Constructing Efficient Multi-Asset Class Portfolios: Top-Down or Bottom-Up?

A Dissertation
presented to

The **Development Finance Centre (DEFIC)**
Graduate School of Business
University of Cape Town

In partial fulfilment
of the requirements for the degree
Master of Commerce in Developmental Finance

By

Lebohang Pule
(PLXLEB002)

October, 2017

Supervisor: Dr. Ailie Charteris
Co-Supervisor: Dr. Johan de Kock

PLAGIARISM DECLARATION

Declaration

1. I know that plagiarism is wrong. Plagiarism is to use another's work and pretend that it is one's own.

2. I have used the Harvard convention for citation and referencing. Each contribution to, and quotation in, this thesis from the work(s) of other people has been attributed, and has been cited and referenced.

3. This thesis is my own work.

4. I have not allowed, and will not allow, anyone to copy my work with the intention of passing it off as his or her own work.

5. I acknowledge that copying someone else's assignment or essay, or part of it, is wrong, and declare that this is my own work.

Signature ______________________________

Lebohang Pule

ABSTRACT

This dissertation concerns itself with the problem of constructing multi asset class portfolios. The investment process is aimed at solving two problems. The first problem is estimating the future returns of individual securities, which is an exercise fraught with uncertainty as the future is fundamentally unpredictable. This uncertainty means that the investor must allocate his portfolio to a number of assets instead of just one, in case his predicted future returns do not materialize. This leads the investor to the second problem of how best to construct the portfolio. It is this part of the investment process which is the subject of this dissertation which examines whether it is best to construct multi-asset class portfolios using a top-down or bottom-up approach. In the top-down approach one begins by creating independent single asset class portfolios which are then combined to create a multi-asset class portfolio. The bottom-up approach constructs the portfolio by considering all the securities available to the investor (irrespective of asset class) at the same time. The Mean-Variance and Black-Litterman models are reviewed in detail. Portfolios are then created using these portfolio construction methods in order to compare the two approaches. In constructing these portfolios, the commonly encountered problem of missing data in financial return series is also examined. The main result is that the top-down and bottom-up approaches create similar efficient frontiers, though the bottom-up approach results in an extended frontier which allows investors to obtain efficient portfolios with either a higher expected return or a lower volatility.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ACKNOWLEDGEMENT

I would like to thank my two supervisors as well as the staff of the UCT Graduate School of Business. Without their indulgence I would have never completed this dissertation.

I would also like to thank Selloane and Teboho Pule, without their persistent encouragement I would not have completed this degree.

1 INTRODUCTION

1.1 Research Area

The investment process begins with the investor postulating on the future investment performance of all financial securities available to him. These postulations or investment beliefs allow the investor to choose the subset of financial securities which he believes will deliver the highest returns. The investor is then tasked with constructing a portfolio which expresses all his investment beliefs and maximizes whichever criteria generally make one portfolio superior to another.

In a reality without uncertainty, there would be no portfolio selection problem for the investor to grapple with. He would simply invest the entire portfolio in the single security he is certain will yield the highest return. It is the uncertainty of the security returns which gives rise to the portfolio selection problem and indeed has guided its development. It was not until Markowitz' ground breaking work in 1952 that the very role of uncertainty, which today is central to portfolio construction, was well-defined.

Markowitz (Markowitz, 1952) articulated a new theory which brought rigour to the measurement of both the uncertainty inherent in security return forecasts and the concept of diversification which until then was understood only in normative terms and conveyed using cliché's advising investors not to put their "eggs in one basket." Markowitz described portfolios which minimized this uncertainty while achieving the highest expected returns as efficient.

This dissertation focuses on the problem of constructing multi-asset class portfolios which are efficient. It specifically asks whether it is best to construct single asset class portfolios which are then combined or to construct one single portfolio with many securities across different asset classes. This is a problem which Development Finance Institutions as well as other financial institutions that hold portfolios with securities across multiple asset classes must grapple with.

1.2 Problem Statement

Development Finance Institutions often find themselves facing the task of investing large portfolios across multiple asset classes in order to meet some financial objective. This objective may be the financing of pension liabilities or the management of an endowment meant to serve developmental objectives over a long period of time.

In South Africa, such multi-asset class portfolios are called Balanced Funds. Faced with constructing a balanced fund to meet some objective, it is common for these institutions to begin the portfolio construction at the asset class level. This was already the case internationally in the 1990's according to Brinson, Hood and Beebower (Brinson, 1995). An allocation to each asset class is decided after which point a portfolio is constructed for each asset class. This asset class portfolio is constructed by finding the combination of asset classes which results in the highest expected return given some target risk.

The construction of each asset class portfolio is independent of the other portfolios. Indeed the portfolios are often managed by different portfolio managers who aim to produce returns larger than their respective benchmark portfolios. The different single asset class portfolios are constructed by finding the combination of securities (within the particular asset class) which results in the highest expected return given some target risk. In this dissertation, this approach is called *top-down* portfolio construction.

This approach seems to be validated by the results of Brinson and his colleagues (Brinson, 1995) who analysed the returns generated by pension plans over a 10 year period. They found that the asset allocation decision was responsible for the vast majority of the returns generated by portfolios.

An alternative would be to consider all the securities available to be invested, regardless of their asset class characterization, and to find the combination of all securities which results in the highest expected return given some target risk. This dissertation asks if this approach results in better portfolios. In this dissertation, this approach is called *bottom-up* portfolio construction.

Faced with these two approaches, development finance as well as other financial institutions, must decide which approach is best.

The question this dissertation aims to answer is:

1. Are multi-asset class portfolios constructed using the bottom-up approach more efficient than those constructed using the top-down approach?

1.3 Purpose and Significance of the Research

The question posed by this dissertation is at the core of what every investor is trying to do: produce the most economically optimal portfolios. The question of whether to construct portfolios using the bottom-up or top-down approach should not only drive the way portfolios are optimized but the manner in which the investment firms managing these portfolios operate. If the top-down approach produces the most efficient portfolios, then investment firms and divisions should allocate large multi-asset class portfolios to a number of specialist portfolio managers, each managing their respective asset class. If, however, the bottom-up approach produces the most efficient portfolios, then investment firms should concentrate on analysing securities across multiple asset classes. The results of this analysis should then be used to construct large portfolios combining securities across the asset classes.

1.4 Research Questions and Scope

Operational considerations are important. One allure of the top-down approach is that the mammoth task of investing across multiple asset classes is divided amongst a number of experts looking after their respective fields. This dissertation does not concern itself with these operational considerations, however. This dissertation focuses only on which approach leads to the most optimal portfolios.

There are many inputs to the portfolio construction problem. The primary inputs are the expected returns, risk metrics as well as the relationships between different securities. There are various methods to estimate these parameters. This dissertation does not concern itself with deciding which of these methods is best. It assumes that the investor has his preferences or proprietary methods to estimate these.

What this dissertation does concern itself with is the portfolio construction methodologies the investor would use in order to choose his bottom-up or top-down constructed portfolio. This study considers the mean-variance portfolio optimization model as well as the Black-Litterman model. Much attention is paid to these models as well as the practical issues the investor must deal with when using these models to construct a portfolio.

1.5 Dissertation Plan

This dissertation comprises eight chapters which are grouped into an introduction and two major parts. Part I with the methodologies used in the dissertation while Part II discusses the analysis conducted as well as its findings.

This dissertation is empirical in nature focusing on the use of existing portfolio construction methods to decide on the best process to construct portfolios. As a result the layout of this dissertation is somewhat unconventional. Instead of dedicating a chapter to surveying the literature available on the methodologies used in this dissertation, relevant literature is surveyed as the methodologies in question are introduced. Attention is then turned to empirical examinations of the methodologies.

2 PART I: METHODOLOGIES USED IN RESEARCH

Part I of this dissertation deals with the methodologies used in order to examine the research problem. This study dedicates a large amount of attention to this section as it is core to the portfolio construction process every investor must face. By dedicating so much attention to the methodologies used to construct portfolios, this dissertation attempts to shed light on the considerations and nuances all investors must face regardless of the construction approach they choose.

Chapter 2 (Data Sources, Uses and Analysis) deals with the data used in this dissertation, the problems often encountered with financial data and the approaches used in order to fix those problems. The indices used to represent each asset class are presented with their constituents. These indices are the benchmarks against which single-asset class portfolios will be constructed when using the top-down approach. The index constituents define the universe of securities available to the investor using the bottom-up approach. The primary data problem faced when compiling this dissertation was that of missing data which this chapter deals with. The current constituents of both bond and equity indices may not have a long enough history to support required quantitative analysis. One input to portfolio construction models is the covariance matrix whose correct calculation requires longer historical data the larger the number of assets or variables in question. This illustrates a method using linear regression to estimate missing return data.

Chapter 3 (Portfolio Selection Models) introduces the portfolio selection models used in this dissertation. The mean-variance optimization model is introduced first before illustrating some of its short comings. Attention is then turned to the Black-Litterman model by way of a brief derivation from the literature.

Chapter 4 (Application To The Equity Market) details the procedure an investor must follow in order to use the portfolio optimization models to construct a portfolio of equities. This chapter shows that it is more appropriate for the investor to model equity returns rather than prices when constructing portfolios. It is then shown that log-returns possess properties which make them more ideal to model than linear returns. The portfolio optimization models are defined in terms of linear returns, so the chapter ends off converting log-return statistics to linear return statistics which are then used as input to the portfolio optimization models.

Chapter 5 (Application To The Bond Market) illustrates the nuances of the bond market which make the portfolio construction exercise very different to that of equities. Unlike equities, bonds are instruments with a predefined maturity date as well as a maturity price. Bond prices therefore tend towards the redemption price over time. This affects the nature of bond price returns over time. The closer a bond is to redemption, the less volatile the price and therefore returns. This chapter shows that it is more appropriate to model bond yields. The bond yield statistics are then used to model potential bond price returns which can be used to construct the portfolios of bonds.

3 DATA SOURCES, USES AND ANALYSIS

This chapter outlines the data used in this dissertation. The data set used is presented, followed by a discussion of the problems encountered with the data and their solutions.

This dissertation concerns itself with the optimization of domestic multi-asset class portfolios constructed using South African Equities, Bonds, and Property. We describe here the market indices and securities used to represent these asset classes.

The market data used in this dissertation was sourced from the Bloomberg Terminal. Weekly share, index and government bond yield data spanning the period 01 December 2000 to 25 November 2016 was used to generate the results contained here.

It is often difficult to find historical financial data for all the securities a portfolio manager may be considering for investment. This chapter concludes with a section illustrating how linear regression can be used in order to estimate missing historical returns.

3.1 Equity Market Exposure

We use the FTSE/JSE Top40 Index and its constituents to represent exposure to the South African general equity market.

The FTSE/JSE Top40 Index is comprised of the forty largest companies listed on the Johannesburg Stock Exchange as ranked by investable market value (FTSE Russel, 2016).

Despite being comprised of the forty largest investable companies listed on the Johannesburg Stock Exchange, this dissertation will assume that the FTSE/JSE Top40 Index is representative of the South African Equity Market. Figure 1 shows, over the five year period to 30 November 2016, the performance of the FTSE/JSE Top40 Index was similar to that of the FTSE/JSE All Share Index which is comprised of a much larger universe of companies listed on the Johannesburg Stock Exchange.

Figure 1: Performance Of FTSE/JSE Top40 Index vs. FTSE/JSE All Share Index

This dissertation uses the FTSE/JSE Top40 Index as it was comprised at close of business on 30 November 2016. Table 1 shows the listed shares which were part of the Top40 Index on 30 November 2016.

Top40 Index Constituents		30 Nov 16	
Ticker	Name	Share Price (Rands)	Weight (%)
AGL	Anglo American PLC	212.00	5.47
ANG	AngloGold Ashanti Ltd	155.34	1.24
APN	Aspen Pharmacare Holdings Ltd	289.93	2.03
BAT	Brait SE	85.15	0.55
BGA	Barclays Africa Group Ltd	157.31	1.30
BID	Bid Corp Ltd	246.57	1.63
BIL	BHP Billiton PLC	233.89	9.64
BTI	British American Tobacco PLC	781.35	4.32
BVT	Bidvest Group Ltd/The	162.82	1.07
CFR	Cie Financiere Richemont SA	91.18	8.92
DSY	Discovery Ltd	111.42	0.70
FFA	Fortress Income Fund Ltd	16.23	0.34
FFB	Fortress Income Fund Ltd	30.54	0.42
FSR	FirstRand Ltd	50.50	2.99
GFI	Gold Fields Ltd	44.60	0.72
GRT	Growthpoint Properties Ltd	24.88	1.30
IMP	Impala Platinum Holdings Ltd	44.20	0.52
INL	Investec Ltd	89.97	0.47
INP	Investec PLC	91.10	1.12
ITU	Intu Properties PLC	47.19	0.86
LHC	Life Healthcare Group Holdings Ltd	31.04	0.60
MEI	Mediclinic International PLC	124.70	0.90
MND	Mondi Ltd	283.28	0.65
MNP	Mondi PLC	284.90	2.04
MRP	Mr Price Group Ltd	145.05	0.67
MTN	MTN Group Ltd	113.20	3.88
NED	Nedbank Group Ltd	230.00	0.96
NPN	Naspers Ltd	2,054.86	17.21
NTC	Netcare Ltd	31.24	0.89
OML	Old Mutual PLC	33.10	3.05
RDF	Redefine Properties Ltd	10.35	0.92
REI	Reinet Investments SCA	27.66	0.79
REM	Remgro Ltd	209.11	2.16
RMH	RMB Holdings Ltd	62.40	0.81
SBK	Standard Bank Group Ltd	151.00	3.55
SGL	Sibanye Gold Ltd	29.17	0.42
SHP	Shoprite Holdings Ltd	186.94	1.58
SLM	Sanlam Ltd	61.67	2.11
SNH	Steinhoff International Holdings NV	65.08	3.24
SOL	Sasol Ltd	379.35	4.09
TBS	Tiger Brands Ltd	394.09	1.24
VOD	Vodacom Group Ltd	144.73	1.43
WHL	Woolworths Holdings Ltd	64.93	1.20
		Total	100.00

Table 1: Top40 Index Constituents As At 30 November 2016

3.2 Property Market Exposure

For the South African property market, the FTSE/JSE JSAPY Index as well as its constituents are used. This index is comprised of the twenty largest liquid property companies with a primary listing on the Johannesburg Stock Exchange. The companies are weighted by their market capitalization. Table 2 shows the constituents of the Property Index as at 30 November 2016.

JSAPY Index Constituents		30 Nov 16	
Ticker	Name	Share Price (Rands)	Weight (%)
GRT	Growthpoint Properties Ltd	24.88	19.59
RDF	Redefine Properties Ltd	10.35	13.88
NEP	New Europe Property Investments PLC	160.97	10.88
RES	Resilient REIT Ltd	103.50	8.28
HYP	Hyprop Investments Ltd	113.01	8.26
FFB	Fortress Income Fund Ltd	30.54	6.38
FFA	Fortress Income Fund Ltd	16.23	5.19
ROC	Rockcastle Global Real Estate Co Ltd	35.30	4.74
SAC	SA Corporate Real Estate Ltd	5.50	3.70
VKE	Vukile Property Fund Ltd	18.05	3.41
ATT	Attacq Ltd	16.49	2.98
AWA	Arrowhead Properties Ltd	8.70	2.44
IPF	Investec Property Fund Ltd	15.30	1.89
EMI	Emira Property Fund Ltd	13.80	1.79
REB	Rebosis Property Fund Ltd	10.85	1.35
MSP	MAS Real Estate Inc	20.90	1.22
APF	Accelerate Property Fund Ltd	6.44	1.09
PIV	Pivotal Fund Ltd/The	16.00	1.06
OCT	Octodec Investments Ltd	20.54	0.96
TDH	Tradehold Ltd	23.75	0.47
STP	Stenprop Ltd	19.05	0.42
		Total	100.00

Table 2: JSAPY Index Constituents As At 30 November 2016

Looking at the constituents of the JSAPY Index, it is clear that one constituent is also a member of the FTSE/JSE Top40 Index. For an investor using these two indices to give him exposure to the general equity and property markets, there is the risk that the resulting portfolio has more property exposure than expected. This dissertation does not address this risk other than to note it.

3.3 Government Bond Exposure

For exposure to the South African bond market we use JSE Govi Index. The Govi Index uses the top ten bonds issued by the South African government as ranked by both liquidity and market capitalization. Table 3 shows the constituents of the Govi Index on 30 November 2016.

GovTR Index Constituents			30 Nov 16
Ticker	Maturity Date	YTM	Weight
R186	21 Dec 26	9.02	23.84
R2048	28 Feb 48	9.76	12.82
R2030	31 Jan 30	9.42	9.98
R213	28 Feb 31	9.44	9.33
R2037	31 Jan 37	9.74	8.93
R207	15 Jan 20	8.26	7.70
R209	31 Mar 36	9.63	7.26
R208	31 Mar 21	8.45	7.02
R2023	28 Feb 23	8.73	7.02
R204	21 Dec 18	8.00	6.10
		Total	100.00

Table 3: Govi Index Constituents As At 30 November 2016

The returns of the bonds in Table 3 and their weights in the Index on 30 November 2016 are used to construct Index returns which are then used to construct portfolios.

3.4 Data Problems and Fixes

The main data problem we encountered was missing financial data. For each index, we use the constituents available on 30 November 2016 as the universe of securities available to the investor. The problem is that not all of these assets have the same length of historical data available. Stambaugh (Stambaugh, 1997) provides an excellent resource for methodologies which can be used to solve this problem.

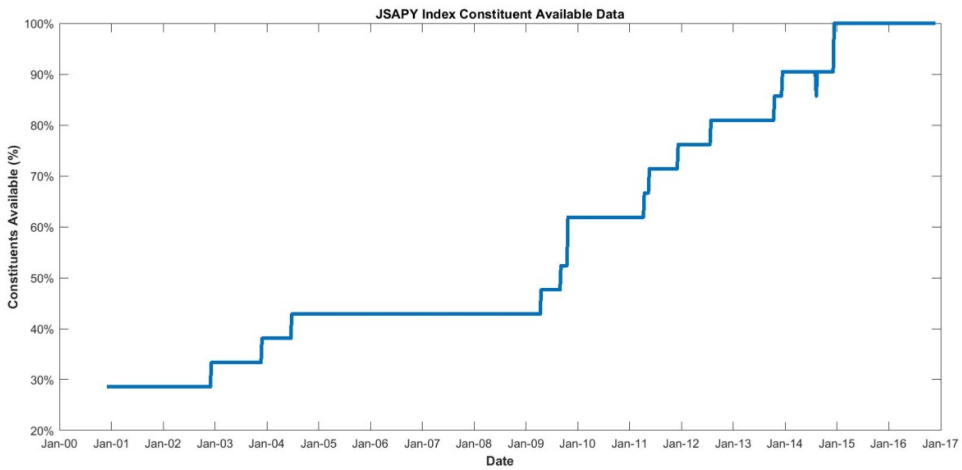

Figure 2: Proportion Of Current JSAPY Constituents Listed At Each Point In Time

Using the JSAPY Index as an example, Figure 2 shows the percentage of the current constituents which were available at each point in time over the period we have used for historical returns. It's clear that the vast majority of the shares were not listed (and therefore have no data) during the early stages of our history.

The reason we have used such a long period of time for historical returns is due to the necessary calculation of a covariance matrix. Covariance matrices must be positive semi-definite in order to rule out the possibility of negative portfolio variances (and therefore undefined portfolio volatilities). In order for a covariance matrix to be positive semi-definite, the number of observations used to construct the matrix must be significantly larger than the number of variables or shares in our case.

To get around the issue of missing historical data we have used multiple linear regression to estimate the returns for the periods over which the shares were not listed. The rational here being that the regression helps us ensure that the relationship between the returns remains intact.

In the context of a multiple linear regression set up

$$Y = \beta_0 1 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \cdots + \beta_n X_n + \epsilon$$

we are estimating what the missing returns would have been, so the dependent variable or Y above is the missing returns. We are assuming that the stock with missing returns has a stable or stationary relationship with the returns of the stocks which are not missing. So we use the returns of the stocks with a full history of returns as the explanatory variables (X_i in the equation above) when finding the linear regression relationship. To find the linear relationship ($\beta_i's$), we use that period in the data where returns are available for the stock with missing returns in earlier periods.

