Technische Universität Dresden
Fachrichtung Mathematik

Institut für Geometrie

Pure Measures, Traces and a General Theorem of Gauß

Dissertation
zur Erlangung des akademischen Grades

Doctor rerum naturalium
(Dr. rer. nat.)

vorgelegt von

Name:	Schönherr	Vorname:	Moritz
geboren am:	29.06.1987	in:	Anklam

Tag der Einreichung: 18.08.2017

Betreuer: Prof. Dr. rer. nat. habil. Friedemann Schuricht
(Technische Universität Dresden)

Danksagung

Das Verfassen einer Dissertation und alle damit verbundenen Schritte sind ein langfristiges Projekt, welches ohne die richtigen Rahmenbedingungen nicht umsetzbar wäre. Deshalb möchte ich an dieser Stelle all jenen danken, die mich in den letzten Jahren dabei unterstützt haben. Zu allererst möchte ich meinem Betreuer Prof. Friedemann Schuricht danken, sowohl für seine fachliche Unterstützung und Leitung, als auch für die Herstellung der finanziellen Rahmenbedingung. Mit seinem andauernden Interesse und vielen fachlichen Gesprächen hat er die vorliegende Arbeit in eine gute Richtung gelenkt. Prof. Marco Degiovanni möchte ich dafür danken, dass er sich für das langwierige Begutachten der Arbeit Zeit nimmt. Die Stellensituation an deutschen Universitäten ist nicht immer einfach für den Nachwuchs. Mein Vater hat mich in schwierigen Phasen finanziell unterstützt und mir die Fortsetzung meiner Arbeit ermöglicht. Ohnehin hat die frühe Prägung durch meine Eltern einen wesentlichen Teil dazu beigetragen, dass ich ebenso wie meine Schwester Sarah die Mathematik als Studienfach gewählt habe. Ihr möchte ich für Hilfe bei der Korrektur danken. Meiner lieben Rebekka will ich für Motivation in zähen Phasen, die Unterstützung im Alltag und die vielen schönen Momente danken, die den Trott des Schreibens unterbrochen haben. Als Ingenieurin konnte sie mir viele Dinge, die losgelöst von der konkreten Anwendung schienen in einen angewandten Kontext rücken und half mir bei der Korrektur der Arbeit. Darüber hinaus gibt es viele Personen, die mir geholfen haben. Nicht zuletzt mein Kollege Jan Mankau, durch viele fachliche Gespräche und kritische Nachfragen, meine Freunde Almuth und Arian durch Korrekturhilfen, alle meine Schwestern durch ermutigende Worte und alle Freunde und Verwandte, mit denen ich die Zeit verleben durfte.

Contents

List of Figures

Introduction

The Divergence Theorem is a well-known result in mathematics. One of the first appearances of a statement that translates the integral over a volume into an integral over the bounding surface of said volume can be found in a treatise of Gauß (cf. [22]). Nevertheless, it is assumed that Lagrange already knew of a similar technique. The first formal proof of a special case of this theorem is attributed to Ostrogradsky. Throughout the nineteenth century many famous scientists, including Green, proved increasingly general forms of this statement (cf. [26]).

The continued interest in this theorem results from its many applications in mathematics and science. For a physicist it arises e.g. in the context of conservative equations, i.e. the conservation of mass and energy. Early applications already included magnetism, heat transfer and elastic bodies (cf. [26]). Mathematicians use it for partial integration in higher dimensions and the analysis of partial differential equations. In Continuum Mechanics, it describes a balance of forces.

Since the inception of the Theorem of Gauß, as it is called by German mathematicians, there have been attempts to generalise its statement to more abstract settings. On the one hand, it is desirable to integrate on very general domains which do not need to have a well-defined normal vector at each point on the boundary (cf. [5]). On the other hand, applications in mechanics often need to employ the theorem for highly non-smooth vector fields. Especially in the field of mechanical engineering, where e.g. cogs exert forces concentrated on lines or even points on each other, vector fields whose distributional divergence is a measure arise naturally. These phenomena were already known to Heinrich Hertz. The notion of Hertzian Contact Stress was introduced to sidestep the problem of dealing with these concentrated loads (cf. [25]). A general Divergence Theorem which is capable of describing these situations would enable a more rigorous analysis of these problems.

The Divergence Theorem in its simplest form is stated for smooth vector fields F on domains $\Omega \subset \mathbb{R}^n$ with smooth boundary. It has the following form

$$\int_\Omega \operatorname{div} F \,\mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} F \cdot \nu \,\mathrm{d}\mathcal{H}^{n-1}\,,$$

where ν is the outward pointing normal vector of Ω.

The challenges in generalising this statement are threefold. First and foremost, the volume integral on the left-hand side only makes sense if the divergence of F is an integrable function with respect to Lebesgue measure. It is a well-established fact that this integral can be exchanged for $\operatorname{div} F(\Omega)$ in the case where $\operatorname{div} F$ is a Radon measure on Ω (cf. [8]). Second, the integral on the right-hand side of the equation needs some notion of normal vector ν to the set. This normal vector exists $\mathcal{H}^{n-1}$-almost everywhere for sets Ω of finite perimeter (cf. [20], [2]). For domains with possibly infinite perimeter, a substitute is yet to be found. Third and last, for the area integral to be meaningful, the vector field F must be integrable with respect to area measure. For F that do not fulfill this requirement, multiple strategies can be found in the literature.

One way is to compute the normal trace as an essentially bounded function on the boundary via mollification ([13]). This approach has the drawback that geometry and the information encoded in the vector field are combined, thus making the interpretation of the trace itself more difficult. Other techniques exchange the area integral with a continuous linear functional on a function space on $\partial\Omega$, but do not provide a representation of this functional as an integral ([32],[10]). Most of the results found in the literature hold true for essentially bounded vector fields having divergence measure (cf [12], [10]). In [31] it is shown that the area integral can be substituted by

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{\Omega_\delta \setminus \Omega} F \cdot D(\mathrm{dist}_\Omega) \, \mathrm{d}\mathcal{L}^n$$

even in the case where F is only integrable and Ω is an arbitrary closed set.

In the literature, many of the cases mentioned above have been discussed in detail. A prominent source is the paper by Anzelotti [3], where vector fields with integrable divergence and sets with Lipschitz boundary are considered. The case of essentially bounded vector fields having divergence measure on sets with Lipschitz deformable boundary has been discussed by Chen and Frid in [8] and [9]. In [10], Chen and Frid proved a Gauß formula for vector measures having divergence measure and sets with Lipschitz deformable boundary. Sets of finite perimeter and essentially bounded vector fields have been discussed by Chen and Torres in [12]. Their trace is an essentially bounded function on the boundary and is obtained by mollification. In [32], Silhavy proved a Gauß Theorem for open sets and measures having divergence measure, the normal trace being a functional on the Lipschitz continuous functions on the boundary. Schuricht [31] investigated arbitrary closed sets and unbounded divergence measure fields and proved the limit formula given above. The listed sources also contain a large part of the theory for vector fields having divergence measure. In this thesis, Evans [20] and Ambrosio [2] are used for the theory of functions of bounded variation. A good compilation of important results on finitely additive set functions can be found in Rao [30]. Other important sources are Alexandroff [1], Leader [28], Bochner [6], Dunford [18], Yosida [33] and Kolmogoroff [27].

The main question addressed in this thesis is: Is it possible to generalise the area integral in the Divergence Theorem in such a way that integral calculus is available for the area part, even for domains with unbounded perimeter? This is investigated for essentially bounded vector fields having divergence measure as well as for unbounded vector fields. It is shown that this is possible using so-called pure measures, which are necessarily only finitely additive. The properties of these measures are analysed in detail.

Consequently, the structure of this thesis is as follows.

In the fist chapter, a theory of finitely additive measures is laid out. Since some results from lattice theory are needed in the course of the analysis, they are presented at the beginning of this chapter. An important result on successive decomposition of lattices into normal sublattices is proved. Afterwards, the basic definitions of measure theory are recalled and pure measures are introduced. Concrete examples for these measures were essentially only known on $\mathbb{N}$ up to now. A new example of a pure measure on $\mathbb{R}^n$ is presented, which is in essence the density of a set at zero. Using a slightly adapted notion of support of a measure, a sufficient condition for a measure to be pure is derived.

The second chapter covers the theory of integration for the measures introduced in the previous chapter. Using the sublattice decomposition technique, an improved characterisation of the dual of the space of essentially bounded functions is given. As the spaces of p-integrable functions with respect to a finitely additive measure are not necessarily complete, the second section presents the completions of these spaces and the corresponding dual spaces in a concise form.

In the third chapter, it is shown that the new example for pure measures is prototypical in the sense that many measures share its structure. These new measures are called density measures. The space of all density measures of a closed set is introduced and its extremal points are analysed. It is shown that the latter are extensions of the Dirac measure to essentially bounded functions and that they concentrate along one-dimensional directions. Furthermore, a direct correspondence of density measures and σ-measures which are singular with respect to Lebesgue measure is shown.

The fourth chapter contains an exposition on functions of bounded variation and vector fields having divergence measure, which facilitates the proof of Gauß formulas later on. The first section on functions of bounded variation contains an important proposition on mollification of sets having finite perimeter. The section on vector fields having divergence measure contains several useful product formulas which are repeatedly used in the subsequent analysis.

The main results on Gauß formulas are given in the last chapter. The first section contains a general Divergence Theorem for sets of finite perimeter and essentially bounded vector fields. In particular, the existence of so-called normal measures is proved and some of their properties are presented. It is shown that, in general, unbounded vector fields cannot be integrated with respect to these normal measures. The second part of the last chapter contains a Theorem of Gauß for unbounded vector fields having divergence measure and bounded open sets with path-connected boundary. This theorem gives the normal trace of Silhavy (cf. [32]) a representation as the sum of a Radon measure and a finitely additive measure. The analysis conveys an interesting new measure, which vanishes in the regular case.

Chapter 1

Theory of Finitely Additive Measures

This chapter contains a basic theory of finitely additive measures and some useful tools from lattice theory. The first section presents these tools and a proposition on successive decomposition of vector lattices into sublattices. The spaces of measures defined in the subsequent section turn out to be boundedly complete vector lattices. This enables the decomposition of measures which are weakly absolutely continuous with respect to Lebesgue measure into pure and σ-additive parts. In the literature, explicit examples of pure measures can essentially be found only on $\mathbb{N}$ (cf. [30]). Here, a new example on $\mathbb{R}^n$ is given. This example is essentially the density of a set at a point. In Chapter 3 this enables the identification of a large class of pure measures. In addition, a new notion of support of a measure is introduced, called core. This is necessary because pure measures can have their core outside of the set on which they live. It turns out that every weakly absolutely continuous measure whose core has Lebesgue measure zero is necessarily pure.

1.1 Lattice Theory

First, some results on vector lattices are gathered. These are useful in the decomposition of finitely additive measures. This decomposition technique was used in special cases by Alexandroff (cf. [1]) and Yosida (cf. [33]). By embedding it into a lattice setting, the technique becomes much more tractable (cf. [30]). The following exposition is a very short summary of the relevant statements. A general treatment can be found in Birkhoff [4].

First, the basic definitions for vector lattices from Rao [30, p. 24ff] (cf.

[4, p. 347]) is given.

Definition 1.1. *Let L be a vector space and $\leq$ a partial order on L which is compatible with $+$ and the multiplication with a scalar on L. If for all $l_1, l_2 \in L$ the supremum and infimum of $\{l_1, l_2\}$ exist, then L is called a* ***vector lattice****. For $l, l_1, l_2 \in L$ write*

$$\begin{aligned} l_1 \vee l_2 &:= \sup\{l_1, l_2\} \\ l_1 \wedge l_2 &:= \inf\{l_1, l_2\} \\ l^+ &:= l \vee 0 \\ l^- &:= -l \vee 0 \\ |l| &:= l^+ + l^- \end{aligned}$$

$l_1, l_2 \in L$ are called ***orthogonal****, if $|l_1| \wedge |l_2| = 0$, written $l_1 \perp l_2$. If for a family $\{l_i\}_{i\in\mathcal{I}} \subset L$ the supremum exists, write*

$$\bigvee_{i\in\mathcal{I}} l_i := \sup_{i\in\mathcal{I}} l_i \,.$$

If the infimum of $\{l_i\}_{i\in\mathcal{I}}$ exists, it is denoted by

$$\bigwedge_{i\in\mathcal{I}} l_i := \inf_{i\in\mathcal{I}} l_i \,.$$

A set $L' \subset L$ is called ***bounded from above****, if there exists $l \in L$, such that $l' \leq l$ for all $l' \in L'$.*

A vector lattice is called ***boundedly complete****, if for every $\{l_i\}_{i\in\mathcal{I}} \subset L$ which is bounded from above the supremum $\bigvee_{i\in\mathcal{I}} l_i$ exists.*

For a vector lattice L and $l_1, l_2 \in L$

$$|l_1 + l_2| \leq |l_1| + |l_2|$$

with equality if $l_1 \perp l_2$ (cf. [30, p. 25]). The following example foreshadows the partial order that turns spaces of measures into vector lattices.

Example 1.2. Let M be any set. Let L be the set of all functions

$$f : M \to \mathbb{R}$$

then there is a natural partial order on L turning M into a vector lattice, i.e.

$$f_1 \leq f_2 \iff f_1(x) \leq f_2(x) \text{ for all } x \in M \,.$$

In the following, L denotes a boundedly complete vector lattice.

In order to obtain results for an orthogonal decomposition of vector lattices (and their elements), one has to define appropriate sub-structures (cf. [30, p. 28]).

Definition 1.3. *A linear subspace L' of L is called a* ***sublattice*** *of L if $l_1 \vee l_2 \in L'$ and $l_1 \wedge l_2 \in L'$ for all l_1, l_2 in L'.*

A sublattice L' of L is called ***normal****, if*

1. *for all $l' \in L'$ and all $l \in L$*

$$|l| \leq |l'| \implies l \in L'$$

2. *if for $\{l_i\}_{i \in \mathcal{I}} \subset L'$ the supremum exists in L, then $\bigvee_{i \in \mathcal{I}} l_i \in L'$.*

In order to decompose a vector lattice into normal sublattices, a notion of orthogonality is needed (cf. [30, p. 29]).

Definition 1.4. *For a subset L' of L, the set*

$$(L')^\perp := \{l \in L \mid \forall l' \in L' : l \perp l'\}$$

is called ***orthogonal complement*** *of L'.*

The following statements from [30, p. 29f] illustrates that normal sublattices and orthogonality interact in a similar way as closed linear subspaces and orthogonality in Hilbert spaces do.

Proposition 1.5. *Let $S \subset L$, then $S^\perp$ is a normal sublattice of L. If S is a normal sublattice, then $(S^\perp)^\perp = S$.*

A useful characterisation of the orthogonal complement of a normal sublattice is the following.

Proposition 1.6. *Let S be a normal sublattice of L. Then $l \in S^\perp$ if and only if for every $s \in S$*

$$0 \leq |s| \leq |l| \implies s = 0\,.$$

Proof. Assume first that $l \in S^\perp$. Then for every $s \in S$

$$0 \leq |s| \leq |l| \implies 0 = |s| \wedge |l| = |s| \implies s = 0\,.$$

Now assume for every $s \in S$

$$0 \leq |s| \leq |l| \implies s = 0\,.$$

Since S is a normal sublattice

$$0 \leq |s| \wedge |l| \leq |s| \implies |s| \wedge |l| \in S\,.$$

By assumption

$$|s| \wedge |l| \leq |l| \implies |s| \wedge |l| = 0\,.$$

Thus $s \perp l$. $\square$

As in the setting of Hilbert spaces, a boundedly complete vector lattice can be represented as the direct sum of a normal sublattice and its orthogonal complement (cf. [30, p. 29]).

Proposition 1.7. *Riesz Decomposition Theorem*
Let S be a normal sublattice of L, then for every $l \in L$ there exist unique elements $s \in S, s^\perp \in S^\perp$ such that

$$l = s + s^\perp\,.$$

Furthermore, if $l \geq 0$, then $s = \bigvee_{s' \in S} l \wedge |s'|$. For general $l \in L$

$$s = \bigvee_{s' \in S} l^+ \wedge |s'| - \bigvee_{s' \in S} l^- \wedge |s'|\,.$$

The following proposition enables the successive decomposition of a lattice into sublattices. This is used in the analysis of measures. In particular, this proposition enables a better characterisation of the dual of the space of essentially bounded functions.

Proposition 1.8. *Let L_1, L_2 be two normal sublattices of L. Then $L_1 \cap L_2$ is a normal sublattice of L_2. Furthermore, the orthogonal complement of $L_1 \cap L_2$ in L_2 is $L_1^\perp \cap L_2$.*

Proof. Let $l_1 \in L_1 \cap L_2$ and $l_2 \in L_2$ with

$$|l_2| \leq |l_1|\ .$$

Since L_1 is a normal sublattice of L,

$$l_2 \in L_1\,.$$

Whence $l_2 \in L_1 \cap L_2$.

Now, let $\{l_i\}_{i \in \mathcal{I}} \subset L_1 \cap L_2$ be such that $\bigvee_{i \in \mathcal{I}} l_i \in L$. Since L_1 and L_2 are normal,

$$\bigvee_{i \in \mathcal{I}} l_i \in L_1 \quad \text{and} \quad \bigvee_{i \in \mathcal{I}} l_i \in L_2\,.$$

This implies $\bigvee_{i \in \mathcal{I}} l_i \in L_1 \cap L_2$. Thus $L_1 \cap L_2$ is a normal sublattice of L_2.

