] \subset [x',y']$ and $[z,y'] \subset [x',y']$ are stable.

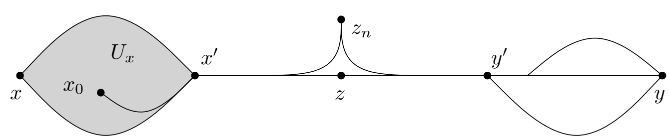

Figure 3.7: Theorem 3.2.16: As depicted, $(x,y) \in N(2,3)$. A typical point $x_0 \in U_x$ gives normal geodesics $[x_0,y]$. For some $z_n \in T_0$ sufficiently close to z, we have that $(z_n,x) \in N(1,2)$ and $(z_n,y) \in N(1,3)$, and hence $(x,y) \in S_2(z_n) \times S_3(z_n)$.

Take a sequence of points $z_n \in T_0$ converging to z. Any subsequential limit of geodesics $[x,z_n]$ converges to some geodesic $[x,z]$, which, by the choice of $[x',y']$, contains $[x',z]$. Since $[x',z]$ is stable, for large enough n the geodesics $[x,z_n]$ intersect $[x',z]$, and therefore (viewing $[x,z_n]$ as parametrized from x to z_n) necessarily coincide with one of the geodesics $[x,x']$, and then continue along $[x',y']$ before branching off towards z_n. It follows that for such n, we have that $(x,z_n) \in N(j,1)$. Similarly, since $[z,y']$ is stable, for large enough n the geodesics $[z_n, y]$ all go through y', and hence $(z_n, y) \in N(1, k)$.

By property (iii) of points in T, we note that for any $u \in T$ and $i \in \{1, 2, 3\}$, $S_i(u)$ (as defined in Section 3.3.4) is equal to $\{v : (u, v) \in N(1, i)\}$. Furthermore, by properties (iv),(v) of points in T, we have that $\dim_{\mathrm{P}} S_i(u) = 6 - 2i$, and moreover, $S_3(u)$ is countable.

The above argument shows that for every $(x, y) \in N(j, k)$ we have that $(z_n, x) \in N(1, j)$ and $(z_n, y) \in N(1, k)$ for some $z_n \in T_0$. Thus

$$N(j, k) \subset \bigcup_{u \in T_0} S_i(u) \times S_j(u).$$

Since T_0 is countable, it follows by Lemma 3.5.7(ii) that $\dim_{\mathrm{P}} N(j, k) \leq (6 - 2j) + (6 - 2k)$, giving the requisite upper bound. Moreover, we find that $N(3, 3)$ is countable.

Altogether, since $\dim A \leq \dim_{\mathrm{P}} A$, we conclude that $N(j, k)$ has Hausdorff and packing dimension $2(6 - j - k)$. ■

Proof of Corollaries 3.2.18 and 3.2.19. Noting that $N(j, k) \subset P(jk)$, for all $j, k \in \mathbb{N}$, we observe that Theorems 3.2.15 and 3.2.16 immediately yield Corollaries 3.2.18 and 3.2.19. ■

3.6 Related models

Our results have implications for the geodesic structure of models related to the Brownian map.

An infinite volume version of the Brownian map, the *Brownian plane* (P, D), has been introduced by Curien and Le Gall [53]. The random metric space (P, D) is homeomorphic to the plane $\mathbb{R}^2$ and arises as the local Gromov-Hausdorff scaling limit of the UIPQ (discussed in Section 3.2.1). The Brownian plane has an additional scale invariance property which makes it more amenable to analysis, see the recent works of Curien and Le Gall [54, 55]. As discussed in [95], almost surely there are isometric neighbourhoods of the roots of (M, d) and (P, D). Using this fact and scale invariance, properties of the Brownian plane can be deduced from those of the Brownian map.

In a series of works, Bettinelli [33, 34, 35] investigates *Brownian surfaces* of positive genus. In [33] subsequential Gromov-Hausdorff convergence of uniform random bipartite quadrangulations of the g-torus $\mathbb{T}_g$ is established (also general orientable surfaces with a boundary are analyzed in [35]), and it is an ongoing work of Bettinelli and Miermont [37, 38] to confirm that a unique scaling limit exists. Some properties hold independently of which subsequence is extracted. For instance, a scaling limit of bipartite quadrangulations of $\mathbb{T}_g$ is homeomorphic to $\mathbb{T}_g$ (see [34]) and has Hausdorff dimension 4 (see [33]). Also, a confluence of geodesics is observed at typical points of the surface (see [35]). Our results imply further properties of geodesics in such surfaces, although in these settings there are additional technicalities to be addressed.

Part III

Susceptibility of Random Graphs

Chapter 4

Thresholds for Contagious Sets in Random Graphs

4.1 Overview

For fixed $r \geq 2$, we consider bootstrap percolation with threshold r on the Erdős–Rényi graph $\mathcal{G}_{n,p}$. We identify a threshold for p above which there is with high probability a set of size r that can infect the entire graph. This improves a result of Feige, Krivelevich and Reichman, which gives bounds for this threshold, up to multiplicative constants.

As an application of our results, we obtain an upper bound for the threshold for K_4-percolation on $\mathcal{G}_{n,p}$, as studied by Balogh, Bollobás and Morris. This bound is proved to be sharp in Chapter 6.

These thresholds are closely related to the survival probabilities of certain time-varying branching processes, and we derive asymptotic formulae for these survival probabilities which are of interest in their own right.*

4.2 Background and main results

4.2.1 Bootstrap percolation

The *r-bootstrap percolation process* on a graph $G = (V, E)$ evolves as follows. Initially, some set $V_0 \subset V$ is *infected*. Subsequently, any vertex that has at least r infected neighbours becomes infected, and remains infected. Formally

*This chapter is joint work with Omer Angel [12], to appear in the *Annals of Applied Probabilty*.

the process is defined by

$$V_{t+1} = V_t \cup \Big\{ v : |N(v) \cap V_t| \geq r \Big\},$$

where $N(v)$ is the set of neighbours of a vertex v. The sets V_t are increasing, and so converge to some set V_∞ of eventually infected vertices. We denote the infected set by $\langle V_0, G\rangle_r = V_\infty$. A *contagious set* for G is a set $I \subset V$ such that if we put $V_0 = I$ then we have that $\langle I, G\rangle_r = V$, that is, the infection of I results in the infection of all vertices of G.

Bootstrap percolation was introduced by Chalupa, Leath and Reich [50] (see also [39, 104, 116, 131, 134]), in the context of statistical physics, for the study of disordered magnetic systems. Since then it has been applied diversely in physics, and in other areas, including computer science, neural networks, and sociology, see [7, 5, 9, 57, 58, 63, 64, 65, 68, 69, 70, 89, 112, 126, 133, 136, 137] and further references therein.

Special cases of r-bootstrap percolation have been analyzed extensively on finite grids and infinite lattices, see for instance [6, 20, 21, 23, 26, 47, 46, 72, 79, 80, 120] (and references therein). Other special graphs of interest have also been studied, including hypercubes and trees, see [19, 22, 25, 66]. Recent work has focused on the case of random graphs, see for example [9, 10, 27, 83], and in particular, on the Erdős–Rényi random graph $\mathcal{G}_{n,p}$. See [84, 132] (and [17, 18, 118] for related results).

The main questions of interest in this field revolve around the size of the eventual infected set V_∞. In most works, the object of study is the probability that a *random* initial set is contagious, and its dependence on the size of V_0. For example, in [84, Theorem 3.1], the critical size for a random set to be contagious for $\mathcal{G}_{n,p}$ is identified for all $r \geq 2$ and p in a range depending on r.

More recently, and in contrast with the above results, Feige, Krivelevich and Reichman [62] study the *existence* of small contagious sets in $\mathcal{G}_{n,p}$, in a range of p. We call a graph *susceptible* (or say that it *r-percolates*) it if contains a contagious set of the smallest possible size r. In [62, Theorem 1.2], the threshold for p above which $\mathcal{G}_{n,p}$ is likely to be susceptible is approximated, up to multiplicative constants. Our main result identifies sharp thresholds for the susceptibility of $\mathcal{G}_{n,p}$, for all $r \geq 2$.

Let $p_c(n,r)$ denote the infimum over $p > 0$ so that $\mathcal{G}_{n,p}$ is susceptible with probability at least 1/2.

Theorem 4.2.1. *Fix $r \geq 2$ and $\alpha > 0$. Let*

$$p = p(n) = \left(\frac{\alpha}{n \log^{r-1} n}\right)^{1/r}$$

and denote

$$\alpha_r = (r-1)! \left(\frac{r-1}{r}\right)^{2(r-1)}.$$

If $\alpha > \alpha_r$, then with high probability $\mathcal{G}_{n,p}$ is susceptible. If $\alpha < \alpha_r$, then there exists $\beta = \beta(\alpha, r)$ so that for $G = \mathcal{G}_{n,p}$, with high probability for every I of size r we have $|\langle I, G\rangle_r| \leq \beta \log n$. In particular, as $n \to \infty$,

$$p_c(n,r) = \left(\frac{\alpha_r}{n \log^{r-1} n}\right)^{1/r} (1 + o(1)).$$

Thus r-bootstrap percolation undergoes a sharp transition. For small p sets of size r infect at most $O(\log n)$ vertices, whereas for larger p there are contagious sets of size r.

We remark that for $\alpha < \alpha_r$, with high probability $\mathcal{G}_{n,p}$ has susceptible subgraphs of size $\Theta(\log n)$. Moreover, our methods identify the largest β so that there are susceptible subgraphs of size $\beta \log n$ (see Proposition 4.3.1 below).

4.2.2 Graph bootstrap percolation and seeds

Let H be some finite graph. Following Bollobás [39], *H-bootstrap percolation* is a rule for adding edges to a graph G. Eventually no further edges can be added, and the process terminates. An edge is added whenever its addition creates a copy of H within G. Informally, the process completes all copies of H that are missing a single edge. Formally, we let $G_0 = G$, and G_{i+1} is G_i together with every edge whose addition creates a subgraph which is isomorphic to H. (Note that these are not necessarily induced subgraphs, so having more edges in G can only increase the final result. The vertex set is fixed, and no vertices play any special role.) For a finite graph G, this procedure terminates once $G_{\tau+1} = G_\tau$, for some $\tau = \tau(G)$. We denote the resulting graph G_τ by $\langle G \rangle_H$. If $\langle G \rangle_H$ is the complete graph on the vertex set V, the graph G is said to *H-percolate* (or that it is *H-percolating*).

Balogh, Bollobás and Morris [24] study the model in the case that $H = K_k$ and $G = \mathcal{G}_{n,p}$. The case $H = K_4$ is the minimal case of interest. Indeed, *all* graphs K_2-percolate, and a graph K_3-percolates if and only if it is connected. Hence by a classical result of Erdős and Rényi [60], $\mathcal{G}_{n,p}$ will K_3-percolate precisely for $p > n^{-1} \log n + \Theta(n^{-1})$. Critical thresholds are defined as

$$p_c(n, H) = \inf \left\{ p > 0 : \mathbf{P}(\langle \mathcal{G}_{n,p} \rangle_H = K_n) \geq 1/2 \right\}.$$

It is expected that this property has a sharp threshold for $H = K_k$ for any k, in the sense that for some $p_c = p_c(k)$ we have that $\mathcal{G}_{n,p}$ is K_k-percolating with high probability for $p > (1+\delta)p_c$ and is K_k-percolating with probability tending to 0 for $p = (1-\delta)p_c$. Some bounds on $p_c(n, K_k)$, $k \geq 4$, are obtained in [24]. One of the main results is that $p_c(n, K_4) = \Theta(1/\sqrt{n \log n})$.

We improve the upper bound on $p_c(n, K_4)$ given in [24].

Theorem 4.2.2. *Let $p = \sqrt{\alpha/(n \log n)}$. If $\alpha > 1/3$ then $\mathcal{G}_{n,p}$ is K_4-percolating with high probability. In particular as $n \to \infty$, we have that*

$$p_c(n, K_4) \leq \frac{1 + o(1)}{\sqrt{3n \log n}}.$$

This bound is shown to be asymptotically sharp in Chapter 6.

One way for a graph G to K_{r+2}-percolate is if there is some ordering of the vertices so that vertices $1, \dots, r$ form a clique, and every later vertex is connected to at least r of the previous vertices according to the order. In this case we call the clique formed by the first r vertices a *seed* for G. When $r = 2$, the seed is a clique of size 2, so we call it a *seed edge*.

Lemma 4.2.3. *Fix $r \geq 2$. If G has a seed for K_{r+2}-bootstrap percolation, then $\langle G \rangle_{K_{r+2}} = K_n$.*

Proof. We prove by induction that for $k \geq r$ the subgraph induced by the first k vertices percolates. For $k = r$, the definition of a seed implies that the subgraph is complete. Given that the first $k-1$ vertices span a percolating graph, some number of steps will add all edges among them. Finally, vertex k has r neighbours among these, and so every edge between vertex k and a previous vertex can also be added by K_{r+2}-bootstrap percolation. ■

In light of this, Theorem 4.2.2 above is a direct corollary of the following result.

Theorem 4.2.4. *Let $p = \sqrt{\alpha/(n \log n)}$. As $n \to \infty$, the probability that $\mathcal{G}_{n,p}$ has a seed edge tends to* 1 *if $\alpha > 1/3$ and tends to* 0 *if $\alpha < 1/3$.*

The case of K_4-bootstrap percolation, corresponding to $r = 2$, appears to be special: We conjecture that existence of a seed edge is the easiest way for a graph to K_4-percolate. This is similar to other situations where a threshold of interest on $\mathcal{G}_{n,p}$ coincides with that of a more fundamental event. For instance, with high probability, $\mathcal{G}_{n,p}$ is connected (equivalently, K_3-percolating) if and only if it has no isolated vertices (see [60]); $\mathcal{G}_{n,p}$ contains a Hamiltonian cycle if and only if its minimum degree is at least 2 (Komlós and Szemerédi [92]).

Essentially, if G K_4-percolates, then either there is a seed edge, or some other small structure that serves as a seed (i.e., K_4-percolates and exhausts G by adding doubly connected vertices), or else, there are at least two large structures within G that K_4-percolate independently. Since $p_c \to 0$, having multiple large percolating structures within G is less likely. This is further investigated in Chapter 6.

For $r > 2$, having a seed is no longer the easiest way for a graph to K_4-percolate. Indeed, by [24], the critical probability for K_{r+2}-bootstrap percolation is $n^{-(2r)/(r^2+3r-2)}$ up to (unknown) poly-logarithmic factors (note that r in [24] is $r+2$ here). The threshold for having a seed is of order $n^{-1/r}(\log n)^{1/r-1}$, which is much larger (see Theorem 4.6.1).

4.2.3 A non-homogeneous branching process

Given an edge $e = (x_0, x_1)$, we can explore the graph to determine if it is a seed edge. The number of vertices that are connected to both of its endpoints is roughly Poisson with mean np^2. In our context, the interesting p are $o(n^{-1/2})$, and therefore the number of such vertices has small mean, which we denote by $\varepsilon = np^2$. If there are any such vertices, denote them $x_2, \dots$. We then seek vertices connected to x_2 and at least one of x_0, x_1. The number of such vertices is roughly $\mathrm{Poi}(2\varepsilon)$. Indeed, the number of vertices connected to the kth vertex and at least one of the previous vertices is (approximately) $\mathrm{Poi}(k\varepsilon)$.

This leads us to the case $r = 2$ of the following non-homogeneous branching process defined by parameters $r \in \mathbb{N}$ and $\varepsilon > 0$. The process starts with a single individual. The first $r-2$ individuals have precisely one child each. For $n \geq r-1$, the nth individual has a Poisson number of children with mean $\binom{n}{r-1}\varepsilon$, where here $\varepsilon = np^r$. Thus for $r = 2$ the nth individual has a mean of $n\varepsilon$ children. The process may die out (e.g., if individual $r-1$ has no children). However, if the process survives long enough the mean number of children exceeds one and the process becomes super-critical. Thus the probability of survival is strictly between 0 and 1. Formally, this may be defined in terms of independent random variables $Z_n = \mathrm{Poi}\left(\binom{n}{r-1}\varepsilon\right)$ by $X_t = \sum_{n=r-1}^{t} Z_n - 1$. Survival is the event $\{X_t \geq 0, \forall t\}$.

Theorem 4.2.5. *As $\varepsilon \to 0$, we have that*

$$\mathbf{P}(X_t > 0, \forall t) = \exp\left[-\frac{(r-1)^2}{r} k_r (1 + o(1))\right]$$

where

$$k_r = k_r(\varepsilon) = \left(\frac{(r-1)!}{\varepsilon}\right)^{1/(r-1)}.$$

Note that $\varepsilon\binom{k_r}{r-1} \approx 1$. Hence k_r is roughly the time at which the process becomes super-critical.

In closing, we mention that this process is closely related to the *binomial chain* representation of the r-bootstrap percolation dynamics, discussed in more detail in Section 5.3 below. (One difference here is that we keep track of the number of vertices in each "generation." This, in fact, can be recovered from the binomial chain process as well, see [84, Chapter 10]. Also, whereas here the process starts with r infected vertices $x_0, x_1, \dots x_{r-1}$, in Chapter 5 we study the situation where a set of ℓ vertices is initially infected, where possibly $\ell/k_r \sim c > 0$. Hence Theorem 5.4.2 below, in the particular case that $\ell/k_r \to 0$, is a (slightly more general) version of the above theorem, noting that setting $\varepsilon = np^r$ above, $k_r(\varepsilon)$ coincides with k_r in Theorem 5.4.2.)

4.2.4 Outline of the proof

In Section 4.3, we obtain a recurrence (4.3.1) for the number of graphs which r-percolate with the minimal number of edges. Using this, we estimate the asymptotics of such graphs, and thereby identify a quantity $\beta_*(\alpha)$, so that for $\alpha < \alpha_r$ (and p as in Theorem 4.2.1), with high probability no r-percolation on $\mathcal{G}_{n,p}$ grows to size $\beta \log n$, for any $\beta \geq \beta_*(\alpha) + \delta$. Let $\beta_r(\alpha) = k_r(np^r)/\log n$, where $k_r = k_r(\varepsilon)$ is as in Section 4.2.3. Moreover, we find that $\beta_*(\alpha) = \beta_r(\alpha)$ if and only if $\alpha = \alpha_r$, suggesting that α_r is indeed the critical value of α.

In Section 4.4, we show by the second moment method that, if $\alpha > \alpha_r$, then $\mathcal{G}_{n,p}$ r-percolates with high probability. The main difficulty towards establishing this fact is that contagious sets are far from independent. One way to see (very roughly) that this is the case is as follows: For super-critical $\alpha > \alpha_r$, it is reasonable to presume that the expected number of contagious sets of size r is n^μ, for some $\mu(\alpha) \downarrow 0$ as $\alpha \downarrow \alpha_r$. Let $r = 2$ (the cases $r > 2$ are similar), and suppose that some pair x, y infects a set V containing $\beta \log n$ vertices. Let x', y' be some other pair, such that $\{x, y\} \cap \{x', y'\} = \emptyset$. One way that x', y' can infect a set V' of size $\beta \log n$ is by first infecting some set V_1 where $|V \cap V_1| = 2$, and then infecting some $V_2 \subset V$ such that $|V_1 \cup V_2| = \beta \log n$. Note that this only implies the existence of at least three edges in $\mathcal{G}_{n,p}$ with at most one endpoint in V. To see this, observe that the first infected vertex $u \in V \cap V_1$ necessary has at least two neighbours not in V, however the second vertex infected $v \neq u \in V \cap V_1$ may only have one such neighbour if $(u, v) \in E(\mathcal{G}_{n,p})$. As a result, it is perhaps not straightforward to obtain an upper bound for the conditional probability that x', y' infects $\beta \log n$ vertices, given that x, y infects $\beta \log n$ vertices, that is much smaller than p^3. Since there are $O(n^2)$ such pairs x', y', and since $p = \sqrt{\alpha/(n \log n)}$, it would appear that correlations are too high for a simple application of the second moment method.

To overcome this difficultly, we observe that if x', y' infects a set $V' = V_1 \cup V_2$ as above, and moreover $|V \cap V'| > 2$, then either (i) the second and third vertices $v, w \in V \cap V'$ that are infected (after the first vertex $u \in V \cap V'$, with two neighbours not in V, is infected) both have a neighbour not in V, resulting in a total of at least four edges in $\mathcal{G}_{n,p}$ with at least one endpoint not in V, or else (ii) the vertices u, v, w induce a triangle. For this reason, we instead study contagious sets which infect *triangle-free* subgraphs of $\mathcal{G}_{n,p}$. To give some intuition for why this restriction should not effect the threshold (up to smaller order terms), note that the threshold p_c' for the existence of a contagious set of size r that induces a graph with at least one edge is much larger, $p_c' \gg p_c$. Therefore, although for p close to p_c there are many triangles in $\mathcal{G}_{n,p}$, we do not expect $\mathcal{G}_{n,p}$ to require a triangle in order to infect at large set of size $\beta \log n$.

More specifically, we modify the recurrence (4.3.1) to obtain a recursive lower bound for graphs which r-percolate without using triangles, and show that this restriction does not significantly effect the asymptotics. Using Mantel's [102] theorem, we establish the approximate independence of correspondingly restricted r-percolations, which we call $\hat{r}$-percolations, with relative ease.

A secondary obstacle is the need for a lower bound on the asymptotics of graphs which $\hat{r}$-percolate, with a significant proportion of vertices in the *top level* (i.e., vertices v of a graph $G = (V, E)$ such that $v \in V_t - V_{t-1}$ where $V_t = V$). Such bounds are required to estimate the growth of super-critical $\hat{r}$-percolations on $\mathcal{G}_{n,p}$, which have grown larger than the critical size $\beta_r(\alpha) \log n$. Using a lower bound for the overall number of graphs which $\hat{r}$-percolate, we obtain a lower bound for the number of such graphs with $i = \Omega(k)$ vertices in the top level. This estimate, together with the approximate independence result, is sufficient to show that with high probability $\mathcal{G}_{n,p}$ has subgraphs of size $\beta \log n$ which r-percolate, for some $\beta \geq \beta_*(\alpha) + \delta$ (where for $\alpha > \alpha_r$, $\beta_r(\alpha) < \beta_*(\alpha)$).

Finally, to conclude, we show by the first moment method that for any given $A > 0$, with high probability an r-percolation which grows to size $(\beta_*(\alpha) + \delta) \log n$ continues to grow to size $A \log n$. Having established the existence of a subgraph of $\mathcal{G}_{n,p}$ of size $A \log n$, for a sufficiently large value of A (depending on the difference $\alpha - \alpha_r$), it is straightforward (by sprinkling) to show that with high probability $\mathcal{G}_{n,p}$ r-percolates.

4.3 Lower bound for $p_c(n,r)$

In this section, we prove the sub-critical case of Theorem 4.2.1, by the first moment method. Throughout this section we fix some $r \geq 2$. More precisely, we prove the following

Proposition 4.3.1. *Let*

$$\alpha_r = (r-1)! \left(\frac{r-1}{r} \right)^{2(r-1)}, \qquad p = \theta_r(\alpha, n) = \left(\frac{\alpha}{n \log^{r-1} n} \right)^{1/r}.$$

Define $\beta_*(\alpha)$ to be the unique positive root of

$$r + \beta \log \left(\frac{\alpha \beta^{r-1}}{(r-1)!} \right) - \frac{\alpha \beta^r}{r!} - \beta(r-2).$$

For any $\alpha < \alpha_r$ and $\delta > 0$, with high probability, for every $I \subset [n]$ of size r, we have that $|\langle I, \mathcal{G}_{n,p} \rangle_r| \leq (\beta_*(\alpha) + \delta) \log n$.

The methods of Section 4.4 can be used to show that with high probability there are sets I of size r which infect $(\beta_* - \delta) \log n$ vertices. For $\alpha < \alpha_r$, we have (see Lemma 4.3.9) the following upper bound

$$\beta_*(\alpha) < \left(\frac{(r-1)!}{\alpha} \right)^{1/(r-1)}.$$

(In fact, it can be shown by elementary calculus that α can be replaced with α_r on the right hand side, resulting in the slightly improved upper bound of

$\beta_*(\alpha) < (r/(r-1))^2$.) This is asymptotically optimal for $\alpha \sim \alpha_r$.

In closing, we mention that Proposition 4.3.1 can alternatively be established using the large deviations estimates developed in the next Chapter 5, see Theorem 5.4.2. These two approaches are completely different, and so are of independent interest: Theorem 5.4.2 is proved using variational calculus, whereas Proposition 4.3.1 is proved by combinatorial arguments.

4.3.1 Small susceptible graphs

As discussed in Section 4.2.4, a key idea is to study the number of subgraphs of size $k = \Theta(\log n)$ which are susceptible with the minimal number of edges. If none exist, then there can be no contagious set in G. Thus an important step is developing estimates for the number of such susceptible graphs of size k.

For a graph G and initial infected set V_0, recall that $V_t = V_t(V_0, G)$ is the set of vertices infected up to and including step t. We let $\tau = \inf\{t : V_t = V_{t+1}\}$. We put $I_0 = V_0$ and $I_t = V_t - V_{t-1}$, for $t \geq 1$. We refer to I_t as the set of vertices infected in *level* i. In particular, the *top level* of G is I_τ.

For a graph G, we let $V(G)$ and $E(G)$ denote its vertex and edge sets, and put $|G| = |V(G)|$.

We call a graph *minimally susceptible* if it is susceptible and has exactly $r(|G| - r)$ edges. If a graph G is susceptible, it has at least $r(|G| - r)$ edges, since each vertex in I_t, $t \geq 1$, is connected to r vertices in V_{t-1}.

For $k \in \mathbb{N}$, let $[k] = \{1, 2, \ldots, k\}$.

Definition 4.3.2. Let $m_r(k)$ denote the number of minimally susceptible graphs G with vertex set $[k]$ such that $[r]$ is a contagious set for G. Let $m_r(k, i)$ denote the number of such graphs with i vertices infected in the top level (so that $m_r(k) = \sum_{i=1}^{k-r} m_r(k, i)$).

We note that $m_r(k, k-r) = 1$, and claim that for $i < k - r$,

$$m_r(k, i) = \binom{k-r}{i} \sum_{j=1}^{k-r-i} a_r(k-i, j)^i m_r(k-i, j), \tag{4.3.1}$$

where

$$a_r(x, y) = \binom{x}{r} - \binom{x-y}{r}. \tag{4.3.2}$$

To see this, note that removing the top level from a minimally susceptible graph G of size k leaves a minimally susceptible graph G' of size $k - i$. If the top level of G' has size j, then all vertices in the top level of G are connected to r vertices of G', with at least one in the top level of G'. Thus each vertex has $a_r(k - i, j)$ options for the connections. The $\binom{k-r}{i}$ term accounts for the set of possible labels of the top level of G.

To study asymptotics of m it is convenient to define

$$\sigma_r(k, i) = \frac{m_r(k, i)}{(k - r)!} \left(\frac{(r - 1)!}{k^{r-1}}\right)^k. \tag{4.3.3}$$

Substituting this in (4.3.1) gives

$$\sigma_r(k, i) = \sum_{j=1}^{k-r-i} A_r(k, i, j)\sigma_r(k - i, j) \quad \text{for } i < k - r, \tag{4.3.4}$$

where

$$A_r(k, i, j) = \frac{j^i}{i!} \left(\frac{k - i}{k}\right)^{(r-1)k} \left(\frac{(r - 1)!}{(k - i)^{r-1}} \frac{a_r(k - i, j)}{j}\right)^i. \tag{4.3.5}$$

We make the following observation.

Lemma 4.3.3. *Let $A_r(k, i, j)$ be as in (4.3.5). Put $A_r(i, j) = j^i e^{-(r-1)i}/i!$. For any $i < k - r$ and $j \le k - r - i$, we have that $A_r(k, i, j)$ is increasing in k and converges to $A_r(i, j)$.*

Proof. It is well known that for $m > 0$ we have $(1 - m/k)^k$ is increasing and tends to e^{-m}. Thus

$$\frac{j^i}{i!} \left(\frac{k - i}{k}\right)^{(r-1)k} \to A_r(i, j).$$

The lemma follows by (4.3.5) and the following claim, a formula which will also be of later use.

Claim 4.3.4. *For all integers $x \ge r$ and $1 \le y \le x-r$, we have that*

$$\frac{(r-1)!}{x^{r-1}}\frac{a_r(x,y)}{y} = \frac{1}{y}\sum_{\ell=1}^{y}\left(\frac{x-\ell}{x}\right)^{r-1}.$$

Proof. For an integer $m \ge r$, let $(m)_r = m!/(m-r)!$ denote the rth falling factorial of the integer m. Since

$$(m)_r - (m-1)_r = r(m-1)^{r-1}.$$

it follows that

$$\frac{(r-1)!}{x^{r-1}}\frac{a_r(x,y)}{y} = \frac{(x)_r-(x-y)_r}{ryx^{r-1}} = \frac{1}{y}\sum_{\ell=1}^{y}\left(\frac{x-\ell}{x}\right)^{r-1}$$

as required. ∎

Since each term on the right of Claim 4.3.4 is increasing to 1, the same holds for their average. The proof is complete. ∎

4.3.2 Upper bounds for susceptible graphs

Our first task is to derive bounds on the number of minimally susceptible graphs of size k with i vertices in the top level. This relies on the recurrence (4.3.1).

Lemma 4.3.5. *Fix $r \ge 2$. For all $k > r$ and $i \le k-r$, we have that*

$$m_r(k,i) \le \frac{e^{-i-(r-2)k}}{\sqrt{i}}(k-r)!\left(\frac{k^{r-1}}{(r-1)!}\right)^{k}.$$

Equivalently, $\sigma_r(k,i) \le i^{-1/2}e^{-i-(r-2)k}$.

Proof. Since $m_r(k,k-r)=1$, it is straightforward to verify that the claim holds in the case that $i=k-r$.

For the remaining cases $i<k-r$, we prove the claim by induction on k. Applying the inductive hypothesis to the right hand sum of (4.3.4), bounding

$A_r(k,i,j)$ therein by $A_r(i,j)$ using Lemma 4.3.3, and extending the sum to all j we have

$$\sigma_r(k,i) \le \sum_{j=1}^{\infty} A_r(i,j) j^{-1/2} e^{-j-(r-2)(k-i)}.$$

Thus it suffices to prove that this sum is at most $i^{-1/2}e^{-i-(r-2)k}$. Using the definition of $A_r(i,j)$ and cancelling the $e^{-(2-r)k}$ factors, we need the following

Claim 4.3.6. *For any $i \ge 1$ we have*

$$\sum_{j=1}^{\infty} \frac{j^i e^{-i}}{i!} j^{-1/2} e^{-j} \le i^{-1/2} e^{-i}.$$

This is proved in Section 4.7.1.

We remark that Claim 4.3.6 is fundamentally a pointwise bound on the Perron eigenvector of the infinite operator A_2. (Other values of r follow since the influence of r cancels out.) This eigenvector decays roughly as e^{-i}, but with some lower order fluctuations. It appears that the $\sqrt{i}$ correction can be replaced by various other slowly growing functions of i. However, Claim 4.3.6 fails for certain i without the $\sqrt{j}$ term. ■

4.3.3 Susceptible subgraphs of $\mathcal{G}_{n,p}$

With Lemma 4.3.5 at hand, we obtain upper bounds for the growth probabilities of r-percolations on $\mathcal{G}_{n,p}$.

A set I of size r is called *k-contagious* in the graph $\mathcal{G}_{n,p}$, if there is some t so that $|V_t(I,\mathcal{G}_{n,p})| = k$, i.e., there is some time at which there are exactly k infected vertices. The set I is called (k,i)-contagious if in addition the number of vertices infected at step t is i, i.e., $|I_t(I,\mathcal{G}_{n,p})| = i$. Let $P_r(k,i) = P_r(p,k,i)$ denote the probability that a given $I \subset [n]$, with $|I| = r$ is (k,i)-contagious. Let $P_r(k) = \sum_i P_r(k,i)$ denote the probability that such an I is k-contagious. Finally, let $E_r(k,i)$ and $E_r(k)$ denote the expected number of such subsets I.

We remark that $P_r(k)$ is not the same as the probability of *survival* to size k, which is given by $\sum_{\ell \geq k} \sum_{i > \ell - k} P_r(\ell, i)$.

Lemma 4.3.7. *Let $\alpha > 0$, and let $p = \theta_r(\alpha, n)$ (as defined in Proposition 4.3.1) and $\varepsilon = np^r = \alpha / \log^{r-1} n$. For $i \leq k - r$ and $k \leq n^{1/(r(r+1))}$, we have that*

$$P_r(k, i) \leq (1 + o(1)) \frac{e^{-\varepsilon \binom{k-i}{r}} \varepsilon^{k-r}}{(k-r)!} m_r(k, i)$$

where $o(1)$ depends on n, but not on i, k.

Proof. Let $I \subset [n]$, with $|I| = r$, be given, and put

$$\ell_r(k, i) = \frac{e^{-\varepsilon \binom{k-i}{r}} \varepsilon^{k-r}}{(k-r)!} m_r(k, i)$$

so that the lemma states $P_r(k, i) \leq (1 + o(1)) \ell_r(k, i)$. This follows by a union bound: If I is (k, i)-contagious, then I is a contagious set for a minimally susceptible subgraph $G \subset \mathcal{G}_{n,p}$ (perhaps not induced) of size k with i vertices infected in the top level, and all vertices in $v \in V(G)^c$ are connected to at most $r - 1$ vertices below the top level of G (so that $V(G) = V_t(I, \mathcal{G}_{n,p})$, for some t). There are $\binom{n}{k-r}$ choices for the vertices of G and $m_r(k, i)$ choices for its edges. For any such v and G, the probability that v is connected to r vertices below the top level of G is bounded from below by

$$\binom{k-i}{r} p^r (1-p)^{k-i-r} > \binom{k-i}{r} p^r (1-p)^k.$$

Hence

$$P_r(k, i) < \binom{n}{k-r} m_r(k, i) p^{r(k-r)} \left(1 - \binom{k-i}{r} p^r (1-p)^k\right)^{n-k}.$$

By the inequalities $\binom{n}{k} \leq n^k / k!$ and $1 - x < e^{-x}$, it follows that

$$\log \frac{P_r(k, i)}{\ell_r(k, i)} < \varepsilon \binom{k-i}{r} \left(1 - (1-p)^k \left(1 - \frac{k}{n}\right)\right).$$

By the inequality $(1-x)^y \geq 1-xy$, and since $k \leq n^{1/(r(r+1))}$, the right hand side is bounded by

$$\varepsilon k^{r+1}(p + (1-pk)/n) \leq \varepsilon n^{1/r}(p + 1/n) \ll 1$$

as $n \to \infty$. Hence $P_r(k, i) \leq (1 + o(1))\ell_r(k, i)$, as claimed. ■

As a corollary we get a bound for $E_r(k, i)$.

Lemma 4.3.8. *Let $\alpha, \beta_0 > 0$. Put $p = \theta_r(\alpha, n)$. For all $k = \beta \log n$ and $i = \gamma k$, such that $\beta \leq \beta_0$, we have that*

$$E_r(k, i) \lesssim n^{\mu} \log^{r(r-1)} n$$

where

$$\mu = \mu_r(\alpha, \beta, \gamma) = r + \beta \log \left(\frac{\alpha \beta^{r-1}}{(r-1)!} \right) - \frac{\alpha \beta^r}{r!}(1-\gamma)^r - \beta(r - 2 + \gamma). \quad (4.3.6)$$

Here $\lesssim$ denotes inequality up to a constant depending on α, β_0, but not on β, γ.

Proof. Let $r \geq 2$ and $\alpha, \beta_0 > 0$ be given. Put $\varepsilon = np^r$. By Lemmas 4.3.5 and 4.3.7, for all $k = \beta \log n$ and $i = \gamma k$, with $\beta \leq \beta_0$, we have that

$$E_r(k, i) \leq (1 + o(1)) \binom{n}{r} \left(\frac{\varepsilon k^{r-1}}{(r-1)!} \right)^k \varepsilon^{-r} e^{-i-(r-2)k-\varepsilon\binom{k-i}{r}} \lesssim n^{\mu} \log^{r(r-1)} n.$$

The $\sqrt{i}$ term from Lemma 4.3.5 is safely dropped for this upper bound. ■

4.3.4 Sub-critical bounds

In this section, we prove Proposition 4.3.1.