Once the linear relationship has been found, we can then use it to estimate the stock's missing returns. Figure 3 below shows the estimated relationship for the stock RES. This relationship

$$Y = 0.0019 + 0.0847x_1 + 0.1912x_2 + 0.1874x_3 + 0.3179x_4 + 0.0931x_5 + 0.0558x_6 + 0.0309x_7$$

is used to estimate the missing returns.

```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7

Estimated Coefficients:
                   Estimate        SE        tStat       pValue
                   _________    __________    ______    __________

    (Intercept)    0.0019152    0.00080974    2.3652      0.018283
    x1              0.084627      0.028027    3.0195     0.0026216
    x2               0.19118      0.037581     5.087    4.6442e-07
    x3               0.18741      0.031895    5.8756    6.4375e-09
    x4               0.31788      0.034152    9.3077    1.5453e-19
    x5              0.093109      0.029396    3.1674     0.0016032
    x6              0.055831      0.019551    2.8556     0.0044186
    x7              0.030898      0.011878    2.6014     0.0094758


Number of observations: 728, Error degrees of freedom: 720
Root Mean Squared Error: 0.0218
R-squared: 0.831,  Adjusted R-Squared 0.829
F-statistic vs. constant model: 506, p-value = 5.32e-273
```

Figure 3: JSAPY Missing Data Regression Results

Figure 3 also shows how well the regression explains the known returns of RES, and therefore how well we can expect it to estimate the missing returns. The model fits quite well giving us confidence in the estimated returns. Figure 4 and Figure 5 show the residuals of the regression analysis and give us comfort that the residuals have no systematic relationship to their lags and are a close fit to the normal distribution.

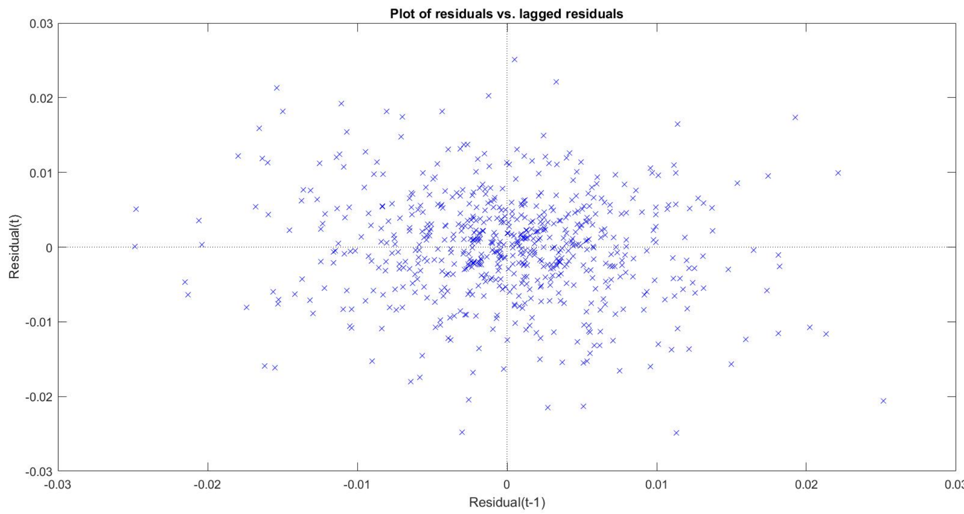

Figure 4: Lagged Scatter Plot Of JSAPY Regression Residuals

In the linear regression equation

$$Y = \beta_0 1 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \cdots + \beta_n X_n + \epsilon$$

the variable ϵ is meant to capture the part of the returns which are not explained by our explanatory variables. These are called residuals. For our linear regression to be a good estimator of the missing returns, it is important that these residuals have no discernible pattern or relationship. If they do have a discernible pattern it means that our regression model is missing some explanatory variable. In the case of stock RES, Figure 4 shows that there is no pattern and therefore we are likely not missing some explanatory variable. Figure 5 is a test for another important attribute of the residuals, that they follow the normal distribution. We can see that there is a reasonable fit.

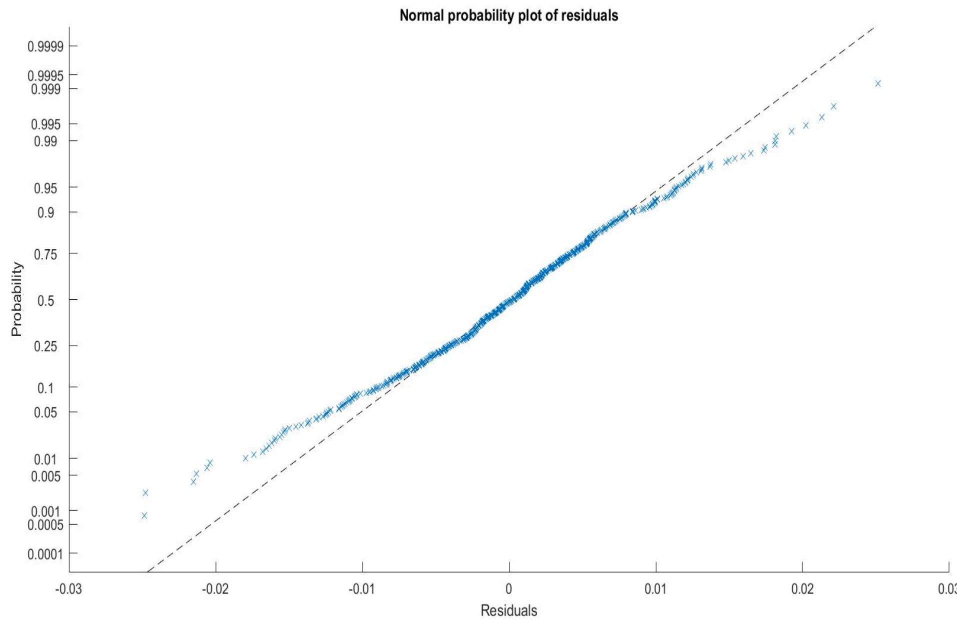

Figure 5: Normal Probability Plot Of JSAPY Regression Residuals

4 PORTFOLIO SELECTION MODELS

This chapter introduces the mean-variance and Black-Litterman portfolio optimization models. These are the models used to construct portfolios examined in this dissertation. The mean-variance optimization model is introduced first before illustrating its main shortfalls using a portfolio of property shares. The Black-Litterman model is then introduced together with a derivation from the literature.

4.1 Mean-Variance Portfolio Optimization

In 1952 Markowitz (Markowitz, 1952) published a paper which would change the way investors think about diversification and putting together portfolios. Markowitz begins by describing how portfolio returns and risks can be described using the return and risk parameters of the individual assets. Efficient portfolios, which attain the highest possible return given a level of risk, are then introduced.

Markowitz' model assumes that the investor wishes to attain two objectives. The first is to maximize the expected return of his portfolio. The second to minimize the risk of his portfolio as defined by the variance of portfolio returns.

The mean-variance framework sets this problem up by maximizing the portfolio return while penalizing the volatility of the portfolio returns

$$max_{\omega}\{\omega\mu - \lambda\omega'\Sigma\omega\}$$

This scheme works by finding the portfolio weights, ω, that maximize the portfolio return, $\omega\mu$, without being overly penalized for the portfolio volatility, $\omega'\Sigma\omega$, which is scaled by a risk aversion factor, λ. The maximization is conducted under the condition that the sum of the portfolio weights adds up to the total portfolio

$$\omega' 1 = 1.$$

The mean-variance model considers only the first two moments of the asset returns and uses the variance of returns as a measure of risk. This is equivalent to assuming that the asset returns in question follow one of the elliptical statistical distributions (Miskolczi, 2016). The other assumption is that the variance of returns is an appropriate measure of risk for the assets in question. The literature is rich with extensions to the mean-variance model which make changes to the model to make it more appropriate for non-elliptical distributions or use a risk measure other than the variance of returns.

Rockafellar and Uryasev (Rockafellar, 2000) show how to optimize portfolios using conditional value at risk (CVaR) as the risk measure instead of the variance of returns. CVaR is the expected loss that an investor may experience at a certain confidence level. When considering assets whose returns exhibit skeweness and kurtosis, an asymmetric risk measure such as CVaR is more appropriate than the variance of returns (Chen, 2012). Yilmaz (Yilmaz, 2015) the mean-CVaR method to a highly asymmetrical portfolio of options held in a dispersion trading strategy and shows that using the CVaR risk measure produces better returns than the variance of returns. Instead of changing the risk measure, Aracioglu et al (Aracioglu, 2011) retain variance as a risk measure but add skewness and kurtosis as well. Using the Istanbul stock exchange they use a polynomial goal programming model to maximize returns and skewness while minimizing risk and kurtosis.

4.2 Problem with Mean-Variance Optimization

By the early 1990's, investors generally accepted the mean-variance maxim that their objective is to maximize their expected returns for a specified level of risk. The mean-variance optimization process did not, however, dominate investor's asset allocation processes (Black, 1992). This was despite the growing use of computers which made applying mean-variance optimization to many assets easier. This was due to some fundamental problems with the mean-variance optimization process.

Michaud (Michaud, 1989) tells how despite being well known, the mean-variance optimizer was not necessarily ubiquitous in its use due to its flaws. The first flaw of the mean-variance optimizer is that it takes risk and return estimates which are uncertain and treats them as certain when optimizing, thus maximizing the error embedded in the estimates. The portfolios produced by the optimizer are also highly unstable, relatively small changes in the inputs create large changes to the portfolios. Best and Grauer (Best, 1991) also demonstrate how sensitive the mean-variance optimizer can be to small changes in asset returns. Their most significant result being that a small change to just one asset's expected return results in changes to more than half the assets holdings.

The sections below illustrate some of the problems with the mean-variance approach using our chosen data set.

4.3 The Black-Litterman Model

Fischer Black and Robert Litterman begin their influential paper (Black, 1992) lamenting the many problems investors face when using the mean-variance optimization model. When no constraints are imposed, the model almost always allocates large short positions in many assets. When constraints are imposed to rule our these short positions, the model produces *corner* solutions or portfolios which have zero allocations in many assets and large allocations in some assets (which may have small market capitalizations). These limitations of the mean-variance model are identified as being due to investors typically having views for a few asset classes while the mean-variance model requires expected returns for all assets. This forces investors to augment their views with auxiliary return assumptions. The second cause of the limitations is that the mean-variance model is extremely sensitive to return assumptions. These two causes therefore compound each other. Black and Litterman give us a model which solves these two problems by distinguishing between the views of the investor and the expected returns that are optimized. They further use equilibrium risk premiums to provide a center of gravity for expected returns which dampens sensitivity to expected return assumptions.

The original paper (Black, 1992) by Black and Litterman outlined the Black-Litterman model and demonstrated how it addresses the limitations of the mean-variance model. The paper does not give the reader much guidance on how to use and interpret the portfolios produced by the model, however. He and Litterman (He, 2002) address this in their paper. In 1998

Bevan and Winkelmann (Bevan, 1998) published a Goldman Sachs fixed income research note detailing their experience using the Black-Litterman over three years. The paper explains how the Black-Litterman model fits into the firm's investment process before describing how they set the main parameters used in the model. Read together, the papers by He et al (He, 2002) and Bevan et al (Bevan, 1998) are pivotal to understanding and interpreting the portfolios produced by the Black-Litterman model.

He and Litterman show that the Black-Litterman model produces portfolios which are a blend between the equilibrium portfolios and a weighted sum of portfolios representing the views of the investor. The stronger the view held by the investor the more weight that view-portfolio carries. If a view-portfolio is similar to the equilibrium portfolio it is penalized since it does not add any new information. If a view-portfolio has a high covariance to other view-portfolios it is penalized in order to avoid double counting views. Below we outline the Black-Litterman portfolio optimization model using the approaches of He and Litterman (He, 2002), Meucci (Meucci, 2010) as well as Satchell et al (Satchell, 2000).

We begin by assuming an investor has access to N assets with which to create his portfolio. We assume furthermore that the compound equilibrium returns of the assets over a period of s years are normally distributed

$$R_s \sim N(\mu_s, \Sigma_s).$$

This is the same assumption made by the Mean-Variance optimization model. Where the Black-Litterman model differs is that it assumes that the investors' equilibrium return are themselves an uncertain estimation of the true equilibrium returns.

The Black-Litterman model assumes that there is some error between the investor's estimation of the equilibrium returns and the true equilibrium returns

$$\mu = \mu_{investor} + \epsilon$$

where

$$\epsilon \sim N(\pi_s, \tau\Sigma_s)$$

and τ represents the investor's confidence in his estimation of the equilibrium returns. If τ is set to zero then the investor is supremely confident in his estimation of the equilibrium returns.

Black and Litterman assume that the investor has no uncertainty regarding his estimate of the equilibrium returns from which it follows that

$$R_s \sim N(\pi_s, \Sigma_s).$$

The Black-Litterman model assumes that investors choose their portfolios via the Mean-Variance approach which means that they solve

$$\omega_{market} = max_{\omega}\{ \omega' \pi_s - \lambda \, \omega' \Sigma_s \, \omega \}.$$

If this is true, then the e of the equilibrium returns of the market can be found by reverse optimization and are

$$\pi_s \equiv 2\lambda\Sigma \, \omega_{market}.$$

Equation 1: Reverse Optimization Equation

This is a useful scheme which allows investors to find the equilibrium returns implied by any benchmark or market index using a risk aversion parameter, the asset's return covariance matrix as well as the benchmark or index weights.

Another area where the Black-Litterman model differs to the Mean-Variance model is that it allows the investor to input his views about how the assets returns will differ from the equilibrium returns. If we let P_{KxN} be the view portfolio matrix and ν be the vector of expected returns on these view portfolios. The rows of P_{KxN} represent the weightings of each security in the view portfolios

$$\begin{pmatrix} \omega_{11} & \cdots & \omega_{1N} \\ \vdots & \ddots & \vdots \\ \omega_{K1} & \cdots & \omega_{KN} \end{pmatrix}$$

The returns of the view portfolios are $P\mu$, and since there is uncertainty in the investor's views

$$P\mu \sim N(\nu, \Omega).$$

Here Ω allows the investor to input his uncertainty in his returns. If the portfolio manager is using the views of his analysts, he could use this parameter to represent his confidence in a particular analyst's views. In the original paper (Black, 1992), Black and Litterman do not give any guidance as to how this parameter could be set. In his book, Meucci (Meucci, 2005) suggests that the user set this parameter using the uncertainty inherent in the assets covariances together with a scaling factor representing his confidence in the views

$$\Omega \;=\; \frac{1}{c}\, P\Sigma\, P'.$$

The larger the user's confidence, c, the smaller Ω and therefore the larger the user's confidence in the views.

Meucci (Meucci, 2010) and He (He, 2002) show that using Baye's formula, the distribution of μ given the views, ν, and their error, Ω can be shown to be

$$\mu|\nu;\Omega \sim\; N(\mu_{BL},\Sigma_{BL}^{\mu})$$

where

$$\mu_{BL} \;=\; [\;(\tau\Sigma)^{-1} \;+\; P'\Omega\, P\;]^{-1}[(\tau\Sigma)^{-1}\pi \;+\; P'\Omega^{\nu}\;]$$

and

$$\Sigma_{\mu_{;BL}} \;=\; [\,(\tau\Sigma^{-1} \;+\; P'\Omega^{-1}P]^{-1} \;+\Sigma.$$

To get the distribution of $R|\nu;\Omega$, we notice that

$$R \;=\mu \;+\; Z$$

where

$$Z \sim\; N(0,\Sigma)$$

Which means that the posterior security return model $R|\nu;\Omega \;=\mu|\nu;\Omega \;+\; Z$
i.e.

$$R|\nu;\Omega \sim\; N(\mu_{BL},\Sigma_{BL})$$

where

$$\Sigma_{BL} \;=\Sigma \;+\Sigma\_\mu_{;BL}.$$

The Black-Litterman expected returns and covariance matrix can be represented simpler as follows

$$\mu_{BL} \;=\pi \;+\tau\Sigma\, P'(\tau\, P\Sigma\, P' + \Omega)^{-1}(\nu \;-\; P\pi)$$

and

$$\Sigma_{BL} \;=\; (1+\tau)\Sigma \;-\tau^{2}\Sigma\, P'(\tau\, P\,\Sigma\, P' + \Omega)^{-1}\, P\,\Sigma.$$

These two moments can then be used in the usual mean-variance framework:

$$\omega_{BL} \equiv\; ma\;\;_{\omega}\,\{\,\omega'\mu_{BL} \;-\lambda\,\omega'\Sigma_{BL}\omega\}.$$

4.4 Making Sense of Black-Litterman Portfolios

This section is a summary of the main result from He and Litterman's paper (He, 2002).

The expected returns of different securities are not independent but depend on each other through a complex set of relations governed by the securities' correlations and volatilities.

Here we consider the complex manner in which the view portfolios and the market/reference portfolio returns relate to the optimal Black-Litterman portfolio weights.

As noted in the previous section, the optimal Black-Litterman portfolio is found by solving the mean-variance trade off

$$\omega_{BL} \equiv \ max_{\omega}\{ \omega' \mu_{BL} - \lambda \, \omega' \Sigma_{BL} \omega\}$$

whose solution is

$$\omega_{BL} \ = \frac{1}{2\lambda} \Sigma_{BL} \mu_{BL}$$

$$= \frac{1}{2\backslash lambda} \Sigma_{BL}^{-1} \Sigma_{BL_{;\mu}} [\ (\tau\Sigma)^{-1}\pi \ + \ P'\Omega^{-1}\nu]$$

noting that

$$\Sigma_{BL}^{-1} \ = \left(\Sigma \ + \Sigma_{BL_{;\mu}}\right)^{-1}$$

$$= \Sigma_{BL_{;\mu}}^{\ -1} - \Sigma_{BL_{;\mu}}^{\ -1} \left(\Sigma_{BL_{;\mu}}^{\ -1} + \Sigma_{BL}^{-1}\right)^{-1} \Sigma_{BL_{;\mu}}^{\ -1}$$

$$\therefore \Sigma_{BL}^{-1} \, \Sigma_{BL_{;\mu}}^{\ -1} \ = \frac{\tau}{1+\tau} [\, I \ - \ P' A^{-1} P \{ {}^{\Sigma}\!/_{1+\tau} \} \,]$$

where

$$A \ = {}^{\Omega}\!/_{\tau} \ + \ {}^{P\Sigma}\!/_{(1+\tau)P'}$$

using the above, the optimal Black-Litterman portfolio weights can be simplified to

$$\omega_{BL} \ = \frac{1}{1+\tau} [\omega_{market} \ + \ P'\Lambda]$$

where

$$\Lambda \ = \frac{1}{2\lambda} \tau \Omega^{-1} \nu \ - \ A^{-1} P \frac{\Sigma}{1+\tau} \omega_{market} \ - \ A^{-1} P \frac{\Sigma}{1+\tau} P' \tau \Omega^{-1} \frac{\nu}{2\lambda}$$

The two above equations prove a key insight of the Black-Litterman model. Namely that the Black-Litterman optimal portfolio weights are derived by starting at the market portfolio ω_{market} and adding a weighted sum of the view portfolios, scaled by a factor $1/_{1+\tau}$.

The weight of each view portfolio is given by Λ which provides the key workings behind the Black-Litterman portfolios.

$$\Lambda = \frac{1}{2\lambda}\tau\Omega^{-1}\nu - A^{-1}P\frac{\Sigma}{1+\tau}\omega_{market} - A^{-1}P\frac{\Sigma}{1+\tau}P'\tau\Omega^{-1}\frac{\nu}{2\lambda}$$

First Term: The stronger the view (the higher the expected return ν or the lower the view uncertainty Ω) the more weight the portfolio carries.

Second Term: The higher the view portfolio's covariance to the market equilibrium portfolio (i.e. if it does not add any new information) the more that view portfolio is penalized.

Third Term: The higher a view portfolio's covariance to other view portfolios, the more it is penalized. This avoids double counting the same view.

To summarize, the Black-Litterman model starts off at the market/equilibrium portfolio and then tilts towards portfolios which have strong views and away from redundant portfolios which are already represented by either other views or the market.

4.5 Extensions to the Black-Litterman Model and Other Literature

One limitation of the Black-Litterman model in its original format as outlined by Black and Litterman (Black, 1992) as well as He and Litterman (He, 2002) is the unintuitive portfolios it produces. Both Da Silva (Da Silva, 2009) as well as Meucci (Meucci, 2010) analyse this problem and find that the model deviates from the benchmark or equilibrium portfolio even when the investor has no views at all or alternatively no confidence in his views. When the investor has infinite confidence in his views the model does not tilt completely towards the view portfolios. Da Silva attributes this to the Black-Litterman model being based on the mean-variance paradigm and shows that in the mean-variance paradigm, optimal portfolios are found by making multiple comparisons between each asset and the global minimum variance (GMV) portfolio

$$\omega_{active} = \frac{1}{2\lambda}\, \Sigma^{-1} \{ I - 1.\, \omega_{GMV} \}\, \mu.$$

Here ω represents the portfolio's weights in each asset while μ represents the expected returns of each asset. Long positions are taken in assets with returns higher than the GMV portfolio and short positions are taken in assets with the reverse. One clear example of portfolios which are inconsistent with the active manager's views is the case where the manager has no views at all. In this scenario the manager would expect the model to produce the benchmark portfolio. The Black-Litterman model in its original form does not. The primary reason for this is because of the use of reverse optimization to obtain the expected returns implied by the benchmark portfolio

$$\Pi = \gamma.\, \Sigma.\, \omega_{benchmark}.$$

This results in the Black-Litterman model choosing portfolios by comparing the benchmark portfolio to the GMV portfolio

$$\omega_{BL,\Pi} = \frac{\gamma}{2\lambda} \{ \omega_{Bench} - \omega_{GMV} \}.$$

We can see then that the Black-Litterman model will generate active portfolios even if the investor does not have active views. In this formulation, the Black-Litterman model will produce no active weights only if the benchmark weights are the same as the GMV portfolio's. Furthermore, inspecting the returns implied by the benchmark portfolio we can see that the higher an asset's volatility the higher the return implied by its benchmark weight. Where the investor has no active views then, the Black-Litterman model will take active positions in the riskiest of assets. Da Silva et al solve this problem by replacing the reverse optimization expected returns.