Let $l_2 \in L_2$ such that $l_2 \in (L_1 \cap L_2)^\perp$. Since L_1 is a normal sublattice of L, there exist $l_1 \in L_1, l_1^\perp \in L_1^\perp$ such that $l_2 = l_1 + l_1^\perp$. Now, using additivity of the total variation on orthogonal elements (cf. [30, p. 25])

$$0 \leq \sup\{|l_1|, |l_1^\perp|\} \leq |l_1| + |l_1^\perp| = |l_2|\,.$$

Hence, $l_1, l_1^\perp \in L_2$ and $l_1, l_1^\perp \in (L_1 \cap L_2)^\perp$. Since $l_2 \in (L_1 \cap L_2)^\perp$,

$$0 = |l_2| \wedge |l_1| = |l_1| \wedge |l_1| + |l_1| \wedge |l_1^\perp| = |l_1| \wedge |l_1|\,.$$

This implies $l_1 = 0$. Hence

$$(L_1 \cap L_2)^\perp \subset L_1^\perp \cap L_2\,.$$

On the other hand, if $l_1^\perp \in L_1^\perp \cap L_2$, then for all $l_1 \in L_1 \cap L_2$

$$|l_1| \wedge |l_1^\perp| = 0\,,$$

whence

$$L_1^\perp \cap L_2 \subset (L_1 \cap L_2)^\perp\,.$$

□

1.2 Finitely Additive Measures

In the following, a self-contained exposition of a theory of measures is presented. Furthermore, the new example for pure measures is given and the notion of the core of a measure is used to characterise the pure measures. Many of the following statements hold true for arbitrary topological spaces Ω. Nevertheless, in the following let $n \in \mathbb{N}_{>0}$ and $\Omega \subset \mathbb{R}^n$ with the usual relative topology.

The following definition of measures is an adapted version of the definition of charges given in Rao [30, p. 35].

Definition 1.9. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{M} \subset 2^\Omega$ and $\mu : \mathcal{M} \to \mathbb{R}$. Then μ is called **measure** on Ω with respect to $\mathcal{M}$, if for all $M_1, M_2, ..., M_m \in \mathcal{M}$ such that $M_i \cap M_j = \emptyset$ for $i \neq j$ and $\bigcup_{k=0}^{m} M_k \in \mathcal{M}$*

$$\mu\left(\bigcup_{k=0}^{m} M_k\right) = \sum_{k=0}^{m} \mu(M_k)\,.$$

If for all $\{M_k\}_{k\in\mathbb{N}} \subset \mathcal{M}$ *such that* $M_i \cap M_j = \emptyset$ *for* $i \neq j$ *and* $\bigcup_{k\in\mathbb{N}} M_k \in \mathcal{M}$

$$\mu\left(\bigcup_{k\in\mathbb{N}} M_k\right) = \sum_{k=0}^{\infty} \mu(M_k)$$

then μ *is called* ***σ-measure****.*

Note that this entails the unconditional convergence of the series on the right hand side.

A set function $\mu : \mathcal{M} \to \mathbb{R}$ *is called* ***bounded****, if*

$$\sup_{M\in\mathcal{M}} |\mu(M)| < \infty\,.$$

Remark 1.10. One could also take $m \in \mathbb{N}$, $m > 1$ and

$$\mu : \mathcal{M} \to \mathbb{R}^m$$

in the above definitions.

In order to obtain a vector space structure on the set of measurable functions, mainly systems of sets of the following types are considered (cf. [30, p. 2]).

Definition 1.11. *Let* $\Omega \subset \mathbb{R}^n$*. Then* $\mathcal{A} \subset 2^\Omega$ *is called* ***algebra****, if for all* $A_1, A_2 \in \mathcal{A}$*, the sets* $A_1 \cap A_2 \in \mathcal{A}$*,* $A_1^c \in \mathcal{A}$ *and* $A_1 \cup A_2 \in \mathcal{A}$ *and* $\emptyset \in \mathcal{A}$*.*

If in addition for all $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}$ *the set* $\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{A}$*, then* $\mathcal{A}$ *is called* ***σ-algebra****.*

In the following, $\mathcal{A}$ denotes an algebra on Ω. The spaces of measures considered in this thesis are defined in accordance with [30].

Definition 1.12. *Let* $\Omega \subset \mathbb{R}^n$ *and* $\mathcal{A} \subset 2^\Omega$ *be an algebra. The set of all bounded measures* $\mu : \mathcal{A} \to \mathbb{R}$ *is denoted by*

$$\mathrm{ba}(\Omega, \mathcal{A})\,.$$

The set of all bounded σ*-measures* $\sigma : \mathcal{A} \to \mathbb{R}$ *is denoted by*

$$\mathrm{ca}(\Omega, \mathcal{A})\,.$$

There is a natural partial order on $\mathrm{ba}(\Omega, \mathcal{A})$ (cf. [30, p. 43]).

Definition 1.13. *Let* $\Omega \subset \mathbb{R}^n$ *and* $\mathcal{A} \subset 2^\Omega$ *be an algebra. For* $\mu, \lambda \in \mathrm{ba}(\Omega, \mathcal{A})$ *one writes*

$$\mu \leq \lambda$$

if and only if for every $A \in \mathcal{A}$

$$\mu(A) \leq \lambda(A)\,.$$

The following proposition links the theory of measures with the theory of boundedly complete vector lattices. This is essential for the subsequent results on the decomposition of measures. The proposition is taken from [30, p. 43f].

Proposition 1.14. *Let* $\Omega \subset \mathbb{R}^n$ *and* $\mathcal{A} \subset 2^\Omega$ *be an algebra. Then* $\mathrm{ba}(\Omega, \mathcal{A})$ *together with the partial order* $\leq$ *is a boundedly complete vector lattice.*

The following definitions are standard in measure theory (cf. [30, p. 45]).

Definition 1.15. *Let* $\Omega \subset \mathbb{R}^n$ *and* $\mathcal{A} \subset 2^\Omega$ *be an algebra. For* $\mu \in \mathrm{ba}(\Omega, \mathcal{A})$ *define*

$$\begin{aligned}
&\mu^+ := \mu \vee 0 = \sup\{\mu, 0\} \\
&\mu^- := (-\mu) \vee 0 = \sup\{-\mu, 0\} \\
&|\mu| := \mu^+ + \mu^-\,.
\end{aligned}$$

Call μ^+ ***positive part*** *of* μ, μ^- ***negative part*** *of* μ *and* $|\mu|$ ***total variation*** *of* μ.

Furthermore, for $A \in \mathcal{A}$ *define* $\mu\lfloor A : \mathcal{A} \to \mathbb{R}$ *by*

$$(\mu\lfloor A)(A') := \mu(A \cap A') \text{ for all } A' \in \mathcal{A}\,.$$

The total variation can be characterised in the following way (cf. [30, p. 46]).

Proposition 1.16. *Let* $\Omega \subset \mathbb{R}^n$ *and* $\mathcal{A} \subset 2^\Omega$ *be an algebra. Then for every* $\mu \in \mathrm{ba}(\Omega, \mathcal{A})$ *and* $A \in \mathcal{A}$

$$|\mu|(A) = \sup \sum_{k=1}^{m} |\mu(A_k)|\,.$$

where the supremum is taken over all finite partitions $\{A_k\}_{k=0}^m \subset \mathcal{A}$ *of* A.

The following proposition can be found in Rao [30, p. 44]. It states that in the space of bounded measures, the norm is compatible with the partial order.

Proposition 1.17. *Let* $\Omega \subset \mathbb{R}^n$ *and* $\mathcal{A} \subset 2^\Omega$ *be an algebra. Then* $\mathrm{ba}(\Omega, \mathcal{A})$ *together with* $\leq$ *and the norm*

$$\|\mu\| := |\mu|(\Omega) \text{ for } \mu \in \mathrm{ba}(\Omega, \mathcal{A})$$

is a Banach lattice, i.e. it is a Banach space and a vector lattice such that for all $\mu, \lambda \in \mathrm{ba}(\Omega, \mathcal{A})$

$$|\mu| \leq |\lambda| \implies \|\mu\| \leq \|\lambda\| .$$

The following proposition is an application of Riesz's decomposition Theorem (Proposition 1.7) (cf. [30, p. 241]). In particular, every bounded measure can be uniquely decomposed into a σ-measure and a pure measure. Recall the definition of orthogonal complement from page 7.

Proposition 1.18. *Let* $\Omega \subset \mathbb{R}^n$ *and* $\mathcal{A} \subset 2^\Omega$ *be an algebra. Then* $\mathrm{ba}(\Omega, \mathcal{A})$ *is a boundedly complete vector lattice and* $\mathrm{ca}(\Omega, \mathcal{A})$ *one of its normal sublattices. Hence, every* $\mu \in \mathrm{ba}(\Omega, \mathcal{A})$ *can uniquely be decomposed into* $\mu_c \in \mathrm{ca}(\Omega, \mathcal{A})$ *and* $\mu_p \in \mathrm{ca}(\Omega, \mathcal{A})^\perp$ *such that*

$$\mu = \mu_c + \mu_p$$

and for every $\sigma \in \mathrm{ca}(\Omega, \mathcal{A})$

$$0 \leq \sigma \leq |\mu_p| \implies \sigma = 0 .$$

Definition 1.19. *Let* $\Omega \subset \mathbb{R}^n$ *and* $\mathcal{A} \subset 2^\Omega$ *be an algebra. Then every measure* $\mu_p \in \mathrm{ca}(\Omega, \mathcal{A})^\perp$ *is called* ***pure****. Notice that* μ_p *is not* σ*-additive, by definition.*

One important example of measures that are pure are density measures. The following new example presents a particular density measure, namely a density at zero. In the literature, examples of pure measure are only known for $\Omega = \mathbb{N}$ (cf. [30, p. 247]), they are defined on very small algebras (cf. [30, p. 246]) or they are constructed in such a way that the measure cannot be computed explicitly, even on simple sets (cf. [33, p. 57f]). The example given here is constructed on $\Omega = \mathbb{R}^n$ and lives on the Borel subsets of Ω.

Example 1.20. Let $\Omega := B_1(0) \subset \mathbb{R}^n$ be open. Then there exists $\mu \in \mathrm{ba}(\Omega, \mathcal{B}(\Omega))$, $\mu \geq 0$ such that for every $B \in \mathcal{B}(\Omega)$

$$\mu(B) = \lim_{\delta \downarrow 0} \frac{\mathcal{L}^n(B \cap B_\delta(0))}{\mathcal{L}^n(B_\delta(0))}$$

if this limit exists. This measure is non-unique. Its existence is shown in Proposition 3.7 (take $\lambda := \mathcal{L}^n$ and $C = \{0\}$).

It is shown in Example 1.28 that μ is indeed pure. Figure 1.1 shows the family $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{B}(\Omega)$

$$A_k := \left[\frac{1}{k+2}, \frac{1}{k+1}\right) \times [-1,1]^{n-1} .$$

For this family

$$\sum_{k\in\mathbb{N}} \mu(A_k \cap \Omega) = 0 \neq \mu\left(\left(0, \frac{1}{2}\right) \times [-1,1]^{n-1} \cap \Omega\right) = \mu\left(\bigcup_{k=1}^{\infty} A_k \cap \Omega\right) .$$

Hence, μ is not a σ-measure.

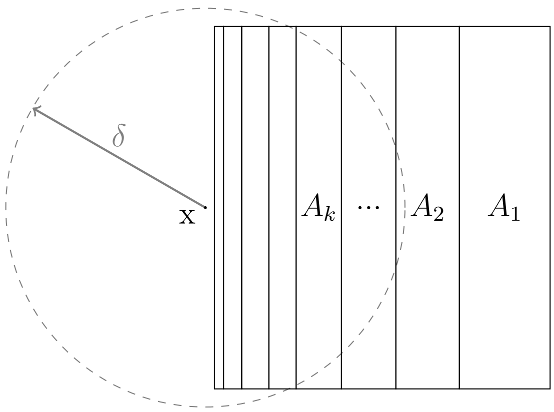

Figure 1.1: A family of sets on which μ is not σ-additive

Measures that do not charge sets of Lebesgue measure zero are of special interest, because these measures lend themselves naturally to the integration of functions that are only defined outside of a set of measure zero. When treating non σ-additive measures, one carefully has to distinguish the following two notions (cf. [30, p. 159]).

Definition 1.21. *Let $\Omega \subset \mathbb{R}^n, \mathcal{A} \subset 2^\Omega$ be an algebra and $\lambda \in \mathrm{ba}(\Omega, \mathcal{A})$. Then $\mu \in \mathrm{ba}(\Omega, \mathcal{A})$ is called*

1. ***absolutely continuous*** *with respect to λ, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $A \in \mathcal{A}$*

$$|\lambda|(A) < \delta \implies |\mu(A)| < \varepsilon .$$

In this case, write $\mu << \lambda$.

2. ***weakly absolutely continuous*** *with respect to* λ*, if for every* $A \in \mathcal{A}$

$$|\lambda|(A) = 0 \implies \mu(A) = 0 \,.$$

In this case, write $\mu <<^w \lambda$*.*

The set of all weakly absolutely continuous measures in $\mathrm{ba}(\Omega, \mathcal{A})$ *is denoted by*

$$\mathrm{ba}\,(\Omega, \mathcal{A}, \lambda) \ .$$

The following proposition shows that there is no pure measure which is absolutely continuous with respect to some σ-measure (cf. [30, p. 163]).

Proposition 1.22. *Let* $\Omega \subset \mathbb{R}^n, \mathcal{A} \subset 2^\Omega$ *be an algebra and* $\sigma \in \mathrm{ca}(\Omega, \mathcal{A})$*. Then for every* $\mu \in \mathrm{ba}(\Omega, \mathcal{A})$

$$\mu << \sigma \implies \mu \in \mathrm{ca}(\Omega, \mathcal{A}) \,.$$

Remark 1.23. The preceding proposition shows that one should focus on the notion of weak absolute continuity when studying measures that are continuous with respect to some σ-measure.

Example 1.24. μ from Example 1.20 is even weakly absolutely continuous with respect to $\mathcal{L}^n$. This is evident from the construction in Proposition 3.7 (take $\lambda := \mathcal{L}^n$ and $C := \{0\}$).

Proposition 1.25. *Let* $\mu_1, \mu_2 \in \mathrm{ba}(\Omega, \mathcal{A})$ *be such that* $\mu_1 <<^w \mu_2$*. If* $A \in \mathcal{A}$ *such that* $|\mu_2|(A) = 0$*, then* $|\mu_1|(A) = 0$*.*

Proof. Since $|\mu_2|$ is monotone,

$$|\mu_2(A')| \le |\mu_2|(A') \le |\mu_2|(A) = 0$$

for all $A' \in \mathcal{A}$ such that $A' \subset A$. Since

$$\mu_1^+(A) = \sup_{\substack{A' \in \mathcal{A} \\ A' \subset A}} \mu_1(A') = 0$$

and a similar equation holds for μ_1^-

$$|\mu_1|(A) = \mu_1^+(A) + \mu_1^-(A) = 0 \,.$$

□

The following proposition is the key to decompose measures into σ-measures which are weakly absolutely continuous with respect to some measure and pure measures.

Proposition 1.26. *Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be an algebra and $\lambda \in \mathrm{ba}(\Omega, \mathcal{A})$.*

Then $\mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$ *is a normal sublattice of* $\mathrm{ba}(\Omega, \mathcal{A})$ *and thus a boundedly complete vector lattice.*

Proof. $\mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$ is obviously a linear space. Let $\{\mu_i\}_{i\in\mathcal{I}} \subset \mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$ be such that there exists $\mu \in \mathrm{ba}(\Omega, \mathcal{A})$ with

$$\mu_i \leq \mu \text{ for all } i \in \mathcal{I}\,.$$

By Proposition 1.14, $\mathrm{ba}(\Omega, \mathcal{A})$ is boundedly complete (cf. [30, p. 44]). Hence, there exists $\mu' \in \mathrm{ba}(\Omega, \mathcal{A})$ such that

$$\mu_i \leq \mu' \text{ for all } i \in \mathcal{I}$$

and if this holds true for another $\mu'' \in \mathrm{ba}(\Omega, \mathcal{A})$ then $\mu' \leq \mu''$.

Assume $\mu' \notin \mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$. Then there exists $A \in \mathcal{A}$ such that

$$|\lambda|(A) = 0 \text{ but } \mu'(A) \neq 0\,.$$

Now, $|\mu' \lfloor A| \in \mathrm{ba}(\Omega, \mathcal{A})$. Whence $\mu' - |\mu' \lfloor A| \in \mathrm{ba}(\Omega, \mathcal{A})$. Since $\mu_i(A) = 0$

$$\mu_i \leq \mu' - |\mu' \lfloor A| < \mu' \text{ for all } i \in \mathcal{I}\,,$$

in contradiction to the minimality of μ'. Hence $\mu' \in \mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$.

Now let $\mu' \in \mathrm{ba}(\Omega, \mathcal{A})$ and $\mu \in \mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$ such that $|\mu'| \leq |\mu|$. Let $A \in \mathcal{A}$ be such that $|\lambda|(A) = 0$. Then

$$|\mu'(A)| \leq |\mu'|(A) \leq |\mu|(A) = 0$$

by Proposition 1.25. Hence $\mu' \in \mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$. Therefore, $\mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$ is a normal sublattice and thus a boundedly complete vector lattice. $\square$

The proposition above enables the decomposition of measures into pure parts and σ-measures, analogously to Proposition 1.18.

