ON REPRESENTATION THEORY OF FINITE-DIMENSIONAL HOPF ALGEBRAS

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ABSTRACT

ON REPRESENTATION THEORY OF FINITE-DIMENSIONAL HOPF ALGEBRAS

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Dr. Martin Lorenz, Chair

Representation theory is a field of study within abstract algebra that originated around the turn of the 19th century in the work of Frobenius on representations of finite groups over the field of complex numbers. Subsequently, representations of finite groups over an arbitrary base field $\Bbbk$ have been thoroughly explored and the theory can by now be considered to be well understood, especially in the case where the characteristic of $\Bbbk$ does not divide the order of the group G in question. The principal tool in the investigation of the representations of G over $\Bbbk$ is the group algebra $\Bbbk[G]$. By Maschke's Theorem, the stated hypothesis on the characteristic of $\Bbbk$ amounts to semisimplicity of $\Bbbk[G]$, a property that allows for a greatly simplified description of the representations of G over $\Bbbk$.

More recently, Hopf algebras – a class of algebras that includes group algebras, enveloping algebras of Lie algebras, and many other interesting algebras that are often referred to under the collective name of "quantum groups" – have come to the fore. The representation theory of Hopf algebras is currently under rapid development, in part because it covers the two main classical flavors of representation theory: representations of groups and of Lie algebras. The principal aim of this dissertation is to generalize certain results from group representation theory to the setting of Hopf algebras. Specifically, our focus is on the following two areas:

(1) Frobenius divisibility and Kaplansky's sixth conjecture, and

(2) the adjoint representation and the Chevalley property

As for (1), a classical result of Frobenius [17] states that degrees of all irreducible representations of a given finite group G over the complex numbers divide the order of the group G ("Frobenius divisibility"). Approximately 80 years after Frobenius, Kaplansky [25] formulated ten conjectures on Hopf algebras, the sixth of which proposes the following generalized version of Frobenius divisibility: the degree of every irreducible representation of a semisimple Hopf algebra H over an algebraically closed field of characteristic 0 divides the dimension of H. Kaplansky's conjecture remains open despite numerous attempts at proving it in the intervening 40 years since the conjecture was stated. This dissertation describes a new approach to Frobenius divisibility that is based on some general observations on symmetric algebras, a class of algebras that includes all semisimple algebras; this material comes from the joint article [23] with my advisor, Martin Lorenz. I then use this approach to provide simpler proofs of many of the known partial results towards Kaplansky's sixth conjecture, thereby unifying many previously disparate results under a common framework.

Turning to (2), the adjoint representation of a Hopf algebra is a natural generalization of the conjugation action of a group G on its group algebra, $\Bbbk[G]$. It has been shown by Lorenz and Passman (see [31]) that if the latter representation is completely reducible, then it actually has the "Chevalley property": all its tensor powers are completely reducible as well. This dissertation takes the first steps towards the goal of generalizing this result to the context of finite-dimensional Hopf algebras. My main tool is the investigation of conjugacy classes for finite-dimensional Hopf algebras that are not necessarily semisimple, generalizing results in the literature for semisimple Hopf algebras in characteristic 0.

Throughout this dissertation, numerous explicit examples are constructed to illustrate my theoretical findings. In order to aid with this task, I have written some Python code that is specifically tailored to computations with Hopf algebras.

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CHAPTER 1

INTRODUCTION

1.1 Background

The objective of Representation Theory is to investigate the different ways in which a given algebraic object – such as an algebra, a group, or a Lie algebra – can act on a vector space. The benefits of such an action are at least twofold: the structure of the acting object gives rise to symmetries of the vector space on which it acts; conversely, the highly developed machinery of linear algebra can be brought to bear on the acting object itself to help uncover some of its hidden properties. Besides being a subject of great intrinsic beauty, representation enjoys the additional benefit of having applications in myriad contexts other than algebra, ranging from number theory, geometry and combinatorics to general physics [52], quantum field theory [54], and the study of molecules in chemistry [9].

We now give some brief background on the main "flavors" of Representation Theory insofar as they are relevant to this dissertation.

1.1.1 Groups

Historically, the first application of representation theory in its current form was to the study of groups, especially finite groups. In detail, a ***representation*** of an arbitrary group G over a field $\Bbbk$ is a group homomorphism,

$$\rho_V\colon G \to \mathrm{GL}(V),$$

where V is a $\Bbbk$-vector space and $\mathrm{GL}(V)$ denotes the group of invertible linear transformations of V. The dimension of V, which may be infinite, is called the ***degree*** of the representation. The representation ρ_V is called ***irreducible*** if $V \neq 0$ and no subspace of V other than 0 and V itself is stable under all transformations in the image of ρ_V. Such representations play a crucial role in group representation theory. Indeed, if the group G is finite and the characteristic of the base field $\Bbbk$ does not divide the order $|G|$, then every representation of G over $\Bbbk$ can be decomposed uniquely (up to isomorphism) into a direct sum of irreducible representations. This reduces the problem of describing all representations of G over $\Bbbk$ to the case of irreducible representations. The following key result from classical group representation theory severely narrows the possibilities for the irreducible representations of G; for a proof, see [49].

Frobenius' Theorem [17]**.** *Let G be a finite group and let $\Bbbk$ be an algebraically closed field whose characteristic does not divide the order $|G|$. Then the degree of every irreducible representation of G over $\Bbbk$ is a divisor of $|G|$.*

Later on many strengthening of this theorem arose, the simplest of which is the following result of Schur [48].

Theorem 1.1.1. *Let G be a finite group and let $\Bbbk$ be an algebraically closed field whose characteristic does not divide $|G|$. Then the degree of an irreducible representation of G over $\Bbbk$ divides $|G/\mathscr{Z}G|$, where $\mathscr{Z}G$ is the center of G.*

While the complete description of all irreducible representations of a given group is generally a formidable task, it has in fact been achieved for many groups of great interest. Foremost among them are the symmetric groups, where a description of the irreducible representations can be given in combinatorial terms, using the so-called Young graph of partitions; see Okounkov and Vershik [43] (who elaborate on earlier work of Frobenius [18], Schur [48], and Young [56]). Additionally, it is worth noting that Representation Theory was the main tool used in the proof of the celebrated Classification Theorem of finite simple groups [51].

1.1.2 Algebras and Other Structures

As has first been observed by Emmy Noether, group representation theory can be embedded into the more general representation theory of associative algebras. This is accomplished by associating to each group G and each base field $\Bbbk$ an associative algebra, the so-called group algebra $\Bbbk[G]$. The precise definition of $\Bbbk[G]$, while not difficult, is omitted here, but we do at least mention that a ***representation*** of a $\Bbbk$-algebra A is a homomorphism of $\Bbbk$-algebras,

$$\rho_V \colon A \to \operatorname{End}_{\Bbbk}(V),$$

where V is a $\Bbbk$-vector space. Equivalently, representations of A can be described in the language of (left) A-modules. The operative fact concerning the group algebra $\Bbbk[G]$, in the context of representation theory, is that representations of $\Bbbk[G]$ are in natural one-to-one correspondence with the representations of G over $\Bbbk$. In particular, irreducible representations of G over $\Bbbk$ correspond to irreducible $\Bbbk[G]$-modules in the usual ring theoretic sense.

Similar reductions to the case of algebras exist for the representations of other algebraic structures as well. For example, in the case of a Lie algebra $\mathfrak{g}$, the algebra in question is the so-called enveloping algebra of $\mathfrak{g}$; for a quiver Γ, the vehicle is the path algebra of Γ.

Exactly as for group algebras, a representation ρ_V of an algebra A is called ***irreducible*** if $V \neq 0$ and no subspace of V other than 0 and V itself is stable under all transformations in $\rho_V(A)$; this is equivalent to irreducibility of V as A-module. Similarly, representations of A are called ***equivalent*** or ***isomorphic*** if the corresponding A-modules are isomorphic in the usual sense. One of the main goals of representation theory is to find, for a given algebra A, a description of the set of all equivalence classes of irreducible representatioins of A. This set, or a full representative set of non-equivalent irreducible representations of A, will be denoted by

$$\operatorname{Irr} A\,.$$

For the most part, we shall be concerned with finite-dimensional representations, that is, $\dim_{\Bbbk} V < \infty$. In this case, the ***character*** of V is the linear form $\chi_V \in A^*$ that is defined by

$$\chi_V(a) = \operatorname{trace}(\rho_V(a)) \qquad (a \in A).$$

A particularly prominent role will be played by to the so-called ***regular representation*** of A; it is obtained by taking $V = A$ and letting A act on itself by left multiplication. This representation will be denoted by A_{reg}. If A is finite-dimensional, which will usually be the case below, then we may consider the ***regular character***,

$$\chi_{\text{reg}} := \chi_{A_{\text{reg}}}\,.$$

The regular character will be an important tool in this thesis.

1.1.3 Hopf algebras

While the representation theory of associative algebras provides a useful setting in which to study many aspects of group representation theory, it turns out that general associative algebras fail to capture certain features of group representations; the same can be said for representations of Lie algebras as well. Additional structure is needed in order to access these features, and this structure is naturally provided by the class of ***Hopf algebras***. The formal definition of Hopf algebras is rather unwieldy; so we refrain from spelling it out until later. In brief, a Hopf $\Bbbk$-algebra is a $\Bbbk$-algebra H – so there is a multiplication, $m\colon H \otimes H \to H$, and a unit map, $u\colon \Bbbk \to H$ – but there are three additional linear structure maps: the comultiplication $\Delta\colon H \to H \otimes H$, the counit $\epsilon\colon H \to \Bbbk$ and the antipode $S\colon H \to H$. All these maps must satisfy certain axioms and compatibility conditions. A remarkable feature of the Hopf algebra axioms is their "self-duality." This makes it possible to equip the linear dual $H^* = \operatorname{Hom}_{\Bbbk}(H, \Bbbk)$ of any finite-dimensional Hopf algebra H

with the structure of a Hopf algebra in its own right, employing the transposes of the structure maps of H as the structure maps of H^*.

The comultiplication, counit and antipode of a Hopf algebra H impart additional structure on the category of finite-degree representations of H; this category will be denoted by

$$\mathfrak{Rep} H$$

Specifically, the comultiplication Δ allows to form the tensor product representation $V \otimes W$ of any two given V and W in $\mathfrak{Rep} H$, and ϵ yields a distinguished object of $\mathfrak{Rep} H$, the so-called trivial representation $\mathbb{1}$. With these data, $\mathfrak{Rep} H$ becomes a ***monoidal category***. Furthermore, using the antipode S, one can give a module structure to the linear dual $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ of any V in $\mathfrak{Rep} H$. For more information on monoidal categories with a notion of a dual, see [4] or [16]. Here, we just mention that, as a consequence of the monoidal structure of $\mathfrak{Rep} H$, we can assign a ring, $\mathscr{R}(H)$, to this category. The underlying additive group of $\mathscr{R}(H)$ has generators $[V]$, one for each isomorphism class of objects V in $\mathfrak{Rep} H$, with defining relations $[U] + [W] = [V]$ for each short exact sequence $0 \to U \to V \to W \to 0$ in $\mathfrak{Rep} H$. The multiplication of $\mathscr{R}(H)$ is given by $[V][W] = [V \otimes W]$ and the identity element is $[\mathbb{1}]$. The map $[V] \mapsto \chi_V$ yields a well-defined ring homomorphism

$$\chi \colon \mathscr{R}(H) \to H^*$$

The $\Bbbk$-subalgebra of H^* that is generated by the image $\chi(\mathscr{R}(H))$ will be denoted by $R(H)$ below; it is generally called the ***representation algebra*** of H. The representation algebra $R(H)$ plays a prominent role in the representation theory of H, as we shall repeatedly see throughout this dissertation.

1.2 Summary of Research

As was indicated in the previous section, Hopf algebras provide a common setting in which to treat the representation theories of groups and Lie algebras. The focus of my research, to be described in this section, was on representations of finite degree. Since "most" irreducible representations of Lie algebras have infinite degree, my principal motivation came from group representation theory. In brief, one would like to develop the representation theory of finite-dimensional Hopf algebras to a level that has been achieved for finite groups. One of the challenges one encounters in this endeavor is the absence of a distinguished $\Bbbk$-basis in a general Hopf algebra H – for a group algebra $\Bbbk[G]$, such a basis is provided by the elements of the group G. Thus, the element-based approach that is common in classical group representation theory needs to replaced by more abstract ring theoretic and categorical methods.

The focus of this dissertation is on possible generalizations of results from group representation theory to more general classes of Hopf algebras. Specifically, we consider the following two areas:

(1) Frobenius divisibility and Kaplansky's sixth conjecture, and

(2) the adjoint representation and the Chevalley property

Throughout this section, H will denote a finite-dimensional Hopf algebra over a field $\Bbbk$. We will occasionally use the Sweedler comultiplication notation for $h \in H$,

$$\Delta(h) = h_{(1)} \otimes h_{(2)} .$$

1.2.1 Frobenius Divisibility

The terminology "Frobenius divisibility," due to Etingof [15], is motivated by Frobenius' Theorem on finite group representations as quoted in Section 1.1.1. The term is now also used in connection with similar divisibility results for the degrees of irreducible representations of other algebraic objects. Such results play a major role in classification attempts, since they place severe restrictions on the potential candidates for irreducible representations.

In [25] Kaplansky proposed ten conjectures on Hopf algebras, the sixth of which was aimed at a generalization of Frobenius' Theorem. The original formulation of the conjecture was missing some crucial hypotheses, and hence easily seen to be false, but the revised version below has remained open for 40 years.

Kaplansky's Sixth Conjecture. *Let H be a semisimple and cosemisimple Hopf algebra over an algebraically closed field. Then the degree of every irreducible representation of H divides the dimension of H.*

Using a variety of different techniques, many partial results of this conjecture have been proven; some of the main findings are listed below:

- Nichols and Zoeller [40] have shown the conjecture to be true for irreducible representations of dimension two.
- Montgomery and Witherspoon [36] have confirmed the conjecture when the dimension of H is a prime power.
- Etingof and Gelaki [14] have shown the conjecture to be true when H is "factorizable." This class of Hopf algebras includes the famous Drinfel'd double. An immediate corollary shows that the conjecture in fact holds for the more general class of "quasi-triangular" Hopf algebras.
- S. Zhu [57] proved that $\dim_{\Bbbk} H$ is divisible by the degree of any irreducible representation of H whose character belongs to the center of H^*.

One of the earliest classification results on Hopf algebras, due to Y. Zhu [58], states that Hopf algebras of prime dimension over an algebraically closed field of characteristic 0 are in fact group algebras. The crucial tool in the proof of this result is a different but related Frobenius divisibility result for Hopf algebras:

The Class Equation ([24], [58])**.** *Let H be a semisimple Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic 0. If $e \in R(H)$ is a primitive idempotent, then $\dim_{\Bbbk}(H^*e)$ divides $\dim_{\Bbbk} H$.*

Finite-dimensional Hopf algebras are known to have the structure of a Frobenius algebra in a way that is deeply intertwined with their Hopf structure [29]. A detailed discussion of Frobenius algebras will be given in Section 2.3.1; in particular, we will describe the so-called Casimir element, which is part of the data associated to any Frobenius algebra. It was shown in [10] that a Hopf algebra satisfies Kaplansky's sixth conjecture if and only if its Casimir element satisfies a monic polynomial over the integers. My advisor and I have shown that many of the aforementioned partial results on Kaplansky's conjecture as well as the class equation can be derived directly from a more general integrality result about Casimir elements of Frobenius algebras. This opens a new approach to Kaplansky's long-standing conjecture, in addition to providing unified proofs of the earlier known results. Additionally, the result can be used to provide a purely algorithmic approach for determining if a given semisimple algebra satisfies Frobenius Divisibility. This algorithm has been implemented in Python code, for more information on the code see chapter 5.

Next, I look at partial generalizations of Theorem 1.1.1. Adapting an argument due to Tate, I show that some of the known Frobenius Divisibility results for Hopf algebras actually can be strengthened by using a suitable notion of the ***Hopf center*** of an irreducible character.

1.2.2 The Adjoint Representation of a Hopf Algebra

As is the case with group algebras, any Hopf algebra acts on itself in multiple ways. One such action of particular importance is the so-called ***adjoint action***. Using the Sweedler notation, the action of $h \in H$ on $k \in H$ is given by

$$^{h}k = \sum h_{(1)} k S(h_{(2)})\,.$$

This action is a natural generalization of the conjugation action of a group on itself, extended to an action on the group algebra. The Hopf algebra H, equipped with the adjoint action, will be denoted by ${}^{\mathrm{ad}}H$. In this way, we obtain a representation

$$H \to \mathrm{End}_{\Bbbk}({}^{\mathrm{ad}}H)$$

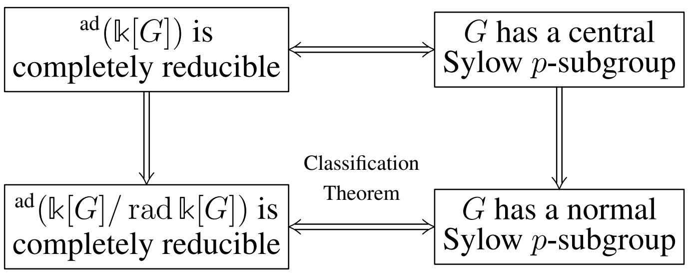
that will be referred to as the ***adjoint representation*** of H. Much is known about the adjoint representation for a semisimple Hopf algebra over a field of characteristic 0, notably through the works of Zhu [59], Witherspoon [55], Burciu [5], and

Cohen and Westreich [8]. Many of these papers use of an extension of the adjoint representation to the Drinfel'd double of H; see Section 2.5 for the exact description. The representation of the Drinfel'd double that is afforded by the extended action will be called the ***extended adjoint representation***. Without the assumption of semisimplicity almost nothing in known about the adjoint representation of H and its extended version. This dissertation fills this gap and it uses the resulting theory to prove new results.

One of the main motivating problems is the question as to when ${}^{\mathrm{ad}}H$ is completely reducible, that is, ${}^{\mathrm{ad}}H$ is a sum of irreducible subrepresentations. Here are the classical examples:

- For a finite-dimensional Lie algebra $\mathfrak{g}$ and $\operatorname{char} \Bbbk = 0$, it is a standard fact that the adjoint representation ${}^{\mathrm{ad}}U(\mathfrak{g})$ is completely reducible if and only if $\mathfrak{g}$ is reductive, that is, $\mathfrak{g}/\mathscr{Z}\mathfrak{g}$ is semisimple, where $\mathscr{Z}\mathfrak{g}$ denotes the center of $\mathfrak{g}$.

- For a finite group G the answer is provided by the top row of the following diagram:



(1.1)

Here, the vertical implications are trivial. However, while the top equivalence is relatively elementary ([31, Exercise 3.32]), the proof of the bottom equivalence relies on the Ito-Michler Theorem [34], whose proof in turn uses the classification of finite simple groups; see [31] for further details. It is a long-term goal and hope that the more general approach via Hopf algebras might shed additional light on the representation theory of finite groups, in particular on the above equivalence, ideally rendering the Classification Theorem superfluous for its proof.

While we have not yet been able to fully determine when ${}^{\mathrm{ad}}H$ is completely reducible for an arbitrary finite-dimensional Hopf algebra H, this dissertation takes the first steps in this direction. One such result is the calculation of the Hopf annihilator of ${}^{\mathrm{ad}}H$, that is, the largest Hopf ideal that is contained in the usual annihilator of ${}^{\mathrm{ad}}H$. The first step in the proof of the top equivalence in (1.1) is that complete reduciblity of ${}^{\mathrm{ad}}H$ implies that $\mathrm{char}\,\Bbbk$ does not divide the dimension of any conjugacy classes. This result has successfully been generalized to cosemisimple unimodular involutory Hopf algebras in this dissertation.

CHAPTER 2

PRELIMINARIES

This chapter introduced the background, notation, definitions, and results used throughout this thesis. The reader is assumed to already be familiar with the basics of the representation theory of associative algebras and groups. The chapter begins with a detailed definition of a Hopf algebra and a brief introduction to their representation theory including several important theorems. We then go on to give the definition of a Frobenius algebra and discuss much of their basic structure to the extent needed here. A celebrated theorem of Larson and Sweedler states that all finite-dimensional Hopf algebras are Frobenius algebras. This dissertation will make extensive use of this fact. Therefore, the next section is devoted to developing the Frobenius structure of finite-dimensional Hopf algebras and its connections with the Hopf structure in explicit detail. The chapter concludes with several more complex constructions that will be used throughout the dissertation, notably the Drinfel'd double and the extended adjoint representation.

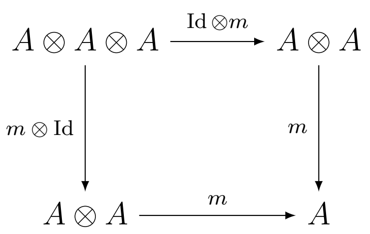
Throughout this chapter, $\Bbbk$ denotes an arbitrary field and all tensor products are assumed to be over $\Bbbk$ unless otherwise denoted.

2.1 Hopf algebras

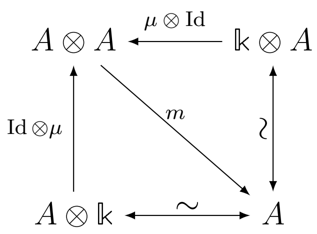
2.1.1 Coalgebras

The familiar axioms of a $\Bbbk$-algebra with multiplication $m\colon A \otimes A \to A$ and unit $\mu\colon \Bbbk \to A$ can be expressed by the commutativity of the following diagrams in the category of $\Bbbk$-vector spaces:

Associativity

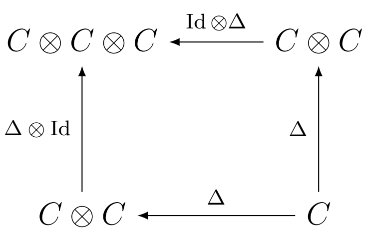


Unit

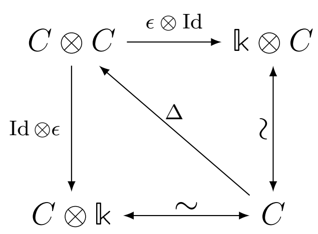


Reversing the direction of all arrows in the above diagrams, we obtain commutative diagrams describing the defining axioms of coalgebras. In detail, a $\Bbbk$-***coalgebra*** is a $\Bbbk$-vector space, C, that is equipped with two linear maps, the comultiplication $\Delta\colon C \to C \otimes C$ and the counit $\epsilon\colon C \to \Bbbk$, which satisfy the ***coassociativity*** and ***counit*** axioms:

Coassociativity



Counit



For example, if A is a finite-dimensional $\Bbbk$-algebra, then the $\Bbbk$-linear dual $C = A^*$ becomes a $\Bbbk$-coalgebra by taking the dual maps $\Delta = m^*$ and $\epsilon = \mu^*$.

Without special notation, computations using the comultiplication Δ quickly become unwieldy. This dissertation will make use of an abbreviated notation known as Sweedler notation (after Moss Sweedler, one of the first to research Hopf algebras). In this notation the element $\Delta(c) = \sum_i c^i_{(1)} \otimes c^i_{(2)}$ will be abbreviated by

$$\Delta(c) = c_{(1)} \otimes c_{(2)} ,$$

where summation is assumed. Using this notation, the counit axiom can be written simply as

$$\langle \epsilon, c \rangle = \langle \epsilon, c_{(1)} \rangle c_{(2)} = c_{(1)} \langle \epsilon, c_{(2)} \rangle$$

and the coassociativity axiom can be expressed as

$$c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = (\mathrm{Id} \otimes \Delta)\Delta(c) = (\Delta \otimes \mathrm{Id})\Delta(c) = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} .$$

We will write the map $(\mathrm{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}) \circ \Delta$ more simply as Δ^2 and it is also customary to write $\Delta^2(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$. Inductively, for any number n of iterations of the comultiplication, one obtains a linear map $\Delta^n : C \to C^{\otimes(n+1)}$ that will be written as

$$\Delta^n(c) = c_{(1)} \otimes c_{(2)} \otimes \ldots \otimes c_{(n+1)} .$$

A subcoalgebra of a coalgebra C is defined exactly as expected: it is a subspace D of C such that $\Delta(D) \subseteq D \otimes D$. Given two coalgebras C and D, a $\Bbbk$-linear map $\phi : C \to D$ is a ***morphism of coalgebras*** if $\Delta_D \circ \phi = (\phi \otimes \phi) \circ \Delta_C$ and $\epsilon_C = \epsilon_D \circ \phi$. A ***coideal*** of a coalgebra C is a subspace $I \subseteq C$ such that $\Delta(I) \subseteq I \otimes H + H \otimes I$ and $I \subseteq \mathrm{Ker}(\epsilon)$. These are exactly the conditions necessary to make the coalgebra structure maps descend to the vector space C/I, thus giving it the structure of a coalgebra. As in the case of associative algebras, it remains true that coideals are exactly the kernels of coalgebra morphisms.