In order to produce portfolios which are consistent with the investors' views, Meucci (Meucci, 2010) uses the following parameters for the Black-Litterman model

$$\mu_{BL} = \pi + \Sigma P' (P \Sigma P' + \Omega)^{-1} (\nu - P\pi)$$

for the expected returns, and

$$\Sigma_{BL} = \Sigma - \Sigma P' (P \Sigma P' + \Omega)^{-1} P \Sigma$$

for the covariance structure. Remembering that Ω represents the investors uncertainty around his views, its clear that when the investor's has no confidence in his views, $\Omega \to \infty$, then the two parameters reduce to the equilibrium parameters. This means that no views or no confidence in the views would result in the model choosing the benchmark or equilibrium portfolio. When the investor has supreme confidence in his views ($\Omega \rightarrow 0$) then the two parameters are tilted completely away from the equilibrium parameters.

The literature is littered with studies which either provide a different derivation of the Black-Litterman model or modify the original model to allow application to non-normal stocks or factors. The standard Black-Litterman model operates within the mean-variance paradigm since it simply takes returns blended between equilibrium and investor view returns into the standard mean-variance optimizer. This limits the Black-Litterman model to the first few moments of asset returns when constructing portfolios. This is a standard limitation of the mean-variance model. Martellini and Ziemann (Martellini, 2007) extend the Black-Litterman model by allowing it to incorporate the higher moments of the assets used to construct portfolios. This is accomplished by using a four-moment capital asset pricing model (instead of the standard CAPM) to derive Black-Litterman parameters. Meucci (Meucci, 2009) takes this further by showing that the normal market assumption in the original model does not restrict the Black-Litterman model to normally distributed assets, as long as the risk factors underlying the asset return's randomness are normal. Meucci demonstrates his extension by creating a Black-Litterman portfolio of call options using a traders views on the slope of the interest rate yield curve.

Mankert and Seiler (Mankert, 2011) derive the Black-Litterman model using a sampling theory approach. Sampling theory studies sample data in the hope that it will provide insight into the underlying stochastic distribution. Using maximum-likelihood, estimates which maximize the probability of observing data are calculated. Mankert and Seiler assume that both the investor and the market have observed samples they think are representative of future returns. Maximum likelihood is used to estimate the investor and market's return statistics. Maximum likelihood is then used again to find the blended statistics incorporating both samples which gives the standard Black-Litterman formulae. This derivation provides an interesting interpretation of the τ parameter in the Black-Litterman model. Here τ is the ratio of the sizes of the investor and market's samples. Within the sampling theory paradigm, the larger the sample size the more certainty around the underlying distribution. The ratio therefore provides a measure of certainty between the investor and the market. The more confident the investor is, the higher τ should be.

Another section of the literature on the Black-Litterman model is centred on allowing the Black-Litterman model to interact with the asset data as it changes. Zhou (Zhou, 2009) extends the Black-Litterman model by allowing it to combine equilibrium returns, the investors views and learnings from the market in a Bayesian setting. Fabozzi et al (Fabozzi, 2006) provide a regression based derivation of the Black-Litterman model. They then show how to incorporate factor models and cross-sectional rankings into the standard Black-Litterman framework. Portfolios are then created using a cross-sectional momentum strategy which buys stocks that have outperformed over a six to twelve month horizon sells those which have not.

5 APPLICATION TO THE EQUITY MARKET

This chapter details the procedure investors follow when constructing portfolios of shares. The chapter begins by illustrating that it is not appropriate for investors to model share prices as they do not follow a stationary statistical distribution. It is shown that it is more appropriate to model returns when considering equity markets and log-returns in particular. The chapter concludes by demonstrating how to change log-returns into linear returns since the portfolio optimization models are defined in terms of linear returns.

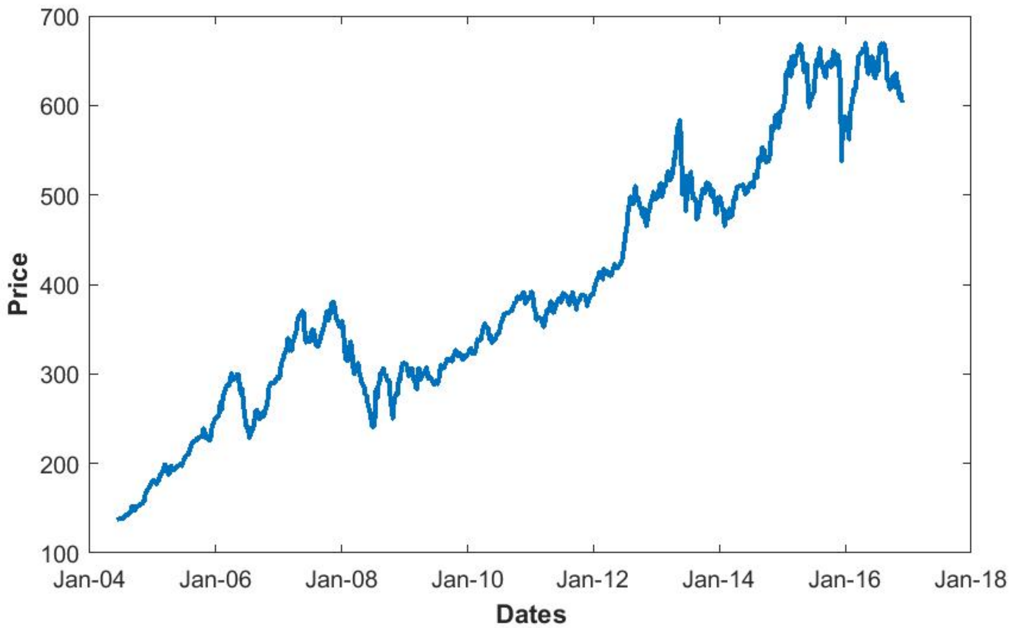

Figure 6: Stock Performance 7 January 2000 to 23 October 2016

Figure 6 shows the price of a stock over a period of 16 years. It is this stock price behaviour that this chapter will show is no stationary and therefore not appropriate to model for putting together a portfolio.

5.1 Estimating Model Parameters

In order to perform the mean-variance optimization we will require $E[R_{T,t}]$, the vector of expected returns from time T to T+t, as well as $\Sigma_{T,t}$, the covariance matrix of the security returns from time T to T+t. In order to estimate these parameters, we must model the joint distribution of the individual security returns.

The period from time T to T+t is typically at some point in the future. The investor, therefore, has only past information with which to estimate $E[R_{T,t}]$ as well as $\Sigma_{T,t}$.

The investor is interested in knowing the future prices of the securities, at some point in the future S_{T+_t}. Since he cannot know the future value of these securities with certainty, the investor is interested in the multivariate distribution of the future returns, $f(R_{T,t})$.

In order to estimate the distribution of $f(R_{T,t})$ we will need as many realizations of the security returns as possible. We may, therefore, use returns for a period smaller than t in an effort to maximize the number of non-overlapping return observations in question.

Armed with the joint distribution of the security returns, we can then simulate the security prices to the required time horizon, T+t. The security returns implied by these prices are then used to estimate the distribution of the securities $f(R_{T,t})$ and any required moments.

Investors have a choice of frameworks with which to forecast future security returns. These range from the study of charts and price patterns to the study of fundamental data such as a company's financial statements or the wider macro-economic environment and its impact on the industry within which a company operates. Regardless, the forecasting of future security returns is not our concern here. We therefore assume that the investor has some means of forecasting future return and focus our attention on the estimation of $\Sigma_{T,t}$.

Key to estimating the required parameters $E[R_{T,t}]$ and $\Sigma_{T,t}$. is accounting for the time component. The expected value and covariance of security returns over a period of length τ will be different to those over a period of length υ.

$$E[R_{T,\tau}] \neq E[R_{T,\upsilon}]$$

$$\Sigma_{T,\tau} \neq \Sigma_{T,\upsilon}$$

5.1.1 Share Prices Are Not Appropriate To Model

We are interested in a way of finding the returns of the shares over any abstract term τ, $R_{T,\tau}$ }. This can be achieved if we an simulate the prices of the appropriate security to any term $T + \tau$. Our starting point is naturally to consider estimating the distribution of the security prices.

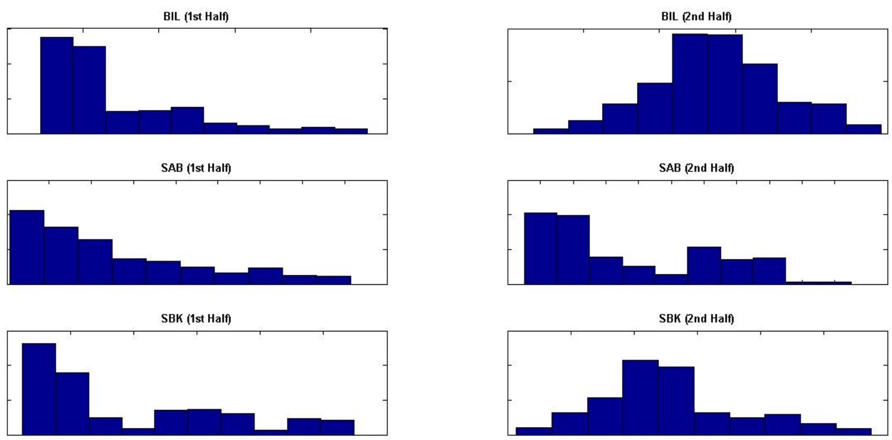

Figure 7: Histogram Of Price Returns Split Into Two Halves

Figure 7 takes some securities' price series and splits them it into two halves. A histogram of each price series is then plotted.

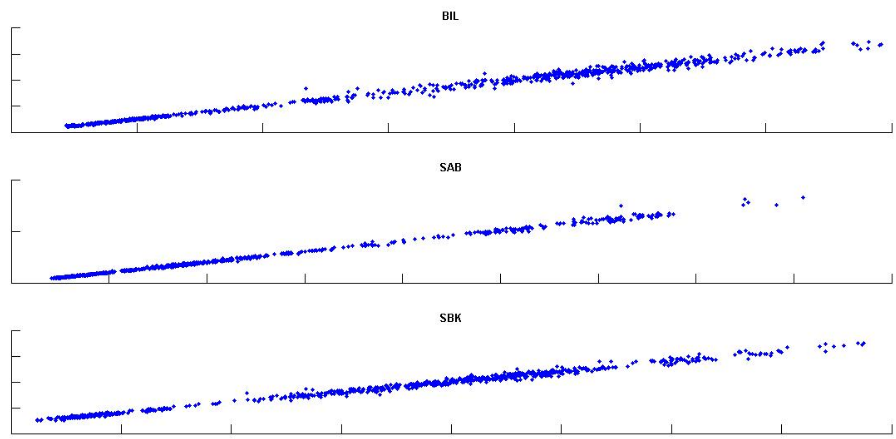

Figure 8: Stock Prices Against Lagged Prices Of The Same Series

In Figure 8 each price point S_t is plotted against a lagged point of the same price series $S_{t+\tau}\}$.

The two histograms serve to illustrate whether the two half-series follow the same statistical distribution. That is, whether the series is governed by one statistical distribution we can use or whether it changes over time. As can be seen in the figure, the distribution of each half-series is different to its associated half. There is not, therefore, one statistical distribution governing the security prices.

The scatter plot of the security price series against a lagged point of the same price series reveals that there is a strong relationship between the prices at time T, S_T and those at time $S_{t+\tau}$. The prices are therefore not independently distributed and therefore are not appropriate for us to model in order to find our required parameters.

5.1.2 Security Price Returns Are Appropriate To Model

We now turn our attention to the returns of the security prices. We subject the return series to the same tests we applied to the security price series.

Despite previously formulating the mean-variance framework in terms of linear returns, we focus on compound returns for reasons we will explain in the next section.

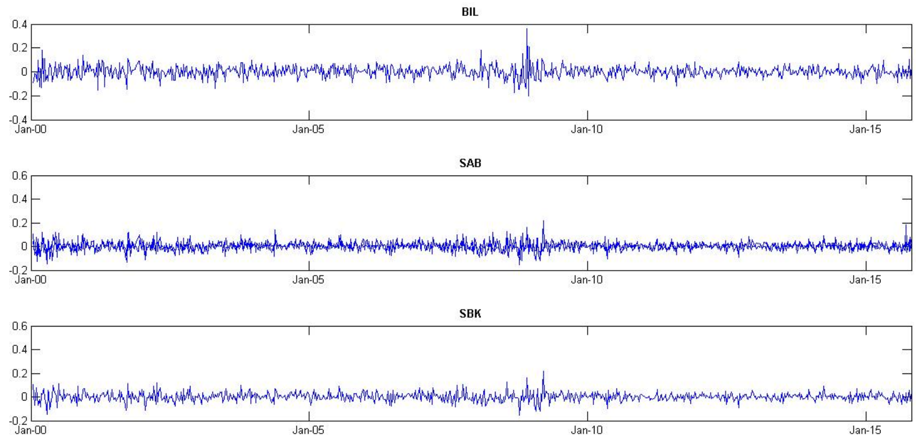

Figure 9: Individual Stock Compound Price Returns

As before, Figure 10 splits the log-return series into two halves and plot a histogram for each while Figure 11 plots a scatter plot of each return against a lagged return of the same security.

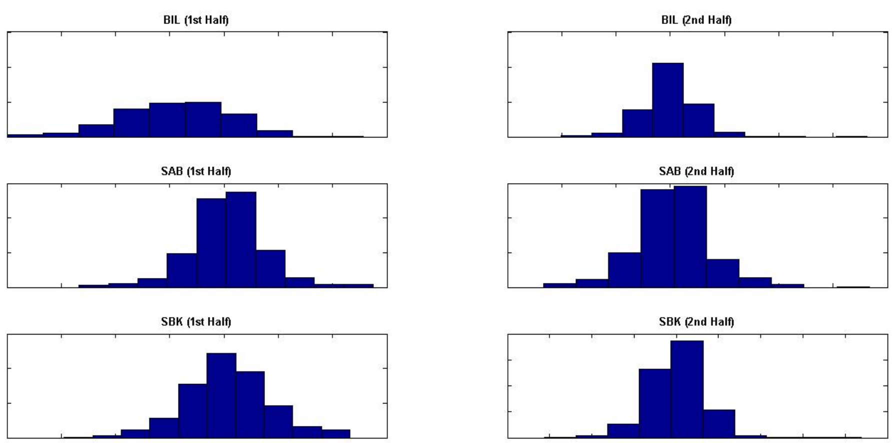

Figure 10: Histograms Of Compound Stock Price Return Series Split In Two

This time we notice that the two histograms are the same for each half series. This means that a single statistical distribution governs the return series over time.

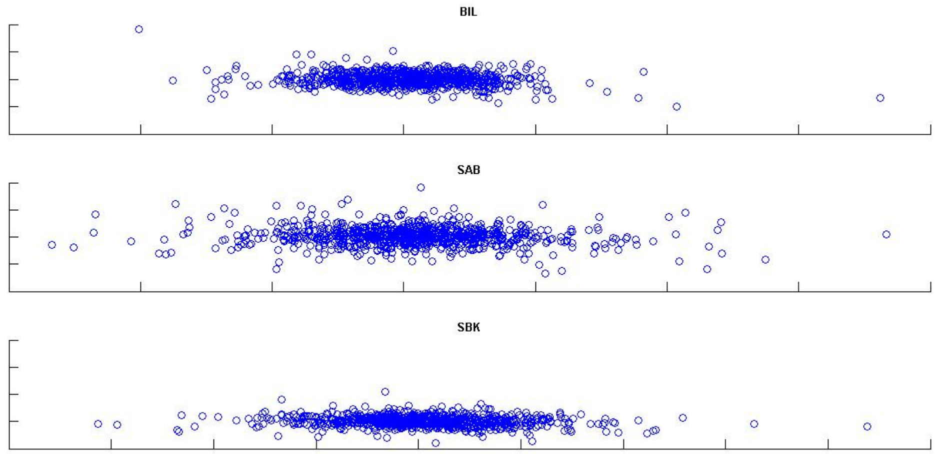

Figure 11: Compound Stock Price Return Series Against Lagged Points On Series

Figure 11 shows that there is no intelligible relationship between the security returns and their lagged counterpart. The returns are independently and identically distributed over time. They are therefore appropriate for us to model in order to obtain the parameters we require for the mean-variance optimization.

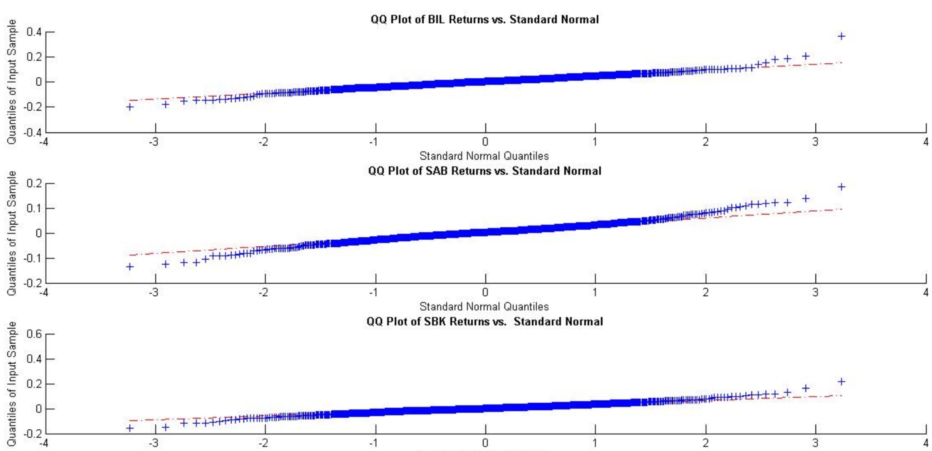

Figure 12: Compound Price Return Series QQ-Plot

Figure 12 shows that the log-returns series is a good fit to the normal distribution. There are, however, a few points that are not on the line and therefore deviate from the normal distribution. These are outliers, however, and are to be expected in any series.

5.1.3 Why Model Compound Returns

As previously mentioned, we are interested in finding a way to estimate the covariance of returns over any abstract term τ. The returns available to model may be for some other non-overlapping period υ however. It is therefore often necessary to have a methodology by which we can find returns over τ using returns over a term υ.

The log-returns offer a convenience over linear returns since there is an analytical formula for projecting statistics calculated for returns over some term τ to some other term υ. We show how to do this below.

The log-return over a term of τ is defined as

$$C_{T,\tau} = ln\{ {}^{S_{T+\backslash tau}}/_{S_T} \}$$

$$= ln(S_{T+\tau}) + ln(S_T)$$

$$= \sum_{i=1} C_{T+\tau-(i-1)\varepsilon,\varepsilon}$$

where ε is such that there are exactly τ periods of length ε between time T and time [T + τ].

As we discovered, the non-overlapping log-returns can be considered to be independently and identically distributed to the normal distribution. We can therefore represent each of the τ returns between time T and T + τ by the following random variable

$$C_\varepsilon \sim N(\mu_\varepsilon, \Sigma_\varepsilon).$$

The return between time T and [T + τ] can therefore be represented by

$$C_{T,\tau} = \sum_{i=1} C_\varepsilon \, .$$

Now, each C_ε can be represented by its moment generating function:

$$M_{C_\varepsilon}(t) = exp\{ t'\mu_\varepsilon + \frac{1}{2} t'\Sigma_\varepsilon\} \, t \}.$$

This follows from the fact that for any $X \sim N(\mu, \Sigma)$, the moment generating function is defined as

$$M_X(t) = exp\{ t'\mu + \frac{1}{2}t'\Sigma\, t \}.$$

Since each C_ε is independently and identically distributed

$$\begin{aligned} M_{C_{T,\tau}}(t) &= \prod_{i=1}^{\tau} M_{C_\varepsilon}(t) \\ &= \prod_{i=1}^{\tau} [\, exp\{ t'\mu_\varepsilon + \frac{1}{2}t'\Sigma_\varepsilon t\}\,] \\ &= [exp\{ t'\mu_\varepsilon + \frac{1}{2}t'\Sigma_\varepsilon t\}]^\tau \\ &= exp\,\{ t'(\tau\mu_\varepsilon) + \frac{1}{2}t'(\tau\Sigma\_\varepsilon)t \} \end{aligned}$$

This moment generating function corresponds to that belonging to the Normal distribution. i.e.

$$C_{T,\tau} \sim N(\tau\mu_\varepsilon, \tau\Sigma\_\varepsilon)$$

The covariance matrix of $C_{T,\tau}$ can therefore be written as

$$\Sigma_{C_{T,\tau}} = \tau\Sigma_\varepsilon.$$

While that of $C_{T,\nu}$ }, the return over some different term ($\tau \neq \nu$), can be similarly written as

$$\Sigma_{C_{T,\tau}} = \nu\Sigma_\varepsilon.$$

We can therefore conclude that

$$\Sigma_{C_{T,\nu}} = \frac{\nu}{\tau}\Sigma_{C_{T,\tau}}.$$

A similar result follows for the expected returns over some arbitrary period, ν. Though as previously mentioned, we assume that the investor arrives at his expected return estimations through some proprietary process of investment analysis rather than using historical averages.

$$E[C_{T,\nu}] = \frac{\nu}{\tau}E[C_{T,\tau}].$$

The above results mean that we can use the statistics calculated from the returns over a term ν to calculate the statistics of the returns over some different term τ. This result will hold as long as the returns in question are normally distributed, which the log-returns have proven to be.