Theorem 1.27. *Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be an algebra and $\lambda \in \mathrm{ba}\,(\Omega, \mathcal{A})$.*

Then for every $\mu \in \mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$ there exist unique $\mu_c \in \mathrm{ca}(\Omega, \mathcal{A}) \cap \mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$, $\mu_p \in \mathrm{ca}(\Omega, \mathcal{A})^\perp \cap \mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$ such that

$$\mu = \mu_c + \mu_p\,.$$

Proof. Since $\mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$ and $\mathrm{ca}(\Omega, \mathcal{A})$ are normal sublattices of $\mathrm{ba}(\Omega, \mathcal{A})$, Proposition 1.8 yields that

$$\mathrm{ca}(\Omega, \mathcal{A}) \cap \mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$$

is a normal sublattice of $\mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$ whose orthogonal complement is

$$\mathrm{ca}(\Omega, \mathcal{A})^{\perp} \cap \mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)\ .$$

This, together with Riesz's decomposition Proposition 1.7, yields the statement of the proposition. $\square$

Example 1.28. Since the measure μ from Example 1.20 is positive and $\mu_c \perp \mu_p$, using the additivity of the total variation on orthogonal element (cf. [30, p. 25]) yields

$$0 \le |\mu_c| \le |\mu_c| + |\mu_p| = |\mu| = \mu\,.$$

Hence, for every $\delta > 0$

$$|\mu_c|\,(B_\delta\,(0)^c) = 0\,.$$

Thus

$$|\mu_c|\,(\Omega \setminus \{0\}) = \lim_{\delta \downarrow 0} |\mu_c|\,(B_\delta\,(0)^c) = 0\,.$$

But $|\mu_c|\,(\{0\}) \le \mu(\{0\}) = 0$. Hence

$$|\mu_c|\,(\Omega) = 0$$

and $\mu = \mu_p$ is pure.

When λ is a σ-measure, the structure of μ_c is well known by the Radon Nikodym theorem (cf. [24, p. 128ff]).

Proposition 1.29. *Radon-Nikodym Theorem*
Let $\Omega \subset \mathbb{R}^n$ and $\Sigma \subset 2^\Omega$ be a σ-algebra. Furthermore, let $\sigma \in \mathrm{ca}(\Omega, \Sigma)$ and $\mu \in \mathrm{ca}(\Omega, \Sigma)$ be such that $\mu <<^w \sigma$. Then there exists $f \in \mathcal{L}^1\,(\Omega, \Sigma, \sigma)$ such that

$$\mu(A) = \int_A f\,\mathrm{d}\sigma$$

for every $A \in \Sigma$.

The structure of μ_p is described by the following proposition taken from [30, p. 244] (cf. [33, p. 56]).

Remark 1.30. The following results are stated for σ-measures $\sigma \ge 0$. They also hold for arbitrary σ-measures σ when using $|\sigma|$.

Proposition 1.31. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a σ-algebra and $\sigma \in \mathrm{ca}(\Omega, \Sigma)$, $\sigma \geq 0$. Then $\mu \in \mathrm{ba}(\Omega, \Sigma, \sigma)$ is pure if and only if there exists a decreasing sequence $\{A_k\}_{k\in\mathbb{N}} \subset \Sigma$ such that*

$$\sigma(A_k) \xrightarrow{k\to\infty} 0$$

and for all $k \in \mathbb{N}$

$$|\mu_p|(A_k^c) = 0\,.$$

Intuitively speaking, weakly absolutely continuous measures are pure if and only if they concentrate in the vicinity of a set of measure zero. Reviewing Example 1.20, the support (cf. [2, p.30]) of the measure can be seen to lie outside of $\Omega \setminus \{0\}$. Yet the construction of the measure would still work on this set. Hence, it is possible for a pure measure to have support outside of its domain of definition. This necessitates the following definition of core.

Definition 1.32. *Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be an algebra containing every relatively open set in Ω. Furthermore let $\mu \in \mathrm{ba}(\Omega, \mathcal{A})$. Then the set*

$$\operatorname{core}\mu := \{x \in \mathbb{R}^n \mid |\mu|(V \cap \Omega) > 0, \forall V \subset \mathbb{R}^n, V \text{ open}, x \in V\}$$

*is called **core** of μ.*

*Let $d \in [0, n]$ be the Hausdorff dimension of $\operatorname{core}\mu$. Then d is called **core dimension of** μ and μ is called d**-dimensional**.*

Remark 1.33. Note that there is a slight difference to the notion of *support* of a measure as defined in classic measure theory (cf. [21, p. 60]). The core of a measure is not necessarily contained in Ω, the support of a σ-measure is.

Example 1.34. The measure μ from Example 1.20 has

$$\operatorname{core}\mu = \{0\}$$

and is thus 0-dimensional.

Now, an example for a density measure with a larger core is given. Note that in this thesis

$$C_\delta := \mathrm{dist}_\Omega^{-1}((-\infty, \delta)) \text{ for } C \subset \mathbb{R}^n\,.$$

Example 1.35. Let $\Omega \subset \mathbb{R}^n$ be open, $d \in [0, n)$ and $C \subset \overline{\Omega}$ be closed with Hausdorff dimension d. Then there exists a pure measure $\mu \in \mathrm{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$, $\mu \geq 0$ such that for every $B \in \mathcal{B}(\Omega)$

$$\mu(B) = \lim_{\delta\downarrow 0} \frac{\mathcal{L}^n(B \cap C_\delta \cap \Omega)}{\mathcal{L}^n(C_\delta \cap \Omega)} =: \mathrm{dens}_C(B)\,,$$

if this limit exists. Here, C_δ is the open δ-neighbourhood of C. Furthermore

$$\operatorname{core}\mu = C$$

and μ is thus d-dimensional.

The existence of this measure is evident by Proposition 3.7 (take $\lambda := \mathcal{L}^n$).

Proposition 1.36. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A} \subset 2^\Omega$ be an algebra containing every relatively open set and $\mu \in \mathrm{ba}(\Omega, \mathcal{A})$. Then $\operatorname{core}\mu$ is a closed set in $\mathbb{R}^n$.*

Proof. Set $B := \operatorname{core}\mu$ and let $x \in B^c$. Then there is an open neighbourhood $V \subset \mathbb{R}^n$ of x such that

$$|\mu|(V \cap \Omega) = 0\,.$$

Now let $x' \in V$ and $V' \subset \mathbb{R}^n$ be an open neighbourhood of x'. Then

$$|\mu|(V \cap V' \cap \Omega) \leq |\mu|(V \cap \Omega) = 0\,.$$

Thus, $x' \in B^c$. Since x was arbitrary, it follows that for every $x \in B^c$ there exists an open neighbourhood $V \subset \mathbb{R}^n$ of x such that $V \subset B^c$, whence B^c is open and B closed. □

On bounded domains, the core is non-empty.

Proposition 1.37. *Let $\Omega \subset \mathbb{R}^n$ be bounded, $\mathcal{A} \subset 2^\Omega$ be an algebra containing every relatively open set in Ω and $\mu \in \mathrm{ba}(\Omega, \mathcal{A}), \mu \neq 0$. Then $\operatorname{core}\mu$ is non-empty and for every $\delta > 0$*

$$|\mu|\left(\Omega \cap \left((\operatorname{core}\mu)_\delta\right)^c\right) = 0\,.$$

Proof. Set $B := \operatorname{core}\mu$. Assume $\operatorname{core}\mu$ was empty. Then, by compactness of $\overline{\Omega}$ there exists an open covering $\{V_k\}_{k=0}^m$ of $\overline{\Omega}$ such that for $k = 0, ..., m$

$$|\mu|(V_k \cap \Omega) = 0\,.$$

But then

$$|\mu|(\Omega) \leq \sum_{k=0}^m |\mu|(V_k \cap \Omega) = 0$$

in contradiction to $\mu \neq 0$.

Now, let $\delta > 0$. For every $x \in \overline{(B_\delta)^c}^{\mathbb{R}^n}$ there is a $0 < \delta_x < \frac{\delta}{2}$ such that

$$|\mu|\left(B\left(x, \delta_x\right) \cap \Omega\right) = 0\,.$$

Otherwise, $x \in \operatorname{core}\mu$. Now

$$\left\{B\left(x, \delta_x\right)\right\}_{x \in \overline{(B_\delta)^c}^{\mathbb{R}^n}}$$

is an open covering of

$$\overline{(B_\delta)^c}^{\mathbb{R}^n} \cap \Omega$$

Since Ω is relatively compact in $\mathbb{R}^n$, there exists a finite open sub-covering

$$\{B(x_l, \delta_{x_l})\}_{l=0}^m$$

of

$$(B_\delta)^c \cap \Omega .$$

Hence

$$|\mu|((B_\delta)^c \cap \Omega) \leq \sum_{l=0}^{m} |\mu|(B(x_l, \delta_{x_l}) \cap \Omega) = 0 .$$

□

Remark 1.38. If Ω is unbounded, the statement of the preceding proposition need not be true. The measures in Example 10.4.1 in [30, p. 245] can be shown to have empty core, since they concentrate near infinity.

The core itself does not give all information on the way in which a pure measure concentrates. Hence, the sequences from Proposition 1.31 is investigated further.

Definition 1.39. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a σ-algebra, $\sigma \in \mathrm{ca}(\Omega, \Sigma)$, $\sigma \geq 0$ and $\mu_p \in \mathrm{ba}(\Omega, \Sigma, \sigma)$ be pure. Then every $A \in \Sigma$ such that*

$$|\mu_p|(A^c) = 0$$

is called ***aura*** *of μ_p.*

Any decreasing sequence $\{A_k\}_{k\in\mathbb{N}} \subset \Sigma$ of auras for μ_p such that

$$\sigma(A_k) \xrightarrow{k\to\infty} 0$$

is called ***aura sequence****.*

Now, it is shown that any aura sequence can be restricted to neighbourhoods of the core.

Proposition 1.40. *Let $\Omega \subset \mathbb{R}^n$ be bounded and $\Sigma \subset 2^\Omega$ be a σ-algebra containing every relatively open set in Ω. Furthermore, let $\sigma \in \mathrm{ca}(\Omega, \Sigma)$ with $\sigma \geq 0$ and $\mu_p \in \mathrm{ba}(\Omega, \Sigma, \sigma)$ be pure. Then for every aura sequence $\{A_k\}_{k\in\mathbb{N}} \subset \Sigma$ of μ_p the sequence*

$$\{A'_k\}_{k\in\mathbb{N}} := \left\{A_k \cap (\mathrm{core}\,\mu_p)_{\frac{1}{k}}\right\} \subset \Sigma$$

is an aura sequence of μ_p with

$$\mathrm{core}\,\mu_p = \bigcap_{k\in\mathbb{N}} \overline{A'_k}^{\mathbb{R}^n} .$$

Proof. Let $C := \operatorname{core}\mu_p$. Note that $|\mu_p|$ is pure and let $\{A_k\}_{k\in\mathbb{N}} \subset \Sigma$ be any aura sequence of μ_p. Then for every $k \in \mathbb{N}$, $x \in \left(\overline{A_k}^{\mathbb{R}^n}\right)^c$ and any open neighbourhood $V \subset \left(\overline{A_k}^{\mathbb{R}^n}\right)^c$ of x

$$|\mu_p|(V \cap \Omega) \le |\mu_p|\left(\left(\overline{A_k}^{\mathbb{R}^n}\right)^c \cap \Omega\right) \le |\mu_p|(A_k^c \cap \Omega) = 0\,.$$

Hence

$$C \subset \overline{A_k}^{\mathbb{R}^n} \text{ for every } k \in \mathbb{N}\,.$$

Thus,

$$C \subset \bigcap_{k\in\mathbb{N}} \overline{A_k}^{\mathbb{R}^n}\,.$$

For $k \in \mathbb{N}$ set

$$A'_k := A_k \cap C_{\frac{1}{k}}\,.$$

Then for every $k \in \mathbb{N}$

$$|\mu_p|\left(A'^c_k\right) \le |\mu_p|(A_k^c) + |\mu_p|\left(\left(C_{\frac{1}{k}}\right)^c \cap \Omega\right) = 0\,,$$

by Proposition 1.37.

Furthermore

$$0 \le \sigma(A'_k) \le \sigma(A_k) \xrightarrow{k\to\infty} 0\,.$$

Obviously

$$\bigcap_{k\in\mathbb{N}} \overline{A_k \cap C_{\frac{1}{k}}}^{\mathbb{R}^n} \subset \bigcap_{k\in\mathbb{N}} C_{\frac{1}{k}} = C\,.$$

It remains to show that

$$C \subset \bigcap_{k\in\mathbb{N}} \overline{A_k \cap C_{\frac{1}{k}}}^{\mathbb{R}^n}\,.$$

Let $x \in C$. Then $x \in \overline{A_k}^{\mathbb{R}^n}$ for every k. Hence, for every k there is a sequence $\{x_l^k\}_{l\in\mathbb{N}} \subset A_k$ such that

$$x_l^k \xrightarrow{l\to\infty} x\,.$$

In particular, there is an $l_0^k \in \mathbb{N}$ such that

$$\|x_l^k - x\| < \frac{1}{k} \text{ for } l \ge l_0^k\,.$$

Hence, for every $k \in \mathbb{N}$,

$$x \in \overline{A_k \cap C_{\frac{1}{k}}}^{\mathbb{R}^n}\,.$$

Since $x \in C$ was arbitrary, this finally implies

$$C \subset \bigcap_{k \in \mathbb{N}} \overline{A_k \cap C_{\frac{1}{k}}}^{\mathbb{R}^n} .$$

□

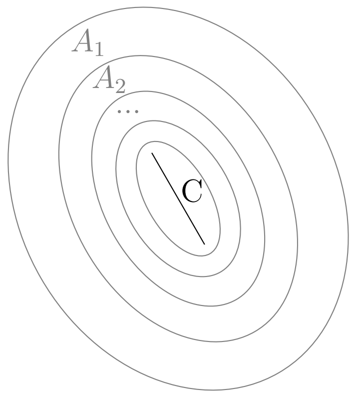

Figure 1.2: An aura sequence $\{A_k\}_{k\in\mathbb{N}}$ of a 1-dimensional measure with core $C = \bigcap\limits_{k\in\mathbb{N}} A_k$

The following lemma identifies a big class of pure measures. In particular, if the core of a measure is a Lebesgue null set, the measure is necessarily pure.

Proposition 1.41. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ and $\mu \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$.*
If $\mathrm{core}\,\mu \cap \Omega$ is a $\mathcal{L}^n$-null set then μ is pure.

Proof. Let $B := \mathrm{core}\,\mu$. Then by the definition of the core, for every $\delta > 0$

$$|\mu|\,(B_\delta^c \cap \Omega) = 0 .$$

Now let $B_k := B_{\frac{1}{k}} \cap \Omega$ for $k \in \mathbb{N}$ and $\sigma \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)\,, \sigma \geq 0$ be a σ-measure such that

$$0 \leq \sigma \leq |\mu| .$$

Then for every $k \in \mathbb{N}$

$$0 \leq \sigma((B_k)^c) \leq |\mu|((B_k)^c) = 0 .$$

On the other hand, since $\mathrm{core}\,\mu \cap \Omega$ is a $\mathcal{L}^n$-null set,

$$\sigma(\Omega \cap B) = 0 .$$

Hence

$$\sigma(\Omega) = \sigma(\Omega \cap B) + \sigma\left(\bigcup_{k\in N} B_k^c\right) = \lim_{k\to\infty} \sigma\left(B_k^c\right) = 0\,.$$

This implies $\sigma = 0$.

Since σ was arbitrary, μ is pure by Proposition 1.6 and Proposition 1.18.

□

Remark 1.42. Note that $\operatorname{core}\mu \subset \overline{\Omega}$. If $\Omega \subset \mathbb{R}^n$ is open such that $\mathcal{L}^n(\partial\Omega) > 0$, then there is $\mu \in \operatorname{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that $\operatorname{core}\mu = \partial\Omega$. Hence $\operatorname{core}\mu$ is not a null set, but $\operatorname{core}\mu \cap \Omega = \emptyset$. Thus, μ is necessarily pure.

The following proposition is taken from [30, p. 70]. It shows that there are many degrees of freedom when choosing an extension of a measure to a larger class of sets. Since all pure measures used below are constructed using an extension argument, they are in general not unique.

Proposition 1.43. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A} \subset 2^\Omega$ be an algebra on Ω. Let $\mu \in \operatorname{ba}(\Omega, \mathcal{A}), \mu \geq 0$. Let $A \in 2^\Omega \setminus \mathcal{A}$ and $\mathcal{A}' \subset 2^\Omega$ the smallest algebra such that $\mathcal{A}, \{A\} \subset \mathcal{A}'$. Then for any $c \in [0, \infty)$ such that*

$$\sup\{\mu(A') \mid A' \in \mathcal{A}, A' \subset A\} \leq c \leq \inf\{\mu(A') \mid A' \in \mathcal{A}, A \subset A'\}$$

there exists an extension $\mu' \in \operatorname{ba}(\Omega, \mathcal{A}')\,, \mu' \geq 0$ of μ to all of $\mathcal{A}'$ such that

$$\mu'(A) = c\,.$$

Chapter 2

Integration Theory

The usefulness of measures hinges on the fact that the dual spaces of important function spaces can be represented by integration with respect to some class of measures. The most prominent result is the Riesz Representation Theorem (cf. [21, p. 106]), which links $C_0(\Omega)^*$ with the set of Radon measures on Ω. The dual spaces of L^p-spaces with $1 \leq p < \infty$ are in essence spaces of σ-measures which are absolutely continuous with respect to Lebesgue measure (cf. [29, p. 253]). Less known is a result of Alexandroff, characterising the dual of $C_b(\Omega)$ as the space of finitely additive Radon measures, which are inner regular with respect to the relatively closed sets (cf. [1, p. 582]).