The case of $\gamma = 0$ in Lemma 4.3.8 (corresponding to values of i such that $i/k \ll 1$) is of particular importance for the growth of sub-critical r-percolations. For this reason, we let $\mu^*(\alpha, \beta) = \mu(\alpha, \beta, 0)$. The next result in particular shows that $\beta_*(\alpha)$, as in Proposition 4.3.1, is well-defined.

Lemma 4.3.9. *Let $\alpha > 0$. Let α_r be as in Proposition 4.3.1. Put*

$$\beta_r(\alpha) = \left(\frac{(r-1)!}{\alpha}\right)^{1/(r-1)}.$$

(i) The function $\mu_r^*(\alpha,\beta)$ is decreasing in β, with a unique zero at $\beta_*(\alpha)$.

(ii) We have that

$$\mu_r^*(\alpha,\beta_r(\alpha)) = r - \beta_r(\alpha)\frac{(r-1)^2}{r}$$

and hence $\beta_*(\alpha) = \beta_r(\alpha)$ (resp. $>$ or $<$) if $\alpha = \alpha_r$ (resp. $>$ or $<$).

The quantity $\beta_*(\alpha)$ also plays a crucial role in analyzing the growth of super-critical r-percolations on $\mathcal{G}_{n,p}$, see Section 4.4.5 below.

Proof. For the first claim, we note that by setting $\gamma = 0$ in (4.3.6) we obtain

$$\mu_r^*(\alpha,\beta) = r + \beta\log\left(\frac{\alpha\beta^{r-1}}{(r-1)!}\right) - \frac{\alpha\beta^r}{r!} - \beta(r-2). \tag{4.3.7}$$

Therefore

$$\frac{\partial}{\partial\beta}\mu_r^*(\alpha,\beta) = 1 + \log\left(\frac{\alpha\beta^{r-1}}{(r-1)!}\right) - \frac{\alpha\beta^{r-1}}{(r-1)!}.$$

Since $\alpha\beta_r(\alpha)^{r-1}/(r-1)! = 1$, the above expression is equal to 0 at $\beta = \beta_r(\alpha)$ and negative for all other $\beta > 0$. Hence $\mu_*(\alpha,\beta)$ is decreasing in β, as claimed. Moreover, since $\lim_{\beta\to 0^+}\mu_r^*(\alpha,\beta) = r$ and $\lim_{\beta\to\infty}\mu_r^*(\alpha,\beta) = -\infty$, $\beta_*(\alpha)$ is well-defined.

We obtain the expression for $\mu_r^*(\alpha,\beta_r(\alpha))$ in the second claim by (4.3.7) and the equality $\alpha\beta_r(\alpha)^{r-1}/(r-1)! = 1$. The conclusion of the claim thus follows by the first claim, noting that $\beta_r(\alpha)$ is decreasing in α and $\mu_r^*(\alpha_r,\beta_r(\alpha_r)) = 0$ since $\beta_r(\alpha_r) = (r/(r-1))^2$. ■

We are ready to prove the main result of this section.

Proof of Proposition 4.3.1. Let $\alpha < \alpha_r$ and $\delta > 0$ be given. First, we show that with high probability, $\mathcal{G}_{n,p}$ contains no m-contagious set, for $m = \beta\log n$ with $\beta \in [\beta_*(\alpha)+\delta, \beta_r(\alpha)]$.

Claim 4.3.10. *For all $\beta \le \beta_r(\alpha)$, we have that $\mu_r(\alpha, \beta, \gamma) \le \mu_r^*(\alpha, \beta)$.*

This is proved in Section 4.7.2.

By Lemmas 4.3.8 and 4.3.9 and Claim 4.3.10, we find by summing over all $O(\log n)$ relevant k that the probability that such a set exists is bounded (up to a multiplicative constant) by

$$n^{\mu_*(\alpha,\beta_*(\alpha)+\delta)} \log^{r(r-1)+1} n \ll 1.$$

It thus remains to show that with high probability, $\mathcal{G}_{n,p}$ has no m-contagious set I, for some $m \ge \beta_r \log n$. To this end, note that if such a set I exists, then there is some t so that

$$|V_t(I, \mathcal{G}_{n,p})| < \beta_r \log n \le |V_{t+1}(I, \mathcal{G}_{n,p})|.$$

Letting $k = |V_t(I, \mathcal{G}_{n,p})|$, we find that for some $k < \beta_r \log n$ there is a k-contagious set I, with $m - k$ further vertices with r neighbours in $V_t(I, \mathcal{G}_{n,p})$.

The expected number of k-contagious sets with i vertices infected in the top level is $E_r(k, i)$. Let $p_r(k, i)$ be the probability that for a given set of size k with i vertices identified as the top level, there are at least $\beta_r \log n - k$ vertices r-connected to the set with at least one neighbour in the top level. Hence the probability that $\mathcal{G}_{n,p}$ has a m-contagious set I for some $m \ge \beta_r \log n$ is at most

$$\sum_{i<k<\beta_r(\alpha)\log n} E_r(k, i) p_r(k, i).$$

The proposition now follows from the following claim, which is proved in Section 4.7.3.

Claim 4.3.11. *For all $k < \beta_r(\alpha) \log n$ and $i \le k - r$, we have that*

$$E_r(k, i) p_r(k, i) \lesssim n^{\mu_r^*(\alpha,\beta_r(\alpha))} \log^{r(r-1)} n$$

where $\lesssim$ denotes inequality up to a constant, independent of i, k.

Indeed, by Claim 4.3.11, it follows, by summing over all $O(\log^2 n)$ relevant

i,k, that the probability that $\mathcal{G}_{n,p}$ has an m-contagious set for some $m \geq \beta_r(\alpha)\log n$ is bounded (up to a constant) by

$$n^{\mu_r^*(\alpha,\beta_r)} \log^{r(r-1)+2} n \ll 1$$

where the last inequality follows by Lemma 4.3.9, since $\alpha < \alpha_r$ and hence $\mu_r^*(\alpha,\beta_r(\alpha)) < 0$. ■

4.4 Upper bound for $p_c(n,r)$

In this section, we prove Theorem 4.2.1. In light of Proposition 4.3.1, it remains to prove that for $\alpha > \alpha_r$, with high probability $\mathcal{G}_{n,p}$ is susceptible. Fundamentally this is done using the second moment method. As discussed in the introduction, the main obstacle is showing that contagious sets are sufficiently independent for the second moment method to apply. To this end, we restrict to a special type of contagious sets, which infect k vertices with no triangles.

As in the previous section, we fix $r \geq 2$ throughout.

4.4.1 Triangle-free susceptible graphs

Recall that a graph is called *triangle-free* if it contains no subgraph which is isomorphic to K_3.

Definition 4.4.1. Let $\hat{m}_r(k,i)$ denote the number of triangle-free graphs that contribute to $m_r(k,i)$ (see Section 4.3.1). Put $\hat{m}_r(k) = \sum_{i=1}^{k-r} \hat{m}_r(k,i)$.

Following Section 4.3.1, we obtain a recursive lower bound for $\hat{m}_r(k,i)$. We note that $\hat{m}_r(k,k-r) = m_r(k,k-r) = 1$. For $i < k-r$ we claim that

$$\hat{m}_r(k,i) \geq \binom{k-r}{i} \sum_{j=1}^{k-r-i} \hat{a}_r(k-i,j)^i \hat{m}_r(k-i,j) \tag{4.4.1}$$

where

$$\hat{a}_r(x,y) = \max\{0, a_r(x,y) - 2ryx^{r-2}\}. \tag{4.4.2}$$

Note that (in contrast to the recursion for $m(k, i)$), this is only a lower bound. To see (4.4.1), we argue that of the $a_r(k-i, j)$ ways to connect a vertex in the top level to lower levels, at most $2rj(k-i)^{r-2}$ create a triangle. This is so since the number of ways of choosing r vertices from $k-i$, including at least one of the top j and including at least one edge is at most

$$jr\binom{k-i-2}{r-2} + jr(k-i-r)\binom{k-i-3}{r-3} < 2jr(k-i)^{r-2},$$

where the first (resp. second) term accounts for case that an edge selected contains (resp. does not contain) a vertex among the top j.

Setting

$$\hat{\sigma}_r(k, i) = \frac{\hat{m}_r(k, i)}{(k-r)!}\left(\frac{(r-1)!}{k^{r-1}}\right)^k,$$

(4.4.1) reduces to

$$\hat{\sigma}_r(k, i) \geq \sum_{j=1}^{k-r-i} \hat{A}_r(k, i, j)\hat{\sigma}_r(k-i, j) \tag{4.4.3}$$

where

$$\hat{A}_r(k, i, j) = \frac{j^i}{i!}\left(\frac{k-i}{k}\right)^{(r-1)k}\left(\frac{(r-1)!}{(k-i)^{r-1}}\frac{\hat{a}_r(k-i, j)}{j}\right)^i. \tag{4.4.4}$$

The following observation indicates that restricting to susceptible graphs which are triangle-free does not have a significant effect on the asymptotics.

Lemma 4.4.2. *Let $\hat{A}_r(k, i, j)$ be as in* (4.4.4) *and let $A_r(i, j)$ be as defined in Lemma 4.3.3. For any fixed $i, j \geq 1$, we have that $\hat{A}_r(k, i, j) \to A_r(i, j)$, as $k \to \infty$.*

Proof. Fix $i, j \geq 1$. From their definitions we have that

$$\frac{\hat{A}_r(k, i, j)}{A_r(k, i, j)} = \left(\frac{\hat{a}_r(k, i, j)}{a_r(k, i, j)}\right)^i.$$

Since $a_r(k, i, j)$ is of order k^i and $\hat{a}_r(k, i, j) - a(k, i, j) = O(k^{i-1})$, we have

$\hat{a}_r(k,i,j)/a_r(k,i,j) \to 1$. Since i is fixed, it follows by Lemma 4.3.3 that

$$\lim_{k\to\infty} \hat{A}_r(k,i,j) = \lim_{k\to\infty} A_r(k,i,j) = A_r(i,j). \qquad \blacksquare$$

In order to get asymptotic lower bounds on $\hat{m}_r(k,i)$ it is useful to further restrict to graphs with bounded level sizes.

Definition 4.4.3. For $\ell \geq r$, let $\hat{m}_{r,\ell}(k) \leq \hat{m}_r(k)$ be the number of graphs that contribute to $\hat{m}_r(k)$ which have level sizes bounded by ℓ (i.e., $|I_i| \leq \ell$ for all i). Let $\hat{m}_{r,\ell}(k,i)$ be the number of such graphs with exactly $i \leq \ell$ vertices in the top level. Hence $\hat{m}_{r,\ell}(k) = \sum_{i=1}^{\ell} \hat{m}_{r,\ell}(k,i)$.

Observe that for fixed k, $\hat{m}_{r,\ell}(k)$ is increasing in ℓ, and equals $m_r(k)$ for $\ell \geq k-r$.

Lemma 4.4.2 will be used to prove asymptotic lower bounds for $\hat{m}$. When i is small, the resulting bounds are not sufficiently strong. Thus we also make use of the following lower bound on $\hat{m}_{r,\ell}(k,i)$ for values of i which are small compared with k. This is also used as a base case for an inductive proof of lower bounds using Lemma 4.4.2.

Lemma 4.4.4. *For all relevant i,k and $\ell \geq r$ such that $k > r(r^2+1)+i+2$, we have that*

$$\hat{m}_{r,\ell}(k,i) \geq \binom{k-r}{i} \hat{b}_r(k,i)^i \hat{m}_{r,\ell}(k-i)$$

where

$$\hat{b}_r(k,i) = \binom{k-i-r-1}{r-1}\left(1 - \frac{r^3}{k-i-r-2}\right).$$

In particular $\hat{m}_{r,\ell}(k,i) > 0$ for such k.

Proof. Let i,k,ℓ as in the lemma be given. We obtain the lemma by considering the subset $\mathcal{H}$ of graphs contributing to $\hat{m}_{r,\ell}(k,i)$, constructed as follows. To obtain a graph $H \in \mathcal{H}$, select a subset $U \subset [k]-[r]$ of size i for the vertices in the top level of H, and a minimally susceptible, triangle-free graph H' on $[k]-U$ so that $[r]$ is a contagious set for H' with all level sizes bounded by ℓ and j vertices in the top level, for some $1 \leq j \leq \min\{k-r-i, \ell\}$. Let v denote the vertex in the top level of H' of largest index. For each

$u \in U$, select a subset $V_u \subset [k] - U$ of size r which contains v and none of the neighbours of v in H', and so that no $v', v'' \in V_u$ are neighbours in H'. Finally, let H be the minimally susceptible graph on $[k]$ with subgraph H' such that each vertex $u \in U$ is connected to all vertices in V_u. By the choice of H' and V_u, H contributes to $\hat{m}_{r,\ell}(k,i)$. By the choice of v, for any choice of U, H' and V_u, a unique graph H is obtained. Hence $|\mathcal{H}| \leq \hat{m}_{r,\ell}(k,i)$.

To conclude, we claim that, for each $u \in U$, the number of possibilities for V_u is bounded from below by

$$\binom{k-i-r-1}{r-1} - r(k-i-r-1)\binom{k-i-r-3}{r-3} \geq \hat{b}_r(k,i).$$

To see this, note that of the $r(k-i)$ edges in H', there are $r(r+1)$ that are either incident to v or else connect a neighbour of v in H' to another vertex below the top level of H'. Therefore

$$\hat{m}_{r,\ell}(k,i) \geq \binom{k-r}{i}\hat{b}_r(k,i)\sum_j \hat{m}_{r,\ell}(k-i,j) = \binom{k-r}{i}\hat{b}_r(k,i)\hat{m}_{r,\ell}(k-i)$$

(where the sum is over $1 \leq j \leq \min\{k-r-i, \ell\}$) as claimed.

By the choice of i, k, $\hat{b}_r(k,i) > 0$. Hence $\hat{m}_{r,\ell}(k,i) > 0$, since $\hat{m}_{r,\ell}(k) > 0$ for all relevant k, ℓ, as is easily seen (e.g., by considering minimally susceptible, triangle-free graphs of size $k = nr + m$, for some $n \geq 1$ and $m \leq r$, which have m vertices in the top level and r vertices in all levels below, and all vertices in level $i \geq 1$ are connected to all r vertices in level $i-1$). ■

Lemma 4.4.5. *As $k \to \infty$, we have that*

$$m_r(k) \geq \hat{m}_r(k) \geq e^{-o(k)} e^{-(r-2)k}(k-r)! \left(\frac{k^{r-1}}{(r-1)!}\right)^k.$$

Comparing this with Lemma 4.3.5, we see that the number of triangle-free susceptible graphs of size k is not much smaller than the number of susceptible graphs (up to an error of $e^{o(k)}$).

Proof. The idea is to use spectral analysis of the linear recursion (4.4.5).

However, some work is needed to write the recursion in a usable form.

Put

$$\hat{\sigma}_{r,\ell}(k,i) = \frac{\hat{m}_{r,\ell}(k,i)}{(k-r)!}\left(\frac{(r-1)!}{k^{r-1}}\right)^k$$

Restricting (4.4.3) to $j \le \ell$, it follows that

$$\hat{\sigma}_{r,\ell}(k,i) \ge \sum_{j=1}^{\ell} \hat{A}_r(k,i,j)\hat{\sigma}_{r,\ell}(k-i,j) \quad \text{for } i \le \ell. \tag{4.4.5}$$

In order to express (4.4.5) in matrix form, we introduce the following notations. For an $\ell \times \ell$ matrix M, let M_j, be the $\ell \times \ell$ matrix whose jth row is that of M and all other entries are 0. Let

$$\psi(M) = \begin{bmatrix} M_1 & M_2 & \cdots & M_{\ell-1} & M_\ell \\ I_\ell & & & & \\ & I_\ell & & & \\ & & \ddots & & \\ & & & I_\ell & \end{bmatrix}$$

where I_ℓ is the $\ell \times \ell$ identity matrix and all empty blocks are filled with 0's. For all relevant k, put

$$\hat{\Sigma}_k = \hat{\Sigma}_k(r,\ell) = \begin{bmatrix} \hat{\sigma}_k \\ \hat{\sigma}_{k-1} \\ \vdots \\ \hat{\sigma}_{k-\ell+1} \end{bmatrix}$$

where $\hat{\sigma}_k = \hat{\sigma}_k(r,\ell)$ is the $1 \times \ell$ vector with entries $(\hat{\sigma}_k)_j = \hat{\sigma}_{r,\ell}(k,j)$.

Using this notation, (4.4.5) can be written as

$$\hat{\Sigma}_k \ge \psi(\hat{A}_k)\hat{\Sigma}_{k-1}, \tag{4.4.6}$$

where $\hat{A}_k = \hat{A}_k(r,\ell)$ is the $\ell \times \ell$ matrix with entries $(\hat{A}_k)_{i,j} = \hat{A}_r(k,i,j)$.

By Lemma 4.4.4, we have that all coordinates of $\hat{\Sigma}_k$ are positive for all k large enough. Let $A = A(r,\ell)$ denote the $\ell \times \ell$ matrix with entries

$A_{i,j} = A_r(i,j)$ (as defined in Lemma 4.3.3). For $\varepsilon > 0$, let $A_\varepsilon = A_\varepsilon(r,\ell)$, be the $\ell \times \ell$ matrix with entries $(A_\varepsilon)_{i,j} = A_{i,j} - \varepsilon$. By Lemma 4.4.2, for k large enough each entry of $\hat{A}_k$ is greater than the same entry of A_ε. Since $A > 0$, for some $\varepsilon_{r,\ell} > 0$, we have that $A_\varepsilon > 0$ for all $\varepsilon \in (0, \varepsilon_{r,\ell})$. Hence, by Lemma 4.4.2 and (4.4.6), for any such $\varepsilon > 0$, there is a k_ε so that

$$\hat{\Sigma}_{k_\varepsilon + k} \geq \psi(A_\varepsilon)^k \hat{\Sigma}_{k_\varepsilon} > 0 \quad \text{for } k \geq 0,$$

with entries of Σ_{k_ε} positive. Therefore, up to a factor of $e^{-o(k)}$, the growth rate of $\hat{\sigma}_{r,\ell}(k) = \sum_i \hat{\sigma}_{r,\ell}(k,i)$ is given by the Perron eigenvalue $\lambda = \lambda(r,\ell)$ of $\psi(A)$.

Let $D_\lambda = \mathrm{diag}(\lambda^{-i} : 1 \leq i \leq \ell)$. We claim that the Perron eigenvalue of $\psi(A)$ is characterized by the property that the Perron eigenvalue of $D_\lambda A$ is 1. To see this, one simply verifies that if $D_\lambda A v = v$, then

$$v_\lambda = \begin{bmatrix} \lambda^{\ell-1} v \\ \lambda^{\ell-2} v \\ \vdots \\ v \end{bmatrix}$$

satisfies $\psi(A) v_\lambda = \lambda v_\lambda$. If v has non-negative entries, then 1 is the Perron eigenvalue of $D_\lambda A$ and λ the Perron eigenvalue of $\psi(A)$.

If $\lambda < e^{-(r-2)}(e\ell)^{-1/\ell}$, we claim that every row sum of $D_\lambda A$ is greater than 1. Indeed, for all such λ, the sum of row $i \leq \ell$ is (using the bound $i! \leq ei(i/e)^i$)

$$(e^{r-1}\lambda)^{-i} \sum_{j=1}^{\ell} \frac{j^i}{i!} > (e^{r-1}\lambda)^{-i} \frac{\ell^i}{i!} > \frac{1}{ei} \left((e\ell)^{1/\ell} \frac{\ell}{i} \right)^i.$$

Twice differentiating the log of the right hand side with respect to i, we obtain $-(i-1)/i^2$. Therefore, noting that for $i = \ell$ the right hand side above equals to 1, and for $i = 1$ it equals $(\ell/e)(e\ell)^{1/\ell} \geq 1$ for all relevant ℓ, the claim follows.

Since the spectral radius of a matrix is bounded below by its minimum row sum, it follows that for such λ, the spectral radius of $D_\lambda A$ is greater than 1. Since the spectral radius of $D_\lambda A$ is decreasing in λ, the Perron eigenvalue $\lambda(r, \ell)$ of $\psi(A)$ is at least $e^{-(r-2)}(e\ell)^{-1/\ell}$, and hence $\liminf_{\ell\to\infty} \lambda(r, \ell) \geq e^{-(r-2)}$. Taking $\ell \to \infty$, we find that

$$\hat{m}_r(k) \geq e^{-o(k)} e^{-(r-2)k} (k-r)! \left(\frac{k^{r-1}}{(r-1)!} \right)^k$$

as required. ■

We require a lower bound for the number of minimally susceptible graphs of size k with $i = \Omega(k)$ vertices in the top level in order to estimate the growth of super-critical r-percolations on $\mathcal{G}_{n,p}$.

Lemma 4.4.6. *Let $\varepsilon \in (0, 1/(r+1))$. For all sufficiently large k and $i \leq (\varepsilon/r)^2 k$, we have that*

$$\hat{m}_r(k, i) \geq e^{-i\varepsilon-(r-2)k-o(k)} (k-r)! \left(\frac{(k-i)k^{r-2}}{(r-1)!} \right)^k$$

where $o(k)$ depends on k, ε, but not on i.

Although the proof is somewhat involved, the general scheme is straightforward. We use Lemmas 4.4.4 and 4.4.5 to obtain a sufficient bound for i, k in a range for which $i/k \ll 1$. Then, for all other relevant i, k we proceed by induction, using (4.4.1). The inductive step (Claim 4.4.7 below) of the proof appears in Section 4.7.4.

Proof. Fix some k_r so that

$$k_r > \max\left\{ e^{r/\varepsilon}, \frac{r(r^2+1)+2}{1-(\varepsilon/r)^2} \right\}.$$

Note that, for all $k > k_r$ and $i \leq (\varepsilon/r)^2 k$, we have that $k/\log^2 k < (\varepsilon/r)^2 k$ and that Lemma 4.4.4 applies to $\hat{m}_r(k, i)$ (setting $\ell = k - r$, so that $\hat{m}_{r,\ell}(k, i) = \hat{m}_r(k, i)$).

For all relevant i,k, let

$$\hat{\rho}_r(k,i) = \frac{\hat{m}_r(k,i)}{(k-r)!}\left(\frac{(r-1)!}{(k-i)k^{r-2}}\right)^k. \tag{4.4.7}$$

By Lemma 4.4.5 there is some $f_r(k) \ll k$ such that

$$\hat{m}_r(k) \geq e^{-(r-2)k-f_r(k)}(k-r)!\left(\frac{k^{r-1}}{(r-1)!}\right)^k.$$

Without loss of generality, we assume f_r is non-decreasing.

By Lemma 4.4.4, we find that for all $k > k_r$ and relevant i, $\hat{\rho}_r(k,i)$ is bounded from below by

$$\frac{e^{-(r-2)(k-i)-f_r(k-i)}}{i!}\hat{b}_r(k,i)^i\left(\frac{(k-i)^{r-1}}{(r-1)!}\right)^{k-i}\left(\frac{(r-1)!}{(k-i)k^{r-2}}\right)^k.$$

By the bound $\binom{n}{k} \geq (n-k)^k/k!$,

$$\hat{b}_r(k,i) \geq \frac{(k-i-2r)^{r-1}}{(r-1)!}\left(1-\frac{r^3}{k-i-r-2}\right).$$

Therefore the lower bound for $\hat{\rho}_r(k,i)$ above is bounded from below by (using the inequality $i! < i^i$)

$$C_r(k,i)g_r(k,i)e^{-(r-2)k-f_r(k-i)-i\log i}$$

where

$$C_r(k,i) = \left(1-\frac{2r}{k-i}\right)^{(r-1)i}\left(1-\frac{r^3}{k-i-r-2}\right)^i$$

and

$$g_r(k,i) = e^{(r-2)i}\left(\frac{k-i}{k}\right)^{(r-2)k}.$$

If $r=2$, then $g_r \equiv 1$. We note that, for $r > 2$,

$$\frac{\partial}{\partial i}g_r(k,i) = -\frac{(r-2)i}{k-i}g_r(k,i) < 0$$

and so, for any such r, for any relevant k, $g_r(k,i)$ is decreasing in i. By the inequality $(1-x)^y > 1-xy$, for any $k > k_r$ and $i \le (\varepsilon/r)^2 k$,

$$\begin{aligned} C_r(k,i) &> 1 - \frac{r^2 i}{k-i} - \frac{r^3 i}{k-i-r-2} \\ &> 1 - \frac{2\varepsilon^2}{1-(\varepsilon/r)^2} - \frac{r\varepsilon^2}{1-(\varepsilon/r)^2-(r+2)/k} \\ &> 1 - \frac{2/(r+1)^2}{1-1/r^4} - \frac{1/r}{1-1/r^4-(r+2)/k_r} \\ &> 0 \end{aligned}$$

since $k_r > e^{r/\varepsilon} > e^{r(r+1)}$, $r \ge 2$, and $\varepsilon < 1/(r+1)$ (and noting that the second last line is increasing in r). Altogether, for some $\xi'(r) > 0$, we have that

$$\hat{\rho}_r(k,i) \ge \xi'(r) e^{-(r-2)k-h_r(k)} \quad \text{for } k > k_r \text{ and } i \le k/\log^2 n \tag{4.4.8}$$

where

$$h_r(k) = f_r(k) - \log g_r\left(k, \frac{k}{\log^2 k}\right) + \frac{k}{\log^2 k} \log\left(\frac{k}{\log^2 k}\right). \tag{4.4.9}$$

We note that $h(k) \ll k$ as $k \to \infty$.

Claim 4.4.7. *For some $\xi = \xi(r,\varepsilon) > 0$, for all $k > k_r$ and $i \le (\varepsilon/r)^2 k$, we have that $\hat{\rho}_r(k,i) \ge \xi e^{-i\varepsilon-(r-2)k-h_r(k)}$.*

Claim 4.4.7 is proved in Section 4.7.4.

Since $h_r(k) \ll k$ and ξ depends only on r, ε, the lemma follows by Claim 4.4.7 and (4.4.7). ■

4.4.2 $\hat{r}$-bootstrap percolation on $\mathcal{G}_{n,p}$

We define *$\hat{r}$-percolation,* a restriction of r-percolation, which informally halts upon requiring a triangle. Formally, recall the definitions of $I_t(I,G)$ and $V_t(I,G)$ given in Section 4.3.1. Let $\hat{I}_t = I_t$ if G contains a triangle-free subgraph H such that $V_t(I,H) = V_t(I,G)$, and otherwise put $\hat{I}_t = \emptyset$. Put $\hat{V}_t = \bigcup_{s\le t} \hat{I}_s$.

Definition 4.4.8. Let $\hat{P}_r(k,i) = \hat{P}_r(p,k,i)$, for some $p = p(n)$, denote the probability that for a given $I \subset [n]$, with $|I| = r$, we have that $|\hat{V}_t(I,\mathcal{G}_{n,p})| = k$ and $|\hat{I}_t(I,\mathcal{G}_{n,p})| = i$, for some t. Let $\hat{E}_r(k,i)$ denote the expected number of such subsets I. We put $\hat{P}_r(k) = \sum_{i=1}^{k-2} \hat{P}_r(k,i)$ and $\hat{E}_r(k) = \sum_{i=1}^{k-r} \hat{E}_r(k,i)$.

Using Lemma 4.4.6, we obtain lower bounds on the growth probabilities of $\hat{r}$-percolations on $\mathcal{G}_{n,p}$.

Lemma 4.4.9. *Let $\alpha > 0$. Put $p = \theta_r(\alpha,n)$ and $\varepsilon = np^r = \alpha/\log^{r-1} n$. For $i \le k-r$ and $k \le n^{1/(r(r+1))}$, we have that*

$$\hat{P}_r(k,i) \ge (1-o(1))\frac{e^{-\varepsilon\binom{k-i}{r}}\varepsilon^{k-r}}{(k-r)!}\hat{m}_r(k,i)$$

where $o(1)$ depends on n, but not on i,k.

Proof. Let $I \subset [n]$, with $|I| = r$, be given. Put

$$\hat{\ell}_r(k,i) = \frac{e^{-\varepsilon\binom{k-i}{r}}\varepsilon^{k-r}}{(k-r)!}\hat{m}_r(k,i).$$

If for some $V \subset [n]$ with $|V| = k$ and $I \subset V$ we have that the subgraph $G_V \subset \mathcal{G}_{n,p}$ induced by V is minimally susceptible and triangle-free, I is a contagious set for G_V with i vertices in the top level, and all vertices in $v \in V^c$ are connected to at most $r-1$ vertices below the top level of G_V, then it follows that $|\hat{V}_t(I,\mathcal{G}_{n,p})| = k$ and $|\hat{I}_t(I,\mathcal{G}_{n,p})| = i$ for some t. Hence

$$\hat{P}_r(k,i) > \binom{n-r}{k-r}\hat{m}_r(k,i)p^{r(k-r)}(1-p)^{k^2}\left(1-\binom{k-i}{r}p^r\right)^n.$$

By the inequalities $\binom{n}{k} \ge (n-k)^k/k!$ and $(1-x/n)^n \ge e^{-x}(1-x^2/n)$, it follows that

$$\frac{\hat{P}_r(k,i)}{\hat{\ell}_r(k,i)} > \left(1-\frac{k}{n}\right)^k (1-p)^{k^2}\left(1-\binom{k-i}{r}^2\frac{\varepsilon^2}{n}\right).$$

For all large n, the right hand side is bounded from below by

$$\left(1-\frac{k}{n}\right)^k \left(1-\frac{1}{n^{1/r}}\right)^{k^2} \left(1-\frac{k^{2r}}{n}\right) \sim 1$$

since $k \le n^{1/(r(r+1))} \ll n^{1/(2r)}$, as $r \ge 2$. It follows that $\hat{P}_r(k,i) \ge (1-o(1))\hat{\ell}_r(k,i)$, where $o(1)$ depends on n, but not on i,k, as required. ■

4.4.3 Super-critical bounds

In this section we show that, for $\alpha > \alpha_r$, the expected number of super-critical $\hat{r}$-percolations on $\mathcal{G}_{n,p}$ which grow larger than the critical size of $\beta_*(\alpha)\log n > \beta_r(\alpha)\log n$ is large. The importance of $\beta_*(\alpha)$ is established in Section 4.4.5 below. Subsequent sections establish the existence of sets I of size r so that $\hat{r}$-percolation initialized at I grows larger than $\beta_*(\alpha)\log n$.

Lemma 4.4.10. *Let $\alpha, \beta_0 > 0$ and $\varepsilon \in (0, 1/(r+1))$. Put $p = \theta_r(\alpha, n)$. For all sufficiently large $k = \beta\log n$ and $i = \gamma k$, with $\beta \le \beta_0$ and $\gamma \le (\varepsilon/r)^2$, we have that*

$$\hat{E}_r(k,i) \ge n^{\mu_\varepsilon - o(1)}$$

where

$$\mu_\varepsilon = \mu_{r,\varepsilon}(\alpha,\beta,\gamma) = r + \beta\log\left(\frac{\alpha\beta^{r-1}(1-\gamma)}{(r-1)!}\right) - \frac{\alpha\beta^r}{r!}(1-\gamma)^r - \beta(r-2+\varepsilon\gamma)$$

and $o(1)$ depends on $\alpha,\varepsilon,\beta_0$, but not on β,γ.

Proof. Put $\delta = np^2$. By Lemmas 4.4.6 and 4.4.9, for large $k = \beta\log n$ and $i = \gamma k$, with $\beta \le \beta_0$ and $\gamma \le (\varepsilon/r)^2$,

$$\hat{E}_r(k,i) \ge \xi(n)\binom{n}{r}\left(\frac{\delta(k-i)k^{r-2}}{(r-1)!}\right)^k \delta^{-r} e^{-i\varepsilon-(r-2)k-\delta\binom{k-i}{r}-o(k)} = n^{\mu_\varepsilon - o(1)}$$

where $\xi(n) \sim 1$ depends only on n, and $o(k)$ depends only on r, ε, β_0. ■

We note that, for any $\alpha, \varepsilon > 0$,

$$\mu_{r,\varepsilon}(\alpha, \beta, 0) = \mu_r^*(\alpha, \beta). \tag{4.4.10}$$

We now state the main result of this section.

Lemma 4.4.11. *Let $\varepsilon < 1/(r+1)$. Put $\alpha_{r,\varepsilon} = (1+\varepsilon)\alpha_r$ and $p = \theta_r(\alpha_{r,\varepsilon}, n)$. For some $\delta(r, \varepsilon) > 0$ and $\zeta(r, \varepsilon) > 0$, if $k_n/\log n \in [\beta_*(\alpha_{r,\varepsilon}), \beta_*(\alpha_{r,\varepsilon}) + \delta]$, for all large n, then $\hat{E}_r(k_n) \gg n^\zeta$ as $n \to \infty$.*

The proof appears in Section 4.7.5. The argument is technical but straightforward: the basic idea is to show that, for some $\zeta > 0$ and all large n, for all relevant k there is some i so that $\hat{E}_r(k, i) > n^\zeta$. For $k > \beta_* \log n$, values of i with this property are on the order of k. We shall thus require Lemma 4.4.6.

4.4.4 $\hat{r}$-percolations are almost independent

For a set $I \subset [n]$, with $|I| = r$, let $\hat{\mathcal{E}}_k(I)$ denote the event that $\hat{r}$-percolation on $\mathcal{G}_{n,p}$ initialized by I grows to size k, i.e., we have that $|\hat{V}_t(I)| = k$ for some t. Hence $\hat{P}_r(k) = \mathbf{P}(\hat{\mathcal{E}}_k(I))$. In this section we show that for sets $I \neq I'$ of size r and suitable values of k, p, the events $\hat{\mathcal{E}}_k(I)$ and $\hat{\mathcal{E}}_k(I')$ are approximately independent. Specifically, we establish the following

Lemma 4.4.12. *Let $\alpha, \beta > 0$ and put $p = \theta_r(\alpha, n)$. Fix sets $I \neq I'$ such that $|I| = |I'| = r$ and $|I \cap I'| = m$. For $\beta \log n \leq k \leq n^{1/(r(r+1))}$, we have that $\mathbf{P}(\hat{\mathcal{E}}_k(I') | \hat{\mathcal{E}}_k(I))$ is bounded from above by*

$$(k/n)^{r-m} + O\big(k^{2r}(kp)^{r(r-m)}\big) + \begin{cases} (1+o(1))\hat{P}_r(k) & \text{if } m = 0, \\ o((n/k)^m)\hat{P}_r(k) & \text{if } 1 \leq m < r, \end{cases}$$

where $o(1)$ depends only on n.

For sets $I \subset V$ of sizes r and k, let $\hat{\mathcal{E}}(I, V)$ be the event that for some t we have $\hat{V}_t(I) = V$. By symmetry these events all have the same probability. Since for a fixed I and different sets V these events are disjoint, we have $\hat{P}_r(k) = \binom{n-r}{k-r}\mathbf{P}(\hat{\mathcal{E}}(I, V))$.

Lemma 4.4.13. *Fix sets $I \subset V$ with $|I| = r$ and $|V| = k$.*

(i) For any set of edges $E \subset [n]^2 - V^2$, the conditional probability that $E \subset E(\mathcal{G}_{n,p})$, given $\hat{\mathcal{E}}(I, V)$, is at most $p^{|E|}$.

(ii) For any $u \notin V$ and set of vertices $W \subset [n]$ such that $|W| = r$ and $|V \cap W| < r$, the conditional probability that $(u, w) \in \mathcal{G}_{n,p}$ for all $w \in W$, given $\hat{\mathcal{E}}(I, V)$, is at least $p^r(1-p)^k$.

Proof. Let $\mathcal{G}_V$ denote the subgraph of $\mathcal{G}_{n,p}$ induced by V. The event $\hat{\mathcal{E}}(I, V)$ occurs if and only if for some t and triangle-free subgraph $H \subset \mathcal{G}_V$, we have that $V_t(I, H) = V_t(I, \mathcal{G}_V) = V$ and all vertices in V^c are connected to at most $r - 1$ vertices below the top level of H (i.e., $V - I_t(I, H)$). This event is increasing in the set of edges of $\mathcal{G}_V$, and decreasing in edges outside V. By the FKG inequality,

$$\mathbf{P}(E \subset E(\mathcal{G}_{n,p}) | \hat{\mathcal{E}}(I, V)) \le \mathbf{P}(E \subset E(\mathcal{G}_{n,p})) = p^{|E|}.$$

For claim (ii), let G be a possible value for $\mathcal{G}_V$ on $\hat{\mathcal{E}}(I, V)$, with a subgraph H as above and $i \le k - r$ vertices infected in the top level (i.e., $I_t(I, H) = i$). The conditional probability that u is connected to all vertices in W, given $\hat{\mathcal{E}}(I, V)$ and $\mathcal{G}_V = G$, is equal to

$$\frac{p^r \sum_{\ell=0}^{r-1-\ell_0} \binom{k-i-\ell_0}{\ell} p^\ell (1-p)^{k-i-\ell_0-\ell}}{\sum_{\ell=0}^{r-1} \binom{k-i}{\ell} p^\ell (1-p)^{k-i-\ell}}$$

where $\ell_0 < r$ is the number of vertices in W below the top level of H. Bounding the numerator by the $\ell = 0$ term and the denominator by 1, the above expression is at least $p^r(1-p)^{k-i-\ell_0} \ge p^r(1-p)^k$. Hence, summing over the possibilities for G we obtain the second claim. ∎

The following result, a special case of Turán's Theorem [127], plays an key role in establishing the approximate independence of $\hat{r}$-percolations.