It is for this reason that we prefer to model log-returns instead of the linear-returns we used to set up the mean-variance framework.

The equations above are very useful and will allow us to calculate the statistics of returns over a term ν using return data for a period τ. We must not proceed without caution however. Meucci (Meucci, 2005) cautions that these formulae which project statics from a period τ to a different period ν hold only if the the distribution of returns over the period ν are known. These formulae assume that the returns follow the normal distribution over both term τ and ν. If the distribution of returns changes over time then these formulae will not hold. While useful, these formulae come with model/distribution estimation risk.

5.1.4 From Log-Return to Linear-Return Statistics

We start off with weekly log-return statistics for the constituents of the JSAPY as shown on Table 4 and Table 5. Using the formulae in the previous section, we're able to convert the weekly log-return statistics into annual log-return statistics.

Using the log-return statistics, we simulate 100 000 sample returns using the Multivariate Normal Distribution. From these returns we're able to estimate a linear-return covariance matrix which we need for the portfolio construction models.

We can convert our log-return statistics into linear-return statistics using the fact that the linear-returns follow the log-normal distribution if the log-returns follow the normal distribution.

5.2 Mean-Variance Portfolios

We calculate efficient portfolios for the 21 shares which make up the JSAPY Index. Table 4 and Table 5 show the expected returns we're using as well as the covariance of the returns.

JSAPY Index Constituents		30 Nov 16	
Ticker	Name	Exp Returns	Weight
GRT	Growthpoint Properties Ltd	0.17	19.59
RDF	Redefine Properties Ltd	0.10	13.88
NEP	New Europe Property Investments PLC	0.19	10.88
RES	Resilient REIT Ltd	0.05	8.28
HYP	Hyprop Investments Ltd	0.14	8.26
FFB	Fortress Income Fund Ltd	0.11	6.38
FFA	Fortress Income Fund Ltd	0.14	5.19
ROC	Rockcastle Global Real Estate Co Ltd	0.04	4.74
SAC	SA Corporate Real Estate Ltd	0.19	3.70
VKE	Vukile Property Fund Ltd	0.23	3.41
ATT	Attacq Ltd	0.21	2.98
AWA	Arrowhead Properties Ltd	0.13	2.44
IPF	Investec Property Fund Ltd	0.11	1.89
EMI	Emira Property Fund Ltd	0.20	1.79
REB	Rebosis Property Fund Ltd	0.16	1.35
MSP	MAS Real Estate Inc	0.08	1.22
APF	Accelerate Property Fund Ltd	0.10	1.09
PIV	Pivotal Fund Ltd/The	0.17	1.06
OCT	Octodec Investments Ltd	0.06	0.96
TDH	Tradehold Ltd	0.19	0.47
STP	Stenprop Ltd	0.09	0.42
		Total	100.00

Table 4: Expected Returns Of The JSAPY Constituents

JSAPY Annual log-Return Covariance Matrix	GRT	RDF	NEP	RES	HYP	FFB	FFA	ROC	SAC	VKE	ATT	AWA	IPF	EMI	REB	MSP	APF	PIV	OCT	TDH	STP
GRT	0.2180	0.1799	0.1830	0.1895	0.1806	0.2197	0.1626	0.1985	0.1636	0.1674	0.1512	0.1592	0.1649	0.1677	0.1498	0.1556	0.1580	0.1481	0.1739	0.1247	0.1204
RDF	0.1799	0.2275	0.1730	0.1909	0.1791	0.2309	0.1586	0.1768	0.1670	0.1668	0.1472	0.1569	0.1568	0.1682	0.1525	0.1443	0.1552	0.1372	0.1773	0.1233	0.1122
NEP	0.1830	0.1730	0.2604	0.1931	0.1799	0.2556	0.1687	0.2281	0.1648	0.1702	0.1573	0.1652	0.1719	0.1678	0.1558	0.1831	0.1733	0.1575	0.1865	0.1902	0.1585
RES	0.1895	0.1909	0.1931	0.2357	0.1960	0.2427	0.1666	0.2049	0.1735	0.1755	0.1559	0.1652	0.1696	0.1816	0.1591	0.1611	0.1603	0.1568	0.1878	0.1401	0.1329
HYP	0.1806	0.1791	0.1799	0.1960	0.2213	0.2329	0.1600	0.1938	0.1685	0.1665	0.1566	0.1591	0.1614	0.1702	0.1595	0.1477	0.1508	0.1549	0.1771	0.1222	0.1240
FFB	0.2197	0.2309	0.2556	0.2427	0.2329	0.4303	0.2101	0.2721	0.2003	0.2100	0.1938	0.2060	0.2057	0.2120	0.1914	0.2186	0.2042	0.1864	0.2189	0.2049	0.1781
FFA	0.1626	0.1586	0.1687	0.1666	0.1600	0.2101	0.1535	0.1764	0.1494	0.1479	0.1337	0.1426	0.1450	0.1490	0.1356	0.1414	0.1430	0.1339	0.1564	0.1312	0.1193
ROC	0.1985	0.1768	0.2281	0.2049	0.1938	0.2721	0.1764	0.2645	0.1710	0.1765	0.1686	0.1737	0.1817	0.1804	0.1644	0.1903	0.1808	0.1670	0.1926	0.2008	0.1671
SAC	0.1636	0.1670	0.1648	0.1735	0.1685	0.2003	0.1494	0.1710	0.2062	0.1540	0.1352	0.1535	0.1457	0.1603	0.1507	0.1360	0.1438	0.1522	0.1657	0.1088	0.1462
VKE	0.1674	0.1668	0.1702	0.1755	0.1665	0.2100	0.1479	0.1765	0.1540	0.1872	0.1363	0.1471	0.1483	0.1579	0.1397	0.1404	0.1452	0.1380	0.1655	0.1259	0.1191
ATT	0.1512	0.1472	0.1573	0.1559	0.1566	0.1938	0.1337	0.1686	0.1352	0.1363	0.1340	0.1322	0.1357	0.1369	0.1273	0.1307	0.1332	0.1235	0.1375	0.1282	0.1123
AWA	0.1592	0.1569	0.1652	0.1652	0.1591	0.2060	0.1426	0.1737	0.1535	0.1471	0.1322	0.1529	0.1410	0.1487	0.1360	0.1367	0.1418	0.1358	0.1576	0.1113	0.1215
IPF	0.1649	0.1568	0.1719	0.1696	0.1614	0.2057	0.1450	0.1817	0.1457	0.1483	0.1357	0.1410	0.1633	0.1499	0.1366	0.1425	0.1452	0.1359	0.1611	0.1452	0.1189
EMI	0.1677	0.1682	0.1678	0.1816	0.1702	0.2120	0.1490	0.1804	0.1603	0.1579	0.1369	0.1487	0.1499	0.1899	0.1418	0.1407	0.1435	0.1384	0.1602	0.1217	0.1256
REB	0.1498	0.1525	0.1558	0.1591	0.1595	0.1914	0.1356	0.1644	0.1507	0.1397	0.1273	0.1360	0.1366	0.1418	0.1461	0.1281	0.1342	0.1326	0.1550	0.1213	0.1143
MSP	0.1556	0.1443	0.1831	0.1611	0.1477	0.2186	0.1414	0.1903	0.1360	0.1404	0.1307	0.1367	0.1425	0.1407	0.1281	0.2126	0.1419	0.1289	0.1539	0.1515	0.1316
APF	0.1580	0.1552	0.1733	0.1603	0.1508	0.2042	0.1430	0.1808	0.1438	0.1452	0.1332	0.1418	0.1452	0.1435	0.1342	0.1419	0.1603	0.1406	0.1746	0.1583	0.1245
PIV	0.1481	0.1372	0.1575	0.1568	0.1549	0.1864	0.1339	0.1670	0.1522	0.1380	0.1235	0.1358	0.1359	0.1384	0.1326	0.1289	0.1406	0.1623	0.2056	0.1238	0.1231
OCT	0.1739	0.1773	0.1865	0.1878	0.1771	0.2189	0.1564	0.1926	0.1657	0.1655	0.1375	0.1576	0.1611	0.1602	0.1550	0.1539	0.1746	0.2056	0.3274	0.1308	0.1326
TDH	0.1247	0.1233	0.1902	0.1401	0.1222	0.2049	0.1312	0.2008	0.1088	0.1259	0.1282	0.1113	0.1452	0.1217	0.1213	0.1515	0.1583	0.1238	0.1308	0.4854	0.1298
STP	0.1204	0.1122	0.1585	0.1329	0.1240	0.1781	0.1193	0.1671	0.1462	0.1191	0.1123	0.1215	0.1189	0.1256	0.1143	0.1316	0.1245	0.1231	0.1326	0.1298	0.1526

Table 5: JSAPY Expected Returns Covariance Matrix

We set our Mean-Variance optimizer to allow short positions but no leverage. This means each stock can have a weight within the range [-1,+1]. The weights of each portfolio must sum up to one. Figure 13 shows the efficient frontier we obtain while Table 6 shows the allocations to each share.

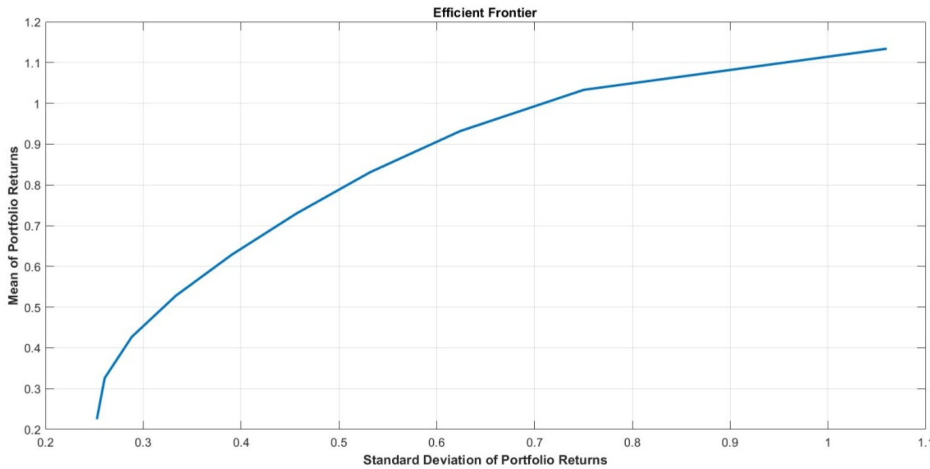

Figure 13: JSAPY Efficient Frontier

Table 6 highlights the extreme allocations the Mean-Variance optimizer has chosen. Almost all the portfolios make allocations as extreme as the bounds we have set will allow. This is the first problem with the mean-variance optimizer, its tendency to create portfolios with extreme long and short positions.

Portfolio Return	Portfolio Risk	GRT	RDF	NEP	RES	HYP	FFB	FFA	ROC	SAC	VKE	ATT	AWA	IPF	EMI	REB	MSP	APF	PIV	OCT	TDH	STP
		Efficient Portfolio Allocations																				
22.5%	25.3%	-3.0%	12.5%	-19.7%	-7.7%	-29.1%	-18.3%	32.6%	-62.2%	-71.2%	3.9%	75.3%	25.3%	16.0%	2.2%	45.5%	9.7%	-20.3%	61.9%	-21.5%	0.8%	67.2%
27.3%	25.4%	-2.9%	7.9%	-14.6%	-12.8%	-35.6%	-17.1%	30.6%	-65.4%	-65.0%	8.1%	96.9%	21.1%	11.1%	7.4%	45.7%	8.8%	-27.7%	64.8%	-20.8%	1.4%	58.1%
32.1%	26.0%	-0.2%	5.2%	-7.6%	-20.3%	-39.0%	-15.7%	30.2%	-70.1%	-61.7%	14.9%	100.0%	17.5%	6.5%	14.1%	48.6%	8.0%	-34.3%	76.8%	-25.2%	2.5%	49.9%
36.9%	27.0%	2.8%	2.8%	-0.2%	-28.1%	-41.8%	-14.4%	30.2%	-75.1%	-59.0%	22.1%	100.0%	14.0%	2.0%	20.9%	51.9%	7.2%	-40.9%	90.4%	-30.4%	3.7%	41.8%
41.6%	28.4%	6.3%	-0.6%	7.3%	-36.2%	-44.0%	-13.1%	30.4%	-80.6%	-55.5%	29.6%	100.0%	11.2%	-2.6%	28.1%	55.9%	6.3%	-47.3%	100.0%	-34.2%	5.1%	34.1%
46.4%	30.3%	10.6%	-6.5%	15.0%	-45.0%	-45.0%	-11.8%	31.4%	-87.2%	-50.2%	37.8%	100.0%	9.9%	-7.2%	35.8%	61.2%	5.1%	-53.6%	100.0%	-34.5%	7.1%	27.2%
51.2%	32.5%	14.8%	-12.4%	22.7%	-53.8%	-46.0%	-10.6%	32.4%	-93.8%	-44.9%	45.9%	100.0%	8.7%	-11.8%	43.6%	66.6%	3.9%	-59.8%	100.0%	-34.8%	9.0%	20.3%
56.0%	35.0%	19.0%	-18.3%	30.3%	-62.7%	-47.0%	-9.4%	33.5%	-100.0%	-39.4%	54.1%	100.0%	7.4%	-16.5%	51.3%	72.1%	2.7%	-66.2%	100.0%	-35.1%	10.9%	13.2%
60.8%	37.8%	21.5%	-23.9%	37.3%	-73.2%	-49.0%	-9.2%	35.5%	-100.0%	-31.9%	63.5%	100.0%	5.9%	-22.1%	59.7%	78.2%	1.2%	-73.6%	100.0%	-35.5%	12.6%	3.1%
65.6%	40.7%	24.1%	-29.6%	44.4%	-83.8%	-50.9%	-9.0%	37.4%	-100.0%	-24.5%	72.8%	100.0%	4.4%	-27.7%	68.1%	84.3%	-0.4%	-81.0%	100.0%	-35.9%	14.3%	-7.0%
70.3%	43.9%	26.6%	-35.2%	51.4%	-94.3%	-52.9%	-8.8%	39.4%	-100.0%	-17.0%	82.2%	100.0%	2.9%	-33.3%	76.4%	90.4%	-1.9%	-88.4%	100.0%	-36.3%	16.1%	-17.1%
75.1%	47.2%	29.0%	-42.4%	59.0%	-100.0%	-56.6%	-8.9%	41.5%	-100.0%	-8.9%	92.5%	100.0%	0.6%	-40.7%	84.6%	97.6%	-4.0%	-96.0%	100.0%	-37.1%	18.0%	-28.2%
79.9%	50.7%	31.4%	-52.5%	68.4%	-100.0%	-60.7%	-9.8%	45.5%	-100.0%	3.7%	100.0%	100.0%	-1.5%	-51.2%	95.2%	100.0%	-6.8%	-100.0%	100.0%	-38.9%	20.4%	-43.2%
84.7%	54.5%	36.3%	-64.4%	81.6%	-100.0%	-63.1%	-11.1%	54.3%	-100.0%	23.9%	100.0%	100.0%	-1.1%	-63.5%	100.0%	100.0%	-10.9%	-100.0%	100.0%	-41.9%	23.6%	-63.8%
89.5%	58.8%	42.2%	-76.5%	95.6%	-100.0%	-64.8%	-12.5%	64.8%	-100.0%	46.1%	100.0%	100.0%	0.2%	-76.1%	100.0%	100.0%	-15.4%	-100.0%	100.0%	-45.3%	27.0%	-85.3%
94.3%	63.5%	55.2%	-90.8%	100.0%	-100.0%	-65.5%	-12.4%	78.8%	-100.0%	69.2%	100.0%	100.0%	-0.9%	-93.8%	100.0%	100.0%	-22.4%	-100.0%	100.0%	-49.6%	32.1%	-100.0%
99.0%	69.0%	77.3%	-100.0%	100.0%	-100.0%	-67.6%	-12.1%	86.4%	-100.0%	94.2%	100.0%	100.0%	-18.3%	-100.0%	100.0%	100.0%	-39.4%	-100.0%	100.0%	-58.5%	37.9%	-100.0%
103.8%	75.9%	100.0%	-100.0%	100.0%	-100.0%	-66.6%	-15.3%	100.0%	-100.0%	100.0%	100.0%	100.0%	-21.3%	-100.0%	100.0%	100.0%	-69.4%	-100.0%	100.0%	-74.6%	47.1%	-100.0%
108.6%	85.0%	100.0%	-100.0%	100.0%	-100.0%	-47.9%	-24.9%	100.0%	-100.0%	100.0%	100.0%	100.0%	8.7%	-100.0%	100.0%	100.0%	-100.0%	-100.0%	100.0%	-100.0%	64.1%	-100.0%
113.4%	106.0%	100.0%	-100.0%	100.0%	-100.0%	100.0%	-100.0%	100.0%	-100.0%	100.0%	100.0%	100.0%	-100.0%	-100.0%	100.0%	100.0%	-100.0%	-100.0%	100.0%	-100.0%	100.0%	-100.0%

Table 6: Return, Risk and Portfolio Allocation Of Efficient Portfolios

The second problem with mean-variance optimization is its sensitivity to the security expected-return assumptions. A relatively small change in one of the expected returns yields portfolios which are very different from the original portfolio.

To illustrate this we selected four shares and changed their expected return by one percent. Table 7 shows the changed returns.