We now discuss two important constructions. Given an algebra A, we can construct it opposite algebra A^{op} in the familiar way. A similar construction is available for coalgebras. Namely, given a coalgebra C, its ***coopposite coalgebra*** C^{cop} is the vector space C with comultiplication given by $\Delta^{cop} = \tau \circ \Delta$, where $\tau : C \otimes C \to C \otimes C$ simply switches the order of the tensor factors, and with $\epsilon^{cop} = \epsilon$. Next, given two coalgebras C and D, we can give their tensor product the structure of a coalgebra by defining $\epsilon_{C\otimes D} = \epsilon_C \otimes \epsilon_D$ and $\Delta_{C\otimes D} = (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}) \circ (\Delta_C \otimes \Delta_D)$ or, in Sweedler notation,

$$\Delta(c \otimes d) = c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)}\,.$$

Example 2.1.1. One of the simplest and most useful examples of a $\Bbbk$-algebra is the matrix ring $M_n(\Bbbk)$. Its dual $M_n(\Bbbk)^*$ gives us one of the simplest and most useful examples of a coalgebra. Let $E_{i,j} \in M_n(\Bbbk)$ be the matrix with a 1 in position (i, j) and a 0 everywhere else and take $\{d_{i,j}\}$ to be the basis of $M_n(\Bbbk)^*$ that is dual to $\{E_{i,j}\}$. In terms of this basis the coalgebra structure of $M_n(\Bbbk)^*$ is given by the equations below:

$$\Delta(d_{i,j}) = \sum_{k=1}^{n} d_{i,k} \otimes d_{k,j}$$

$$\epsilon(d_{i,j}) = \delta_{i,j} \qquad \text{(Kronecker delta)}$$

Example 2.1.2. The vector space $\Bbbk[x]$ admits a coalgebra structure with structure maps as given below:

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \otimes x^k$$

$$\epsilon(x^n) = \delta_{n,0}$$

This is an example of an infinite-dimensional $\Bbbk$-coalgebra.

2.1.2 Convolution Product

Given a $\Bbbk$-algebra A and a $\Bbbk$-coalgebra C, we can give $\mathrm{Hom}_{\Bbbk}(C, A)$ the structure of an algebra. The multiplication is called the ***convolution product***, it is denoted by $*$ and defined by $* = m \circ (\,.\otimes .\,) \circ \Delta\colon\ \mathrm{Hom}_{\Bbbk}(C, A) \otimes \mathrm{Hom}_{\Bbbk}(C, A) \to \mathrm{Hom}_{\Bbbk}(C, A)$. In explicit elementwise form, for $c \in C$ and $f, g \in \mathrm{Hom}_{\Bbbk}(C, A)$,

$$\langle f * g, c\rangle = \langle f, c_{(1)}\rangle\langle g, c_{(2)}\rangle\,.$$

The unit map is given by it is straightforward to verify that $\mu \circ \epsilon\colon C \to \Bbbk \to A$ serves as a unit element for $*$.

2.1.3 Bialgebras and Hopf Algebras

A ***bialgebra*** is a coalgebra in the category of algebras or, equivalently, an algebra in the category of coalgebras. More explicitly a $\Bbbk$-bialgebra is both a $\Bbbk$-algebra and a $\Bbbk$-coalgebra and the coalgebra structure maps are algebra morphisms or, equivalently, the algebra structure maps are coalgebra morphisms.

Example 2.1.3. The vector space $\Bbbk[x]$ is a bialgebra, where $\Bbbk[x]$ has the usual polynomial algebra structure and the coalgebra structure described in Example 2.1.2.

Let H be a $\Bbbk$-bialgebra. Then $\mathrm{End}_{\Bbbk}(H) = \mathrm{Hom}_{\Bbbk}(H, H)$ is an algebra via the convolution product, where the first H is viewed as a coalgebra structure and the second H as an algebra. If there exists a two-sided inverse, $S \in \mathrm{End}_{\Bbbk}(H)$, to the identity morphism, $\mathrm{Id}_H \in \mathrm{End}_{\Bbbk}(H)$, then H is called a ***Hopf Algebra***. The element S is then called the ***antipode*** of H. In Sweedler notation, the defining property of the antipode can be written as

$$S(h_{(1)})h_{(2)} = \epsilon(h)\mu(1) = h_{(1)}S(h_{(2)})\,.$$

It is worth noting that the antipode is always an antialgebra and anticoalgebra map [35]. Explicitly, for any $a, b, h \in H$,

$$S(ab) = S(b)S(a) \quad \text{and} \quad S(h_{(2)}) \otimes S(h_{(1)}) = S(h)_{(1)} \otimes S(h)_{(2)}\,.$$

A remarkable feature of Hopf algebras is that their axioms are self dual. Thus given a finite-dimensional Hopf algebra H, the $\Bbbk$-linear dual H^* also has the structure of a Hopf algebra, where the structure maps of H^* come from applying the dualizing functor $\,.^*$ to the structure maps of H.

Example 2.1.4. Given a group G and a field $\Bbbk$, we can construct the group algebra $\Bbbk[G]$: it is the $\Bbbk$ vector space with basis the elements of G; multiplication is given by linear extension of the group multiplication; and unit element of $\Bbbk[G]$ is given by the unit element of G. The group algebra is in fact a Hopf algebra, with coalgebra structure maps and the antipode as given below, for $g \in G$:

$$\Delta(g) = g \otimes g$$
$$\epsilon(g) = 1$$
$$S(g) = g^{-1}$$

Example 2.1.5. When $|G| < \infty$ we can construct the Hopf algebra $\Bbbk[G]^*$, the $\Bbbk$-linear dual of $\Bbbk[G]$. Let $\{\rho_x \mid x \in G\}$ denote the basis of $\Bbbk[G]^*$ that is dual to the basis G of $\Bbbk[G]$. Then for $g, h \in G$ the Hopf algebra structure of $\Bbbk[G]^*$ is given by:

$$\mu(1) = \sum_{g \in G} \rho_g = \epsilon \qquad \rho_g \rho_h = \delta_{g,h} \rho_g$$
$$\epsilon(\rho_g) = \rho_g(1) = \delta_{g,1} \qquad \Delta(\rho_g) = \sum_{h \in G} \rho_h \otimes \rho_{h^{-1}g}$$
$$S(\rho_g) = \rho_{g^{-1}}$$

Group algebras are particularly nice examples of Hopf algebras in that they are ***cocommutative***, meaning that, $\tau \circ \Delta = \Delta$ where τ is the twist map as in Section 2.1.1. In other words, $\Bbbk[G]^{cop} = \Bbbk[G]$.

An element h of a Hopf algebra is called ***group-like*** if $\Delta(h) = h \otimes h$ and $\epsilon(h) = 1$. In this case $S(h)$ will also be a group-like element, and $S(h)$ is the multiplicative inverse of h. As the name suggests, the collection of all group-like elements of any Hopf algebra H forms a subgroup of the group of units of H.

A Hopf algebra is called ***trivial*** if it is isomorphic to a group algebra or the dual of a group algebra. The smallest non-trivial example of a Hopf algebra is a four-dimensional example named after Sweedler, who first constructed it.

Example 2.1.6. Assume that $\Bbbk$ has characteristic $\neq 2$. The ***Sweedler algebra***, denoted H_4, is the unique non-commutative and non-cocommutative Hopf algebra of dimension 4. The algebra structure of H_4 is defined by $H_4 = \Bbbk\langle g, x \mid g^2 = 1, x^2 = 0, xg = -gx\rangle$. The coalgebra structure and the antipode of H_4 are defined by:

$$\Delta(g) = g \otimes g\ , \qquad \Delta(x) = x \otimes 1 + g \otimes x\ ,$$
$$\epsilon(g) = 1\ , \qquad \epsilon(x) = 0\ ,$$
$$S(g) = g\ , \qquad S(x) = -gx\ .$$

2.1.4 Hopf Subalgebras and Quotient Hopf Algebras

A ***Hopf subalgebra*** of a Hopf algebra H, by definition, is a subalgebra of H that is also a subcoalgebra and is stable under the antipode. Likewise, a ***Hopf ideal*** of H is an ideal of H that is also a coideal and is stable under the antipode.

We will commonly use the following technique to construct Hopf subalgebras. Given a subalgebra, A, of H we define $\mathscr{H}(A)$ to be the subalgebra of A that is generated by all Hopf subalgebras of H that are contained in A. It is a simple exercise to see that this subalgebra is in fact a Hopf subalgebra, and thus is the unique largest Hopf subalgebra of H contained in A. Of particular interest will be the largest Hopf subalgebra contained in the center of H. For brevity, we will simply write

$$\zeta(H) = \mathscr{H}\mathscr{Z}(H)\,.$$

A similar process can be done starting with an ideal I of H to construct the largest Hopf ideal, $\mathscr{H}(I)$, contained in I: define $\mathscr{H}(I)$ to be the sum of all Hopf ideals of H that are contained in I. Given an H-module, M, we can use this construction to construct the largest Hopf ideal of H contained in the annihilator of M; this ideal will be called the ***Hopf kernel*** of M and denoted by

$$\mathscr{H}\operatorname{Ker}(M)\,.$$

A representation of H that is given by an H-module M with Hopf kernel $\mathscr{H}\operatorname{Ker}(M) = 0$ will be called ***inner faithful***.

Example 2.1.7. Given a Hopf algebra H, we always have the augmentation ideal, $H^+ := \operatorname{Ker}(\epsilon)$. This is in fact a Hopf ideal. The quotient Hopf algebra $\overline{H} = H/H^+$ is isomorphic to $\Bbbk$ as an algebra and the coalgebra structure is given by $\Delta_{\overline{H}}(c) = c \otimes 1$. Given a Hopf subalgebra K in H, we can form the left ideal HK^+ of H. It is easy to see that HK^+ is also a coideal of H. In Section 2.2.4, we will discuss under which circumstances HK^+ is in fact a Hopf ideal of H.

Example 2.1.8. For any group G, the Hopf subalgebras of the group algebra $\Bbbk[G]$ are exactly the various $\Bbbk[H]$, where H is a subgroup of G. The Hopf ideals of $\Bbbk[G]$

are exactly the ideals of the form $\Bbbk[G]\Bbbk[H]^+$, where H is a normal subgroup of G. Furthermore, $\Bbbk[G]/\Bbbk[G]\Bbbk[H]^+ \cong \Bbbk[G/H]$. An inner faithful representation of $\Bbbk[G]$ is just a representation of G such that no group element $g \in G$ other than $g = 1$ acts as the identity transformation.

It is important to note that, while all Hopf ideals of group algebras arise from Hopf subalgebras, this is not always the case in general. This is illustrated in the next example.

Example 2.1.9. Observe the set $G := \{1, g\} \subseteq H_4$ in the Sweedler algebra is a group. The Hopf algebra $\Bbbk[G]$ is the only Hopf subalgebra of the Sweedler algebra other then $\Bbbk$ and H_4. The space $H_4\Bbbk[G]^+$ has basis $\{x + gx, 1 - g\}$ which is not an ideal of H_4 since it does not contain $x - gx = (1 - g)x$. Now look at the Hopf ideal of H_4 given by the $\Bbbk$-span of $\{x, gx\}$. This is in fact the Jacobson radical of H_4. Since this Hopf ideal did not arise from the unique nontrivial Hopf subalgebra $\Bbbk[G]$ it could not have arisen from any Hopf subalgebra. The corresponding quotient Hopf algebra is isomorphic to the group algebra $\Bbbk[C_2]$, where C_2 is the cyclic group of order two.

2.2 Representation Theory

As is the case with groups, Hopf algebras have a representation theory that has additional features not present in the representation theory of general associative algebras. Throughout this dissertation we will focus on left modules. This is only for consistency as the theory could be formulated equally well for right modules.

In the following, H denotes a Hopf $\Bbbk$-algebra and $\mathfrak{Rep} H$ denotes the category of left H-modules that are finite-dimensional over $\Bbbk$. All further assumptions will be explicitly stated when they are needed.

2.2.1 The Representation Ring and Character Algebra

The coalgebra structure of H allows us to endow the category of left H-modules with the structure of a tensor category. The precise definition of a tensor category is not needed and hence will be omitted; the details can be found in *Categories for the Working Mathematician* [32]. The key fact that is needed is that, given two left H-modules V and W, the tensor product $V \otimes W$ has an H-module structure with $h \in H$ acting by

$$h.(v \otimes w) = h_{(1)}.v \otimes h_{(2)}.w\,. \tag{2.1}$$

The counit of a Hopf algebra gives rise to a representation, which will be called the ***trivial representation*** and denoted by $\mathbb{1}$. Explicitly, $\mathbb{1} = \Bbbk$ with $h \in H$ acting by $h.1 = \langle \epsilon, h \rangle$. It is easy to see that, for any $V \in \mathfrak{Rep}H$, the isomorphisms $V \otimes \mathbb{1} \cong V \cong \mathbb{1} \otimes V$ hold in $\mathfrak{Rep}H$. Using the trivial representation we get a notion of the ***invariants*** of any $V \in \mathfrak{Rep}H$,

$${}^{\mathrm{Inv}}V = \{v \in V | h.v = \langle \epsilon, h \rangle v\}\,.$$

The antipode of the Hopf algebra allows us to give the $\Bbbk$-linear dual of a module the structure of an H-module. Given $V \in \mathfrak{Rep}H$, the vector space V^* becomes an H-module with action defined by

$$\langle h.f, v \rangle = \langle f, S(h).v \rangle\,. \tag{2.2}$$

We will work with the ***representation ring*** $\mathscr{R}(H)$ of H, which is defined as the abelian group with generators the isomorphism classes $[V]$ of representations $V \in \mathfrak{Rep}H$ and with relations $[U] + [W] = [V]$ for each short exact sequence $0 \to U \to V \to W \to 0$ in $\mathfrak{Rep}H$. The multiplication of $\mathscr{R}(H)$ comes from the tensor product of representations: $[V][W] = [V \otimes W]$. By extension of scalars from $\mathbb{Z}$ to $\Bbbk$, we obtain the $\Bbbk$-algebra $\mathscr{R}_\Bbbk(H) := \Bbbk \otimes \mathscr{R}(H)$; this algebra will be called the ***representation algebra*** of H.

It is a standard fact that the representation algebra $\mathscr{R}_\Bbbk(H)$ embeds into the linear dual H^* via the ***character map*** and this embedding is a homomorphism of $\Bbbk$-algebras for the convolution algebra structure of H^*; see [31, Proposition 12.10]. Explicitly, for any $V \in \mathfrak{Rep}H$, the ***character*** χ_V is the linear form on H that is defined by $\langle \chi_V, h \rangle = \mathrm{trace}(h_V)$, where $h_V \in \mathrm{End}_\Bbbk(V)$ denotes the operator given by the action of $h \in H$. The character map is given by

$$\begin{array}{rcl} \chi\colon \mathscr{R}_{\Bbbk}(H) & \hookrightarrow & H^* \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [V] & \longmapsto & \chi_V \end{array}$$

The image of the character map in H^* is called ***character algebra*** of H and is denoted $R(H)$. A $\Bbbk$-basis of $R(H)$ is given by the irreducible characters of H, that is, the characters of a full set of non-isomorphic irreducible finite-dimensional representations of H. If H is semisimple and $\Bbbk$ is a splitting field for H (e.g., if $\Bbbk$ is algebraically closed), then the character algebra $R(H)$ coincides with the subspace of all ***trace forms*** on H, that is, the linear forms on H that vanish on the subspace $[H, H]$ spanned by the Lie commutators $[h, k] = hk - kh$ for $h, k \in H$. Thus, the space of trace forms is isomorphic to $(H/[H, H])^*$; it can equivalently be thought of as the set of cocommutative elements of H^*.

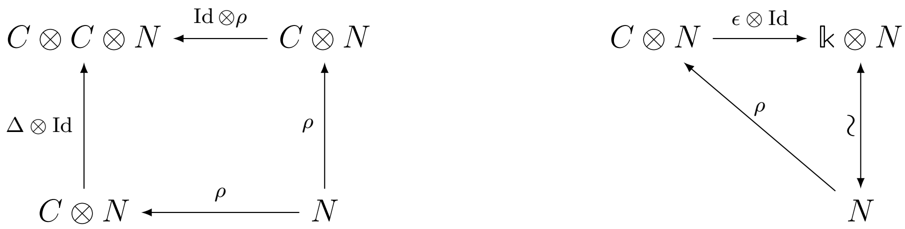
Example 2.2.1. Let G be a finite group. Then the set of irreducible characters of $\Bbbk[G]^*$ are the elements of G and hence $\Bbbk[G] = R(\Bbbk[G]^*)$.

2.2.2 Comodules

Given an algebra $A = (A, m, \mu)$, the familiar axioms of a left A-module M can be expressed by the existence of a $\Bbbk$-linear "action" map $a\colon A \otimes M \to M$ such that the following diagrams are commutative:

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{\mathrm{Id} \otimes a} & A \otimes M \\ {\scriptstyle m \otimes \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle a} \\ A \otimes M & \xrightarrow{a} & M \end{array} \qquad \begin{array}{ccc} A \otimes M & \xleftarrow{\mu \otimes \mathrm{Id}} & \Bbbk \otimes M \\ & {\scriptstyle a}\searrow & \updownarrow{\scriptstyle \wr} \\ & & M \end{array}$$

Dually, if $C = (C, \Delta, \epsilon)$ is a $\Bbbk$-coalgebra, then a $\Bbbk$-vector space N is called a left C-***comodule*** if there is a $\Bbbk$-linear "coaction" map $\rho\colon N \to C \otimes N$ such that the following diagrams commute:



As with coalgebras it is customary to use a version of the Sweedler notation when dealing with comodules: for $n \in N$,

$$\rho(n) = n_{(-1)} \otimes n_{(0)} .$$

For right comodules, defined by the obvious modification of the above, we will instead use the notation

$$\rho(n) = n_{(0)} \otimes n_{(1)} \in N \otimes C .$$

Let M be a right C-comodule. Then M can be viewed as a left a left module over the convolution algebra C^* via

$$c^*.m = m_{(0)}\langle c^*, m_{(1)}\rangle \qquad (c^* \in C^*, m \in M) .$$

If C is finite-dimensional, then all left C^*-modules arise in this fashion; so there is equivalence between the categories of right C-comodules and left C^*-modules. Thus, as one would expect, there are analogs of all constructions and properties of modules for comodules. For example, a subcomodule of a left comodule M is a subspace $V \subseteq M$ such that $\rho(V) \subseteq C \otimes V$. When C is a bialgebra, one subcomodule of particular interest is given by the ***coinvariants***,

$$^{\mathrm{coInv}}M := \{m \in M | \rho(m) = 1 \otimes m\} .$$

A comodule M is called ***simple*** if M contains no subcomodules other than 0 and M itself, and M is called ***indecomposable*** if it can not be expressed as the direct sum of two nonzero subcomodules. A coalgebra C is called ***cosemisimple*** if all C-comodules are direct sums of simple comodules. We will not go into detail on cosemisimplicity as, in the case where C is finite-dimensional, it is equivalent to the dual algebra C^* being semisimple by our remark above. A Hopf algebra with cosemisimple coalgebra structure is called cosemisimple.

Example 2.2.2. The group algebra $\Bbbk[G]$ is cosemisimple. All simple comodules are one dimensional and have the form $\Bbbk g$ for $g \in G$ with structure map given by $\rho(g) = g \otimes g$.

Example 2.2.3. The coalgebra $M_n(\Bbbk)^*$ of Example 2.1.1 is also cosemisimple. Up to isomorphism it has exactly one simple left comodule. This module can be viewed as $\Bbbk\{e_{i,1}\}$ with $\rho(e_{i,1}) = \sum_k e_{i,k} \otimes e_{k,1}$. If $\Bbbk$ is algebraically closed, then dualizing the Artin-Wedderburn structure theorem of semisimple algebras, one sees that all cosemisimple coalgebras over $\Bbbk$ are isomorphic to a direct sum of duals of matrix algebras.

2.2.3 Module Algebras

Quantum invariant theory is concerned with actions of Hopf algebras on associative algebras. In detail, let A be an associative $\Bbbk$-algebra and assume that A is a left module over the Hopf $\Bbbk$-algebra H with action map $H \otimes A \to A$, $h \otimes a \mapsto h.a$. If the multiplication $m\colon A \otimes A \to A$ and the unit map $\mu\colon A \to \Bbbk = \mathbb{1}$ are maps of H-modules, then we say A is a ***H-module algebra***. These axioms can be stated in Sweedler notation as folows: for $b, c \in A$ and $h \in H$, we have

$$h.(bc) = (h_{(1)}.b)(h_{(2)}.c) \qquad \text{and} \qquad h.1_A = \langle \epsilon, h \rangle 1_A \,.$$

Example 2.2.4. Let A be an associative algebra and let G be a subgroup of the autmorphism group $\mathrm{Aut}_{\mathrm{alg}}(A)$. Then the G-action on A extends to an action of the group algebra $\Bbbk[G]$, making A a $\Bbbk[G]$-module algebra.

2.2.4 Adjoint Representation

Let V be an H-bimodule, that is, a left module over the algebra $H \otimes H^{\mathrm{op}}$. Then we can form a left adjoint module of V, denoted by ${}^{\mathrm{ad}}V$, by defining

$${}^{h}v := h_{(1)} v S(h_{(2)}) \qquad (h \in H, v \in V)\,.$$

Naturally, there is also an analogous right adjoint action, given by

$$v^h = S(h_{(1)}) v h_{(2)}\,.$$

However, by passing to the opposite-cooposite Hopf algebra, this action can be reduced to the above action; so we will mainly focus on the left-handed version here. In fact, we will mostly be interested in the case where $V = H$ is the regular H-bimodule, with left and right actions given by multiplication. As in the Introduction, the corresponding representation of H will be called the adjoint representation. The algebra H, with the (left) adjoint H-action, is an example of an H-module algebra as introduced in the previous section.

For any H-bimodule V, the ordinary H-actions on V and the adjoint action are related by:

$$hv = ({}^{h_{(1)}}v)h_{(2)}\,. \tag{2.3}$$

The H-invariants of ${}^{\mathrm{ad}}V$ are given by:

$$^{\mathrm{Inv}}({}^{\mathrm{ad}}V) = \mathscr{Z}V := \{v \in V \mid hv = vh \text{ for all } h \in H\}\,. \tag{2.4}$$

Indeed, for $v \in \mathscr{Z}V$ and $h \in H$, we have ${}^{h}v = h_{(1)}S(h_{(2)})v = \langle \epsilon, h\rangle v$; so $v \in {}^{\mathrm{Inv}}({}^{\mathrm{ad}}V)$. Conversely, if $v \in {}^{\mathrm{Inv}}({}^{\mathrm{ad}}V)$ and $h \in H$, then (2.3) gives $hv = ({}^{h_{(1)}}v)h_{(2)} = \langle \epsilon, h_{(1)}\rangle v h_{(2)} = vh$.

Example 2.2.5. For G a group the adjoint action of $\Bbbk[G]$ is given by the $\Bbbk$-linear extension of G acting on itself by conjugation.

Example 2.2.6. To he adjoint action of H_4 is given on algebra generators by:

$$\begin{array}{llll} {}^{g}g = g & {}^{g}x = -x & {}^{g}1 = 1 & {}^{g}gx = -gx \\ {}^{x}g = -2gx & {}^{x}x = 0 & {}^{x}1 = 0 & {}^{x}gx = 0 \end{array}$$

If a Hopf subalgebra K of H is stable under the left and right adjoint actions of H, then we say K is ***normal***. In this case, the coideal HK^{+} of H is also a two-sided ideal of H, and hence it is a Hopf ideal. If H is finite dimensional, then the converse holds as well: for any Hopf subalgebra K of H, the Hopf ideal HK^{+} is an ideal of H if and only if K is normal; see [35, Corollary 3.4.4].

2.2.5 Integrals, Semisimplicity and Cosemisimplicity

A $\Bbbk$-algebra, A, is said to be ***augmented*** if A is equipped with a given algebra map $\epsilon\colon A \to \Bbbk$, called the ***augmentation map*** of A. Thus, a Hopf algebra is always an augmented algebra, the augmentation map ϵ being the counit. A ***left integral*** in an augmented algebra (A, ϵ) is an element $\Lambda \in A$ such that $a\Lambda = \epsilon(a)\Lambda$ for all $a \in A$. Right integrals are defined similarly by the condition $\Lambda a = \Lambda\epsilon(a)$ for all $a \in A$. Throughout this thesis, Λ will always be used to denote an integral, superscripts of L or R will be used to distinguish left and right integrals. The space of all left integral of an augmented algebra A will be denoted $\int_A^L$ and similarly the space of right integrals will be denoted $\int_A^R$. The following theorem of Larson and Sweedler shows that finite-dimensional Hopf algebras always have integrals, and they are unique up to scalar multiples [29].