Lemma 4.4.14 (Mantel's Theorem [102])**.** *If a graph G is triangle-free, then we have that $e(G) \le \lfloor v(G)^2/4 \rfloor$.*

In other words, a triangle-free graph has edge-density at most 1/2. The number $2r-1$ is key, since $\lfloor (2r-1)^2/4 \rfloor = r(r-1)$, and thus

$$r(2r-1) - \lfloor (2r-1)^2/4 \rfloor = r^2. \tag{4.4.11}$$

Lemma 4.4.15. *Let $\alpha > 0$ and $k \le n^{1/(r(r+1))}$. Put $p = \theta_r(\alpha, n)$. Fix sets $I \subset V$ and I' such that $|I| = |I'| = r$, $|V| = k$ and $\ell = |V \cap I'| < r$. Let $\hat{\mathcal{E}}_{k,q}(I')$ denote the event that for some t we have that $\hat{V}_t(I') = V'$ for some V' such that $|V'| = k$ and $|V \cap V'| = q$. Then*

$$\mathbf{P}(\hat{\mathcal{E}}_{k,q}(I')|\hat{\mathcal{E}}(I,V)) \le \begin{cases} (1+o(1))\hat{P}_r(k) & q = 0, \\ o((n/k)^\ell)\hat{P}_r(k) & 1 \le q < 2r-1, \\ k^{2r-1}(kp)^{r(r-\ell)} & q \ge 2r-1, \end{cases}$$

where $o(1)$ depends only on n.

Proof. **Case i** ($q < 2r-1$). We claim that

$$\mathbf{P}(\hat{\mathcal{E}}_{k,q}(I')|\hat{\mathcal{E}}(I,V)) \le \left(\left(\frac{n}{k}\right)^\ell \left(\frac{k^2}{np^{q/4}}\right)^q \right) \sum_{i=1}^{k-r} \hat{Q}_r(k,i) \tag{4.4.12}$$

where $\hat{Q}_r(k,i)$ is equal to

$$\binom{n}{k-r} \hat{m}_r(k,i) p^{r(k-r)} \left(1 - \left(\binom{k-i}{r} - \binom{q}{r}\right) p^r (1-p)^{2k}\right)^{n-2k}.$$

To see this, note that if $\hat{\mathcal{E}}_{k,q}(I)$ occurs then for some V' such that $|V'| = k$, $I' \subset V'$, and $|V \cap V'| = q$, we have that I' is a contagious set for a triangle-free subgraph $H' \subset \mathcal{G}_{n,p}$ on V' with i vertices in the top level, for some $i \le k-r$, and all vertices in $(V \cup V')^c$ are connected to at most $r-1$ vertices below the top level of H'. There are at most

$$\binom{k}{q-\ell}\binom{n-(q-\ell)}{k-r-(q-\ell)} \le \left(\frac{n}{k}\right)^\ell \left(\frac{k^2}{n}\right)^q \binom{n}{k-r}$$

such subsets V'. By Lemmas 4.4.13 and 4.4.14, for any such V' and i as above, the conditional probability that such a subgraph H' exists, given

$\hat{\mathcal{E}}(I,V)$, is bounded by $\hat{m}(k,i)p^{r(k-r)-q^2/4}$, since at most $q^2/4$ edges of H' join vertices in $V\cap V'$. By Lemma 4.4.13, for any $u\in(V\cup V')^c$ and set V'' of r vertices below the top level of H' with at most $r-1$ vertices in $V\cap V'$, the conditional probability that u is connected to all vertices in V'' is at least $p^r(1-p)^k$. Hence any such u is connected to all vertices in such a V'' with conditional probability at least $\left(\binom{k-i}{r}-\binom{q}{r}\right)p^r(1-p)^{2k}$. The claim follows.

To conclude, let $\hat{\ell}_r(k,i)$ be as in the proof of Lemma 4.4.9, which recall shows that $\hat{P}_r(k,i)\ge(1-o(1))\hat{\ell}_r(k,i)$ as $k\to\infty$, where $o(1)$ depends only on n. We have, by the inequalities $\binom{n}{k}\le n^k/k!$ and $1-x<e^{-x}$, that

$$\log\frac{\hat{Q}_r(k,i)}{\hat{\ell}_r(k,i)}<\varepsilon\binom{k-i}{r}\left(1-(1-p)^{2k}\left(1-\frac{2k}{n}\right)\right)+\varepsilon q^r.$$

By the inequality $(1-x)^y\ge 1-xy$, and since $k\le n^{1/(r(r+1))}$, it follows that the right hand side is at most $\varepsilon n^{1/r}(p+1/n)+\varepsilon q^r\sim 0$, and so

$$\hat{Q}_r(k,i)\le(1+o(1))\hat{\ell}_r(k,i)\le(1+o(1))\hat{P}_r(k,i)$$

where $o(1)$ depends only on n. Hence

$$\sum_{i=1}^{k-r}\hat{Q}_r(k,i)\le(1+o(1))\sum_{i=1}^{k-r}\hat{P}_r(k,i)=(1+o(1))\hat{P}_r(k).$$

Finally, case (i) follows by (4.4.12) and noting that

$$\frac{np^{q/4}}{k^2}>\frac{np^{r/2}}{k^2}\ge n^{1/2-2/(r(r+1))}\left(\frac{\alpha}{\log^{r-1}n}\right)^{1/2}\gg 1$$

since $q<2r$, $k\le n^{1/(r(r+1))}$ and $r\ge 2$.

Case ii ($q\ge 2r-1$). Put $q_*=2r-1-\ell$. If $\hat{\mathcal{E}}_{k,q}(I')$ occurs, then for some $\{v_j\}_{j=1}^{q_*}\subset V-I'$ and non-decreasing sequence $\{t_j\}_{j=1}^{q_*}$, we have that $v_j\in\hat{I}_{t_j}(I')$ and $\hat{V}_j=\hat{V}_{t_j-1}(I')$ satisfy $|\hat{V}_{q_*}|<k$ and $\hat{V}_j\cap(V-I')\subset\bigcup_{i<j}\{v_i\}$. Informally, t_j is the jth time that $\hat{r}$-percolation initialized by I' infects a vertex in $V-I'$. It follows that $\mathcal{G}_{n,p}$ contains a triangle-free subgraph on $\{v_j\}_{j=1}^{q_*}\cup\hat{V}_{q_*}$. Since $v_j\in\hat{I}_{t_j}(I')$, note that v_j is r-connected to $\hat{V}_j$. Hence,

by Lemma 4.4.14 and (4.4.11), there are at least

$$rq_* - \lfloor (2r-1)^2/4 \rfloor = r(r-\ell)$$

edges between $\{v_j\}_{j=1}^{q_*}$ and $\hat{V}_{q_*} - V$. Thus, by Lemma 4.4.13, the conditional probability of $\hat{\mathcal{E}}_{k,q}(I')$, given $\hat{\mathcal{E}}(I, V)$, is bounded by $k^{q_*}(kp)^{r(r-\ell)} \leq k^{2r-1}(kp)^{r(r-\ell)}$, as claimed. ■

Using Lemma 4.4.15 we establish the main result of this section.

Proof of Lemma 4.4.12. Fix a sequence of sets $\{V_\ell\}_{\ell=m}^{r}$ such that $I \subset V_\ell$ and $\ell = |V_\ell \cap I'|$. By symmetry, we have that

$$\begin{aligned}
\mathbf{P}(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}_k(I)) &= \binom{n-r}{k-r}^{-1} \sum_{\ell=m}^{r} \binom{n-r-(\ell-m)}{k-r-(\ell-m)} \mathbf{P}(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}(I, V_\ell)) \\
&\leq \sum_{\ell=m}^{r} (k/n)^{\ell-m} \mathbf{P}(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}(I, V_\ell)).
\end{aligned}$$

If $\ell = m$, then by Lemma 4.4.15, summing over $q \in [\ell, k]$, we get

$$\mathbf{P}(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}(I, V_m)) \leq \begin{cases} (1+o(1))\hat{P}_r(k) + k^{2r}(kp)^{r^2} & m = 0, \\ o((n/k)^m)\hat{P}_r(k) + k^{2r}(kp)^{r(r-m)} & 1 \leq m < r. \end{cases}$$

Likewise, for any $m < \ell < r$,

$$\begin{aligned}
(\tfrac{k}{n})^{\ell-m} \mathbf{P}(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}(I, V_\ell)) &\leq (\tfrac{k}{n})^{\ell-m} \left(o((\tfrac{n}{k})^{\ell})\hat{P}_r(k) + k^{2r}(kp)^{r(r-\ell)} \right) \\
&= o((\tfrac{n}{k})^{m})\hat{P}_r(k) + k^{2r}(kp)^{r(r-m)}(np^r k^{r-1})^{m-\ell} \\
&\leq o((\tfrac{n}{k})^{m})\hat{P}_r(k) + k^{2r}(kp)^{r(r-m)}(\alpha\beta^{r-1})^{m-\ell} \\
&= o((\tfrac{n}{k})^{m})\hat{P}_r(k) + O(k^{2r}(kp)^{r(r-m)}).
\end{aligned}$$

Finally, for $\ell = r$ we bound $\mathbf{P}(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}(I, V_r)) \leq 1$. Summing over $\ell \in [m, r]$ we obtain the result. ■

4.4.5 Terminal r-percolations

In this section, we establish the importance of $\beta_*(\alpha)$ to the growth of supercritical r-percolations. Essentially, we find that an r-percolation on $\mathcal{G}_{n,p}$, having grown larger than $\beta_*(\alpha)\log n$, with high probability continues to grow.

Definition 4.4.16. We say that $I \subset [n]$ is a *terminal* (k,i)*-contagious set* for $\mathcal{G}_{n,p}$ if $|V_\tau(I, \mathcal{G}_{n,p}, r)| = k$ and $|I_\tau(I, \mathcal{G}_{n,p}, r)| = i$.

Lemma 4.4.17. *Let $\alpha > \alpha_r$ and $\beta_r^*(\alpha) < \beta_1 < \beta_2$. Put $p = \theta_r(\alpha, n)$. With high probability, $\mathcal{G}_{n,p}$ has no terminal m-contagious set, with $m = \beta \log n$, for all $\beta \in [\beta_1, \beta_2]$.*

Proof. If r-percolation initialized by $I \subset [n]$ terminates at size k with i vertices in the top level, then I is a contagious set for some subgraph $H \subset \mathcal{G}_{n,p}$ of size k with i vertices in the top level, and all vertices in $V(H)^c$ are connected to at most $r-1$ vertices in $V(H)$. Hence the probability that a given I is as such is bounded by

$$\binom{n}{k-r} m_r(k,i) p^{r(k-r)} \left(1 - \binom{k}{r} p^r (1-p)^r\right)^{n-k}.$$

For $k \le \beta_2 \log n$ and relevant i, we have that

$$1 - \binom{k}{r} p^r (1-p)^r = 1 - \binom{k}{r} p^r + O(n^{-1})$$

where $O(n^{-1})$ depends on α, β_2, but not on $k/\log n$ and i/k. Put $\varepsilon = np^r$. By Lemma 4.3.5 (and the inequalities $\binom{n}{k} \le n^k/k!$ and $1 - x < e^{-x}$), it follows that the expected number of terminal (k,i)-contagious sets, with $k = \beta \log n$ and $i = \gamma k$, for some $\beta \le \beta_2$, is bounded (up to a constant) by

$$\binom{n}{r} \left(\frac{\varepsilon k^{r-1}}{(r-1)!}\right)^k \varepsilon^{-r} e^{-i-(r-2)k-\varepsilon\binom{k}{r}} \lesssim n^{\mu_r^*(\alpha,\beta)-\beta\gamma} \log^{r(r-1)} n$$

where $\lesssim$ denotes inequality up to a constant depending on α, β_2, but not on β, γ.

By Lemma 4.3.9, we have that $\mu_r^*(\alpha, \beta) \le \mu_r^*(\alpha, \beta_1) < 0$ for all $\beta \in [\beta_1, \beta_2]$. Hence, summing over the $O(\log^2 n)$ relevant values of i, k, we find that the probability that $\mathcal{G}_{n,p}$ contains a terminal m-contagious set for some $m = \beta \log n$, with $\beta \in [\beta_1, \beta_2]$, is bounded (up to a constant) by

$$n^{\mu_r^*(\alpha,\beta_1)} \log^{r(r-1)+2} n \ll 1$$

as required. ■

4.4.6 Almost sure susceptibility

Finally, we complete the proof of Theorem 4.2.1. Using Lemmas 4.4.11, 4.4.12 and 4.4.17, we argue that if $\alpha > \alpha_r$, then with high probability $\mathcal{G}_{n,p}$ contains a large susceptible subgraph. By adding independent random graphs with small edge probabilities, we deduce that percolation occurs with high probability.

Proof of Theorem 4.2.1. Proposition 4.3.1 gives the sub-critical case $\alpha < \alpha_r$. Assume therefore that $\alpha > \alpha_r$. Let $\mathcal{G}_*, \mathcal{G}_i$, for $i \ge 0$, be independent random graphs with edge probabilities $p_* = \theta_r(\alpha_r + \varepsilon, n)$ and $p_i = 2^{-i(r-1)/r} p_\varepsilon$, where $p_\varepsilon = \theta_r(\varepsilon, n)$. Moreover, let $\varepsilon > 0$ be sufficiently small so that $\mathcal{G} = \mathcal{G}_* \cup \bigcup_{i\ge 0} \mathcal{G}_i$ is a random graph with edge probabilities at most $p = \theta_r(\alpha, n)$. Thus, to show that $\mathcal{G}_{n,p}$ is susceptible, it suffices to show that $\mathcal{G}$ is susceptible.

Claim 4.4.18. *Let $A > 0$. With high probability, the graph $\mathcal{G}_*$ contains a susceptible subgraph on some set $U_0 \subset [n]$ of size $|U_0| \ge A \log n$.*

Proof. Using Lemmas 4.4.11 and 4.4.12, we show by the second moment method that, with high probability, $\mathcal{G}_*$ contains a susceptible subgraph of size at least $(\beta_r^*(\alpha) + \delta_0) \log n$, for some $\delta_0 > 0$. By Lemma 4.4.17, this gives the claim.

Recall that Lemma 4.4.11 provides $\delta, \zeta > 0$ so that if $k_n / \log n \in [\beta_*(\alpha) + \delta/2, \beta_*(\alpha) + \delta]$, then $\hat{E}_r(k_n) \gg n^\zeta$. Fix such a sequence k_n. For each n, fix

$I_n \subset [n]$ with $|I_n| = r$. By Lemma 4.4.12, it follows that

$$\begin{aligned}\sum_I \frac{\mathbf{P}(\hat{\mathcal{E}}_{k_n}(I)|\hat{\mathcal{E}}_{k_n}(I_n))}{\hat{E}_r(k_n)} &\le 1 + o(1) + \binom{n}{r}^{-1} \sum_{m=1}^{r-1} \binom{n-m}{r-m} o((n/k_n)^m) \\ &\quad + n^{-\zeta} \sum_{m=0}^{r-1} \binom{n-m}{r-m} (O(k_n^{2r}(k_n p_*)^{r(r-m)}) + (k_n/n)^{r-m}) \\ &\le 1 + o(1) + \sum_{m=1}^{r-1} o((r/k_n)^m) \\ &\quad + n^{-\zeta} \sum_{m=0}^{r-1} (O(k_n^{2r}((k_n p_*)^r n)^{r-m}) + (k_n)^{r-m}) \\ &= 1 + o(1) + O(n^{-\zeta} \log^{3r} n) \\ &\sim 1\end{aligned}$$

where the sum is over $I \neq I_n$ with $|I| = r$, and $|I \cap I_n| = m$ for some $0 \le m < r$. Hence, by the second moment method, with high probability some $\hat{r}$-percolation on $\mathcal{G}_*$ grows to size k_n and thus $\mathcal{G}_*$ contains a suceptible subgraph of size k_n, as required. As discussed, the claim follows by the choice of k_n and Lemma 4.4.17. ∎

Claim 4.4.19. *There is some $A = A(\varepsilon)$ so that if U_0 is a set of size $|U_0| \ge A \log n$, then with high probability, r-percolation on $\bigcup_{i\ge 1} \mathcal{G}_i$ initialized at U_0 infects a set of vertices of order $n/\log n$.*

Proof. Let $A = 2r(16r/\varepsilon)^{1/(r-1)}$. Moreover assume that n is sufficiently large and ε is sufficiently small so that $A \ge 2$ and $A(2^{1-r}\varepsilon/\log n)^{1/r} \le 1/2$.

We define a sequence of disjoint sets U_i as follows. Given U_i, we consider all vertices not in $U_0, \dots, U_i$, and add to U_{i+1} some $2^{i+1} A \log n$ vertices that are r-connected in $\mathcal{G}_{i+1}$ to U_i (say, those of lowest index).

We first argue that, as long as at most $n/2$ vertices are included in $\bigcup_{j=1}^{i} U_j$ and $2^i \le n/\log^2 n$, the probability that we can find $2^{i+1} A \log n$ vertices to populate U_{i+1} is at least $1 - n^{-1}$. Indeed, a vertex not in $\bigcup_{j=1}^{i} U_j$

is at least r-connected in $\mathcal{G}_{i+1}$ to U_i with probability bounded from below by

$$\binom{|U_i|}{r} p_{i+1}^r (1-p_{i+1})^{|U_i|-r} \geq \left(\frac{|U_i|p_{i+1}}{r}\right)^r (1-|U_i|p_{i+1}) \geq \frac{1}{2}\left(\frac{|U_i|p_{i+1}}{r}\right)^r,$$

since, for all large n,

$$|U_i|p_{i+1} = 2^{-(r-1)/r}(A\log n)\left(\frac{2^i\varepsilon}{n\log^{r-1} n}\right)^{1/r} \leq A\left(\frac{2^{1-r}\varepsilon}{\log n}\right)^{1/r} \leq \frac{1}{2}.$$

Hence the expected number of such vertices is at least

$$\frac{n}{2}\frac{1}{2}\left(\frac{|U_i|p_{i+1}}{r}\right)^r = \frac{\varepsilon}{4r}\left(\frac{A}{2r}\right)^{r-1}(2^i A\log n) = 2^{i+2}A\log n$$

by the choice of A. Therefore by Chernoff's bound, such a set U_{i+1} of size $2^{i+1}A\log n$ can be selected with probability at least $1-\exp(-2^{i-1}A\log n) \geq 1-n^{-1}$, since $A \geq 2$ and $i \geq 0$, as required. Since the number of levels before reaching $n/2$ vertices is $O(\log n)$, the claim follows. ■

By Claims 4.4.18 and 4.4.19, with high probability $\mathcal{G}_* \cup \bigcup_{i\geq 1}\mathcal{G}_i$ contains a susceptible subgraph on some $U \subset [n]$ of order $n/\log n$. To conclude, we observe that given this, by adding $\mathcal{G}_0$ we have that $\mathcal{G} = \mathcal{G}_* \cup \bigcup_{i\geq 0}\mathcal{G}_i$ is susceptible with high (conditional) probability. Indeed, the expected number of vertices in U^c which are connected in $\mathcal{G}_0$ to at most $r-1$ vertices of U is bounded from above by

$$n\sum_{j=0}^{r-1}\binom{|U|}{j}p_0^j(1-p_0)^{|U|-j} \ll n(|U|p_0)^r e^{-p_0(|U|-r)} \ll n^r e^{-n^{(1-1/r)/2}} \ll 1.$$

Hence $\mathcal{G}$ is susceptible with high probability, as required. ■

4.5 Time dependent branching processes

In this section, we prove Theorem 4.2.5, giving estimates for the survival probabilities for a family of non-homogenous branching process which are closely related to contagious sets in $\mathcal{G}_{n,p}$.

Recall that in our branching process, the nth individual has a Poisson number of children with mean $\binom{n}{r-1}\varepsilon$. This does not specify the order of the individuals, i.e. which of these children is next. While the order would affect the resulting tree, the choice of order clearly does not affect the probability of survival. In light of this, we can use the breadth first order: Define generation 0 to be the first $r-1$ individuals, and let generation k be all children of individuals from generation $k-1$. All individuals in a generation appear in the order before any individual of a later generation. Let Y_t be the size of generation t, and $S_t = \sum_{i \le t} Y_i$.

Let $\Psi_r(k,i)$ be the probability that for some t we have $S_t = k$ and $Y_t = i$.

Lemma 4.5.1. *We have that*

$$\Psi_r(k,i) = \frac{e^{-\varepsilon\binom{k-i}{r}}\varepsilon^{k-r}}{(k-r)!} m_r(k,i).$$

Proof. We first give an equivalent branching process. Instead of each individual having a number of children, children will have r parents. We start with r individuals (indexed $0,\dots,r-1$), and every subset of size r of the population gives rise to an independent $\mathrm{Poi}(\varepsilon)$ additional individuals. Thus the initial set of r individuals produces $\mathrm{Poi}(\varepsilon)$ further individuals, indexed $r,\dots$. Individual k together with each subset of $r-1$ of the previous individuals has $\mathrm{Poi}(\varepsilon)$ children, so overall individual k has $\mathrm{Poi}\left(\binom{k}{r-1}\varepsilon\right)$ children where k is the maximal parent.

Let X_S be the number of children of a set S of individuals. A graph contributing to $m_r(k,i)$ requires $\mathrm{Poi}(\varepsilon)$ variables to equal X_S, so the probability is $\prod e^{-\varepsilon}\varepsilon^{X_s}/X_S!$. Up to generation t this considers $\binom{k-i}{r}$ sets, and $\sum X_S = k-r$, giving the terms involving ε in the claim. The combinatorial terms $\prod X_S!$ and $(k-r)!$ come from possible labelings of the graph. ∎

Proof of Theorem 4.2.5. Up to the $o(1)$ term appearing in the statement of the theorem, the survival of (X_t) is equivalent to the probability p_S that for some t we have that $S_t \ge k_r$, where $(S_t)_{t\ge 0}$ is as defined above Lemma 4.5.1

and $k_r = k_r(\varepsilon)$ is as in the theorem. By Lemma 4.5.1,

$$p_S \geq \sum_i \Psi_r(k_r, i) \geq \frac{e^{-\varepsilon\binom{k_r}{r}}\varepsilon^{k_r-r}}{(k_r-r)!}\sum_i m_r(k_r, i) \geq \frac{e^{-\varepsilon\binom{k_r}{r}}\varepsilon^{k_r-r}}{(k_r-r)!} m_r(k_r).$$

By Lemma 4.4.5, as $\varepsilon \to 0$, the right hand side is bounded from below by

$$e^{-o(k_r)}e^{-(r-2)k_r-\varepsilon\binom{k_r}{r}}\left(\varepsilon\frac{k_r^{r-1}}{(r-1)!}\right)^{k_r}\varepsilon^{-r} = e^{-\frac{(r-1)^2}{r}k_r(1+o(1))}.$$

On the other hand, we note that the formula for $\Psi_r(k, i)$ in Lemma 4.5.1 agrees with the upper bound for $P_r(k, i)$ in Lemma 4.3.7 (up to the $1 + o(1)$ factor). Hence, using the bounds in Lemma 4.3.5 and slightly modifying of the proof of Proposition 4.3.1 (since here we have Poisson random variables instead of Binomial random variables), it can be shown that

$$p_S \leq e^{o(k_r)}\frac{e^{-\varepsilon\binom{k_r}{r}}\varepsilon^{k_r-r}}{(k_r-r)!} m_r(k_r) = e^{-\frac{(r-1)^2}{r}k_r(1+o(1))}$$

completing the proof. ■

4.6 Graph bootstrap percolation

Fix $r \geq 2$ and a graph H. We say that a graph G is (H, r)-*susceptible* if for some $H' \subset G$ we have that H' is isomorphic to H and $V(H)$ is a contagious set for G. We call such a subgraph H' a *contagious copy of* H. Hence a seed, as discussed in Section 4.2.2, is a contagious clique. Let $p_c(n, H, r)$ denote the infimum over $p > 0$ such that $\mathcal{G}_{n,p}$ is (H, r)-susceptible with probability at least $1/2$.

By the arguments in Sections 4.3 and 4.4, with only minor changes, we obtain the following result. We omit the proof.

Theorem 4.6.1. *Fix $r \geq 2$ and $H \subset K_r$ with $e(H) = \ell$. Put*

$$\alpha_{r,\ell} = (r-1)!\left(\frac{(r-1)^2}{r^2-\ell}\right)^{r-1}.$$

As $n \to \infty$,

$$p_c(n, H, r) = \left(\frac{\alpha_{r,\ell}}{n \log^{r-1} n}\right)^{1/r} (1 + o(1)).$$

We obtain Theorem 4.2.4, from which Theorem 4.2.2 follows, as a special case.

Proof of Theorem 4.2.4. The result follows by Theorem 4.6.1, taking $r = 2$ and $\ell = 1$, in which case $\alpha_{2,1} = 1/3$. ■

4.7 Technical lemmas

We collect in this section several technical results used above.

4.7.1 Proof of Claim 4.3.6

Proof of Claim 4.3.6. By the bound $i! > \sqrt{2\pi i}(i/e)^i$, it suffices to verify that

$$\frac{(e/i)^i}{\sqrt{2\pi}} \Lambda(i) \leq 1 \quad \text{for } i \geq 1, \tag{4.7.1}$$

where $\Lambda(i) = \mathrm{Li}(-i+1/2, 1/e)$ and $\mathrm{Li}(s, z) = \sum_{j=1}^{\infty} z^j j^{-s}$ is the polylogarithm function.

Let Γ denote the gamma function. From the relationship between Li and the Herwitz zeta function, it can be shown that $\Lambda(i)/\Gamma(i + 1/2) \sim 1$, as $i \to \infty$, and hence $(e/i)^i \Lambda(i) \to \sqrt{2\pi}$, as $i \to \infty$. It appears (numerically) that $(e/i)^i \Lambda(i)$ increases monotonically to $\sqrt{2\pi}$, however this is perhaps not simple to verify (or in fact true). Instead, we find a suitable upper bound for $\Lambda(i)$.

Claim 4.7.1. *For all $i \geq 1$, we have that*

$$\Lambda(i) < \Gamma(i + 1/2)(1 + ab^i)$$

where $a = \zeta(3/2)$ and $b = e/(2\pi)$, and ζ is the Riemann zeta function.

Proof. For all $|u| < 2\pi$ and $s \notin \mathbb{N}$, we have the series representation

$$\mathrm{Li}(s, e^u) = \Gamma(1-s)(-u)^{s-1} + \sum_{\ell=0}^{\infty} \frac{\zeta(s-\ell)}{\ell!} u^\ell.$$

Hence

$$\Lambda(i) = \Gamma(i+1/2) + \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \zeta(1/2 - i - \ell). \tag{4.7.2}$$

Recall the functional equation for ζ,

$$\zeta(x) = 2^x \pi^{x-1} \sin(\pi x/2) \Gamma(1-x) \zeta(1-x).$$

Therefore, since $\zeta(1/2+x) > 0$ is decreasing in $x \geq 1$ we have that, for all relevant i, ℓ,

$$|\zeta(1/2 - i - \ell)| \leq a \sqrt{\frac{2}{\pi}} \frac{\Gamma(\ell+i+1/2)}{(2\pi)^{\ell+i}} < a \frac{\Gamma(\ell+i+1/2)}{(2\pi)^{\ell+i}}. \tag{4.7.3}$$

Applying (4.7.2),(4.7.3) (and the inequalities $\Gamma(x+\ell) < (x+\ell-1)^\ell \Gamma(x)$, $\ell! > \sqrt{2\pi\ell}(\ell/e)^\ell$, and $(1+x/\ell)^\ell < e^\ell$), we find that, for all $i \geq 1$,

$$\begin{aligned}
\frac{\Lambda(i)}{\Gamma(i+1/2)} - 1 &< \frac{a}{(2\pi)^i} \sum_{\ell=0}^{\infty} \frac{(\ell+i-1/2)^\ell}{(2\pi)^\ell \ell!} \\
&< \frac{ab^i}{e^i} \left(1 + \sum_{\ell=1}^{\infty} \frac{1}{\sqrt{2\pi\ell}} \left(\frac{e}{2\pi}\left(1 + \frac{i-1/2}{\ell}\right)\right)^\ell\right) \\
&< ab^i \left(\frac{1}{e} + \frac{1}{\sqrt{2e\pi}} \sum_{\ell=1}^{\infty} \left(\frac{e}{2\pi}\right)^\ell\right) \\
&< ab^i
\end{aligned}$$

establishing the claim. ∎

By Claim 4.7.1, the formula

$$\Gamma(i+1/2) = \sqrt{\pi} \frac{i!}{4^i} \binom{2i}{i},$$

and the bounds
$$\binom{2i}{i} < \frac{4^i}{\sqrt{\pi i}}\left(1 - \frac{1}{9i}\right)$$
and
$$i! < \sqrt{2\pi i}\left(\frac{i}{e}\right)^i \left(1 - \frac{1}{12i}\right)^{-1}$$
(valid for all $i \geq 1$), we find that
$$\frac{(e/i)^i}{\sqrt{2\pi}}\Lambda(i) < \frac{4}{3}\frac{9i-1}{12i-1}(1+ab^i) \quad \text{for } i \geq 1. \tag{4.7.4}$$

Differentiating the right hand side of (4.7.4), and dividing by the positive term $4/(3(12i-1)^2)$, we obtain
$$3 + ab^i \left(3 + \log(b)(108i^2 - 12i + 1)\right)$$
which, for $i \geq 11$, is bounded from below by
$$3 + 108ab^i \log(b) i^2 > 3 - 237b^i i^2 > 0.$$

Hence, for $i \geq 11$, the right hand side of (4.7.4) increases monotonically to 1 as $i \to \infty$. It follows that (4.7.1) holds for all $i \geq 11$. Inequality (4.7.1), for $i \leq 10$, can be verified numerically (e.g., by interval arithmetic), completing the proof of Claim 4.3.6. ∎

4.7.2 Proof of Claim 4.3.10

Proof of Claim 4.3.10. By (4.3.6), we have that
$$\frac{\partial^2}{\partial\gamma^2}\mu_r(\alpha,\beta,\gamma) = -\frac{\alpha\beta^r}{(r-2)!}(1-\gamma)^{r-2} < 0.$$

The result thus follows, noting that
$$\frac{\partial}{\partial\gamma}\mu_r(\alpha,\beta,\gamma) = -\beta\left(1 - \frac{\alpha\beta^{r-1}}{(r-1)!}(1-\gamma)^{r-1}\right)$$

and hence for any $\xi < 1$ and $\gamma \in (0, 1)$,

$$\frac{\partial}{\partial\gamma}\mu_r(\alpha, \xi\beta_r(\alpha), \gamma) = -\xi\beta_r(\alpha)\left(1 - (\xi(1-\gamma))^{r-1}\right) < 0. \qquad \blacksquare$$

4.7.3 Proof of Claim 4.3.11

Proof of Claim 4.3.11. By Lemma 4.3.8, for all $k = \beta \log n$ and $i = \gamma k$ as in the lemma, we have that

$$E_r(k, i) \lesssim n^{\mu_r(\alpha,\beta,\gamma)} \log^{r(r-1)} n. \tag{4.7.5}$$

We find a suitable upper bound for $p_r(k, i)$ as follows. For $\beta < \beta_r(\alpha)$, put $\ell_\beta = \xi_\beta \log n$, where $\xi_\beta = \beta_r(\alpha) - \beta$. For a given set V of size k with i vertices identified as the top level, there are $a_r(k, i)$ ways to select r vertices in V with at least one in the top level. Hence, for $k = \beta \log n$ with $\beta < \beta_r(\alpha)$, it follows that

$$p_r(k, i) \le \binom{n}{\ell_\beta}(a_r(k, i)p^r)^{\ell_\beta}.$$

By Claim 4.3.4, we have that $a_r(k, i) < ik^{r-1}/(r-1)!$. Hence, applying the bound $\binom{n}{\ell} \le (ne/\ell)^\ell$, we find that

$$p_r(k, i) \le \left(\frac{e\alpha\beta^r\gamma}{\xi_\beta(r-1)!}\right)^{\ell_\beta}.$$

Hence, by Lemma 4.3.8,

$$E_r(k, i)p_r(k, i) \lesssim n^{\bar{\mu}_r(\alpha,\beta,\gamma)} \log^{r(r-1)} n \tag{4.7.6}$$

where

$$\bar{\mu}_r(\alpha, \beta, \gamma) = \mu_r(\alpha, \beta, \gamma) + \xi_\beta \log\left(\frac{e\alpha\beta^r\gamma}{\xi_\beta(r-1)!}\right). \tag{4.7.7}$$

Therefore, by (4.7.5),(4.7.6), we obtain Claim 4.3.11 by the following fact.

Claim 4.7.2. *For any $\gamma \in (0,1)$, we have that*

$$\min\{\mu_r(\alpha,\beta,\gamma), \bar{\mu}_r(\alpha,\beta,\gamma)\} \le \mu_r^*(\alpha,\beta_r)$$

for all $\beta \in (0,\beta_r(\alpha)]$.

Proof. For convenience, we simplify notations as follows. Put $\beta_r = \beta_r(\alpha)$. We parametrize β using a variable δ: for $\delta \in (0,1]$, let $\beta_\delta = \delta\beta_r$. For $\gamma \in (0,1)$, let $\mu_r(\delta,\gamma) = \mu_r(\alpha,\beta_\delta,\gamma)$, $\bar{\mu}_r(\delta,\gamma) = \bar{\mu}_r(\alpha,\beta_\delta,\gamma)$, and $\delta_\gamma = \delta_\gamma(r) = 1-\sqrt{\gamma/r}$. Finally, put $\mu_r^* = \mu_r(1,0) = \mu_r^*(\alpha,\beta_r)$. In this notation, Claim 4.7.2 states that

$$\min\{\mu_r(\delta,\gamma), \bar{\mu}_r(\delta,\gamma)\} \le \mu_r^*, \qquad \text{for } \delta \in (0,1].$$

Since $\alpha\beta_r^{r-1}/(r-1)! = 1$, it follows that $\alpha\beta_\delta^{r-1}/(r-1)! = \delta^{r-1}$. Therefore, by (4.3.6),(4.7.7), we have that

$$\mu_r(\delta,\gamma) = r - \beta_r\left(\frac{\delta^r}{r}(1-\gamma)^r + \delta(r-2+\gamma) - (r-1)\delta\log\delta\right) \tag{4.7.8}$$

and

$$\bar{\mu}_r(\delta,\gamma) = \mu_r(\delta,\gamma) + \beta_r(1-\delta)\log\left(\frac{e\gamma\delta^r}{1-\delta}\right). \tag{4.7.9}$$

We obtain Claim 4.7.2 by the following subclaims (as we explain below the statements).

Sub-claim 4.7.3. *For any fixed $\gamma \in (0,1)$, we have that $\mu_r(\delta,\gamma)$ and $\bar{\mu}_r(\delta,\gamma)$ are convex and concave in $\delta \in (0,1)$, respectively.*

Sub-claim 4.7.4. *For $\gamma \in (0,1)$, we have that*

(i) $\mu_r(1,\gamma) < \mu_r^*$,
(ii) $\mu_r(\delta_\gamma,\gamma) < \mu_r^*$, *and*
(iii) $e\gamma\delta_\gamma^r/(1-\delta_\gamma) < 1$.

Indeed, by Sub-claim 4.7.4(ii),(iii), we have that $\bar{\mu}_r(\delta_\gamma,\gamma) < \mu_r(\delta_\gamma,\gamma) < \mu_r^*$. Therefore, noting that $\lim_{\delta\to1^-}\bar{\mu}_r(\delta,\gamma) = \mu_r(1,\gamma)$, $\lim_{\delta\to0^+}\mu_r(\delta,\gamma) = r$, and $\lim_{\delta\to0^+}\bar{\mu}_r(\delta,\gamma) = -\infty$ (see (4.7.8),(4.7.9)), we then obtain Claim 4.7.2 by applying Sub-claims 4.7.3 and 4.7.4(i).