JSAPY Index Constituents		30 Nov 16	
Ticker	**Name**	**Exp Returns**	**Weight**
GRT	Growthpoint Properties Ltd	0.17	19.59
RDF	Redefine Properties Ltd	0.10	13.88
NEP	New Europe Property Investments PLC	0.18	10.88
RES	Resilient REIT Ltd	0.05	8.28
HYP	Hyprop Investments Ltd	0.14	8.26
FFB	Fortress Income Fund Ltd	0.11	6.38
FFA	Fortress Income Fund Ltd	0.13	5.19
ROC	Rockcastle Global Real Estate Co Ltd	0.04	4.74
SAC	SA Corporate Real Estate Ltd	0.19	3.70
VKE	Vukile Property Fund Ltd	0.23	3.41
ATT	Attacq Ltd	0.21	2.98
AWA	Arrowhead Properties Ltd	0.13	2.44
IPF	Investec Property Fund Ltd	0.12	1.89
EMI	Emira Property Fund Ltd	0.20	1.79
REB	Rebosis Property Fund Ltd	0.16	1.35
MSP	MAS Real Estate Inc	0.08	1.22
APF	Accelerate Property Fund Ltd	0.10	1.09
PIV	Pivotal Fund Ltd/The	0.17	1.06
OCT	Octodec Investments Ltd	0.05	0.96
TDH	Tradehold Ltd	0.19	0.47
STP	Stenprop Ltd	0.09	0.42
		Total	**100.00**

Table 7: Changed JSAPY Expected Returns

Portfolio Return	Portfolio Risk	GRT	RDF	NEP	RES	HYP	FFB	FFA	ROC	SAC	VKE	ATT	AWA	IPF	EMI	REB	MSP	APF	PIV	OCT	TDH	STP
		Changed Efficient Portfolio Allocations																				
22.7%	25.3%	-3.0%	12.5%	-19.7%	-7.7%	-29.1%	-18.3%	32.6%	-62.2%	-71.2%	3.9%	75.3%	25.3%	16.0%	2.2%	45.5%	9.7%	-20.3%	61.9%	-21.5%	0.8%	67.2%
27.4%	25.4%	-2.8%	8.1%	-14.9%	-12.8%	-35.6%	-16.9%	28.8%	-65.3%	-65.1%	8.2%	96.4%	21.2%	12.3%	7.4%	45.7%	9.0%	-27.4%	65.7%	-21.4%	1.4%	58.2%
32.1%	25.9%	-0.3%	5.7%	-8.4%	-20.1%	-39.1%	-15.4%	25.8%	-69.9%	-61.9%	15.0%	100.0%	17.8%	9.5%	13.9%	48.4%	8.3%	-33.6%	78.4%	-26.2%	2.4%	50.0%
36.7%	26.9%	2.7%	3.8%	-1.6%	-27.9%	-42.0%	-13.9%	22.8%	-74.8%	-59.2%	22.2%	100.0%	14.5%	6.9%	20.6%	51.6%	7.6%	-39.7%	93.0%	-32.1%	3.5%	42.0%
41.4%	28.4%	6.5%	-0.3%	5.5%	-36.3%	-43.8%	-12.4%	20.3%	-80.6%	-55.1%	30.1%	100.0%	12.5%	4.3%	27.9%	56.0%	6.7%	-45.6%	100.0%	-35.2%	5.0%	34.6%
46.1%	30.2%	10.9%	-6.2%	12.7%	-45.2%	-44.8%	-11.0%	18.1%	-87.4%	-49.5%	38.5%	100.0%	11.7%	1.8%	35.7%	61.5%	5.7%	-51.4%	100.0%	-35.8%	6.9%	27.7%
50.7%	32.4%	15.3%	-12.1%	19.9%	-54.1%	-45.7%	-9.5%	15.9%	-94.1%	-44.0%	46.9%	100.0%	10.9%	-0.7%	43.5%	66.9%	4.7%	-57.2%	100.0%	-36.4%	8.8%	20.9%
55.4%	34.8%	19.5%	-17.9%	27.0%	-63.3%	-46.7%	-8.2%	13.8%	-100.0%	-38.2%	55.5%	100.0%	10.0%	-3.2%	51.4%	72.5%	3.6%	-63.1%	100.0%	-37.1%	10.7%	13.7%
60.1%	37.6%	22.2%	-23.5%	33.5%	-74.0%	-48.6%	-7.8%	12.2%	-100.0%	-30.4%	65.2%	100.0%	9.0%	-6.5%	59.9%	78.8%	2.2%	-70.1%	100.0%	-37.8%	12.4%	3.6%
64.7%	40.5%	24.8%	-29.1%	40.0%	-84.8%	-50.6%	-7.4%	10.7%	-100.0%	-22.7%	74.9%	100.0%	7.9%	-9.8%	68.3%	85.0%	0.8%	-77.0%	100.0%	-38.6%	14.1%	-6.5%
69.4%	43.6%	27.5%	-34.7%	46.5%	-95.5%	-52.5%	-7.0%	9.1%	-100.0%	-14.9%	84.6%	100.0%	6.8%	-13.1%	76.8%	91.3%	-0.6%	-84.0%	100.0%	-39.3%	15.8%	-16.6%
74.1%	46.9%	30.0%	-42.5%	53.6%	-100.0%	-56.7%	-6.9%	7.1%	-100.0%	-6.1%	95.6%	100.0%	4.8%	-18.4%	85.1%	99.1%	-2.6%	-91.0%	100.0%	-40.7%	17.8%	-28.1%
78.7%	50.5%	33.5%	-52.6%	63.4%	-100.0%	-60.9%	-7.7%	7.2%	-100.0%	8.3%	100.0%	100.0%	6.0%	-24.9%	97.5%	100.0%	-5.5%	-98.7%	100.0%	-42.6%	20.9%	-44.0%
83.4%	54.4%	39.8%	-65.0%	76.6%	-100.0%	-63.1%	-8.7%	9.4%	-100.0%	31.3%	100.0%	100.0%	9.1%	-31.9%	100.0%	100.0%	-9.7%	-100.0%	100.0%	-46.6%	24.5%	-65.8%
88.1%	58.8%	46.5%	-77.7%	90.2%	-100.0%	-65.0%	-9.6%	11.7%	-100.0%	55.7%	100.0%	100.0%	12.4%	-38.9%	100.0%	100.0%	-14.2%	-100.0%	100.0%	-50.9%	28.3%	-88.5%
92.7%	63.7%	60.7%	-90.3%	100.0%	-100.0%	-65.1%	-10.0%	12.1%	-100.0%	78.7%	100.0%	100.0%	10.9%	-50.1%	100.0%	100.0%	-22.9%	-100.0%	100.0%	-56.8%	32.8%	-100.0%
97.4%	69.2%	86.0%	-100.0%	100.0%	-100.0%	-63.4%	-9.3%	10.6%	-100.0%	100.0%	100.0%	100.0%	6.2%	-67.6%	100.0%	100.0%	-36.8%	-100.0%	100.0%	-65.3%	39.7%	-100.0%
102.1%	76.1%	100.0%	-100.0%	100.0%	-100.0%	-53.3%	-13.4%	29.7%	-100.0%	100.0%	100.0%	100.0%	26.1%	-92.9%	100.0%	100.0%	-64.7%	-100.0%	100.0%	-82.9%	51.4%	-100.0%
106.7%	85.0%	100.0%	-100.0%	100.0%	-100.0%	-38.7%	-22.2%	47.7%	-100.0%	100.0%	100.0%	100.0%	45.9%	-100.0%	100.0%	100.0%	-100.0%	-100.0%	100.0%	-100.0%	67.4%	-100.0%
111.4%	106.0%	100.0%	-100.0%	100.0%	-100.0%	100.0%	-100.0%	100.0%	-100.0%	100.0%	100.0%	100.0%	-100.0%	-100.0%	100.0%	100.0%	-100.0%	-100.0%	100.0%	-100.0%	100.0%	-100.0%

Table 8: Changed Efficient Allocations

Efficient Allocations Changes

GRT	RDF	NEP	RES	HYP	FFB	FFA	ROC	SAC	VKE	ATT	AWA	IPF	EMI	REB	MSP	APF	PIV	OCT	TDH	STP
0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
0.0%	0.3%	-0.3%	0.0%	0.0%	0.1%	-1.8%	0.0%	-0.1%	0.1%	-0.5%	0.1%	1.2%	0.0%	0.0%	0.1%	0.3%	0.9%	-0.5%	-0.1%	0.1%
-0.1%	0.6%	-0.9%	0.1%	-0.1%	0.3%	-4.5%	0.2%	-0.1%	0.1%	0.0%	0.3%	3.0%	-0.2%	-0.2%	0.3%	0.7%	1.6%	-1.0%	-0.2%	0.1%
-0.1%	1.0%	-1.4%	0.2%	-0.2%	0.4%	-7.3%	0.3%	-0.2%	0.1%	0.0%	0.5%	4.9%	-0.3%	-0.4%	0.4%	1.2%	2.6%	-1.7%	-0.3%	0.2%
0.2%	0.3%	-1.8%	-0.1%	0.2%	0.6%	-10.1%	0.0%	0.4%	0.5%	0.0%	1.4%	6.9%	-0.2%	0.1%	0.5%	1.7%	0.0%	-1.0%	-0.1%	0.5%
0.4%	0.3%	-2.3%	-0.2%	0.2%	0.8%	-13.3%	-0.2%	0.7%	0.7%	0.0%	1.8%	9.0%	-0.1%	0.2%	0.6%	2.2%	0.0%	-1.3%	-0.2%	0.6%
0.5%	0.4%	-2.8%	-0.3%	0.3%	1.0%	-16.5%	-0.3%	0.9%	1.0%	0.0%	2.2%	11.1%	0.0%	0.3%	0.8%	2.7%	0.0%	-1.6%	-0.2%	0.6%
0.5%	0.4%	-3.3%	-0.6%	0.3%	1.2%	-19.7%	0.0%	1.2%	1.3%	0.0%	2.6%	13.2%	0.1%	0.5%	0.9%	3.1%	0.0%	-2.0%	-0.2%	0.5%
0.6%	0.4%	-3.8%	-0.8%	0.3%	1.4%	-23.2%	0.0%	1.5%	1.7%	0.0%	3.0%	15.5%	0.2%	0.6%	1.1%	3.5%	0.0%	-2.3%	-0.2%	0.5%
0.8%	0.4%	-4.4%	-1.0%	0.4%	1.6%	-26.7%	0.0%	1.8%	2.0%	0.0%	3.5%	17.9%	0.3%	0.8%	1.2%	4.0%	0.0%	-2.7%	-0.2%	0.5%
0.9%	0.5%	-4.9%	-1.2%	0.4%	1.9%	-30.2%	0.0%	2.1%	2.4%	0.0%	3.9%	20.2%	0.4%	0.9%	1.3%	4.5%	0.0%	-3.1%	-0.3%	0.5%
1.0%	-0.1%	-5.3%	0.0%	-0.1%	2.0%	-34.4%	0.0%	2.7%	3.1%	0.0%	4.2%	22.3%	0.5%	1.5%	1.4%	5.0%	0.0%	-3.6%	-0.2%	0.1%
2.1%	-0.2%	-5.0%	0.0%	-0.2%	2.1%	-38.2%	0.0%	4.6%	0.0%	0.0%	7.5%	26.3%	2.3%	0.0%	1.3%	1.3%	0.0%	-3.7%	0.5%	-0.8%
3.4%	-0.6%	-5.0%	0.0%	0.0%	2.5%	-45.0%	0.0%	7.4%	0.0%	0.0%	10.2%	31.7%	0.0%	0.0%	1.1%	0.0%	0.0%	-4.7%	1.0%	-2.0%
4.3%	-1.2%	-5.4%	0.0%	-0.1%	2.9%	-53.0%	0.0%	9.6%	0.0%	0.0%	12.2%	37.2%	0.0%	0.0%	1.2%	0.0%	0.0%	-5.6%	1.3%	-3.2%
5.5%	0.5%	0.0%	0.0%	0.3%	2.5%	-66.6%	0.0%	9.5%	0.0%	0.0%	11.9%	43.6%	0.0%	0.0%	-0.5%	0.0%	0.0%	-7.2%	0.6%	0.0%
8.8%	0.0%	0.0%	0.0%	4.1%	2.7%	-75.8%	0.0%	5.8%	0.0%	0.0%	24.4%	32.4%	0.0%	0.0%	2.6%	0.0%	0.0%	-6.8%	1.7%	0.0%
0.0%	0.0%	0.0%	0.0%	13.3%	1.9%	-70.3%	0.0%	0.0%	0.0%	0.0%	47.4%	7.1%	0.0%	0.0%	4.7%	0.0%	0.0%	-8.3%	4.3%	0.0%
0.0%	0.0%	0.0%	0.0%	9.1%	2.6%	-52.3%	0.0%	0.0%	0.0%	0.0%	37.2%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	3.3%	0.0%
0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%

Table 9: Allocation Changes Due To Return Changes

Table 8 and Table 9 show the extent of the changes to the efficient portfolios. Despite changing each of the four stocks by only one percent, the allocation to some stocks changes by as much as 75 percent!

6 APPLICATION TO THE BOND MARKET

This chapter focuses on applying the methodologies to the Bond Market. Figure 13 shows the evolution of the yield curve over the period in our data set.

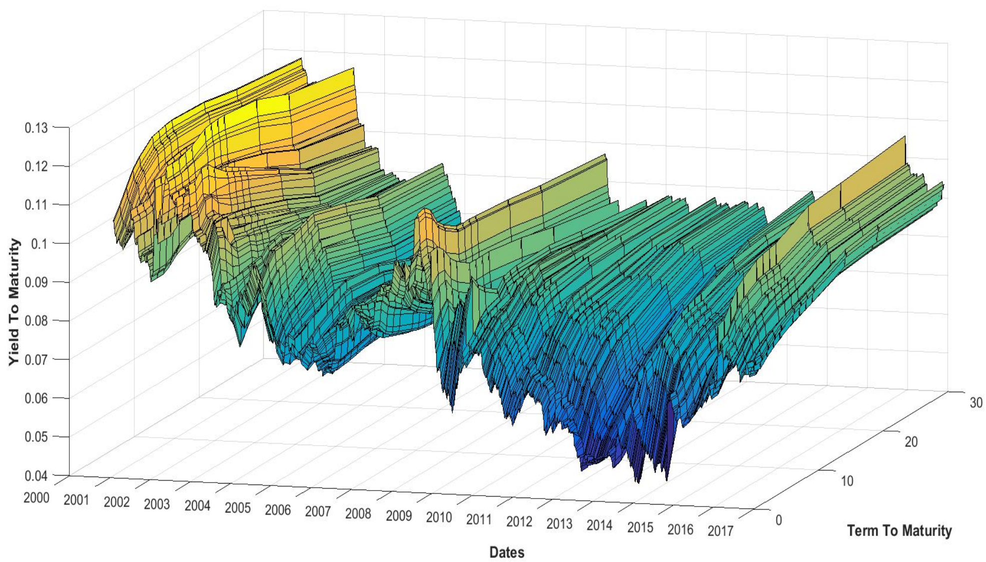

Figure 14: Evolution Of The Yield Curve

The Bond Market is different to the Equity Market in a number of ways, as a result the methods required to construct bond portfolios are quite different to those used for the Equity Market. This chapter outlines these differences.

In order to perform portfolio optimization on a portfolio of bonds, we will require the bonds expected returns as well the covariance matrix of their expected returns. In contrast with the Equity market, we cannot find these parameters by looking at the historical returns of the bonds in question. We show this in the sections that follow.

6.1 Bonds Are Different

Figure 15: Bloomberg Extract Of Information For The R206 Series South African Government Bond

Figure 15 shows the details of the R206 series South African government bond. This bond was first issued in July of 2005 and paid a fixed coupon of 7.5 percent. It was to be redeemed in January of 2014 at par.

A bond is an instrument which pays fixed coupons at regular intervals and is redeemed at a fixed price on a predetermined maturity date.

Unlike equities, which can trade in perpetuity, bonds have an expiry date and a predefined expiry price. The price of a bond will, therefore, converge towards that fixed expiry price as time gets closer to the expiry date. As a result, bond price returns converge to zero as the expiry date gets closer.

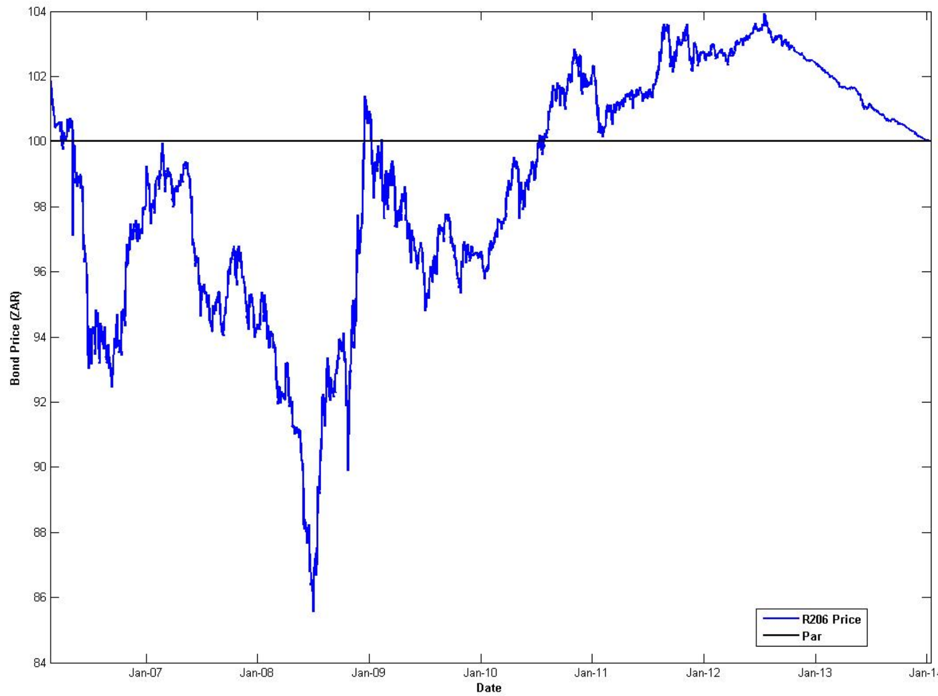

Figure 16: Price Series Of The R206 Series South African Government Bond

Figure 16 shows the price of the R206 series South African government bond from the day it was first issued to the date of its maturity. The price of the bond is subject to market forces and fluctuates over time but converges towards the fixed redemption price as the bond nears its maturity date.

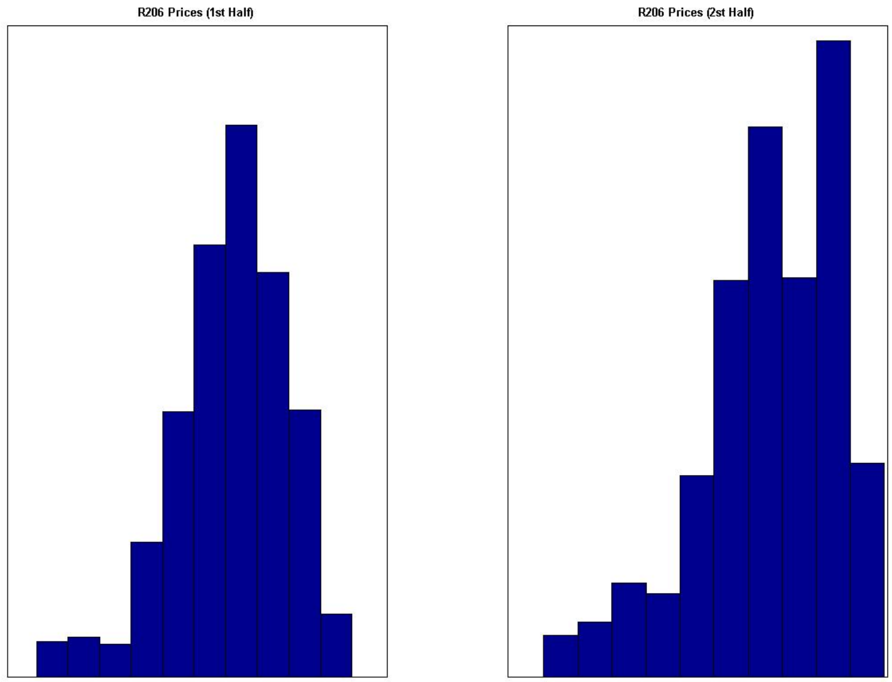

Figure 17: Histogram Of The R206 Series South African Government Bond Price Series

Figure 17 divides the R206 government bond's price series in two and plots the respective series' histograms. The distributions of the two half series are different. This shows that the distribution of the price series changes over time.

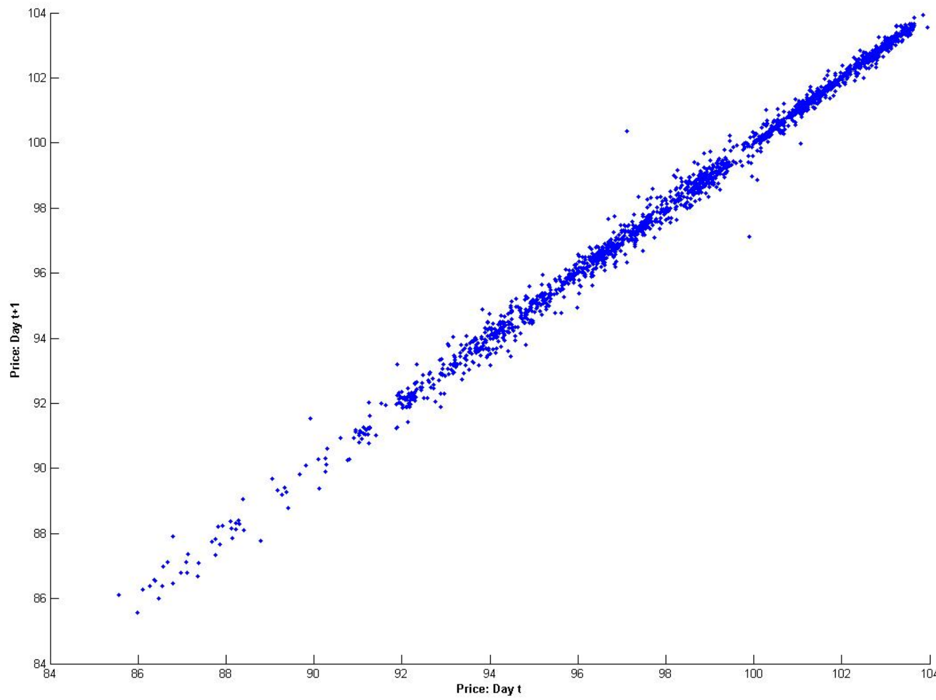

Figure 18: Scatter Plot Of R206 Price Series Against A Lagged Version Of The Same Series

Figure 18 shows the R206 government bond's price series against the same series lagged by a single week. There is a clear relationship between the price of the bond and the next price the following week. Each week's prices are not identically nor independently distributed to the following week's prices.

We now turn to the bond price returns as we did with the stock market. The figure below shows the bond price returns over time.