Theorem 2.2.7. *Let H be a finite-dimensional Hopf algebra. Then the spaces $\int_H^L$ and $\int_H^R$ are one dimensional.*

As H^* is also a Hopf algebra it is an augmented algebra with augmentation map μ^*. Thus Theorem 2.2.7 also implies the existence of integrals of H^*. These integrals will commonly be denoted with the lowercase Greek letter λ and superscripts of L and R will be used to distinguish left and right integrals.

Observe for all $h \in H$ we have $h\Lambda^R$ is also a right integral and since the space of right integrals is one dimensional this gives $h\Lambda^R = \langle \alpha, h \rangle \Lambda^R$ for some $\alpha \in H^*$. The element α is easily seen to be an algebra map; thus it is a group-like element of H^*. The element α is called the ***distinguished group-like element of*** H^*. A Hopf algebra is called ***unimodular*** if the distinguished group like element of H^* is the counit of H or equivalently if H contains a central integral. Identifying H with H^{**} we also get a distinguished group-like element of H.

The integral of a Hopf algebra can be used to easily determine when the Hopf algebra is semisimple via the following theorem.

Maschke's Theorem for Hopf Algebras [29]. *A Hopf algebra H is semisimple iff H is finite dimensional and $\epsilon(\int_H^L) \neq 0$ or, equivalently, $\epsilon(\int_H^R) \neq 0$.*

The theorem, though due to Larson and Sweedler, is generally named after Heinrich Maschke, who proved the special case of group algebras: $\Bbbk[G]$ is semisimple if and only if G is finite and $\operatorname{char}\Bbbk$ does not divide the order $|G|$ [49].

It is an immediate consequence of Maschke's Theorem that a semisimple Hopf algebra is unimodular. Another immediate consequence is that semisimple Hopf algebras are separable: $F\otimes H$ remains semisimple for all field extensions $F/\Bbbk$. This follows, because any integral of H is also an integral of $F\otimes H$. If $\operatorname{char}\Bbbk = 0$, then a theorem of Larson and Radford [27] gives that H is also cosemisimple. Moreover, in this case, the antipode of H and H^* must satisfy $S\circ S = \mathrm{Id}$. A Hopf algebra with the latter property is called ***involutory*** [28]. It was shown in [27] that, for an involutory unimodular Hopf algebra, the character of the regular representation is an integral of H^*. We will make frequent use of this fact in this thesis.

Example 2.2.8. For a finite group G, the standard integral of $\Bbbk[G]$ is $\sum_{g\in G} g$. It is easily seen to be central and hence group algebras are unimodular. Applying Maschke's Theorem for Hopf algebras, we get back Maschke's original result that $\Bbbk[G]$ is semisimple iff G is finite and $\operatorname{char}\Bbbk$ does not divide $|G|$. Even in the case where $\operatorname{char}\Bbbk$ divides $|G|$, the group algebra $\Bbbk[G]$ is still involutory as, clearly, the antipode $g\mapsto g^{-1}$ composed with itself is the identity.

Example 2.2.9. The standard integral of $\Bbbk[G]^*$ is ρ_1 where 1 is the identity element of $\Bbbk[G]$. Since $\Bbbk[G]^*$ is commutative, it is clearly unimodular.

Example 2.2.10. The Sweedler algebra H_4 is our first example of a non-unimodular Hopf algebra. The space of left integrals is spanned by $gx+x$, and the space of right integral by $-gx+x$. The distinguished group like element of the dual is defined on algebra generators by $\alpha(x) = 0$ and $\alpha(g) = -1$. Since $\epsilon(gx+x) = 0$, Maschke's Theorem for Hopf algebras tells us that H_4 is not semisimple. In fact, as we have pointed out in Example 2.1.9, the Jacobson radical of H_4 is the 2-dimensional ideal of H_4 that is generated by the element x .

2.2.6 Chevalley Property

For a non-semisimple Hopf algebra the tensor product of two completely reducible modules fails to be completely reducible in general. It is of interest to know when the tensor product inherits complete reduciblity from its factors. Following [3], a Hopf algebra H (not necessarily finite-dimensional) is said to have the ***Chevalley property*** if the tensor product of any two finite-dimensional completely reducible H-modules is again completely reducible. A classical result of Chevalley [6] states that group algebras of arbitrary groups over fields of characteristic zero do in fact have this property; see also [22, Theorem VII.2.2]. We will say that a left H-module M has the ***Chevalley property*** if all tensor powers $M^{\otimes n}$ are completely reducible or, equivalently, the H-module $\mathsf{T}(M) := \bigoplus_{n \in \mathbb{Z}_+} M^{\otimes n}$ is completely reducible. The Chevalley property for H, in the above sense, is evidently equivalent to the Chevalley property for the direct sum of all finite-dimensional irreducible H-modules.

2.3 Frobenius and Symmetric Structure

Frobenius algebras were first studied by Georg Frobenius in [19] and later in more detail by Tadasi Nakayama in [37], [38] and [39]. Hopf algebras carry the structure of a Frobenius algebra in a way that is deeply intertwined with their Hopf structure. This thesis will make extensive use of this fact in proving our main results. This section reviews the necessary background material on Frobenius algebras, including the special case of symmetric algebras, and gives explicit descriptions of these structures for finite-dimensional Hopf algebras.

2.3.1 Frobenius and Symmetric Algebras

Every $\Bbbk$-algebra A carries the "regular" (A, A)-bimodule structure: the left action of $a \in A$ on A is given by the left multiplication operator, a_A, and the right action by right multiplication, ${}_A a$. This structure gives rise to an (A, A)-bimodule structure on the linear dual $A^* = \operatorname{Hom}_{\Bbbk}(A, \Bbbk)$, for which the following notation is customary in the Hopf literature:

$$a\rightharpoonup f\leftharpoonup b \stackrel{\text{def}}{=} f \circ b_A \circ {}_A a \qquad \text{or} \qquad \langle a\rightharpoonup f\leftharpoonup b, c\rangle = \langle f, bca\rangle,$$

for $a, b, c \in A$ and $f \in A^*$. The algebra A is said to be ***Frobenius*** if $A \cong A^*$ as left A-modules. It is a standard fact that this is equivalent to an isomorphism $A \cong A^*$ as right A-modules. Note that even a mere $\Bbbk$-linear isomorphism $A^* \cong A$ forces A to be finite-dimensional; so Frobenius algebras will necessarily have to be finite-dimensional. Fixing a left A-module isomorphism $A \xrightarrow{\sim} A^*$, the image of $1_A \in A$ has special significance; it is called the ***Frobenius form*** afforded by the given isomorphism. Throughout this thesis, Frobenius forms will commonly be denoted with the symbol λ. Then the given isomorphism can be realized as follows:

$$\begin{array}{ccc} A & \xrightarrow{\sim} & A^* \\ \Cup & & \Cup \\ a & \longmapsto & a \rightharpoonup \lambda \end{array} \tag{2.5}$$

A Frobenius algebra can equivalently be defined as a finite-dimensional $\Bbbk$-algebra A equipped with a non-degenerate bilinear form $\langle\, .\, ,\, .\,\rangle : A \times A \to \Bbbk$ that is associative in the sense that $\langle ab, c\rangle = \langle a, bc\rangle$ for all $a, b, c \in A$. Indeed, such a bilinear form gives rise to an isomorphism $A \cong A^*$ as left A-modules via $a \mapsto \langle\, .\, , a\rangle$, or as right A-modules via $a \mapsto \langle a,\, .\,\rangle$. Conversely, given a left A-module isomorphism $A \xrightarrow{\sim} A^*$ with corresponding Frobenius form λ, we obtain an associative non-degenerate bilinear form by $(a, b) \mapsto \lambda(ab)$.

For a Frobenius algebra A, the isomorphism between A and A^* need not be unique and thus there can be multiple Frobenius forms. However, if λ is one such form, then the complete set of possible Frobenius forms is given by $\{u \rightharpoonup \lambda \mid u \in A^\times\}$. Because of this lack of uniqueness we will think of a Frobenius algebra as a pair (A, λ) consisting of the algebra A together with a fixed Frobenius morphism λ. A ***morphism*** $(A, \lambda) \to (B, \mu)$ of Frobenius algebras is a $\Bbbk$-algebra map $f\colon A \to B$ such that $\lambda = \mu \circ f$.

The algebra A is said to be ***symmetric*** if A satisfies the stronger condition that $A \cong A^*$ as (A, A)-bimodules. This amounts to the identity $a \rightharpoonup \lambda = \lambda \leftharpoonup a$ for the corresponding Frobenius form λ, which in turn spells out to λ being a trace form, that is, $\langle \lambda, ab \rangle = \langle \lambda, ba \rangle$ for all $a, b \in A$. Symmetry of A is also equivalent to the corresponding bilinear form $\langle\, .\,,\, .\,\rangle : A \times A \to \Bbbk$ being symmetric. When we say that (A, λ) is symmetric, it is being understood that λ is a trace form. We note that just because an algebra A admits a symmetric structure does not mean that all Frobenius forms are trace forms. For a symmetric algebra (A, λ), the Frobenius morphism $u \rightharpoonup \lambda$ is a trace form iff u is a central unit of A.

Let (A, λ) be a Frobenius algebra. Non-degeneracy of λ gives that there exists $\eta \in \mathrm{Aut}_{\Bbbk}(A)$ such that $a \rightharpoonup \lambda = \lambda \leftharpoonup \eta(a)$ for all $a \in A$. In fact it is easy to check that $\eta \in \mathrm{Aut}_{\Bbbk-\mathrm{alg}}(A)$. The map η is called the ***Nakayama automorphism*** of (A, λ).

Let (A, λ) be a Frobenius algebra. Then the canonical $\Bbbk$-linear isomorphism $\mathrm{End}_{\Bbbk}(A) \cong A \otimes A^*$ gives rise to an isomorphism $\mathrm{End}_{\Bbbk}(A) \cong A \otimes A$ by virtue of (2.5). The element $c_\lambda \in A \otimes A$ that corresponds to $\mathrm{Id}_A \in \mathrm{End}_{\Bbbk}(A)$ under this isomorphism is called the ***Casimir element*** of (A, λ):

$$\begin{array}{ccccc}
\mathrm{End}_{\Bbbk}(A) & \xrightarrow[\text{can.}]{\sim} & A \otimes A^* & \xrightarrow[\text{via } \lambda \text{ as in (2.5)}]{\sim} & A \otimes A \\
\Cup & & & & \Cup \\
\mathrm{Id}_A & \longmapsto & & & c_\lambda := \sum_i x_i \otimes y_i
\end{array}$$

Explicitly, this means that

$$a = \sum_i x_i \langle \lambda, a y_i \rangle = \sum_i \langle \lambda, x_i a \rangle y_i \qquad (a \in A). \tag{2.6}$$

Choosing the x_i to be $\Bbbk$-linearly independent, as we may, this condition states that the x_i form a $\Bbbk$-basis of A such that $\langle \lambda, x_i y_j \rangle = \delta_{i,j}$ for all i, j. The family (x_i, y_i) will be called ***dual bases*** of (A, λ). Using the identity $a = \sum_i x_i \langle \lambda, a y_i \rangle = \sum_i y_i \langle \lambda, a x_i \rangle$ for all $a \in A$, we compute

$$\sum_i a x_i \otimes y_i = \sum_{i,j} x_j \langle \lambda, a x_i y_j \rangle \otimes y_i = \sum_{i,j} x_j \otimes y_i \langle \lambda, y_j a x_i \rangle = \sum_j x_j \otimes y_j a\,.$$

Thus,

$$c_\lambda (a \otimes 1) = (1 \otimes a) c_\lambda\,. \tag{2.7}$$

Now assume that (A, λ) is a symmetric algebra. Then we additionally get the identity

$$\sum_i x_i \otimes y_i = \sum_i y_i \otimes x_i \,. \tag{2.8}$$

Applying the switch automorphism τ to (2.7) and using the fact that c_λ is stable under τ by the above identity, we also obtain $(1 \otimes b)c_\lambda = c_\lambda(b \otimes 1)$ for all $b \in A$. Hence,

$$c_\lambda(a \otimes b) = (b \otimes a)c_\lambda \,. \tag{2.9}$$

The following operator was originally introduced by D.G. Higman [21]:

$$\begin{array}{rcccl} \gamma_\lambda\colon & A & \xrightarrow{\text{can.}} A/[A,A] \longrightarrow & \mathscr{Z}A & \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & \\ & a & \longmapsto & \sum_i x_i a y_i & \end{array} \tag{2.10}$$

Part (a) of the following lemma justifies the claims, implicit in (2.10), that γ_λ is a center-valued trace function on A. We will refer to γ_λ as the ***Casimir trace*** of the symmetric algebra (A, λ).

Lemma 2.3.1. *Let (A, λ) be a symmetric algebra. Then $a\gamma_\lambda(bc) = \gamma_\lambda(cb)a$ for all $a, b, c \in A$. Furthermore,*

$$c_\lambda^2 = (\mathrm{Id} \otimes \gamma_\lambda)(c_\lambda) = (\gamma_\lambda \otimes \mathrm{Id})(c_\lambda) \in \mathscr{Z}(A \otimes A) = \mathscr{Z}A \otimes \mathscr{Z}A \,.$$

Proof. Spelling out (2.9) results in $\sum_i ax_i \otimes by_i = \sum_i x_i b \otimes y_i a$. Multiplying this identity in $A \otimes A$ on the right with $c \otimes 1$ and then applying the multiplication map $A \otimes A \to A$ gives $\sum_i ax_i cby_i = \sum_i x_i bcy_i a$ or, equivalently, $a\gamma_\lambda(cb) = \gamma_\lambda(bc)a$ as claimed. From (2.9) we also obtain $c_\lambda^2(a \otimes b) = c_\lambda(b \otimes a)c_\lambda = (a \otimes b)c_\lambda^2$, proving that $c_\lambda^2 \in \mathscr{Z}(A \otimes A)$. Finally,

$$c_\lambda^2 = \sum_i (x_i \otimes y_i)c_\lambda \underset{(2.9)}{=} \sum_i (x_i \otimes 1)c_\lambda(y_i \otimes 1) = (\gamma_\lambda \otimes \mathrm{Id})(c_\lambda)$$

and the identity $c_\lambda^2 = (\mathrm{Id} \otimes \gamma_\lambda)(c_\lambda)$ also follows by applying the switch operator τ and using the fact that c_λ is τ-invariant by (2.8). $\square$

The Casimir element c_λ can be used to give a convenient trace formula for endomorphisms $f \in \mathrm{End}_\Bbbk(A)$:

$$\mathrm{trace}(f) = \sum_i \langle \lambda, f(x_i)y_i \rangle = \sum_i \langle \lambda, x_i f(y_i) \rangle \, . \tag{2.11}$$

To see this, note that (2.6) gives $f(a) = \sum_i f(x_i)\langle \lambda, ay_i \rangle$ for all $a \in A$. Thus,

$$\begin{array}{cccccc} \mathrm{trace}\colon & \mathrm{End}_\Bbbk(A) & \xrightarrow[\text{can.}]{\sim} & A \otimes A^* & \xrightarrow[\text{evaluation}]{\qquad\qquad} & \Bbbk \\ & \Cup & & \Cup & & \Cup \\ & f & \longmapsto & \sum_i f(x_i) \otimes (y_i \rightharpoonup \lambda) & \longmapsto & \sum_i \langle \lambda, f(x_i)y_i \rangle \end{array}$$

This proves the first equality; the second follows from (2.8) and the fact that λ is a trace form.

Example 2.3.2. The $\Bbbk$-algebra $(M_n(\Bbbk), \mathrm{Tr})$ is a symmetric algebra where Tr is the ordinary matrix trace. Let $E_{i,j}$ be the matrix with a 1 in position (i, j) and 0 everywhere else. Then the Casimir element is given by $\sum_{i,j} E_{i,j} \otimes E_{j,i}$. The Higman trace is identical to the ordinary matrix trace:

$$\gamma_{\mathrm{Tr}}(E_{k,\ell}) = \sum_{i,j} E_{i,j}E_{k,\ell}E_{j,i} = \delta_{k,\ell} \, \mathrm{Id}_{n\times n} \, . \tag{2.12}$$

It is easy to check that the regular character of $M_n(\Bbbk)$ is given by

$$\chi_{\text{reg}} = n \, \mathrm{Tr} \tag{2.13}$$

In particular, if $\mathrm{char}\, \Bbbk \nmid n$, then we may also take χ_{reg} as our Frobenius trace form. Additionally $(M_n(\Bbbk), U \rightharpoonup \mathrm{Tr})$ is a Frobenius algebra iff $\det(U) \neq 0$, and this structure is symmetric iff U is a scalar matrix.

More generally, every finite-dimensional semisimple $\Bbbk$-algebra A is symmetric [11, 9.8]. Indeed, Wedderburn's Structure Theorem allows us to assume that A is simple. Thus, $K = \mathscr{Z} A$ is an extension field of $\Bbbk$ and $A \otimes_K \overline{K}$ is a matrix algebra over an algebraic closure $\overline{K}$ of K. By the above example, the matrix trace is then a Frobenius trace form λ for $A \otimes_K \overline{K}$, and this form clearly does not vanish on A. By simplicity λ does not vanish on any nonzero ideal of A, proving non-degeneracy.

2.3.2 Frobenius Structure of Finite-Dimensional Hopf Algebras

In [29] it was shown that all finite-dimensional Hopf algebras are in fact Frobenius algebras. This structure was later explored in more depth by Schneider and Oberst in [41]. For an English reference, see the lecture notes [47] of a course Schneider gave at Universidad Nacional de Córdoba. Here is a brief summary of the essentials. Throughout, H denotes a finite-dimensional Hopf $\Bbbk$-algebra.

First, the Frobenius form can to chosen to be any left integral $0 \neq \lambda^L \in H^*$ or any right integral $0 \neq \lambda^R \in H^*$. The Nakayama automorphism of (H, λ^L) is given by

$$\eta := m \circ (S^2 \otimes \mu \circ \alpha) \circ \Delta$$

or, in Sweedler notation, $h \mapsto S^2(h_{(1)})\alpha(h_{(2)})$. For (H, λ^R) the Nakayama automorphism is given by $m \circ (\mu \circ \alpha \otimes S^{-2}) \circ \Delta$ or, in Sweedler notation, $h \mapsto \alpha(h_{(1)})S^{-2}(h_{(2)})$. The Casimir element of (H, λ^L) is given by

$$c_{\lambda^L} = \Lambda^R_{(2)} \otimes S(\Lambda^R_{(1)}),$$

where $\langle \lambda^L, \Lambda^R \rangle = 1$. The Casimir element of (H, λ^R) is given by $S(\Lambda^L_{(2)}) \otimes \Lambda^L_{(1)}$ where $\langle \lambda^R, \Lambda^L \rangle = 1$.

From the descriptions of the Nakayama automorphisms, we obtain:

$$\begin{aligned} (H, \lambda^L) \text{ is symmetric} \quad &\iff \quad (H, \lambda^R) \text{ is symmetric} \\ &\iff \quad H \text{ is unimodular and involutory.} \end{aligned}$$

It is possible for H admit the structure of a symmetric algebra even if λ^L and λ^R are not trace forms. A Hopf algebra admits the structure of a symmetric algebra exactly when H is unimodular and S^2 is an inner automorphism. When $S^2(h) = uhu^{-1}$ then $(H, u \rightharpoonup \lambda^L)$ and $(H, u^{-1} \rightharpoonup \lambda^R)$ are symmetric algebras.

Example 2.3.3. For a finite group G, the Hopf algebra $(\Bbbk[G], \rho_1)$ is a symmetric algebra. The Casimir element is given by $\sum_{g \in G} g \otimes g^{-1}$.

Example 2.3.4. For a finite group G, the Hopf algebra $(\Bbbk[G]^*, \sum_{g \in G} g)$ is a symmetric algebra. The Casimir element is given by $\sum_{g \in G} \rho_g \otimes \rho_g$.

Example 2.3.5. Let $\{\rho_1, \rho_g, \rho_x, \rho_{gx}\}$ be a basis of H_4^* that is dual to the basis $\{1, g, x, gx\}$ of H_4. A left integral of H_4^* is given by ρ_{gx} and a right integral is given by ρ_x. So (H_4, ρ_{gx}) is a Frobenius algebra; it has Nakayama automorphism given by $g \mapsto -g$ and $x \mapsto x$ and Casimir element $gx \otimes g - 1 \otimes x - x \otimes 1 + g \otimes x$. Additionally, (H_4, ρ_x) is a Frobenius algebra, with Nakayama automorphism $g \mapsto -g$ and $x \mapsto -x$ and Casimir element $gx \otimes g - 1 \otimes x + x \otimes 1 - g \otimes x$. The Sweedler algebra does not admit a symmetric structure as it is not unimodular.

2.3.3 Frobenius Structure of the Character Algebra

For details concerning the following, see [31, Chapter 12]. Let H be a semisimple Hopf algebra H and assume that $\Bbbk$ is a splitting field for H. Then the representation algebra $\mathscr{R}_{\Bbbk}(H)$ is also a symmetric algebra, with Frobenius trace form

$$\mathrm{Inv} : \mathscr{R}_{\Bbbk}(H) \to \Bbbk\,, \quad [V] \mapsto \dim_{\Bbbk} {}^{\mathrm{Inv}}V\,,$$

where ${}^{\mathrm{Inv}}V$ is the space of H invariants of V. The Casimir element is then given by

$$c_{\mathrm{Inv}} = \sum_{V \in \mathrm{Irr}\, H} [V] \otimes [V^*]\,.$$

Pushing this structure through the character map to gives a symmetric structure on $R(H)$. In this case the Frobenius trace form is given by $\Lambda|_{R(H)}$, where $\Lambda \in H$ is the idempotent integral and we have identified H with H^{**}. The Casimir element becomes $\sum_{\chi \in \mathrm{Irr}(H)} \chi \otimes S(\chi)$. Thus the character map $\chi : (\mathscr{R}_{\Bbbk}(H), \mathrm{Inv}) \to (H^*, \Lambda)$ is a morphism of symmetric algebras. Also of interest is that $\gamma_\Lambda(1)$ is the character of the adjoint representation:

$$\gamma_\Lambda(1) = \sum_{\chi \in \mathrm{Irr}\, H} \chi S(\chi) = \chi_{\mathrm{ad}}\,.$$

2.4 The Drinfel'd Double, Almost Cocommutativity, Quasitriangularity and Factorizablity

Cocommutative Hopf algebras possess many nice properties. However, assuming cocommutativity generally greatly restricts the structure of the available Hopf algebras. For example, the only cocommutative semisimple Hopf algebras over an algebraically closed base field are group algebras of finite groups. The notion of almost cocommutativity, due to Drinfel'd [13], is significantly weaker than cocommutativity while still preserving some of the desirable features of cocommutativity. In detail, a Hopf algebra H having an invertible antipode S is called ***almost cocommutative*** if there exists

$$R := R^1 \otimes R^2 \in (H \otimes H)^\times$$

such that $R\Delta(h)R^{-1} = \tau \circ \Delta(h)$. (Here and in the following, we have suppressed the summation symbol.) Because there may be multiple such R, we will frequently write (H, R) to specify an almost commutative Hopf algebra. In this case, for any $V, W \in \mathfrak{Rep} H$, the map $V \otimes W \xrightarrow{\sim} W \otimes V$, $v \otimes w \mapsto R^{-1}.(w \otimes v)$ is an isomorphism in $\mathfrak{Rep} H$. Thus, the representation ring $\mathscr{R}(H)$ is commutative.

A stronger notion than almost cocommutativity, but still weaker than cocommutativity, is quasitriangularity. In detail, a Hopf algebra H is called ***quasitriangular*** if it is almost cocommutative and the element R above satisfies the following properties:

$$(\mathrm{Id} \otimes \Delta)(R) = i_{13}(R)i_{12}(R) \tag{2.14}$$

$$(\Delta \otimes \mathrm{Id})(R) = i_{13}(R)i_{23}(R) \tag{2.15}$$

Here $i_{12} : H \otimes H \to H \otimes H \otimes H$ is defined via $a \otimes b \mapsto a \otimes b \otimes 1$ and i_{13} and i_{23} are defined likewise. At first glance, these conditions seem arbitrary, but there are in fact strong connections with physics via the Yang-Baxter equation. Given a quasitriangular Hopf algebra, we can construct a map $\Phi = \Phi_R \in \mathrm{Hom}_{\Bbbk}(H^*, H)$ as follows. Let $b := \tau(R)R = b_1 \otimes b_2$ and define $\Phi(f) = b_1 f(b_2)$ for $f \in H^*$. Following Drinfel'd [13], we also put

$$C(H) := \{f \in H^* \mid f(xy) = f(yS^2(x)) \text{ for all } x, y \in H\}\,.$$

The usefulness of the map Φ comes from the following theorem.