In this chapter, a basic integration theory for finitely additive measures is presented. The exposition closely adheres to Rao [30], other sources can be found in Dunford [18] and Bochner [7]. The characterisation of the dual space of L^∞ as the space $\mathrm{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ (cf. [18, p. 296],[30, p. 139]) is improved upon by using the decomposition results from the previous chapter. The $\mathcal{L}^p$-spaces introduced in this chapter are in general not complete but their completion is known and has an interesting structure (cf. [30], [28]). This and the dual spaces are covered in the second section.

2.1 Integration Theory for Finitely Additive Measures

This section lays out the theory of integration for finitely additive measures used in this thesis. As usual, the integral is at first defined for simple functions. Then, convergence in measure is introduced. A function is defined to be measurable, if some sequence of simple functions converges in measure to it. Integrability is then introduced using L^1-Cauchy sequences of simple functions. Once $\mathcal{L}^p$-spaces are introduced, an improved characterisation of the dual of $\mathcal{L}^\infty$ is given.

The following definition of simple functions is taken from [30, p. 90].

Definition 2.1. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A} \subset 2^\Omega$ an algebra. A function $h : \Omega \to \mathbb{R}$ is called **simple**, if there exists $m \in \mathbb{N}, \{a_k\}_{k=0}^m \subset \mathbb{R}$ and $\{A_k\}_{k=0}^m \subset \mathcal{A}$ such that*

$$h = \sum_{k=0}^{m} a_k \chi_{A_k} .$$

Remark 2.2. Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A} \subset 2^\Omega$ an algebra and $h : \Omega \to \mathbb{R}$. Then h is simple if and only if

$$\mathcal{R}(h) := \{y \in \mathbb{R} \mid \exists x \in \Omega : h(x) = y\}$$

is finite and for every $y \in \mathcal{R}(h)$ the set $h^{-1}(y) \in \mathcal{A}$.

In this case

$$h = \sum_{y \in \mathcal{R}(h)} y \cdot \chi_{h^{-1}(y)} .$$

Measurability is not defined through the regularity of preimages but by approximability by simple functions in measure. In this definition, the measure is needed on possibly non-measurable sets. Hence, an outer measure has to be used. This outer measure is defined as in the case of σ-measures (cf. [30, p. 86], [24, p. 42]).

Definition 2.3. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A} \subset 2^\Omega$ be an algebra. For $\mu \in \mathrm{ba}(\Omega, \mathcal{A})$, $\mu \geq 0$ the **outer measure** of μ is defined for $B \in 2^\Omega$ by*

$$\mu^*(B) := \inf_{\substack{A \in \mathcal{A}, \\ B \subset A}} \mu(A) .$$

For reasons of completeness, the following proposition gathers some properties of outer measures(cf. [30, pp. 86-87]).

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A} \subset 2^\Omega$ be an algebra. Furthermore, let $\mu, \lambda \in \mathrm{ba}(\Omega, \mathcal{A})$ be positive. Then for every $B_1, B_2 \in 2^\Omega$ and $A \in \mathcal{A}$*

1. $\mu^*(\emptyset) = 0$
2. $\mu^*(B_1) \leq \mu^*(B_2)$, *if* $B_1 \subset B_2$
3. $\mu^*(A) = \mu(A)$
4. $\mu^*(B_1 \cup B_2) \leq \mu^*(B_1) + \mu^*(B_2)$

5. $(\mu + \lambda)^* = \mu^* + \lambda^*$

Now, convergence in measure can be defined. The definition is taken from [30, p. 92] (cf. [24, p. 91]).

Definition 2.5. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A} \subset 2^\Omega$ be an algebra and $\mu : \mathcal{A} \to \mathbb{R}$ be a measure. A sequence $\{f_k\}_{k\in\mathbb{N}}$ of functions $f_k : \Omega \to \mathbb{R}$ is said to converge **in measure** to a function $f : \Omega \to \mathbb{R}$ if for every $\varepsilon > 0$*

$$\lim_{k\to\infty} |\mu|^* \{x \in \Omega \mid |f_k(x) - f(x)| > \varepsilon\} = 0 \,.$$

In this case, write

$$f_k \xrightarrow{\mu} f \,.$$

Note that the limit in measure is not unique, yet. Therefore, the following notion of equality almost everywhere is needed. The definition is taken from [30, p. 88].

Definition 2.6. *Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ and $\mu : \mathcal{A} \to \mathbb{R}$ be a measure.*

*Then $f : \Omega \to \mathbb{R}$ is called **null function**, if for every $\varepsilon > 0$*

$$|\mu|^* \left(\{x \in \Omega \mid |f(x)| > \varepsilon\}\right) = 0 \,.$$

*Two functions $f_1 : \Omega \to \mathbb{R}$, $f_2 : \Omega \to \mathbb{R}$ are called **equal almost everywhere (a.e.)** with respect to μ, if $f_1 - f_2$ is a null function.*

In this case, write

$$f_1 = f_2 \ \mu\text{-a.e.}$$

Remark 2.7. If $f : \Omega \to \mathbb{R}$ is a null function, then it need not be true that

$$|\mu|^* \left(\{x \in \Omega \mid f(x) \neq 0\}\right) = 0 \,. \tag{2.1}$$

Take e.g. the density measure μ introduced in Example 1.20 and $f(x) := |x|$. Then f is a null function but

$$|\mu|^* (\{x \in \mathbb{R}^n | f(x) \neq 0\} = \mu(B_1(0) \setminus \{0\}) = 1 > 0 \,.$$

This entails that the notion of equality almost everywhere that was defined above does not imply the existence of a null set such that $f_1 = f_2$ outside of that set. Take e.g. the density measure introduced in Example 1.20, $f_1(x) := |x|$ and $f_2(x) := 2f_1(x)$.

On the other hand, if μ is a σ-measure and $\mathcal{A}$ a σ-algebra, then Equation (2.1) is equivalent to f being a null function (cf. [30, p. 89]).

The limit in measure turns out to be unique in the sense of almost equality. This is stated in the following proposition taken from [30, p. 92].

Proposition 2.8. *Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be an algebra and $\mu : \mathcal{A} \to \mathbb{R}$ be a measure. Furthermore let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions $f_k : \Omega \to \mathbb{R}$ and $f, \tilde{f} : \Omega \to \mathbb{R}$ be functions such that*

$$f_k \xrightarrow{\mu} f\,.$$

Then

$$f_k \xrightarrow{\mu} \tilde{f} \iff f = \tilde{f} \ \mu\text{-a.e.}$$

Now, the notion of measurability is introduced. The definition is similar to the definition of T_1-measurability in [30, p. 101].

Definition 2.9. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A} \subset 2^\Omega$ be an algebra and $\mu : \mathcal{A} \to \mathbb{R}$ be a measure. A function $f : \Omega \to \mathbb{R}$ is called **measurable** if there exists a sequence $\{h_k\}_{k\in\mathbb{N}}$ of simple functions $h_k : \Omega \to \mathbb{R}$ such that*

$$h_k \xrightarrow{\mu} f\,.$$

The following proposition shows that this notion of measurability coincides with the usual one in the case of σ-measures and σ-algebras, if the null sets are added to the σ-algebra, i.e. if the completed σ-algebra is used.

Proposition 2.10. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a σ-algebra, $\sigma \in \mathrm{ca}(\Omega, \Sigma)$ and $f : \Omega \to \mathbb{R}$ be measurable. Then for every $B \in \mathcal{B}(\mathbb{R})$ there exists $A \in \Sigma$ such that $f^{-1}(B)\Delta A$ is a null set.*

Proof. Let $C \subset \mathbb{R}$ be closed and $\{h_k\}_{k\in\mathbb{N}}$ be a sequence of simple functions with

$$h_k \xrightarrow{\sigma} f\,.$$

Fix $\varepsilon > 0$. Let $\{h_k^\varepsilon\}_{k\in\mathbb{N}}$ be a subsequence of $\{h_k\}_{k\in\mathbb{N}}$ with

$$|\sigma|^* \left(\left\{ x \in \Omega \mid |h_k^\varepsilon(x) - f(x)| \geq \frac{1}{k} \right\} \right) < \frac{\varepsilon}{2^{k+1}}\,.$$

Then there exist sets $A_k^\varepsilon \in \Sigma$ such that

$$\left\{ x \in \Omega \mid |h_k^\varepsilon(x) - f(x)| \geq \frac{1}{k} \right\} \subset A_k^\varepsilon \quad \text{and} \quad |\sigma| \left(A_k^\varepsilon\right) < \frac{\varepsilon}{2^{k+1}}\,.$$

Set

$$A_\varepsilon := \bigcup_{k\in\mathbb{N}} A_k^\varepsilon\,.$$

Then

$$|\sigma|(A_\varepsilon) \leq \sum_{k\in\mathbb{N}} |\sigma|(A_k^\varepsilon) \leq \varepsilon$$

and for every $x \in (A_\varepsilon)^c$

$$h_k^\varepsilon(x) \xrightarrow{k\to\infty} f(x)\,.$$

Then

$$A^\varepsilon := f^{-1}(C) \cap A_\varepsilon^c = \bigcap_{k\in\mathbb{N}} (h_k^\varepsilon)^{-1}(C_{\frac{1}{k}}) \cap A_\varepsilon^c \in \Sigma$$

where $C_{\frac{1}{k}}$ is a neighbourhood of C with radius $\frac{1}{k}$. Now let $\{\varepsilon_l\}_{l\in\mathbb{N}} \subset (0,\infty)$ be a sequence with $\varepsilon_l \xrightarrow{l\to\infty} 0$ and set

$$A := \bigcup_{l\in\mathbb{N}} A^{\varepsilon_l} \in \Sigma\,.$$

Then for any $l \in \mathbb{N}$

$$|\sigma|^*(f^{-1}(C) \setminus A) \leq |\sigma|^*(f^{-1}(C) \setminus A^{\varepsilon_l}) \leq |\sigma|(A_{\varepsilon_l}) \leq \varepsilon_l \xrightarrow{l\to\infty} 0\,.$$

This shows the statement for closed sets. Since the closed sets generate $\mathcal{B}(\mathbb{R})$, f is measurable in the usual sense with respect to the σ-algebra generated by Σ and the σ-null sets. This yields the statement for arbitrary $B \in \mathcal{B}(\mathbb{R})$. $\square$

In the general case, the following statement from [30, p. 101] holds true.

Proposition 2.11. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A} \subset 2^\Omega$ be an algebra, $\mu : \mathcal{A} \to \mathbb{R}$ a measure and $f : \Omega \to \mathbb{R}$ be a function. Then f is measurable if and only if for every $\varepsilon > 0$ there exists $\{A_k\}_{k=0}^m \subset \mathcal{A}$ such that $\bigcup_{k=0}^m A_k = \Omega$,*

$$|\mu|(A_0) < \varepsilon$$

and for every $1 \leq k \leq m$ and $x_1, x_2 \in A_k$

$$|f(x_1) - f(x_2)| < \varepsilon\,.$$

Now, the integral for simple functions is defined. The definition is standard in integration theory (cf. [30, p. 96]).

Definition 2.12. *Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be an algebra and $\mu : \mathcal{A} \to \mathbb{R}$ be a measure. A simple function $h : \Omega \to \mathbb{R}$ is called **integrable** if for every $y \in \mathcal{R}(h) \setminus \{0\}$*

$$|\mu\left(h^{-1}(y)\right)| < \infty\,.$$

*In this case the **integral** of h is defined by*

$$\int_\Omega h \,\mathrm{d}\mu := \sum_{y\in\mathcal{R}(h)} y\cdot\mu\left(h^{-1}(y)\right) .$$

Here, the convention $0\cdot\infty = 0$ is used.

The integral for measurable functions can now be defined via $\mathcal{L}^1$-Chauchy sequences. This is of course well-defined (cf. [30, p. 102]).

Definition 2.13. *Let $\Omega\subset\mathbb{R}^n$, $\mathcal{A}\subset 2^\Omega$ be an algebra and $\mu:\mathcal{A}\to\mathbb{R}$ be a measure. A function $f:\Omega\to\mathbb{R}$ is said to be **integrable** if there exists a sequence $\{h_k\}_{k\in\mathbb{N}}$ of integrable simple functions $h_k:\Omega\to\mathbb{R}$ such that*

1. $h_k \xrightarrow{\mu} f$.
2. $\lim\limits_{k,l\to\infty}\int_\Omega |h_k-h_l|\,\mathrm{d}|\mu| = 0$.

In this case, denote

$$\int_\Omega f\,\mathrm{d}\mu := \lim_{k\to\infty}\int_\Omega h_k\,\mathrm{d}\mu\,.$$

*The sequence $\{h_k\}_{k\in\mathbb{N}}$ is called **determining sequence** for the integral of f.*

Remark 2.14. In particular, integrable functions are measurable. This notion of integral is also called Daniell-Integral in the literature (cf. [30]).

The $\mathcal{L}^p$-spaces are defined in the usual way (cf. [30, p. 121]).

Definition 2.15. *Let $\Omega\subset\mathbb{R}^n$, $\mathcal{A}\subset 2^\Omega$, $\mu:\mathcal{A}\to\mathbb{R}$ be a measure and $p\in[1,\infty)$. Then the set of all measurable functions $f:\Omega\to\mathbb{R}$ such that $|f|^p$ is $|\mu|$-integrable is denoted by*

$$L^p\left(\Omega,\mathcal{A},\mu\right) .$$

If $\mathcal{A}=\mathcal{B}(\Omega)$, write

$$L^p\left(\Omega,\mu\right) .$$

For $f_1,f_2\in L^p\left(\Omega,\mathcal{A},\mu\right)$

$$f_1 = f_2 \;\; \mu\text{-a.e.}$$

defines an equivalence relation. The set of all equivalence classes of this relation is denoted by

$$\mathcal{L}^p\left(\Omega,\mathcal{A},\mu\right) .$$

If $\mathcal{A}=\mathcal{B}(\Omega)$, write

$$\mathcal{L}^p\left(\Omega,\mu\right) .$$

The following definition of norms is in accordance with [30, p. 121].

Definition 2.16. *Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be an algebra and $\mu : \mathcal{A} \to \mathbb{R}$ a measure. Then for every $p \in [1, \infty)$ and $f \in L^p(\Omega, \mathcal{A}, \mu)$ write*

$$\|f\|_p := \left(\int_\Omega |f|^p \,\mathrm{d}\,|\mu| \right)^{\frac{1}{p}} .$$

Furthermore, for measurable $f : \Omega \to \mathbb{R}$ define

$$\operatorname{ess\,sup} f := \inf \{ K \in \mathbb{R} \mid |\mu|^* (\{ x \in \Omega | f(x) > K \}) = 0 \}$$

and

$$\|f\|_\infty := \operatorname{ess\,sup} |f| .$$

The set of all measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\|f\|_\infty < \infty$$

is denoted by

$$L^\infty(\Omega, \mathcal{A}, \mu) .$$

As in the case $p \in [1, \infty)$,

$$\mathcal{L}^\infty(\Omega, \mathcal{A}, \mu)$$

denotes the set of all equivalence classes in $L^\infty(\Omega, \mathcal{A}, \mu)$ with respect to equality almost everywhere.

In the case $\mathcal{A} = \mathcal{B}(\Omega)$, only write

$$L^\infty(\Omega, \mu) \text{ and } \mathcal{L}^\infty(\Omega, \mu) \text{ respectively.}$$

The mappings introduced above are indeed norms on their respective spaces, as the following proposition from [30, p. 125ff] shows.

Proposition 2.17. *Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be an algebra and $\mu : \mathcal{A} \to \mathbb{R}$ be a measure.*

Then $\|\cdot\|_p$ is a norm on $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ for every $p \in [1, \infty]$.

Remark 2.18. For measures which are not σ-additive, the normed spaces

$$\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$$

need not be complete, even if $\mathcal{A} = \mathcal{B}(\Omega)$. See Remark 4.6.8 in Rao [30, p. 125] for reference.

Before proceeding to the characterisation of the dual of $\mathcal{L}^\infty$, a new integral symbol is introduced, which gives formulas for normal traces and integrals over pure measures a more pleasing shape.

Definition 2.19. *Let $\Omega \subset \mathbb{R}^n$ be bounded and $C \subset \overline{\Omega}$ be closed. Then for every $\mu \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that*

$$\operatorname{core}\mu \subset C,$$

every $f \in \mathcal{L}^1\,(\Omega, \mu)$ and $\delta > 0$ write

$$\rlap{\kern 0.3em -}\int_C f \,\mathrm{d}\mu := \int_{C_\delta \cap \Omega} f \,\mathrm{d}\mu\,.$$

Remark 2.20. This notion of integral is well-defined since the definition of $\operatorname{core}\mu$ yields

$$|\mu|\,((C_\delta)^c) = 0$$

for any $\delta > 0$.