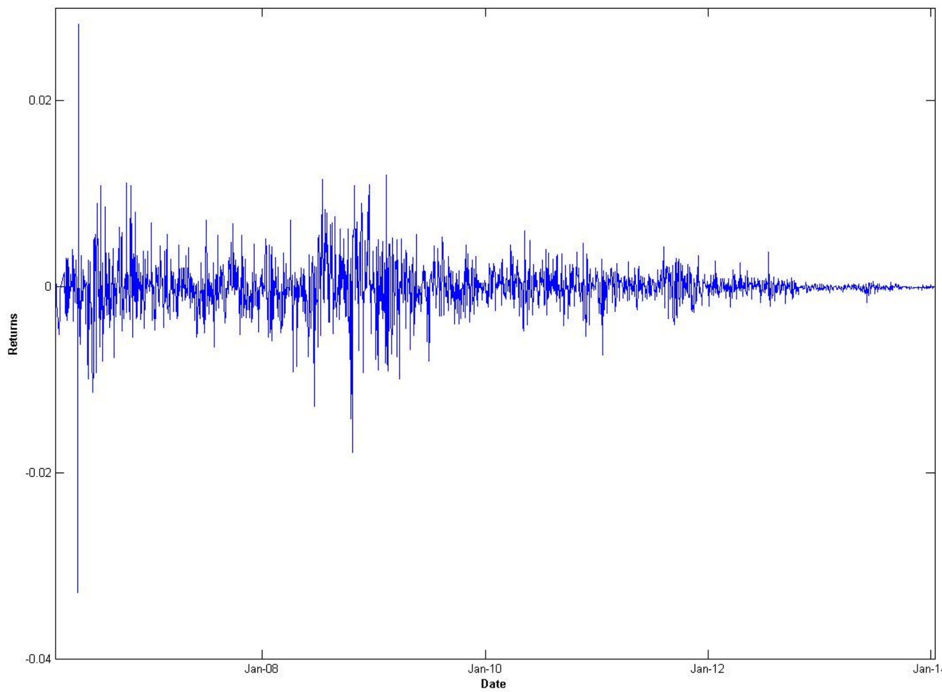

Figure 19: R206 Series Bond Price Returns Over Time

Figure 19 shows how the bond price returns become smaller and smaller as the expiration date gets closer. This means that the distribution of the bond price returns changes as the expiration date gets closer. Put another way, the distribution of the bond price returns is a function of the bond's time to expiration. From Figure 19 it is also clear that the volatility of the bond price returns is also a function of the bond's time to maturity.

Figure 20 shows the one year volatility of bond price returns over time. It is clear that the volatility of the bond price returns changes over time, become lower as the bond approaches its redemption date.

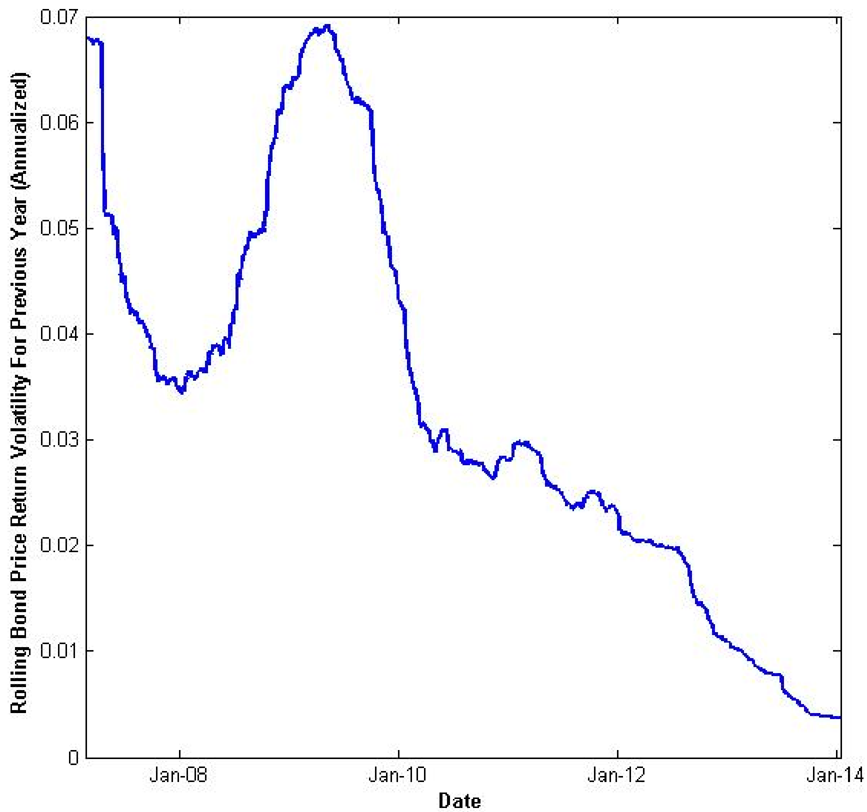

Figure 20: R206 Series Bond Price Return Volatility Over Time

Since the distribution of the bond price returns changes as the bond becomes closer to maturity, we cannot use the past price returns of a bond to estimate it's likely future distribution. Neither bond price nor their returns are appropriate to model for portfolio optimization.

Put another way, a bond changes in nature as it nears its maturity. A bond with five years to maturity does not behave as it did when it had ten years to maturity. For this reason we cannot use bond prices or their returns to model the future return distribution of the bond.

6.2 Yields Are Well Behaved

In the previous section we showed that the distribution of a bond's price returns is a function of the bond's time to maturity. We take this notion further and consider whether bonds with the same time to maturity have the same distribution.

The fixed income market already has a way of representing bonds by their time to maturity and it's called the yield curve. Martelline et al (Meucci, 2005) define a bond's yield to maturity as the single rate that sets the value of a bond's cash flows equal to the bond's price

$$P = \sum_{t=1}^{T} CF_t / (1 + YTM_{linear})^t$$

We can understand the yield to maturity further by considering the simplest bond of all, the zero-coupon bond. A zero-coupon bond is a bond which pays no coupons and is redeemed at maturity for a single unit of currency. Consider the zero-coupon bond maturing at time T

$$Z_t^T = 1 / (1 + YTM_{linear})^{(T-t)}$$

$$1 / Z_t^T = (1 + YTM_{linear})^{T-t}$$

using the fact that the zero-coupon bond is redeemed for a unit of currency at maturity

$$Z_T^T / Z_t^T = (1 + YTM_{linear})^{T-t}.$$

Now, Z_T^T / Z_t^T is the linear return of the zero-coupon bond from time t to time T. The yield to maturity is therefore the annualized return of the zero-coupon bond over its remaining life.

If we defined the yield to maturity in terms of log returns

$$YTM_{log} = -\frac{1}{(T-t)} ln\{ Z_T^T / Z_t^T \}.$$

Changes in yield to maturities of bonds with the same time to maturity, M, observed at different points in time are equal to

$$YTM_t^M - YTM_{t-\tau}^{M+\tau} = -\frac{1}{(M-t)} . ln\{ Z_t^M / Z_{t-\tau}^{M+\tau} \}$$

this is the log-return of zero-coupon bonds with the same time to maturity, observed at different points in time. This result extends to the more general coupon paying bonds.

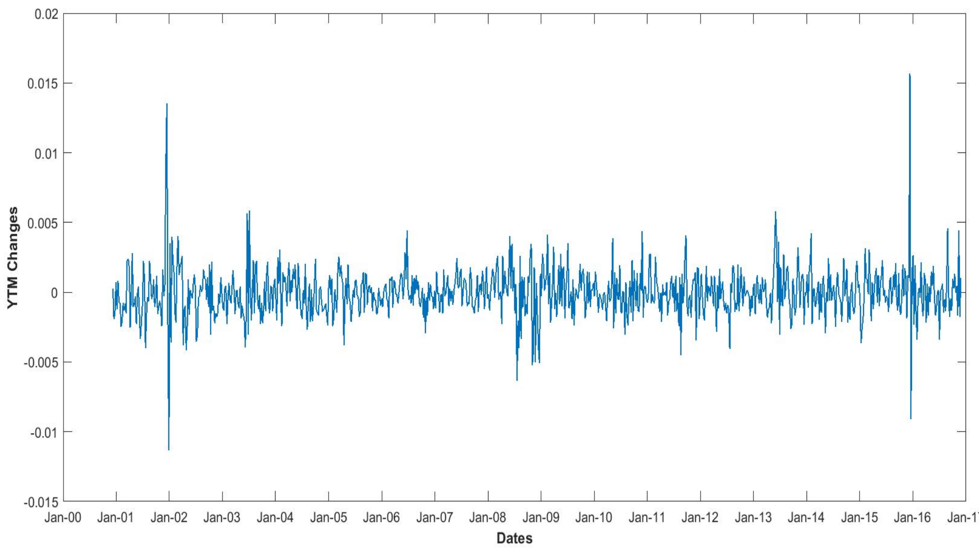

Figure 21: 10-Year Yield-To-Maturity Changes Over Time

Figure 21 shows the weekly yield to maturity changes for the 10Y point on the bond curve. The yield to maturity changes are reminiscent of the equity log-returns we saw earlier.

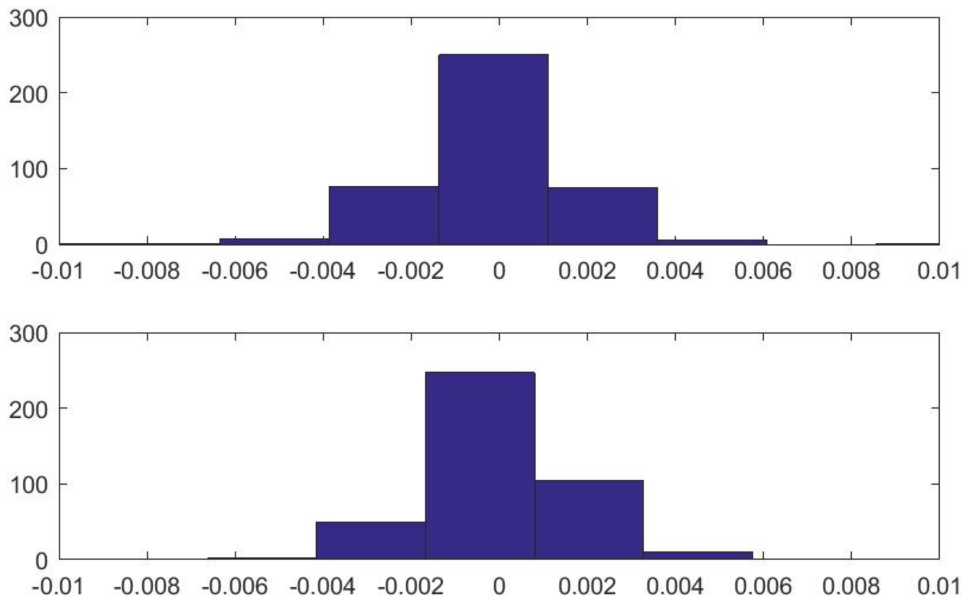

Figure 22: Histogram Of The 10-Year Yield-To-Maturity Changes

Figure 22 shows histograms of two halves of the 10Y YTM changes. The histograms of the two half-series confirm that the distribution of the YTM changes is the same over the two halves.

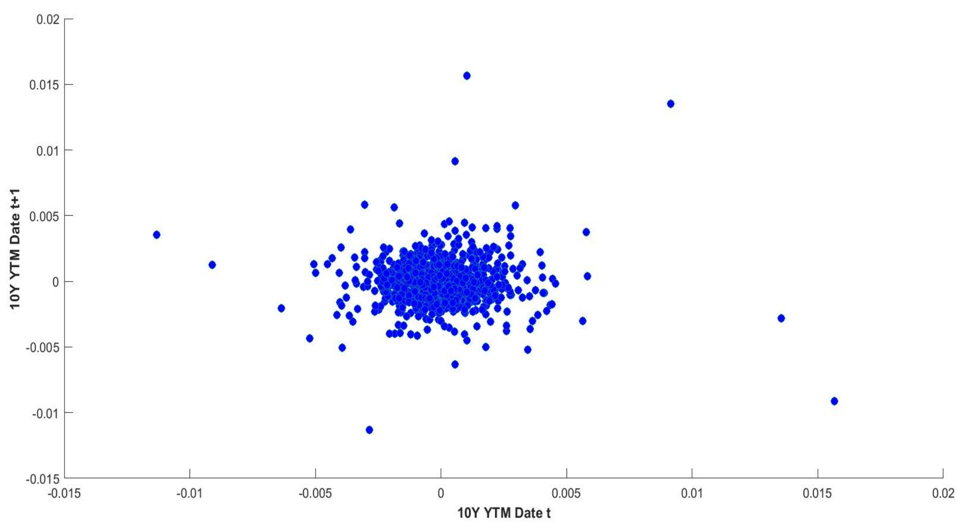

Figure 23: 10-Year Yield-To-Maturity Changes vs. Lagged 10-Year Yield-To-Maturity Changes

Figure 23 and Figure 22 show that we can use historical yield to maturity changes to estimate the likely future distribution of yield to maturities as they are time homogeneous.

6.3 Modelling Bond Returns

This section outlines the strategy we follow to model the bond returns. This strategy follows from the results of the previous sections. Namely that the bond returns are not appropriate to model but that log-yield differences for specific terms to maturity are appropriate.

- We begin with historical yield to maturities for constant maturity bonds, which we source from Bloomberg. These yields are constructed by taking the yield to maturities of South African Government bonds in issue and using an interpolation model to find the yield to maturity of specific term to maturities.
- South African Government bonds are quoted using linear-yield to maturities. So we convert these to log yields.
- Using the term to maturity of the bonds in the Govi index as at 30 November 2016, we use interpolation to find the historical log-yields for bonds with their term to maturity.
- We then calculate the historical log-yield changes for bonds with the maturities of the bonds in question.
- These historical log-yields as well as the bonds' current (as at 30 November 2016) yield to maturities are then used to potential one week yield to maturity changes for the bonds in question.
- Using these potential one week yield to maturity changes we were then able to calculate potential one week price returns.

Once we had the potential one week price returns of the bonds, we proceed as we did with the equity returns. We estimate the one week log-return covariance matrix from which we can estimate an annual covariance matrix. This annual covariance matrix is used to simulate one hundred thousand simulated returns which are used to estimate the linear return covariance matrix. Table 10 shows this covariance matrix.

	Annual Linear-Return Covariance Matrix									
	R204	R207	R208	R2023	R186	R2030	R213	R209	R2037	R2048
R204	0.00068	0.00098	0.00126	0.00158	0.00199	0.00241	0.00257	0.00290	0.00270	0.00298
R207	0.00098	0.00156	0.00199	0.00242	0.00307	0.00372	0.00397	0.00448	0.00417	0.00461
R208	0.00126	0.00199	0.00273	0.00337	0.00433	0.00526	0.00562	0.00635	0.00592	0.00654
R2023	0.00158	0.00242	0.00337	0.00445	0.00579	0.00706	0.00755	0.00856	0.00799	0.00883
R186	0.00199	0.00307	0.00433	0.00579	0.00795	0.00973	0.01035	0.01173	0.01097	0.01197
R2030	0.00241	0.00372	0.00526	0.00706	0.00973	0.01214	0.01300	0.01487	0.01389	0.01538
R213	0.00257	0.00397	0.00562	0.00755	0.01035	0.01300	0.01399	0.01610	0.01502	0.01677
R209	0.00290	0.00448	0.00635	0.00856	0.01173	0.01487	0.01610	0.01890	0.01767	0.01983
R2037	0.00270	0.00417	0.00592	0.00799	0.01097	0.01389	0.01502	0.01767	0.01654	0.01850
R2048	0.00298	0.00461	0.00654	0.00883	0.01197	0.01538	0.01677	0.01983	0.01850	0.02186

Table 10: Annual Linear-Return Covariance Matrix Of Govi Bonds

The linear return covariance matrix allows us to calculate the expected returns implied by the weights of the different bonds in the Govi Index using Equation 1. Table 11 show these implied returns. These two parameters are used in the portfolio optimizations.

	30 Nov 16	
Bond	Weight	Exp Returns
R204	23.8%	0.40%
R207	12.8%	0.62%
R208	10.0%	0.86%
R2023	9.3%	1.14%
R186	8.9%	1.53%
R2030	7.7%	1.91%
R213	7.3%	2.05%
R209	7.0%	2.36%
R2037	7.0%	2.20%
R2048	6.1%	2.47%
	100.0%	

Table 11: Annual Linear-Return Matrix Of Govi Bonds

6.4 Mean-Variance Results

In this section we show the mean-variance efficient portfolios which result from the statistics estimated in the previous section.

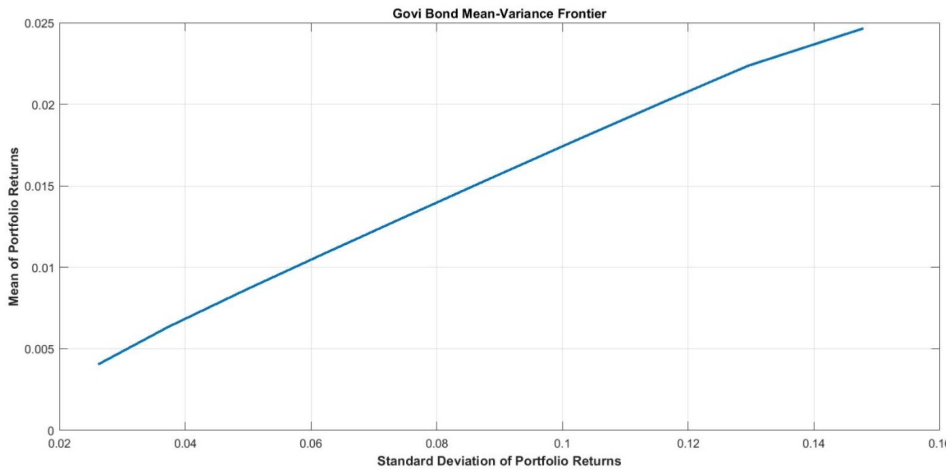

Figure 24: Govi Bond Mean-Variance Efficient Frontier

Figure 24 shows the efficient frontier while Figure 25 and Table 12 show the allocations of the efficient portfolios.

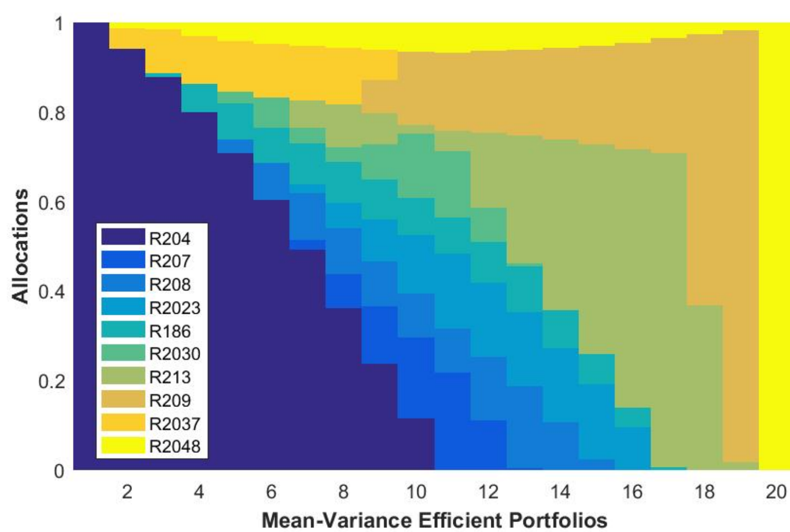

Figure 25: Govi Bond Mean-Variance Allocations

		Govi Mean-Variance Efficient Portfolio Allocations									
Portfolio Return	Portfolio Risk	R204	R207	R208	R2023	R186	R2030	R213	R209	R2037	R2048
0.4%	2.6%	100.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
0.5%	3.1%	94.1%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	4.6%	1.2%
0.6%	3.7%	87.8%	0.0%	0.0%	0.0%	0.9%	0.0%	0.0%	0.0%	9.7%	1.6%
0.7%	4.2%	80.0%	0.0%	0.0%	0.0%	6.2%	0.0%	0.0%	0.0%	10.7%	3.0%
0.8%	4.8%	70.8%	0.0%	3.0%	0.0%	8.1%	2.6%	0.0%	0.0%	11.5%	4.1%
0.9%	5.4%	60.4%	0.0%	8.2%	0.0%	7.8%	6.7%	0.0%	0.0%	12.0%	4.8%
1.1%	6.0%	49.1%	2.2%	10.6%	1.9%	9.2%	3.5%	6.0%	0.0%	12.2%	5.2%
1.2%	6.7%	36.2%	7.6%	10.2%	5.7%	9.2%	3.2%	9.7%	0.0%	12.6%	5.6%
1.3%	7.3%	23.8%	12.8%	10.0%	9.3%	8.9%	8.0%	6.9%	7.4%	6.7%	6.1%
1.4%	7.9%	11.6%	18.0%	9.8%	13.1%	8.3%	14.5%	2.0%	16.3%	0.0%	6.5%
1.5%	8.5%	0.0%	21.7%	10.0%	16.7%	8.0%	14.8%	4.8%	17.4%	0.0%	6.7%
1.6%	9.2%	0.0%	11.1%	14.1%	16.7%	9.0%	7.8%	16.7%	18.3%	0.0%	6.4%
1.7%	9.8%	0.0%	0.5%	18.2%	16.7%	10.0%	0.7%	28.5%	19.2%	0.0%	6.1%
1.8%	10.4%	0.0%	0.0%	10.7%	16.6%	8.5%	0.0%	38.0%	20.6%	0.0%	5.6%
1.9%	11.1%	0.0%	0.0%	2.5%	16.7%	6.7%	0.0%	46.9%	22.1%	0.0%	5.1%
2.0%	11.7%	0.0%	0.0%	0.0%	9.5%	4.5%	0.0%	57.7%	23.7%	0.0%	4.5%
2.1%	12.4%	0.0%	0.0%	0.0%	0.0%	0.7%	0.0%	70.0%	25.7%	0.0%	3.6%
2.2%	13.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	36.8%	60.5%	0.0%	2.6%
2.4%	13.7%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	1.8%	96.4%	0.0%	1.8%
2.5%	14.8%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	100.0%

Table 12: Govi Bond Mean-Variance Allocations

6.5 Black-Litterman-Variance Results

In this section we show the Black-Litterman efficient portfolios which result from the statistics estimated in the previous section.