Theorem 2.4.1 ([13]). *Let (H,R) be a quasitriangular Hopf algebra and let* $\Phi = \Phi_R \colon H^* \to H$ *be as above. Then, for all* $f \in H^*$ *and* $g \in C(H)$*,*

$$\Phi(g) \in \mathscr{Z}H \qquad \textit{and} \qquad \Phi(fg) = \Phi(f)\Phi(g)\,.$$

A quasitriangular Hopf algebra (H, R) is called ***factorizable*** if the map Φ_R is bijective. An important example of a factorizable Hopf algebra is the ***Drinfel'd double*** $D(H)$ of any finite-dimensional Hopf algebra H. As a coalgebra,

$$D(H) = (H^*)^{\mathrm{cop}} \otimes H\,.$$

To avoid confusion with the ordinary tensor product of Hopf algebras, $H^* \otimes H$, the element $f \otimes h \in D(H)$ is commonly denoted by $f \bowtie h$ and we will also write $D(H) = (H^*)^{\mathrm{cop}} \bowtie H$. Then the multiplication of $D(H)$ is given by the following formula; see [35, Lemma 10.3.11]:

$$(f \bowtie h)(g \bowtie k) = f(h_{(1)} \rightharpoonup g \leftharpoonup S^{-1}(h_{(3)})) \bowtie h_{(2)}k\,. \tag{2.16}$$

The subspaces $\epsilon \bowtie H$ and $(H^*)^{\mathrm{cop}} \bowtie 1$ are in fact Hopf subalgebras of $D(H)$ that are isomorphic to H and $(H^*)^{\mathrm{cop}}$, respectively. We will often view H and $(H^*)^{\mathrm{cop}}$ as Hopf subalgebras of $D(H)$ in this way. Let (h_i) be a $\Bbbk$-basis of H and let (f_i) the dual basis of H^*. Then the factorizable structure of $D(H)$ come from the element

$$R := \sum_i (\epsilon \bowtie h_i) \otimes (f_i \bowtie 1) = \sum_i h_i \otimes f_i \in D(H) \otimes D(H)\,.$$

For a more in-depth review on the Drinfel'd double, see any of the standard references on Hopf algebras. We mention that, for any quasitriangular Hopf algebra (H, R), there is a surjective Hopf algebra map $D(H) \twoheadrightarrow H$, $f \bowtie h \mapsto \sum f(R^1)R^2h$, where $R = \sum R^1 \otimes R^2$ as usual.

Example 2.4.2. Any cocommutative Hopf algebra H is quasitriangular, with $R = 1 \otimes 1$. In particular, all group algebras are quasitriangular. For a finite group G, the multiplication of $D(\Bbbk[G])$ is given by

$$(\rho_h \bowtie g)(\rho_k \bowtie \ell) = \delta_{h,gkg^{-1}} \rho_h \bowtie g\ell$$

for $g, h, k, \ell \in G$

2.5 The Extended Adjoint Representation

Earlier, in Section 2.2.4, the adjoint representation ${}^{\mathrm{ad}}H$ was introduced. In this section, we describe an extension of the adjoint action of H on itself to an action of the Drinfel'd double $D(H)$ on H. We also present some important results pertaining to this extended action. Throughout this section, H will be a semisimple Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic 0. Then the space $C(H) \subseteq H^*$, as introduced earlier in connection with Theorem 2.4.1, becomes the space of trace forms, which is identical to the character algebra $R(H)$:

$$C(H) := \{f \in H^* \mid f(xy) = f(yx) \text{ for all } x, y \in H\} = R(H)\,.$$

Following Y. Zhu [59], the extended $D(H)$-action on H is defined by

$$(f \bowtie h).k = ({}^{h}k) \leftharpoonup S^{-1}(f)\,. \tag{2.17}$$

We will call this action the ***extended adjoint action*** and denote the corresponding module by ${}^{\mathrm{Ad}}H$. Naturally, ${}^{\mathrm{Ad}}H$ provides a powerful tool in studying ${}^{\mathrm{ad}}H$. The main result of Zhu's paper can be summarized as follows:

Theorem 2.5.1. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic 0. Then the algebra $\mathrm{End}_{D(H)}({}^{\mathrm{Ad}}H)$ is isomorphic to the character algebra $C(H)$.*

Using the inclusion of H into $D(H)$ we get the isomorphism ${}^{\mathrm{Ad}}H \cong \Bbbk_\epsilon\!\uparrow_H^{D(H)}$; this will be explained in detail later (§4.3.4). From this view point, the extended adjoint representation was studied further by Burciu [5]. The following result of his is of interest.

Theorem 2.5.2. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic 0. Then the Hopf kernel of the adjoint representation ${}^{\mathrm{ad}}H$ is the Hopf ideal $\zeta(H)^+H$ arising from the largest central Hopf subalgebra, $\zeta(H)$.*

Finally, let us turn to conjugacy classes of a Hopf algebra H. For almost cocommutative semisimple Hopf algebras over the complex numbers, a notion of conjugacy class sums was first studied by Witherspoon [55]. Later, Cohen and Westreich

[8] gave a definition of conjugacy classes in a slightly more general setting as follows:

Definition. A ***conjugacy class*** of a semisimple Hopf algebra H over an algebraically closed field $\Bbbk$ of characteristic 0 is an irreducible subrepresentation of ${}^{\mathrm{Ad}}H$.

Example 2.5.3. For finite group algebras $\Bbbk[G]$, the action of $D(H)$ on ${}^{\mathrm{Ad}}\Bbbk[G]$ is explicitly given by

$$(\rho_h \bowtie k).g = \delta_{h^{-1},kgk^{-1}} kgk^{-1}$$

for $g, h, k \in G$. The above definition of conjugacy classes is closely related to the usual definition of conjugacy classes in a group: the $\Bbbk$-subspaces of $\Bbbk[G]$ that are generated by the conjugacy classes of G are exactly the conjugacy classes of $\Bbbk[G]$.

2.6 Freeness and Faithful (Co)flatness

Given a subalgebra K of a finite-dimensional algebra H, we are interested in the regular representation of H restricted to K. In general, not much can be said about this K-representation, but in the case that K is a Hopf subalgebra, it has a very simple structure:

Nichols-Zoeller Theorem 2.6.1. *Let H be a finite-dimensional Hopf algebra and let K a Hopf subalgebra of H. Then H is a free as left K-module (with the regular multiplication action).*

Example 2.6.1. Let G be a finite group and let H be a subgroup of G. Then, clearly, $\Bbbk H$ is a Hopf subalgebra of $\Bbbk[G]$. In this case, the Nichols-Zoeller Theorem is an immediate consequence of Lagrange's Theorem: G is the disjoint union of the cosets Hg with $g \in H\backslash G$, and hence $\Bbbk[G] = \bigoplus_{g \in H\backslash G} \Bbbk[H]g$. This even holds when G and H are not necessarily finite.

In general, when H is infinite dimensional, the Nichols-Zoeller Theorem fails: infinite-dimensional Hopf algebras need not be free over all Hopf subalgebras; see, for example, Oberst and Schneider [42, Proposition 10]. Instead, it is better to ask for the weaker property of H being faithfully flat as (left) module over Hopf subalgebras or, alternatively, of H being ***faithfully coflat***. In order to define what it means for H is be faithfully coflat over K, we need to introduce the notion of a ***cotensor product***. Let C be a coalgebra then, given a right C-comodule V and a left C-comodule W, the cotensor product of V and W is the $\Bbbk$-vector space that is defined by

$$V\Box_C W := \{\sum_i v_i \otimes w_i | \sum_i \rho(v_i) \otimes w^i = \sum_i v_i \otimes \rho(w_i)\} .$$

We say that V if a coflat right C-comodule if the functor $V\Box_C-$ from left C-comodules to $\Bbbk$-vector spaces preserves exact sequences; similarly for left coflatness. If the functor $V\Box_C-$ is additionally faithful, in the sense that the $\Bbbk$-linear map $V\Box_C f\colon V\Box_C W \to V\Box_C W'$ is nonzero for any nonzero map $f\colon W \to W'$ of left C-comodules, then we say that V is faithfully coflat. In general, Hopf algebras are not faithfully (co)flat over Hopf subalgebras. However, many theorems are known for determining when this occurs. For example, H is faithfully flat over all its Hopf subalgebras when the coradical of H is cocommutative [33].

CHAPTER 3

FROBENIUS DIVISIBILITY

In this chapter, I present my main results on Frobenius Divisibility. The chapter is organized into four main sections: the first discusses general Frobenius Divisibility for semisimple algebras; the second then uses these results to construct an algorithm for studying the degrees of irreducible representations; the third section applies our general results to Hopf algebras to give alternate proofs of many of the main known results, putting them into a common framework; and lastly, section four extends some of these results by considering the Hopf center of characters.

3.1 Semisimple Algebras

Let A be a finite-dimensional semisimple $\Bbbk$-algebra. Consider the Wedderburn isomorphism

$$\begin{array}{ccc} A & \xrightarrow{\sim} & \prod_{S\in\operatorname{Irr} A} \operatorname{End}_{D(S)}(S) \\ \uplus & & \uplus \\ a & \longmapsto & (a_S) \end{array} \tag{3.1}$$

where $D(S) = \operatorname{End}_A(S)$ is the Schur division algebra of S and $a_S \in \operatorname{End}_{D(S)}(S)$ is given by the action of a on S. The ***primitive central idempotent*** $\boldsymbol{e}(S) \in \mathscr{Z}A$ is the element corresponding to $(0, \ldots, 0, \operatorname{Id}_S, 0, \ldots, 0) \in \prod_{S\in\operatorname{Irr} A} \operatorname{End}_{D(S)}(S)$ under the Wedderburn isomorphism; so

$$\boldsymbol{e}(S)_T = \delta_{S,T}\,\mathrm{Id}_S \qquad (S, T \in \operatorname{Irr} A).$$

The following proposition gives a formula for $\boldsymbol{e}(S)$ using the ***character*** χ_S of S, defined by $\langle \chi_S, a\rangle = \operatorname{trace}(a_S)$ for $a \in A$, together with data coming from the structure of A as a symmetric algebra (§2.3.1).

Proposition 3.1.1. *Let A be a finite-dimensional semisimple $\Bbbk$-algebra with Frobenius trace form λ. Then, for each $S \in \operatorname{Irr} A$, we have the following formula in $A = \Bbbk \otimes A$,*

$$\gamma_\lambda(1)\,\boldsymbol{e}(S) = d(S)\,(\chi_S \otimes \mathrm{Id}_A)(c_\lambda) = d(S)\,(\mathrm{Id}_A \otimes \chi_S)(c_\lambda),$$

where $d(S) = \dim_{D(S)} S$. In particular, $\gamma_\lambda(1)_S = 0$ if and only if $\chi_S = 0$ or $d(S)\,1_\Bbbk = 0$.

Proof. First, $(\chi_S \otimes \mathrm{Id}_A)(c_\lambda) = (\mathrm{Id}_A \otimes \chi_S)(c_\lambda)$ by (2.8). So we only need to show that $\gamma_\lambda(1)\,\boldsymbol{e}(S) = d(S)\,(\mathrm{Id}_A \otimes \chi_S)(c_\lambda)$, which amounts to the condition

$$\langle \lambda, \gamma_\lambda(1)\,\boldsymbol{e}(S)\,a\rangle = d(S)\,\langle \lambda, (\mathrm{Id}_A \otimes \chi_S)(c_\lambda)a\rangle$$

for all $a \in A$ by nondegeneracy of λ. But

$$\langle \lambda, (\mathrm{Id}_A \otimes \chi_S)(c_\lambda)a\rangle \underset{(2.8)}{=} \langle \lambda, \textstyle\sum_i x_i \langle \chi_S, y_i\rangle a\rangle = \sum_i \langle \chi_S, y_i\rangle \langle \lambda, x_i a\rangle = \langle \chi_S, a\rangle,$$

and so we need to show that

$$\langle \lambda, \gamma_\lambda(1)\,\boldsymbol{e}(S)\,a\rangle = d(S)\,\langle \chi_S, a\rangle \qquad (a \in A) \tag{3.2}$$

For this, we use the regular character,

$$\langle \chi_{\text{reg}}, \boldsymbol{e}(S)\,a\rangle \underset{(2.11)}{=} \langle \lambda, \gamma_\lambda(1)\,\boldsymbol{e}(S)\,a\rangle.$$

On the other hand, by Wedderburn's Structure Theorem, the regular representation of A has the form $A_{\text{reg}} \cong \bigoplus_{T \in \operatorname{Irr} A} T^{\oplus d(T)}$, which gives $\chi_{\text{reg}} = \sum_{T \in \operatorname{Irr} A} d(T)\chi_T$. Since $\boldsymbol{e}(S) \rightharpoonup \chi_T = \chi_T \leftharpoonup \boldsymbol{e}(S) = \delta_{S,T}\chi_S$, we obtain

$$\boldsymbol{e}(S) \rightharpoonup \chi_{\text{reg}} = \chi_{\text{reg}} \leftharpoonup \boldsymbol{e}(S) = d(S)\chi_S \tag{3.3}$$

Therefore, $\langle \chi_{\text{reg}}, \boldsymbol{e}(S)a\rangle = d(S)\langle \chi_S, a\rangle$, proving (3.2). Finally, (3.2) also shows that $\gamma_\lambda(1)\,\boldsymbol{e}(S) = 0$ if and only if $d(S)\chi_S = 0$, which implies the last assertion in the proposition. □

3.1.1 The Casimir Square

Continuing to assume that A is a finite-dimensional semisimple $\Bbbk$-algebra, we now describe the Casimir square $c_\lambda^2 \in \mathscr{Z}A \otimes \mathscr{Z}A$ in terms of the following isomorphism coming from the Wedderburn isomorphism (3.1):

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\sim} & \prod_{S,T \in \operatorname{Irr} A} \operatorname{End}_{D(S)}(S) \otimes \operatorname{End}_{D(T)}(T) \\ \cup\!\!\!\!\!\!\cup & & \cup\!\!\!\!\!\!\cup \\ a \otimes b & \longmapsto & \big(a_S \otimes b_T\big) \end{array} \tag{3.4}$$

We will write $t_{S,T} \in \operatorname{End}_{D(S)}(S) \otimes \operatorname{End}_{D(T)}(T)$ for the (S,T)-component of the image of $t \in A \otimes A$ under the above isomorphism; so $(a \otimes b)_{S,T} = a_S \otimes b_T$. Recall that $S \in \operatorname{Irr} A$ is absolutely irreducible if and only if $D(S) = \Bbbk$.

Theorem 3.1.2. *Let A be a finite-dimensional semisimple $\Bbbk$-algebra with Frobenius trace form λ. Then $(c_\lambda)_{S,T} = 0$ for $S \neq T \in \operatorname{Irr} A$. If S is absolutely irreducible, then $(\dim_\Bbbk S)^2\,(c_\lambda^2)_{S,S} = \gamma_\lambda(1)_S^2$.*

Proof. For $S \neq T$, the identity $(a \otimes b)c_\lambda = c_\lambda(b \otimes a)$ in (2.9) gives

$$(c_\lambda)_{S,T} = \big((\boldsymbol{e}(S) \otimes \boldsymbol{e}(T))c_\lambda\big)_{S,T} = \big(c_\lambda(\boldsymbol{e}(T) \otimes \boldsymbol{e}(S))\big)_{S,T} = (c_\lambda)_{S,T}(0_S \otimes 0_T) = 0.$$

It remains to consider $(c_\lambda^2)_{S,S}$ for S absolutely irreducible. Then, for $c \in \mathscr{Z}(A)$, the operator $c_S \in \operatorname{End}_\Bbbk(S)$ is a scalar and $\chi_S(c) = d(S)c_S$ with $d(S) = \dim_\Bbbk S$. Therefore, writing $\rho_S(a) = a_S$ for $a \in A$, we calculate

$$\begin{aligned} d(S)(\rho_S \circ \gamma_\lambda)(a) &= (\chi_S \circ \gamma_\lambda)(a) = \chi_S(\textstyle\sum_i x_i a y_i) = \chi_S(\textstyle\sum_i a y_i x_i) \\ &= \chi_S(a\,\gamma_\lambda(1)) = \chi_S(a)\,\gamma_\lambda(1)_S \end{aligned} \tag{3.5}$$

and further

$$\begin{aligned} d(S)^2\,(c_\lambda^2)_{S,S} &= d(S)^2\,(\rho_S \otimes \rho_S)\big((\gamma_\lambda \otimes \operatorname{Id})(c_\lambda)\big) \\ &= d(S)^2\,\big((\rho_S \circ \gamma_\lambda) \otimes \rho_S\big)(c_\lambda) \\ &\underset{(3.5)}{=} d(S)\,(\chi_S \otimes \rho_S)(c_\lambda)\,\gamma_\lambda(1)_S \\ &= (\operatorname{Id}_\Bbbk \otimes \rho_S)\big(d(S)\,(\chi_S \otimes \operatorname{Id})(c_\lambda)\big)\,\gamma_\lambda(1)_S \\ &= \rho_S\big(\boldsymbol{e}(S)\gamma_\lambda(1)\big)\,\gamma_\lambda(1)_S = \gamma_\lambda(1)_S^2\,, \end{aligned}$$

which completes the proof of the theorem. □

3.1.2 Integrality and Divisibility

Theorem 3.1.2 is a useful tool in proving certain divisibility results for the degrees of irreducible representations. A semisimple $\Bbbk$-algebra A is said to be ***split*** if $D(S) = \Bbbk$ holds for all $S \in \operatorname{Irr} A$. All split semisimple $\Bbbk$-algebras are finite dimensional, and any finite-dimensional semisimple $\Bbbk$-algebra over an algebraically closed field $\Bbbk$ is split.

Corollary 3.1.3. *Let A be a split semisimple $\Bbbk$-algebra with Frobenius trace form λ. Assume that* $\operatorname{char} \Bbbk = 0$ *and that* $\gamma_\lambda(1) \in \mathbb{Z}$. *Then the following are equivalent:*

(i) *The degree of every irreducible representation of A divides $\gamma_\lambda(1)$;*

(ii) *the Casimir element c_λ is integral over $\mathbb{Z}$.*

Proof. Theorem 3.1.2 gives the formula

$$(c_\lambda^2)_{S,S} = \left(\frac{\gamma_\lambda(1)}{\dim_\Bbbk S}\right)^2. \tag{3.6}$$

If (i) holds, then the isomorphism (3.4) sends $\mathbb{Z}[c_\lambda^2]$ to $\prod_{S \in \operatorname{Irr} H} \mathbb{Z}$, because $(c_\lambda)_{S,T} = 0$ for $S \neq T$ by Theorem 3.1.2. Thus, $\mathbb{Z}[c_\lambda]$ is a finitely generated $\mathbb{Z}$-module and (ii) follows. Conversely, (ii) implies that c_λ^2 also satisfies a monic polynomial over $\mathbb{Z}$ and all $(c_\lambda^2)_{S,S}$ satisfy the same polynomial. Therefore, the fractions $\frac{\gamma_\lambda(1)}{\dim_\Bbbk S}$ must be integers, proving (i). □

Next, for a given homomorphism $(A, \lambda) \to (B, \mu)$ of symmetric algebras, we may consider the induced module $\operatorname{Ind}_A^B S = B \otimes_A S$ for each $S \in \operatorname{Irr} A$

Corollary 3.1.4. *Let A be a split semisimple algebra over a field $\Bbbk$ of characteristic 0 and let λ be a Frobenius trace form for A. Furthermore, let (B, μ) be a symmetric $\Bbbk$-algebra such that $\gamma_\mu(1) \in \Bbbk$ and let $\phi\colon (A, \lambda) \to (B, \mu)$ be a homomorphism of symmetric algebras. If the Casimir element c_λ is integral over $\mathbb{Z}$, then so is the scalar $\frac{\gamma_\mu(1)}{\dim_\Bbbk \operatorname{Ind}_A^B S}$ for each $S \in \operatorname{Irr} A$.*

Proof. It suffices to show that

$$\frac{\gamma_\mu(1)}{\dim_\Bbbk \operatorname{Ind}_A^B S} = \frac{\gamma_\lambda(1)_S}{\dim_\Bbbk S}. \tag{3.7}$$

Indeed, by Theorem 3.1.2, the square of the fraction on the right equals $(c_\lambda^2)_{S,S}$, which is integral over $\mathbb{Z}$ if c_λ is. To check (3.7), let us put $e := \boldsymbol{e}(S)$ for brevity. Then $S^{\oplus \dim_{\Bbbk} S} \cong Ae$ and so $\operatorname{Ind}_A^B S^{\oplus \dim_{\Bbbk} S} \cong B\phi(e)$. Since $\phi(e) \in B$ is an idempotent, $\dim_{\Bbbk} B\phi(e) = \operatorname{trace}({}_B\phi(e))$. Therefore,

$$\begin{aligned}\dim_{\Bbbk} \operatorname{Ind}_A^B S^{\oplus \dim_{\Bbbk} S} &= \operatorname{trace}({}_B\phi(e)) = \langle \mu, \phi(e)\, \gamma_\mu(1)\rangle = \langle \mu, \phi(e)\rangle\, \gamma_\mu(1) \\ &= \langle \lambda, e\rangle\, \gamma_\mu(1) = \frac{(\dim_{\Bbbk} S)^2}{\gamma_\lambda(1)_S}\gamma_\mu(1).\end{aligned}$$

The desired equality (3.7) is immediate from this. □

We will say a finite-dimensional $\Bbbk$-algebra A satisfies the ***Frobenius Divisibility Property***, often abbreviated **FD**, if the dimension of every $S \in \operatorname{Irr} A$ divides $\dim_{\Bbbk} A$.

Example 3.1.5. Let A be a split semisimple $\Bbbk$-algebra and assume that $\operatorname{char} \Bbbk = 0$. Then the Wedderburn isomorphism (3.1) takes the form $A = \prod_{S \in \operatorname{Irr} A} A_S$ with $A_S \cong \operatorname{End}_{\Bbbk}(S)$. Thus, the regular character χ_{reg} of A can be written as follows:

$$\chi_{\text{reg}} = \sum_{S \in \operatorname{Irr} A} (\chi_{\text{reg}})_S = \sum_{S \in \operatorname{Irr} A} (\dim_{\Bbbk} S) \operatorname{Tr}_S, \tag{3.8}$$

where $(\chi_{\text{reg}})_S$ and Tr_S denote the regular character and the ordinary trace function of $A_S \cong \operatorname{End}_{\Bbbk}(S)$, repectively, and both are understood to vanish on all components A_T with $T \neq S$; the last equality in (3.8) follows from (2.13). Instead of χ_{reg}, we will often work with the following element of A^*, which will be referred to as ***normalized regular character*** of A:

$$\hat{\chi}_{\text{reg}} := \frac{\chi_{\text{reg}}}{\dim_{\Bbbk} A}.$$

Recall from Example 2.3.2 that $\gamma_{\operatorname{Tr}_S}(a) = \operatorname{Tr}_S(a) \operatorname{Id}$ for all $a \in A_S$. Using $\hat{\chi}_{\text{reg}}$ as our Frobenius trace form, we obtain for $a = \sum_S a_S \in A$,

$$\begin{aligned}\gamma_{\hat{\chi}_{\text{reg}}}(a) &= \sum_{S \in \operatorname{Irr} A} \gamma_{\frac{\dim_{\Bbbk} S}{\dim_{\Bbbk} A} \operatorname{Tr}_S}(a_S) = \sum_{S \in \operatorname{Irr} A} \tfrac{\dim_{\Bbbk} A}{\dim_{\Bbbk} S} \operatorname{Tr}_S(a_S) \\ &= \sum_{S \in \operatorname{Irr} A} \tfrac{\dim_{\Bbbk} A}{\dim_{\Bbbk} S} \operatorname{Tr}_S(a_S)\boldsymbol{e}(S)\end{aligned}$$

In particular,

$$\gamma_{\hat{\chi}_{\text{reg}}}(1) = \sum_{S \in \operatorname{Irr}(A)} (\dim_{\Bbbk} A)\boldsymbol{e}(S) = (\dim_{\Bbbk} A)1_A\,.$$

Thus, by Corollary 3.1.3, A satisfies **FD** iff $c_{\hat{\chi}_{\text{reg}}}$ is integral.