The following proposition is a specialised version of the proposition from [30, p. 139] (cf. [33, p. 53]).

Proposition 2.21. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a σ-algebra and $\sigma : \Sigma \to \mathbb{R}$ be a σ-measure.*

Then for every $u^* \in (\mathcal{L}^\infty\,(\Omega, \Sigma, \sigma))^*$ there exists a unique $\mu \in \mathrm{ba}(\Omega, \Sigma, \sigma)$ such that

$$\langle u^*, f\rangle = \int_\Omega f \,\mathrm{d}\mu$$

for every $f \in \mathcal{L}^\infty\,(\Omega, \Sigma, \sigma)$ and

$$\|u^*\| = \|\mu\| = |\mu|\,(\Omega)\,.$$

On the other hand, every $\mu \in \mathrm{ba}(\Omega, \Sigma, \sigma)$ defines $u^* \in \mathcal{L}^\infty\,(\Omega, \Sigma, \sigma)^*$.

Hence, $\mathcal{L}^\infty\,(\Omega, \Sigma, \sigma)^*$ and $\mathrm{ba}(\Omega, \Sigma, \sigma)$ can be identified.

Using the decomposition Theorem 1.27 that was proved earlier, one obtains a more refined characterisation of the dual of $\mathcal{L}^\infty\,(\Omega, \Sigma, \sigma)$. In particular, every element of the dual space is the sum of a σ-measure with $\mathcal{L}^n$-density and a pure measure. In contrast to the literature, this makes the intuitive idea of the dual of $\mathcal{L}^\infty$ being $\mathcal{L}^1$ plus something which is not weakly absolutely continuous with respect to Lebesgue measure precise.

Theorem 2.22. *Let $\Omega \subset \mathbb{R}^n$ and $\Sigma \subset 2^\Omega$ be a σ-algebra and $\sigma : \Sigma \to \mathbb{R}$ be a σ-measure. Then for every $u^* \in \mathcal{L}^\infty(\Omega, \Sigma, \sigma)^*$ there exists a unique pure $\mu_p \in \mathrm{ba}(\Omega, \Sigma, \sigma)$ and a unique $h \in \mathcal{L}^1(\Omega, \Sigma, \sigma)$ such that*

$$\langle u^*, f \rangle = \int_\Omega f h \,\mathrm{d}\mathcal{L}^n + \int_\Omega f \,\mathrm{d}\mu_p$$

for every $f \in \mathcal{L}^\infty(\Omega, \Sigma, \sigma)$.

Proof. Let $u^* \in \mathcal{L}^\infty(\Omega, \Sigma, \sigma)^*$. Then by Proposition 2.21 there exists $\mu \in \mathrm{ba}(\Omega, \Sigma, \sigma)$ such that for all $f \in \mathcal{L}^\infty(\Omega, \Sigma, \sigma)$

$$\langle u^*, f \rangle = \int_\Omega f \,\mathrm{d}\mu \,.$$

Now, by proposition 1.27, there exist unique $\mu_c, \mu_p \in \mathrm{ba}(\Omega, \Sigma, \sigma)$ such that

$$\mu = \mu_c + \mu_p$$

and μ_c is a σ-measure and μ_p is pure. By the Radon-Nikodym Theorem (Proposition 1.29) there is $h \in \mathcal{L}^1(\Omega, \Sigma, \sigma)$ such that

$$\mu_c(A) = \int_A h \,\mathrm{d}\sigma$$

for every $A \in \Sigma$. Since the integral is obviously linear in μ

$$\int_\Omega f \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu_c + \int_\Omega f \,\mathrm{d}\mu_p = \int_\Omega f h \,\mathrm{d}\sigma + \int_\Omega f \,\mathrm{d}\mu_p$$

for every $f \in \mathcal{L}^\infty(\Omega, \Sigma, \sigma)$, whence the statement of the proposition follows. □

2.2 Completion of $\mathcal{L}^p$-Spaces

In this section, $\mathscr{L}$-spaces are presented as the completion of $\mathcal{L}$-spaces (cf. [30, p. 178ff], [6, p. 778], [28, p. 528]). The key point is that the completion of $\mathcal{L}$-spaces over μ are spaces of measures which are absolutely continuous with respect to μ.

The following definition is useful in the ensuing characterisation of $\mathscr{L}$-spaces. It was first introduced by Kolmogoroff (cf. [27, p. 663]) and is an abstraction of many notions of integral, including the Lebesgue integral.

Definition 2.23. *Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be an algebra and $\zeta : \mathcal{A} \to \mathbb{R}$ be a set function and $A \in \mathcal{A}$. Denote the class of all finite partitions of A by sets in $\mathcal{A}$ by*

$$\mathcal{P}^{\mathcal{A}}(A)\,.$$

This class is directed by the partial order defined for $P_1, P_2 \in \mathcal{P}^{\mathcal{A}}(A)$ by

$$P_1 \leq P_2 \iff \forall A_2 \in P_2 : \exists A_1 \in P_1 : A_2 \subset A_1\,.$$

Write

$$\int_A^{\mathcal{R}} \zeta := \lim_{P \in \mathcal{P}^{\mathcal{A}}(A)} \sum_{A \in P} \zeta(A)\,,$$

if this limit exists in the sense of nets, and call it ***refinement integral*** *of ζ on A and call ζ* ***refinement integrable*** *on A.*

If ζ is refinement integrable on Ω just call it ***refinement integrable****.*

Remark 2.24. Note that ζ is not required to be additive in the definition above.

The following proposition characterises the refinement integral as a kind of additivisation and is taken from [27, p. 664].

Proposition 2.25. *Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be an algebra and $\zeta : \mathcal{A} \to \mathbb{R}$ be a refinement integrable set function.*

Then

$$\int^{\mathcal{R}} \zeta : \mathcal{A} \to \mathbb{R} : A \mapsto \int_A^{\mathcal{R}} \zeta$$

is well-defined and a measure.

Remark 2.26. The set function $\int^{\mathcal{R}} \zeta$ is also called **refinement integral**.

Remark 2.27. Note that trivially

$$\zeta = \int^{\mathcal{R}} \zeta + \lambda\,,$$

where $\int^{\mathcal{R}} \lambda = 0$. The set function λ is in general not additive.

Thus, $\int^{\mathcal{R}} \zeta$ can be regarded as *additivisation* of ζ in a similar sense as Df is the linearisation of a point function f.

From now on, only measures are considered. The following convention is useful in the next statements.

Definition 2.28. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a σ-algebra and $\mu, \lambda \in \mathrm{ba}(\Omega, \Sigma)$ such that $\lambda <<^w \mu$. For $A \in \Sigma$ set*

$$\frac{\lambda}{\mu}(A) := \begin{cases} \frac{\lambda(A)}{\mu(A)} & \textit{if } \mu(A) \neq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

The following definition of norm is taken from [30, p. 180-183] (cf. [28, p. 528ff]).

Definition 2.29. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a σ-algebra and $\mu \in \mathrm{ba}(\Omega, \Sigma)$, $\mu \geq 0$. For any measure $\lambda : \Sigma \to \mathbb{R}$ such that $\lambda << \mu$ define*

$$\|\lambda\|_p := \left(\int_\Omega^{\mathcal{R}} \left| \frac{\lambda}{\mu} \right|^p \mu \right)^{\frac{1}{p}} = \left(\sup_{P \in \mathcal{P}^\Sigma(\Omega)} \sum_{\substack{A \in P \\ \mu(A) \neq 0}} \left| \frac{\lambda(A)}{\mu(A)} \right|^p \mu(A) \right)^{\frac{1}{p}}$$

where the supremum is taken over all finite partitions $\{A_k\}_{k=0}^m \subset \Sigma$ of Ω.

Furthermore define

$$\|\lambda\|_\infty := \sup \left\{ x \in \mathbb{R} \mid \exists A \in \Sigma : x = \left| \frac{\lambda(A)}{\mu(A)} \right| \right\}$$

Remark 2.30. Note that λ is demanded to not only be weakly absolutely continuous but absolutely continuous. This is needed to obtain the density of so-called simple measures.

Now, $\mathscr{L}^p$-spaces can be defined as the class of all measures which have finite norm. The definition is taken from [30, p. 185] (cf. [28, p. 530]).

Definition 2.31. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a σ-algebra and $\mu \in \mathrm{ba}(\Omega, \Sigma)$, $\mu \geq 0$. For $p \in [1, \infty]$ denote*

$$\mathscr{L}^p(\Omega, \Sigma, \mu) := \{\lambda \in \mathrm{ba}(\Omega, \Sigma) \mid \lambda << \mu, \|\lambda\|_p < \infty\}$$

In contrast to the $\mathcal{L}^p$-spaces defined in the previous section, these spaces are complete. The following proposition is taken from [30, p. 185].

Proposition 2.32. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a σ-algebra and $\mu \in \mathrm{ba}(\Omega, \Sigma)$, $\mu \geq 0$. Then for any $p \in [1, \infty]$*

$$\mathscr{L}^p(\Omega, \Sigma, \mu)$$

equipped with $\|\cdot\|_p$ is a Banach space.

The goal of this section is to present $\mathscr{L}^p$-spaces as the completions of $\mathcal{L}^p$-spaces. The following proposition shows that $\mathcal{L}^p$ can be regarded as a subspace of $\mathscr{L}^p$ and is taken from [30, p. 182].

Proposition 2.33. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a σ-algebra, $\mu \in \mathrm{ba}(\Omega, \Sigma)$ with $\mu \geq 0$, $p \in [1, \infty)$ and $f \in \mathcal{L}^p(\Omega, \Sigma, \mu)$.*

Then for the measure $\lambda : \Sigma \to \mathbb{R}$ defined by

$$\lambda(A) := \int_A f \, \mathrm{d}\mu \text{ for } A \in \Sigma$$

holds

$$\|f\|_p = \|\lambda\|_p \, .$$

In particular, $\lambda \in \mathscr{L}^p(\Omega, \Sigma, \mu)$.

Remark 2.34. Below, $\mathscr{L}^p(\Omega, \Sigma, \mu)$ is identified as the completion of $\mathcal{L}^p(\Omega, \Sigma, \mu)$.

An example of a measure for which $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ is not complete can be found in Remarks 4.6.8 and 7.2.15 in Rao [30, p. 125,p. 192]. Note that this example is constructed on $\Omega = \mathbb{N}$. It seems to be an open problem to find such a measure for $\Omega \subset \mathbb{R}^n$.

The same holds true for $p = \infty$ (cf. [30, p. 184]).

Proposition 2.35. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a σ-algebra, $\mu \in \mathrm{ba}(\Omega, \Sigma), \mu \geq 0$ and $f \in \mathcal{L}^\infty(\Omega, \Sigma, \mu)$.*

Then for the measure $\lambda : \Sigma \to \mathbb{R}$ defined by

$$\lambda(A) := \int_A f \, \mathrm{d}\mu \text{ for } A \in \Sigma$$

holds

$$\|f\|_\infty = \|\lambda\|_\infty \, .$$

In particular $\lambda \in \mathscr{L}^\infty(\Omega, \Sigma, \mu)$.

In order to obtain the completeness of $\mathscr{L}^p$-spaces, the simple functions must be embedded in $\mathscr{L}^p$. The following definition is taken from [30, p. 188] (cf. [28, p. 533]).

Definition 2.36. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be an σ-algebra and $\mu \in \mathrm{ba}(\Omega, \Sigma)$, $\mu \geq 0$.*

*A measure $\lambda \in \mathrm{ba}(\Omega, \Sigma)$ is called **simple measure** (with respect to μ) if there exists a partition $\{A_k\}_{k=0}^m \subset \Sigma$ of Ω and $\{a_k\}_{k=0}^m \subset \mathbb{R}$ such that*

$$\lambda = \sum_{k=0}^m a_k \cdot \mu \lfloor A_k \, .$$

The simple measure turn out to be dense in $\mathscr{L}^p$. Note that the simple functions are dense in $\mathcal{L}^p$, by definition. The following proposition is taken from [30, p. 190] (cf. [28, p. 533]).

Proposition 2.37. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a σ-algebra and $\mu \in \mathrm{ba}(\Omega, \Sigma)$, $\mu \geq 0$. Then for every $p \in [1, \infty)$, the simple measures are dense in $\mathscr{L}^p(\Omega, \Sigma, \mu)$.*

Remark 2.38. Note that the statement of the preceding proposition does in general not hold true for $p = \infty$ (cf. [30, p. 190]).

This proposition has several useful applications. An important one is the following, which is taken from [30, p. 192].

Corollary 2.39. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a σ-algebra and $\mu \in \mathrm{ba}(\Omega, \Sigma)$, $\mu \geq 0$.*

Then for every $p \in [1, \infty)$, $\mathscr{L}^p(\Omega, \Sigma, \mu)$ is the completion of $\mathcal{L}^p(\Omega, \Sigma, \mu)$.

The Radon-Nikodym Theorem is a direct consequence of this and is taken from [30, p. 191] (cf. [18, p. 315]).

Corollary 2.40. *Radon-Nikodym Theorem*
Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset 2^\Omega$ be a σ-algebra and $\mu \in \mathrm{ba}(\Omega, \Sigma)$, $\mu \geq 0$. Then for every $\lambda \in \mathrm{ba}(\Omega, \Sigma)$ such that $\lambda << \mu$ and every $\varepsilon > 0$ there exists a simple function $h : \Omega \to \mathbb{R}$ such that

$$\left| \lambda(A) - \int_A h \,\mathrm{d}\mu \right| < \varepsilon$$

for every $A \in \Sigma$.

Knowledge on the dual of a Banach space is beneficial in many situations. For $\mathscr{L}^p$-spaces they are known and have a similarly good structure as the original space. The following characterisation of the dual spaces of $\mathscr{L}^p$-spaces can be found in [30, p. 193] and [28, p. 536]. It is completely analogue to the statement for $\mathcal{L}^p$-spaces over σ-measures.

Proposition 2.41. *Let $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset 2^\Omega$ be a σ-algebra and $\mu \in \mathrm{ba}(\Omega, \Sigma)$, $\mu \geq 0$. Then for $p \in [1, \infty)$ the dual of $\mathscr{L}^p(\Omega, \Sigma, \mu)$ is isomorphic to $\mathscr{L}^{p'}(\Omega, \Sigma, \mu)$, where p' is the Hölder-conjugate of p.*

For $v \in \mathscr{L}^p(\Omega, \Sigma, \mu)$ and $v^* \in \mathscr{L}^{p'}(\Omega, \Sigma, \mu)$ the dual pairing is given by

$$\langle v^*, v \rangle = \int_\Omega^{\mathcal{R}} \frac{v}{\mu} \cdot v^* \,.$$

Furthermore,

$$| \langle v^*, v \rangle | \leq \|v^*\|_{p'} \|v\|_p \,.$$

Chapter 3

Pure Measures

This chapter discusses pure measures in detail. In the first section, a large class of pure measures is introduced. These measures share some structure with the density at zero from Example 1.20. Their structure is investigated and the properties of the set $\mathrm{Dens}(C)$ of all those measures is analysed. The action of this set on a fixed essentially bounded function is shown to be compatible with the essential supremum and the essential infimum. The extremal points in the sense of the Krein-Milman Theorem (cf. [19, p. 154], [34, p. 157]) are characterised, since they span $\mathrm{Dens}(C)$. The second section discusses the connection between pure measures and σ-measures which are singular with respect to Lebesgue measure. It is proven that every pure measure can be represented by a Radon measure on its core, if only continuous functions are considered. Vice versa, in regular settings, every Radon measure on a Lebesgue null set can be extended to a pure measure on all of the domain. Finally, some examples on traces show that pure measures are suitable for the representation of trace operators. Most of this chapter is comprised of original work that has not been treated in the literature, to the authors knowledge.

3.1 Density Measures

In this section, measures with a similar structure as the measure in Example 1.20 are investigated. These measures represent a large class of pure measures. The properties of this class is analysed and its extremal points identified.

It turns out that the signed distance function plays an important role.

Definition 3.1. *Let $\Omega \subsetneq \mathbb{R}^n$ be non-empty. The function*

$$\mathrm{dist}_\Omega : \mathbb{R}^n \to (-\infty, \infty)$$

defined by

$$\mathrm{dist}_\Omega(x) := \begin{cases} \inf_{y\in\Omega} |x-y| & \text{if } x \notin \Omega \\ -\inf_{y\in\Omega^c} |x-y| & \text{if } x \in \Omega\,. \end{cases}$$

*is called **signed distance function**.*

For sets $B \subset \mathbb{R}^n$ write

$$\mathrm{dist}_\Omega(B) := \inf_{x\in B} \mathrm{dist}_\Omega(x)\,.$$

Furthermore, neighbourhoods of sets prove useful. Therefore, set

$$\Omega_\delta := \mathrm{dist}_\Omega^{-1}((-\infty, \delta))$$

for $\delta \in \mathbb{R}$.

Remark 3.2. Note that dist_Ω is Lipschitz continuous, since it is the sum of two Lipschitz continuous functions. If $\Omega \subset \mathbb{R}^n$ is bounded, by the Coarea formula (cf. [20, p. 112])

$$\mathcal{H}^{n-1}(\partial(\Omega_\delta)) < \infty$$

for a.e. $\delta \in \mathcal{R}(\mathrm{dist}_{\partial\Omega})$, the range of $\mathrm{dist}_{\partial\Omega}$.