Proof of Sub-claim 4.7.3. By (4.7.8), for any $\gamma \in (0,1)$, we have that

$$\frac{\partial^2}{\partial \delta^2}\mu_r(\delta,\gamma) = \frac{(r-1)\beta_r}{\delta}(1-\delta^{r-1}(1-\gamma)^r) > 0$$

for all $\delta \in (0,1)$. Moreover, by (4.7.8),(4.7.9), the above expression, and noting that

$$\frac{\partial^2}{\partial \delta^2}(1-\delta)\log\left(\frac{e\gamma\delta^r}{1-\delta}\right) = -\frac{r-(r-1)\delta^2}{\delta^2(1-\delta)},$$

it follows that, for any $\gamma \in (0,1)$,

$$\begin{aligned}\frac{\partial^2}{\partial \delta^2}\bar{\mu}_r(\delta,\gamma) &= -\frac{\beta_r}{\delta^2(1-\delta)}\Big(r-(r-1)\delta^2-\delta(1-\delta)(1-\delta^{r-1}(1-\gamma)^r)\Big)\\ &= -\frac{\beta_r}{\delta^2(1-\delta)}\left(1+(r-1)(1-\delta)(1+\delta^r(1-\gamma)^r)\right)\\ &< 0\end{aligned}$$

for all $\delta \in (0,1)$. The claim follows. ■

Proof of Sub-claim 4.7.4. Note that $\mu_r^* = r-\beta_r(r-1)^2/r$. Since, by (4.7.8),

$$\mu_r(1,\gamma) = r-\beta_r\left(\frac{(1-\gamma)^r}{r}+r-2+\gamma\right)$$

claim (i) follows immediately by the inequality $(1-\gamma)^r > 1-r\gamma$.

Next, we note that by (4.7.8), to establish claim (ii) we need to show that $f_r(\delta_\gamma,\gamma) > (r-1)^2/r$, where

$$f_r(\delta,\gamma) = \frac{\delta^r}{r}(1-\gamma)^r + \delta(r-2+\gamma) - (r-1)\delta\log\delta.$$

We deal with the cases $\gamma \in (0,1/r)$ and $\gamma \in [1/r,1)$ separately. By the inequality $\log\delta \le 1-\delta$, we have that

$$f_r(\delta,\gamma) > \delta(r-2+\gamma) - (r-1)\delta(1-\delta).$$

The right hand side is equal to $(r-1)^2/r$ when $\delta=\delta_\gamma$ and $\gamma = 1/r$ or $\gamma = 1$. Setting $\delta=\delta_\gamma$ in the right hand side, and differentiating twice with respect to γ, we obtain $-(1+3\gamma)/(4\sqrt{\gamma^3 r}) < 0$. It follows that $f_r(\delta, \gamma) > (r-1)^2/r$ for all $\gamma \in [1/r, 1)$. For the case $\gamma \in (0, 1/r)$, we note that by the bound $(1-\gamma)^r > 1 - \gamma r$,

$$f_r(\delta, \gamma) > \frac{\delta^r}{r}(1-\gamma r) + \delta(r-2+\gamma) - (r-1)\delta \log \delta.$$

Setting $\zeta = \sqrt{\gamma/r}$, $f_r(\delta_\gamma, \gamma)$ is thus bounded from below by

$$\frac{(1-\zeta)^r}{r}(1-(r\zeta)^2) + (1-\zeta)(r-2+r\zeta^2) - (r-1)(1-\zeta)\log(1-\zeta).$$

Hence it suffices to show that this expression is bounded from below by $(r-1)^2/r$ for all $\zeta \in (0, 1/r)$. To this end, we note that it is equal to $(r-1)^2/r$ when $\zeta = 0$, and claim that it is increasing in $\zeta \leq 1/r$. Indeed, differentiating with respect to ζ, we obtain

$$(1-\zeta)^{r-1}\Big(r(r+2)\zeta^2 - 2r\zeta - 1\Big) - 3r\zeta^2 + 2r\zeta + 1 + (r-1)\log(1-\zeta).$$

Note that $r(r+2)\zeta^2 - 2r\zeta - 1 < 0$ for all $\zeta \in [0, 1/r]$. Hence, since $(1-\zeta)^{r-1} \leq (1+(r-1)\zeta)^{-1}$ and $\log(1-\zeta) \geq -\zeta(1+\zeta)$ for all relevant $\zeta \leq 1/2$, the above expression is bounded from below by

$$\frac{(r-1)\zeta^2\,(2(1-2\zeta)r+\zeta)}{1+(r-1)\zeta} > 0.$$

It follows that $f_r(\delta_\gamma, \gamma) > (r-1)^2/r$ for all $\gamma \in (0, 1/r)$. Altogether, claim (ii) is proved.

Finally, for claim (iii), let $g_r(\delta, \gamma) = e\gamma\delta^r/(1-\delta)$. In this notation, claim (iii) states that $g_r(\delta_\gamma, \gamma) < 1$. To verify this inequality, we note that

$$\frac{\partial}{\partial \delta} g_r(\delta, \gamma) = \frac{e\gamma\delta^{r-1}}{(1-\delta)^2}(r-(r-1)\delta)$$

and hence

$$\frac{\partial}{\partial \delta} g_r(\delta_\gamma, \gamma) = e\delta_\gamma^{r-1}(r+(r-1)\sqrt{\gamma r}).$$

Therefore, noting that

$$\frac{\partial}{\partial\gamma} g_r(\delta,\gamma)\mid_{\delta=\delta_\gamma}= \frac{e\delta_\gamma^r}{1-\delta_\gamma} = e\delta_\gamma^{r-1}\left(\sqrt{\frac{r}{\gamma}}-1\right)$$

and recalling that

$$\frac{\partial}{\partial\gamma}\delta_\gamma = -\frac{1}{2\sqrt{\gamma r}}$$

it follows that

$$\frac{\partial}{\partial\gamma} g_r(\delta_\gamma,\gamma) = \frac{e\delta_\gamma^{r-1}}{2}\left(\sqrt{\frac{r}{\gamma}}-(r+1)\right).$$

Therefore, for any $r \ge 2$, $g_r(\delta_\gamma,\gamma)$ is maximized at $\gamma = r/(r+1)^2$. By the inequality $(1-x/n)^n < e^{-x}$, we find that

$$g_r(r/(r+1)^2) = \frac{er}{r+1}\left(1-\frac{1}{r+1}\right)^r < \frac{r}{r+1}\left(1-\frac{1}{r+1}\right)^{-1} = 1$$

giving the claim. ■

As discussed, Sub-claims 4.7.3 and 4.7.4 imply Claim 4.7.2. ■

To conclude, we recall that Claim 4.7.2 implies Claim 4.3.11. ■

4.7.4 Proof of Claim 4.4.7

Proof of Claim 4.4.7. We recall the relevant quantities defined in the proof of Lemma 4.4.6, see (4.4.7),(4.4.8),(4.4.9). We have that

$$\hat{\rho}_r(k,i) \ge \xi' e^{-(r-2)k-h_r(k)} \quad \text{for } k > k_r \text{ and } i \le k/\log^2 n$$

where

$$h_r(k) = f_r(k) - \log g_r\left(k, \frac{k}{\log^2 k}\right) + \frac{k}{\log^2 k}\log\left(\frac{k}{\log^2 k}\right),$$

$f_r(k)$ is non-decreasing and $f_r(k) \ll k$, and $g_r(k,i) = e^{(r-2)i}\left(\frac{k-i}{k}\right)^{(r-2)k}$. Claim 4.4.7 states that for some $\xi > 0$, for all large k and $i \le (\varepsilon/r)^2 k$, we have that $\hat{\rho}_r(k,i) \geq \xi e^{-i\varepsilon-(r-2)k-h_r(k)}$.

Sub-claim 4.7.5. *For all $k > k_r$, we have that $h_r(k)$ is increasing in k.*

Proof. Since $f_r(k)$ is non-decreasing and $k/\log^2 k$ is increasing, it suffices by (4.4.9) to show that $g_r(k, k/\log^2 k)$ is decreasing for $k > k_r$ (and assuming $r > 2$, as else $g_r \equiv 1$ and so there is nothing to prove). To this end, we note that

$$\frac{\partial}{\partial i} g_r(k,i) = -\frac{(r-2)i}{k-i} g_r(k,i),$$

$$\frac{\partial}{\partial k}\frac{k}{\log^2 k} = \frac{\log k - 2}{\log^3 k},$$

and

$$\frac{\partial}{\partial k} g_r(k,i) = \frac{r-2}{k-i}\left((k-i)\log\left(\frac{k-i}{k}\right) + i\right) g_r(k,i)$$

Hence, differentiating $g_r(k, k/\log^2 k)$ with respect to k, and dividing by

$$-\frac{(r-2)k}{k(1-\log^{-2} k)\log^3 k} g_r(k, k/\log^2 k) < 0$$

we obtain

$$(\log^3 k)(1-\log^{-2} k)\log\left(\frac{\log^2 k}{\log^2 k - 1}\right) - \frac{\log^3 k - \log k + 2}{\log^2 k}.$$

By the inequality $\log x > 2(x-1)/(x+1)$ (valid for $x > 1$), the above expression is bounded from below by

$$\frac{\log^3 k - 4\log^2 k - \log k + 2}{(\log^2 k)(2\log^2 k - 1)} > \frac{\log k - 5}{2\log^2 k - 1} > 0$$

for all $k > k_r$, since $k_r > e^{r/\varepsilon} > e^{r(r+1)}$ and $r > 2$. The claim follows. ∎

By Sub-claim 4.7.5, fix some $k_* = k_*(r,\varepsilon) > k_r$ so that $k/\log^2 k$ is larger than $9(r/\varepsilon)^4$ and $(r+2)!/(1-\varepsilon)$ for all $k \geq k_*$, and $h_r(k)$ is increasing for all $k \geq (1-(\varepsilon/r)^2)k_*$. By (4.4.8), select some $\xi(r,\varepsilon) \leq \xi'$ so that the claim holds for all $k > k_r$ and relevant i, provided either $i \leq k/\log^2 k$ or $k \leq k_*$.

We establish the remaining cases, $k > k_*$ and $k/\log^2 k < i \le (\varepsilon/r)^2$, by induction. To this end, let $k > k_*$ be given, and assume that the claim holds for all $k' < k$ and relevant i. By (4.4.1) it follows that

$$\hat{\rho}_r(k,i) \ge \sum_{j=1}^{k-r-i} \hat{B}_r(k,i,j)\hat{\rho}_r(k-i,j) \quad (i < k-r) \tag{4.7.10}$$

where

$$\hat{B}_r(k,i,j) = \frac{j^i}{i!}\left(\frac{k-i}{k}\right)^{(r-2)k}\left(\frac{k-i-j}{k-i}\right)^{k-i}\left(\frac{(r-1)!}{(k-i)^{r-1}}\frac{\hat{a}_r(k-i,j)}{j}\right)^i.$$

Sub-claim 4.7.6. *For all $(r+2)! \le i, j \le k/r^2$, we have that*

$$\hat{B}_r(k,i,j) \ge \frac{j^i}{i!}\left(\frac{k-i}{k}\right)^{(r-2)k}\left(\frac{k-i-j}{k-i}\right)^{k+(r-2)i}.$$

Proof. By the formula for $\hat{B}_r(k,i,j)$ above, it suffices to show that

$$\frac{(r-1)!}{(k-i)^{r-1}}\frac{\hat{a}_r(k-i,j)}{j} > \left(\frac{k-i-j}{k-i}\right)^{r-1}.$$

To this end, we note that by (4.4.2) and Claim 4.3.4 the left hand side is bounded from below by

$$\frac{1}{j}\sum_{\ell=1}^{j}\left(\frac{k-i-\ell}{k-i}\right)^{r-1} - \frac{2r!}{k-i}$$

Since, for any integer m, $(1-y/x)^m - (1-(y+1/2)/x)^m$ is decreasing in y, for $y < x$, it follows that

$$\frac{1}{j}\sum_{\ell=1}^{j}\left(\frac{k-i-\ell}{k-i}\right)^{r-1} \ge \left(\frac{k-i-(j+1)/2}{k-i}\right)^{r-1}.$$

Thus, applying the inequalities $1-xy \le (1-x)^y \le 1/(1+xy)$, we find that

$$\frac{(r-1)!}{(k-i)^{r-1}}\frac{\hat{a}_r(k-i,j)}{j} - \left(\frac{k-i-j}{k-i}\right)^{r-1}$$

is bounded from below by

$$1 - \frac{(j+1)(r-1)}{2(k-i)} - \frac{2r!}{k-i} - \frac{1}{1+j(r-1)/(k-i)}$$

which equals

$$\frac{((r-1)j-(r+4r!-1))(k-i)-((r-1)j+(r+4r!-1))(r-1)j}{2(k-i)(k-i+(r-1)j)}.$$

It thus remains to show that the numerator in the above expression is non-negative, for all i, j as in the claim. To see this, we observe that $r + 4r! - 1 < (r-1)(r+2)!$ for all $r \geq 2$. Hence, for $(r+2)! \leq i, j \leq k/r^2$ and $r \geq 2$, the numerator divided by $(r-1)k > 0$ is bounded from below by

$$(j-(r+2)!)\left(1-\frac{1}{r^2}\right) - (j+(r+2)!)\frac{1}{r^2} = \left(1-\frac{2}{r^2}\right)(j-(r+2)!) \geq 0$$

as required. The claim follows. ■

Applying Sub-claim 4.7.6, the inductive hypothesis, and the bound $i! < 3\sqrt{i}(i/e)^i$ to (4.7.10), it follows that

$$\hat{\rho}_r(k,i) > \xi \frac{e^{-(r-2)k+(r-1)i-h_r(k-i)}}{3\sqrt{i}} \left(\frac{k-i}{k}\right)^{(r-2)k} \sum_{j \in \mathcal{J}_{r,\varepsilon}} \psi_{r,\varepsilon}(i/k, j/i)^k \tag{4.7.11}$$

where $\mathcal{J}_{r,\varepsilon}(k,i)$ is the set of j satisfying $(r+2)! \leq j \leq (\varepsilon/r)^2(k-i)$, and

$$\psi_{r,\varepsilon}(\gamma,\delta) = \delta^\gamma e^{-\delta\gamma\varepsilon}\left(1-\frac{\delta\gamma}{1-\gamma}\right)^{1+\gamma(r-2)}.$$

Sub-claim 4.7.7. *Put $\delta_\varepsilon = 1-\varepsilon$ and $\delta_{r,\varepsilon} = \delta_\varepsilon + (\varepsilon/r)^2$. For any fixed $\gamma \leq (\varepsilon/r)^2$, we have that $\psi_{r,\varepsilon}(\gamma,\delta)$ is increasing in δ, for $\delta \in [\delta_\varepsilon, \delta_{r,\varepsilon}]$.*

Proof. Differentiating $\psi_{r,\varepsilon}(\gamma,\delta)$ with respect to δ, we obtain

$$\frac{\psi_{r,\varepsilon}(\gamma,\delta)\gamma}{\delta(1-\gamma-\delta\gamma)}\left(\varepsilon\gamma\delta^2 - (1+\varepsilon+\gamma(r-1-\varepsilon))\delta + 1 - \gamma\right).$$

Hence, to establish the claim, it suffices to show that

$$\varepsilon\gamma\delta_{r,\varepsilon}^2 - (1+\varepsilon+\gamma(r-1-\varepsilon))\delta_{r,\varepsilon} + 1 - \gamma$$

is positive for relevant γ. Moreover, since the above expression is decreasing in γ, we need only verify the case $\gamma = (\varepsilon/r)^2$. Setting γ as such in the above expression, and then dividing by ε^2/r^6, we obtain

$$r^6 - (1-\varepsilon)r^5 - (1+3\varepsilon^2-\varepsilon^3)r^4 - r^3\varepsilon^2 + \varepsilon^2(1+3\varepsilon-2\varepsilon^2)r^2 + \varepsilon^5.$$

For $\varepsilon < 1/r$ and $r \geq 2$, this expression is bounded from below by

$$r(r^5 - r^4 - (1+3/r^2)r^3 - 1) \geq r > 0$$

as required, giving the claim. ■

By the choice of k_* and since $k > k_*$, for all relevant $k/\log^2 k \leq i \leq (\varepsilon/r)^2 k$, we have that $\delta_\varepsilon i \geq (r+2)!$ and

$$\frac{\delta_{r,\varepsilon} i}{k-i} \leq (\varepsilon/r)^2 \frac{1-\varepsilon+(\varepsilon/r)^2}{1-(\varepsilon/r)^2} \leq (\varepsilon/r)^2$$

where the second inequality follows since

$$\frac{\partial}{\partial\varepsilon}\frac{1-\varepsilon+(\varepsilon/r)^2}{1-(\varepsilon/r)^2} = -r^2\frac{(r^2+\varepsilon^2-4\varepsilon)}{(r-\varepsilon)^2(r+\varepsilon)^2} < 0$$

for all $r \geq 2$. Hence, for all such i,k, we have that $j \in \mathcal{J}_{r,\varepsilon}(k,i)$ for all $j \in [\delta_\varepsilon, \delta_{r,\varepsilon}]$. Therefore, for any such i,k, by (4.7.11) and Sub-claim 4.7.7, we have that

$$\begin{aligned}
\hat{\rho}_r(k,i) &> \xi\frac{e^{-(r-2)k+(r-1)i-h_r(k-i)}}{3\sqrt{i}}\left(\frac{k-i}{k}\right)^{(r-2)k}\sum_{\delta_\varepsilon i\leq j\leq\delta_{r,\varepsilon} i}\psi_{r,\varepsilon}(i/k,j/i)^k \\
&> \xi\frac{(\delta_{r,\varepsilon}-\delta_\varepsilon)\sqrt{i}}{3}e^{-(r-2)k+(r-1)i-h_r(k-i)}\left(\frac{k-i}{k}\right)^{(r-2)k}\psi_{r,\varepsilon}(i/k,\delta_\varepsilon)^k \\
&> \xi e^{-(r-2)k+(r-1)i-h_r(k-i)}\left(\frac{k-i}{k}\right)^{(r-2)k}\psi_{r,\varepsilon}(i/k,\delta_\varepsilon)^k
\end{aligned}$$

where the last inequality follows since for any such i, k, by the choice of k_* and since $k > k_*$, we have that $\delta_{r,\varepsilon} - \delta_\varepsilon = (\varepsilon/r)^2 > 3/\sqrt{i}$.

Sub-claim 4.7.8. *Fix $k/\log^2 k \le i \le (\varepsilon/r)^2 k$, and define $\zeta_r(k,i)$ such that*

$$\hat{\rho}_r(k,i) = \xi e^{-\zeta_r(k,i)\varepsilon i-(r-2)k-h_r(k)}.$$

We have that $\zeta_r(k,i) < 1$.

Proof. Letting $\gamma = i/k$, it follows by the bound for $\hat{\rho}_r(k,i)$ above, and since $k > k_*$ and hence $h_r(k-i) < h_r(k)$ by the choice of k_*, that $\zeta_r(k,i)$ is bounded from above by

$$\delta_\varepsilon - \frac{r-1}{\varepsilon} - \frac{r-2}{\varepsilon\gamma}\log(1-\gamma) - \frac{1}{\varepsilon}\log\delta_\varepsilon - \frac{1+\gamma(r-2)}{\varepsilon\gamma}\log\left(1-\frac{\delta_\varepsilon\gamma}{1-\gamma}\right).$$

Recall that $\delta_\varepsilon = 1-\varepsilon$. Applying the bound $-\log(1-x) \le x/(1-x)$ for $x = \gamma$ and $x = \delta_\varepsilon\gamma/(1-\gamma)$, and the bound $-\log(1-x) \le x + (1+x)x^2/2$ for $x = \varepsilon$ (valid for any $x < 1/3$, and so for all relevant $\varepsilon < 1/(r+1)$ with $r \ge 2$), we find that the expression above is bounded from above by

$$\nu(\varepsilon,\gamma) = 2 - \frac{\varepsilon(1-\varepsilon)}{2} - \frac{1-(r-1)\gamma}{\varepsilon(1-\gamma)} + \frac{(1-\varepsilon)(1+(r-2)\gamma)}{\varepsilon(1-(2-\varepsilon)\gamma)}.$$

Therefore, noting that

$$\frac{\partial}{\partial\gamma}\nu(\varepsilon,\gamma) = \frac{r-2}{\varepsilon(1-\gamma)^2} + \frac{(1-\varepsilon)(r-\varepsilon)}{\varepsilon(1-(2-\varepsilon)\gamma)^2} > 0,$$

to establish the subclaim, it suffices to verify that $\nu(\varepsilon,(\varepsilon/r)^2) < 1$ for all $r \ge 2$ and $\varepsilon < 1/(r+1)$. Furthermore, since

$$\nu(\varepsilon,(\varepsilon/r)^2) = 2 - \frac{\varepsilon(1-\varepsilon)}{2} - \frac{r^2-\varepsilon^2(r-1)}{\varepsilon(r^2-\varepsilon^2)} + \frac{(1-\varepsilon)(r^2+\varepsilon^2(r-2))}{\varepsilon(r^2-2\varepsilon^2+\varepsilon^3)}$$

and hence

$$\frac{\partial}{\partial r}\nu(\varepsilon,(\varepsilon/r)^2) = -\frac{\varepsilon(r(r-4)+\varepsilon^2)}{(r^2-\varepsilon^2)^2} - \frac{\varepsilon(1-\varepsilon)(r(r-2\varepsilon)+\varepsilon^2(2-\varepsilon))}{(r^2-2\varepsilon^2+\varepsilon^3)^2} < 0$$

for all $k \geq 4$ and $\varepsilon < 1$, we need only verify the cases $r \leq 4$.

To this end, let $\eta(r, \varepsilon)$ denote the difference of the numerator and denominator of $\nu(\varepsilon, (\varepsilon/r)^2)$ (in its factorized form), namely

$$-\varepsilon^7 + 3\varepsilon^6 + (r^2 - 4)\varepsilon^5 - 2(2r^2 - 2r + 1)\varepsilon^4 + (5r^2 - 6r + 8)\varepsilon^3 \\ + r^2(r^2 - 2r - 2)\varepsilon^2 - r^2(r - 2)^2\varepsilon.$$

For all $\varepsilon < 1/3$, we have that

$$\eta(2, \varepsilon) = -\varepsilon^2(1 - \varepsilon)(2 - \varepsilon)(2 + \varepsilon)(2 - 2\varepsilon + \varepsilon^2) < -\varepsilon^2 < 0.$$

Similarly,

$$\eta(3, \varepsilon) = -\varepsilon(9 - 9\varepsilon - 35\varepsilon^2 + 26\varepsilon^3 - 5\varepsilon^4 - 3\varepsilon^5 + \varepsilon^6) < -\varepsilon < 0$$

and

$$\eta(4, \varepsilon) = -\varepsilon(64 - 96\varepsilon - 64\varepsilon^2 + 50\varepsilon^3 - 12\varepsilon^4 - 3\varepsilon^5 + \varepsilon^6) < -\varepsilon < 0.$$

It follows that $\nu(\varepsilon, (\varepsilon/r)^2) < 1$ for all $\varepsilon < 1/3$ and $k \leq 4$, and hence for all $k \geq 2$, giving the subclaim. ∎

By Sub-claim 4.7.8, we find that $\hat{\rho}_r(k, i) = \xi e^{-\varepsilon i - (r-2)k - h_r(k)}$ for all i, k such that $k/\log^2 k \leq i \leq (\varepsilon/r)^2 k$, completing the induction, and thus giving Claim 4.4.7. ∎

4.7.5 Proof of Lemma 4.4.11

Proof of Lemma 4.4.11. Put $\alpha_{r,\varepsilon} = (1 + \varepsilon)\alpha_r$. Let $\beta_r = \beta_r(\alpha_{r,\varepsilon})$ and $\beta_* = \beta_*(\alpha_{r,\varepsilon})$. For $\beta > 0$ and $\gamma \in [0, 1)$, let $\mu_{r,\varepsilon}(\beta, \gamma) = \mu_{r,\varepsilon}(\alpha_{r,\varepsilon}, \beta, \gamma)$ and $\mu_r^*(\beta) = \mu_r^*(\alpha_{r,\varepsilon}, \beta)$. Let $\gamma_{r,\varepsilon}^*(\beta)$ denote the maximizer of $\mu_{r,\varepsilon}(\beta, \gamma)$ over $\gamma \in [0, 1)$, which is well-defined, since for all $\gamma \in (0, 1)$,

$$\frac{\partial^2}{\partial \gamma^2}\mu_{r,\varepsilon}(\beta, \gamma) - \frac{\beta}{(1 - \gamma)^2} - \frac{\alpha_{r,\varepsilon}\beta^r}{(r - 2)!}(1 - \gamma)^{r-2} < 0 \qquad (4.7.12)$$

and $\lim_{\gamma\to 1^-}\mu_{r,\varepsilon}(\beta,\gamma)=-\infty$. Finally, put $\gamma_{r,\varepsilon}(\beta)=\min\{\gamma^*_{r,\varepsilon}(\beta),(\varepsilon/r)^2\}$.

We show that $\mu_{r,\varepsilon}(\beta,\gamma_{r,\varepsilon}(\beta))$ is bounded away from 0 for $\beta\in[\beta^*_r,\beta^*_r+\delta]$, for some $\delta>0$. By Lemma 4.4.10, the result follows.

Claim 4.7.9. *For $\gamma\in(0,1)$, let*

$$\beta_{r,\varepsilon}(\gamma)=\frac{(1/(1-\gamma)+\varepsilon)^{1/(r-1)}}{1-\gamma}\beta_r$$

and put

$$\beta_{r,\varepsilon}=\lim_{\gamma\to 0^+}\beta_{r,\varepsilon}(\gamma)=(1+\varepsilon)^{1/(r-1)}\beta_r.$$

We have that

(i) $\gamma^*_{r,\varepsilon}(\beta)=0$, *for all* $\beta\le\beta_{r,\varepsilon}$,

(ii) for $\beta>\beta_{r,\varepsilon}$, $\gamma=\gamma^*_{r,\varepsilon}(\beta)$ *if and only if* $\beta=\beta_{r,\varepsilon}(\gamma)$, *and*

(iii) $\gamma^*_{r,\varepsilon}(\beta)$ *is increasing in* β, *for* $\beta\ge\beta_{r,\varepsilon}$.

Proof. By (4.7.12), we have that $\mu_{r,\varepsilon}(\beta,\gamma)$ is concave in γ. Therefore, since

$$\frac{\partial}{\partial\gamma}\mu_{r,\varepsilon}(\beta,\gamma)-\beta\left(\frac{1}{1-\gamma}+\varepsilon-\frac{\alpha_{r,\varepsilon}\beta^{r-1}}{(r-1)!}(1-\gamma)^{r-1}\right)$$

and hence, for any $\xi>0$,

$$\frac{\partial}{\partial\gamma}\mu_{r,\varepsilon}(\xi\beta_r,\gamma)=-\xi\beta_r\left(\frac{1}{1-\gamma}+\varepsilon-\xi^{r-1}(1-\gamma)^{r-1}\right),$$

the first two claims follow. The third claim is a consequence of the second claim and the fact that $\beta_{r,\varepsilon}(\gamma)$ is increasing in γ. ∎

By the following claims, we obtain the lemma (as we discuss below the statements).

Claim 4.7.10. *For $\beta>0$ and $\gamma\in[0,1)$, let*

$$\omega_{r,\varepsilon}(\beta,\gamma)=\mu_{r,\varepsilon}(\beta,\gamma)-\mu^*_r(\beta).$$

We have that

(i) $\omega_{r,\varepsilon}(\beta,\gamma_{r,\varepsilon}(\beta))=0$, *for all* $\beta\le\beta_{r,\varepsilon}$, *and*

(ii) $\omega_{r,\varepsilon}(\beta, \gamma_{r,\varepsilon}(\beta))$ *is increasing in* β*, for* $\beta \geq \beta_{r,\varepsilon}$*.*

Claim 4.7.11. *We have that* $\beta_{r,\varepsilon} < \beta_*$*.*

Indeed, the claims together imply that $\omega_{r,\varepsilon}(\beta_*, \gamma_{r,\varepsilon}(\beta_*)) > 0$. Therefore, since $\mu_r^*(\beta_*) = 0$, we thus have that $\mu_{r,\varepsilon}(\beta_*, \gamma_{r,\varepsilon}(\beta_*)) > 0$. Therefore, by the continuity of $\mu_{r,\varepsilon}(\beta, \gamma_{r,\varepsilon}(\beta))$ in β, it follows that $\mu_{r,\varepsilon}(\beta, \gamma_{r,\varepsilon}(\beta)) > 0$ for all $\beta \in [\beta_*, \beta_* + \delta]$, for some $\delta > 0$. As discussed the lemma follows, applying Lemma 4.4.10.

Proof of Claim 4.7.10. The first claim follows by (4.4.10) and Claim 4.7.9(i).

For the second claim, we show that (a) $\omega_{r,\varepsilon}(\beta, \gamma^*_{r,\varepsilon}(\beta))$ is increasing in β, for $\beta \geq \beta_{r,\varepsilon}$ such that $\gamma^*_{r,\varepsilon}(\beta) \leq (\varepsilon/r)^2$, and (b) $\omega_{r,\varepsilon}(\beta, (\varepsilon/r)^2)$ is increasing in β, for $\beta \geq \beta_{r,\varepsilon}$. By Claim 4.7.9(iii), this implies the claim.

Since $\gamma^*_{r,\varepsilon}(\beta)$ maximizes $\mu_{r,\varepsilon}(\beta, \gamma)$, and so $\partial \omega_{r,\varepsilon}(\beta, \gamma^*_{r,\varepsilon}(\beta))/\partial\gamma = 0$, it follows that

$$\frac{\partial}{\partial \beta}\omega_{r,\varepsilon}(\beta, \gamma^*_{r,\varepsilon}(\beta)) = \frac{\partial}{\partial \beta}\omega_{r,\varepsilon}(\beta, \gamma)\big|_{\gamma=\gamma^*_{r,\varepsilon}(\beta)}.$$

Hence, by Claim 4.7.9(ii), to establish (a) we show that for all $\gamma \leq (\varepsilon/r)^2$, $\partial\omega_{r,\varepsilon}(\beta_{r,\varepsilon}(\gamma), \gamma)/\partial\beta > 0$. To this end, we observe that

$$\frac{\partial}{\partial \beta}\omega_{r,\varepsilon}(\beta, \gamma) = \log(1-\gamma) - \varepsilon\gamma + \frac{\alpha_{r,\varepsilon}\beta^{r-1}}{(r-1)!}(1-(1-\gamma)^r). \qquad (4.7.13)$$

Setting $\beta = \beta_{r,\varepsilon}(\gamma)$, the above expression simplifies as

$$\log(1-\gamma) - \varepsilon\gamma + \frac{1/(1-\gamma)+\varepsilon}{(1-\gamma)^{r-1}}(1-(1-\gamma)^r).$$

By the inequalities $(1-x)^y \leq 1/(1+xy)$ and $\log(1-x) \geq -x/(1-x)$, this expression is bounded from below by

$$-\frac{\gamma}{1-\gamma} - \varepsilon\gamma + (1+(r-1)\gamma)\left(\frac{1}{1-\gamma}+\varepsilon\right)\left(1-\frac{1}{1+\gamma r}\right)$$

which factors as

$$\frac{\gamma(1+\varepsilon(1-\gamma))}{(1-\gamma)(1+\gamma r)}(r-1+\gamma r(r-2)) > 0$$

and (a) follows.

Similarly, we note that by (4.7.13), for any $\beta \geq \beta_{r,\varepsilon}$ and $\gamma > 0$,

$$\begin{aligned}\frac{\partial}{\partial\beta}\omega_{r,\varepsilon}(\beta,\gamma) &\geq \log(1-\gamma) - \varepsilon\gamma + \frac{\alpha_{r,\varepsilon}\beta_{r,\varepsilon}^{r-1}}{(r-1)!}(1-(1-\gamma)^r)\\ &= \log(1-\gamma) - \varepsilon\gamma + (1+\varepsilon)(1-(1-\gamma)^r).\end{aligned}$$

Hence, using the same bounds for $(1-x)^y$ and $\log(1-x)$ as above, we find that for all such $\beta \geq \beta_{r,\varepsilon}$, $\partial\omega_{r,\varepsilon}(\beta,(\varepsilon/r)^2)/\partial\beta$ is bounded from below by

$$\frac{\varepsilon^2(r^3(r-1)(1+\varepsilon) - 2r^2\varepsilon^2 - r(2r-1)\varepsilon^3 + \varepsilon^5)}{(r-\varepsilon)(r+\varepsilon)(r+\varepsilon^2)r^2}.$$

For $\varepsilon < 1/r$, the numerator is bounded from below by

$$\varepsilon^2\left(r^3(r-1) - 2 - \frac{2r-1}{r^2}\right) = \frac{\varepsilon^2}{r}\left(r^6 - r^5 - 2r^2 - 2r + 1\right) > 0$$

since $r \geq 2$. Hence $\partial\omega_{r,\varepsilon}(\beta,(\varepsilon/r)^2)/\partial\beta > 0$, giving (b), and thus completing the proof of the second claim. ■

Proof of Claim 4.7.11. By Lemma 4.3.9, the claim is equivalent to the inequality $\mu_r^*(\beta_{r,\varepsilon}) > 0$. To verify this, we note that

$$\beta_r = \left(\frac{(r-1)!}{\alpha_{r,\varepsilon}}\right)^{1/(r-1)} = \left(\frac{1}{1+\varepsilon}\right)^{1/(r-1)}\left(\frac{r-1}{r}\right)^2,$$

and hence by (4.3.7), for any $\xi > 0$, we have that

$$\begin{aligned}\mu_r^*(\xi^{1/(r-1)}\beta_r) &= r - \xi^{1/(r-1)}\beta_r\left(r-2+\frac{\xi}{r} - \log\xi\right)\\ &= r - \left(\frac{r}{r-1}\right)^2\left(\frac{\xi}{1+\varepsilon}\right)^{1/(r-1)}\left(r-2+\frac{\xi}{r}-\log\xi\right).\end{aligned}$$

In particular,

$$\mu_r^*(\beta_{r,\varepsilon}) = r - \left(\frac{r}{r-1}\right)^2\left(r-2+\frac{1+\varepsilon}{r} - \log(1+\varepsilon)\right).$$

Therefore, by the bound $\log(1+x) > x/(1+x)$, we find that

$$\mu_r^*(\beta_{r,\varepsilon}) > \frac{\varepsilon r(r-1-\varepsilon)}{(1+\varepsilon)(r-1)^2} > 0$$

as required. ■

As discussed, Lemma 4.4.11 follows by Claims 4.7.10 and 4.7.11. ■

Chapter 5

Minimal Contagious Sets in Random Graphs

5.1 Overview

Bootstrap percolation with threshold r on a graph $G = (V, E)$ is the following process: Initially some subset $I \subset V$ is declared *active*. Subsequently, any vertex with at least r active neighbours is activated. If all vertices in V are eventually activated, we call I *contagious* for G.

We take G to be the Erdős–Rényi random graph $\mathcal{G}_{n,p}$. We obtain lower bounds for the size of the smallest contagious sets in $\mathcal{G}_{n,p}$, improving those recently obtained by Feige, Krivelevich and Reichman. A key step is to identify the large deviations rate function for the number of vertices eventually activated by small sets that are unlikely to be contagious. This complements the central limit theorems of Janson, Łuczak, Turova and Vallier, which describe the typical behaviour. As a further application, our large deviation estimates play a key role in Chapter 6 to locate the sharp threshold for K_4-bootstrap percolation on $\mathcal{G}_{n,p}$, refining an approximation due to Balogh, Bollobás and Morris.*

5.2 Background and main results

Let $G = (V, E)$ be a graph. Given an initial set of *activated* vertices $V_0 \subset V$, the *r-bootstrap percolation process* activates all vertices with at least r active neighbours. Formally, let V_{t+1} be the union of V_t and the set of all vertices

*This chapter is joint work with Omer Angel [11], currently under review for publication.

with at least r neighbours in V_t. The sets V_t are increasing, and therefore converge to some set of eventually active vertices, denoted by $\langle V_0, G\rangle_r$. A set $I \subset V$ is called *contagious* for G if it activates all of V, that is, $\langle I, G\rangle_r = V$. Let $m(G,r)$ denote the size of a minimal contagious set for G.