Example 3.1.6. Consider the group algebra $\Bbbk[G]$ of any finite group G and assume that $\Bbbk$ is a splitting field for $\Bbbk[G]$ with $\operatorname{char} \Bbbk = 0$; so $\Bbbk[G]$ is split semisimple. The Frobenius form $\lambda = \rho_1$ of Example 2.3.3 is the same as the one specified in Example 3.1.5. The corresponding Casimir element is

$$c_\lambda = \sum_{g \in G} g \otimes g^{-1}.$$

Thus, $c_\lambda \in \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G]$, a subring of $\Bbbk[G] \otimes \Bbbk[G]$ that is finitely generated over $\mathbb{Z}$. Therefore, condition (ii) in Corollary 3.1.3 is satisfied. Moreover, $\gamma_\lambda(1) = |G| \in \mathbb{Z}$ as required in Corollary 3.1.3, and so the corollary yields that the degrees of all irreducible representations of $\Bbbk[G]$ divide $\gamma_\lambda(1) = |G|$, as stated in Frobenius' classical theorem.

3.2 Algorithms

Throughout this section A, will denote a split semisimple $\Bbbk$-algebra and we assume that $\operatorname{char} \Bbbk = 0$. It is usually prohibitively complex to check if A satisfies **FD** by finding all the irreducible representations. However, using the approach presented in Section 3.1.2, it becomes a far simpler matter; in fact, the test for **FD** can be done completely algorithmically. The first step in the algorithm is to compute the normalized regular character introduced in Example 3.1.5,

$$\lambda = \hat{\chi}_{\text{reg}} := \frac{\chi_{\text{reg}}}{\dim_{\Bbbk} A}\,.$$

The next step is to compute the corresponding Casimir element c_λ. To this end, let $(b_i)_{i=1}^{\dim_{\Bbbk} A}$ be a chosen $\Bbbk$-basis of A and write $c_\lambda = \sum_i b_i \otimes (\sum_k \alpha_{i,k} b_k)$ for suitable scalars $\alpha_{i,k} \in \Bbbk$. Then, for all $i, j \leq \dim_{\Bbbk} A$,

$$\langle \lambda, b_j (\sum_k \alpha_{i,k} b_k) \rangle = \delta_{i,j}\,.$$

This is a system of $(\dim_{\Bbbk} A)^2$ linear equations for the $(\dim_{\Bbbk} A)^2$ unknown scalars $\alpha_{i,k}$. To obtain the coefficient matrix of the system, we need to know the products $b_j b_k$, that is, the multiplication matrix of A. Solving the system, which is possible by non-degeneracy of λ, we find the Casimir element c_λ.

Recall from Example 3.1.5 that $\gamma_\lambda(1) = \dim_{\Bbbk} A$. Therefore, using Theorem 3.1.2 we obtain the following formula for the image of c_λ^2 under the multiplication map $m\colon A \otimes A \to A$ of A:

$$C := m(c_\lambda^2) = \sum_{S \in \operatorname{Irr} A} \tfrac{(\dim_{\Bbbk} A)^2}{(\dim_{\Bbbk} S)^2} \boldsymbol{e}(S)\,. \tag{3.9}$$

Observe that

$$\begin{aligned} A \text{ satisfies } \mathbf{FD} &\iff \tfrac{(\dim_{\Bbbk} A)^2}{(\dim_{\Bbbk} S)^2} \in \mathbb{Z} \text{ for all } S \in \operatorname{Irr} A \\ &\iff C \text{ is integral over } \mathbb{Z}\,. \end{aligned}$$

The advantage of working with C rather than c_λ^2 is that $C \in A$, a space of smaller dimension than $A \otimes A$. So computations with C are simpler and faster. Writing $a_S = a\boldsymbol{e}(S)$ for $a \in A$, Equation (3.9) gives $(C^2 - \tfrac{(\dim_{\Bbbk} A)^2}{(\dim_{\Bbbk} S)^2} C)_S = 0$. Hence,

$$\prod_{S \in \operatorname{Irr} A} (C^2 - \tfrac{(\dim_{\Bbbk} A)^2}{(\dim_{\Bbbk} S)^2} C) = 0.$$

Let $D = \{d \in \mathbb{N} \mid d | \dim_{\Bbbk} A, d \leq \sqrt{\dim_{\Bbbk} A}\}$. Then C is integral over $\mathbb{Z}$ iff C satisfies the following polynomial in $\mathbb{Z}[x]$:

$$p(x) := \prod_{d \in D} x^2 - \tfrac{(\dim_{\Bbbk} A)^2}{d^2} x.$$

Hence, we can tell if A satisfies **FD** simply by evaluating C on $p(x)$. While this approach works well for many examples, the computational difficulty scales with the number of divisors of $\dim_{\Bbbk} A$. For $\dim_{\Bbbk} A$ large and highly composite, the polynomial $p(x)$ can be of large degree, which can often lead to rounding errors if floating point numbers are being used. This necessitates the use of symbolic tools, which while more accurate come at the cost of a large computational efficiency trade off.

For an alternative algorithm, let $L_C \in \operatorname{End}_{\Bbbk}(A)$ be the operator given by (left) multiplication with C. By Equation (3.9), the eigenvalues of L_C are of the form $\frac{(\dim_{\Bbbk} A)^2}{(\dim_{\Bbbk} S)^2}$ with $S \in \operatorname{Irr} A$. The multiplicity of the eigenvalue $\frac{(\dim_{\Bbbk} A)^2}{(\dim_{\Bbbk} S)^2}$ is $(\dim_{\Bbbk} S)^2$ times the number of non-isomorphic irreducible representations having dimension equal to $\dim_{\Bbbk} S$. One problem is that the algorithms for finding the eigenvalues of an operator scales quickly with dimension. One way to alleviate this problem is to instead work with $\mathscr{Z}A$. The best way to compute $\mathscr{Z}A$ will vary case by case, but it can always be computed as the image of the Higman trace. Then since $C \in \mathscr{Z}A$ we let L'_C be the multiplication operator of C on $\mathscr{Z}A$. Then L'_C has the same eigenvalues of C but the eigenvalue $\frac{(\dim_{\Bbbk} A)^2}{(\dim_{\Bbbk} S)^2}$ now has multiplicity equal to the number of non-isomorphic irreducible representations of dimension $\dim_{\Bbbk} S$.

If the multiplicities are not needed we can take advantage of the fact that we have a finite collection of possible eigenvalues of L_C (or equivalently L'_C). There is an irreducible representation of dimension n iff:

$$\det(L_C - \frac{(\dim_{\Bbbk} A)^2}{n^2} \operatorname{Id}) = 0.$$

As with the other methods, rounding errors can become a problem when $\dim_{\Bbbk} A$ is large if measures are not taken to curb them.

I have implemented these algorithms in Python code. The code takes the matrix corresponding to the multiplication of the algebra as input and can run any of the algorithms mentioned above. For more details on this code, see Section 5.

3.3 Hopf Algebras

3.3.1 Frobenius Divisibility for Hopf Algebras

We first offer an extension, due to Cuadra and Meir [10, Theorem 3.4], of Frobenius' Divisibility Theorem to the context of Hopf algebras. The proof will be identical to the one given in Example 3.1.6 for finite group algebras.

Theorem 3.3.1. *Let H be a split semisimple Hopf algebra over a field $\Bbbk$ of characteristic 0 and let $\Lambda \in H$ be the unique integral of H such that $\langle \epsilon, \Lambda \rangle = 1$. Then the* *following are equivalent:*

(i) **FD** *holds for H, that is,* $\dim_k S$ *divides* $\dim_{\Bbbk} H$ *for all* $S \in \operatorname{Irr} H$*;*

(ii) *The element* $\dim_{\Bbbk}(H)\Lambda_{(1)} \otimes S(\Lambda_{(2)})$ *is integral over* $\mathbb{Z}$*.*

Proof. Choose the Frobenius trace form λ for H to be the normalized regular character. Since $\langle \chi_{\text{reg}}, \Lambda \rangle = 1$, the integral $\Lambda' := \dim_{\Bbbk}(H)\Lambda$ satisfies $\langle \lambda, \Lambda' \rangle = 1$ as required in §2.3.2. Thus, the Casimir element for λ is given by

$$c_\lambda = \dim_{\Bbbk}(H)\Lambda_{(1)} \otimes S(\Lambda_{(2)})$$

and $\gamma_\lambda(1) = \dim_{\Bbbk} H$. Thus, the theorem is a consequence of Corollary 3.1.3. □

3.3.2 The Class Equation

We now prove the celebrated class equation due to Kac [24, Theorem 2] and Y. Zhu [58, Theorem 1]. The proof given here is based on [30]. To set the stage, let us assume that H is a semisimple Hopf algebra over an algebraically closed field $\Bbbk$ with $\operatorname{char} \Bbbk = 0$ and consider $\mathscr{R}_{\Bbbk}(H)$, with the Frobenius structure discussed in Section 2.3.3.

Theorem 3.3.2. *Let H be a semisimple Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic 0. Then* $\dim_{\Bbbk} \operatorname{Ind}_{\mathscr{R}_{\Bbbk}(H)}^{H^*} M$ *divides* $\dim_{\Bbbk} H$ *for every M in* $\operatorname{Irr} \mathscr{R}_{\Bbbk}(H)$*.*

Proof. This is an application of Corollary 3.1.4 to the morphism of symmetric algebras that is given by the character map $\chi : (\mathscr{R}_{\Bbbk}(H), \text{Inv}) \to (H^*, \Lambda)$, where $\Lambda \in H$ be the unique integral of H such that $\langle \epsilon, \Lambda \rangle = 1$; see Section 2.3.3. We have already shown that $\gamma_\lambda(1) = \dim_{\Bbbk} H$. The last condition needed is that c_{Inv} is integral. However, this is true since $c_{\text{Inv}} \in \mathscr{R}(H)^{\otimes 2}$ a finitely generated $\mathbb{Z}$-module. Therefore, Corollary 3.1.4 applies and yields that the fraction $\frac{\dim_{\Bbbk} H}{\dim_{\Bbbk} \operatorname{Ind}_{\mathscr{R}_{\Bbbk}(H)}^{H^*} M}$ is integral over $\mathbb{Z}$, proving the theorem. □

Frobenius' Divisibility Theorem for finite group algebras $\Bbbk[G]$ also follows from part (a) above applied to $H = (\Bbbk[G])^*$, because $\chi_{\Bbbk} \colon \mathscr{R}_{\Bbbk}(H) \xrightarrow{\sim} H^* = \Bbbk[G]$ in this case.

3.3.3 Factorizable Hopf Algebras

Schneider [46] proved the following result for factorizable semisimple Hopf algebras H over an algebraically closed field $\Bbbk$ of characteristic 0: if $S \in \operatorname{Irr} H$, then $(\dim_{\Bbbk} S)^2$ divides $\dim_{\Bbbk} H$. In this papragraph, we will give an alternate proof of Schneider's result using our approach to Frobenius divisibility via Frobenius algebras.

As in Section 2.4, we let $\Phi = \Phi_R \colon H^* \xrightarrow{\sim} H$ denote the isomorphism associated to $H = (H, R)$; it is defined by

$$\Phi(f) = b_1 f(b_2) \qquad (f \in H^*)\,,$$

with $b := \tau(R)R = b_1 \otimes b_2 \in (H \otimes H)^{\times}$. Recall also that $R(H) \subseteq H^*$ is the character algebra, that is, the image of the character map $\chi : \mathscr{R}_{\Bbbk}(H) \to (H^*, \Lambda)$. It follows directly from Theorem 2.4.1 that $\Phi|_{R(H)}$ is an algebra map whose image is contained in $\mathscr{Z}H$. Since $R(H)$ and $\mathscr{Z}H$ have the same dimension, we obtain an isomorphism of algebras

$$\Phi|_{R(H)} \colon R(H) \xrightarrow{\sim} \mathscr{Z}H\,.$$

Furthermore:

Proposition 3.3.3. *Let $H = (H, R)$ be a semisimple factorizable Hopf algebra and assume that $\Bbbk$ is algebraically closed with $\operatorname{char} \Bbbk = 0$. Let $\Lambda \in H$ be the unique integral of H such that $\langle \epsilon, \Lambda \rangle = 1$ and let $\lambda \in H^*$ be the normalized regular character. Then the map $\Phi|_{R(H)} : (R(H), \Lambda) \to (H, \lambda)$ is a morphism of Frobenius algebras.*

Proof. It just needs to be shown that $\lambda \circ \Phi = \Lambda$. To do this, we will need to some known facts about Φ. First, $b_1\epsilon(b_2) = (\operatorname{Id} \otimes \epsilon)(\tau(R)R) = 1$, because $(\operatorname{Id} \otimes \epsilon)(R) = (\operatorname{Id} \otimes \epsilon)(\tau(R)) = 1$ by [35, 10.1.11]. It follows that $\epsilon(\Phi(f)) = f(b_1\epsilon(b_2)) = f(1)$ for any $f \in H^*$. Using Theorem 2.4.1 again, we obtain

$$\Phi(\lambda)h = \Phi(\lambda)\Phi(\Phi^{-1}(h)) = \Phi(\lambda\Phi^{-1}(h)) = \Phi(\lambda)(\Phi^{-1}(h))(1) = \Phi(\lambda)\epsilon(h).$$

So Φ preserves integrals. In particular $\Phi(\lambda)$ is an integral of H, and we also have $\epsilon(\Phi(\lambda)) = \lambda(1) = 1 = \epsilon(\Lambda)$. Since the space of integrals is 1-dimensional, it follows that $\Phi(\lambda) = \Lambda$, as desired. $\square$

We are now ready to prove the main result of the section.

Theorem 3.3.4. *Let H be a semisimple factorizable Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic 0. Then $(\dim_{\Bbbk} S)^2$ divides $\dim_{\Bbbk} H$ for every $S \in \operatorname{Irr} H$.*

Proof. Recall that the character map $\chi : (\mathscr{R}_{\Bbbk}(H), \mathrm{Inv}) \hookrightarrow (H^*, \Lambda)$ is a monomorphism of symmetric algebras with image $R(H)$. Thus, by Proposition 3.3.3, the composition of $\Phi|_{R(H)}$ with χ is a monomorphism of symmetric algebras,

$$\phi\colon (\mathscr{R}_{\Bbbk}(H), \mathrm{Inv}) \hookrightarrow (H, \lambda)\,,$$

with $\operatorname{Im} \phi = \mathscr{Z} H$. If $\boldsymbol{e}(S) \in \mathscr{Z} H$ is the central primitive idempotent associated to S, then $\phi^{-1}(\boldsymbol{e}(S))$ is a primitive idempotent in $\mathscr{R}_{\Bbbk}(H)$; so

$$I_S := \mathscr{R}_{\Bbbk}(H)\phi^{-1}(\boldsymbol{e}(S))$$

is a 1-dimensional ideal of the representation algebra $\mathscr{R}_{\Bbbk}(H)$. Applying Corollary 3.1.4 to the map ϕ, we obtain that $\dim_{\Bbbk}(\operatorname{Ind}^H_{\mathscr{R}_{\Bbbk}(H)} I_S)$ divides $\gamma_\lambda(1) = \dim_{\Bbbk} H$. Finally, $\dim_{\Bbbk}(\operatorname{Ind}^H_{\mathscr{R}_{\Bbbk}(H)} I_S) = \dim_{\Bbbk}(H\boldsymbol{e}(S)) = (\dim_{\Bbbk} S)^2$, finishing the proof. $\square$

3.4 Hopf Centers and Frobenius Divisibility

3.4.1 Hopf Commutators

Given two elements h and k of an arbitrary Hopf algebra H, we define the ***Hopf commutator*** $[h, k]$ by

$$[h, k] = h_{(1)}k_{(1)}S(h_{(2)})S(k_{(2)}).$$

Note the following relations with the adjoint action:

$$^{h}k = [h, k_{(1)}]\, k_{(2)} \qquad \text{and} \qquad [h, k] = {}^{h}k_{(1)}\, S(k_{(2)}) \tag{3.10}$$

Commutators are useful for determining when two Hopf subalgebras of H commute as shown in the following lemma.

Lemma 3.4.1. *Let K and L be Hopf subalgebras of H. The following conditions are equivalent:*

(i) $kl = lk$ *for all* $k \in K$ *and* $l \in L$*;*

(ii) ${}^{k}l = \epsilon(k)l$ *for all* $k \in K$, $l \in L$*;*

(iii) ${}^{l}k = \epsilon(l)k$ *for all* $k \in K$, $l \in L$*;*

(iv) $[l, k] = \epsilon(l)\epsilon(k)$ *for all* $k \in K$, $l \in L$.

Proof. Assuming (i) we compute ${}^{k}l = k_{(1)}lS(k_{(2)}) = k_{(1)}S(k_{(2)})l = \epsilon(k)l$; so (ii) holds. Conversely, (ii) gives $kl = ({}^{k_{(1)}}l)\,k_{(2)} = \epsilon(k_{(1)})lk_{(2)} = lk$, proving (i). The formulae in (3.10) immediately yield the equivalence of (ii) and (iv). Thus (i), (ii) and (iv) are all equivalent. By symmetry, (i) is also equivalent to (iii). $\square$

3.4.2 Hopf Centers of Representations

For the remainder of this section, we work over an *algebraically closed* field $\Bbbk$, which can be of arbitrary characteristic unless explicitly specified otherwise. Given an arbitrary Hopf $\Bbbk$-algebra H and a finite-dimensional irreducible representation $V \in \operatorname{Irr} H$, we have a surjective algebra map

$$\rho_V \colon H \twoheadrightarrow \operatorname{End}_{\Bbbk}(V)\,.$$

We define the ***Hopf center*** of V to be the largest Hopf subalgebra of H that is contained in the subalgebra $\rho_V^{-1}(\Bbbk\,\mathrm{Id}_S)$ of H; the Hopf center of V will be denoted by

$$\mathscr{H}\mathscr{Z}(V) \qquad \text{or} \qquad \mathscr{H}\mathscr{Z}_H(V)\,.$$

Clearly, $\mathscr{H}\mathscr{Z}(V)$ is a normal Hopf subalgebra of H. Moreover, since $\rho_V(\mathscr{Z}H) \subseteq \Bbbk\,\mathrm{Id}_V$, we have $\zeta(H) \subseteq \mathscr{H}\mathscr{Z}(V)$, where $\zeta(H)$ is the Hopf center of H, that is, the largest Hopf subalgebra of H that is contained in the ordinary center, $\mathscr{Z}H$. For inner faithful representations, more can be said:

Lemma 3.4.2. *Let H be finite-dimensional and let $V \in \operatorname{Irr} H$ be inner faithful. Then*

$$\mathscr{H}\mathscr{Z}(V) = \zeta(H)\,.$$

Proof. It suffices to show that $K := \mathscr{H}\mathscr{Z}(V) \subseteq \mathscr{Z}H$. By Lemma 3.4.1, this means that

$$[h,k] = \epsilon(h)\epsilon(k) \text{ for all } h \in H, k \in K\,.$$

To this end, consider the representation $V^{\otimes n}$ of H and recall that the H-action on $V^{\otimes n}$ comes from the algebra map $\rho_V^{\otimes n} \circ \Delta^{n-1} : H \to \operatorname{End}_{\Bbbk}(V)^{\otimes n} \cong \operatorname{End}_{\Bbbk}(V^{\otimes n})$. Let us write $h_{V^{\otimes n}} \in \operatorname{End}_{\Bbbk}(V^{\otimes n})$ for the image of $h \in H$ under this map. We claim that, for all $h \in H$ and $k \in K$,

$$[h,k]_{V^{\otimes n}} = \epsilon(h)\epsilon(k)\operatorname{Id}_{V^{\otimes n}}\,.$$

It will then follow that the element $[h,k] \in H$ acts on $\mathsf{T}(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ as the scalar operator $\epsilon(h)\epsilon(k)\operatorname{Id}_{\mathsf{T}(V)}$. Since inner faithfulness of S is equivalent to faithfulness of $\mathsf{T}(V)$ in the usual sense by [45], we obtain the desired conclusion, $[k,h] = \epsilon(k)\epsilon(h)$.

To prove the claim, we proceed by induction on n. The base case $n = 0$ states the obvious identity $\epsilon([h,k]) = \epsilon(h)\epsilon(k)$. For the inductive step, note that

$$[h,k]_{V^{\otimes n}} = \rho_V(h_{(1)}k_{(1)}S(h_{(2n)})S(k_{(2n)})) \otimes \ldots \otimes \rho_V(h_{(n)}k_{(n)}S(h_{(n+1)})S(k_{(n+1)})).$$

Since $\Delta^{2n-1}(k) \in \mathscr{H}\mathscr{Z}(V)^{\otimes 2n}$, we can move $S(k_{(n+1)})$ past $S(h_{(n+1)})$ to rewrite the right hand side above in the following form:

$$\begin{aligned}
&\rho_V(h_{(1)}k_{(1)}S(h_{(2n)})S(k_{(2n)})) \otimes \ldots \otimes \rho_V(h_{(n)}k_{(n)}S(k_{(n+1)})S(h_{(n+1)})) \\
&\quad = \rho_V(h_{(1)}k_{(1)}S(h_{(2n-2)})S(k_{(2n)})) \otimes \ldots \otimes \rho_V(h_{(n-1)}k_{(n-1)}S(k_{(n)})S(h_{(n)})) \otimes \operatorname{Id}_V \\
&\quad = [h,k]_{V^{\otimes n-1}} \otimes \operatorname{Id}_V = \epsilon(h)\epsilon(k)\operatorname{Id}_{V^{\otimes n-1}} \otimes \operatorname{Id}_V = \epsilon(h)\epsilon(k)\operatorname{Id}_{V^{\otimes n}}\,,
\end{aligned}$$

where the penultimate equality uses our inductive hypothesis. This completes the proof. □

3.4.3 Frobenius Divisibility Modulo Centers

The following theorem is a generalization of Theorem 1.1.1 for Hopf algebras, extending the work in [50]. The main part of the proof given below is an adaptation of an argument due to Tate.

Theorem 3.4.3. *Let $\mathfrak{C}$ be class of finite-dimensional Hopf $\Bbbk$-algebras that is closed under tensor products and taking under (Hopf) homomorphic images. Assume that all $H \in \mathfrak{C}$ satisfy* **FD***, that is,* $\dim_{\Bbbk} V$ *divides* $\dim_{\Bbbk} H$ *for all $V \in \operatorname{Irr} H$. Then, in fact,* $\dim_{\Bbbk} V$ *divides* $\frac{\dim_{\Bbbk} H}{\dim_{\Bbbk} \mathscr{H}\mathscr{Z}(V)}$ *for all $V \in \operatorname{Irr} H$ and $H \in \mathfrak{C}$.*

Proof. We first show that $\dim_{\Bbbk} V$ divides $\frac{\dim_{\Bbbk} H}{\dim_{\Bbbk} \zeta(H)}$. Note that $V^{\otimes n}$ is an irreducible representation of $H^{\otimes n}$ for each $n \geq 0$, because $\rho_V^{\otimes n}$ maps $H^{\otimes n}$ onto $\operatorname{End}_{\Bbbk}(V)^{\otimes n} \cong \operatorname{End}_{\Bbbk}(V^{\otimes n})$. Since $\zeta(H)$ is commutative, the multiplication map $\mu_n := m^{\otimes n-1}|_{\zeta(H)^{\otimes n}} : \zeta(H)^{\otimes n} \to \zeta(H)$ is a morphism of Hopf algebras, and hence $\operatorname{Ker} \mu_n$ is a Hopf ideal of $\zeta(H)^{\otimes n}$. Furthermore, the following diagram commutes:

$$\begin{array}{ccc} \Bbbk^{\otimes n} & \xrightarrow{\sim} & \Bbbk \\ {\scriptstyle \rho_V^{\otimes n}} \uparrow & & \uparrow {\scriptstyle \rho_V} \\ \zeta(H)^{\otimes n} & \xrightarrow{\mu_n} & \zeta(H) \end{array}$$

Thus $\rho_V^{\otimes n}(\operatorname{Ker} \mu_n) = 0$ and so $V^{\otimes n}$ is an irreducible representation of the the following Hopf algebra, which belongs to $\mathfrak{C}$:

$$H_n := H^{\otimes n}/(\operatorname{Ker} \mu_n)H^{\otimes n}.$$

Consequently, $\dim_{\Bbbk} V^{\otimes n} = (\dim_{\Bbbk} V)^n$ divides $\dim_{\Bbbk} H_n$. Finally, putting $d := \dim_{\Bbbk} H$ and $\delta := \dim_k \zeta(H)$ for brevity, we know by the Nichols-Zoeller Theorem that $H^{\otimes n}$ is free of rank $(\frac{d}{\delta})^n$ as module over $\zeta(H)^{\otimes n}$. Therefore,

$$\dim_{\Bbbk}(\operatorname{Ker} \mu_n)H^{\otimes n} = (\dim_{\Bbbk} \operatorname{Ker} \mu_n)\left(\frac{d}{\delta}\right)^n = (\delta^n - \delta)\left(\frac{d}{\delta}\right)^n = d^n - \frac{d^n}{\delta^{n-1}},$$

and so $\dim_{\Bbbk} H_n = \frac{d^n}{\delta^{n-1}}$. Therefore $(\dim_{\Bbbk} V)^n$ divides $\frac{d^n}{\delta^{n-1}}$ for all n. In other words, $q := \frac{d}{\delta \dim_{\Bbbk} V}$ satisfies $q^n \in \frac{1}{\delta}\mathbb{Z}$ for all n, and so $\mathbb{Z}[q] \subseteq \frac{1}{\delta}\mathbb{Z}$. It follows that the fraction q is integral over $\mathbb{Z}$, and hence $q \in \mathbb{Z}$, proving that $\dim_{\Bbbk} V$ divides $\frac{d}{\delta}$.