Now, density measures can be defined. The basic definition essentially demands the measure to be a probability measure whose core is a Lebesgue null set. By scaling, any bounded positive measure whose support has no volume can be seen as a density measure.

Definition 3.3. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$, $C \subset \overline{\Omega}$ be closed and $\mathcal{L}^n(C \cap \Omega) = 0$. A measure $\mu \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ is called a **density measure** for C, if $\mu \geq 0$ and for all $\delta > 0$*

$$\mu(C_\delta \cap \Omega) = \mu(\Omega) = 1\,.$$

The set of all density measures for C is denoted by

$$\mathrm{Dens}\,(C)\,.$$

Remark 3.4. If $\mathcal{L}^n(\Omega \cap C_\delta) = 0$ for some $\delta > 0$ or $C = \emptyset$, then

$$\mathrm{Dens}\,(C) = \emptyset\,.$$

The following proposition shows that density measures indeed have core on C and that they are pure.

Proposition 3.5. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ and $C \subset \overline{\Omega}$ be closed with $\mathcal{L}^n(C \cap \Omega) = 0$. Then for every $\mu \in \mathrm{Dens}\,(C)$*

$$\mathrm{core}\,\mu \subset C$$

and μ is pure.

Proof. Let $x \in \mathbb{R}^n \setminus C$. Let

$$\delta := \frac{1}{2}\,\mathrm{dist}_C(x)\,.$$

Then for every $0 < \tilde{\delta} < \delta$

$$\mu(B_{\tilde{\delta}}(x)) \leq \mu(\Omega \setminus C_\delta) = 0\,.$$

Hence

$$x \notin \mathrm{core}\,\mu\,,$$

and thus

$$\mathrm{core}\,\mu \subset C\,.$$

Finally

$$\mathcal{L}^n(\mathrm{core}\,\mu \cap \Omega) \leq \mathcal{L}^n(C \cap \Omega) = 0\,.$$

By Proposition 1.41, μ is pure. $\square$

Density measures can be characterised in a way that justifies their name. In essence, they are densities of other measures on their core.

Proposition 3.6. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ and $C \subset \overline{\Omega}$ be closed with $\mathcal{L}^n(C \cap \Omega) = 0$. A measure $\mu \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ is a density measure for C if and only if there exists a measure $\lambda \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ with $\lambda \geq 0$ satisfying*

$$\lambda(C_\delta \cap \Omega) > 0 \text{ for all } \delta > 0\,,$$

such that for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$

$$\int_\Omega f \,\mathrm{d}\mu \leq \limsup_{\delta \downarrow 0} ⨍_{C_\delta \cap \Omega} f \,\mathrm{d}\lambda\,.$$

Then for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$

$$⨍_C f \,\mathrm{d}\mu = \lim_{\delta \downarrow 0} ⨍_{C_\delta \cap \Omega} f \,\mathrm{d}\lambda \tag{3.1}$$

if this limit exists.

Proof. Let $\mu \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$.

Assume there exists $\lambda \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ with $\lambda \geq 0$ satisfying

$$\lambda(C_\delta \cap \Omega) > 0 \text{ for all } \delta > 0$$

such that for $f \in \mathcal{L}^\infty\,(\Omega, \mathcal{L}^n)$

$$\int_\Omega f \,\mathrm{d}\mu \leq \limsup_{\delta \downarrow 0} \fint_{C_\delta \cap \Omega} f \,\mathrm{d}\lambda\,.$$

Note that since

$$\int_\Omega -f \,\mathrm{d}\mu \leq \limsup_{\delta \downarrow 0} \fint_{C_\delta \cap \Omega} -f \,\mathrm{d}\lambda$$

for $f \in \mathcal{L}^\infty\,(\Omega, \mathcal{L}^n)$,

$$\liminf_{\delta \downarrow 0} \fint_{C_\delta \cap \Omega} f \,\mathrm{d}\lambda \leq \int_\Omega f \,\mathrm{d}\mu\,.$$

Then for $\delta > 0$

$$\mu(\Omega) = \mu(C_\delta \cap \Omega) = \lim_{\delta \downarrow 0} \frac{\lambda(C_\delta \cap \Omega)}{\lambda(C_\delta \cap \Omega)} = 1\,.$$

Furthermore, for every $B \in \mathcal{B}(\Omega)$

$$\mu(B) \geq \liminf_{\delta \downarrow 0} \frac{\lambda(B \cap C_\delta)}{\lambda(C_\delta \cap \Omega)} \geq 0\,.$$

Thus, μ is a density measure for C. Equation (3.1) follows with Proposition 3.5 and the previous estimates.

Now assume μ to be a density measure for C. Set $\lambda = \mu$. Note that $\lambda(C_\delta \cap \Omega) > 0$ for every $\delta > 0$. Then for all $f \in \mathcal{L}^\infty\,(\Omega, \mathcal{L}^n)$

$$\int_\Omega f \,\mathrm{d}\mu = \lim_{\delta \downarrow 0} \int_{C_\delta \cap \Omega} f \,\mathrm{d}\mu \leq \limsup_{\delta \downarrow 0} \fint_{C_\delta \cap \Omega} f \,\mathrm{d}\lambda\,.$$

□

Now, existence is proved. It turns out that every measure $\lambda \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$, which does not vanish near C, induces a density measure.

Proposition 3.7. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ and $C \subset \overline{\Omega}$ be closed with $\mathcal{L}^n(C \cap \Omega) = 0$. Furthermore, let $\lambda \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ with $\lambda \geq 0$ be such that for all $\delta > 0$*

$$\lambda(C_\delta \cap \Omega) > 0\,.$$

Then there exists a density measure $\mu \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for every $f \in \mathcal{L}^\infty\,(\Omega, \mathcal{L}^n)$

$$\liminf_{\delta \downarrow 0} \fint_{C_\delta \cap \Omega} f \,\mathrm{d}\lambda \leq \fint_C f \,\mathrm{d}\mu \leq \limsup_{\delta \downarrow 0} \fint_{C_\delta \cap \Omega} f \,\mathrm{d}\lambda\,.$$

Remark 3.8. In particular, if $\mathcal{L}^n(C_\delta \cap \Omega) > 0$ for every $\delta > 0$, then $\mathrm{Dens}\,(C) \neq \emptyset$. In order to see this, note that $\lambda = \mathcal{L}^n\lfloor\Omega$ satisfies the assumptions of the preceding proposition. Furthermore, every density measure arises in this way (cf. Proposition 3.6).

Proof. Let $\lambda \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ be such that for every $\delta > 0$

$$\lambda(C_\delta \cap \Omega) > 0\,.$$

Then

$$p : \mathcal{L}^\infty\,(\Omega, \mathcal{L}^n) \to \mathbb{R} : f \mapsto \limsup_{\delta\downarrow 0} \fint_{C_\delta\cap\Omega} f\,\mathrm{d}\lambda$$

is a positively homogeneous, subadditive functional. Set $X := \mathcal{L}^\infty\,(\Omega, \mathcal{L}^n)$ and

$$X_0 := \left\{ f \in X \mid \lim_{\delta\downarrow 0} \fint_{C_\delta\cap\Omega} f\,\mathrm{d}\lambda \text{ exists} \right\}\,.$$

Then X_0 is a linear subspace of X and

$$u_0^* : X_0 \to \mathbb{R} : f \mapsto \lim_{\delta\downarrow 0} \fint_{C_\delta\cap\Omega} f\,\mathrm{d}\lambda$$

is a continuous linear functional which is bounded by p. The subadditive version of the Hahn-Banach theorem [18, p. 62] yields the existence of a linear extension u^* of u_0^* to all of X which is bounded by p. Note that for every $f \in \mathcal{L}^\infty\,(\Omega, \mathcal{L}^n)$

$$\langle u^*, f\rangle \leq p(f) \leq \|f\|_\infty$$

since $\lambda <<^w \mathcal{L}^n$. Hence, u^* is a continuous linear functional on $\mathcal{L}^\infty\,(\Omega, \mathcal{L}^n)$. By Proposition 2.21, there exists $\mu \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for every $f \in \mathcal{L}^\infty\,(\Omega, \mathcal{L}^n)$

$$\langle u^*, f\rangle = \int_\Omega f\,\mathrm{d}\mu\,.$$

Note that for every $f \in \mathcal{L}^\infty\,(\Omega, \mathcal{L}^n)$

$$\int_\Omega -f\,\mathrm{d}\mu \leq p(-f) = \limsup_{\delta\downarrow 0} \fint_{C_\delta\cap\Omega} -f\,\mathrm{d}\lambda$$

which implies

$$\liminf_{\delta\downarrow 0} \fint_{C_\delta\cap\Omega} f\,\mathrm{d}\lambda \leq \int_\Omega f\,\mathrm{d}\mu\,.$$

Now it is easy to see that for every $B \in \mathcal{B}(\Omega)$

$$0 \leq \liminf_{\delta\downarrow 0} \fint_{C_\delta\cap\Omega} \chi_B\,\mathrm{d}\lambda \leq \mu(B)\,.$$

Hence, $\mu \geq 0$. Furthermore,

$$1 = \liminf_{\delta \downarrow 0} \fint_{C_\delta \cap \Omega} \chi_\Omega \, \mathrm{d}\lambda \leq \mu(\Omega) \leq \limsup_{\delta \downarrow 0} \fint_{C_\delta \cap \Omega} \chi_\Omega \, \mathrm{d}\lambda = 1 \, .$$

Finally, let $\tilde{\delta} > 0$. Then

$$1 = \liminf_{\delta \downarrow 0} \fint_{C_\delta \cap \Omega} \chi_{C_{\tilde{\delta}} \cap \Omega} \, \mathrm{d}\lambda \leq \mu(C_{\tilde{\delta}} \cap \Omega) \leq \limsup_{\delta \downarrow 0} \fint_{C_\delta \cap \Omega} \chi_{C_{\tilde{\delta}} \cap \Omega} \, \mathrm{d}\lambda = 1 \, .$$

Thus, μ is a density measure of C. □

Example 3.9. Let $\Omega \subset \mathbb{R}^2$ be a cusped set as in Figure 3.1 below and $C = \{x\}$, where $x \in \mathbb{R}^2$ is the point at the cusp. Then for every $\delta > 0$

$$\mathcal{L}^n(C_\delta \cap \Omega) > 0 \, .$$

Hence there exists a density measure $\mu \in \mathrm{Dens}(C)$ such that for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$

$$\fint_C f \, \mathrm{d}\mu = \lim_{\delta \downarrow 0} \fint_{C_\delta \cap \Omega} f \, \mathrm{d}\mathcal{L}^n \, ,$$

if this limit exists. This example is in essence identical to Example 1.20.

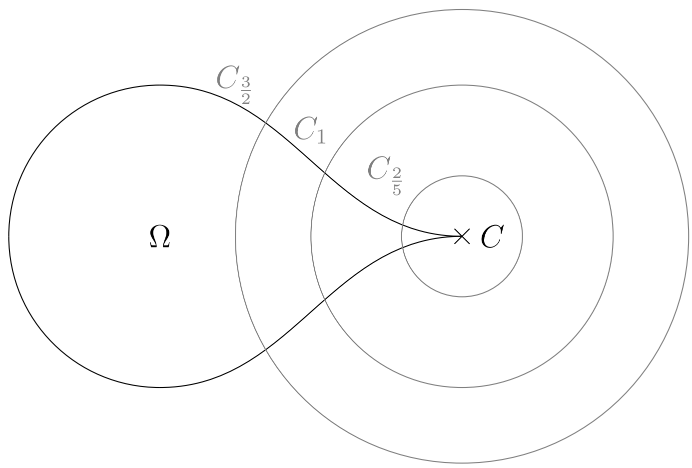

Figure 3.1: Existence of a density measure at a cusp

The integral with respect to a density measure can be estimated by the essential supremum and the essential infimum of the integrand near the core.

Proposition 3.10. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ and $C \subset \overline{\Omega}$ be closed with $\mathcal{L}^n(C \cap \Omega) = 0$. Furthermore, let $\mu \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ be a density measure of C. Then for every $f \in \mathcal{L}^\infty\,(\Omega, \mathcal{L}^n)$*

$$\lim_{\delta \downarrow 0} \operatorname*{ess\,inf}_{C_\delta \cap \Omega} f \leq \fint_C f \,\mathrm{d}\mu \leq \lim_{\delta \downarrow 0} \operatorname*{ess\,sup}_{C_\delta \cap \Omega} f$$

Proof. It suffices to prove the right-hand side of the inequality.

Let $f \in \mathcal{L}^\infty\,(\Omega, \mathcal{L}^n)$. Since $\mu \geq 0$, for every $\delta > 0$

$$\int_\Omega f \,\mathrm{d}\mu = \int_{C_\delta \cap \Omega} f \,\mathrm{d}\mu \leq \int_{C_\delta \cap \Omega} \operatorname*{ess\,sup}_{C_\delta \cap \Omega} f \,\mathrm{d}\mu = \operatorname*{ess\,sup}_{C_\delta \cap \Omega} f\,.$$

$\operatorname*{ess\,sup}_{C_\delta \cap \Omega} f$ is increasing in $\delta > 0$ and bounded. Passing to the limit yields the statement. □

If $\mathrm{Dens}\,(C) \neq \emptyset$ is ensured, then the inequalities in the preceding proposition are sharp.

Proposition 3.11. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ and $C \subset \overline{\Omega}$ be non-empty, closed with $\mathcal{L}^n(C \cap \Omega) = 0$ such that for every $\delta > 0$*

$$\mathcal{L}^n(C_\delta \cap \Omega) > 0\,.$$

Furthermore, let $f \in \mathcal{L}^\infty\,(\Omega, \mathcal{L}^n)$. Then

$$\sup_{\mu \in \mathrm{Dens}\,(C)} \fint_C f \,\mathrm{d}\mu = \lim_{\delta \downarrow 0} \operatorname*{ess\,sup}_{C_\delta \cap \Omega} f$$

and

$$\inf_{\mu \in \mathrm{Dens}\,(C)} \fint_C f \,\mathrm{d}\mu = \lim_{\delta \downarrow 0} \operatorname*{ess\,inf}_{C_\delta \cap \Omega} f\,.$$

Proof. Let $f \in \mathcal{L}^\infty\,(\Omega, \mathcal{L}^n)$ and $\varepsilon > 0$. Set

$$M_\varepsilon := \{x \in \Omega \mid f(x) \geq \lim_{\delta \downarrow 0} \operatorname*{ess\,sup}_{C_\delta \cap \Omega} f - \varepsilon\}$$

and

$$\lambda_\varepsilon := \mathcal{L}^n \lfloor M_\varepsilon\,.$$

Then $\lambda_\varepsilon \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ is positive and such that for every $\delta > 0$

$$\lambda_\varepsilon(C_\delta \cap \Omega) > 0\,.$$

Hence by Proposition 2.21 , there exists a density measure $\mu_\varepsilon \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ of Ω such that

$$\int_\Omega f \,\mathrm{d}\mu_\varepsilon \geq \liminf_{\delta\downarrow 0} \fint_\Omega f \,\mathrm{d}\lambda_\varepsilon \geq \lim_{\delta\downarrow 0} \operatorname*{ess\,sup}_{C_\delta\cap\Omega} f - \varepsilon\,.$$

Hence

$$\sup_{\mu\in\mathrm{Dens}\,(C)} \int_\Omega f \,\mathrm{d}\mu \geq \sup_{\varepsilon>0} \int_\Omega f \,\mathrm{d}\mu_\varepsilon \geq \lim_{\delta\downarrow 0} \operatorname*{ess\,sup}_{C_\delta\cap\Omega} f\,.$$

On the other hand, Proposition 3.10 yields

$$\sup_{\mu\in\mathrm{Dens}\,(C)} \int_\Omega f \,\mathrm{d}\mu \leq \lim_{\delta\downarrow 0} \operatorname*{ess\,sup}_{C_\delta\cap\Omega} f\,.$$

The statement for $\operatorname{ess\,inf}$ follows analogously. □

The set of all density measures is a weak* compact convex set, as the following proposition shows.

Proposition 3.12. *Let* $\Omega \in \mathcal{B}(\mathbb{R}^n)$, $C \subset \overline{\Omega}$ *be non-empty, closed such that* $\mathcal{L}^n(C \cap \Omega) = 0$. *Then* $\mathrm{Dens}\,(C)$ *is a convex weak* compact subset of* $\mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ *as the dual of* $\mathcal{L}^\infty(\Omega, \mathcal{L}^n)$.

Proof. W.l.o.g. $\mathrm{Dens}\,(C) \neq \emptyset$.

Let $\mu_1, \mu_2 \in \mathrm{Dens}\,(C)$ and $a_1, a_2 \in [0, 1]$ such that $a_1 + a_2 = 1$. Then for every $\delta > 0$

$$a_1\mu_1(C_\delta \cap \Omega) + a_2\mu_2(C_\delta \cap \Omega) = a_1\mu_1(\Omega) + a_2\mu_2(\Omega) = a_1 + a_2 = 1\,.$$

and

$$a_1\mu_1 + a_2\mu_2 \geq a_1\mu_1 \geq 0\,.$$

Hence, $\mathrm{Dens}\,(C)$ is a convex set.

For $\mu \in \mathrm{Dens}\,(C)$

$$\|\mu\| = |\mu|\,(\Omega) = \mu(\Omega) = 1\,.$$

Hence, $\mathrm{Dens}\,(C)$ is a bounded set.