Bootstrap percolation is most often attributed to Chalupa, Leath and Reich [50] (see also Pollak and Riess [116]), who introduced the model on the Bethe lattice (the infinite d-regular tree T_d) as a simple model for a magnetic system undergoing a phase transition. Since then the process has been analyzed on various graphs and found many applications in mathematics, physics and several other fields, see for example the extensive surveys in the introductory sections of articles [24, 27, 84] and the references therein. More recently, bootstrap percolation has been studied on random graphs, see for instance [24, 27, 83, 84].

Recall that the Erdős–Rényi [60] graph $\mathcal{G}_{n,p}$ is the random subgraph of K_n obtained by including edges independently with probability p. In this work, we obtain improved bounds for $m(\mathcal{G}_{n,p}, r)$, for all $r \geq 2$.

Theorem 5.2.1. *Fix $r \geq 2$. Suppose that $\vartheta = \vartheta(n)$ satisfies $1 \ll \vartheta \ll n$. Let*

$$\alpha_r = (r-1)!\left(\frac{r-1}{r}\right)^{2(r-1)}, \qquad p = p(n,\vartheta) = \left(\frac{\alpha_r}{n\vartheta^{r-1}}\right)^{1/r}.$$

Then, with high probability,

$$m(\mathcal{G}_{n,p}, r) \geq \frac{r\vartheta}{\log(n/\vartheta)}(1+o(1))$$

where $o(1)$ depends only on n.

We denote $\psi = \psi(n,\vartheta) = \vartheta/\log(n/\vartheta)$, so that the theorem states that with high probability $m(\mathcal{G}_{n,p}, r) \geq r\psi(1+o(1))$. Of course, this bound is only of interest if $r\psi > 1$, as else we have the trivial bound $m(G,r) \geq r$, which holds for any graph G.

Janson, Łuczak, Turova and Vallier [84] (see also Vallier [132]) showed that for p as in Theorem 5.2.1, $\ell_r = \frac{r}{r-1}\vartheta$ is the critical size for a *random* set (selected independently of $\mathcal{G}_{n,p}$) to be contagious (see Section 5.3). Theorem 5.2.1 is a consequence of our key result, Theorem 5.4.2 below, which identifies the large deviations rate function associated with the number of vertices activated by sets smaller than ℓ_r.

More recently, Feige, Krivelevich and Reichman [62] studied small contagious sets in $\mathcal{G}_{n,p}$. Although it is unlikely for a random set of size $\ell < \ell_r$ to be contagious, there typically *exist* contagious sets in $\mathcal{G}_{n,p}$ that are much smaller than ℓ_r. In [62] it is shown that if p is as in Theorem 5.2.1 and moreover

$$\frac{\log^2 n}{\log\log n} \ll \vartheta \ll n,$$

then, with high probability,

$$c_r \le \frac{m(\mathcal{G}_{n,p}, r)}{\psi(n,\vartheta)} \le C_r \tag{5.2.1}$$

where $c_r < r$ and, as $r \to \infty$, $c_r \to 2$ and $C_r = \Omega(r^{r-2})$. (Note that d in [62] corresponds to $(\alpha_r(n/\vartheta)^{(r-1)})^{1/r}$ in this context.) The lower bound in (5.2.1) holds in fact for all ϑ. (Although this is not stated in [62, Theorem 1.1], it follows from the proof, see [62, Corollaries 2.1 and 4.1].)

The inequality $c_r < r$ is not shown in [62], so we briefly explain it here: In [62, Lemma 4.2 and Corollary 4.1], it is observed that a graph of size k with a contagious set of size ℓ contagious has at least $r(k-\ell)$ edges. From this it follows easily that with high probability

$$m(\mathcal{G}_{n,p}, r) \ge \xi \frac{r-1}{r} \frac{n}{d^{r/(r-1)} \log d},$$

provided that $\xi^{r-1} e^{r+2}/(2r)^r < 1$. Since $(r-1)! > e((r-1)/e)^{r-1}$, this leads to the bound $m(\mathcal{G}_{n,p}, r) \ge c\psi(n,\vartheta)$, where for all $r \ge 2$,

$$c < 2\left(\frac{r}{r-1}\right)^3 \left(\frac{2r}{e^4}\right)^{1/(r-1)} < r.$$

Therefore, since $c_r < r$, we find that Theorem 5.2.1 improves the lower bound in (5.2.1) for all $r \ge 2$. To obtain this significant improvement, we in a sense (see Section 5.4) track the full trajectory of activation in percolating graphs, rather than using only a rough estimate for graphs arrived at by such trajectories. Using (discrete) variational calculus, we identify the optimal trajectory from a set of size ℓ in $\mathcal{G}_{n,p}$ to an eventually active set of k vertices. This leads to refined bounds for the structure of percolating subgraphs of $\mathcal{G}_{n,p}$ with unusually small contagious sets, and so an improved bound for $m(\mathcal{G}_{n,p}, r)$. Moreover, we note that this improvement increases as r increases. Since $c_r \to 2$, our bound is larger by a factor of roughly $r/2$ for large r. This is due to the fact that the crude bound of $r(k-\ell)$ for the number of edges in a graph of size k with a contagious set of size ℓ is an increasingly inaccurate estimate for the combinatorics of such graphs as $r \to \infty$.

Hence, in particular, we find that $m(\mathcal{G}_{n,p}, r)/\psi(n, \vartheta)$ grows at least linearly in r. It seems plausible that this is the truth, and that moreover, the bound in Theorem 5.2.1 is asymptotically sharp. In any case, as it stands now, a substantial gap remains between the linear lower bound of Theorem 5.2.1 and the super-exponential upper bound in (5.2.1). The upper bound in (5.2.1) has the advantage of being proved by a procedure that with high probability locates a contagious set in polynomial time. That being said, this set is possibly much larger than a *minimal* contagious set, especially for large r. In closing, we state the open problems of (i) identifying $m(\mathcal{G}_{n,p}, r)$ up to a factor of $1 + o(1)$ and (ii) efficiently locating contagious sets that are as close as possible to minimal.

5.2.1 Thresholds for contagious sets

The critical threshold $p_c(n, r, q)$ for the existence of contagious sets of size q in $\mathcal{G}_{n,p}$ is defined as the infimum over $p > 0$ for which such a set exists with probability at least $1/2$. If $q = r$, we simply write $p_c(n, r)$. In [62] it is shown that $p_c(n, r) = \Theta((n \log^{r-1} n)^{-1/r})$. In the previous Chapter 4, we identified the sharp threshold for contagious sets of the smallest possible size r as

$$p_c(n, r) = \left(\frac{\alpha_r}{n \log^{r-1} n}\right)^{1/r} (1 + o(1)). \tag{5.2.2}$$

Moreover, (5.2.2) holds if the 1/2 in the definition of p_c is replaced with any probability in $(0,1)$. As a consequence of Theorem 5.2.1, we obtain lower bounds for $p_c(n,r,q)$ for $q \geq r$.

Corollary 5.2.2. *Fix $r \geq 2$. Suppose that $r \leq q = q(n) \ll n/\log n$. As $n \to \infty$,*

$$p_c(n,r,q) \geq \left(\frac{\alpha_{r,q}}{n \log^{r-1} n}\right)^{1/r} (1+o(1)),$$

where $\alpha_{r,q} = \alpha_r (r/q)^{r-1}$.

Indeed, by Theorem 5.2.1, we see that if $p = (\alpha/(n \log^{r-1} n))^{1/r}$, where $\alpha = (1-\delta)\alpha_{r,q}$ for some $\delta > 0$, then with high probability $m(\mathcal{G}_{n,p}, r) > q$. In particular, we obtain an alternative proof of the lower bound in (5.2.2).

In closing, we remark that determining whether the inequalities in Corollary 5.2.2 are asymptotically sharp, even for fixed $q > r$, is of interest. The proof in Chapter 4 of the special case $q = r$ is fairly involved. Although the upper bound in (5.2.2) is proved using the standard second moment method, the application is not straightforward (see Section 4.2.4 for a brief overview). Roughly speaking, for $p > p_c$, we show that the expected number of *triangle-free* percolating subgraphs of $\mathcal{G}_{n,p}$ is large. We then use Mantel's theorem to deduce the existence of such sets (see Section 4.4.4). This strategy is not sufficient, however, for $q > r$.

5.3 Binomial chains

Fix some $r \geq 2$. To analyze the spread of activation from an initially active set I in $\mathcal{G}_{n,p}$, we consider the *binomial chain* construction, as used by Janson, Łuczak, Turova and Vallier [84]. This representation of the bootstrap percolation dynamics is due to Scalia-Tomba [118] (see also Sellke [123]). We refer to [84, Section 2] for a detailed description, and here only present the properties relevant to the current chapter. The main idea is to reveal the graph one vertex at a time. As a vertex is revealed, we mark its neighbours. Once a vertex has been marked r times, we know it will be activated, and add it to the list of active vertices.

Formally, sets $A(t)$ and $U(t)$ of *active* and *used* vertices at time $t \geq 0$ are defined as follows: Let $A(0) = I$ and $U(0) = \emptyset$. For $t > 0$, choose some unused, active vertex $v_t \in A(t-1) - U(t-1)$, and give each neighbour of v_t a *mark*. Then let $A(t)$ be the union of $A(t-1)$ and the set of all vertices in $\mathcal{G}_{n,p}$ with at least r marks, and put $U(t) = U(t-1) \cup \{v_t\}$. The process terminates at time $t = \tau$, where $\tau = \min\{t \geq 0 : A(t) = U(t)\}$, that is, when all active vertices have been used. It is easy to see that $A(\tau) = \langle I, \mathcal{G}_{n,p}\rangle_r$.

Let $S(t) = |A(t)| - |I|$. By exploring the edges of $\mathcal{G}_{n,p}$ one step at a time, revealing the edges from v_t only at time t, the random variables $S(t)$ can be constructed in such a way that $S(t) \sim \mathrm{Bin}(n - |I|, \pi(t))$, where $\pi(t) = \mathbf{P}(\mathrm{Bin}(t,p) \geq r)$, see [84, Section 2]. Moreover, for $s < t$, we have that $S(t) - S(s) \sim \mathrm{Bin}(n - |I|, \pi(t) - \pi(s))$. Finally, it is shown that $|\langle I, \mathcal{G}_{n,p}\rangle_r| \geq k$ if and only if $\tau \geq k$ if and only if $S(t) + |I| > t$ for all $t < k$. Thus to determine the size of the eventually active set $\langle I, \mathcal{G}_{n,p}\rangle_r$, it suffices to analyze $S(t)$.

Making use of this construction, many results are developed in [84]. We close this section by mentioning two such results that are closely related to our key result, Theorem 5.4.2 below. The following quantities play an important role in [84] and in the present article. We denote

$$k_r = k_r(\vartheta) = \left(\frac{r}{r-1}\right)^2 \vartheta, \qquad \ell_r = \ell_r(\vartheta) = \frac{r-1}{r} k_r. \tag{5.3.1}$$

For $\varepsilon \in [0,1]$, we define $\delta_\varepsilon \in [0, \varepsilon]$ implicitly by

$$\frac{\delta_\varepsilon^r}{r} = \delta_\varepsilon - \varepsilon_r, \qquad \varepsilon_r = \frac{r-1}{r}\varepsilon. \tag{5.3.2}$$

It is easily verified that $\varepsilon_r \leq \delta_\varepsilon \leq \varepsilon$, for all $\varepsilon \in [0,1]$. (We note that $\ell_r, k_r, \delta_\varepsilon$ correspond to $a_{\mathsf{c}}, t_{\mathsf{c}}, \varphi(\varepsilon)$ in [84].)

As mentioned already, ℓ_r is identified in [84] as the critical size for a random set (selected independently of $\mathcal{G}_{n,p}$) to be contagious. More specifically, suppose that

$$p = p(n, \vartheta) = \left(\frac{\alpha_r}{n\vartheta^{r-1}}\right)^{1/r} = \left(\frac{(r-1)!}{n k_r^{r-1}}\right)^{1/r} \tag{5.3.3}$$

and $I \subset [n]$ is such that $|I|/\ell_r \to \varepsilon$. If $\varepsilon < 1$ then with high probability I activates less than εk_r vertices. On the other hand, if $\varepsilon > 1$, then with high probability I activates all except possibly very few vertices. In the sub-critical case $\varepsilon < 1$, $|\langle I, \mathcal{G}_{n,p}\rangle_r|$ is asymptotically normal with mean $\mu \sim \delta_\varepsilon k_r$.

More precisely, the following results are proved in [84].

Theorem 5.3.1 ([84, Theorem 3.1])**.** *Fix $r \geq 2$. Let p be as in* (5.3.3)*, where $\vartheta = \vartheta(n)$ satisfies $1 \ll \vartheta \ll n$. Suppose that $I = I(n) \subset [n]$ is independent of $\mathcal{G}_{n,p}$ and such that $|I|/\ell_r \to \varepsilon$, as $n \to \infty$. If $\varepsilon \in [0, 1)$, then with high probability $|\langle I, \mathcal{G}_{n,p}\rangle_r| = (\delta_\varepsilon + o(1))k_r$. On the other hand, if $\varepsilon > 1$, then with high probability $|\langle I, \mathcal{G}_{n,p}\rangle_r| = n(1 - o(1))$.*

(If $np \gg \log n + (r-1)\log\log n$, then, in fact, with high probability I is contagious, that is $|\langle I, \mathcal{G}_{n,p}\rangle_r| = n$, see [84, Theorem 3.1](iii).) Moreover, the following central limit theorem is established.

Theorem 5.3.2 ([84, Theorem 3.8(i)])**.** *Fix $r \geq 2$. Let p be as in* (5.3.3)*, where $\vartheta = \vartheta(n)$ satisfies $1 \ll \vartheta \ll n$. Suppose that $I = I(n) \subset [n]$ is independent of $\mathcal{G}_{n,p}$ and such that $|I|/\ell_r \to \varepsilon \in (0, 1)$, as $n \to \infty$. Then $|\langle I, \mathcal{G}_{n,p}\rangle_r|$ is asymptotically normal with mean $\mu \sim \delta_\varepsilon k_r$ and variance $\sigma^2 = \delta'_\varepsilon k_r$, where $\delta'_\varepsilon = \delta_\varepsilon^r(1 - \delta_\varepsilon^{r-1})^{-2}/r$.*

(See (3.13) and (3.22) in [84] for the definition of μ.) In particular, note that the mean and variance of $|\langle I, \mathcal{G}_{n,p}\rangle_r|$ are of the same order as k_r.

In [84, Section 6] a heuristic is provided for the criticality of ℓ_r, which we recount here. By the law of large numbers, with high probability $S(t) \approx \mathbf{E}S(t)$. A calculation shows that if $|I| > \ell_r$ then $|I| + \mathbf{E}S(t) \geq t$ for all $t < n - o(n)$, whereas if $|I| < \ell_r$ then already for $t = k_r$ we get $|I| + \mathbf{E}S(k_r) < k_r$.

In particular, for $t \leq k_r$, since $\vartheta \ll n$ we have that

$$pt \leq pk_r = O((\vartheta/n)^{1/r}) \ll 1. \tag{5.3.4}$$

It follows that $\pi(t) \sim (tp)^r/r!$. We therefore have for $t = xk_r$ that

$$\mathbf{E}S(xk_r) = (n - |I|)\pi(t) \sim \frac{x^r}{r}k_r \cdot \frac{k_r^{r-1}np^r}{(r-1)!} = \frac{x^r}{r}k_r. \tag{5.3.5}$$

If $|I| < \ell_r$, then for $x = 1$ we have

$$|I| + \mathbf{E}S(k_r) < \ell_r + k_r/r = k_r.$$

5.4 Optimal activation trajectories

Recall $k_r, \ell_r, \delta_\varepsilon, \varepsilon_r$ as defined in (5.3.1) and (5.3.2), and let p be as in (5.3.3). By Theorems 5.3.1 and 5.3.2, ℓ_r is the critical size for a random (equivalently, given) set to be contagious for $\mathcal{G}_{n,p}$. Moreover, a set of size $\varepsilon\ell_r < \ell_r$ typically activates approximately $\delta_\varepsilon k_r$ vertices. In this section, we study the probability that such a set activates more than $\delta_\varepsilon k_r$ vertices.

Definition 5.4.1. We let $P(\ell, k)$ denote the probability that for a given set $I \subset [n]$ (independent of $\mathcal{G}_{n,p}$), with $|I| = \ell$, we have that $|\langle I, \mathcal{G}_{n,p}\rangle_r| \geq k$.

Theorem 5.4.2. *Fix $r \geq 2$. Let p be as in* (5.3.3)*, where $\vartheta = \vartheta(n)$ satisfies $1 \ll \vartheta \ll n$. Let $\varepsilon \in [0, 1)$ and $\delta \in [\delta_\varepsilon, 1]$. Suppose that $\ell/\ell_r \to \varepsilon$ and $k/k_r \to \delta$, as $n \to \infty$. Then, as $n \to \infty$, we have that $P(\ell, k) = \exp[\xi k_r(1 + o(1))]$, where $\xi = \xi(\varepsilon, \delta)$ is equal to*

$$-\frac{\delta^r}{r} + \begin{cases} (\delta - \varepsilon_r)\log(er^{-1}\delta^r/(\delta - \varepsilon_r)), & \delta \in [\delta_\varepsilon, \varepsilon); \\ (\varepsilon/r)\log\left(e\varepsilon^{r-1}\right) - (r-2)(\delta - \varepsilon) + (r-1)\log(\delta^\delta/\varepsilon^\varepsilon), & \delta \in [\varepsilon, 1], \end{cases}$$

and $o(1)$ depends only on n.

Since the mean and variance of $|\langle I, \mathcal{G}_{n,p}\rangle_r|$ are of the same order (see Theorem 5.3.2), the event that $|\langle I, \mathcal{G}_{n,p}\rangle_r| \geq \delta k_r$, for some $\delta \in (\delta_\varepsilon, 1]$, represents a large deviation from the typical behaviour.

We note that by (5.3.2), we have that $\xi(\varepsilon, \delta_\varepsilon) = 0$ for all $\varepsilon \in [0, 1)$, in line with Theorem 5.3.1. Note that $t = k_r$ is the point at which the binomial chain $S(t)$ becomes super-critical (since $np^r\binom{t}{r-1} \approx (t/k_r)^{r-1}$), so we have that $P(\varepsilon\ell_r, \delta k_r) = e^{o(k_r)}P(\varepsilon\ell_r, k_r)$ for $\delta > 1$.

We remark that the main novelty of Theorem 5.4.2 is that it gives bounds for $P(\ell, k)$ when $\ell/k \to c > 0$. The case $\varepsilon = 0$ and $\delta = 1$ in Theorem 5.4.2 (essentially) follows by Theorem 4.2.5 proved in the previous Chapter 4 (where the initial set is of size $\ell = r$). That being said, the proof of Theorem 5.4.2 takes a completely different approach. In Chapter 4 the equality (in the case $\ell = r$) is proved by combinatorial arguments, whereas here we use variational calculus to obtain a more general result.

Before proving Theorem 5.4.2, we observe that Theorem 5.2.1 follows as a simple consequence. For this proof, we only require the special case $\xi(0,1) = -(r-1)^2/r$. In Chapter 6, Theorem 5.4.2 is used to its full extent, in the case of $r = 2$, to locate the sharp threshold for K_4-percolation, as introduced by Balogh, Bollobás and Morris [24].

Proof of Theorem 5.2.1. Let $\delta > 0$ be given. The theorem states that with high probability $m(\mathcal{G}_{n,p}, r) \geq (1-\delta)r\psi$, where $\psi = \vartheta/\log(n/\vartheta)$. Let k_r be as in (5.3.1) and put $\ell_\delta = (1-\delta)r\psi$. Since $\vartheta \ll n$,

$$\ell_\delta/k_r = O\left(1/\log(n/\vartheta)\right) \ll 1.$$

Hence by Theorem 5.4.2, noting that $\xi(0,1)k_r = -r\vartheta$, the expected number of subsets $I \subset [n]$ such that $|I| = \ell_\delta$ and $|\langle I, \mathcal{G}_{n,p}\rangle_r| \geq k_r$ is bounded by

$$\binom{n}{\ell_\delta} e^{-r\vartheta(1+o(1))} \leq \left(\frac{ne}{\ell_\delta}\right)^{\ell_\delta} e^{-r\vartheta(1+o(1))} = e^{-r\vartheta\nu},$$

where

$$\nu = 1 + o(1) - (1-\delta)\frac{\log(ne/\ell_\delta)}{\log(n/\vartheta)}.$$

Since

$$\log(ne/\ell_\delta) \leq \log(n/\vartheta) + O\left(\log\log(n/\vartheta)\right)$$

we have that $\nu > 0$ for all large n. Therefore, with high probability $\mathcal{G}_{n,p}$ has no contagious set of size at most ℓ_δ. The result follows. ■

We turn to the proof of Theorem 5.4.2. The overall idea is to identify the optimal trajectory for the spread of activation from a set I with $|I| = \varepsilon\ell_r = \varepsilon_r k_r$ to a set of size δk_r, where $\delta \in [\delta_\varepsilon, 1]$. Intuitively, we expect this to follow a trajectory $S(xk_r) + \varepsilon_r k_r = f(x)k_r$ for some function $f : [0,\delta] \to \mathbb{R}$ that starts at $f(0) = \varepsilon_r$ and ends at $f(\delta) \geq \delta$. Recall (see Section 5.3) that the binomial chain $S(t)$ is non-decreasing and $|\langle I, \mathcal{G}_{n,p}\rangle_r| \ge k$ if and only if $S(t) + |I| > t$ for all $t < k$. Hence, in order for $|\langle I, \mathcal{G}_{n,p}\rangle_r| \ge \delta k_r$, we require f to be non-decreasing and $f(x) > x$ for all $x \in [0, \delta)$. Moreover, since this event is very unlikely, and since until reaching size $\delta k_r \le k_r$ the binomial chain $S(t)$ is sub-critical (noting that $np^r\binom{t}{r-1} \approx (t/k_r)^{r-1}$), it is reasonable to further expect that $f(\delta) = \delta$ and f to be convex. Thus possibly we have that $f(x) = x$ for all x in some interval $[\varepsilon', \delta]$.

To identify f, we use a discrete analogue of the Euler–Lagrange equation, due to Guseinov [77], to deduce that the optimal trajectory between points above the diagonal is of the form $ax^r + b$. In light of this, in the case that $\delta > \varepsilon$, we expect the trajectory to meet the diagonal at $\varepsilon' = \varepsilon$ (and then coincide with it on $[\varepsilon, \delta]$), since then $f'(x)$ is continuous at ε'. On the other hand, if $\delta \le \varepsilon$, we expect the trajectory to intersect the diagonal only at $x = \delta$. Since, as discussed near (5.3.5), we have $S(xk_r) \approx (x^r/r)k_r$ for $x \le 1$, the typical trajectory is $x^r/r + \varepsilon_r$. By (5.3.2) this trajectory intersects the diagonal at $x = \delta_\varepsilon$, in line with Theorem 5.3.1. See Figure 5.1.

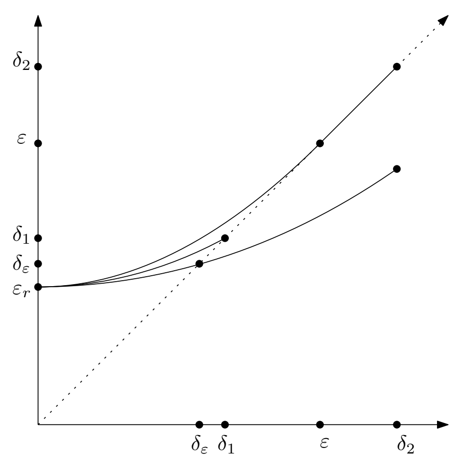

Figure 5.1: Three activation trajectories: The trajectory ending at (δ_i, δ_i), $i \in \{1, 2\}$, is optimal among those from $(0, \varepsilon_r)$ to endpoints (δ, δ'), with $\delta' \ge \delta_i$. Note that for $\delta_1 < \varepsilon$, the optimal trajectory intersects the diagonal only at δ_1, whereas for $\delta_2 > \varepsilon$, it coincides with the diagonal between ε and δ_2. The typical trajectory $x^r/r + \varepsilon_r$ intersects the diagonal at δ_ε.

For a function $g : \{0, 1, \ldots, m\} \to \mathbb{R}$, where $m \in \mathbb{N}$, let $\Delta g(i) = g(i + 1) - g(i)$ for all $0 \le i < m$.

Lemma 5.4.3 ([77, Theorem 5]). *Let $a, b \in \mathbb{R}$. Let $x_0 < x_1 < \cdots < x_m \in \mathbb{R}$ be $m \in \mathbb{N}$ evenly spaced points. Put $\mathbb{X} = \{x_0, x_1, \ldots, x_m\}$. Suppose that $\sigma(s, t, w)$ is a function from $\mathbb{X} \times \mathbb{X} \times \mathbb{R}$ to $\mathbb{R}$ with continuous first order partial derivative σ_w. Let $\mathcal{F}$ denote the set of functions $f : \mathbb{X} \to \mathbb{R}$ such that $f_0 = a$ and $f_m = b$, where $f_i = f(x_i)$ for $0 \le i \le m$. For $f \in \mathcal{F}$, let*

$$S(f) = \sum_{i=0}^{m-1} \sigma\left(x_i, x_{i+1}, \frac{\Delta f_i}{\Delta x_i}\right) \Delta x_i.$$

If some function $\hat{f}$ is a local extremum of S on $\mathcal{F}$, then $\hat{f}$ satisfies

$$\sigma_w\left(x_i, x_{i+1}, \frac{\Delta f_i}{\Delta x_i}\right) \equiv c \tag{5.4.1}$$

for some $c \in \mathbb{R}$ and all $0 \le i < m$.

We remark that this is a special case of [77, Theorem 5] that suffices for our purposes. In [77] a more general result is established that allows for functions $\sigma = \sigma(x_i, x_{i+1}, f_i, f_{i+1}, \Delta f_i/\Delta x_i)$, that is, depending also on the values f_i and points x_i that are not necessarily evenly spaced. The conclusion there is a discrete version of the Euler–Lagrange equation, which simplifies to (5.4.1) in the special case we consider.

Proof of Theorem 5.4.2. Recall $\ell_r, k_r, \delta_\varepsilon, \varepsilon_r$, as defined at (5.3.1) and (5.3.2). In particular, recall that $\varepsilon_r = \frac{r-1}{r}\varepsilon$, so $\varepsilon\ell_r = \varepsilon_r k_r$. We show that

$$P(\varepsilon\ell_r, \delta k_r) = \exp[\xi k_r(1 + o(1))], \tag{5.4.2}$$

where

$$\xi = \int_0^\delta \left(f_*'(x) \log\left(\frac{e x^{r-1}}{f_*'(x)}\right) - x^{r-1} \right) dx$$

and f_* is defined by

$$f_*(x) = \frac{\delta - \varepsilon_r}{\delta^r} x^r + \varepsilon_r,$$

if $\delta \in [\delta_\varepsilon, \varepsilon]$, and by

$$f_*(x) = \begin{cases} x^r/(r\varepsilon^{r-1}) + \varepsilon_r, & \text{if } x \le \varepsilon; \\ x, & \text{if } x > \varepsilon, \end{cases}$$

if $\delta \in (\varepsilon, 1]$. See Figure 5.1. (We note that $(\delta - \varepsilon_r)/\delta^r$ is increasing in $\delta \in [0,1]$ and equal to $1/(r\varepsilon^{r-1})$ when $\delta = \varepsilon$, since $\varepsilon_r = \frac{r-1}{r}\varepsilon$. Therefore we can express f_* as $f_*(x) = (\eta - \varepsilon_r)(x/\eta)^r + \varepsilon_r$ for $x \in [0, \eta]$ and $f_*(x) = x$ otherwise, where $\eta = \min\{\delta, \varepsilon\}$.) The theorem follows.

Fix a vertex set I of size $\varepsilon_r k_r$. (For simplicity, we ignore the insignificant detail of rounding to integers, here and in the arguments that follow.) Recall (see Section 5.3) that the binomial chain $S(t)$ is a non-decreasing process and $|\langle I, \mathcal{G}_{n,p}\rangle_r| \ge \delta k_r$ if and only if $S(t) + \varepsilon_r k_r > t$ for all $t < \delta k_r$, where $S(t) \sim \mathrm{Bin}(n - \varepsilon_r k_r, \pi(t))$ and $\pi(t) = \mathbf{P}(\mathrm{Bin}(t,p) \ge r)$.

We first show that we can restrict to the event that $S(t)$ is never too large. Indeed, for any $c > 0$, by (5.3.5) and Chernoff's bound, it follows that

$$\mathbf{P}(S(\delta k_r) \ge (1+c)k_r) \ll \left(\frac{e^c}{(1+c)^{1+c}}\right)^{(1+o(1))\delta^r k_r/r} \ll e^{\nu}$$

where $\nu = c\log(e/c) \cdot \delta^r k_r/r$. Noting that $c\log(e/c) \downarrow -\infty$ as $c \to \infty$, there is some sufficiently large $C > 0$ so that

$$\mathbf{P}(S(\delta k_r) + \varepsilon_r k_r > C k_r) \le e^{-k_r}. \tag{5.4.3}$$

Therefore, to establish (5.4.2), we may assume that $S(\delta k_r) + \varepsilon_r k_r \le C k_r$. Since $S(t)$ is non-decreasing, the same holds for $S(t) + \varepsilon_r k_r$, for all $t \le \delta k_r$.

Let $x_0 < x_1 < \cdots < x_m \in \mathbb{R}$ be evenly spaced points such that $x_0 = 0$ and $x_m = \delta$, where

$$m = \min\left\{\log(\delta k_r), (n/k_r)^{1/(2r)}\right\}. \tag{5.4.4}$$

Note that since $1 \ll k_r \ll n$, we have that $m \gg 1$.

For a function $f : \{x_0, x_1, \ldots, x_m\} \to \mathbb{R}$ let $f_i = f(x_i)$ and

$$p(f,i) = \mathbf{P}(S(x_{i+1}k_r) + \varepsilon_r k_r = f_{i+1}k_r | S(x_i k_r) + \varepsilon_r k_r = f_i k_r).$$

Recall (see Section 5.3) that $S(t) - S(s) \sim \mathrm{Bin}(n - \varepsilon_r k_r, \pi(t) - \pi(s))$. Hence

$$p(f,i) = \mathbf{P}(\mathrm{Bin}(n - \varepsilon_r k_r, \Delta\pi(x_i k_r)) = \Delta(f_i k_r)). \tag{5.4.5}$$

Let $\mathcal{F}$ denote the set of non-decreasing functions $f : \{x_0, x_1, \ldots, x_m\} \to \mathbb{R}$ such that $f_0 = \varepsilon_r$, $f_i \geq x_i$ and $f_m = \delta'$, where $\delta' \in [\delta, C]$. Let $\mathcal{F}' \subset \mathcal{F}$ denote the subset of functions f which additionally satisfy $f_i k_r \in \mathbb{N}$ for all i. (As already mentioned, we will ignore the small detail of rounding to integers whenever the issue is immaterial, and hence often not differentiate between the sets $\mathcal{F}$ and $\mathcal{F}'$.)

Claim 5.4.4. *We have that*

$$P(\varepsilon\ell_r, \delta k_r) = e^{o(k_r)} \prod_{i=0}^{m-1} p(\hat{f}, i) \tag{5.4.6}$$

where $\hat{f}$ maximizes $\prod_i p(f,i)$ on $\mathcal{F}$.

Proof. By (5.4.4) there are at most $e^{o(k_r)}$ functions $f \in \mathcal{F}'$. Therefore, since $\mathcal{F}' \subset \mathcal{F}$, we have that

$$\mathbf{P}(|\langle I, \mathcal{G}_{n,p}\rangle_r|/k_r \in [\delta, C]) \leq \sum_{f \in \mathcal{F}'} \prod_{i=0}^{m-1} p(f,i) \leq e^{o(k_r)} \prod_{i=0}^{m-1} p(\hat{f}, i).$$

Applying (5.4.3), it follows that

$$P(\varepsilon\ell_r, \delta k_r) \leq e^{o(k_r)} \prod_{i=0}^{m-1} p(\hat{f}, i).$$

Next, to obtain the matching lower bound, we consider the function $\hat{f} + 1/m$. Note that $S(t) = 0$ for all $t < r$ (since $\pi(t) = 0$ for all such t), and $S(r) \sim \mathrm{Bin}(n - \varepsilon_r k_r, p^r)$. Thus, for convenience, assume for this part of the argument that $x_0 = r/k_r \ll 1$ (rather than 0) and so $\hat{f}(r) = \varepsilon_r$, as clearly this has no effect on our calculations up to the $e^{o(k_r)}$ error. Since, by (5.4.5), $p(f,i)$ depends on the difference Δf_i but not on the specific values of f_i and f_{i+1}, we have that $p(\hat{f},i) = p(\hat{f}+1/m,i)$ for all i. On the other hand, recalling that $1 \ll k_r \ll n$, and hence $m \gg 1$ and $np^r = (r-1)!/k_r^{r-1} \ll 1$, we find (using the inequality $\binom{n}{k} \geq (n/k)^k$) that

$$\mathbf{P}(S(r) = k_r/m) \geq \binom{n-\varepsilon_r k_r}{k_r/m} p^{\frac{rk_r}{m}} (1-p^r)^n = e^{o(k_r)} \left(\frac{m}{k_r^r}\right)^{\frac{k_r}{m}} = e^{o(k_r)}.$$

Moreover, since $S(t)$ is non-decreasing and all $\Delta x_i = \delta/m \leq 1/m$, if $S(x_i k_r) + \varepsilon_r k_r = \hat{f}_i k_r + 1/m$ for all i, it follows that $S(t) > t$ for all $t < \delta k_r$. Altogether, we conclude that

$$P(\varepsilon \ell_r, \delta k_r) \geq e^{o(k_r)} \prod_{i=0}^{m-1} p(\hat{f}, i),$$

completing the proof of the claim. ■

Therefore, to establish (5.4.2), it remains to identify $\hat{f}$. To this end, we first obtain the following estimate in order to put the problem of maximizing $\prod_i p(f,i)$ in a convenient form for the application of Lemma 5.4.3.

Claim 5.4.5. *For all* $0 \leq i < m$,

$$n\Delta\pi(x_i k_r) = \frac{\Delta(x_i^r k_r)}{r}(1+o(1)). \tag{5.4.7}$$

In particular, note that $\Delta\pi(x_i k_r) = O(k_r/n) \ll 1$, since $k_r \ll n$.

Proof. Since $x_0 = 0$, the case $i = 0$ follows by (5.3.5). Hence assume that $i \geq 1$. It is easy to show (see [84, Section 8]) that, for all $t > 0$ such that $pt \leq 1$,

$$\pi(t) = \frac{(pt)^r}{r!}(1 + O(pt + t^{-1})).$$

By (5.3.4), for all i, we have that $x_i k_r p \leq k_r p \ll 1$. By (5.3.3) we have that $n(k_r p)^r / r! = k_r / r$. Hence, for $i \geq 1$,

$$n\Delta\pi(x_i k_r) = \frac{\Delta(x_i^r k_r)}{r}\left[1 + O\left(\frac{x_{i+1}^r}{\Delta(x_i^r)}(k_r p + (x_i k_r)^{-1})\right)\right].$$

Recall that $x_i = i\delta/m$. Note that $k_r p = O((k_r/n)^{1/r})$. We therefore have

$$\frac{x_{i+1}^r}{\Delta(x_i^r)}(k_r p + (x_i k_r)^{-1}) \leq O\left(\frac{(k_r/n)^{1/r} + m/k_r}{1-(1-1/m)^r}\right).$$

By the bound $(1-1/m)^r \leq 1/(1+r/m)$, and recalling $1 \ll k_r \ll n$ and the definition of m at (5.4.4), the right hand side is bounded by

$$O(m((k_r/n)^{1/r} + m/k_r)) \ll 1.$$

We conclude that (5.4.7) holds for all i, as claimed. ■

Recall that for any $f \in \mathcal{F}$, all $f_i \leq Ck_r$. Hence, by (5.4.5) and (5.4.7), and the inequalities $1 \ll k_r \ll n$, $e^{-x/(1-x)} \leq 1 - x \leq e^{-x}$ and

$$\left(\frac{ne}{k}\right)^k \geq \binom{n}{k} \geq \frac{(n-k)^k}{k!} \geq \frac{1}{ek}\left(\frac{n-k}{k}\right)^k \left(\frac{ne}{k}\right)^k,$$

it is straightforward to verify that

$$p(f,i) = e^{o(k_r)}\left(e\frac{n\Delta(x_i k_r)}{\Delta(f_i k_r)}\right)^{\Delta\pi(f_i k_r)} e^{-n\Delta\pi(x_i k_r)}$$

for any $f \in \mathcal{F}$ and $0 \leq i < m$. Applying (5.4.7), for any such f and i, we obtain

$$p(f,i) = \exp[\sigma_i k_r(1+o(1))], \tag{5.4.8}$$

where

$$\sigma_i = (x_{i+1} - x_i)\left(\frac{\Delta f_i}{\Delta x_i}\log\left(ex_i^{r-1}\frac{\Delta x_i}{\Delta f_i}\right) - x_i^{r-1}\right). \tag{5.4.9}$$

We express σ in this way to relate to Lemma 5.4.3, which we now apply.