To obtain the stronger assertion, that $\dim_{\Bbbk} V$ divides $\frac{\dim_{\Bbbk} H}{\dim_{\Bbbk} \mathscr{H}\mathscr{Z}(V)}$, consider the canonical map $\overline{} : H \twoheadrightarrow \overline{H} = H/\mathscr{H}\operatorname{Ker} V$. Then $\overline{H} \in \mathfrak{C}$ and V can be viewed as an inner-faithful irreducible representation of $\overline{H}$. By the foregoing, $\dim_{\Bbbk} V$ divides $\frac{\dim_{\Bbbk}(\overline{H})}{\dim_{\Bbbk} \zeta(\overline{H})}$, and hence it suffices to show that $\frac{\dim_{\Bbbk}(\overline{H})}{\dim_{\Bbbk} \zeta(\overline{H})}$ divides $\frac{\dim_{\Bbbk} H}{\dim_{\Bbbk} \mathscr{H}\mathscr{Z}(V)}$. But $\overline{\mathscr{H}\mathscr{Z}(V)} \subseteq \mathscr{H}\mathscr{Z}_{\overline{H}}(V) = \zeta(\overline{H})$, where the last equality holds by Lemma 3.4.2. Thus, the canonical Hopf epimorphism $H \twoheadrightarrow \overline{H} \twoheadrightarrow \overline{H}/\overline{H}\zeta(\overline{H})^+$ factors through the epimorphism $H \twoheadrightarrow H/H\mathscr{H}\mathscr{Z}(V)^+$, giving an epimorphism

$$H/H\mathscr{H}\mathscr{Z}(V)^+ \twoheadrightarrow \overline{H}/\overline{H}\zeta(\overline{H})^+,$$

and hence a Hopf monomorphism $(\overline{H}/\overline{H}\zeta(\overline{H})^+)^* \hookrightarrow (H/H\mathscr{H}\mathscr{Z}(V)^+)^*$. The Nichols-Zoeller Theorem now gives the desired conclusion that $\dim_{\Bbbk} \overline{H}/\overline{H}\zeta(\overline{H})^+ = \frac{\dim_{\Bbbk}(\overline{H})}{\dim_{\Bbbk} \zeta(\overline{H})}$ divides $\dim_{\Bbbk} H/H\mathscr{H}\mathscr{Z}(V)^+ = \frac{\dim_{\Bbbk} H}{\dim_{\Bbbk} \mathscr{H}\mathscr{Z}(V)}$, finishing the proof. □

Corollary 3.4.4. *Assume that* $\operatorname{char} \Bbbk = 0$. *Let H be a semisimple quasitriangular Hopf algebra and let $V \in \operatorname{Irr} H$. Then* $\dim_{\Bbbk} V$ *divides* $\dim_{\Bbbk} H / \dim_{\Bbbk} \mathscr{H}\mathscr{Z}(V)$.

Proof. The class of semisimple quasitriangular Hopf $\Bbbk$-algebras is closed under tensor products and quotients. Additionally, semisimple quasitriangular Hopf $\Bbbk$-algebras satisfy **FD** by [14]. □

Corollary 3.4.5. *Assume that* $\operatorname{char} \Bbbk = 0$. *Let H be a semisimple Hopf $\Bbbk$-algebra and let $V \in \operatorname{Irr} H$ be such that $\chi_V \in \mathscr{Z}(H^*)$. Then* $\dim_{\Bbbk} V$ *divides* $\frac{\dim_{\Bbbk} H}{\dim_{\Bbbk} \mathscr{H}\mathscr{Z}(V)}$.

Proof. Let K be a normal Hopf subalgebra of H with $K \subseteq \mathscr{H}\operatorname{Ker}(V)$. Then, as in the last part of the proof of Theorem 3.4.3, V descends to a representation of $\overline{H} = H/HK^+$, and the character χ_V belongs to the (Hopf) subalgebra $\overline{H}^* = (HK^+)^\perp$ of H^*. Therefore, $\chi_V \in \mathscr{Z}(\overline{H}^*)$.

Also, viewing $V^{\otimes n}$ as a representation of $H^{\otimes n}$ as in the first part of the proof of Theorem 3.4.3, we have $\chi_{V^{\otimes n}} = \chi_V^{\otimes n} \in \mathscr{Z}((H^{\otimes n})^*)$. Lastly, by [57], we know that the degree of any central irreducible character must divide the dimension of the Hopf algebra. With these three observations, the proof of Theorem 3.4.3 goes through. □

CHAPTER 4

THE ADJOINT REPRESENTATIONS

This chapter contains my results on the adjoint representation ${}^{\mathrm{ad}}H$ of a Hopf algebra H and its extended version, ${}^{\mathrm{Ad}}H$. The first section determines the Hopf kernel of ${}^{\mathrm{ad}}H$ in a large range of settings; the second section discusses the consequences of this result for the Chevalley property of ${}^{\mathrm{ad}}H$; the third section defines the notion of a conjugacy class in our context and generalizes many of the known results on conjugacy classes and the extended adjoint representation; and lastly, the fourth section presents an example of the conjugacy classes of a non-trivial Hopf algebra over a field of characteristic 3.

4.1 The Hopf Kernel of the Adjoint Representation

4.1.1 Hopf Centralizers

Given a Hopf subalgebra K of H, we let $C_H(K)$ denote the centralizer of K in H, that is,

$$C_H(K) = \{h \in H \mid hk = kh \text{ for all } k \in K\} \tag{4.1}$$

$$= \{h \in H \mid {}^{k}h = \epsilon(k)h \text{ for all } k \in K\}\,. \tag{4.2}$$

Here, the second equality follows from (2.4) or Lemma 3.4.1. As in §2.1.4, we let $\mathscr{H}C_H(K)$ denote the largest Hopf subalgebra of H that is contained in the subalgebra $C_H(K)$. We will refer to $\mathscr{H}C_H(K)$ as the ***Hopf centralizer*** of K in H. The following proposition is a generalization of [7, Theorem 2.2]. The first part of the proof is simply an adaptation of the argument given in [7]. The entire proof is included here for the sake of completeness.

Proposition 4.1.1. *Let K be a Hopf subalgebra of H. Then*

$$\begin{aligned}\mathscr{H}C_H(K) &= \{x \in H \mid \Delta(x) \in H \otimes C_H(K)\}\\ &= \{x \in H \mid \Delta(x) \in C_H(K) \otimes H\}\\ &= \{x \in H \mid \Delta(x) \in C_H(K) \otimes C_H(K)\}\,.\end{aligned}$$

Proof. Put $M := \{x \in H \mid \Delta(x) \in H \otimes C_H(K)\}$; this is a subalgebra of H, since $H \otimes C_H(K)$ is a subalgebra of $H^{\otimes 2}$. Note also that $M = (\epsilon \otimes \mathrm{Id})(\Delta M) \subseteq C_H(K)$. Moreover, any Hopf subalgebra $L \subseteq C_H(K)$ is contained in M, because $\Delta(L) \subseteq L \otimes L \subseteq H \otimes C_H(K)$. Thus, in order to show that $M = \mathscr{H}C_H(K)$, it suffices to show that M is a Hopf subalgebra of H.

The first step is to show that:

$$\Delta(M) \subseteq C_H(K) \otimes C_H(K)\,.$$

In view of (4.2), it is enough to show that that, for all $x \in M$ and $k \in K$, we have ${}^k x_{(1)} \otimes x_{(2)} = \epsilon(k)\Delta(x)$. This is done in the following computation:

$$\begin{aligned}{}^k x_{(1)} \otimes x_{(2)} &= k_{(1)}x_{(1)}S(k_{(2)}) \otimes x_{(2)}\\ &= k_{(1)}x_{(1)}S(k_{(4)}) \otimes k_{(2)}S(k_{(3)})x_{(2)}\\ &= k_{(1)}x_{(1)}S(k_{(4)}) \otimes k_{(2)}x_{(2)}S(k_{(3)})\\ &= \Delta({}^k x) = \epsilon(k)\Delta(x)\,.\end{aligned}$$

It follows that $M = \{x \in H \mid \Delta(x) \in C_H(K) \otimes C_H(K)\}$. By symmetry, we also have

$$\{x \in H \mid \Delta(x) \in C_H(K) \otimes H\} = \{x \in H \mid \Delta(x) \in C_H(K) \otimes C_H(K)\}\,.$$

Thus, all three sets on the right hand side of the proposition coincide.

Next, note that $(\mathrm{Id}\otimes\Delta)(\Delta(x)) = (\Delta\otimes\mathrm{Id})(\Delta(x)) \in H\otimes H\otimes C_H(K)$ for $x\in M$. This shows that $\Delta(M)\subseteq H\otimes M$ and, in the same way, one also obtains that $\Delta(M)\subseteq M\otimes H$. Therefore, $\Delta(M)\subseteq (H\otimes M)\cap(M\otimes H) = M\otimes M$, proving that M is a subcoalgebra of H. It just remains to show that $S(M)\subseteq M$. Since $\Delta(S(M))\subseteq S(M)\otimes S(M)$ it suffices to show that $S(M)\subseteq C_H(K)$. This is done in the following computation, for $m\in M$ and $k\in K$:

$$\begin{aligned} kS(m) &= \epsilon(m_{(1)})kS(m_{(2)}) = S(m_{(1)})m_{(2)}kS(m_{(3)}) \\ &= S(m_{(1)})km_{(2)}S(m_{(3)}) = S(m)k \end{aligned}$$

This completes the proof. □

Corollary 4.1.2. *If N is a normal Hopf subalgebra of H, then $\mathscr{H}C_H(N)$ is also normal.*

Proof. We first observe that, for any H-bimodule V, the N-invariants ${}^{N}({}^{\mathrm{ad}}V)$ are an H-submodule of ${}^{\mathrm{ad}}V$. This follows from the following computation, for $n\in N$, $h\in H$ and $v\in {}^{N}({}^{\mathrm{ad}}V)$:

$${}^{nh}v = {}^{(h_{(1)}n^{h_{(2)}})}v = {}^{h_{(1)}}\big(\epsilon(n^{h_{(2)}})v\big) = \epsilon(n)\,{}^{h}v$$

Here, the first equality uses the analog of (2.3) for the right adjoint action of H on N. Similarly, one shows that N-invariants for the right adjoint action, $(V^{\mathrm{ad}})^N$, are an H-submodule of V^{ad}. Finally, by (2.4),

$$(V^{\mathrm{ad}})^N = {}^{N}({}^{\mathrm{ad}}V) = \{v\in V \mid nv = vn \text{ for all } n\in N\}\,.$$

So this space is stable under the left and right adjoint actions of H on V. In particular, $C_H(N) = {}^{N}({}^{\mathrm{ad}}H)^N$ is stable under both adjoint actions of H. Now let $x\in\mathscr{H}C_H(N)$. Then $\Delta(x) = x_{(1)}\otimes x_{(2)}\in H\otimes C_H(N)$ and so:

$$\Delta({}^{h}x) = h_{(1)}x_{(1)}S(h_{(4)})\otimes {}^{h_{(2)}}x_{(2)} \in H\otimes C_H(N)$$

In view of Proposition 4.1.1, this shows that ${}^{h}x\in\mathscr{H}C_H(N)$. Closure under the right adjoint action follows by a similar argument. □

4.1.2 Hopf Kernels

This section will focus on the largest Hopf ideal that annihilates the adjoint representation ${}^{\mathrm{ad}}H$, that is, the Hopf kernel of ${}^{\mathrm{ad}}H$, in the terminology of §2.1.4. The example below shows that the Hopf kernel of ${}^{\mathrm{ad}}H$ is generally strictly smaller than the usual kernel, $\mathrm{Ker}({}^{\mathrm{ad}}H)$, that is, the ordinary annihilator of the H-module ${}^{\mathrm{ad}}H$.

Example 4.1.3. Recall from Example 2.2.6 that the adjoint action of the algebra generators of the Sweedler algebra H_4 is given by:

$$\begin{array}{llll} {}^{g}1 = 1\ , & {}^{g}g = g\ , & {}^{g}x = -x\ , & {}^{g}(gx) = -gx\ , \\ {}^{x}1 = 0\ , & {}^{x}g = -2gx\ , & {}^{x}x = 0\ , & {}^{x}(gx) = 0\ . \end{array}$$

A simple computation shows that $\mathrm{Ker}({}^{\mathrm{ad}}H_4) = \Bbbk\Lambda$ with $\Lambda = x + gx$. However, $\Bbbk\Lambda$ is not a Hopf ideal and so we must have $\mathscr{H}\,\mathrm{Ker}({}^{\mathrm{ad}}H_4) = 0$; see also Corollary 4.1.5 below.

Recall from Section 2.1.4 that all Hopf ideals of the group algebra $\Bbbk[G]$ have the form $\Bbbk[G]\Bbbk[D]^+$ for some normal subgroup D of G. It follows that the Hopf kernel of ${}^{\mathrm{ad}}(\Bbbk[D])$, for the adjoint $\Bbbk[G]$-action, is given by $\Bbbk[G]\Bbbk[C_G(D)]^+$. The following theorem establishes the corresponding statement for more general classes of Hopf algebras. We note that (b) includes all cocommutative Hopf algebras (e.g., group algebras and enveloping algebras of Lie algebras) as well as all pointed Hopf algebras.

Theorem 4.1.4. *Let H be a Hopf algebra with N a normal Hopf subalgebra. Assume that H satisfies one of the following conditions:*

(a) *H is finite-dimensional or*

(b) *the coradical of H is cocommutative.*

Then the Hopf kernel of the adjoint H-action on N is given by $\mathscr{H}\,\mathrm{Ker}({}^{\mathrm{ad}}N) = H(\mathscr{H}C_H(N))^+$.

Proof. The beginning of the proof is the same for (a) and (b). For the inclusion $H(\mathscr{H}C_H(N))^+ \subseteq \mathscr{H}\operatorname{Ker}({}^{\mathrm{ad}}N)$, recall that $\mathscr{H}C_H(N)$ is a normal Hopf subalgebra of H by Corollary 4.1.2. For $n \in N$ and $z \in (\mathscr{H}C_H(N))^+$, we have ${}^{z}n = z_{(1)}nS(z_{(2)}) = z_{(1)}S(z_{(2)})n = 0$. Thus $H(\mathscr{H}C_H(N))^+$ is a Hopf ideal of H that is contained in the annihilator of the adjoint representation ${}^{\mathrm{ad}}N$, which gives the desired inclusion.

Let $\overline{}$ denote the canonical surjections of H onto $\overline{H} := H/\mathscr{H}\operatorname{Ker}({}^{\mathrm{ad}}N)$, and define the left coaction $\rho := (\overline{} \otimes \mathrm{Id}) \circ \Delta \colon H \to \overline{H} \otimes H$ and denote the coinvariants of ρ by C. For $c \in C$, we compute $((\rho \otimes \mathrm{Id}) \circ \Delta)(c) = ((\mathrm{Id} \otimes \Delta) \circ \rho)(c) = \overline{1} \otimes \Delta(c)$. Therefore, $\Delta(C) \subseteq C \otimes H$. Moreover, since $\overline{}$ factors through the quotient $H/H\mathscr{H}C_H(N)^+$ by the first paragraph of the proof, we certainly have $\mathscr{H}C_H(N) \subseteq C$. For the reverse inclusion, note that the adjoint H-action on N descends to an action of $\overline{H}$ on N. For $c \in C$ and $n \in N$, formula (2.3) gives $cn = ({}^{\overline{c_{(1)}}}n)c_{(2)} = ({}^{\overline{1}}n)c = nc$. Thus, $C \subseteq C_H(N)$ and hence $\Delta(C) \subseteq C \otimes H \subseteq C_H(N) \otimes H$. Proposition 4.1.1 now gives $C \subseteq \mathscr{H}C_H(N)$ as desired. Therefore, $\mathscr{H}C_H(N) = C$.

The next step is to show that H is a faithfully coflat $\overline{H}$-comodule. In the finite-dimensional case, this is clear, since H is a free $\overline{H}$-comodule by the Nichols-Zoeller Theorem applied to the dual. For case (b), Theorem 1.3 of [33] states that H is faithfully coflat over all quotient left H-module coalgebras, and thus H is certainly faithfully coflat over all quotient Hopf algebras.

Now, in [53, Theorem 2], it is shown that $I = H({}^{coH/I}H)^+$ holds for any left H-module coideal, I, such that H is a faithfully coflat left H/I-comodule. Applying this for $I = \mathscr{H}\operatorname{Ker}({}^{\mathrm{ad}}N)$ and using that the coinvariants are given by $\mathscr{H}C_H(N)$ gives the desired result of $\mathscr{H}\operatorname{Ker}({}^{\mathrm{ad}}N) = H(\mathscr{H}C_H(N))^+$. □

The key fact in the proofs of both cases of Theorem 4.1.4 is that H is a faithfully coflat $H/\mathscr{H}\operatorname{Ker}({}^{\mathrm{ad}}N)$-comodule. The result still holds only assuming this weaker but less natural assumption. In the special case of $N = H$, Theorem 4.1.4 implies the following result.

Corollary 4.1.5. *Let H be a Hopf algebra satisfying one of the following conditions:*

(a) *H is finite-dimensional or*

(b) *the coradical of H is cocommutative.*

Then the Hopf kernel of the adjoint H-action on H is given by $\mathscr{H}\operatorname{Ker}({}^{\mathrm{ad}}H) = H\zeta(H)^+$.

4.2 The Chevalley Property

Throughout this section, we let H denote a finite-dimensional Hopf algebra.

4.2.1 Background

Recall that an H-module M is said to have the ***Chevalley property*** if the H-module $\mathsf{T}(M) = \bigoplus_{n\geq 0} M^{\otimes n}$ is completely reducible. If M has the Chevalley property, then M must evidently be completely reducible. The converse fails in general.

We put

$$H_M := H/\mathscr{H}\operatorname{Ker}(M) = H/\operatorname{Ker}(\mathsf{T}(M))\,.$$

Here, the second equality holds by a result of Rieffel [45]. The following lemma reduces the question of weather a module has the Chevalley property to a question of semisimplicity of the Hopf algebra H_M .

Lemma 4.2.1. *Let M be an H-module. Then M has the Chevalley property if and only if H_M is a semisimple algebra.*

Proof. Clearly, complete reducibility of M as H-module is equivalent to complete reducibility as H_M-module. Also, semisimplicity of H_M certainly implies that the H_M-module $\mathsf{T}(M)$ is completely reducible. Conversely, if $\mathsf{T}(M)$ is completely reducible, then H_M has a faithful completely reducible module, and hence H_M is a semisimple algebra. □

If $M = \bigoplus_{S \in \mathrm{Irr}\, H} S$ is the direct sum of all irreducible H-modules, then $H_M = H/\mathscr{H}(\mathrm{rad}\, H)$, where $\mathrm{rad}\, H$ denotes the Jacobson radical of H. Thus, the Hopf algebra H has the Chevalley property if and only if the Jacobson radical of H is a Hopf ideal.

Example 4.2.2. Let $\Bbbk[G]$ be the group algebra of a finite group G and put $p = \mathrm{char}\, \Bbbk$ (≥ 0). It is a standard fact that, for M the direct sum of all irreducible $\Bbbk[G]$-modules as above,

$$H_M \cong \Bbbk[G/O_p(G)] ,$$

where $O_p(G)$ denotes the largest normal p-subgroup of G (which is understood to be $\{1\}$ if $p = 0$). In particular, $\Bbbk[G]$ has the Chevalley property if and only if either $p = 0$ or $p > 0$ and G has a normal Sylow p-subgroup.

4.2.2 The Chevalley Property for the Adjoint Representation

Corollary 4.1.5, in conjunction with Lemma 4.2.1, immediately implies the following result.

Corollary 4.2.3. *The adjoint representation* $^{\mathrm{ad}}H$ *has the Chevalley property if and only if* $H/H\zeta(H)^+$ *is a semisimple algebra.*

Corollary 4.2.3 gives in particular that $^{\mathrm{ad}}\Bbbk[G]$ has the Chevalley property if and only if the order of $\overline{G} = G/\mathscr{Z}G$ is not divisible by $\mathrm{char}\, \Bbbk$ or, equivalently, $\Bbbk[\overline{G}]$ is semisimple. By an earlier remark in §1.2.2—see especially eqrefAd:Diag—this condition s also equivalent to complete reducibility of $^{\mathrm{ad}}\Bbbk[G]$. Consequently, complete reducibility of $^{\mathrm{ad}}\Bbbk[G]$ actually implies that $^{\mathrm{ad}}\Bbbk[G]$ also has the Chevalley property. I do not know to what extent this fact generalizes to arbitrary finite-dimensional Hopf algebras.

4.2.3 Unimodularity

This subsection will show that the Chevalley property of $^{\mathrm{ad}}H$ implies the unimodularity of H. The result will follow directly from the following more general lemma.

Lemma 4.2.4. *H is unimodular if and only if $H/H\zeta(H)^+$ is unimodular.*

Proof. Let Γ be the integral of $\zeta(H)$ and let $\overline{} : H \twoheadrightarrow \overline{H} := H/H\zeta(H)^+$ be the canonical surjection. Furthermore, define $\phi : \overline{H} \to M := H\Gamma = \Gamma H$ via $\phi(\overline{h}) = h\Gamma$. Using the Nichols-Zoeller Theorem it is easy to see that ϕ is a (left and right) H-module isomorphism. Therefore, letting ${}^H(\,.\,)$ and $(\,.\,)^H$ denote invariants of left and right H-modules, respectively, we have ${}^H\overline{H} \xrightarrow{\sim} {}^HM$ and $\overline{H}^H \xrightarrow{\sim} M^H$ via ϕ. Here, ${}^H\overline{H}$ and $\overline{H}^H$ are of course the spaces of left and right integrals of $\overline{H}$, respectively. Finally, it is also easy to see, again with Nichols-Zoeller, that ${}^HH = {}^HM$ and $H^H = M^H$. Thus, ${}^H\overline{H} \xrightarrow{\sim} {}^HH$ and $\overline{H}^H \xrightarrow{\sim} H^H$ via ϕ. Therefore, ${}^H\overline{H} = \overline{H}^H$ if and only if ${}^HH = H^H$. □

Corollary 4.2.5. *If the adjoint representation ${}^{\mathrm{ad}}H$ has the Chevalley property, then H is unimodular.*

Proof. By Corollary 4.2.3, our assumption on ${}^{\mathrm{ad}}H$ means that $H/H\zeta(H)^+$ is semisimple and thus unimodular. Now the result follows from Lemma 4.2.4. □

4.3 Conjugacy Classes

Recall that the adjoint representation ${}^{\mathrm{ad}}\Bbbk[G]$ for a finite group G is completely reducible if and only if the group algebra $\Bbbk[G/\mathscr{Z}G]$ is semisimple (§1.2.2). The conjugacy classes of G play a key role in the proof of this fact; indeed, the order of $G/\mathscr{Z}G$ is not divisible by $\operatorname{char}\Bbbk$ if and only if this holds for the sizes of all conjugacy classes of G. Thus it is natural to proceed in our exploration of the adjoint representation by studying a generalization of the notion of a conjugacy class to the context of Hopf algebras.