Now let $\lambda \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \setminus \mathrm{Dens}\,(C)$. Then either $\lambda(\Omega) \neq 1$ or there is a $\delta > 0$ such that $\lambda(C_\delta \cap \Omega) \neq 1$ or there is $B \in \mathcal{B}(\Omega)$ such that $\lambda(B) < 0$.

Consider the first case. Set $\varepsilon := \frac{1}{2}|\lambda(\Omega) - 1|$. Then

$$V(\lambda) := \{\mu \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \mid |\mu(\Omega) - \lambda(\Omega)| < \varepsilon\}$$

is a weak* open set such that

$$V(\lambda) \cap \mathrm{Dens}\,(C) = \emptyset\,.$$

In the second case set $\varepsilon := \frac{1}{2}|\lambda(C_\delta \cap \Omega) - 1|$ and

$$V(\lambda) := \{\mu \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \mid |\mu(C_\delta \cap \Omega) - \lambda(C_\delta \cap \Omega)| < \varepsilon\}$$

is a weak* open set and

$$V(\lambda) \cap \mathrm{Dens}\,(C) = \emptyset\,.$$

In the third and final case set $\varepsilon := \frac{1}{2}|\lambda(B)|$ and

$$V(\lambda) := \{\mu \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \mid |\mu(B) - \lambda(B)| < \varepsilon\}\,.$$

Also in this case

$$V(\lambda) \cap \mathrm{Dens}\,(C) = \emptyset\,.$$

Since λ was arbitrary, the complement of $\mathrm{Dens}\,(C)$ is weak* open and thus, $\mathrm{Dens}\,(C)$ is weak* closed. The statement of the proposition follows by the Banach-Alaoglu/Alaoglu-Bourbaki Theorem (cf. [35, p. 777]). □

Now, the action of $\mathrm{Dens}\,(C)$ on a fixed essentially bounded function can be characterised.

Corollary 3.13. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$, $C \subset \overline{\Omega}$ be non-empty, closed such that $\mathcal{L}^n(C \cap \Omega) = 0$ and for every $\delta > 0$*

$$\mathcal{L}^n(C_\delta \cap \Omega) > 0\,.$$

Furthermore, let $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$.

Then

$$\langle \mathrm{Dens}\,(C), f\rangle = \left[\lim_{\delta \downarrow 0} \operatorname*{ess\,inf}_{C_\delta \cap \Omega} f, \lim_{\delta \downarrow 0} \operatorname*{ess\,sup}_{C_\delta \cap \Omega} f\right]\,.$$

Proof. Since $\mathrm{Dens}\,(C)$ is a weak* compact convex subset of $\mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$

$$\langle \mathrm{Dens}\,(C), f\rangle$$

is a convex compact subset of $\mathbb{R}$. In order to see this, note that

$$f \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)^*\,.$$

Since continuous images of compact sets are again compact,

$$\langle \mathrm{Dens}\,(C), f\rangle$$

is compact. The convexity follows from the convexity of Dens(C). By Proposition 3.10 and Proposition 3.11

$$\left(\lim_{\delta\downarrow 0} \operatorname*{ess\,inf}_{C_\delta\cap\Omega} f, \lim_{\delta\downarrow 0} \operatorname*{ess\,sup}_{C_\delta\cap\Omega} f\right) \subset \langle \mathrm{Dens}\,(C), f\rangle$$
$$\subset \left[\lim_{\delta\downarrow 0} \operatorname*{ess\,inf}_{C_\delta\cap\Omega} f, \lim_{\delta\downarrow 0} \operatorname*{ess\,sup}_{C_\delta\cap\Omega} f\right]$$

This, together with the fact that $\langle \mathrm{Dens}\,(C), f\rangle$ is closed, implies the statement. $\square$

Recall that for a convex set M in a locally convex topological vector space $m \in M$ is an extremal point if for every $m_1, m_2 \in M$ with $m_1 \neq m_2$ and $a_1, a_2 \in [0,1]$ with $a_1 + a_2 = 1$

$$m = a_1 m_1 + a_2 m_2 \implies a_1 = 1 - a_2 \in \{0,1\}\,.$$

The importance of extremal points follows from the theorem of Krein-Milman (cf. [19, p. 154], [34, p. 157]). In particular, every compact convex set is the closure of the convex hull of its extremal points. Note that the theorem also implies that the set of extremal points is non-empty. Hence, the extremal points of Dens(C) can be regarded as spanning Dens(C). The following proposition gives a sufficient and necessary condition for a density measure to be an extremal point.

Proposition 3.14. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$, $C \subset \overline{\Omega}$ be non-empty, closed such that $\mathcal{L}^n(C\cap\Omega) = 0$ and $\mu \in$ Dens(C).*

Then μ is an extremal point of Dens(C) *if and only if for every $B \in \mathcal{B}(\Omega)$ either $\mu(B) = 0$ or $\mu(B^c) = 0$.*

Proof. Let $\mu \in$ Dens(C) be such that for every $B \in \mathcal{B}(\Omega)$ either $\mu(B) = 0$ or $\mu(B^c) = 0$. Assume $\mu = a_1\mu_1 + a_2\mu_2$ for $\mu_1, \mu_2 \in$ Dens(C) and $a_1, a_2 \in (0,1)$ such that $a_1 + a_2 = 1$ and $\mu_1, \mu_2 \neq \mu$. Then there is $B \in \mathcal{B}(\Omega)$ such that

$$\mu_1(B) \neq \mu_2(B)\,.$$

Suppose $\mu(B) = 0$. Then $\mu_1(B) = \mu_2(B) = 0$, a contradiction to the assumption.

Hence $\mu(B) = 1$ and $\mu(B^c) = 0$.

This implies
$$\mu_1(B^c) = \mu_2(B^c) = 0$$

and thus
$$\mu_1(B) = 1 = \mu_2(B)\,,$$

a contradiction to the assumption.

Hence $\mu_1 = \mu_2 = \mu$ and μ is an extremal point of $\operatorname{Dens}(C)$.

Now, assume μ to be an extremal point of $\operatorname{Dens}(C)$ and assume, there exists $B \in \mathcal{B}(\Omega)$ such that $\mu(B), \mu(B^c) > 0$. Set

$$\begin{aligned}\mu_1 &:= \frac{1}{\mu(B)}\mu\lfloor B\\ \mu_2 &:= \frac{1}{\mu(B^c)}\mu\lfloor B^c .\end{aligned}$$

Then μ_1 and μ_2 are density measures and

$$\mu = \mu(B)\mu_1 + \mu(B^c)\mu_2 ,$$

and μ is not an extremal point of $\operatorname{Dens}(C)$ in contradiction to the assumption. $\square$

A simple consequence is that the core of extremal points contains exactly one point. This is the same in the case of Radon measure, where the Dirac-measures are the extremal points of the unit ball (cf. [19, p. 156]).

Corollary 3.15. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$, $C \subset \overline{\Omega}$ be non-empty, closed, $\mathcal{L}^n(C\cap\Omega) = 0$ and $\mu \in \operatorname{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ be an extremal point of* $\operatorname{Dens}(C)$*.*

Then $\operatorname{core}\mu$ *is a singleton.*

Proof. Assume there were $x, y \in \operatorname{core}\mu$ such that $x \neq y$. Let $\delta > 0$ be such that $\delta < \frac{1}{2}|x - y|$. Then either

$$\mu(B_\delta(x)) = 0 \text{ or } \mu(B_\delta(y)^c) = 0$$

in contradiction to $x, y \in \operatorname{core}\mu$. $\square$

Another obvious corollary gives the values of extremal points on sets B whose boundary does not meet the core of the extremal point.

Corollary 3.16. *Let $\Omega \subset \mathbb{R}^n$, $C \subset \overline{\Omega}$ be non-empty, closed, $\mathcal{L}^n(C \cap \Omega) = 0$ and $\mu \in \operatorname{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ be an extremal point of* $\operatorname{Dens}(C)$ *with* $\operatorname{core}\mu = \{x\}$ *for some $x \in \overline{\Omega}$.*

Then for every $B \in \mathcal{B}(\Omega)$

$$\mu(B) = \begin{cases} 1 & \text{if } x \in \operatorname{int} B,\\ 0 & \text{if } x \notin \overline{B} .\end{cases}$$

The question arises, what happens on sets whose boundary meets the core. The following proposition gives a partial answer to this. It states that extremal points concentrate along one-dimensional directions.

Proposition 3.17. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$, $C \subset \overline{\Omega}$ be closed with $\mathcal{L}^n(C \cap \Omega) = 0$ and $\mu \in \mathrm{Dens}\,(C)$ be an extremal point. Then there exist unique $x \in C$ and $v \in \mathbb{R}^n$ with $\|v\| = 1$ such that for every $\alpha \in \left(0, \frac{\pi}{2}\right)$*

$$\mu(K(x, v, \alpha) \cap \Omega) = 1\,,$$

where

$$K(x, v, \alpha) := \{y \in \mathbb{R}^n | y \neq x, \sphericalangle(y - x, v) < \alpha\}\,.$$

Proof. By Corollary 3.15, there is a unique $x \in C$ such that

$$\mathrm{core}\,\mu = \{x\}\,.$$

Let $\{\alpha_k\}_{k\in\mathbb{N}} \subset \left(0, \frac{\pi}{2}\right)$ be such that

$$\lim_{k\to\infty} \alpha_k = 0\,.$$

Let $S^n := \partial B_1(0)$ and for every $k \in \mathbb{N}$ and $v \in S^n$

$$V_v^k := \{v' \in S^n | \sphericalangle(v, v') < \alpha_k\}\,.$$

Then for each $k \in \mathbb{N}$

$$\left\{V_v^k\right\}_{v\in S^n}$$

is an open covering of S^n. Assume that for every $v \in S^n$

$$\mu(K(x, v, \alpha_k) \cap \Omega) = 0\,.$$

Since S^n is compact, there exists a finite set $M \subset S^n$ such that

$$S^n \subset \bigcup_{v\in M} V_v^k\,.$$

But then

$$B_1(x) \cap \Omega \subset \left(\{x\} \cup \bigcup_{v\in M} K(x, v, \alpha_k)\right) \cap \Omega\,.$$

Hence

$$\mu(\Omega) = \mu(B_1(x) \cap \Omega) \leq \mu(\{x\} \cap \Omega) + \sum_{v\in M} \mu\left(K(x, v, \alpha_k) \cap \Omega\right) = 0\,,$$

in contradiction to

$$\mu(\Omega) = 1\,.$$

Hence, for every $k \in \mathbb{N}$, there exists $v_k \in S^n$ such that

$$\mu(K(x, v_k, \alpha_k) \cap \Omega) = 1\,.$$

Since S^n is compact, up to a subsequence

$$v_k \xrightarrow{k\to\infty}: v \in S^n\,.$$

Now let $\alpha > 0$ and $k_0 \in \mathbb{N}$ be such that for every $k \in \mathbb{N}$, $k \geq k_0$

$$\sphericalangle(v_k, v) < \frac{\alpha}{2} \text{ and } \alpha_k < \frac{\alpha}{2}\,.$$

Then

$$K(x, v, \alpha) \supset K(x, v_k, \alpha_k)$$

for every $k \geq k_0$ and thus

$$\mu(K(x, v, \alpha) \cap \Omega) \geq \mu(K(x, v_k, \alpha_k) \cap \Omega) = 1\,.$$

In order to prove that v is unique, assume there exists $v' \in \mathbb{R}^n$, $v' \neq v$ such that the statement of the proposition holds. Set

$$\alpha := \frac{1}{3}\sphericalangle(v, v')$$

and note that

$$K(x, v, \alpha) \cap K(x, v', \alpha) = \emptyset\,.$$

But then

$$\mu(\Omega \cap (K(x, v, \alpha) \cup K(x, v', \alpha))) = \mu(\Omega \cap K(x, v, \alpha)) + \mu(\Omega \cap K(x, v', \alpha)) = 2$$

a contradiction to $\mu(\Omega) = 1$. $\square$

Remark 3.18. The proposition above shows that extremal points in Dens(C) concentrate around one dimensional directions. Figure 3.2 illustrates this. Note that it is only necessary for an extremal point of Dens(C) to concentrate in this way. A sufficient condition might be that it concentrates on a cusp but this is still an open problem.

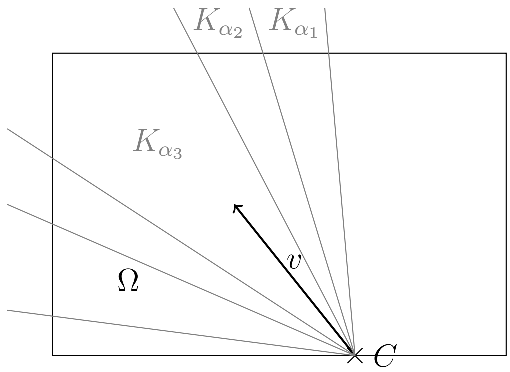

Figure 3.2: The cones on which an extremal point of $\mathrm{Dens}(C)$ is concentrated

Remark 3.19. The extremal points of $\mathrm{Dens}(C)$ are called **directionally concentrated** density measures.

Integration with respect to bounded density measures as laid out in Section 2.1 is well-suited for essentially bounded functions $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$ but in general it is not suited for unbounded functions. The following example illustrates this.

Example 3.20. Let $n = 1$, $\Omega = B_1(0) \subset \mathbb{R}$ and $C := \{0\}$. Let

$$f(x) := \frac{1}{\sqrt{|x|}} \left(\chi_{(-\infty,0)}(x) - \chi_{[0,\infty)}(x)\right)$$

for $x \in \mathbb{R}$. Then

$$\lim_{\delta \downarrow 0} ⨍_{B_\delta(0)} f \, \mathrm{d}\mathcal{L}^n = 0\,.$$

Let $\mu \in \mathrm{Dens}(C)$ be a density measure of C. Then for every $\varepsilon > 0$ and every simple $h \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$

$$|\mu|(\{|f-h| > \varepsilon\}) \geq |\mu|(\{|f| > \|h\|_\infty + \varepsilon\}) = 1\,. \tag{3.2}$$

Hence there is no sequence of simple function that converge in measure to f and thus f is not μ-integrable.

This chapter is closed with some suggestions of further uses for density measures. For example, the trace of a function of bounded variation can be computed using density measures.

Example 3.21. Let $\Omega \subset \mathbb{R}^n$ be bounded with Lipschitz boundary. For $x \in \partial\Omega$ let $\mu_x \in \mathrm{Dens}(\{x\})$ be such that

$$⨍_{\{x\}} f \, \mathrm{d}\mu_x \leq \limsup_{\delta \downarrow 0} ⨍_{B_\delta(x) \cap \Omega} f \, \mathrm{d}\mathcal{L}^n$$

for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$. Then for every $f \in \mathcal{BV}(\Omega) \cap \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$ and $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$

$$\mathrm{T}^\Omega(f)(x) = \fint_{\{x\}} f \, \mathrm{d}\mu_x \,,$$

where T^Ω is the usual trace operator for functions of bounded variation (cf. [20, p. 181]).

It is also possible to use density measures to define a set-valued gradient for Lipschitz continuous functions.

Example 3.22. Let $C = \{x\} \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous. Note that by Rademachers Theorem (cf. [20, p. 81]), Df exists almost everywhere and is essentially bounded. Set

$$\partial_d f(x) := \langle \mathrm{Dens}(\{x\}), Df \rangle \,.$$

Then $\partial_d f(x)$ is a weak* compact, convex set which is contained in $B_L(0)$, where L is the Lipschitz constant of f. In plus, the linearity of the integral implies that for every $f_1, f_2 \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R})$

$$\partial_d(f_1 + f_2)(x) \subset \partial_d f_1(x) + \partial_d f_2(x) \,.$$

and

$$\partial_d(f_1 f_2)(x) \subset f_1(x)\partial_d(f_2)(x) + f_2(x)\partial_d(f_1)(x) \,.$$

Note that the definition of ∂_d hints at similarities to a characterisation of Clarkes Generalised Gradient in [14, p. 63].

3.2 Singular σ-Measures and Pure Measures

In this section, the possibility to identify σ-measures that are singular with respect to Lebesgue measure and density measures is investigated. It is shown that any pure measure gives rise to a Radon measure on its core and, in regular settings, every Radon measure which is singular with respect to Lebesgue measure gives rise to a pure measure on all of its domain. Finally some examples show that pure measure are suitable for the representation of traces of functions. These results are original work, to the authors knowledge.

The following proposition states that every pure measure induces a Radon measure on its core.

Proposition 3.23. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ be bounded and $\mu \in \mathrm{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$. Then there exists a Radon measure σ supported on $\operatorname{core}\mu \subset \overline{\Omega}$ such that for every $\phi \in C(\overline{\Omega})$*

$$\int_\Omega \phi \, \mathrm{d}\mu = \int_{\operatorname{core}\mu} \phi \, \mathrm{d}\sigma \,.$$

Proof. First, note that for every $\phi \in C(\overline{\Omega})$

$$\left| \int_\Omega \phi \, \mathrm{d}\mu \right| \leq \|\phi\|_C \cdot |\mu| (\Omega)$$

Furthermore, note that every $\phi \in C(\overline{\Omega})$ can be extended to a function $\overline{\phi} \in C_0 \left(\overline{\Omega}\right)$ and every element of $C_0 \left(\overline{\Omega}\right)$ can be restricted to Ω to obtain an element of $C(\overline{\Omega})$. Hence

$$u^* : C_0 \left(\overline{\Omega}\right) \to \mathbb{R} : \phi \mapsto \int_\Omega \phi \, \mathrm{d}\mu$$

is a continuous linear operator and by the Riesz Representation Theorem (cf. [21, p. 106]) there is a Radon measure σ on $\overline{\Omega}$ such that for every $\phi \in C(\overline{\Omega})$

$$\int_\Omega \phi \, \mathrm{d}\mu = \int_{\overline{\Omega}} \phi \, \mathrm{d}\sigma \, .$$

Now let $x \in \overline{\Omega} \backslash \operatorname{core} \mu$. Then there exists a $\delta > 0$ such that $B_\delta(x) \cap \operatorname{core} \mu = \emptyset$.