	R204	R207	R208	R2023	R186	R2030	R213	R209	R2037	R2048	View Returns
Yield Curve Views	-0.10%	-0.08%	-0.07%	-0.05%	-0.03%	-0.02%	0.00%	0.02%	0.03%	0.05%	
	-0.20%	-0.20%	-0.20%	-0.20%	-0.20%	-0.20%	-0.20%	-0.20%	-0.20%	-0.20%	
Implied Returns	0.32%	0.37%	0.39%	0.40%	0.37%	0.29%	0.17%	0.02%	-0.12%	-0.31%	
	0.50%	0.68%	0.88%	1.11%	1.40%	1.70%	1.83%	2.11%	1.99%	2.19%	
View Portfolios	13.91%	15.86%	16.92%	17.00%	15.73%	12.59%	7.34%	0.66%	-28.56%	-71.44%	0.60%
	3.50%	4.73%	6.11%	7.73%	9.71%	11.85%	12.70%	14.66%	13.81%	15.21%	1.67%

Table 13: Govi Bond Black-Litterman Views

Table 13 shows the views we used for our Black-Litterman example. The first view is that the yield curve will steepen with bonds up to the R2030 series bond rallying and later bonds selling off in relative terms. The second view is that the entire yield curve will experience a 20bps parallel rally.

	30 Nov 16											
Bond	Weight	BL Exp Returns	BL Covariance Matrix									
R204	23.8%	0.44%	0.00063	0.00090	0.00115	0.00144	0.00181	0.00219	0.00234	0.00263	0.00245	0.00271
R207	12.8%	0.67%	0.00090	0.00144	0.00182	0.00221	0.00278	0.00337	0.00361	0.00407	0.00379	0.00421
R208	10.0%	0.92%	0.00115	0.00182	0.00250	0.00307	0.00393	0.00478	0.00510	0.00577	0.00538	0.00596
R2023	9.3%	1.19%	0.00144	0.00221	0.00307	0.00406	0.00526	0.00642	0.00686	0.00778	0.00725	0.00804
R186	8.9%	1.59%	0.00181	0.00278	0.00393	0.00526	0.00724	0.00885	0.00941	0.01066	0.00997	0.01088
R2030	7.7%	1.94%	0.00219	0.00337	0.00478	0.00642	0.00885	0.01104	0.01183	0.01352	0.01263	0.01398
R213	7.3%	2.06%	0.00234	0.00361	0.00510	0.00686	0.00941	0.01183	0.01272	0.01464	0.01366	0.01524
R209	7.0%	2.31%	0.00263	0.00407	0.00577	0.00778	0.01066	0.01352	0.01464	0.01720	0.01608	0.01802
R2037	7.0%	2.16%	0.00245	0.00379	0.00538	0.00725	0.00997	0.01263	0.01366	0.01608	0.01505	0.01681
R2048	6.1%	2.31%	0.00271	0.00421	0.00596	0.00804	0.01088	0.01398	0.01524	0.01802	0.01681	0.01988
	100.0%											

Table 14: Govi Bond Black-Litterman Statistics

Using the view portfolios in Table 13 we calculated the Black-Litterman expected returns and covariance matrix as in Table 14 which we used to produce Black-Litterman efficient portfolios.

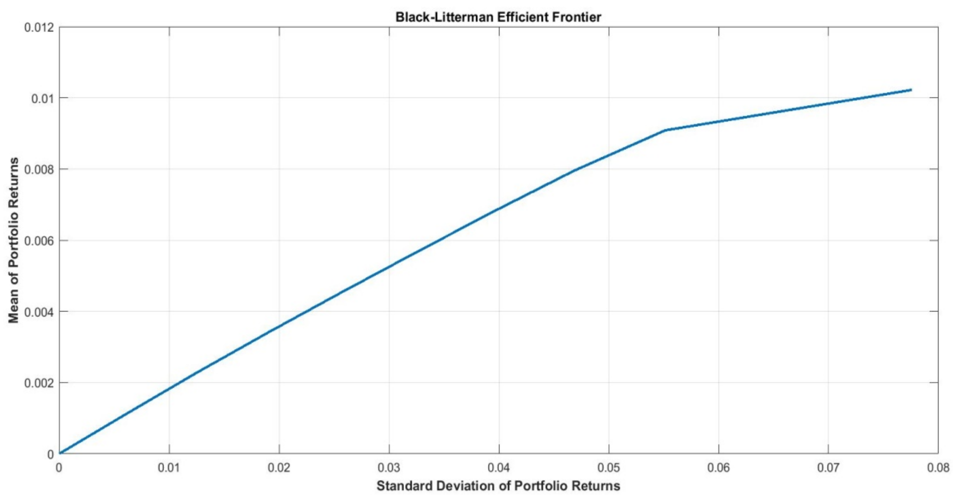

Figure 26: Govi Bond Black-Litterman Efficient Frontier

Figure 26 shows the efficient frontier while Figure 27 and Table 15 show the allocations of the efficient portfolios.

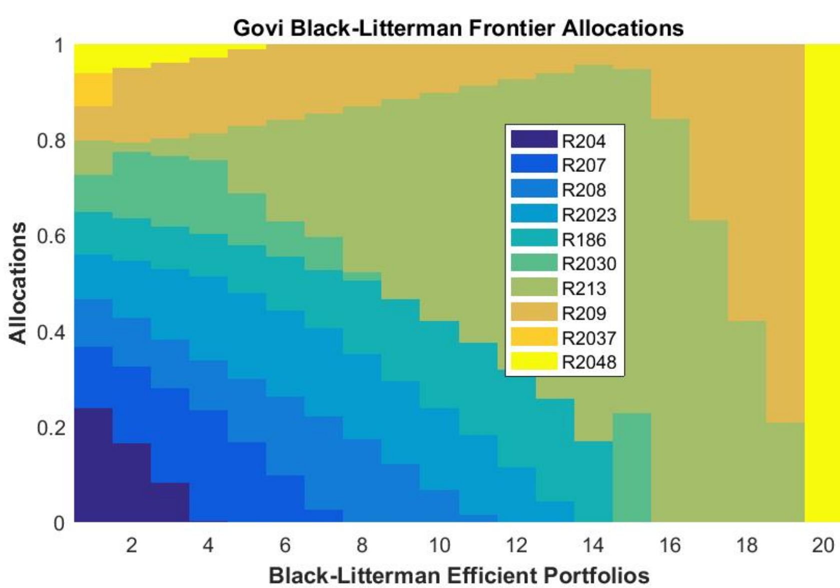

Figure 27: Govi Mean-Variance Portfolio Allocations

		Govi Black-Litterman Efficient Portfolio Allocations									
Portfolio Return	Portfolio Risk	R204	R207	R208	R2023	R186	R2030	R213	R209	R2037	R2048
1.3%	6.9%	23.8%	12.8%	10.0%	9.3%	8.9%	7.7%	7.3%	7.0%	7.0%	6.1%
1.3%	7.2%	16.4%	16.2%	10.1%	12.0%	8.8%	14.0%	2.0%	15.5%	0.0%	5.0%
1.4%	7.5%	8.3%	19.7%	10.3%	14.7%	8.9%	14.7%	3.8%	15.8%	0.0%	3.9%
1.5%	7.8%	0.2%	23.3%	10.4%	17.5%	9.0%	15.4%	5.6%	16.0%	0.0%	2.7%
1.5%	8.1%	0.0%	16.7%	13.3%	17.9%	10.0%	10.8%	14.1%	16.2%	0.0%	1.1%
1.6%	8.4%	0.0%	9.7%	16.4%	18.2%	11.1%	7.5%	21.3%	15.8%	0.0%	0.0%
1.6%	8.7%	0.0%	2.5%	19.6%	18.4%	12.2%	6.8%	25.9%	14.5%	0.0%	0.0%
1.7%	9.0%	0.0%	0.0%	17.3%	17.8%	15.3%	1.7%	34.9%	12.9%	0.0%	0.0%
1.7%	9.3%	0.0%	0.0%	12.1%	17.4%	17.1%	0.0%	42.0%	11.5%	0.0%	0.0%
1.8%	9.6%	0.0%	0.0%	6.8%	17.1%	18.2%	0.0%	47.8%	10.1%	0.0%	0.0%
1.8%	9.9%	0.0%	0.0%	1.5%	16.9%	19.3%	0.0%	53.6%	8.8%	0.0%	0.0%
1.9%	10.2%	0.0%	0.0%	0.0%	11.6%	20.3%	0.0%	60.7%	7.4%	0.0%	0.0%
1.9%	10.5%	0.0%	0.0%	0.0%	4.4%	21.3%	0.0%	68.2%	6.0%	0.0%	0.0%
2.0%	10.9%	0.0%	0.0%	0.0%	0.0%	17.0%	0.0%	78.6%	4.4%	0.0%	0.0%
2.0%	11.2%	0.0%	0.0%	0.0%	0.0%	0.0%	22.7%	72.0%	5.3%	0.0%	0.0%
2.1%	11.5%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	84.4%	15.6%	0.0%	0.0%
2.2%	11.9%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	63.2%	36.8%	0.0%	0.0%
2.2%	12.3%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	42.0%	58.0%	0.0%	0.0%
2.3%	12.7%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	20.8%	79.2%	0.0%	0.0%
2.3%	14.1%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	100.0%

Table 15: Govi Black-Litterman Allocations

7 PART II: FINDINGS, ANALYSIS AND DISCUSSION

The research question this dissertation attempts to answer concerns the construction of multi-asset class portfolios, called Balanced Funds in South Africa. When constructing these multi-asset class portfolios, it is common for these institutions to begin the portfolio construction at the asset class level. An allocation to each asset class is decided after which point a portfolio is constructed for each asset class. This asset class portfolio is constructed by finding the combination of asset classes which results in the highest expected return given some target risk.

The individual asset class portfolios are then constructed independently of each other. Indeed the portfolios are often managed by different portfolio managers who aim to produce returns larger than their respective benchmark portfolios. The different single asset class portfolios are constructed by finding the combination of securities (within the particular asset class) which results in the highest expected return given some target risk. In this dissertation, this approach is called *top-down* portfolio construction.

An alternative would be to consider all the securities available to be invested, regardless of their asset class characterization, and to find the combination of all securities which results in the highest expected return given some target risk. This dissertation asks if this approach results in better portfolios. In this dissertation, this approach is called *bottom-up* portfolio construction.

The final part of this dissertation details the methodology used to create bottom-up and top-down portfolios and compare them.

The first section of Chapter 6 (Research Strategy) details the different comparisons conducted between the two different construction approaches. We first compare unconstrained bottom-up and top-down portfolios. Next we compare constrained bottom-up and top-down portfolios. Both of these comparisons are conducted using the mean-variance optimization model. Next we examine bottom-up portfolios constructed in the Top-Down world of active management. The Black-Litterman model is used for this comparison as it is more suited to active management. The second section of Chapter 6 (Research Findings) presents the results of the comparisons outlined in the first section.

The dissertation is then concluded by Chapter 7 (Research Conclusions) which provides a brief summary of the major points and results of the dissertation.

8 RESEARCH STRATEGY AND FINDINGS

8.1 Research Strategy

8.1.1 Bottom-Up vs. Top-Down Unconstrained Portfolios

We begin by comparing the efficient frontiers of unconstrained Bottom-Up and Top-Down portfolios. Both sets of portfolios are estimated using the Mean-Variance optimization routine.

	JSAPY	Top40	Govi	Exp Returns
JSAPY	0.18273	0.16862	0.00448	28.9%
Top40	0.16862	0.20781	0.00082	32.7%
Govi	0.00448	0.00082	0.00526	0.7%

Table 16: Top-Down Index Statistics

Table 16 shows the covariance matrix as well as the expected returns used. The covariance matrix is estimated using historical index returns which we've estimated using the current security (shares or bonds) weights of the respective indices and the security's historical returns. We chose this method rather than using actual historical index returns since the historical indices contained different shares and/or bonds. The distribution of the historical indices must therefore be different to the current index.

The Index returns are estimated from the bottom-up. For each Index we calculated the returns implied by the share/bond weights in the respective indices using Equation 1.

The only constraint to the Mean-Variance optimizer is for the weights of the different indices to sum to 100 percent and the allocation to each index to be between -100 and 100 percent.

	Weight	Exp Returns
GRT	3.9%	28.5%
RDF	2.8%	27.7%
NEP	2.2%	31.9%
RES	1.7%	29.5%
HYP	1.7%	28.3%
FFB	1.3%	39.0%
FFA	1.0%	25.6%
ROC	0.9%	33.7%
SAC	0.7%	25.5%
VKE	0.7%	26.4%
ATT	0.6%	24.5%
AWA	0.5%	26.0%
IPF	0.4%	25.7%
EMI	0.4%	26.3%
REB	0.3%	23.9%
MSP	0.2%	26.2%
APF	0.2%	26.2%
PIV	0.2%	23.7%
OCT	0.2%	27.5%
TDH	0.1%	22.7%
STP	0.1%	23.8%
	20.0%	

	Weight	Exp Returns
AGL	2.7%	35.3%
ANG	0.6%	24.6%
APN	1.0%	33.8%
BAT	0.3%	29.1%
BGA	0.7%	31.1%
BID	0.8%	-16.7%
BIL	4.8%	36.1%
BTI	2.2%	26.1%
BVT	0.5%	30.5%
CFR	4.5%	31.2%
DSY	0.3%	29.6%
FFA	0.2%	26.0%
FFB	0.2%	38.1%
FSR	1.5%	32.8%
GFI	0.4%	26.3%
GRT	0.6%	28.5%
IMP	0.3%	29.9%
INL	0.2%	30.7%
INP	0.6%	30.4%
ITU	0.4%	25.4%
LHC	0.3%	29.7%
MEI	0.4%	36.5%
MND	0.3%	30.9%
MNP	1.0%	32.4%
MRP	0.3%	34.9%
MTN	1.9%	34.5%
NED	0.5%	28.5%
NPN	8.6%	41.5%
NTC	0.4%	32.9%
OML	1.5%	30.3%
RDF	0.5%	27.7%
REI	0.4%	24.4%
REM	1.1%	30.8%
RMH	0.4%	33.0%
SBK	1.8%	31.0%
SGL	0.2%	31.7%
SHP	0.8%	32.2%
SLM	1.1%	31.4%
SNH	1.6%	31.8%
SOL	2.0%	31.8%
TBS	0.6%	29.9%
VOD	0.7%	29.7%
WHL	0.6%	32.4%
	50.0%	

	Weight	Exp Returns
R204	7.2%	0.2%
R207	3.8%	0.3%
R208	3.0%	0.4%
R2023	2.8%	0.6%
R186	2.7%	0.8%
R2030	2.3%	1.0%
R213	2.2%	1.1%
R209	2.1%	1.3%
R2037	2.1%	1.2%
R2048	1.8%	1.4%
	30.0%	

Table 17: Bottom-Up Expected Returns

For the Bottom-Up portfolios we use returns that are consistent with those used in the Top-Down scenario.

8.1.2 Constrained Portfolios: Bottom-Up vs. Top-Down

Next, we repeat the experiment of the previous section but using constrained portfolios. These constraints are meant to replicate the situation most investors would be faced with.

The experiment is carried out assuming the investor is not allowed to leverage or short the different assets. That is, the holding of each asset must be within the range [0,100].

Most investors would additionally face constraints on the amount of each asset class they are able to hold. To replicate this, we impose a further constraint on the optimizer such that the exposure to each asset class is subject to a minimum and maximum as outlined in Table 18.

	JSAPY	Top40	Govi
Minimum	10.0%	40.0%	20.0%
Benchmark	20.0%	50.0%	30.0%
Maximum	30.0%	60.0%	40.0%

Table 18: Asset Allocation Constraints

8.1.3 Bottom-Up Portfolios In A Top-Down World

Anecdotal evidence suggests that multi asset class investors are judged based on factors that assume that they construct their portfolios in a Top-Down manner.

Not only are constraints on each asset class imposed, but constraints on tracking errors away from the different asset class benchmark indices are imposed. When choosing his bonds for example, the investor may be constrained by the tracking error of his bond portfolio to the Govi index.

To mirror this, we have constructed Bottom-Up as well as Top-Down portfolios which assume that the investor must have a tracking error which is smaller than 3 percent.

For the Bottom-Up approach, we have assumed that our investor has views on how the bonds and shares in his portfolio will perform. The investor uses the Black-Litterman portfolio optimizer to construct his portfolios using these views. Importantly, the investor imposes a constraint on the optimizer limiting the total exposure in any one asset class based on Table 18. A further constraint is imposed such that the sub-portfolios must have a tracking error which is smaller than 3 percent.

For the Top-Down approach, the investor has the same views as those used in the Bottom-Up approach. To ensure consistency, the investor uses the same Black-Litterman optimizer to construct his sub-portfolios. Efficient Frontiers are constructed for each asset class (Equity, Bonds, and Property). We then take these portfolios and combine them with the asset class level portfolios derived in the previous section.

We then compare the portfolios to see if the Bottom-Down approach produces portfolios which are more efficient than the Top-Down approach.

8.2 Research Findings

8.2.1 Bottom-Up vs. Top-Down Unconstrained Portfolios

Figure 28 and Figure 29 show the Top-Down and Bottom-Up unconstrained Mean-Variance portfolios while Figure 30 shows the efficient frontiers for both the Top-Down as well as the Bottom-Up approach.

Figure 30 shows that the Bottom-Up and Top-Down efficient frontiers coincide. The Bottom-Up approach does not produce portfolios with higher expected returns than the Top-Down approach for the same level of risk.

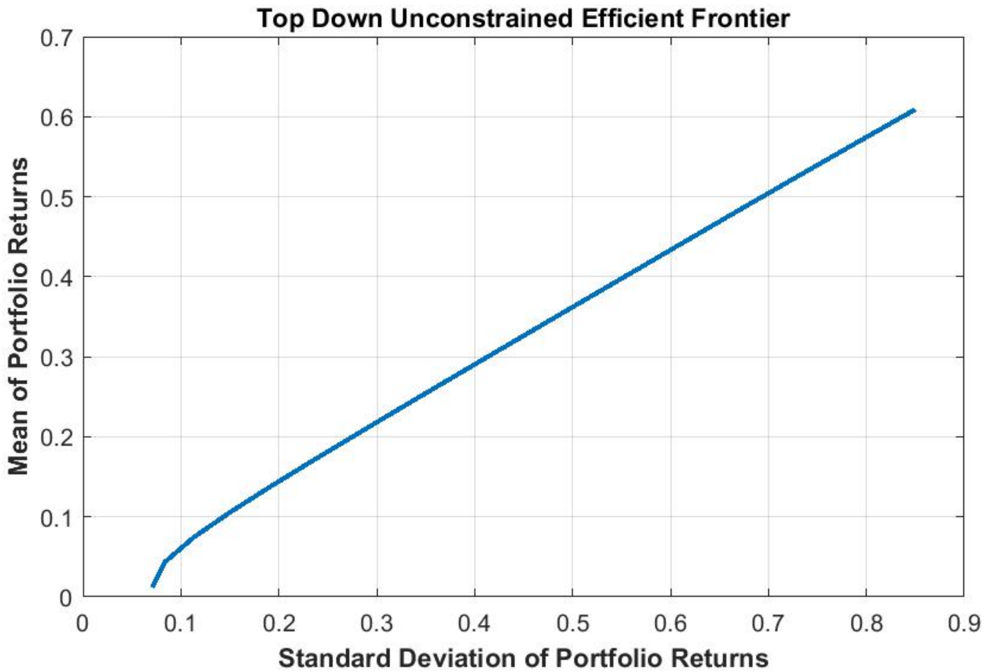

Figure 28: Top-Down Unconstrained Mean-Variance Efficient Frontier

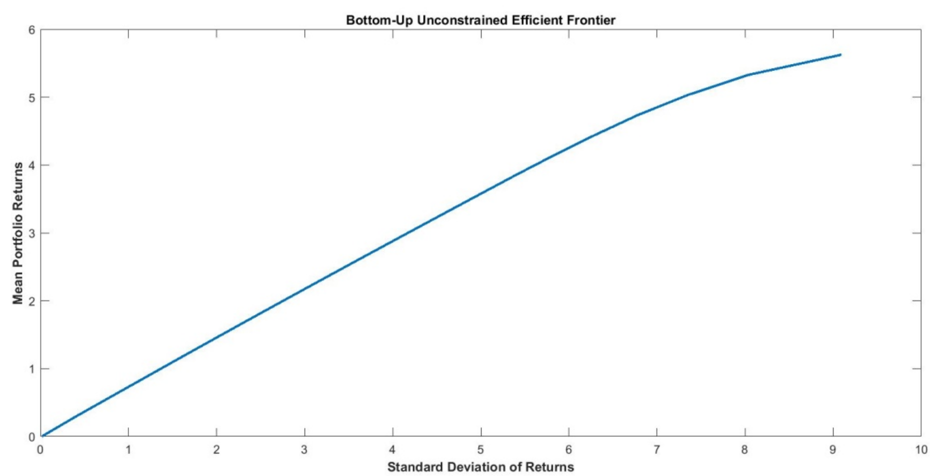

Figure 29: Bottom-Up Unconstrained Mean-Variance Efficient Frontier

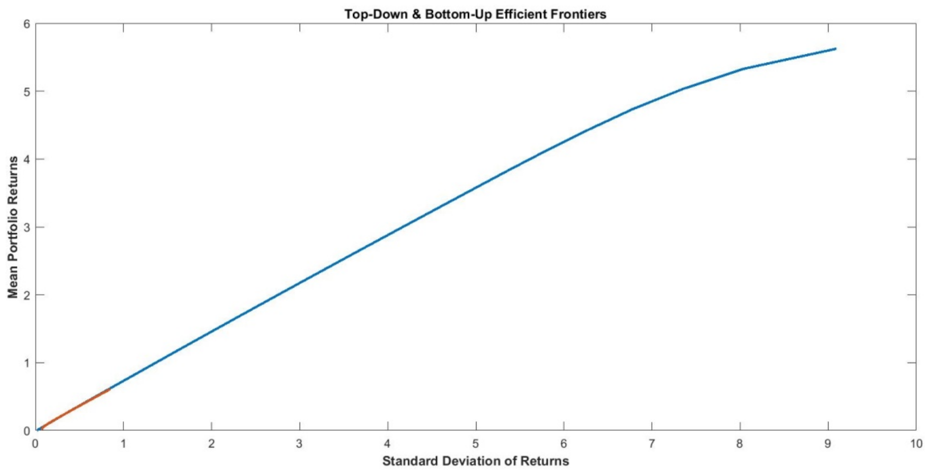

Figure 30: Top-Down and Bottom-Up Unconstrained Mean-Variance Efficient Frontiers

What the Bottom-Up does do is produce portfolios with higher expected returns than the Top-Down portfolios. The likely explanation for this is that the Bottom-Up frontier exposes the investor to portfolios the Top-Down approach does not. This is because the Bottom-Up approach can allocate to any combination of assets while the Top-Down approach is confined to linear combinations of the Index weights. One criticism of this is that the Bottom-Up portfolios may be more concentrated than the Top-Down portfolios. The red part of the efficient frontier Figure 30 shows the part of the frontier the Bottom-Up approach exposes to that the Top-Down does not.