The optimal function $\hat{f}$ is a local extremum of the functional, except that at some x_i we may have $f_i = x_i$, in which case it is only extremal since f_i is at the boundary of its allowed set. Suppose first that $f_i > x_i$ for all $i \in (0,m)$, i.e. except the endpoints. We apply Lemma 5.4.3 with

$$\sigma(s,t,w) = (t-s)(w\log(es^{r-1}/w) - s^{r-1}),$$

so that

$$\sigma_w(s,t,w) = (t-s)(\log(es^{r-1}/w) - 1).$$

We apply this to equally spaced points, so $t-s$ is constant. In this case, Lemma 5.4.3 implies that $\Delta\hat{f}_i/\Delta x_i = cx_i^{r-1}$ for some constant c. Suppose next that $\hat{f}$ takes some values on the diagonal, and suppose $f_j = x_j$ and $f_k = x_k$ are two consecutive places this occurs. The above gives that $\Delta\hat{f}_i/\Delta x_i = cx_i^{r-1}$ for $j \le i < k$. This is impossible unless $k = j+1$. Thus $\hat{f}_i = x_i$ for a single contiguous interval of i's.

Let us summarize our findings so far. Having fixed m and the equally spaced points $(x_i)_{i\le m}$, we wish to maximize $\sum_i \sigma_i$ over non-decreasing sequences $(f_i)_{i\le m}$ with $f_i \ge x_i$. We know that the maximizing function satisfies $f_i = x_i$ for some (possibly empty) interval $x_i \in [\varepsilon', \delta']$ and that $\Delta f_i/x_i^{r-1}$ is constant for $x_i < \varepsilon'$ and another constant for $x_i \ge \delta'$. Next, we observe that if $\Delta f_i/\Delta x_i = cx_i^{r-1}$ for some c and all $j \le i < k$, then f satisfies $f(x) = g(x) + O(1/m)$, for some $g(x) = (c/r)x^r + c'$, and all $x \in [x_j, x_k]$. Moreover, it is easy to verify using (5.4.9) that

$$\sum_{i=0}^{m} \sigma_i = (1+o(1))\left[I(g,0,\delta) - \frac{\delta^r}{r}\right] \tag{5.4.10}$$

where $o(1)$ is as n (and hence m) tends to infinity, and with

$$I(g,s,t) = \int_s^t g'(x)\log\left(\frac{ex^{r-1}}{g'(x)}\right)dx. \tag{5.4.11}$$

In light of this, to establish (5.4.2), it suffices to identify the maximizer $\hat{g}$ of $I(g,0,\delta)$ over continuous, non-decreasing functions g, satisfying $g(x) \ge x$, of the form

$$g(x) = \begin{cases} c_1 x^r + \varepsilon_r, & \text{if } x \in [0,\varepsilon']; \\ x, & \text{if } x \in [\varepsilon', \delta']; \\ c_2(x^r - (\delta')^r) + \delta', & \text{if } x \in [\delta', \delta], \end{cases}$$

where

(i) $c_1 \ge 0$, and hence $\varepsilon' \ge \varepsilon_r$;

(ii) if $\varepsilon' \ge \delta \wedge \varepsilon$ then $c_1 \ge \min\{(\delta - \varepsilon_r)/\delta^r, 1/(r\varepsilon^{r-1})\}$, and hence $\varepsilon' = \delta$;
(iii) if $\varepsilon' < \delta \wedge \varepsilon$ then $c_1 = (\varepsilon' - \varepsilon_r)/(\varepsilon')^r$; and
(iv) $c_2 \ge 1/(r(\delta')^{r-1})$.

(Here $\delta \wedge \varepsilon$ denotes $\min\{\delta, \varepsilon\}$.) Note that (i) holds since g is non-decreasing on $[0, \varepsilon']$; (ii) says that if $g(x) > 0$ for all $x \in [0, \delta \wedge \varepsilon)$, then $g(x) = c_1 x^r + \varepsilon_r$ for some c_1 as above and all $x \in [0, \delta]$; (iii) holds since g is continuous at $x = \varepsilon'$; and (iv) holds since $g(x) \ge x$ on $[\delta', \delta]$.

Indeed, if $\hat{g}$ maximizes $I(g, 0, \delta)$ over such g, then by (5.4.6), (5.4.8) and (5.4.10), we have that

$$P(\varepsilon \ell_r, \delta k_r) = \exp[(I(\hat{g}, 0, \delta) - \delta^r/r) k_r (1 + o(1))]. \tag{5.4.12}$$

Therefore, noting that $\xi = I(f_*, 0, \delta) - \delta^r/r$, (5.4.2) follows once we verify that $\hat{g} = f_*$. To this end, we observe that, if $\delta \le \varepsilon$, then f_* corresponds to g in the case that $\varepsilon' = \delta$ and $c_1 = (\delta - \varepsilon_r)/\delta^r$. On the other hand, if $\delta > \varepsilon$, then f_* corresponds to g in the case that $\varepsilon' = \varepsilon$, $\delta' = \delta$ and $c_1 = 1/(r\varepsilon^{r-1})$. See Figure 5.1. Hence, to complete the proof, we verify that the optimal ε', δ' are $\varepsilon' = \delta \wedge \varepsilon$ and $\delta' = \delta$ (i.e. $\hat{g} = f_*$).

We use of the following observations in the calculations below. For any c, c' and $u \le v$, note that

$$I(x, u, v) = -(r-2)(v-u) + (r-1)\log(v^v/u^u) \tag{5.4.13}$$

and

$$I(cx^r + c', u, v) = c(v^r - u^r)\log(e/(cr)). \tag{5.4.14}$$

First, we show that if the optimal trajectory intersects the diagonal at some $x = \delta'$ it coincides with it thereafter for all $x \in [\delta', \delta]$.

Claim 5.4.6. *For all $\delta' \in [\varepsilon_r, \delta)$ and $c_2 \ge 1/(r(\delta')^{r-1})$, we have that*

$$I(c_2(x^r - (\delta')^r) + \delta', \delta', \delta) < I(x, \delta', \delta). \tag{5.4.15}$$

Proof. Let $g(x) = c_2(x^r - (\delta')^r) + \delta'$. By (5.4.14), it follows that $I(g, \delta', \delta)$ is decreasing in c_2 for $c_2 \ge 1/r$, and hence for all relevant c_2, since we have that

$\delta' \le \delta \le 1$. Therefore it suffices to assume that c_2 is the minimal relevant value $c_2 = 1/(r(\delta')^{r-1})$. In this case, by (5.4.13) and (5.4.14), we have that $I(g, \delta', \delta) - I(x, \delta', \delta)$ is equal to

$$\frac{\delta^r - (\delta')^r}{r(\delta')^{r-1}} \log(e(\delta')^{r-1}) + (r-2)(\delta - \delta') - (r-1)\log(\delta^\delta/(\delta')^{\delta'}).$$

Differentiating this expression with respect to δ' we obtain

$$-1 - \frac{r-1}{r}((r-1)(\delta/\delta')^r + 1)\log(\delta') - (r-2) + (r-1)(\log(\delta') + 1),$$

which simplifies as

$$\frac{(r-1)^2}{r}\log(\delta')(1 - (\delta/\delta')^r) \ge 0$$

for all $\delta' \le \delta \le 1$. Since $I(g, \delta', \delta) - I(x, \delta', \delta) \to 0$ as $\delta' \uparrow \delta$, the claim follows. ∎

Next, we show that the optimal trajectory intersects the diagonal at some point $\varepsilon' \le \delta \wedge \varepsilon$.

Claim 5.4.7. *Suppose that* $c_1 > \min\{(\delta - \varepsilon_r)/\delta^r, 1/(r\varepsilon^{r-1})\}$*. Then*

$$I(c_1 x^r + \varepsilon_r, 0, \delta) < I(f_*, 0, \delta). \tag{5.4.16}$$

Proof. As already noted, by (5.4.14) we see that $I(cx^r + c', u, v)$ is increasing in $c \ge 1/r$. Since $(\delta - \varepsilon_r)/\delta^r$ is increasing in $\delta \in [0, 1]$, it follows by (5.3.2) that for all relevant $\delta \in [\delta_\varepsilon, 1]$,

$$\frac{\delta - \varepsilon_r}{\delta^r} \ge \frac{\delta_\varepsilon - \varepsilon_r}{\delta_\varepsilon^r} = \frac{1}{r}.$$

Therefore, we may assume that $c_1 = \min\{(\delta - \varepsilon_r)/\delta^r, 1/(r\varepsilon^{r-1})\}$. In this case, note that $f_*(x) = c_1 x^r + \varepsilon_r$ for $x \in [0, \delta \wedge \varepsilon]$. Hence, if $\delta \le \varepsilon$, the claim follows immediately. On the other hand, if $\delta > \varepsilon$, the claim follows noting that $f_*(x) = x$ for $x \in [\varepsilon, \delta]$, and $I(c_1 x^r + \varepsilon_r, \varepsilon, \delta) < I(x, \varepsilon, \delta)$ by (5.4.15) (setting $\delta' = \varepsilon$ and $c_2 = 1/(r\varepsilon^{r-1})$). ∎

By (5.4.15) and (5.4.16) the optimal trajectory intersects the diagonal at $\delta \wedge \varepsilon$ and coincides with it thereafter on $[\delta \wedge \varepsilon, \delta]$ (i.e. the optimal δ' is $\delta' = \delta$). Finally, to identify f_* as the optimal trajectory, we show that $\delta \wedge \varepsilon$ is the first place the optimal trajectory intersects the diagonal (i.e. the optimal ε' is $\varepsilon' = \delta \wedge \varepsilon$). By (5.4.15) the only other possibility is that the trajectory meets the diagonal at some $\varepsilon' \in [\varepsilon_r, \delta \wedge \varepsilon)$ and then coincides with it on $[\varepsilon', \delta \wedge \varepsilon]$. We rule this out by the following observation.

Claim 5.4.8. *Let $\varepsilon' \in [\varepsilon_r, \delta \wedge \varepsilon)$. Then*

$$I((\varepsilon' - \varepsilon_r)(x/\varepsilon')^r + \varepsilon_r, 0, \varepsilon') + I(x, \varepsilon', \delta \wedge \varepsilon) < I(f_*, 0, \delta \wedge \varepsilon). \tag{5.4.17}$$

Proof. Let $\eta = \delta \wedge \varepsilon$. By (5.4.13) and (5.4.14), the left hand side above is equal to

$$(\varepsilon' - \varepsilon_r) \log\left(\frac{e}{r}\frac{(\varepsilon')^r}{\varepsilon' - \varepsilon_r}\right) - (r-2)(\eta - \varepsilon') + (r-1)\log(\eta^\eta/(\varepsilon')^{\varepsilon'}).$$

Differentiating this expression with respect to ε' we obtain

$$\log\left(\frac{(\varepsilon')^r}{r(\varepsilon' - \varepsilon_r)}\right) + r(1 - \varepsilon_r/\varepsilon') + (r-2) - (r-1)(\log(\varepsilon') + 1).$$

Since $\varepsilon_r = \frac{r}{r-1}\varepsilon$, this expression simplifies as

$$-\log(r - (r-1)\varepsilon/\varepsilon') + (r-1)(1 - \varepsilon/\varepsilon').$$

By the inequality $\log x < x - 1$ for $x < 1$, the above expression is positive for all $\varepsilon' \in [\varepsilon_r, \eta) \subset [\varepsilon_r, \varepsilon)$. The claim follows, taking $\varepsilon' \uparrow \eta$ and recalling that $f_*(x) = (\eta - \varepsilon_r)(x/\eta)^r + \varepsilon_r$ for $x \in [0, \eta]$. ■

By (5.4.15), (5.4.16) and (5.4.17) it follows that the maximizer $\hat{g}$ of $I(g, 0, \delta)$ (over functions g, as described below (5.4.11)) is $\hat{g} = f_*$. As discussed, (5.4.2) follows by (5.4.12), completing the proof. ■

Chapter 6

Sharp Threshold for K_4-Percolation

6.1 Overview

Graph bootstrap percolation is a variation of bootstrap percolation introduced by Bollobás. Let H be a graph. Edges are added to an initial graph $G = (V, E)$ if they are in a copy of H minus an edge, until no further edges can be added. If eventually the complete graph on V is obtained, G is said to H-percolate. We identify the sharp threshold for K_4-percolation on the Erdős–Rényi graph $\mathcal{G}_{n,p}$. This refines an approximation due to Balogh, Bollobás and Morris, which bounds the threshold up to multiplicative constants.*

6.2 Background and main results

Fix a graph H. Following Bollobás [39], *H-bootstrap percolation* is a cellular automaton that adds edges to a graph $G = (V, E)$ by iteratively completing all copies of H missing a single edge. Formally, given a graph $G_0 = G$, let G_{t+1} be G_t together with every edge whose addition creates a subgraph that is isomorphic to H. For a finite graph G, this procedure terminates once $G_{\tau+1} = G_\tau$, for some $\tau = \tau(G)$. We denote the resulting graph G_τ by $\langle G \rangle_H$. If $\langle G \rangle_H$ is the complete graph on V, the graph G is said to *H-percolate,* or equivalently, that G is *H-percolating.*

Recall that the Erdős–Rényi [60] graph $\mathcal{G}_{n,p}$ is the random subgraph of K_n obtained by including each possible edge independently with probability

*This chapter is independent work of the author [91], currently under review for publication.

p. In this work, we identify the sharp threshold for K_4-percolation on $\mathcal{G}_{n,p}$.

Theorem 6.2.1. *Let $p = \sqrt{\alpha/(n\log n)}$. If $\alpha > 1/3$ then $\mathcal{G}_{n,p}$ is K_4-percolating with high probability. If $\alpha < 1/3$ then with high probability $\mathcal{G}_{n,p}$ does not K_4-percolate.*

In Chapter 4 (joint work with Angel [12]) the super-critical case $\alpha > 1/3$ is established, via a connection with 2-*neighbour bootstrap percolation* (see Section 6.2.1). It thus remains to study the sub-critical case $\alpha < 1/3$. In this case, we also identify the size of the largest K_4-percolating subgraphs of $\mathcal{G}_{n,p}$.

Theorem 6.2.2. *Let $p = \sqrt{\alpha/(n\log n)}$, for some $\alpha \in (0, 1/3)$. With high probability the largest cliques in $\langle\mathcal{G}_{n,p}\rangle_{K_4}$ are of size $(\beta_* + o(1))\log n$, where $\beta_*(\alpha) \in (0, 3)$ satisfies $3/2 + \beta\log(\alpha\beta) - \alpha\beta^2/2 = 0$.*

From the results in Chapter 4, it follows that with high probability $\langle\mathcal{G}_{n,p}\rangle_{K_4}$ has cliques of size at least $(\beta_* + o(1))\log n$. Our contribution is to show that these are typically the largest cliques.

Balogh, Bollobás and Morris [24] study H-bootstrap percolation in the case that $G = \mathcal{G}_{n,p}$ and $H = K_k$. The case $k = 4$ is the minimal case of interest. Indeed, *all* graphs K_2-percolate, and a graph K_3-percolates if and only if it is connected. Therefore the case K_3 follows by a classical result of Erdős and Rényi [60]. If $p = (\log n + \varepsilon)/n$ then $\mathcal{G}_{n,p}$ is K_3-percolating with probability $\exp(-e^{-\varepsilon})(1 + o(1))$, as $n \to \infty$.

Critical thresholds for H-bootstrap percolation are defined in [24] by

$$p_c(n, H) = \inf\left\{p > 0 : \mathbf{P}(\langle\mathcal{G}_{n,p}\rangle_H = K_n) \geq 1/2\right\}.$$

In light of Theorem 6.2.1, we find that $p_c(n, K_4) \sim 1/\sqrt{3n\log n}$, solving Problem 2 in [24]. Moreover, the same holds if the 1/2 in the definition above is replaced by any probability in $(0, 1)$. It is expected that this property has a sharp threshold for $H = K_k$ for all k, in the sense that for some $p_c = p_c(k)$ we have that $\mathcal{G}_{n,p}$ is K_k-percolating with high probability for $p > (1 + \delta)p_c$ and with probability tending to 0 for $p = (1 - \delta)p_c$. Some bounds for $p_c(n, K_k)$ are established in [24]. A main result of [24] is that $p_c(n, K_4) = \Theta(1/\sqrt{n \log n})$. For larger k even the order of p_c is open.

6.2.1 Seed edges

In Chapter 4 (see Theorem 4.2.2), a sharp upper bound for $p_c(n, K_4)$ is established by observing a connection with 2-*neighbour bootstrap percolation* (see Pollak and Riess [116] and Chalupa, Leath and Reich [50]). This process is defined as follows: Let $G = (V, E)$ be a graph. Given some initial set $V_0 \subset V$ of *active* vertices, let V_{t+1} be the union of V_t and the set of all vertices with at least 2 neighbours in V_t. The sets V_t are increasing, and so converge to some set of eventually active vertices, denoted by $\langle V_0, G\rangle_2$. A set I is called *contagious* for G if it activates all of V, that is, $\langle I, G\rangle_2 = V$. (Note that, despite the similar notation, $\langle\cdot\rangle_2$ has a different meaning than $\langle\cdot\rangle_H$ above for graphs H. In the present article, we only use $\langle\cdot\rangle_2$ and $\langle\cdot\rangle_{K_4}$.)

If $G = (V, E)$ has a contagious pair $\{u, v\}$, and moreover $(u, v) \in E$, then clearly G is K_4-percolating (see Lemma 4.2.3). In this case we call (u, v) a *seed edge* and G a *seed graph*. Hence G is a seed graph if some contagious pair of G is joined by an edge.

While it is possible for a graph to be K_4-percolating without containing a seed edge (see Section 6.3), we believe that the two properties are fairly close. In particular, they have the same asymptotic threshold. In Chapter 4, the sharp threshold for the existence of contagious pairs in $\mathcal{G}_{n,p}$ is identified, and is shown to be $1/(2\sqrt{n \log n})$. It is also shown that if $p = \sqrt{\alpha/(n \log n)}$, then for $\alpha > 1/3$ with high probability $\mathcal{G}_{n,p}$ has a seed edge, and so is K_4-percolating. If $\alpha < 1/3$ then the largest seed subgraphs of $\mathcal{G}_{n,p}$ are of size $(\beta_* + o(1)) \log n$ with high probability, where β_* is as defined in Theorem 6.2.2.

6.2.2 Outline

By the results in Chapter 4 discussed in the previous Section 6.2.1, to prove Theorems 6.2.1 and 6.2.2 it remains to establish the following result.

Proposition 6.2.3. *Let $p = \sqrt{\alpha/(n \log n)}$, for some $\alpha \in (0, 1/3)$. For any $\delta > 0$, with high probability $\langle \mathcal{G}_{n,p}\rangle_{K_4}$ contains no clique larger than*

$(\beta_* + \delta)\log n$*, where* β_* *is as defined in Theorem 6.2.2.*

In other words, we need to rule out the possibility that some subgraph of $\mathcal{G}_{n,p}$ is K_4-percolating and larger than $(\beta_* + \delta)\log n$.

For a graph $G = (V, E)$, let $V(G) = V$ and $E(G) = E$ denote its vertex and edge sets. For $H \subset G$, let $\langle H, G\rangle_2$ denote the subgraph of G induced by $\langle V(H), G\rangle_2$ (see Section 6.2.1). It is easy to see that if $H \subset G$ is K_4-percolating, then so is $\langle H, G\rangle_2$. In particular, G is a seed graph if $\langle e, G\rangle_2 = G$ for some seed edge $e \in E(G)$. On the other hand, if a K_4-percolating graph G is not a seed graph, we show that there is some K_4-percolating subgraph $C \subset G$ of minimum degree at least 3 such that $\langle C, G\rangle_2 = G$. We call C the 3-*core* of G. Hence, to establish Proposition 6.2.3, we require bounds for (i) the number of K_4-percolating graphs C of size q with minimum degree at least 3, and (ii) the probability that for a given set $I \subset [n]$ of size q we have that $|\langle I, \mathcal{G}_{n,p}\rangle_2| \geq k$.

We obtain an upper bound of $(2/e)^q q! q^q$ for the number of K_4-percolating 3-cores C of size q. (This is much smaller than the number of seed subgraphs of size q, which in Chapter 4, see Lemmas 4.3.5 and 4.4.5, is shown to be equal to $q!q^q e^{o(q)}$.) Further arguments imply that, for p as in Proposition 6.2.3, with high probability $\mathcal{G}_{n,p}$ has no such subgraphs C larger than $(2\alpha)^{-1}\log n$. This already gives a strong indication that $1/3$ is indeed the critical constant, since as shown by Janson, Łuczak, Turova and Vallier [84, Theorem 3.1] (see Theorem 2.4.1), $(2\alpha)^{-1}\log n$ is the critical size above which a random set is likely to be contagious.

In Chapter 5 (joint work with Angel [11]), large deviation estimates are developed for the probability that small sets of vertices eventually activate a relatively large set of vertices via the r-neighbour bootstrap percolation dynamics. These bounds complement the central limit theorems of [84] (see Theorem 2.4.5). This result, in the case of $r = 2$, plays an important role in the current chapter. For $2 \leq q \leq k$, let $P(q, k)$ denote the probability that for a given set $I \subset [n]$, with $|I| = q$, we have that $|\langle I, \mathcal{G}_{n,p}\rangle_2| \geq k$.

Lemma 6.2.4 (Angel and Kolesnik [11, Theorem 3.2])**.** *Let* $p = \sqrt{\alpha/(n\log n)}$*, for some* $\alpha > 0$*. Let* $\varepsilon \in [0, 1)$ *and* $\beta \in [\beta_\varepsilon, 1/\alpha]$*, where* $\beta_\varepsilon = (1 - \sqrt{1-\varepsilon})/\alpha$*.*

Put $k_\alpha = \alpha^{-1}\log n$ and $q_\alpha = (2\alpha)^{-1}\log n$. Suppose that $q/q_\alpha \to \varepsilon$ and $k/k_\alpha \to \alpha\beta$ as $n \to \infty$. Then $P(q,k) = n^{\xi_\varepsilon + o(1)}$, where $\xi_\varepsilon = \xi_\varepsilon(\alpha,\beta)$ is equal to

$$-\frac{\alpha\beta^2}{2} + \begin{cases} (2\alpha\beta-\varepsilon)(2\alpha)^{-1}\log(e(\alpha\beta)^2/(2\alpha\beta-\varepsilon)), & \beta \in [\beta_\varepsilon, \varepsilon/\alpha); \\ \beta\log(\alpha\beta) - \varepsilon(2\alpha)^{-1}\log(\varepsilon/e), & \beta \in [\varepsilon/\alpha, 1/\alpha]. \end{cases}$$

(This estimate follows by Theorem 5.4.2, setting $r = 2$, $\vartheta = (4\alpha)^{-1}\log n$ and $\delta = \alpha\beta$, in which case, in the notation of [11], we have $k_2 = k_\alpha$, $\ell_2 = q_\alpha$ and $\delta_\varepsilon = \alpha\beta_\varepsilon$.) Applying the lemma and the bound $(2/e)^q q! q^q$ for the number of K_4-percolating 3-cores of size q, we deduce that the expected number of K_4-percolating subgraphs of $\mathcal{G}_{n,p}$ of size $k = \beta\log n$, for some $\beta \in [\beta_\varepsilon, 1/\alpha]$, is bounded by $n^{\mu+o(1)}$, where

$$\mu(\alpha,\beta) = 3/2 + \beta\log(\alpha\beta) - \alpha\beta^2/2,$$

leading to Proposition 6.2.3.

In closing, we remark that the proof of Proposition 4.3.1 in Chapter 4 shows that the expected number of edges in $\mathcal{G}_{n,p}$ that are a seed edge for a subgraph of size at least $k = \beta\log n$, for $\beta \in (0, 1/\alpha]$, is bounded by $n^{\mu+o(1)}$. (Alternatively, we recover this bound from the case $\varepsilon = 0$ in Lemma 6.2.4.) This suggests that perhaps $\mathcal{G}_{n,p}$ is as likely to K_4-percolate due to a seed edge as in any other way. That being said, the precise behaviour in the *scaling window* (the range of p where $\mathcal{G}_{n,p}$ is K_4-percolating with probability in $[\varepsilon, 1-\varepsilon]$) remains an interesting open problem. As mentioned above, the case of K_3-percolation follows by fundamental work of Erdős and Rényi [60]: With high probability $\mathcal{G}_{n,p}$ is K_3-percolating (equivalently, connected) if and only if it has no isolated vertices. It seems possible that K_4-percolation is more complicated. Perhaps, for p in the scaling window, the probability that $\mathcal{G}_{n,p}$ has a seed edge converges to a constant in $(0,1)$, and with non-vanishing probability $\mathcal{G}_{n,p}$ is not a seed graph, however is K_4-percolating due to a small 3-core C of size $O(1)$ such that $|\langle C, \mathcal{G}_{n,p}\rangle_2| = n$. We hope to investigate this in future work.

6.3 Clique processes

If a graph G is K_4-percolating, we will often simply say that G *percolates,* or that it is *percolating.* Following [24], we define the *clique process,* as a way to analyze K_4-percolation on graphs.

Definition 6.3.1. We say that three graphs $G_i = (V_i, E_i)$ *form a triangle* if there are distinct vertices x, y, z such that $x \in V_1 \cap V_2$, $y \in V_1 \cap V_3$ and $z \in V_2 \cap V_3$. If $|V_i \cap V_j| = 1$ for all $i \neq j$, we say that the G_i *form exactly one triangle.*

In [24] the following observation is made.

Lemma 6.3.2. *Suppose that $G_i = (V_i, E_i)$ percolate.*
(i) If $|V_1 \cap V_2| > 1$ then $G_1 \cup G_2$ percolates.
(ii) If the G_i form a triangle then $G_1 \cup G_2 \cup G_3$ percolates.

Moreover, if the G_i form *multiple* triangles (that is, if there are multiple triplets x, y, z as above), then the percolation of $G_1 \cup G_2 \cup G_3$ follows by applying Lemma 6.3.2(ii) twice. Indeed, some G_i, G_j have two vertices in common, and so $G' = G_i \cup G_j$ percolates, and G' has two common vertices with the remaining graph G_k.

By these observations, the K_4-percolation dynamics are classified in [24] as follows (which we modify slightly here in light of the previous observation).

Definition 6.3.3. A *clique process* for a graph G is a sequence $(\mathcal{C}_t)_{t=1}^{\tau}$ of sets of subgraphs of G with the following properties:

(i) $\mathcal{C}_0 = E(G)$ is the edge set of G.

(ii) For each $t < \tau$, $\mathcal{C}_{t+1}$ is constructed from $\mathcal{C}_t$ by either (a) merging two subgraphs $G_1, G_2 \in \mathcal{C}_t$ with at least two common vertices, or (b) merging three subgraphs $G_1, G_2, G_3 \in \mathcal{C}_t$ that form exactly one triangle.

(iii) $\mathcal{C}_\tau$ is such that no further operations as in (ii) are possible.

The reason for the name is that for any $t \leq \tau$ and $H \in \mathcal{C}_t$, $\langle H \rangle_{K_4}$ is the complete graph on $V(H)$.

Lemma 6.3.4. *Let G be a finite graph and $(\mathcal{C}_t)_{t=1}^{\tau}$ a clique process for G. For each $t \le \tau$, $\mathcal{C}_t$ is a set of edge-disjoint, percolating subgraphs of G. Furthermore, $\langle G \rangle_{K_4}$ is the edge-disjoint, triangle-free union of the cliques $\langle H \rangle_{K_4}$, $H \in \mathcal{C}_\tau$. Hence G percolates if and only if $\mathcal{C}_\tau = \{G\}$. In particular, if two clique processes for G terminate at $\mathcal{C}_\tau$ and $\mathcal{C}'_{\tau'}$, then necessarily $\mathcal{C}_\tau = \mathcal{C}'_{\tau'}$.*

6.3.1 Consequences

The following corollaries of Lemma 6.3.4 are proved in [24].

Lemma 6.3.5. *If $G = (V, E)$ percolates then $|E| \ge 2|V| - 3$.*

In light of this, we define the *excess* of a percolating graph $G = (V, E)$ to be $|E| - (2|V| - 3)$. We call a percolating graph *edge-minimal* if its excess is 0. To prove Lemma 6.3.5, the following observations are made in [24].

Lemma 6.3.6. *Suppose that $G_i = (V_i, E_i)$ percolate.*

(i) If the G_i form exactly one triangle, then the excess of $G_1 \cup G_2 \cup G_3$ is the sum of the excesses of the G_i.

(ii) If $|V_1 \cap V_2| = m \ge 2$, then the excess of $G_1 \cup G_2$ is the sum of the excesses of the G_i plus $2m - 3$.

Hence, if G is edge-minimal and percolating, then every step of any clique process for G involves merging three subgraphs that form exactly one triangle. A special class of percolating graphs are *seed graphs,* as discussed in Section 6.2.1. In an edge-minimal seed graph G, every step of some clique process for G involves merging three subgraphs, two of which are a single edge.

Finally, since in each step of any clique process for a graph G either 2 or 3 subgraphs are merged, we have the following useful criterion for percolation.

Lemma 6.3.7. *Let $G = (V, E)$ be a graph of size n, and $1 \le k \le n$. If there is no percolating subgraph $G' \subset G$ of size k', for any $k' \in [k, 3k]$, then G has no percolating subgraph larger than k. In particular, G does not percolate.*

6.4 Percolating graphs

In this section, we analyze the general structure of percolating graphs.

Definition 6.4.1. We say that a graph G is *irreducible* if removing any edge from G results in a non-percolating graph.

Clearly, a graph G is percolating if and only if it has an irreducible percolating subgraph $G' \subset G$ such that $V(G) = V(G')$.

For a graph G and vertex $v \in V(G)$, we let G_v denote the subgraph of G induced by $V - \{v\}$, that is, the subgraph obtained by removing v.

Lemma 6.4.2. *Let G be an irreducible percolating graph. If $v \in V(G)$ is of degree 2, then G_v is percolating.*

Proof. The proof is by induction on the size of G. The case $|V(G)| = 3$, in which case G is a triangle, is immediate. Hence suppose that G, with $|V(G)| > 3$, percolates and some $v \in V(G)$ is of degree 2, and assume that the statement of the lemma holds for all graphs H with $|V(H)| < |V(G)|$.

Let $(\mathcal{C}_t)_{t=1}^{\tau}$ be a clique process for G. Let e_1, e_2 denote the edges incident to v in G. Let $t < \tau$ be the first time in the clique process $(\mathcal{C}_t)_{t=1}^{\tau}$ that a subgraph containing either e_1 or e_2 is merged with other (edge-disjoint, percolating) subgraphs. We claim that $\mathcal{C}_{t+1}$ is obtained from $\mathcal{C}_t$ by merging e_1, e_2 with a subgraph in $\mathcal{C}_t$. To see this, we first observe that if a graph H percolates and $|V(H)| > 2$ (that is, H is not simply an edge), then all vertices in H have degree at least 2. Next, by the choice of t, we note that none of the graphs being merged contain both e_1, e_2. Therefore, since v is of degree 2, if one the graphs contains exactly one e_i, then it is necessarily equal to e_i, being a percolating graph of minimum degree 1. It follows that v is contained in two of the graphs being merged, and hence that $\mathcal{C}_{t+1}$ is the result of merging the edges e_1, e_2 with a subgraph in $\mathcal{C}_t$, as claimed.

To conclude, note that if $t = \tau - 1$ then since G percolates (and so $\mathcal{C}_\tau = \{G\}$) we have that $\mathcal{C}_{\tau-1} = \{e_1, e_2, G_v\}$, and so G_v percolates. On the other hand, if $t < \tau - 1$, then $\mathcal{C}_\tau$ contains 2 or 3 subgraphs, one of which contains e_1 and e_2. If $\mathcal{C}_{\tau-1} = \{G_1, G_2\}$, where $e_1, e_2 \in E(G_1)$, say, then by the inductive hypothesis we have that $(G_1)_v$ percolates. Since G_1, G_2 are edge-disjoint, we have that $v \notin V(G_2)$, as otherwise G_2 would be a percolating graph with an isolated vertex. Hence, by Lemma 6.3.2(i), we find that $(G_1)_v \cup G_2 = G_v$ percolates. Similarly, if $\mathcal{C}_{\tau-1} = \{G_1, G_2, G_3\}$, where $e_1, e_2 \in E(G_1)$, say, then by the inductive hypothesis and Lemma 6.3.2(ii), we find that $(G_1)_v \cup G_2 \cup G_3 = G_v$ percolates.

The induction is complete. ∎

Recall (see Sections 6.2.1 and 6.2.2) that for graphs $H \subset G$, we let $\langle H, G\rangle_2$ denote the subgraph of G induced by $\langle V(H), G\rangle_2$, that is, the subgraph of G induced by the closure of $V(H)$ under the 2-neighbour bootstrap percolation dynamics on G. By Lemma 6.3.2(i), if $H \subset G$ is percolating then so is $\langle H, G\rangle_2$.

The following is an immediate consequence of Lemma 6.4.2.

Lemma 6.4.3. *Let G be an irreducible percolating graph. Then either*

(i) $G = \langle e, G\rangle_2$ *for some edge* $e \in E(G)$*, or else,*

(ii) $G = \langle C, G\rangle_2$ *for some percolating subgraph* $C \subset G$ *of minimum degree at least* 3.

Futhermore,

(iii) the excess of G is equal to the excess of C.

We note that in case (i), G is a seed graph and e is a seed edge for G. An irreducible seed graph is edge-minimal, that is, it has 0 excess. In case (ii), we call C the 3-*core* of G. If $G = C$ we say that G *is* a 3-core.

It is straightforward to verify that all irreducible percolating graphs on $2 < k \le 6$ vertices have a vertex of degree 2. There is however an edge-minimal percolating graph of size $k = 7$ with no vertex of degree 2, see Figure 6.1.

6.4.1 Basic estimates

In this section, we use Lemma 6.4.3 to obtain upper bounds for irreducible percolating graphs. For such a graph G, the relevant quantities are the

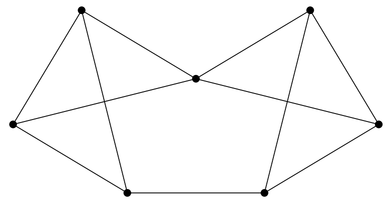

Figure 6.1: The smallest irreducible percolating 3-core.

number of vertices in G of degree 2, the size of its 3-core $C \subset G$, and its number of excess edges.

Definition 6.4.4. Let $I_q^\ell(k,i)$ be the number of labelled, irreducible graphs G of size k with an excess of ℓ edges, i vertices of degree 2, and a 3-core $C \subset G$ of size $2 < q \le k$. If $i = 0$, and hence $k = q$, we let $C^\ell(k) = I_k^\ell(k,0)$.

In the case $\ell = 0$, we will often simply write $I_q(k,i)$ and $C(k)$.

By Lemma 6.4.3(iii), if a graph G contributes to $I_q^\ell(k,i)$ then its 3-core $C \subset G$ has an excess of ℓ edges. Also, as noted above, there are no irreducible 3-cores on $k \le 6$ vertices. Hence $I_q^\ell(k,i) = 0$ if $2 < q \le 6$.

Definition 6.4.5. We define $I_2(k,i)$ to be the number of labelled, edge-minimal seed graphs of size k with i vertices of degree 2.

For convenience, we let $C(2) = 1$ and set $I_2^\ell(k,i) = 0$ and $C^\ell(2) = 0$ for $\ell > 0$ (in light of Lemma 6.4.3(iii)). Moreover, to simplify several statements in this work, if we say that a graph G has a 3-core of size less than $q > 2$, we mean to include also the possibility that $q = 2$.

Definition 6.4.6. We let $I^\ell(k,i) = \sum_q I_q^\ell(k,i)$ denote the number of labelled, irreducible graphs G of size k with an excess of ℓ edges and i vertices of degree 2.