4.3.1 Notation

For the remainder of this thesis, H will continue to denote a finite-dimensional Hopf algebra. Furthermore, Λ will denote a nonzero integral of H and λ denotes a nonzero integral of H^* with $\langle \lambda, \Lambda \rangle = 1$. Superscripts R and L will be used to distinguish right and left integrals. The distinguished group-like element of H will be denoted by x and the distinguished group-like element of H^* by α; these elements are characterized by the following conditions:

$$f\lambda^R = \langle f, x\rangle \lambda^R \qquad (f \in H^*)$$
$$h\Lambda^R = \langle \alpha, h\rangle \Lambda^R \qquad (h \in H).$$

4.3.2 The Subalgebra $C(H)$

The following subalgebra of H^*, introduced by Drinfel'd [13] in his study of quasitriangular Hopf algebras, will play a key role in our analysis of the adjoint representation:

$$C(H) := \{f \in H^* \mid \langle f, ab\rangle = \langle f, bS^2(a)\rangle\}$$

If H is involutory, then $C(H)$ coincides with the algebra of all trace forms on H. In general, since S is invertible, we also have

$$C(H) = \{f \in H^* \mid \langle f, ab\rangle = \langle f, S^{-2}(b)a\rangle\}.$$

In the following lemma, we let $({}^{\text{ad}}H)_\alpha$ denote the $\Bbbk_\alpha$-homogeneous component of ${}^{\text{ad}}H$, that is,

$$({}^{\text{ad}}H)_\alpha = \{h \in H \mid {}^k h = h\langle \alpha, k\rangle \text{ for all } k \in H\}.$$

Recall also that $\mathbb{1} = \Bbbk_\epsilon$ denotes the trivial representation of H.

Lemma 4.3.1. (a) $\mathrm{Hom}_H({}^{\text{ad}}H, \mathbb{1}) = C(H)$.

(b) $({}^{\text{ad}}H)_\alpha = \{h \in H \mid kh = h\langle \alpha, k_{(1)}\rangle k_{(2)} \textit{ for all } k \in H\} = C(H) \rightharpoonup \Lambda^R$
$= C(H) \rightharpoonup \Lambda^L$.

Proof. (a) Let $f \in C(H)$ and observe that the following equalities hold in $\Bbbk = \mathbb{1}$:

$$\langle f, {}^h k\rangle = \langle f, h_{(1)}kS(h_{(2)})\rangle = \langle f, kS(h_{(2)})S^2(h_{(1)})\rangle = \langle \epsilon, h\rangle\langle f, k\rangle = h.\langle f, k\rangle$$

Thus $C(H) \subseteq \mathrm{Hom}_H({}^{\mathrm{ad}}H, \mathbb{1})$. For the other inclusion, let $f \in \mathrm{Hom}_H({}^{\mathrm{ad}}H, \mathbb{1})$. Then:

$$\langle f, ab \rangle = \langle f, a_{(1)} b S^2(a_{(3)}) S(a_{(2)}) \rangle = \langle f, {}^{a_{(1)}}(bS^2(a_{(2)})) \rangle = \langle f, aS^2(b) \rangle$$

This proves the claimed equality.

(b) Put $M := \{h \in H \mid kh = h\langle \alpha, k_{(1)} \rangle k_{(2)} \text{ for all } k \in H\}$. It is a straightforward check that $M \subseteq ({}^{\mathrm{ad}}H)_\alpha$. For the other inclusion, let $m \in ({}^{\mathrm{ad}}H)_\alpha$. Then equation (2.3) gives $hm = ({}^{h_{(1)}}m)h_{(2)} = m\langle \alpha, h_{(1)} \rangle h_{(2)}$. Thus, $({}^{\mathrm{ad}}H)_\alpha = M$, proving the first equality in (b). The following calculations show that $\Lambda^L, \Lambda^R \in M$:

$$h\Lambda^L = \Lambda^L \langle \epsilon, h \rangle = \Lambda^L \langle \alpha, h_{(1)} \rangle \langle \alpha^{-1}, h_{(2)} \rangle = \Lambda^L \langle \alpha, h_{(1)} \rangle h_{(2)}$$

and

$$h\Lambda^R = \Lambda^R \langle \alpha, h \rangle = \Lambda^R \langle \alpha, h_{(1)} \rangle \langle \epsilon, h_{(2)} \rangle = \Lambda^R \langle \alpha, h_{(1)} \rangle h_{(2)} \,.$$

Next, we show that M is a left $C(H)$-module for the action $\rightharpoonup$. To this end, let $f \in C(H)$ and $m \in M$ and calculate

$$\begin{aligned} h(f \rightharpoonup m) &= hm_{(1)} \langle f, m_{(2)} \rangle \\ &= h_{(1)} m_{(1)} \langle f, S^{-1}(h_{(3)}) h_{(2)} m_{(2)} \rangle \\ &= m_{(1)} \langle \alpha, h_{(1)} \rangle h_{(2)} \langle f, S^{-1}(h_{(4)}) m_{(2)} h_{(3)} \rangle \\ &= m_{(1)} \langle \alpha, h_{(1)} \rangle h_{(2)} \langle f, m_{(2)} h_{(3)} S(h_{(4)}) \rangle \\ &= m_{(1)} \langle \alpha, h_{(1)} \rangle h_{(2)} \langle f, m_{(2)} \rangle \\ &= (f \rightharpoonup m) \langle \alpha, h_{(1)} \rangle h_{(2)} \,. \end{aligned}$$

Consequently, $C(H) \rightharpoonup \Lambda^R$ and $C(H) \rightharpoonup \Lambda^R$ are both contained in M. For the reverse inclusions, we will need some technical facts. I claim that

$$S(M) = \{h \in H \mid ah = h\langle \alpha, a_{(2)} \rangle a_{(1)}\} \,.$$

To see this, denote the right hand side of the above equality by M' and let $h \in M$. Then $\langle \alpha^{-1}, a_{(1)} \rangle a_{(2)} h = ha$ and so

$$\begin{aligned} aS(h) &= S(hS^{-1}(a)) = S(\langle \alpha^{-1}, S^{-1}(a_{(2)}) \rangle S^{-1}(a_{(1)}) h) \\ &= S(\langle \alpha, a_{(2)} \rangle S^{-1}(a_{(1)}) h) = S(h) \langle \alpha, a_{(2)} \rangle a_{(1)} \,. \end{aligned}$$

Thus, $S(M) \subseteq M'$. For the other direction let $h \in M'$. Then the following calculation shows that $S^{-1}(h) \in M$ and hence $H \in S(M)$:

$$\begin{aligned} aS^{-1}(h) &= S^{-1}(hS(a)) = S^{-1}(\langle \alpha^{-1}, S(a_{(1)})\rangle S(a_{(2)})h) \\ &= S^{-1}(\langle \alpha, a_{(1)}\rangle S(a_{(2)})h) = S^{-1}(h)\langle \alpha, a_{(1)}\rangle a_{(2)}\,. \end{aligned}$$

Next, we need two identities related to the Frobenius structure of H. Recall from Section 2.3.2 that $c_{\lambda^L} = \Lambda^R_{(2)} \otimes S(\Lambda^R_{(1)})$. Thus, Equation (2.6) gives

$$\langle \lambda^L, hS(\Lambda^R_{(1)})\rangle \Lambda^R_{(2)} = h \qquad (h \in H).$$

Since the Nakayama automorphism of (H, λ^L) is $m \circ (S^2 \otimes \mu \circ \alpha) \circ \Delta$ (Section 2.3.2), we have

$$\langle \lambda^L, ab\rangle = \langle \lambda^L, \langle \alpha^{-1}, b_{(2)}\rangle S^{-2}(b_{(1)})a\rangle \qquad (a, b \in H)\,. \tag{4.3}$$

Now assume that $f \rightharpoonup \Lambda^R \in M$ for $f \in H^*$. Then the following calculation shows that $f \in C(H)$:

$$\begin{aligned} \langle f, ab\rangle &= \langle f, \langle \lambda^L, abS(\Lambda^R_{(1)})\rangle \Lambda^R_{(2)}\rangle \\ &= \langle \lambda^L, abS(f \rightharpoonup \Lambda^R)\rangle \\ &= \langle \lambda^L, aS(f \rightharpoonup \Lambda^R)\langle \alpha, b_{(2)}\rangle b_{(1)}\rangle \\ &= \langle \lambda^L, \langle \alpha, b_{(3)}\rangle \langle \alpha^{-1}, b_{(2)}\rangle S^{-2}(b_{(1)})aS(f \rightharpoonup \Lambda^R)\rangle \\ &= \langle \lambda^L, S^{-2}(b)aS(\Lambda^R_{(1)})\langle f, \Lambda^R_{(2)}\rangle\rangle \\ &= \langle f, \langle \lambda^L, S^{-2}(b)aS(\Lambda^R_{(1)}\rangle \Lambda^R_{(2)}\rangle \\ &= \langle f, S^{-2}(b)a\rangle \end{aligned}$$

Since $H^* \rightharpoonup \Lambda^R = H$, it follows that $C(H) \rightharpoonup \Lambda^R \supseteq M$, proving the equality $C(H) \rightharpoonup \Lambda^R = M$. By nondegeneracy of action of H^* on Λ^R, we also obtain $\dim_{\Bbbk} C(H) = \dim_{\Bbbk} M$. Hence, by nondegeneracy of the action of H^* on Λ^L, we also have $\dim_{\Bbbk}(C(H) \rightharpoonup \Lambda^L) = \dim_{\Bbbk} M$ and thus $C(H) \rightharpoonup \Lambda^L = M$. □

4.3.3 Definition of Conjugacy Classes

The definition of the extended adjoint representation ${}^{\mathrm{Ad}}H$ in (2.17) works for any finite-dimensional Hopf algebra H. For semisimple H and $\Bbbk$ of characteristic

0, a notion of conjugacy classes of H was introduced in [8]: up to isomorphism, they are exactly the irreducible components of ${}^{\mathrm{Ad}}H$. Now, for semisimple H and $\operatorname{char} \Bbbk = 0$, the Drinfel'd double $D(H)$ is semisimple ([35, Corollary 10.3.13]), and hence ${}^{\mathrm{Ad}}H$ is a completely reducible representation of H. As we shall see below, this fails for general finite-dimensional Hopf algebras. Therefore, generalizing the definition of [8], we define the ***conjugacy classes*** of H to be the indecomposable components of ${}^{\mathrm{Ad}}H$—these components are determined up to isomorphism by the Krull-Schmidt Theorem. In Corollary 4.3.5 below, we will give another expression for the conjugacy classes of H.

The following lemma motivates the definition of conjugacy classes. The lemma also shows that ${}^{\mathrm{Ad}}H$ can be completely reducible even if $D(H)$ is not a semisimple algebra, and complete reducibility of ${}^{\mathrm{Ad}}H$ does not force ${}^{\mathrm{ad}}H$ to be completely reducible.

Lemma 4.3.2. *Let G be a finite group. Then ${}^{\mathrm{Ad}}(\Bbbk[G])$ is a completely reducible $D(\Bbbk[G])$-module. The $\Bbbk$-subspaces of $\Bbbk[G]$ that are spanned by the various conjugacy classes of G are pairwise non-isomorphic irreducible $D(\Bbbk[G])$-submodules of ${}^{\mathrm{Ad}}(\Bbbk[G])$ and ${}^{\mathrm{Ad}}\Bbbk[G]$ is the direct sum of these subspaces.*

Proof. For each conjugacy class of $\mathcal{C} \subseteq G$, let $\Bbbk\mathcal{C} \subseteq \Bbbk[G]$ be the subspace generated by $\mathcal{C}$. Clearly, $\Bbbk[G]$ is the direct sum of these subspaces. We will show that, for any $0 \neq c = \sum_{x \in \mathcal{C}} \alpha_x x \in \Bbbk\mathcal{C}$, we have $D(\Bbbk[G]).c = \Bbbk\mathcal{C}$. This will prove that $\Bbbk\mathcal{C}$ is irreducible as $D(\Bbbk[G])$-module. We may assume that $\alpha_y = 1$ for some $y \in \mathcal{C}$. Letting $\{\delta_x \mid x \in G\}$ denote the basis of $\Bbbk[G]^*$ that is dual to the basis G of $\Bbbk[G]$, we have $S(\delta_y).c = c \leftharpoonup \delta_y = y$ by (2.17) and $x.y = xyx^{-1}$ for $x \in G$. This shows that $D(\Bbbk[G]).c = \Bbbk\mathcal{C}$, since the elements δ_y and x generate the algebra $D(\Bbbk[G])$. Finally, since δ_y $(y \in \mathcal{C})$ annihilates all $\Bbbk\mathcal{D}$ for conjugacy classes $\mathcal{D} \neq \mathcal{C}$, the various irreducible $D(\Bbbk[G])$-submodules $\Bbbk\mathcal{C}$ are non-isomorphic. $\square$

It follows from the lemma that the conjugacy classes of $\Bbbk[G]$ are uniquely defined as subspaces of $\Bbbk[G]$, not only up to isomorphism: they are the homogeneous components of the completely reducible $D(\Bbbk[G])$-representation ${}^{\mathrm{Ad}}\Bbbk[G]$.

4.3.4 ${}^{\mathrm{Ad}}H$ as an Induced Representation

Using the inclusion of H into $D(H)$ we can induce $\Bbbk_\alpha$ up to a $D(H)$-module. This module will be denoted $\Bbbk_\alpha \uparrow_H^{D(H)}$; it can be described via the isomorphisms below:

$$\Bbbk_\alpha\uparrow_H^{D(H)} = D(H) \otimes_H \Bbbk_\alpha \cong D(H)(\epsilon \bowtie \Lambda^R) = H^* \bowtie \Lambda^R \tag{4.4}$$

Thus using (2.16) we have that $\Bbbk_\alpha\uparrow_H^{D(H)} \cong H^*$ as $D(H)$-modules, where the action of $D(H)$ on H^* is given by:

$$(f \bowtie h).g = \langle \alpha, h_{(2)} \rangle f(h_{(1)} \rightharpoonup g \leftharpoonup S^{-1}(h_{(3)})) \qquad (f, g \in H^*, h \in H) \tag{4.5}$$

Under the assumption that H is semisimple and $\operatorname{char} \Bbbk = 0$, this module was studied by Burciu [5]. In this case, H is unimodular; so $\alpha = \epsilon$ and (4.5) simplifies to

$$(f \bowtie h).g = f(h_{(1)} \rightharpoonup g \leftharpoonup S^{-1}(h_{(2)})) \qquad (f, g \in H^*, h \in H) \tag{4.6}$$

In Burciu's study of ${}^{\mathrm{Ad}}H$, significant use was made of the fact that ${}^{\mathrm{Ad}}H$ could be constructed by inducing the trivial module $\mathbb{1} = \Bbbk_\epsilon$ up from H to $D(H)$. The following proposition establishes a similar fact in general, with the role of $\mathbb{1}$ being played by $\Bbbk_\alpha$.

Proposition 4.3.3. *The following isomorphism holds in* $\mathfrak{Rep}H$*:*

$${}^{\mathrm{Ad}}H \cong \Bbbk_\alpha\uparrow_H^{D(H)} .$$

Identifying $\Bbbk_\alpha\uparrow_H^{D(H)}$ *with* H^* *via (4.4), this isomorphism is explicitly given by* $h \mapsto (h \rightharpoonup \lambda^L)$ *for* $h \in {}^{\mathrm{Ad}}H = H$*.*

Proof. Let $\phi\colon {}^{\mathrm{Ad}}H \to \Bbbk_\alpha\uparrow_H^{D(H)}$ be the map given by $\phi(h) = h \rightharpoonup \lambda^L$. By nondegeneracy of the action of H on λ^L, the map ϕ is bijective. It only remains to show that ϕ is a $D(H)$-module map.

Recall from (4.3) that $\langle \lambda^L, ab \rangle = \langle \lambda^L, \langle \alpha^{-1}, b_{(2)} \rangle S^{-2}(b_{(1)})a \rangle$. Using this fact, the following computation shows that ϕ is an H-module map:

$$\begin{aligned}\langle \phi({}^h k), a\rangle &= \langle \lambda^L, ah_{(1)}kS(h_{(2)})\rangle \\ &= \langle \lambda^L, \langle \alpha^{-1}, S(h_{(2)})_{(2)}\rangle S^{-2}(S(h_{(2)})_{(1)})ah_{(1)}k\rangle \\ &= \langle \lambda^L, \langle \alpha, h_{(2)}\rangle S^{-1}(h_{(3)})ah_{(1)}k\rangle \\ &= \langle \langle \alpha, h_{(2)}\rangle h_{(1)} \rightharpoonup (k \rightharpoonup \lambda^L) \leftharpoonup S^{-1}(h_{(3)})), a\rangle \\ &= \langle h.\phi(k), a\rangle.\end{aligned}$$

That ϕ is an H^*-module map follows from the computation below:

$$\begin{aligned}\langle \phi(f.h), a\rangle =&\langle \phi(\langle S^{-1}(f), h_{(1)}\rangle h_{(2)}), a\rangle \\ &= \langle S^{-1}(f), h_{(1)}\rangle\langle \lambda^L, ah_{(2)}\rangle \\ &= \langle S^{-1}(f), S(a_{(1)})a_{(2)}h_{(1)}\lambda^L(a_{(3)}h_{(2)})\rangle \\ &= \langle S^{-1}(f), S(a_{(1)})\lambda^L(a_{(2)}h)\rangle \\ &= \langle f(h \rightharpoonup \lambda^L), a\rangle \\ &= \langle f.\phi(h), a\rangle.\end{aligned}$$

□

4.3.5 The Endomorphism Algebra of ${}^{\mathrm{Ad}}H$

Zhu [59] proved that, for a semisimple Hopf algebra H over a field of characteristic 0, the action of the Drinfel'd double on H and the action of the character algebra on H form a commuting pair. We now give the following generalization of Zhu's result for general finite-dimensional Hopf algebras.

Theorem 4.3.4. *The following map is an isomorphism of* $\Bbbk$*-algebras:*

$$\begin{array}{ccc}\Psi\colon C(H) & \xrightarrow{\sim} & \mathrm{End}_{D(H)}({}^{\mathrm{Ad}}H) \\ \cup & & \cup \\ g & \longmapsto & (f.\Lambda^L \mapsto f.(g \rightharpoonup \Lambda^L))\end{array}$$

Proof. We first establish an isomorphism of vector spaces $\mathrm{Hom}_H(\Bbbk_\alpha, {}^{\mathrm{ad}}H) \xrightarrow{\sim} \mathrm{End}_{D(H)}({}^{\mathrm{Ad}}H)$; it is given by Frobenius reciprocity (FR):

$$\begin{array}{ccccccc}
\mathrm{Hom}_H(\Bbbk_\alpha, {}^{\mathrm{ad}}H) & \xrightarrow{\sim} & \mathrm{Hom}_H(\Bbbk_\alpha, \Bbbk_\alpha\uparrow_H^{D(H)}\downarrow_H) & \xrightarrow{\sim} & \mathrm{Hom}_{D(H)}(\Bbbk_\alpha\uparrow_H^{D(H)}, \Bbbk_\alpha\uparrow_H^{D(H)}) & \xrightarrow{\sim} & \mathrm{End}_{D(H)}({}^{\mathrm{Ad}}H) \\
\Cup & & \Cup & & \Cup & & \Cup \\
(1 \mapsto z) & \xmapsto{\text{Prop 4.3.3}} & (1 \mapsto (z \rightharpoonup \lambda^L)) & \xmapsto{\text{FR}} & (f \mapsto f(z \rightharpoonup \lambda^L)) & \xmapsto{\text{Prop 4.3.3}} & (\Lambda^L \mapsto z)
\end{array}$$

The first map is the isomorphism from Proposition 4.3.3, the second is Frobenius reciprocity and the third uses the inverse of the isomorphism from Proposition 4.3.3 and the equality $(\Lambda^L \rightharpoonup \lambda^L) = \epsilon$. Note that ${}^{\mathrm{Ad}}H\downarrow_{H^*}^{D(H)} = H^*.\Lambda^L$; so elements of $\mathrm{End}_{D(H)}({}^{\mathrm{Ad}}H)$ are determined by their values on Λ^L.

Define $\psi : C(H) \to \mathrm{Hom}_H(\Bbbk_\alpha, {}^{\mathrm{ad}}H)$ by

$$\psi(f) = (1 \mapsto (f \rightharpoonup \Lambda^L)).$$

This map is injective since the action of H^* on Λ^L is non-degenerate, and it is an epimorphism in $\mathfrak{Rep}H$ by Lemma 4.3.1(b). The map Ψ in the theorem can be written as ψ composed with the aforementioned bijection of vector spaces arising from Frobenius reciprocity. Thus, being a composition of bijections, Ψ is bijective. That Ψ is an algebra map is shown below:

$$\begin{aligned}
\Psi(f) \circ \Psi(g)(\Lambda^L) &= \Psi(f)(g \rightharpoonup \Lambda^L) = \Psi(f)(S(\eta(g)).\Lambda^L) = S(\eta(g)).\Psi(f)(\Lambda^L) \\
&= S(\eta(g)).(f \rightharpoonup \Lambda^L) = S(\eta(g))S(\eta(f)).\Lambda^L = S(\eta(fg)).\Lambda^L \\
&= \Lambda^L \leftharpoonup \eta(fg) = (fg) \rightharpoonup \Lambda^L .
\end{aligned}$$

Here $\eta : H^* \to H^*$, $f \mapsto \langle x^{-1}, f_{(2)} \rangle f_{(1)}$, is the Nakayama automorphism that is associated to the Frobenius form Λ^L of H^*, with x the distinguished group-like element of H. □

4.3.6 Decomposition of ${}^{\mathrm{Ad}}H$

Theorem 4.3.4 now gives us a way to use $C(H)$ to express the conjugacy classes of H more explicitly. Indeed, it is a standard fact from module theory that, for any (left) module M over a ring R, decompositions

$$M = M_1 \oplus \cdots \oplus M_t$$

with R-submodules M_i correspond to idempotent decompositions of Id_M in $E := \mathrm{End}_R(M)$:

$$\mathrm{Id}_M = \sum_{i=1}^{t} e_i \qquad \text{with} \quad e_i e_j = \delta_{i,j} e_i\,. \tag{4.7}$$

Here, $M_i = e_i(M)$ is indecomposable if and only if e_i is primitive, that is, it is not possible to write $e_i = e + f$ with nonzero idempotents e, f that are orthogonal to each other in the sense that $ef = fe = 0$. Any collection of primitive idempotents $e_1, \dots, e_t \in E$ satisfying (4.7) is called a complete set of primitve idempotents of E. For all this, see [2, Proposition 5.7 and Corollary 5.11] or [26, Theorem 1.4], for example. Thus, we may record the following consequence of Theorem 4.3.4:

Corollary 4.3.5. *The conjugacy classes of H are the $D(H)$-submodules of ${}^{\mathrm{Ad}}H$ of the form $H^*.(e \rightharpoonup \Lambda^L)$, where $e \in C(H)$ is a primitive idempotent.*

Using the Nakayama automorphism that is associated to Λ^L, as discussed in the proof of Theorem 4.3.4, we can see that all conjugacy classes of H have the form $H^* S(\eta(e)).\Lambda^L$ for a primitive idempotent $e \in C(H)$. Since S is an antiautomorphism and η is an automorphism $\{S(\eta(e_i))\}_i$ forms the complete set of primitive idempotents of the algebra $S(\eta(C(H))) = \{f \in H^* \mid \langle f, ab\rangle = \langle f, S^2({}^x b)a\rangle\}$. So all conjugacy classes can be alternatively expressed in the form $H^* e.\Lambda^L$ where e is a primitive idempotent of $S(\eta(C(H)))$.

4.3.7 Complete Reduciblity of ${}^{\mathrm{Ad}}H$

It is natural to ask when the conjugacy classes of H are simple modules or, equivalently, when is ${}^{\mathrm{Ad}}H$ a completely reducible $D(H)$-module. As we have seen, this always holds for finite group algebras (Lemma 4.3.2). The following proposition shows that unimodularity of H is a necessary condition.