8.2.2 Constrained Portfolios: Bottom-Up vs. Top-Down

We now compare constrained Bottom-Up and Top-Down mean-variance portfolios in order to check whether the results of the previous section were due to the Bottom-Up portfolio's ability to choose more concentrated portfolios.

Table 18 details the constraints for each asset class. These constraints are applied for both the Top-Down and Bottom-Up approaches.

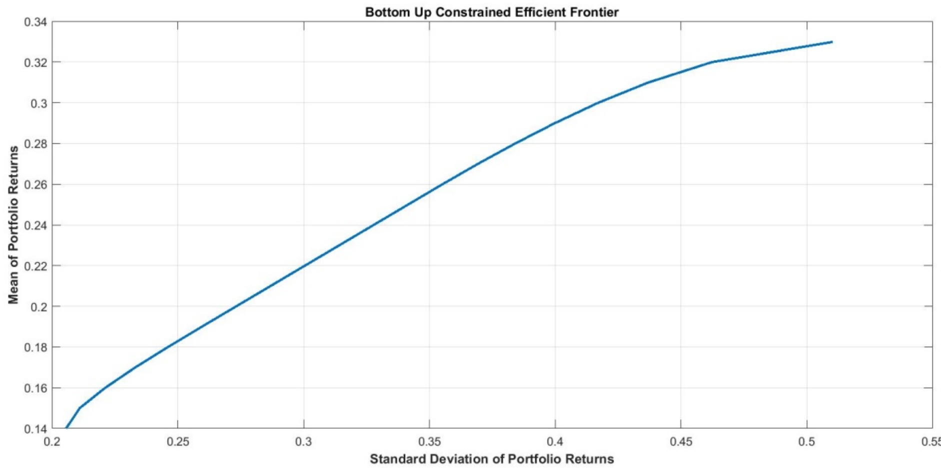

Figure 31: Bottom-Up Constrained Frontier

Figure 31 shows the efficient frontier of the portfolios constructed using the bottom-up approach.

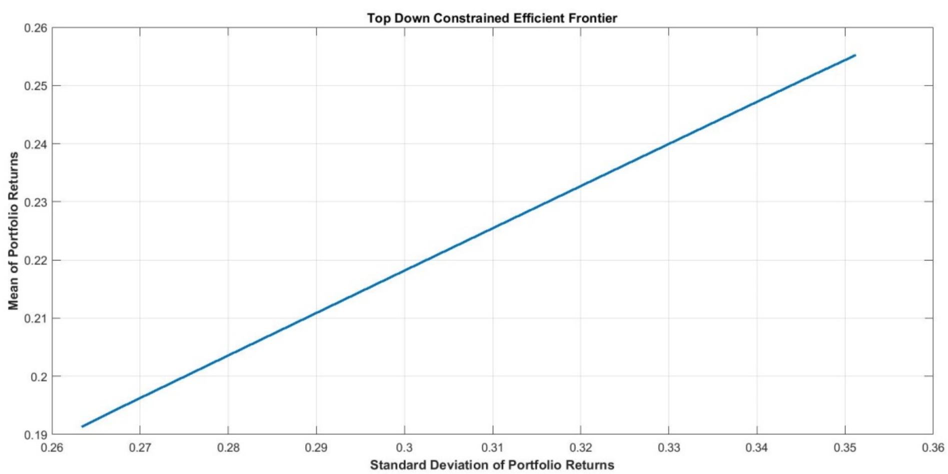

Figure 32: Top-Down Constrained Frontier

Figure 32 shows the efficient frontier of the portfolios constructed using the top-down approach.

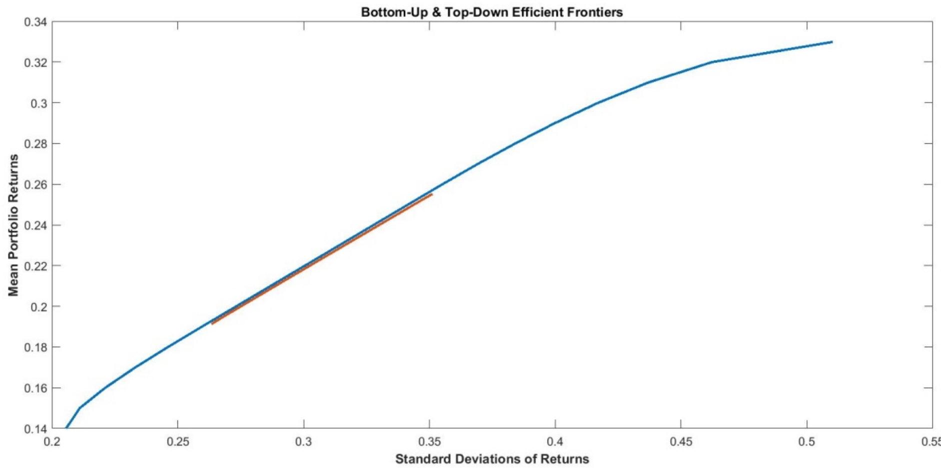

Figure 33: Top-Down and Bottom-Up Constrained Frontier

Figure 33 shows a result similar to that from the previous section, namely that the Bottom-Up portfolios produce portfolios with a wider range of returns than the Top-Down approach. This is despite limiting the portfolios to long only, non-leveraged portfolios constrained to the limits of Table 18.

This result confirms the results of the previous section, namely that constructing the portfolios using the Bottom-Up approach allows the investor to obtain an efficient frontier which matches the Top-Down investors but is extended. That is it allows the investor to obtain efficient portfolios which the Top-Down approach does not.

8.2.3 Bottom-Up Portfolios In A Top-Down World

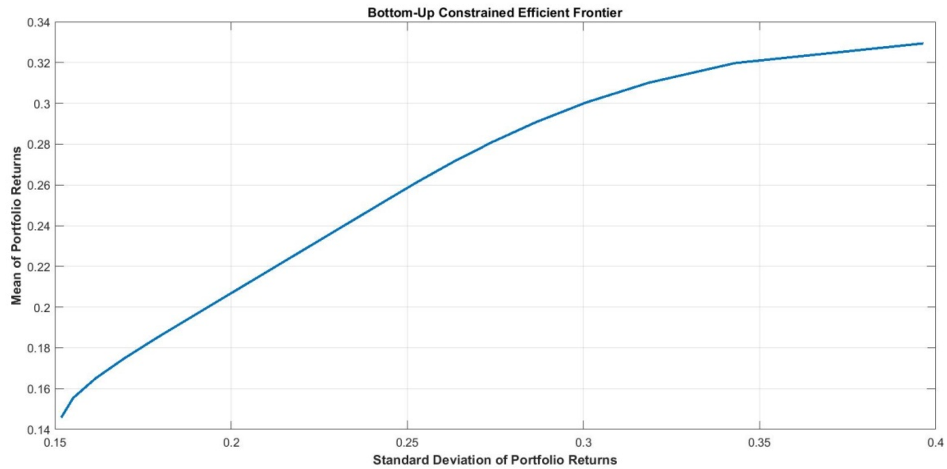

Figure 34: Bottom-Up Constrained Black-Litterman Frontier

Figure 34 shows the efficient frontier of Black-Litterman portfolios which are constrained and constructed using the bottom-up approach.

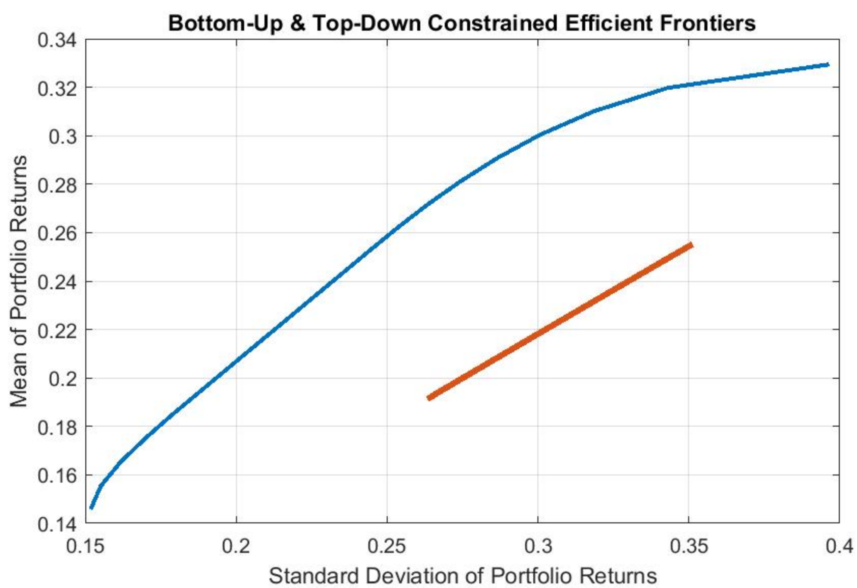

Figure 35: Bottom-Up Constrained Black-Litterman And Top-Down Constrained Frontiers

Figure 35 shows both the bottom-up and top-down efficient frontiers where constraints are put in place. Where there are constraints the bottom-up approach results in more efficient portfolios.

Bottom-Up Constrained BL			Top-Down Constrained			Allocation Differences		
JSAPY	Top40	Govi	JSAPY	Top40	Govi	JSAPY	Top40	Govi
20.0%	40.0%	40.0%	20.0%	40.0%	40.0%	0.0%	0.0%	0.0%
20.0%	40.0%	40.0%	16.2%	44.4%	39.4%	-3.8%	4.4%	-0.6%
20.0%	40.0%	40.0%	16.6%	45.1%	38.3%	-3.4%	5.1%	-1.7%
18.6%	41.4%	40.0%	17.0%	45.8%	37.2%	-1.6%	4.4%	-2.8%
14.5%	45.5%	40.0%	17.5%	46.5%	36.1%	3.0%	0.9%	-3.9%
12.2%	49.0%	38.8%	17.9%	47.1%	35.0%	5.7%	-1.9%	-3.8%
12.7%	51.5%	35.8%	18.3%	47.8%	33.9%	5.6%	-3.6%	-1.9%
13.3%	53.9%	32.8%	18.7%	48.5%	32.8%	5.4%	-5.4%	0.0%
13.8%	56.4%	29.8%	19.1%	49.2%	31.7%	5.3%	-7.2%	1.9%
14.3%	58.9%	26.7%	19.5%	49.9%	30.6%	5.2%	-9.0%	3.9%
16.3%	60.0%	23.7%	19.9%	50.6%	29.5%	3.6%	-9.4%	5.8%
19.3%	60.0%	20.7%	20.3%	51.3%	28.4%	1.0%	-8.7%	7.7%
20.0%	60.0%	20.0%	20.8%	52.0%	27.3%	0.8%	-8.0%	7.3%
20.0%	60.0%	20.0%	21.2%	52.6%	26.2%	1.2%	-7.4%	6.2%
20.0%	60.0%	20.0%	21.6%	53.3%	25.1%	1.6%	-6.7%	5.1%
20.0%	60.0%	20.0%	22.0%	54.0%	24.0%	2.0%	-6.0%	4.0%
20.0%	60.0%	20.0%	22.4%	54.7%	22.9%	2.4%	-5.3%	2.9%
20.0%	60.0%	20.0%	22.8%	55.4%	21.8%	2.8%	-4.6%	1.8%
22.4%	57.6%	20.0%	23.2%	56.1%	20.7%	0.8%	-1.5%	0.7%
20.0%	60.0%	20.0%	20.0%	60.0%	20.0%	0.0%	0.0%	0.0%

Table 19: Bottom-Up Constrained Black-Litterman And Top-Down Allocations

9 RESEARCH CONCLUSIONS

Development Finance Institutions often find themselves facing the task of investing large portfolios of assets across multiple asset classes in order to meet a financial objective. The task may be to manage a large endowment that is meant to finance a development agenda such as the education of poor children or the fight against their malnutrition. Whatever the institution's development agenda, their task will be to allocate their funds to a range of securities across numerous asset classes.

This dissertation deals with how best these institutions can construct their portfolios. Two approaches are considered. The first is what we call the *top-down* approach where the institution begins by deciding the portfolios broad allocation to the asset classes. The allocations to the different asset classes are then managed independently, often by a different portfolio manager per asset class.

The second approach is what we call the *bottom-up* approach. This approach considers all the securities available to the institution , regardless of the asset class they belong to, and finds the combination of securities that would result in the highest return given some target risk.

Formally, the aim of this dissertation is to answer the question:

- Are multi-asset class portfolios constructed using the bottom-up approach more efficient than those constructed using the top-down approach?

The dissertation begins by outlining the methodologies used to answer this question. We discuss the financial data used to answer this question as well as the methods used to deal with the commonly encountered problem of missing data. We then discuss the two portfolio selection models used to answer this question, the mean-variance optimization model as well as the Black-Litterman model. The proper way to apply these models to the equity and bond markets are then illustrated.

We find that when constructing unconstrained portfolios, the Bottom-Up approach produces an efficient frontier which is similar to the Top-Down approach but is extended. The Bottom-Up efficient frontier allows investors to obtain efficient portfolios which have a lower volatility or a higher return than that of the Top-Down approach. The reason for this is that the bottom-up approach allows the investor to choose portfolios that the top-down approach does not. By considering all the assets available to the investor at the same time, regardless of asset class, the bottom-up approach can create portfolios that the top-down approach cannot.

We then constrained both the Top-Down and Bottom-Up portfolios such that they would not be able to leverage or short any assets and the asset class exposures were within a minimum and maximum as outlined in Table 19. This was to be closer to the conditions investors would face in reality. The results here mirrored those of the Unconstrained case. The Bottom-Up efficient frontier again allows investors to achieve higher expected returns and lower volatility portfolios than the Top-Down approach.

Our last case sort to replicate the fact that Top-Down investors make active bets within their asset class portfolios. The Black-Litterman model was used for this case as it is more suited to active management. Here we found that the Bottom-Up approach produced an efficient frontier which is higher than the Top-Down approach obtains. Our conclusion is that even where an investor must play according to Top-Down construction rules, constructing their portfolios in totality rather than piecewise allows them to obtain more efficient portfolios.

These findings have a direct impact on the manner Development Finance Institutions and other investors should allocate their portfolios. By considering all assets at the same time the investors can access portfolios they would otherwise be unable to. These are portfolios which may be more appropriate to solve the investors problems. Even where the investor is subjected to constraints so that he may not have too much or too little exposure to a particular asset class, we find that the bottom-up approach will allow the investor to access efficient portfolios which are less volatile than the those created by the top-down approach. The bottom-up approach will also allow the investor to access efficient portfolios which have higher expected returns than the top-down portfolios.

10 REFERENCES

Aracioglu, B. a. D. F. a. S. H., 2011. Mean-Variance-Skewness-Kurtosis Approach to Portfolio Optimization: An Application in Instanbul Stock Exchange.. *Ege Academic Review,* Volume 11, pp. 9-17.
Best, M. J. a. G. R. R., 1991. On the Sensitivity of Mean-Variance-Efficient Portfolios to Changes In Asset Means: Some Analytical and Computational Results.. *Review of Financial Studies,* 4(2), pp. 315-342.
Bevan, A. a. W. K., 1998. *Using The Black-Litterman Global Asset Allocation Model: Three Years of Practical Experience.,* s.l.: s.n.
Black, F. a. L. R., 1992. Global Portfolio Optimization. *Financial Analysts Journal,* Volume September-October, pp. 28--43.
Brinson, G. P. a. H. R. a. B. G. L., 1995. Determinants of Portfolio Performance. *Financial Analyst Journal,* pp. 133-138.
Chen, A. a. F. F. a. H. D., 2012. Portfolio Revision Under Mean-Variance and Mean-CVaR With Transaction Costs. *Reivew of Quantitative Finance and Accounting,* 39(4), pp. 509-526.
Da Silva, A. a. L. W. a. P. B., 2009. The Black-Litterman Model For Active Portfolio Management. *The Journal Of Portfolio Management,* Winter 2009(1), pp. 61-70.
Fabozzi, F. J. a. F. S. M. a. K. P. N., 2006. Incorporating Trading Strategies in The Black-Litterman Framework. *Journal of Trading,* 1(2), pp. 28-37.
FTSE Russel, 2016. *FTSE/JSE Top40 Index,* s.l.: s.n.
He, G. a. L. B., 2002. *The Intuition Behind Black-Litterman Model Portfolios.,* s.l.: s.n.
Mankert, C. a. S. M., 2011. Mathematical Derivations and Practical Implications for the Use of the Black-Litterman Model. *Journal of Real Estate Portfolio Management,* 17(2), pp. 139-159.
Markowitz, H., 1952. Portfolio Selection. *The Journal Of Finance,* 7(1), pp. 77-91.
Martellini, L. a. Z. V., 2007. Extending Black-Litterman Analysis Beyond the Mean-Variance Framework. *The Journal of Portfolio Management,* 33(4), pp. 33-44.
Meucci, A., 2005. *Risk and Asset Allocation.* Berlin: Springer-Verlag.
Meucci, A., 2009. Enhancing The Black-Litterman and Related Approaches: Views and Stress-test on Risk Factors. *Journal of Asset Management,* 10(2), pp. 89-96.
Meucci, A., 2010. *The Black-Litterman Approach: Original Model and Extensions,* s.l.: s.n.
Michaud, R. O., 1989. The Markowitz Optimization Enigma: Is Optimized Optimal?. *Financial Analyst Journal,* Volume 45, pp. 31-42.
Miskolczi, P., 2016. Differences Between Mean-Variance and Mean-CVAR Portfolio Optimization Models. *Annals of the University of Oradea,* 25(1), pp. 548-557.
Rockafellar, R. T. a. U. S., 2000. Optimization of Conditional VaR. *Journal of Risk,* Volume 2, pp. 21-41.
Satchell, S. a. S. A., 2000. A Demystification of Tthe Black-Litterman Model: Managing Quantitative and Traditional Portfolio Construction. *Journal of Asset Management,* 1(2), pp. 138-150.
Stambaugh, R. F., 1997. Analyzing Investments Whose Histories Differ In Length. *Journal of Financial Economics,* Volume 45, pp. 285-331.
Yilmaz, H., 2015. Mean-CVaR Optimization of Options Portfolios Using Quantile Regression: An Application to Dispersion Trading.. *Journal of Applied Financial Research,* Volume 1, pp. 35-51.

Zhou, G., 2009. Beyond Black-Litterman: Letting The Data Speak. *Journal of Portfolio Management,* Volume Fall, pp. 36-45.