If $\ell = 0$, we will often write $I(k,i)$.

We obtain the following estimate for $I^\ell(k,i)$ in the case that $\ell \le 3$, that is, for graphs with at most 3 excess edges.

Lemma 6.4.7. *For all $k \ge 2$, $\ell \le 3$ and relevant i, we have that*

$$I^\ell(k,i) \le (2/e)^k k! k^{k+2\ell+i}.$$

In particular, $C^{\ell}(k) \leq (2/e)^k k! k^{k+2\ell}$.

The method of proof gives bounds for larger ℓ, however, as it turns out, percolating graphs with a larger excess can be dealt with using less accurate estimates (see Lemma 6.5.3).

The proof is somewhat involved, as there are several cases to consider, depending on the nature of the last step of a clique process for G. We proceed by induction: First, we note that the cases $i > 0$ follow easily, since if G has i vertices of degree 2, then removing such a vertex from G results in a graph with $j \in \{i, i \pm 1\}$ vertices of degree 2. Analyzing this case leads to the constant $2/e$. The case $i = 0$ (corresponding to 3-cores) is the heart of the proof. The following observation allows the induction to go through in this case: If G is a percolating 3-core, then in the last step of a clique process for G either (i) three graphs G_1, G_2, G_3 are merged that form exactly one triangle on $T = \{v_1, v_2, v_3\}$, or else (ii) two graphs G_1, G_2 are merged that share exactly $m \geq 2$ vertices $S = \{v_1, v_2, \ldots, v_m\}$. We note that if some G_j has a vertex v of degree 2, then necessarily $v \in T$ in case (i), and $v \in S$ in case (ii) (as else, G would have a vertex of degree 2). In other words, if a percolating 3-core is formed by merging graphs with vertices of degree 2, then all such vertices belong to the triangle that they form or the set of their common vertices.

Proof. It is easily verified that the statement of the lemma holds for $k \leq 4$. We prove the remaining cases by induction. For $k > 4$, we claim moreover that for all $\ell \leq 3$ and relevant i,

$$I^{\ell}(k, i) \leq A\zeta^k \binom{k}{i} k! k^{k+2\ell} \tag{6.4.1}$$

where $\zeta = 2/e$ and $A = 6/(\zeta^5 5! 5^5)$. The lemma follows, noting that $A < 1$ and $\binom{k}{i} \leq k^i$.

We introduce the constant $A < 1$ in order to push through the induction in the case $i = 0$, corresponding to 3-cores. The last step of a clique process for such a graph G involves merging 2 or 3 subgraphs G_j. Informally, we use the constant A to penalize graphs G such that at least two of the G_j

contain more than 4 vertices, that is, graphs G formed by merging at least two "macroscopic" subgraphs.

By the choice of A, we have that (6.4.1) holds for $k = 5$. Indeed, note that $I(5,i) \le \binom{5}{i}\binom{4}{2}$ for all $i \in \{1,2,3\}$ and $I^\ell(5,i) = 0$ otherwise. Assume that for some $k > 5$, (6.4.1) holds for all $4 < k' < k$, and all $\ell \le 3$ and relevant i.

We begin with the case of graphs G of size k with at least one vertex of degree 2. This case follows easily by a recursive upper bound (and explains the choice of $\zeta = 2/e$).

Case 1 ($i > 0$). Suppose that G is a graph contributing to $I^\ell(k,i)$, where $i > 0$ and $\ell \le 3$. Let $v \in V(G)$ be the vertex of degree 2 in G with the minimal index. By considering which two of the $k - i$ vertices of G of degree larger than 2 are neighbours of v, we find that $I^\ell(k,i)$ is bounded from above by

$$\binom{k}{i}\binom{k-i}{2}\sum_{j=0}^{2}\binom{2}{j}\frac{I^\ell(k-1,i-1+j)}{\binom{k-1}{i-1+j}}.$$

In this sum, $j \in \{0,1,2\}$ is the number of neighbours of v that are of degree 2 in the subgraph of G induced by $V(G) - \{v\}$. Applying the inductive hypothesis, we obtain

$$I^\ell(k,i) \le A\zeta^k\binom{k}{i}k!k^{k+2\ell}\cdot\frac{2}{\zeta}\left(\frac{k-1}{k}\right)^k \le A\zeta^k\binom{k}{i}k!k^{k+2\ell},$$

as required.

The remaining cases deal with 3-cores G of size k, where $i = 0$. First, we establish the case $i = \ell = 0$, corresponding to edge-minimal percolating 3-cores. The cases $i = 0$ and $\ell \in \{1,2,3\}$ are proved by adapting the argument for $i = \ell = 0$.

Case 2 ($i = \ell = 0$). Let G be a graph contributing to $C(k) = I(k,0)$. Then, by Lemma 6.3.6, in the last step of a clique process for G, three edge-minimal percolating subgraphs G_j, $j \in \{1,2,3\}$, are merged which form exactly one triangle on some $T = \{v_1,v_2,v_3\} \subset V(G)$. Moreover, each G_j has at most 2 vertices of degree 2, and if some G_j has such a vertex v

then necessarily $v \in T$ (as else G would have a vertex of degree 2). Also if $k_j = |V(G_j)|$, with $k_1 \geq k_2 \geq k_3$, then (i) $\sum_{j=1}^{3} k_j = k+3$, (ii) $k_1, k_2 \geq 4$ and (iii) $k_3 = 2$ or $k_3 \geq 4$ (since if some $k_j = 3$ or some $k_j = k_{j'} = 2$, $j \neq j'$, then G would have a vertex of degree 2).

Since the inductive hypothesis only holds for graphs with more than 4 vertices, it is convenient to deal with the case $k_1 = 4$ separately: Note that the only irreducible percolating 3-cores of size k with all $k_j \leq 4$ are of size $k \in \{7, 9\}$. These graphs are the graph in Figure 6.1 and the graph obtained from this graph by replacing the bottom edge with a copy of K_4 minus an edge. It is straightforward to verify that (6.4.1) holds if $k \in \{7, 9\}$, and so in the arguments below we assume that $k_1 > 4$. Moreover, since the graph in Figure 6.1 is the only irreducible percolating 3-core on $k = 7$ vertices, we further assume below that $k \geq 8$.

We take three cases, with respect to whether (i) $k_2 = 4$, (ii) $k_2 > 4$ and $k_3 \in \{2, 4\}$, or (iii) $k_3 > 4$.

Case 2(i) ($i = \ell = 0$ and $k_2 = 4$). Note that if $k_2 = 4$ then $k_3 \in \{2, 4\}$. The number of graphs G as above with $k_3 = 2$ and $k_2 = 4$ is bounded from above by

$$\binom{k}{k-3}\binom{k-3}{2}\binom{3}{1} 2!^2 \sum_{j=0}^{2} \binom{2}{j} \frac{I(k-3, j)}{\binom{k-3}{j}}.$$

Here the first binomial selects the vertices for the subgraph of size $k_1 = k-3$, the next two binomials select the vertices for the triangle T, and the rightmost factor bounds the number of possibilities for the subgraph of size $k_1 = k-3$ (recalling that it can have at most 2 vertices of degree 2, and if it contains any such vertex v, then $v \in T$). Applying the inductive hypothesis (recall that we may assume that $k_1 > 4$), the above expression is bounded by

$$A\zeta^k k! k^k \cdot \frac{(k-3)^{k-1}}{k^k} \frac{4}{\zeta^3} \leq A\zeta^k k! k^k \cdot \frac{1}{k} \frac{4}{\zeta^3 e^3}.$$

Here, and throughout this proof, we use the fact that $(\frac{k-x}{k})^{k-y} \leq e^{-x}$ provided that $2y \leq x < k$ and $x > 0$. To see this, note that $(\frac{k-x}{k})^{k-y} \to e^{-x}$

as $k \to \infty$, and

$$\begin{aligned}\frac{\partial}{\partial k}\left(\frac{k-x}{k}\right)^{k-y} &= \left(\frac{k-x}{k}\right)^{k-y}\left(\log\left(\frac{k-x}{k}\right)+\frac{x(k-y)}{k(k-x)}\right)\\ &\geq \left(\frac{k-x}{k}\right)^{k-y}\frac{x(x-2y)}{2k(k-x)} \geq 0,\end{aligned}$$

by the inequality $\log u \geq (u^2-1)/(2u)$ (which holds for $u \in (0,1]$).

Similarly, the number of graphs G as above such that $k_1 = k_2 = 4$ is bounded by

$$\binom{k}{k-5,3,2}\binom{k-5}{2}\binom{3}{1}2!^3\sum_{j=0}^{2}\binom{2}{j}\frac{I(k-5,j)}{\binom{k-5}{j}}.$$

By the inductive hypothesis, this is bounded by

$$A\zeta^k k!k^k \cdot \frac{(k-5)^{k-3}}{k^k}\frac{4}{\zeta^5} \leq A\zeta^k k!k^k \cdot \frac{1}{k^{5/2}\sqrt{k-5}}\frac{4}{\zeta^5 e^5}.$$

Altogether, we find that the number of graphs G contributing to $C(k)$ with $k_2 = 4$, divided by $A\zeta^k k!k^k$, is bounded by

$$\gamma_1 = \frac{1}{8}\frac{4}{\zeta^3 e^3} + \frac{1}{8^{5/2}\sqrt{3}}\frac{4}{\zeta^5 e^5} < 0.07. \tag{6.4.2}$$

Case 2(ii) ($i = \ell = 0$, $k_2 > 4$ and $k_3 \in \{2,4\}$). Note that in this case we may further assume that $k \geq 9$. For a given $k_1, k_2 > 4$, the number of graphs G as above with $k_3 = 2$ (in which case $k_1 + k_2 = k+1$) is bounded by

$$\binom{k}{k_1, k_2-1}\binom{k_1}{2}\binom{k_2-1}{1}2!^2\prod_{j=1}^{2}\sum_{i=0}^{2}\binom{2}{i}\frac{I(k_j,i)}{\binom{k_j}{i}}.$$

Applying the inductive hypothesis, this is bounded by

$$A\zeta^k k!k^k \cdot 2 \cdot 4^2 A\zeta\frac{k_1^{k_1+2}k_2^{k_2+2}}{k^k}.$$

Since $k_2 = k + 1 - k_1$, we have that

$$\frac{\partial}{\partial k_1} k_1^{k_1+2} k_2^{k_2+2} = -k_1^{k_1+1} k_2^{k_2+1}(k_1 k_2 \log(k_2/k_1) - 2(k_1 - k_2)).$$

By the bound $\log x \leq x - 1$, we see that

$$k_1 k_2 \log(k_2/k_1) - 2(k_1 - k_2) \leq -(k_2 + 2)(k_1 - k_2) \leq 0.$$

Hence, setting k_1 to be the maximum relevant value $k_1 = k - 4$ (when $k_2 = 5$), we find

$$\frac{k_1^{k_1+2} k_2^{k_2+2}}{k^k} \leq \frac{5^7 (k-4)^{k-2}}{k^k} \leq \frac{1}{k^2} \frac{5^7}{e^4}$$

for all relevant k_1, k_2. Therefore, summing over the at most $k/2$ possibilities for k_1, k_2, we find that at most

$$A\zeta^k k! k^k \cdot \frac{1}{k} \frac{A\zeta 4^2 5^7}{e^4}$$

graphs G with $k_3 = 2$ and $k_2 > 4$ contribute to $C(k)$.

The case of $k_3 = 4$ is very similar. In this case, for a given $k_1, k_2 > 4$ such that $k_1 + k_2 = k - 1$, the number of graphs G as above is bounded by

$$\binom{k}{k_1, k_2 - 1, 2} \binom{k_1}{2} \binom{k_2 - 1}{1} 2!^3 \prod_{j=1}^{2} \sum_{i=0}^{2} \binom{2}{i} \frac{I(k_j, i)}{\binom{k_j}{i}},$$

which, by the inductive hypothesis, is bounded by

$$A\zeta^k k! k^k \cdot 2 \cdot 4^2 \frac{A}{\zeta} \frac{k_1^{k_1+2} k_2^{k_2+2}}{k^k}.$$

Arguing as in the previous case, we see that the above expression is maximized when $k_2 = 5$ and $k_1 = k - 6$. Hence, summing over the at most $k/2$ possibilities for k_1, k_2, there are at most

$$A\zeta^k k! k^k \cdot \frac{1}{(k-6)k^2} \frac{A 4^2 5^7}{\zeta e^6}$$

graphs G that contribute to $C(k)$ with $k_3 = 4$ and $k_2 > 4$.

We conclude that the number of graphs G that contribute to $C(k)$ with $k_2 > 4$ and $k_3 \in \{2, 4\}$, divided by $A\zeta^k k! k^k$, is bounded by

$$\gamma_2 = \frac{1}{9} \frac{A\zeta 4^2 5^7}{e^4} + \frac{1}{3 \cdot 9^2} \frac{A 4^2 5^7}{\zeta e^6} < 0.15. \tag{6.4.3}$$

Case 2(iii) ($i = \ell = 0$ and $k_3 > 4$). In this case we may further assume that $k \geq 12$. For a given $k_1, k_2, k_3 > 4$ such that $k_1 + k_2 + k_3 = k + 3$, the number of graphs G as above is bounded by

$$\binom{k}{k_1, k_2 - 1, k_3 - 2} \binom{k_1}{2} \binom{k_2 - 1}{1} 2!^3 \prod_{j=1}^{3} \sum_{i=0}^{2} \binom{2}{i} \frac{I(k_j, i)}{\binom{k_j}{i}}.$$

By the inductive hypothesis, this is bounded by

$$A\zeta^k k! k^k \cdot 2^2 4^3 A^2 \zeta^3 \frac{k_1^{k_1+2} k_2^{k_2+2} k_3^{k_3+2}}{k^k}.$$

As in the previous cases considered, the above expression is maximized when $k_2 = k_3 = 5$ and $k_1 = k - 7$. Hence, summing over the at most $k^2/12$ choices for the k_j, we find that at most

$$\frac{1}{((k-7)k)^{3/2}} \frac{A^2 \zeta^3 4^3 5^{14}}{3e^7}$$

graphs G contribute to $C(k)$ with $k_3 > 4$. Hence, the number of such graphs, divided by $A\zeta^k k! k^k$, is bounded by

$$\gamma_3 = \frac{1}{(5 \cdot 12)^{3/2}} \frac{A^2 \zeta^3 4^3 5^{14}}{3e^7} < 0.01. \tag{6.4.4}$$

Finally, combining (6.4.2), (6.4.3) and (6.4.4), we find that

$$\frac{C(k)}{A\zeta^k k! k^k} \leq \gamma_1 + \gamma_2 + \gamma_3 < 0.23 < 1, \tag{6.4.5}$$

completing the proof of Case 2.

It remains to consider the cases $i = 0$ and $\ell \in \{1, 2, 3\}$, corresponding to 3-cores G with a non-zero excess. In these cases, it is possible that only 2 subgraphs are merged in the last step of a clique process for G. We prove the cases $\ell = 1, 2, 3$ separately, however they all follow by adjusting the proof of Case 2.

First, we note that if two graphs G_1, G_2 with at least 2 vertices in common are merged to form an irreducible percolating 3-core G, then necessarily each G_j contains more than 4 vertices. In particular, such a graph G contains at least 8 vertices. This allow us to apply the inductive hypothesis in these cases (recall that we claim that (6.4.1) holds only for graphs with more than 4 vertices), without taking additional sub-cases as in the proof of Case 2.

Case 3 ($i = 0$ and $\ell = 1$). If G contributes to $C^1(k)$, then by Lemma 6.3.6, in the last step of a clique process for G, there are two cases to consider:

(i) Three percolating subgraphs G_j, $j \in \{1, 2, 3\}$, are merged which form exactly one triangle $T = \{v_1, v_2, v_3\}$, such that for some $i_j \le 2$ and $k_j, \ell_j \ge 0$ with $\sum k_j = k+3$ and $\sum \ell_j = 1$, we have that G_j contributes to $I^{\ell_j}(k_j, i_j)$. Moreover, if any $i_j > 0$, the corresponding i_j vertices of G_j of degree 2 belong to T.

(ii) Two percolating subgraphs G_j, $j \in \{1, 2\}$, are merged that share exactly two vertices $S = \{v_1, v_2\}$, such that for some $i_j \le 2$ and k_j with $\sum k_j = k + 2$, we have that the G_j contribute to $I(k_j, i_j)$. Moreover, if any $i_j > 0$, the corresponding i_j vertices of G_j of degree 2 belong to S.

We claim that, by the arguments in Case 2 leading to (6.4.5), the number of graphs G satisfying (i), divided by $A\zeta^k k! k^{k+2}$, is bounded by

$$\gamma_1 + 2\gamma_2 + 3\gamma_3 < 0.40. \tag{6.4.6}$$

To see this, note the only difference between (i) of the present case and Case 2 above is that here one of the G_j has exactly 1 excess edge. Note that if one of the graphs G_j has an excess edge, then necessarily $k_j > 4$. Recall that graphs G that contribute to $C(k)$, as considered in Cases 2(i),(ii),(iii) above, have exactly $1, 2, 3$ subgraphs G_j with $k_j > 4$. Moreover, recall that the number of such graphs G, divided by $A\zeta^k k! k^k$, is bounded by $\gamma_1, \gamma_2, \gamma_3$, respectively, in these cases. Therefore, applying the inductive hypothesis, and noting that if G_j has exactly 1 excess edge then it contributes an extra factor of $k_j^2 < k^2$, it follows that the number of graphs G as in (i) of the present case, divided by $A\zeta^k k! k^{k+2}$, is bounded by $\sum_{j=1}^3 j\gamma_j$, as claimed. (By (6.4.2), (6.4.3) and (6.4.4), this sum is bounded by 0.40.)

On the other hand, arguing along the lines as in Case 2, the number of graphs G satisfying (ii), for a given $k_1, k_2 > 4$ such that $k_1 + k_2 = k + 2$, is bounded by

$$\binom{k}{k_1, k_2 - 2}\binom{k_1}{2} 2!^2 \prod_{j=1}^{2}\sum_{i=0}^{2}\binom{2}{i}\frac{I(k_j, i)}{\binom{k_j}{i}}.$$

By the inductive hypothesis, this is bounded by

$$A\zeta^k k! k^k \cdot 2 \cdot 4^2 A\zeta^2 \frac{k_1^{k_1+2} k_2^{k_2+2}}{k^k}.$$

Arguing as in Case 2, we find that this expression is maximized when $k_2 = 5$ and $k_1 = k - 3$. Hence, summing over the at most $k/2$ choices for k_1, k_2, the number of graphs G satisfying (ii), divided by $A\zeta^k k! k^{k+2}$, is at most

$$\gamma_4 = \frac{1}{8^2}\frac{A\zeta^2 4^2 5^7}{e^3} < 0.04. \tag{6.4.7}$$

Altogether, by (6.4.6) and (6.4.7), we conclude that

$$\frac{C^1(k)}{A\zeta^k k! k^{k+2}} \le \gamma_1 + 2\gamma_2 + 3\gamma_3 + \gamma_4 < 0.44 < 1, \tag{6.4.8}$$

completing the proof of Case 3.

Case 4 ($i = 0$ and $\ell = 2$). This case is nearly identical to Case 3. By Lemma 6.3.6, in the last step of a clique process for a graph G that contributes to $C^2(k)$, either (i) three graphs that form exactly one triangle are merged whose excesses sum to 2, or else (ii) two graphs that share exactly two vertices are merged whose excesses sum to 1. Hence, by the arguments in Case 3 leading to (6.4.8), we find that

$$\frac{C^2(k)}{A\zeta^k k! k^{k+4}} \leq \gamma_1 + 3\gamma_2 + 6\gamma_3 + 2\gamma_4 < 0.66 < 1, \tag{6.4.9}$$

as required.

Case 5 ($i = 0$ and $\ell = 3$). Since $\ell = 3$, it is now possible that in the last step of a clique process for a graph G contributing to $C^\ell(k)$, two graphs are merged that share three vertices. Apart from this difference, the argument is completely analogous to the previous cases.

If G contributes to $C^3(k)$, then by Lemma 6.3.6, in the last step of a clique process for G, there are three cases to consider:

(i) Three percolating subgraphs G_j, $j \in \{1, 2, 3\}$, are merged which form exactly one triangle $T = \{v_1, v_2, v_3\}$, such that for some $i_j \leq 2$ and $k_j, \ell_j \geq 0$ with $\sum k_j = k+3$ and $\sum \ell_j = 3$, we have that G_j contributes to $I^{\ell_j}(k_j, i_j)$. If any $i_j > 0$, the corresponding i_j vertices of G_j of degree 2 belong to T.

(ii) Two percolating subgraphs G_j, $j \in \{1, 2\}$, are merged that share exactly two vertices $S = \{v_1, v_2\}$, such that for some $i_j \leq 2$ and $k_j, \ell_j \geq 0$ with $\sum k_j = k + 2$ and $\sum \ell_j = 2$, we have that the G_j contribute to $I^{\ell_j}(k_j, i_j)$. If any $i_j > 0$, the corresponding i_j vertices of G_j of degree 2 belong to S.

(iii) Two percolating subgraphs G_j, $j \in \{1, 2\}$, are merged that share exactly three vertices $R = \{v_1, v_2, v_3\}$, such that for some $i_j \leq 3$ and k_j with $\sum k_j = k+3$, we have that the G_j contribute to $I(k_j, i_j)$. If any $i_j > 0$, the corresponding i_j vertices of G_j of degree 2 belong to R.

As in Case 4, we find by the arguments in Case 3 leading to (6.4.8) that the number of graphs G satisfying (i) or (ii), divided by $A\zeta^k k! k^{k+6}$, is bounded by

$$\gamma_1 + 4\gamma_2 + 10\gamma_3 + 3\gamma_4 < 0.89. \tag{6.4.10}$$

On the other hand, by the arugments in Case 3 leading to (6.4.7), the number of graphs G satisfying (iii), for a given $k_1, k_2 > 4$ such that $k_1 + k_2 =$

$k+3$, is bounded by

$$\binom{k}{k_1, k_2-3}\binom{k_1}{3}3!^2\prod_{j=1}^{2}\sum_{i=0}^{3}\binom{3}{i}\frac{I(k_j,i)}{\binom{k_j}{i}}.$$

By the inductive hypothesis, this is bounded by

$$A\zeta^k k!k^k \cdot 3!8^2 A\zeta^3 \frac{k_1^{k_1+3}k_2^{k_2+3}}{k^k}.$$

This expression is maximized when $k_2 = 5$ and $k_1 = k-2$. Hence, summing over the at most $k/2$ choices for k_1, k_2, the number of graphs G satisfying (iii), divided by $A\zeta^k k!k^{k+6}$, is at most

$$\gamma_5 = \frac{1}{8^4}\frac{A\zeta^3 3!5^8 8^2}{2e^2} < 0.08. \tag{6.4.11}$$

Therefore, by (6.4.10) and (6.4.11), we have that

$$\frac{C^3(k)}{A\zeta^k k!k^{k+6}} \leq \gamma_1 + 4\gamma_2 + 10\gamma_3 + 3\gamma_4 + \gamma_5 < 0.97 < 1,$$

completing the proof of Case 5.

This last case completes the induction. We conclude that (6.4.1) holds for all $k > 4$, $\ell \leq 3$ and relevant i. As discussed, Lemma 6.4.7 follows. ■

6.4.2 Sharper estimates

In this section, using Lemma 6.4.7, we obtain upper bounds for $I_q^\ell(k,i)$, which improve on those for $I^\ell(k,i)$ given by Lemma 6.4.7, especially when q is significantly smaller than k. These are used in Section 6.6 to rule out the existence of large percolating subgraphs of $\mathcal{G}_{n,p}$ with few vertices of degree 2 and small 3-cores.

Lemma 6.4.8. *Let $\varepsilon > 0$. For some constant $\vartheta(\varepsilon) \geq 1$, the following holds. For all $k \geq 2$, $\ell \leq 3$, and relevant q, i, we have that*

$$I_q^\ell(k,i) \leq \vartheta\psi_\varepsilon(q/k)^k k!k^{k+2\ell+i}$$

where

$$\psi_\varepsilon(y) = \max\{3/(2e) + \varepsilon, (e/2)^{1-2y}y^2\}.$$

This lemma is only useful for $\varepsilon < 1/(2e)$, as otherwise $\psi_\varepsilon(y) \geq 2/e$ for all y, and so Lemma 6.4.7 gives a better bound. Note that, for any $\varepsilon < 1/(2e)$, we have that $\psi_\varepsilon(y)$ is non-decreasing and $\psi_\varepsilon(y) \to 2/e$ as $y \uparrow 1$, in agreement with Lemma 6.4.7. Moreover, $\psi_\varepsilon(y) = 3/(2e) + \varepsilon$ for $y \leq y_*$ and $\psi_\varepsilon(y) = (e/2)^{1-2y}y^2$ for $y > y_*$, where $y_* = y_*(\varepsilon)$ satisfies

$$(e/2)^{1-2y_*}y_*^2 = 3/(2e) + \varepsilon. \tag{6.4.12}$$

We define $\hat{y} = y_*(0) \approx 0.819$, and note that $y_*(\varepsilon) \downarrow \hat{y}$, as $\varepsilon \downarrow 0$.

The general scheme of the proof is as follows: First, we note that the case $i = k - q$ follows easily by Lemma 6.4.7, since $I_q^\ell(k, k-q)$ is equal to $\binom{k}{k-q}\binom{q}{2}^{k-q}C^\ell(q)$. We establish the remaining cases by induction, noting that if a graph G contributes to $I_q^\ell(k,i)$ and $i < k - q$, then there is a vertex $v \in V(G)$ of degree 2 with a neighbour not in the 3-core $C \subset G$. Therefore, either (i) some neighbour of v is of degree 2 in the subgraph of G induced by $V(G) - \{v\}$, or else (ii) there are vertices $u \neq w \in V(G)$ of degree 2 with a common neighbour not in C. This observation leads to an improved bound (when $q < k$) for $I_q^\ell(k,i)$ compared with that for $I^\ell(k,i)$ given by Lemma 6.4.7.

Proof. Let $\varepsilon > 0$ be given. We may assume that $\varepsilon < 1/(2e)$, as otherwise the statement of lemma follows by Lemma 6.4.7, noting that for any q, $I_q^\ell(k,i) \leq I^\ell(k,i)$. We claim that, for some $\vartheta(\varepsilon) \geq 1$ (to be determined below), for all $k \geq 2$, $\ell \leq 3$ and relevant q, i, we have that

$$I_q^\ell(k,i) \leq \vartheta\binom{k}{i}\psi_\varepsilon(q/k)^k k! k^{k+2\ell}. \tag{6.4.13}$$

Case 1 ($i = k - q$). We first observe that Lemma 6.4.7 implies the case $i = k - q$. Indeed, if $q = k$, in which case $i = 0$, then (6.4.13) follows immediately by Lemma 6.4.7, noting that $I_k^\ell(k,0) = C^\ell(k)$ and $\psi(1) = 2/e$.

On the other hand, if $i = k - q > 0$ then

$$I_q^\ell(k, k-q) = \binom{k}{k-q}\binom{q}{2}^{k-q} C^\ell(q),$$

since all $k-q$ vertices of degree 2 in a graph that contributes to $I_q^\ell(k, k-q)$ are connected to 2 vertices in its 3-core of size q. We claim that the right hand side is bounded by

$$\binom{k}{k-q}(e/2)^{k-2q}(q/k)^{2k}k!k^{k+2\ell}.$$

Since $(e/2)^{k-2q}(q/k)^{2k} \le \psi(q/k)^k$, (6.4.13) follows. To see this, note that by Lemma 6.4.7, we have that

$$\frac{\binom{q}{2}^{k-q}C^\ell(q)}{(e/2)^{k-2q}(q/k)^{2k}k!k^{k+2\ell}} \le \left(\frac{q}{k}\right)^{2\ell}\frac{q!}{(q/e)^q}\frac{(k/e)^k}{k!} \le \frac{q!}{(q/e)^q}\frac{(k/e)^k}{k!}.$$

By the inequalities $1 \le i!/(\sqrt{2\pi i}(i/e)^i) \le e^{1/(12i)}$, it is easy to verify that the right hand side above is bounded by 1, for all relevant $q \le k$. Hence (6.4.13) holds also in the case $i = k - q > 0$.

Case 2 ($i < k - q$). Fix some $k_\varepsilon \ge 1/(1-y_*)^2$ such that, for all $k \ge k_\varepsilon$ and relevant q, we have that

$$1 + \frac{2}{k-1}\left(\frac{k-2}{k-1}\right)^k \frac{\psi_\varepsilon(q/(k-2)^{k-2}}{\psi_\varepsilon(q/(k-1))^{k-1}} = 1 + O(1/k) \le 1 + \delta,$$

where

$$\delta = \delta(\varepsilon) = \min\left\{1 - \frac{3/(2e)}{3/(2e)+\varepsilon}, 1 - \frac{3(1-y_*)}{y_*^2}\right\}.$$

Note that, since $3(1-y)/y^2 < 1$ for all $y > (\sqrt{21}-3)/2 \approx 0.791$, and recalling (see (6.4.12)) that $y_* > \hat{y} \approx 0.819$, we have that $\delta > 0$.

Select $\vartheta(\varepsilon) \ge 1$ so that (6.4.13) holds for all $k \le k_\varepsilon$ and relevant q, ℓ, i. By Case 1 and since $\vartheta \ge 1$, we have that (6.4.13) holds for all k, q in the case that $i = k - q$. We establish the remaining cases $i < k - q$ by induction: Assume that for some $k > k_\varepsilon$, (6.4.13) holds for all $k' < k$ and relevant q, ℓ, i.

In any graph G contributing to $I_q^\ell(k,i)$, where $i < k-q$, there is some vertex of degree 2 with at least one of its two neighbours not in the 3-core of G. There are two cases to consider: either

(i) there is a vertex v of degree 2 such that at least one of its two neighbours is of degree 2 in the subgraph of G induced by $V(G) - \{v\}$, or else,

(ii) there is no such vertex v, however there are vertices $u \neq w$ of degree 2 with a common neighbour that is not in the 3-core of G.

Note that, in case (i), removing v results in a graph with $j \in \{i, i+1\}$ vertices of degree 2. On the other hand, in case (ii), removing u and w results in a graph with $j \in \{i-2, i-1, i\}$ vertices of degree 2. By considering the vertices v or u, w as above with minimal labels, we see that, for $i < k-q$, $I_q^\ell(k,i)/\binom{k}{i}$ is bounded by

$$\frac{I_q^\ell(k-1,i+1)}{\binom{k-1}{i+1}}\binom{k-i-q}{2} + \frac{I_q^\ell(k-1,i)}{\binom{k-1}{i}}(k-i-q)(k-i) + (k-i-q)(k-i)^2\sum_{j=0}^{2}\frac{I_q^\ell(k-2,i-2+j)}{\binom{k-2}{i-2+j}}.$$

Applying the inductive hypothesis, it follows that

$$\frac{I_q^\ell(k,i)}{\vartheta\binom{k}{i}\psi_\varepsilon(q/k)^k k! k^{k+2\ell}} \leq \Psi_\varepsilon(q,k)\left[1 + \frac{2}{k-1}\left(\frac{k-2}{k-1}\right)^k \frac{\psi_\varepsilon(q/(k-2)^{k-2}}{\psi_\varepsilon(q/(k-1))^{k-1}}\right]$$

where

$$\Psi_\varepsilon(q,k) = \frac{3}{2}\frac{k-q}{k}\left(\frac{k-1}{k}\right)^k \frac{\psi_\varepsilon(q/(k-1))^{k-1}}{\psi_\varepsilon(q/k)^k}.$$

By the choice of k_ε, and since $k \geq k_\varepsilon$, we have that

$$\frac{I_q^\ell(k,i)}{\vartheta\binom{k}{i}\psi_\varepsilon(q/k)^k k! k^{k+2\ell}} \leq \Psi_\varepsilon(q,k)(1+\delta). \tag{6.4.14}$$

Next, we show that $\Psi_\varepsilon(q,k) < 1-\delta$, completing the induction. To this end, we take cases with respect to whether (i) $q/(k-1) \leq y_*$, (ii) $y_* \leq q/k$, or (iii) $q/k < y_* < q/(k-1)$.

Case 2(i) ($q/(k-1) \leq y_*$). In this case $\psi_\varepsilon(q/m) = 3/(2e) + \varepsilon$, for each

$m \in \{k-1, k\}$. It follows, by the choice of δ, that

$$\Psi_\varepsilon(q,k) \le \left(\frac{k-1}{k}\right)^k \frac{3/2}{3/(2e)+\varepsilon} \le \frac{3/(2e)}{3/(2e)+\varepsilon} < 1-\delta,$$

as required.

Case 2(ii) $(y_* \le q/k)$. In this case $\psi(q/m)^m = (e/2)^{m-2q}(q/m)^{2m}$, for each $m \in \{k-1, k\}$. Hence

$$\Psi_\varepsilon(q,k) = \frac{3}{e}\left(\frac{k}{k-1}\right)^{k-1} \frac{(k-q)(k-1)}{q^2} \le \frac{3(1-y)}{y^2},$$

where $y = q/k$. Since the right hand side is decreasing in y, we find, by the choice of δ, that

$$\Psi_\varepsilon(q,k) \le \frac{3(1-y_*)}{y_*^2} < 1-\delta.$$

Case 2(iii) $(q/k < y_* < q/(k-1))$. In this case, $\psi_\varepsilon(q/k) = 3/(2e)+\varepsilon$ and

$$\psi_\varepsilon(q/(k-1))^{k-1} = (e/2)^{k-1-2q}(q/(k-1))^{2(k-1)}.$$

Hence

$$\Psi_\varepsilon(q,k) = \frac{3}{e}\left(\frac{k}{k-1}\right)^{k-1} \frac{(k-q)(k-1)}{q^2} \frac{(e/2)^{k-2q}(q/k)^{2k}}{(3/(2e)+\varepsilon)^k}.$$

As in the previous case, we consider the quantity $y = q/k$. The above expression is bounded by

$$\frac{3(1-y)}{y^2}\left(\frac{(e/2)^{1-2y}y^2}{3/(2e)+\varepsilon}\right)^k.$$

We claim that this expression is increasing in $y \le y_*$. By (6.4.12) and the choice of δ, it follows that

$$\Psi_\varepsilon(q,k) \le \frac{3(1-y_*)}{y_*^2} < 1-\delta,$$

as required. To establish the claim, simply note that

$$\begin{aligned}\frac{\partial}{\partial y}\frac{1-y}{y^2}((2/e)^y y)^{2k} &= \frac{1}{y^3}((2/e)^y y)^{2k}\,(2(1-y)(1+y\log(2/e))k+y-2)\\ &> \frac{2}{y^3}((2/e)^y y)^{2k}((1-y)^2k-1) \geq 0\end{aligned}$$

for all $y \leq y_*$, since $k \geq k_\varepsilon \geq 1/(1-y_*)^2$.

Altogether, we conclude that $\Psi_\varepsilon(q,k) \leq 1-\delta$, for all relevant q. By (6.4.14), it follows that

$$I_q^\ell(k,i) \leq (1-\delta^2)\vartheta\binom{k}{i}\psi_\varepsilon(q/k)^k k!k^{k+2\ell} < \vartheta\binom{k}{i}\psi_\varepsilon(q/k)^k k!k^{k+2\ell},$$

completing the induction. We conclude that (6.4.13) holds for $k \geq 2$, $\ell \leq 3$ and relevant q, i. Since $\binom{k}{i} \leq k^i$, the lemma follows. ■

6.5 Percolating subgraphs with small cores

With Lemmas 6.2.4, 6.4.7 and 6.4.8 at hand, we begin to analyze percolating subgraphs of $\mathcal{G}_{n,p}$. In this section, we show that for sub-critical p, with high probability $\mathcal{G}_{n,p}$ has no percolating subgraphs larger that $(\beta_* + o(1))\log n$ with a small 3-core. The non-existence of large percolating 3-cores is verified in the next Section 6.6, completing the proof of Proposition 6.2.3. More specifically, we prove the following result.

Proposition 6.5.1. *Let $p = \sqrt{\alpha/(n\log n)}$, for some $\alpha \in (0, 1/3)$. Then, for any $\delta > 0$, with high probability $\mathcal{G}_{n,p}$ has no irreducible percolating subgraph G of size $k = \beta\log n$ with a 3-core $C \subset G$ of size $q \leq (3/2)\log n$, for any $\beta \geq (\beta_* + \delta)\log n$.*

Recall that (as discussed in Section 6.4.1), in statements such as this proposition, we mean also to include the possibility that $q = 2$ (corresponding to a seed graph G) when we say that the 3-core of a graph G is of size less than $q > 2$.