Proposition 4.3.6. *If ${}^{\mathrm{Ad}}H$ is a completely reducible $D(H)$-module, then H is unimodular.*

Proof. The first step is to show that

$$\mathrm{Hom}_{D(H)}({}^{\mathrm{Ad}}H, \mathbb{1}_{D(H)}) = C(H) \cap \int_{H^*}^{L}\,.$$

By Lemma 4.3.1 we have $\mathrm{Hom}_{D(H)}({}^{\mathrm{Ad}}H, \mathbb{1}_{D(H)}) \subseteq \mathrm{Hom}_H({}^{\mathrm{ad}}H, \mathbb{1}_H) = C(H)$. Thus it suffices to show $\mathrm{Hom}_{D(H)}({}^{\mathrm{Ad}}H, \mathbb{1}_{D(H)})$ consists of exactly the left integrals of H^*. Let $\lambda \in \mathrm{Hom}_{D(H)}({}^{\mathrm{Ad}}H, \mathbb{1}_{D(H)})$. Then:

$$\begin{aligned}\langle S^{-1}(f)\lambda, h\rangle &= \langle S^{-1}(f), h_{(1)}\rangle\langle\lambda, h_{(2)}\rangle = \langle\lambda, S^{-1}(f)(h_{(1)})h_{(2)}\rangle \\ &= \langle\lambda, f.h\rangle = f(1)\langle\lambda, h\rangle = S^{-1}(f)(1)\langle\lambda, h\rangle\end{aligned}$$

and λ is a left integral by bijectivity of S^{-1}. For the other direction let $\lambda \in \int_{H^*}^{L} \cap C(H)$. Since $\lambda \in C(H)$ it is an H-module map, so it suffices to show that λ is an H^* module map, this follows from the computation below:

$$\begin{aligned}\langle\lambda, f.h\rangle &= \langle\lambda, S^{-1}(f)(h_{(1)})h_{(2)}\rangle \\ &= \langle S^{-1}(f)\lambda, h\rangle = S^{-1}(f)(1)\langle\lambda, h\rangle = f(1)\langle\lambda, h\rangle = f.\langle\lambda, h\rangle\end{aligned}$$

and so $\mathrm{Hom}_{D(H)}({}^{\mathrm{Ad}}H, \mathbb{1}_{D(H)}) = C(H) \cap \int_{H^*}^{L}$ as desired.

Now note that $\Bbbk 1_H$ is a trivial $D(H)$ submodule of ${}^{\mathrm{Ad}}H$ thus since ${}^{\mathrm{Ad}}H$ is completely reducible we have $\mathrm{Hom}_{D(H)}({}^{\mathrm{Ad}}H, \mathbb{1}_{D(H)}) \neq 0$ thus $C(H) \cap \int_{H^*}^{L} \neq 0$. So let $\lambda^L \in \int_{H^*}^{L} \cap C(H)$.The result then follows by the computation below:

$$\alpha^{-1} = \lambda^L \leftharpoonup \Lambda^L = S^2(\Lambda^L) \rightharpoonup \lambda^L = \langle S^2(\Lambda^L), \lambda^L\rangle\epsilon$$

Therefore $\alpha^{-1} = \epsilon$, since distinct group-like elements are linearly independent, proving that H is unimodular. □

Unimodularity of H says that $\epsilon = \alpha$; so in the isomorphism in Proposition 4.3.3 takes the form

$${}^{\mathrm{Ad}}H \cong \mathbb{1}_H\!\uparrow_H^{D(H)} .$$

Additionally the homogeneous component $({}^{\mathrm{ad}}H)_\alpha$ in Lemma 4.3.1 is now the invariants of ${}^{\mathrm{ad}}H$ or, more simply, the center $\mathscr{Z}H$.

The example of group algebras suggests that the cosemisimplicity of H plays a role in ${}^{\mathrm{Ad}}H$ being a completely reducible $D(H)$-module. Indeed, the proposition below shows that this is always a necessary condition.

Proposition 4.3.7. (a) *If ${}^{\mathrm{Ad}}H$ is a completely reducible $D(H)$-module then H is cosemisimple.*

(b) *If H is cosemisimple and ${}^{\mathrm{ad}}H$ is a completely reducible H-module then ${}^{\mathrm{Ad}}H$ is a completely reducible $D(H)$-module.*

Proof. (a) Assume that ${}^{\mathrm{Ad}}H$ is completely reducible. Then, by Proposition 4.3.6, H is unimodular and $\lambda^L : {}^{\mathrm{Ad}}H \to \mathbb{1}_{D(H)}$ is a $D(H)$-module map. Since ${}^{\mathrm{ad}}H$ is a completely reducible $D(H)$-module, there is a splitting $\sigma : \mathbb{1}_{D(H)} \to {}^{\mathrm{Ad}}H$. Clearly, $\sigma(1)$ must be an H^*-invariant, but the only H^*-invariants of ${}^{\mathrm{Ad}}H$ are scalar multiple multiples of 1. Thus, $\lambda^L(\sigma(1)) \neq 0$ implies that $\lambda(1) \neq 0$. By Maschke's Theorem for Hopf algebras, this says that H is cosemisimple.

(b) Assume that H is cosemisimple and ${}^{\mathrm{ad}}H$ is a completely reducible. It suffices to show that there is a $D(H)$-module projection onto all submodules, M, of ${}^{\mathrm{Ad}}H$. Since ${}^{\mathrm{ad}}H$ is completely reducible there exists an H-module projection $\rho : {}^{\mathrm{ad}}H \to M$. By cosemisimplicity we can choose $\lambda \in \int_{H^*}$ such that $\lambda(1) = 1$. Define

$$\tilde{\rho} : {}^{\mathrm{Ad}}H \to M, \qquad \tilde{\rho}(h) = \lambda_{(1)}.\rho(S(\lambda_{(2)}).h).$$

Equation (2.9) implies that $f\lambda_{(1)} \otimes S(\lambda_{(2)}) = \lambda_{(1)} \otimes S(\lambda_{(2)})f$; so $\tilde{\rho}$ is an H^*-module map. It is is a straightforward check that $\tilde{\rho}$ is the identity on M. To complete the proof, it remains to show that $\tilde{\rho}$ is an H-module map. This is done in the following computation:

$$\begin{aligned}
\tilde{\rho}({}^k h) &= \lambda_{(1)}.\rho(S(\lambda_{(2)}).{}^k h) \\
&= \lambda_{(1)}.\rho(\langle S^{-1}(S(\lambda_{(2)})), k_{(1)}h_{(1)}S(k_{(3)})\rangle {}^{k_{(2)}}h_{(2)}) \\
&= \lambda_{(1)}.{}^{k_{(2)}}\rho(\langle \lambda_{(2)}, k_{(1)}h_{(1)}S(k_{(3)})\rangle h_{(2)}) \\
&= \langle \lambda_{(2)}, k_{(1)}h_{(1)}S(k_{(5)})\rangle \langle S^{-1}(\lambda_{(1)}), k_{(2)}\rho(h_{(2)})_{(1)}S(k_{(4)})\rangle {}^{k_{(3)}}\rho(h_{(2)})_{(2)} \\
&= \langle \lambda_{(2)}, k_{(1)}h_{(1)}S(k_{(5)})\rangle \langle \lambda_{(1)}, k_{(4)}S^{-1}(\rho(h_{(2)})_{(1)})S^{-1}(k_{(2)})\rangle {}^{k_{(3)}}\rho(h_{(2)})_{(2)} \\
&= \langle \lambda, k_{(4)}S^{-1}(\rho(h_{(2)})_{(1)})S^{-1}(k_{(2)})k_{(1)}h_{(1)}S(k_{(5)})\rangle {}^{k_{(3)}}\rho(h_{(2)})_{(2)} \\
&= \langle \lambda, S^{-1}(\rho(h_{(2)})_{(1)})h_{(1)}S(k_{(2)})S^2(k_{(3)})\rangle {}^{k_{(1)}}\rho(h_{(2)})_{(2)} \\
&= \langle \lambda, S^{-1}(\rho(h_{(2)})_{(1)})h_{(1)}\rangle {}^{k}\rho(h_{(2)})_{(2)} \\
&= \langle \lambda_{(1)}, S^{-1}(\rho(h_{(2)})_{(1)})\rangle {}^{k}\rho(\langle \lambda_{(2)}, h_{(1)}\rangle h_{(2)})_{(2)} \\
&= {}^{k}(\lambda_{(1)}.\rho(S(\lambda_{(2)}).h)) = {}^{k}\tilde{\rho}(h).
\end{aligned}$$

□

To summarize, when ${}^{\mathrm{Ad}}H$ is assumed to be completely reducible, then much of the structure discussed in the foregoing simplifies, since H is both unimodular and cosemisimple. In this case, Radford's formula for S^4 [44, Theorem 10.5.6] gives that $S^4 = \mathrm{Id}$. Thus, $S^{-2} = S^2$ and hence $C(H) = S(C(H))$. Next, $C(H)$ is a semisimple algebra, since it is the endomorphism algebra of the completely reducible module ${}^{\mathrm{Ad}}H$. Now the map η, as seen in the proof of Theorem 4.3.4, simply becomes S^2. However, $S^2|_{C(H)} = \mathrm{Id}$. Thus,

$$\mathscr{X}H = (C(H) \rightharpoonup \Lambda) = (\Lambda \leftharpoonup C(H))$$

and the conjugacy classes of H are all of the form $H^*e.\Lambda$, where e ranges over the primitive idempotents of $C(H)$.

4.3.8 Complete Reduciblity of ${}^{\mathrm{ad}}H$ and Class Sums

For the remainder of this thesis, H will be assumed to be cosemisimple and unimodular.

Fix $\lambda \in \int_{H^*}$ with $\langle \lambda, 1 \rangle = 1$ and $\Lambda \in \int_H$ with $\langle \lambda, \Lambda \rangle = 1$. Generalizing [8], we will call $e.\Lambda$ the ***class sum*** of the conjugacy class $(H^*e).\Lambda$, where e is a primitive idempotent of $C(H)$. By Lemma 4.3.1 (b), class sums are always central. The following example shows that this definition is a generalization of the familiar notion of class sums for finite groups.

Lemma 4.3.8. *Let G be a finite group. Then the class sums as defined above are the classical class sums.*

Proof. Let $\mathcal{C}$ be a conjugacy class of G and let $c = \sum_{g \in \mathcal{C}} g$ be the classical class sum of $\mathcal{C}$. By Lemma 4.3.2, $\Bbbk\mathcal{C}$ is a conjugacy class of $\Bbbk[G]$. Letting $\{\delta_x \mid x \in G\}$ denote the basis of $(\Bbbk[G])^*$ that is dual to the basis G of $\Bbbk[G]$, we have $\delta_1 \in \int_{\Bbbk[G]^*}$ with $\langle \delta_1, 1 \rangle = 1$ and $\Lambda = \sum_{g \in G} g \in \int_{\Bbbk[G]}$ with $\langle \delta_1, \Lambda \rangle = 1$.

We know that the class sum of $\Bbbk\mathcal{C}$, in the above sense, is a central element of $\Bbbk[G]$ that belongs to $\Bbbk\mathcal{C}$; so it must be a scalar multiple of c. Then the class sum can be expressed as $\alpha \sum_{g \in \mathcal{C}} \delta_{g^{-1}}.\Lambda$ for some scalar $\alpha \in \Bbbk$ and $\alpha \sum_{g \in \mathcal{C}} d_{g^{-1}}$ must be an idempotent. This forces $\alpha = 1$ and it follows that the class sum in question is c, as desired. □

When $\operatorname{char} \Bbbk = 0$, then H is semisimple and $S^2 = \mathrm{Id}$ by [27], [28]. Thus, if $\Bbbk$ is also algebraically closed, then $C(H)$ is the character algebra $R(H)$. A major tool for studying the character algebra is the Frobenius structure discussed in Section 2.3.3. Much of this structure can still be recovered for positive characteristics.

Theorem 4.3.9. *Assume* ${}^{\mathrm{ad}}H$ *is a completely reducible* H*-module. Then the bilinear form* $\{\,.\,,\,.\,\}\colon C(H) \times C(H) \to \Bbbk$, $\{f, g\} = \langle \Lambda, fg \rangle$, *is nondegenerate, associative and symmetric. If* $\Bbbk$ *is algebraically closed, then the counit of* H *does not vanish on any class sum of* H.

Proof. The form $\{\,.\,,\,.\,\}$ is clearly associative. For nondegeneracy, it suffices to show that Λ does not vanish on any left ideal of $C(H)$. Since $C(H)$ is semisimple, it suffices to show that Λ does not vanish on $C(H)e$ for e a primitive idempotent of $C(H)$. But $\langle \Lambda, H^*e \rangle \neq 0$ by nondegeneracy of the action of H^* on Λ. Now, by hypothesis, H^*e is a completely reducible H-module via the isomorphism of Proposition 4.3.3. It has H-invariants given by $C(H) \cap H^*e = C(H)e$. Observe that $\Lambda : H^*e \to \mathbb{1}_H$ is a nonzero H-module map. Since H^*e is a completely reducible H-module, there exists an H-module splitting $\sigma : \mathbb{1}_H \to H^*e$. Now $\sigma(1)$ is an H invariant so $\sigma(1) \in C(H)e$ and $\langle \Lambda, \sigma(1) \rangle = 1 \neq 0$. Hence $\langle \Lambda, C(H)e \rangle \neq 0$.

To see that $\{\,.\,,\,.\,\}$ is symmetric note that, since H^* is unimodular, the Nakayama automorphism associated to Λ is S^2, but $S^2|_{C(H)} = \mathrm{Id}$ as desired.

For the final statement, let $\Bbbk$ be algebraically closed. Then $C(H)$ is a split semisimple algebra. So Λ does not vanish on any primitive idempotent $e \in C(H)$, since symmetric forms of matrix algebras don't vanish on primitive idempotents. The result now follow from the computation below:

$$\langle \epsilon, e.\Lambda \rangle = \langle \epsilon, \langle S^{-1}(e), \Lambda_{(1)} \rangle \Lambda_{(2)} \rangle = \langle S^{-1}(e), \Lambda \rangle = \langle e, S^{-1}(\Lambda) \rangle = \langle e, \Lambda \rangle \neq 0.$$

□

4.3.9 The Involutory Case

If H is assumed to be involutory, in addition to the assumptions of Theorem 4.3.9, then a nonzero integral for H becomes a symmetric form on H^*. In fact, given $\lambda \in \int_{H^*}$ with $\langle \lambda, 1 \rangle = 1$ and choosing $\Lambda \in \int_H$ such that $\langle \lambda, \Lambda \rangle = 1$, then $\Lambda = \chi_{H^*}$, where χ_{H^*} denoted the character of the regular representation of H^*. With this assumption, the previous theorem can be used to generalize the result for finite groups G that completely reducibility of ${}^{\mathrm{ad}}\Bbbk[G]$ implies that $\mathrm{char}\,\Bbbk$ does not divide the size of any conjugacy class of G.

Corollary 4.3.10. *Assume H is cosemisimple, unimodular and involutory, with ${}^{\mathrm{ad}}H$ a completely reducible H-module and $\Bbbk$ algebraically closed. Then* $\mathrm{char}\,\Bbbk$ *does not divide the dimensions of any conjugacy class of H.*

Proof. By Corollary 4.3.5, it suffices to fix e a primitive idempotent of $C(H)$ and show that $\mathrm{char}\,\Bbbk$ does dot divide $\dim_{\Bbbk}(H^*e)$ or, equivalently, $0 \neq 1_{\Bbbk} \dim_{\Bbbk}(H^*e)$. But, by Proposition 4.3.9, $0 \neq \langle \epsilon, e.\Lambda \rangle = \langle \Lambda, e \rangle = \langle \chi_{H^*}, e \rangle = 1_{\Bbbk} \dim_{\Bbbk}(H^*e)$. $\square$

The assumption of $\Bbbk$ algebraically closed can be weakened to $\Bbbk$ being a splitting field for $C(H)$.

4.4 An Example

This section will give an explicit nontrivial example of the conjugacy classes of a non-semisimple Hopf algebra. All computations for this example were done using Python code; for more information on this code see chapter **??**.

4.4.1 The structure of $\mathfrak{B}_{2,3}$

Let $\Bbbk$ be an algebraically closed field of characteristic 3 and let G be the group defined by the generators and relations below:

$$G = \left\{ a, b, \sigma, \tau \;\middle|\; \begin{matrix} \sigma^2 = \tau^2 = a^3 = b^3 = 1, \sigma a = a^2 \sigma, \tau b = b^2 \tau, \\ [a, b] = [\sigma, \tau] = [a, \sigma] = [b, \tau] = 1 \end{matrix} \right\}$$

Define $\mathfrak{B}_{2,3}$ to be $\Bbbk[G]$ as a $\Bbbk$-algebra. The counit of $\mathfrak{B}_{2,3}$ remains unchanged from that of $\Bbbk[G]$. Also σ and τ remain group-like elements in $\mathfrak{B}_{2,3}$. The comultiplication on the remaining algebra generators of $\mathfrak{B}_{2,3}$ is then given by

$$\Delta(a) = 2(a \otimes a + a\tau \otimes a + a \otimes a^2 - a\tau \otimes a^2)$$
$$\Delta(b) = 2(b \otimes b + b \otimes b\sigma + b^2 \otimes b - b^2 \otimes b\sigma)\,.$$

The antipode is given by

$$S(a) = 2(a^2 + a^2\tau + a - a\tau)$$
$$S(b) = 2(b^2 + b^2\sigma + b - b\sigma)\,.$$

The Hopf Algebra $\mathfrak{B}_{2,3}$ is a special case of a class of Hopf algebras introduced by Galindo and Natale [20, Section 8].

We now list some basic properties of $\mathfrak{B}_{2,3}$. The formula for the comultiplication directly gives that $\mathfrak{B}_{2,3}$ is not cocommutative. By Maschke's Theorem, $\Bbbk[G]$ is not semisimple, and hence $\mathfrak{B}_{2,3}$ is not semisimple, since $\mathfrak{B}_{2,3} = \Bbbk[G]$ as $\Bbbk$-algebra. Since the counit of $\mathfrak{B}_{2,3}$ remains unchanged from that of $\Bbbk[G]$, it also follows that $\mathfrak{B}_{2,3}$ is unimodular. A direct calculation gives that $S^2 = \mathrm{Id}$; so $\mathfrak{B}_{2,3}$ is involutory. Let $\{\rho_{a^ib^j\sigma^k\tau^\ell}\}$ be the basis of $\mathfrak{B}_{2,3}^*$ that is dual to $\{a^ib^j\sigma^k\tau^\ell\}$. Then ρ_1 is an integral of $\mathfrak{B}_{2,3}^*$, and hence $\mathfrak{B}_{2,3}$ is cosemisimple by Maschke's Theorem.

4.4.2 The submodules of $C(\mathfrak{B}_{2,3})$

Since $\mathfrak{B}_{2,3} \cong \Bbbk[G]$ as algebras and $S^2 = \mathrm{Id}$ in both cases, we have $C(\mathfrak{B}_{2,3}) = C(\Bbbk[G])$ as sets. Hence we get the following basis of $C(\mathfrak{B}_{2,3})$,

$$\begin{aligned} C(H) = \Bbbk\{\rho_1\} & \\ & \cup \Bbbk\{\gamma_a := \rho_a + \rho_{a^2},\, \gamma_b := \rho_b + \rho_{b^2}\} \\ & \cup \Bbbk\{\gamma_\sigma = \rho_\sigma + \rho_{a\sigma} + \rho_{a^2\sigma},\, \gamma_\tau := \rho_\tau + \rho_{b\tau} + \rho_{b^2\tau}\} \\ & \cup \Bbbk\{\gamma_{ab} := \rho_{ab} + \rho_{a^2b} + \rho_{ab^2} + \rho_{a^2b^2}\} \\ & \cup \Bbbk\{\gamma_{b\sigma} := \rho_{b\sigma} + \rho_{ab\sigma} + \rho_{a^2b\sigma} + \rho_{b^2\sigma} + \rho_{ab^2\sigma} + \rho_{a^2b^2\sigma}\} \\ & \cup \Bbbk\{\gamma_{a\tau} := \rho_{a\tau} + \rho_{ab\tau} + \rho_{ab^2\tau} + \rho_{a^2\tau} + \rho_{a^2b\tau} + \rho_{a^2b^2\tau}\} \\ & \cup \Bbbk\{\gamma_{\sigma\tau} = \sum_{0\le i\le j} \rho_{a^ib^j\sigma\tau}\} \end{aligned}$$

Thus, $\dim_{\Bbbk} C(\mathfrak{B}_{2,3}) = 9$. In fact, $C(\mathfrak{B}_{2,3})$ is commutative and semisimple; so its primitive idempotents are uniquely determined, and hence so are the conjugacy classes of $\mathfrak{B}_{2,3}$. The table below lists the primitive idempotents of $C(\mathfrak{B}_{2,3})$ and the corresponding conjugacy class sums.

Idempotent	Conjugacy Class sum
ρ_1	1
$2\gamma_a$	$a + a^2$
$2\gamma_b$	$b + b^2$
γ_σ	$\sigma + a\sigma + a^2\sigma$
γ_τ	$\tau + b\tau + b^2\tau$
γ_{ab}	$ab + a^2b + ab^2 + a^2b^2$
$\gamma_{b\sigma}$	$b\sigma + ab\sigma + a^2b\sigma + b^2\sigma + ab^2\sigma + a^2b^2\sigma$
$\gamma_{a\tau}$	$a\tau + ab\tau + a^2b\tau + a^2\tau + ab^2\tau + a^2b^2\tau$
$\gamma_{\sigma\tau}$	$\sum_{0\le i,j\le 2} a^ib^j\sigma\tau$

We then get the following conjugacy classes:

$$
\begin{aligned}
C_{\rho_1} &= \Bbbk\langle 1\rangle \\
C_{\rho_a} &= \Bbbk\langle a, a^2\rangle \\
C_{\rho_b} &= \Bbbk\langle b+b^2, b\sigma - b^2\sigma\rangle \\
C_{\rho_\sigma} &= \Bbbk\langle a, a\sigma, a^2\sigma\rangle \\
C_{\rho_\tau} &= \Bbbk\langle \tau, b\tau, b^2\tau\rangle \\
C_{\rho_{ab}} &= \Bbbk\langle ab+ab^2, a^2b+a^2b^2, ab\sigma - ab^2\sigma, a^2b\sigma - a^2b^2\sigma\rangle \\
C_{\rho_{b\sigma}} &= \Bbbk\left\langle \begin{matrix} b\sigma, b^2\sigma, ab\sigma + a^2b\sigma, ab^2\sigma + a^2b^2\sigma \\ ab\sigma\tau - a^2b\sigma\tau, ab^2\sigma\tau - a^2b^2\sigma\tau \end{matrix}\right\rangle \\
C_{\rho_{a\tau}} &= \Bbbk\left\langle \begin{matrix} a\tau, a^2\tau, ab\tau + ab^2\tau, a^2b\tau + a^2b^2\tau \\ ab\sigma\tau - ab^2\sigma\tau, a^2b\sigma\tau - a^2b^2\sigma\tau \end{matrix}\right\rangle \\
C_{\rho_{\sigma\tau}} &= \Bbbk\left\langle \begin{matrix} \sigma\tau, b\sigma\tau, b^2\sigma\tau, a\sigma\tau + a^2\sigma\tau, a\sigma + a^2\sigma \\ ab\sigma\tau + a^2b\sigma\tau, ab^2\sigma\tau - a^2b^2\sigma\tau, ab\sigma - ab^2\sigma, ab^2\sigma - a^2b^2\sigma \end{matrix}\right\rangle
\end{aligned}
$$

Observe that $\dim_{\Bbbk} C_{\rho_\sigma} = 3 = \operatorname{char} \Bbbk$. Hence, by Corollary 4.3.10, ${}^{\mathrm{ad}}\mathfrak{B}_{2,3}$ is not completely reducible.

CHAPTER 5

BROADER INTERESTS

Currently, the construction and study of explicit new examples or counterexamples in the theory of Hopf algebras is a rather difficult task. This is in part because there is currently no public software that is designed specifically for working with Hopf algebras. To aid in my personal research, I have developed Python code for calculations in finite-dimensional Hopf algebras. This code can be found on my GitHub page at https://github.com/AdamJacoby/Hopf.

At the center of the code is the Hopf algebra class. Let H be a finite-dimensional Hopf algebra, with a fixed basis $(b_i)_{i=1}^{\dim_{\Bbbk} H}$, and give $H \otimes H$ the basis $(b_i \otimes b_j)$ with the lexicographic order. The multiplication of H is then encoded as a matrix of size $\dim_{\Bbbk} H$ by $(\dim_{\Bbbk} H)^2$, where column $i * \dim_{\Bbbk} H + j$ corresponds to $b_i b_j$. The unit, counit, comultiplication and antipode are encoded similarly. The Hopf algebra class has additional attributes for integrals and Casimir elements, with functions included to compute them for the structure data. Classes exist for algebras, coalgebras, modules and module algebras as well.

The code contains many tools for constructing new objects from Hopf algebras; it contains, among others, the following constructions: duals, tensor products, smashed products, Drinfeld twists, dual Drinfeld twists, crossed products, and adjoint modules.

To aid in working within a Hopf algebra, I have also devolved the Hopf algebra element class. Elements can be added, multiplied and exponentiated using the $+$, $*$

and $**$ operations. Functions are also available for the antipode, comultiplication, counit, tensor product and module actions. The primary advantage of working with this class is that elements can be created and referred to by name rather than as a vector, making working with them far more natural and intuitive as the user never has to delve into the vector and matrix notation at the core of the code.

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