First, we justify the definition of β_* in Theorem 6.2.2.

Lemma 6.5.2. *Fix $\alpha \in (0, 1/3)$. For $\beta > 0$, let*

$$\mu(\alpha, \beta) = 3/2 + \beta \log(\alpha\beta) - \alpha\beta^2/2.$$

The function $\mu(\alpha, \beta)$ is decreasing in β, with a unique zero $\beta_*(\alpha) \in (0, 3)$.

In particular, for $\alpha \in (0, 1/3)$, we have that $\beta_* \leq 1/\alpha$.

Proof. Differentiating $\mu(\alpha, \beta)$ with respect to β, we obtain $1 + \log(\alpha\beta) - \alpha\beta$. Since $\log x < x - 1$ for all positive $x \neq 1$, we find that $\mu(\alpha, \beta)$ is decreasing in β. Moreover, since $\alpha < 1/3$, we have that $\mu(\alpha, 3) < (3/2)(3\alpha - 1) < 0$. The result follows, noting that $\mu(\alpha, \beta) \to 3/2 > 0$ as $\beta \downarrow 0$. ■

Recall that the bounds in Sections 6.4.1 and 6.4.2 apply only to graphs with an excess of $\ell \leq 3$ edges. The following observation is useful for dealing with graphs with a larger excess.

Lemma 6.5.3. *Let $p = \sqrt{\alpha/(n \log n)}$, for some $\alpha \in (0, 1/3)$. Then with high probability $\mathcal{G}_{n,p}$ contains no subgraph of size $k = \beta \log n$ with an excess of ℓ edges, for any $\beta \in (0, 2]$ and $\ell > 3$, or any $\beta \in (0, 9]$ and $\ell > 27$.*

Proof. The expected number of subgraphs of size $k = \beta \log n$ in $\mathcal{G}_{n,p}$ with an excess of ℓ edges is bounded by

$$\binom{n}{k}\binom{\binom{k}{2}}{2k-3+\ell} p^{2k-3+\ell} \leq \left(\frac{e^3}{16} knp^2\right)^k \left(\frac{e}{4} kp\right)^{\ell-3} \leq n^\nu \log^\ell n$$

where

$$\nu(\beta, \ell) = -(\ell - 3)/2 + \beta \log(\alpha\beta e^3/16).$$

Note that ν is convex in β and $\nu(\beta, \ell) \to -(\ell - 3)/2$ as $\beta \downarrow 0$. Note also that

$$2 \log(2/3 \cdot e^3/16) \approx -0.356 < 0$$

and

$$9 \log(9/3 \cdot e^3/16) \approx 11.934 < 12.$$

Therefore, since $\alpha < 1/3$, $\nu(2,\ell) < -(\ell-3)/2$ and $\nu(9,\ell) < -(\ell-27)/2$. Hence, the first claim follows by summing over all $k \le 2\log n$ and $\ell > 3$. The second claim follows, summing over all $k \le 9\log n$ and $\ell > 27$. ■

Definition 6.5.4. Let $E(q,k)$ denote the expected number of irreducible percolating 3-cores $C \subset \mathcal{G}_{n,p}$ of size q (or seed edges, if $q = 2$), such that $|\langle C, \mathcal{G}_{n,p}\rangle_2| \ge k$.

Combining Lemmas 6.2.4, 6.4.7 and 6.5.3, we obtain the following estimate. Recall $\beta_\varepsilon, k_\alpha, q_\alpha$ as defined in Lemma 6.2.4, and μ as defined in Lemma 6.5.2.

Lemma 6.5.5. *Let $p = \sqrt{\alpha/(n\log n)}$, for some $\alpha \in (0,1/3)$. Let $\varepsilon \in [0,3\alpha]$ and $\beta \in [\beta_\varepsilon, 1/\alpha]$. Suppose that $q/q_\alpha \to \varepsilon$ and $k/k_\alpha \to \alpha\beta$ as $n \to \infty$. Then $E(q,k) \le n^{\mu_\varepsilon + o(1)}$, where $\mu_\varepsilon(\alpha,\beta) = \mu(\alpha,\beta)$ for $\beta \in [\varepsilon/\alpha, 1/\alpha]$,*

$$\mu_\varepsilon(\alpha,\beta) = \mu(\alpha,\beta) - \beta\log(\alpha\beta) + \frac{\varepsilon}{2\alpha}\log(\varepsilon/e) + \frac{2\alpha\beta-\varepsilon}{2\alpha}\log\left(\frac{e(\alpha\beta)^2}{2\alpha\beta-\varepsilon}\right)$$

for $\beta \in [\beta_\varepsilon, \varepsilon/\alpha]$, and $o(1)$ depends only on n.

We note that $\mu_\varepsilon(\alpha,\varepsilon/\alpha) = \mu(\alpha,\varepsilon/\alpha)$, as is easily verified.

Proof. By the proof of Lemma 6.5.3, the expected number of irreducible percolating 3-cores in $\mathcal{G}_{n,p}$ of size $q \le (3/2)\log n$ with an excess of $\ell > 3$ edges tends to 0 as $n \to \infty$. Therefore, it suffices to show that, for all $\ell \le 3$, we have that $E^\ell(q,k) \le n^{\mu_\varepsilon + o(1)}$, where $E^\ell(q,k)$ is the expected number of irreducible percolating 3-cores $C \subset \mathcal{G}_{n,p}$ of size $q = \varepsilon(2\alpha)^{-1}\log n$ with an excess of ℓ edges, such that $|\langle C, \mathcal{G}_{n,p}\rangle_2| \ge k = \beta\log n$. For such ℓ, by Lemmas 6.2.4 and 6.4.7, we find that

$$\begin{aligned} E^\ell(q,k) &\le \binom{n}{q} C^\ell(q) p^{2q-3+\ell} P(q,k) \\ &\le q^{2\ell} p^{\ell-3} \left(\frac{2}{e} qnp^2\right)^q P(q,k) \le n^{\nu + o(1)} \end{aligned}$$

where

$$\nu = 3/2 + \varepsilon(2\alpha)^{-1}\log(\varepsilon/e) + \xi_\varepsilon(\alpha,\beta) = \mu_\varepsilon(\alpha,\beta)$$

(and $P(q,k)$ and ξ_ε are as in Lemma 6.2.4), as required. ■

Having established Lemma 6.5.5, we aim to prove Proposition 6.5.1 by the first moment method. To this end, we first show that for some $\varepsilon_* \in (0,3\alpha)$, with high probability there are no irreducible percolating 3-cores in $\mathcal{G}_{n,p}$ of size $\varepsilon(2\alpha)^{-1}\log n$, for all $\varepsilon \in (\varepsilon_*, 3\alpha]$. Moreover, we establish a slightly more general result that allows for graphs with $i = O(1)$ vertices of degree 2, which is also used in the next Section 6.6.

Lemma 6.5.6. *Let $p = \sqrt{\alpha/(n\log n)}$, for some $\alpha \in (0,1/3)$. Fix some $i_* \geq 0$. Define $\varepsilon_*(\alpha) \in (0,3\alpha)$ implicitly by $3/2 + \varepsilon_*(2\alpha)^{-1}\log(\varepsilon_*/e) = 0$. Then, for any $\eta > 0$, with high probability $\mathcal{G}_{n,p}$ has no irreducible percolating subgraph G of size $q = \varepsilon(2\alpha)^{-1}\log n$ with i vertices of degree 2, for any $i \leq i_*$ and $\varepsilon \in [\varepsilon_* + \eta, 3\alpha]$.*

Proof. By Lemma 6.5.3, it suffices to consider subgraphs G with an excess of $\ell \leq 3$ edges. By Lemma 6.4.7, the expected number of such subgraphs is bounded by

$$\binom{n}{q} p^{2q-3+\ell} I^\ell(q,i) \leq k^{2\ell+i} p^{\ell-3} \left(\frac{2}{e} knp^2\right)^q \leq n^{\nu+o(1)}$$

where $\nu(\varepsilon) = 3/2 + \varepsilon(2\alpha)^{-1}\log(\varepsilon/e)$. Noting that ν is decreasing in $\varepsilon < 1$, $\nu(\varepsilon) \to 3/2 > 0$ as $\varepsilon \downarrow 0$, and $\nu(3\alpha) = (3/2)\log(3\alpha) < 0$, the lemma follows. ■

Next, we plan to use Lemma 6.5.5 to rule out the remaining cases $\varepsilon \leq \varepsilon_* + \eta$ (where $\eta > 0$ is a small constant, to be determined below). In order to apply Lemma 6.5.5, we first verify that for such ε, we have that β_* is within the range of β specified by Lemma 6.5.5, that is, $\beta_* \geq \beta_\varepsilon$.

Lemma 6.5.7. *Fix $\alpha \in (0,1/3)$. Let $\beta_\varepsilon, \beta_*, \varepsilon_*$ be as defined in Lemmas 6.2.4, 6.5.2 and 6.5.6. Then, for some sufficiently small $\eta(\alpha) > 0$, we have that $\beta_* \geq \beta_\varepsilon$ for all $\varepsilon \in [0, \varepsilon_* + \eta]$.*

Proof. By Lemma 6.5.2 and the continuity of $\mu(\alpha,\beta_\varepsilon)$ in ε, it suffices to show that $\mu(\alpha,\beta_\varepsilon) > 0$, for all $\varepsilon \in [0,\varepsilon_*]$. Let $\delta_\varepsilon = 1 - \sqrt{1-\varepsilon}$, so that $\beta_\varepsilon = \delta_\varepsilon/\alpha$.

Note that

$$\mu(\alpha, \beta_\varepsilon) = 3/2 + (2\alpha)^{-1}(2\delta_\varepsilon \log \delta_\varepsilon - \delta_\varepsilon^2).$$

Therefore, by the bound $\log x \le x - 1$,

$$\frac{\partial}{\partial \varepsilon}\mu(\alpha, \beta_\varepsilon) = (2\alpha)^{-1}(1 + \log(\delta_\varepsilon)/(1 - \delta_\varepsilon)) \le 0.$$

It thus suffices to verify that $\mu(\alpha, \beta_{\varepsilon_*}) > 0$. To this end note that, by the definition of ε_* (see Lemma 6.5.6),

$$\mu(\alpha, \beta_{\varepsilon_*}) = (2\alpha)^{-1}(2\delta_{\varepsilon_*} \log \delta_{\varepsilon_*} - \delta_{\varepsilon_*}^2 - \varepsilon_* \log(\varepsilon_*/e)).$$

By Lemma 6.5.6, we have that $\varepsilon_* = \delta_{\varepsilon_*}(2 - \delta_{\varepsilon_*}) \in (0, 1)$, and so $\delta_{\varepsilon_*} \in (0, 1)$. Hence the lemma follows if we show that $\nu(\delta) > 0$ for all $\delta \in (0, 1)$, where

$$\nu(\delta) = 2\delta \log \delta - \delta^2 - \delta(2 - \delta) \log(\delta(2 - \delta)/e).$$

Note that

$$\nu(\delta)/\delta = \delta \log \delta - (2 - \delta) \log(2 - \delta) + 2(1 - \delta).$$

Differentiating this expression with respect to δ, we obtain $\log(\delta(2 - \delta)) < 0$, for all $\delta < 1$. Noting that $\nu(1) = 0$, the lemma follows. ■

It can be shown that for all sufficiently large $\varepsilon < \varepsilon_*$, we have that $\beta_* < \varepsilon/\alpha$. Therefore, we require the following bound.

Lemma 6.5.8. *Fix $\alpha \in (0, 1/3)$. Let $\varepsilon \in [0, 1)$ and $\beta_\varepsilon, \mu_\varepsilon$ be as defined in Lemmas 6.2.4 and 6.5.5. Then $\mu_\varepsilon(\alpha, \beta) \le \mu(\alpha, \beta)$, for all $\beta \in [\beta_\varepsilon, 1/\alpha]$.*

Proof. Since $\mu(\alpha, \beta) = \mu_\varepsilon(\alpha, \beta)$ for $\beta \in [\varepsilon/\alpha, 1/\alpha]$, we may assume that $\beta < \varepsilon/\alpha$. Let $\delta = \alpha\beta$. Then

$$\alpha(\mu(\alpha, \beta) - \mu_\varepsilon(\alpha, \beta)) = \delta \log \delta - \frac{\varepsilon}{2} \log(\varepsilon/e) - \frac{2\delta - \varepsilon}{2} \log\left(\frac{e\delta^2}{2\delta - \varepsilon}\right).$$

Differentiating this expression with respect to δ, we obtain

$$\varepsilon/\delta - 1 - \log(\delta/(2\delta - \varepsilon)) \le 0,$$

by the inequality $\log x \geq (x-1)/x$. Since $\mu(\alpha, \varepsilon/\alpha) = \mu_\varepsilon(\alpha, \varepsilon/\alpha)$, the lemma follows. ■

Finally, we prove the main result of this section.

of Proposition 6.5.1. Let $\delta > 0$ be given. By Lemma 6.5.2, we may assume without loss of generality that $\beta_* + \delta < 1/\alpha$. If $\mathcal{G}_{n,p}$ has an irreducible percolating subgraph G of size $k \geq (\beta_* + \delta)\log n$ with a 3-core of size $q \leq (3/2)\log n$, then by Lemma 6.4.2 it has such a subgraph of size $k = \beta \log n$ for some $\beta \in [\beta_* + \delta, 1/\alpha]$. Select $\eta(\alpha) > 0$ as in Lemma 6.5.7. By Lemma 6.5.6, with high probability $\mathcal{G}_{n,p}$ has no percolating 3-core of size $q = \varepsilon(2\alpha)^{-1}\log n$, for any $\varepsilon \in [\varepsilon_* + \eta, 3\alpha]$. On the other hand, by the choice of η, Lemmas 6.5.5, 6.5.7 and 6.5.8 imply that for any $\beta \in [\beta_*, 1/\alpha]$, the expected number of irreducible percolating subgraphs of size $k = \beta \log n$ with a 3-core of size $q \leq (\varepsilon_* + \eta)(2\alpha)^{-1}\log n$ is bounded by $n^{\mu + o(1)}$, where $\mu = \mu(\alpha, \beta)$. Hence the result follows by Lemma 6.5.2, summing over the $O(\log^2 n)$ possibilities for k, q. ■

6.6 No percolating subgraphs with large cores

In the previous Section 6.5, it is shown that for sub-critical p, with high probability $\mathcal{G}_{n,p}$ has no percolating subgraphs larger than $(\beta_* + o(1))\log n$ with a 3-core smaller than $(3/2)\log n$. In this section, we rule out the existence of larger percolating 3-cores.

Proposition 6.6.1. *Let $p = \sqrt{\alpha/(n \log n)}$, for some $\alpha \in (0, 1/3)$. Then with high probability $\mathcal{G}_{n,p}$ has no irreducible percolating 3-core C of size $k = \beta \log n$, for any $\beta \in [3/2, 9]$.*

Before proving the proposition we observe that it together with Proposition 6.5.1 implies Proposition 6.2.3. As discussed in Section 6.2.2, our main Theorems 6.2.1 and 6.2.2 follow.

of Proposition 6.2.3. Let $\delta > 0$ be given. By Lemma 6.5.2, without loss of generality we may assume that $\beta_* + \delta < 3$. Hence, by Lemmas 6.3.7

and 6.4.3, if $\mathcal{G}_{n,p}$ has a percolating subgraph that is larger than $(\beta_* + \delta)\log n$, then with high probability it has some irreducible percolating subgraph G of size $k = \beta \log n$ with a 3-core $C \subset G$ of size $q \le k$ (or a seed edge, if $q = 2$), for some $\beta \in (\beta_* + \delta, 9]$. By Proposition 6.6.1, with high probability $q \le (3/2)\log n$. Hence, by Proposition 6.5.1, with high probability $\mathcal{G}_{n,p}$ contains no such subgraph G. Therefore, with high probability, all percolating subgraphs of $\mathcal{G}_{n,p}$ are of size $k \le (\beta_* + \delta)\log n$. ∎

Towards Proposition 6.6.1, we observe that $\mathcal{G}_{n,p}$ has no percolating subgraph with a small 3-core and few vertices of degree 2.

Lemma 6.6.2. *Let $p = \sqrt{\alpha/(n \log n)}$, for some $\alpha \in (0, 1/3)$. Fix some $i_* \ge 1$. With high probability $\mathcal{G}_{n,p}$ has no irreducible percolating subgraph G of size $k \ge (3/2)\log n$ with a 3-core $C \subset G$ of size $q \le (3/2)\log n$ and $i \le i_*$ vertices of degree 2.*

This is a straightforward consequence of Lemma 6.4.8, proved by a direct application of the first moment method and elementary calculus.

Proof. By Lemmas 6.4.3 and 6.5.3, we may assume that if $\mathcal{G}_{n,p}$ has an irreducible percolating subgraph G of size $k = \beta \log n$ with a 3-core of size $q \le (3/2)\log n$, then G has an excess of $\ell \le 3$ edges. By Proposition 6.5.1 and Lemmas 6.5.2 and 6.5.6, we may further assume that $\beta \in [3/2, 3]$ and $q = yk$, where $y\beta \in [0, 3/2 - \varepsilon]$, for some $\varepsilon(\alpha) > 0$. Without loss of generality, we assume that $\varepsilon < 1/(2e)$ and $\log(3/(2e) + \varepsilon) < -1/2$ (which is possible, since $1 + 2\log(3/(2e)) \approx -0.189 < 0$). By Lemma 6.4.8 and since $\alpha < 1/3$, for some constant $\vartheta(\varepsilon) \ge 1$, the expected number of such subgraphs G is bounded by

$$\binom{n}{k} p^{2k-3+\ell} I_q^\ell(k,i) \le \vartheta k^{2\ell+i} p^{\ell-3} (knp^2 \psi_\varepsilon(q/k))^k \ll n^\nu \tag{6.6.1}$$

where

$$\nu(\beta, \psi_\varepsilon(y)) = 3/2 + \beta \log(\beta/3) + \beta \log \psi_\varepsilon(y)$$

and $\psi_\varepsilon(y)$ is as in Lemma 6.4.8, that is,

$$\psi_\varepsilon(y) = \max\{3/(2e) + \varepsilon, (e/2)^{1-2y}y^2\}.$$

Recall that $\psi_\varepsilon(y) = 3/(2e) + \varepsilon$ for $y \le y_*$ and $\psi_\varepsilon(y) = (e/2)^{1-2y}y^2$ for $y > y_*$, where $y_* = y_*(\varepsilon)$ is as defined by (6.4.12). Moreover, $y_* \downarrow \hat{y}$ as $\varepsilon \downarrow 0$, where $\hat{y} \approx 0.819$.

Therefore, to verify that with high probability $\mathcal{G}_{n,p}$ has no subgraphs G as in the lemma, we show that $\nu(\beta, \psi_\varepsilon(y)) < -\delta$ for some $\delta > 0$ and all β, y as above. Moreover, since ν is convex in β, it suffices to consider the extreme points $\beta = 3/2$ and $\beta = \min\{3, 3/(2y)\}$ in the range $y \in [0, 1-\varepsilon']$, where $\varepsilon' = 2\varepsilon/3$.

Since $\psi_\varepsilon(1) = 2/e$, we have that $\nu(3/2, \psi_\varepsilon(1)) = 0$. Hence, for some $\delta_1 > 0$, we have that $\nu(3/2, \psi_\varepsilon(y)) < -\delta_1$ for all $y \in [0, 1-\varepsilon']$. Next, for $\beta = \min\{3, 3/(2y)\}$, we treat the cases (i) $y \in [0, 1/2]$ and $\beta = 3$ and (ii) $y \in [1/2, 1-\varepsilon']$ and $\beta = 3/(2y)$ separately. If $y \le 1/2$, then $\psi_\varepsilon(y) = 3/(2e)+\varepsilon$, in which case, by the choice of ε,

$$\nu(3, \psi_\varepsilon(y)) = \frac{3}{2}(1 + 2\log(3/(2e) + \varepsilon)) < 0.$$

On the other hand, for $y \ge 1/2$, we need to show that

$$\nu(3/(2y), \psi_\varepsilon(y)) = \frac{3}{2}\left(1 + \frac{1}{y}\log\left(\frac{\psi_\varepsilon(y)}{2y}\right)\right) < 0.$$

To this end, we first note that differentiating $\nu(3/(2y), 3/(2e)+\varepsilon)$ twice with respect to y, we obtain

$$\frac{3}{2y^3}\left(3 + 2\log\left(\frac{3/(2e)+\varepsilon}{2y}\right)\right) \ge \frac{3}{2}\left(3 + 2\log\left(\frac{3}{4e}\right)\right) \approx 0.637 > 0.$$

Therefore it suffices to consider the extreme points $y = 1/2$ and $y = 1$. Noting that, by the choice of ε, we have that

$$\nu(3, 3/(2e)) = \frac{3}{2}(1 + 2\log(3/(2e) + \varepsilon)) < 0$$

and

$$\begin{aligned}\nu(3/2, 3/(2e)) &= \frac{3}{2}\left(1 + \log\left(\frac{3/(2e) + \varepsilon}{2}\right)\right) \\ &< \frac{3}{2}(1 + 2\log(3/(2e) + \varepsilon)) < 0,\end{aligned}$$

it follows that $\nu(3/(2y), 3/(2e) + \varepsilon) < 0$ for all $y \in [1/2, 1]$. Next, we observe that differentiating $\nu(3/(2y), (e/2)^{1-2y}y^2)$ with respect to y, we obtain

$$\frac{3}{2y^2}\left(1 - \log(ey/4)\right) \geq 3\log 2 > 0.$$

Therefore, since $\nu(3/(2y), (e/2)^{1-2y}y^2) \to \nu(3/2, \psi_\varepsilon(1)) = 0$ as $y \uparrow 1$, it follows that $\nu(3/(2y), (e/2)^{1-2y}y^2) < 0$ for all $y \in [1/2, 1 - \varepsilon']$. Altogether, there is some $\delta_2 > 0$ so that $\nu(\min\{3, 3/(2y)\}, \psi_\varepsilon(y)) < -\delta_2$ for all $y \in [0, 1 - \varepsilon']$.

Put $\delta = \min\{\delta_1, \delta_2\}$. We conclude that $\nu(\beta, \psi_\varepsilon(y)) < -\delta$, for all relevant β, y. The lemma follows by (6.6.1), summing over the $O(\log^2 n)$ choices for k and q and $O(1)$ relevant values $\ell \leq 3$ and $i \leq i_*$. ■

With Lemma 6.6.2 at hand, we turn to Proposition 6.6.1. The general idea is as follows: Suppose that $\mathcal{G}_{n,p}$ has an irreducible percolating 3-core C of size $k = \beta \log n$, for some $\beta \in [3/2, 9]$. By Lemma 6.5.3, we can assume that the excess of C is $\ell \leq 27$ edges. Hence, in the last step of a clique process for C, either 2 or 3 percolating subgraphs are merged that have few vertices of degree 2 (as observed above the proof of Lemma 6.4.7, in Section 6.4.1). Therefore, by Lemma 6.6.2, each of these subgraphs is smaller than $(3/2)\log n$, or else has a 3-core larger than $(3/2)\log n$. In this way, we see that considering a *minimal* such graph C is the key to proving Proposition 6.6.1. By Lemma 6.5.6, there is some $\beta_1 < 3/2$ so that with high probability $\mathcal{G}_{n,p}$ has no percolating subgraph of size $\beta \log n$ with few vertices of degree 2, for all $\beta \in [\beta_1, 3/2]$. Hence such a graph C, if it exists, is the result of the unlikely event that 2 or 3 percolating graphs, all of which are smaller than $\beta_1 \log n$ and have few vertices of degree 2, are merged to form a percolating 3-core that is larger than $(3/2)\log n$. In other words, "macroscopic" subgraphs are merged to form C.

of Proposition 6.6.1. By Lemma 6.5.6, there is some $\beta_1 < 3/2$ so that with high probability $\mathcal{G}_{n,p}$ has no percolating subgraph of size $\beta \log n$ with i vertices of degree 2, for any $i \leq 15$ and $\beta \in [\beta_1, 3/2]$.

Suppose that $\mathcal{G}_{n,p}$ has an irreducible 3-core C of size $k = \beta \log n$ with an excess of ℓ edges, for some $\beta \in [3/2, 9]$. By Lemma 6.5.3, we may assume that $\ell \leq 27$. Moreover, assume that C is of the minimal size among such subgraphs of $\mathcal{G}_{n,p}$. By Lemma 6.3.6, there are two possibilities for the last step of a clique process for C:

(i) Three irreducible percolating subgraphs G_j, $j \in \{1,2,3\}$, are merged which form exactly one triangle $T = \{v_1, v_2, v_3\}$, such that for some $i_j \leq 2$ and $k_j, \ell_j \geq 0$ with $\sum k_j = k+3$ and $\sum \ell_j = \ell$, we have that the G_j contribute to $I^{\ell_j}(k_j, i_j)$. If any $i_j > 0$, the corresponding i_j vertices of G_j of degree 2 belong to T.

(ii) For some $m \leq (\ell+3)/2 \leq 15$, two percolating subgraphs G_j, $j \in \{1,2\}$, are merged that share exactly m vertices $S = \{v_1, v_2, \ldots, v_m\}$, such that for some $i_j \leq m$ and $k_j, \ell_j \geq 0$ with $\sum k_j = k+m$ and $\sum \ell_j = \ell - (2m-3)$, we have that the G_j contribute to $I^{\ell_j}(k_j, i_j)$. If any $i_j > 0$, the corresponding i_j vertices of G_j of degree 2 belong to S.

Moreover, in either case, by the choice of C, each G_j is either a seed graph or else has a 3-core smaller than $(3/2)\log n$. Hence, by Lemmas 6.4.3 and 6.5.3, we may assume that each $\ell_j \leq 3$. Also, by Lemma 6.6.2 and the choice of β_1, we may further assume that all G_j are smaller than $\beta_1 \log n$.

Case (i). Let k, k_j, ℓ_j be as in (i). Let $k_j - (j-1) = \varepsilon_j k$, so that $\sum \varepsilon_j = 1$. Without loss of generality we assume that $k_1 \geq k_2 \geq k_3$. Hence $\varepsilon_1, \varepsilon_2$ satisfy $1/3 \leq \varepsilon_1 \leq \beta_1/\beta < 1$ and $(1-\varepsilon_1)/2 \leq \varepsilon_2 \leq \min\{\varepsilon_1, 1-\varepsilon_1\}$. The number of 3-cores C as in (i) for these values k, k_j, ℓ_j is bounded by

$$\binom{k}{k_1, k_2-1, k_3-2}\binom{k_1}{2}\binom{k_2-1}{1} 2!^3 \prod_{j=1}^{3}\sum_{i=0}^{2}\binom{2}{i}\frac{I^{\ell_j}(k_j,i)}{\binom{k_j}{i}}.$$

Applying Lemma 6.4.7 and the inequality $k! < ek(k/e)^k$ (and recalling

$\ell_j \le 3$), this is bounded by

$$\binom{k}{k-k_1}\binom{k-k_1}{k_3-2}\frac{k^3}{2}(8ek^7)^3\left(\frac{2}{e^2}\right)^{k+3}\prod_{j=1}^{3}k_j^{2k_j}.$$

By the inequality $\binom{n}{k} < (ne/k)^k$, and noting that

$$k_j^{2k_j} \le (ek)^{2(j-1)}(k_j-(j-1))^{2(k_j-(j-1))},$$

we see that the above expression is bounded by $(2e^{-2}\eta(\varepsilon_1,\varepsilon_2))^k k^{2k} n^{o(1)}$, where

$$\begin{aligned}\eta(\varepsilon_1,\varepsilon_2) &= \left(\frac{e}{1-\varepsilon_1}\right)^{1-\varepsilon_1}\left(\frac{(1-\varepsilon_1)e}{\varepsilon_3}\right)^{\varepsilon_3}\varepsilon_1^{2\varepsilon_1}\varepsilon_2^{2\varepsilon_2}\varepsilon_3^{2\varepsilon_3}\\ &= \frac{e^{1-\varepsilon_1+\varepsilon_3}}{(1-\varepsilon_1)^{\varepsilon_2}}\varepsilon_1^{2\varepsilon_1}\varepsilon_2^{2\varepsilon_2}\varepsilon_3^{\varepsilon_3}.\end{aligned}$$

Since there are $O(1)$ choices for ℓ and the ℓ_j, and since $\alpha < 1/3$, the expected number of 3-cores C in $\mathcal{G}_{n,p}$ of size $k = \beta \log n$ with G_j of size k_j as in (i) is at most

$$\binom{n}{k}p^{2k-3}\left(\frac{2}{e^2}\eta(\varepsilon_1,\varepsilon_2)k^2\right)^k n^{o(1)} = p^{-3}\left(\frac{2}{e}\alpha\beta\eta(\varepsilon_1,\varepsilon_2)\right)^k n^{o(1)} \ll n^\nu \tag{6.6.2}$$

where

$$\nu(\beta,\varepsilon_1,\varepsilon_2) = \frac{3}{2} + \beta\log\left(\frac{2}{3e}\beta\eta(\varepsilon_1,\varepsilon_2)\right).$$

Since there are $O(\log^3 n)$ possibilities for k and the k_j, to show that with high probability $\mathcal{G}_{n,p}$ has no subgraphs C as in (i), it suffices to show that for some $\delta > 0$, we have that $\nu(\beta,\varepsilon_1,\varepsilon_2) < -\delta$ for all relevant β and ε_j. Moreover, since ν is convex in β, we can restrict to the extreme points $\beta = 3/2$ and $\beta = 3/(2\varepsilon_1) > \beta_1/\varepsilon_1$. To this end, observe that when $\beta = 3/2$, we have that $\nu < 0$ if and only if $\eta < 1$. Similarly, when $\beta = 3/(2\varepsilon_1)$, $\nu < 0$ if and only if $\eta < \varepsilon_1 e^{1-\varepsilon_1}$. Since $\varepsilon_1 e^{1-\varepsilon_1} \le 1$ for all relevant ε_1, it suffices to establish the latter claim. To this end, we observe that

$$\begin{aligned}\frac{\partial}{\partial \varepsilon_2}\eta(\varepsilon_1,\varepsilon_2) &= \eta(\varepsilon_1,\varepsilon_2)\log\left(\frac{e\varepsilon_2^2}{(1-\varepsilon_1)(1-\varepsilon_1-\varepsilon_2)}\right)\\ &\geq \eta(\varepsilon_1,\varepsilon_2)\log(e/2) > 0\end{aligned}$$

for all relevant $\varepsilon_2 \geq (1-\varepsilon_1)/2$. Therefore, we need only show that

$$\zeta(\varepsilon_1) = \frac{\eta(\varepsilon_1, \min\{\varepsilon_1, 1-\varepsilon_1\})}{\varepsilon_1 e^{1-\varepsilon_1}} < 1-\delta$$

for some $\delta > 0$ and all relevant ε_1. We treat the cases $\varepsilon_1 \in [1/3, 1/2]$ and $\varepsilon_1 \in [1/2, 1)$ separately.

For $\varepsilon_1 \in [1/3, 1/2]$, we have

$$\zeta(\varepsilon_1) = \frac{\eta(\varepsilon_1,\varepsilon_1)}{\varepsilon_1 e^{1-\varepsilon_1}} = \frac{(e(1-2\varepsilon_1))^{1-2\varepsilon_1}\varepsilon_1^{4\varepsilon_1-1}}{(1-\varepsilon_1)^{\varepsilon_1}}.$$

Hence

$$\frac{\partial}{\partial \varepsilon_1}\zeta(\varepsilon_1) = \zeta(\varepsilon_1)\left(\log\left(\frac{\varepsilon_1^4}{(1-\varepsilon_1)(1-2\varepsilon_1)^2}\right) + \frac{\varepsilon_1^2+\varepsilon_1-1}{\varepsilon_1(1-\varepsilon_1)}\right).$$

The terms $\varepsilon_1^4/((1-\varepsilon_1)(1-2\varepsilon_1)^2)$ and $(\varepsilon_1^2+\varepsilon_1-1)/(\varepsilon_1(1-\varepsilon_1))$ are increasing for $\varepsilon_1 \in [1/3, 1/2]$, as is easily verified. Hence $\zeta(\varepsilon_1)$ is decreasing in ε_1 for $1/3 \leq \varepsilon_1 \leq x_1 \approx 0.439$ and increasing for $x_1 \leq \varepsilon_1 \leq 1/2$. Therefore, since $\zeta(1/3) = (e/6)^{1/3} < 1$ and $\zeta(1/2) = 1/\sqrt{2} < 1$, we have that, for some $\delta_1 > 0$, $\zeta(\varepsilon_1) < 1-\delta_1$ for all $\varepsilon_1 \in [1/3, 1/2]$.

Similarly, for $\varepsilon \in [1/2, 1)$, we have

$$\zeta(\varepsilon_1) = \frac{\eta(\varepsilon_1, 1-\varepsilon_1)}{\varepsilon_1 e^{1-\varepsilon_1}} = (1-\varepsilon_1)^{1-\varepsilon_1}\varepsilon_1^{2\varepsilon_1-1}.$$

Hence

$$\frac{\partial}{\partial \varepsilon_1}\zeta(\varepsilon_1) = \zeta(\varepsilon_1)\left(\log\left(\frac{\varepsilon_1^2}{1-\varepsilon_1}\right) + \frac{\varepsilon_1-1}{\varepsilon_1}\right).$$

Since $\varepsilon_1^2/(1-\varepsilon_1)$ and $(\varepsilon_1-1)/\varepsilon_1$ are increasing in $\varepsilon_1 \in [1/2, 1)$, we find that $\zeta(\varepsilon_1)$ is decreasing in ε_1 for $1/2 \le \varepsilon_1 \le x_2 \approx 0.692$ and increasing for $x_2 \le \varepsilon_1 < 1$. Note that $\zeta(1/2) = 1/\sqrt{2} < 1$ and $\zeta(1) = 1$. Hence, for some $\delta_2 > 0$, $\zeta(\varepsilon_1) < 1 - \delta_2$ for all $\varepsilon_1 \in [1/2, \beta_1/\beta] \subset [1/2, 1)$.

Setting $\delta' = \min\{\delta_1, \delta_2\}$, we find that $\zeta(\varepsilon_1) < 1 - \delta'$ for all relevant ε_1. It follows that, for some $\delta > 0$, we have that $\nu(\beta, \varepsilon_1, \varepsilon_2) < -\delta$, for all relevant $\beta, \varepsilon_1, \varepsilon_2$. Summing over the $O(\log^3 n)$ possibilities for k, k_j and the $O(1)$ possibilities for ℓ, ℓ_j, we conclude by (6.6.2) that with high probability $\mathcal{G}_{n,p}$ has no 3-cores C as in (i).

Case (ii). Let k, k_j, ℓ_j, m be as in (ii). Let $k_1 = \varepsilon_1 k$ and $k_2 - m = \varepsilon_2 k$, so that $\sum \varepsilon_j = 1$. Without loss of generality we assume that $k_1 \ge k_2$. Hence $\varepsilon_1, \varepsilon_2$ satisfy $1/2 \le \varepsilon_1 \le \beta_1/\beta < 1$ and $\varepsilon_2 = 1 - \varepsilon_1$. The number of 3-cores C as in (ii) for these values k, k_j, ℓ_j, m is bounded by

$$\binom{k}{k_2 - m}\binom{k_1}{m} m!^2 \prod_{j=1}^{2} \sum_{i=0}^{m} \binom{m}{i} \frac{I^{\ell_j}(k_j, i)}{\binom{k_j}{i}}.$$

Arguing as in Case (i), by Lemma 6.4.7 and the inequality $k! < ek(k/e)^k$, we see that this is bounded by

$$\binom{k}{k_2 - m} \frac{k^m}{m!} (2^m m! e k^7)^2 \left(\frac{2}{e^2}\right)^{k+m} \prod_{j=1}^{2} k_j^{2k_j}.$$

By the inequality $\binom{n}{k} < (ne/k)^k$, and since

$$k_2^{2k_2} < (ek)^{2m} (k_2 - m)^{2(k_2 - m)},$$

the above expression is bounded by $(2e^{-2}\eta(\varepsilon_1, 1 - \varepsilon_1))^k k^{2k} n^{o(1)}$, where η is as defined in Case (i). Therefore, by the arguments in Case (i), when $\varepsilon_1 \ge 1/2$ and $\varepsilon_2 = 1 - \varepsilon_1$, we find that with high probability $\mathcal{G}_{n,p}$ has no 3-cores C as in (ii).

The proof is complete. ■

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