Then for every $\phi \in C_0 \left(B_\delta(x) \cap \overline{\Omega}\right)$

$$\int_{\overline{\Omega}} \phi \, \mathrm{d}\sigma = \int_\Omega \phi \, \mathrm{d}\mu = 0 \, .$$

Hence

$$|\sigma| (B_\delta(x)) = 0$$

and thus x is not in the support of the σ-measure σ. Since $x \in \overline{\Omega} \setminus \operatorname{core} \mu$ was arbitrary, it is proved that the support of σ is indeed a subset of $\operatorname{core} \mu$. This proves the statement of the proposition. ☐

Remark 3.24. In the setting of the proposition above, σ is said to be a **representation of** μ on $\operatorname{core} \mu$.

The next proposition gives a partial inverse to the statement of the proposition above. In particular, any Radon measure can be extended to a measure on all of its domain.

Proposition 3.25. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ be bounded and $C \subset \overline{\Omega}$ be closed such that for every $x \in C$ and every $\delta > 0$*

$$\mathcal{L}^n(B_\delta(x) \cap \Omega) > 0 \, .$$

Furthermore, let σ be a Radon measure on C. Then there exists $\mu \in \operatorname{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for every $\phi \in C(\overline{\Omega})$

$$\int_\Omega \phi \, \mathrm{d}\mu = \int_C \phi \, \mathrm{d}\sigma \, .$$

In particular,

$$\operatorname{core}\mu \subset C$$

and

$$|\mu|(\Omega) = |\sigma|(C).$$

Remark 3.26. The conditions of the statement are satisfied if, for example, $C \subset \partial_*\Omega \cup \Omega_{int}$.

Proof. Let $\phi \in C(\overline{\Omega})$. Then

$$\|\phi|_C\|_C \leq \|\phi\|_\infty .$$

In order to see this, let $\varepsilon > 0$ and $x \in C$ be such that

$$|\phi(x) - \|\phi|_C\|_C| < \frac{\varepsilon}{2}.$$

Let $\delta > 0$ be such that for all $y \in B_\delta(x) \cap \Omega$

$$|\phi(x) - \phi(y)| < \frac{\varepsilon}{2}.$$

By assumption

$$\mathcal{L}^n(B_\delta(x) \cap \Omega) > 0$$

whence

$$\|\phi\|_\infty \geq |\phi(x)| - \frac{\varepsilon}{2} \geq \|\phi|_C\|_C - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the statement follows.

Set

$$u_0^* : C(\overline{\Omega}) \subset \mathcal{L}^\infty(\Omega, \mathcal{L}^n) \to \mathbb{R} : \phi \mapsto \int_C \phi \, \mathrm{d}\sigma$$

and note that for every $\phi \in C(\overline{\Omega})$

$$|\langle u_0^*, \phi\rangle| \leq \|\phi|_C\|_C |\sigma|(C) \leq \|\phi\|_\infty |\sigma|(C).$$

By the Hahn-Banach theorem (cf. [18, p. 63]) there exists a continuous extension u^* of u_0^* to all of $\mathcal{L}^\infty(\Omega, \mathcal{L}^n)$ such that

$$\|u^*\| = \|u_0^*\| .$$

But $\mathcal{L}^\infty(\Omega, \mathcal{L}^n)^* = \operatorname{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ by Proposition 2.21. Hence, there exists $\mu \in \operatorname{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for every $\phi \in C(\overline{\Omega})$

$$\int_\Omega \phi \, \mathrm{d}\mu = \int_C \phi \, \mathrm{d}\sigma.$$

Let $\phi \in C(\overline{\Omega})$ such that $\|\phi\|_\infty \le 1$. Then

$$\|\phi|_C\|_C \le \|\phi\|_\infty \le 1\,.$$

Hence,

$$|\mu|(\Omega) = \|u_0^*\| = \sup_{\substack{\phi \in C(\overline{\Omega}),\\ \|\phi\|_\infty \le 1}} \int_\Omega \phi \,\mathrm{d}\mu \le \sup_{\substack{\phi \in C(\overline{\Omega}),\\ \|\phi|_C\|_C \le 1}} \int_C \phi \,\mathrm{d}\sigma \le |\sigma|(C)\,.$$

Note that for every $\phi \in C(\overline{\Omega})$

$$\max(\min(\phi, 1), -1) \in C(\overline{\Omega})$$

and that every $\phi \in C_0(C)$ can be extended to all of $\overline{\Omega}$, preserving the norm (cf. [29, p. 25]). Hence, every $\phi \in C_0(C)$ can be extended to $\overline{\phi} \in C_0(\overline{\Omega})$ such that

$$\|\phi\|_C = \|\overline{\phi}\|_C\,.$$

Thus

$$|\sigma|(C) = \sup_{\substack{\phi \in C_0(C),\\ \|\phi\|_C \le 1}} \int_C \phi \,\mathrm{d}\sigma = \sup_{\substack{\overline{\phi} \in C_0(\overline{\Omega}),\\ \|\overline{\phi}\|_C \le 1}} \int_\Omega \overline{\phi} \,\mathrm{d}\mu \le \sup_{\substack{\overline{\phi} \in C_0(\overline{\Omega}),\\ \|\overline{\phi}\|_\infty \le 1}} \int_\Omega \overline{\phi} \,\mathrm{d}\mu \le |\mu|(\Omega)\,.$$

Since changing ϕ outside of C does not change the integral, $\operatorname{core}\mu \subset C$. This finishes the proof. $\square$

The measure from the preceding proposition is pure if the Radon measure is singular with respect to Lebesgue measure.

Corollary 3.27. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ be bounded and $C \subset \overline{\Omega}$ be closed such that for every $x \in C$ and $\delta > 0$*

$$\mathcal{L}^n(B_\delta(x) \cap \Omega) > 0$$

and

$$\mathcal{L}^n(C \cap \Omega) = 0\,.$$

Furthermore, let σ be a Radon measure on C.

Then there exists $\mu \in \operatorname{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for all $\phi \in C_0(\Omega)$

$$\int_\Omega \phi \,\mathrm{d}\mu = \int_C \phi \,\mathrm{d}\sigma\,.$$

Furthermore,

$$|\mu|(\Omega) = |\sigma|(C)$$

and μ is pure.

Proof. The preceding proposition and Proposition 1.41 yield the statement. □

The following example presents another way to construct a density at zero.

Example 3.28. Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ be bounded and $x \in \overline{\Omega}$ such that for every $\delta > 0$

$$\mathcal{L}^n(B_\delta(x) \cap \Omega) > 0\,.$$

Then there exists a pure $\mu \in \mathrm{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for every $\phi \in C(\overline{\Omega})$

$$\int_\Omega \phi \,\mathrm{d}\mu = \phi(x)\,.$$

The next example shows an extension for $\mathcal{H}^{n-1}$.

Example 3.29. Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ be open, bounded and have smooth boundary. Then $\mathcal{L}^n(\partial\Omega) = 0$ and $C = \partial\Omega$ satisfies the assumptions of Proposition 3.25. Hence, there exists $\mu \in \mathrm{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for all $\phi \in C(\overline{\Omega})$

$$\int_{\partial\Omega} \phi \,\mathrm{d}\mathcal{H}^{n-1} = \int_\Omega \phi \,\mathrm{d}\mu\,.$$

The following example shows, that the surface part of a Gauß formula can be expressed as an integral with respect to a pure measure. In Section 5.1 this is extended to vector fields having divergence measure.

Example 3.30. Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ be a bounded set with smooth boundary. Then $C = \partial\Omega \subset \overline{\Omega}$ is a closed set and for every $k \in \mathbb{N}$ such that $1 \leq k \leq n$

$$\nu^k \cdot \mathcal{H}^{n-1}\lfloor\partial\Omega$$

is a Radon measure on C. By Proposition 3.25 there exists $\mu_k \in \mathrm{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that for every $\phi \in C(\overline{\Omega})$

$$\int_{\partial\Omega} \phi \cdot \nu^k \,\mathrm{d}\mathcal{H}^{n-1} = \int_\Omega \phi \,\mathrm{d}\mu_k = \fint_{\partial\Omega} \phi \,\mathrm{d}\mu_k$$

and

$$\mathrm{core}\,\mu_k \subset \partial\Omega\,.$$

Hence, there exists $\mu \in (\mathrm{ba}(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n$ such that for all $\phi \in C^1(\overline{\Omega}, \mathbb{R}^n)$

$$\fint_{\partial\Omega} \phi \,\mathrm{d}\mu = \int_\Omega \phi \,\mathrm{d}\mu = \int_{\partial\Omega} \phi \cdot \nu \,\mathrm{d}\mathcal{H}^{n-1} = \int_\Omega \mathrm{div}\,\phi \,\mathrm{d}\mathcal{L}^n\,,$$

where the Gauß formula for sets with finite perimeter from Evans [20, p. 209] was used. Furthermore,

$$\mathrm{core}\,\mu \subset \partial\Omega$$

and μ is pure by Proposition 1.41.

Chapter 4

Vector Fields Having Divergence Measure

The following chapter contains a short exposition on functions of bounded variation and some useful statements for vector fields having divergence measure. The section on functions of bounded variation contains a useful proposition on the quality of approximation of χ_B by mollification. In the second section on vector fields having divergence measure, product formulas are addressed. These are needed for the subsequent analysis of Gauß formulas.

4.1 Functions of Bounded Variation

In the following, the functions of bounded varation and some of their properties needed for the analysis are presented. See Evans [20] or Ambrosio [2] for more details.

The following basic definition is taken from [20, p. 166].

Definition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in \mathcal{L}^1(\Omega, \mathcal{L}^n)$. Then*

$$|Df|(\Omega) := \sup\left\{\int_\Omega f \operatorname{div}\phi \, \mathrm{d}\mathcal{L}^n \mid \phi \in C_0^1(\Omega, \mathbb{R}^n), |\phi| \le 1\right\}$$

*is called **total variation** of f.*

*If $|Df|(\Omega) < \infty$, f is called **function of bounded varation**.*

The space of all functions of bounded variation is denoted by

$$\mathcal{BV}(\Omega) .$$

The norm on $\mathcal{BV}(\Omega)$ is defined by

$$\|f\|_{\mathcal{BV}} := \|f\|_1 + |Df|(\Omega) .$$

This turns $\mathcal{BV}(\Omega)$ into a Banach space.

Sets of finite perimeter are defined through the regularity of their characteristic function. The following definition is taken from [20, p. 167].

Definition 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open and $B \in \mathcal{B}(\Omega)$. Then B is called* ***set of finite perimeter*** *in Ω, if*

$$\chi_B \in \mathcal{BV}(\Omega) \ .$$

The following proposition on the structure of the distributional derivative of a function of bounded variation is taken from [20, p. 167].

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in \mathcal{BV}(\Omega)$. Then there exists a Radon measure σ on Ω and a σ-measurable $h : \Omega \to \mathbb{R}^n$ such that*

1. $|h(x)| = 1 \quad \sigma$*-a.e.*
2. $\int_\Omega f \operatorname{div} \phi \, \mathrm{d}\mathcal{L}^n = - \int_\Omega \phi \cdot h \, \mathrm{d}\sigma$

for all $\phi \in C_0^\infty(\Omega, \mathbb{R}^n)$.

For sets of finite perimeter, the following convention is used (cf. [20, p. 169]).

Definition 4.4. *Let $\Omega \subset \mathbb{R}^n$ be open, $f \in \mathcal{BV}(\Omega)$ and B be a set of finite perimeter. Then write*

1. $Df := hd\sigma$
2. $\|\partial B\| := \sigma$ *and* $\nu^B := -h$*, with h and σ for χ_B*

with h and σ as in Proposition 4.3.

The following definition of measure theoretic interior and exterior is taken from [20, p. 45].

Definition 4.5. *Let $B \in \mathcal{B}(\mathbb{R}^n)$. The set*

$$B_{int} := \left\{ x \in \mathbb{R}^n | \lim_{\delta \downarrow 0} \frac{\mathcal{L}^n(B \cap B_\delta(x))}{\mathcal{L}^n(B_\delta(x))} = 1 \right\}$$

is called measure theoretic interior of B.

The set

$$B_{ext} := \left\{ x \in \mathbb{R}^n | \lim_{\delta \downarrow 0} \frac{\mathcal{L}^n(B \cap B_\delta(x))}{\mathcal{L}^n(B_\delta(x))} = 0 \right\}$$

is called measure theoretic exterior of B.

The set

$$\partial_* B := \mathbb{R}^n \setminus (B_{int} \cup B_{ext})$$

is called measure theoretic boundary of B.

For Gauß formulas, the part of the measure theoretic boundary which admits a measure theoretic normal is of interest. The following definition of the reduced boundary is taken from [20, p. 194]

Definition 4.6. *Let $\Omega \subset \mathbb{R}^n$ be open and $B \in \mathcal{B}(\Omega)$ be a set of finite perimeter in Ω. The set $\partial^* B$ of all $x \in \mathbb{R}^n$ such that*

1. $\|\partial B\|\left(B_\delta(x)\right) > 0$ *for all* $\delta > 0$
2. $\nu^B(x) = \lim\limits_{\delta \downarrow 0} \fint_{B_\delta(x)} \nu^B \,\mathrm{d}\|\partial B\|$
3. $|\nu^B(x)| = 1$

is called ***reduced boundary*** *of* B.

Remark 4.7. The reduced boundary is the set, where a measure theoretic normal can be defined. The derivative of the characteristic function of a set of finite perimeter satisfies

$$D\chi_B = \|\partial B\| = \nu^B \cdot \mathcal{H}^{n-1} .$$

The measure theoretic normal at $x \in \partial^* B$ is characterised by

$$\lim_{\delta \downarrow 0} \frac{\mathcal{L}^n(B_\delta(x) \cap B \cap \{y \in \mathbb{R}^n \mid \nu^B(x) \cdot (y-x) \geq 0\})}{\mathcal{L}^n(B_\delta(x))} = 0 ,$$

i.e. the set B locally resembles a halfspace with outer normal ν^B (cf. [20, p. 203]).

The following proposition is a useful tool in the proof of existence of normal measures below. It states that the characteristic function of a set of finite perimeter can be approximated $\mathcal{H}^{n-1}$-a.e. by functions with gradients that are bounded in $\mathcal{L}^1$ (cf. [2, p. 163]).

Proposition 4.8. *Let $\Omega \subset \mathbb{R}^n$ be open and $B \in \mathcal{B}(\Omega)$ be a set of finite perimeter in Ω such that* $\operatorname{dist}_{\partial\Omega}(B) > 0$. *Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a mollification kernel (cf. [20, p. 122]). Then the functions $\chi_\delta : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\chi_\delta(x) = \rho_\delta * \chi_B(x) = \frac{1}{\delta^n} \int_B \rho\left(\frac{x-y}{\delta}\right) \mathrm{d}\mathcal{L}^n$$

satisfy

$$\chi_\delta \xrightarrow{\delta \downarrow 0} {\chi_B}^* = \chi_{B_{int}} + \frac{1}{2}\chi_{\partial^* B} \quad \mathcal{H}^{n-1}\text{-a.e.}$$

and

$$\|D\chi_\delta\|_1 \xrightarrow{\delta \downarrow 0} |D\chi_B|(\Omega) .$$

Proof. Begin by proving convergence $\mathcal{H}^{n-1}$-a.e.

1. If $x \in B_{int}$, then for every $\varepsilon > 0$ there exists $\delta_0 > 0$ such that

$$\mathcal{L}^n(B_\delta(x) \setminus B) \leq \varepsilon \delta^n$$

for every $\delta < \delta_0$.

Let $\varepsilon > 0$ be arbitrary. Then

$$\begin{aligned}\chi_\delta(x) &= \frac{1}{\delta^n} \int_{B_\delta(x)} \rho\left(\frac{x-y}{\delta}\right) \mathrm{d}\mathcal{L}^n - \frac{1}{\delta^n} \int_{B_\delta(x) \setminus B} \rho\left(\frac{x-y}{\delta}\right) \mathrm{d}\mathcal{L}^n \\ &\geq 1 - \varepsilon\,,\end{aligned}$$

for $\delta < \delta_0$.

Since $\varepsilon > 0$ was arbitrary and $\chi_\delta \leq 1$

$$\chi_\delta(x) \xrightarrow{\delta \downarrow 0} 1\,.$$

A similar argument yields the statement for $x \in B_{ext}$.

2. If $x \in \partial^* B$, then x lies in the jumpset of χ_B (cf. Example 3.68 [2, p. 163]). Hence

$$\chi_\delta(x) \xrightarrow{\delta \downarrow 0} \frac{1}{2}$$

by proposition 3.69 in [2, p. 164].

Since $\mathbb{R}^n \setminus (B_{int} \cup B_{ext} \cup \partial^* B)$ is a $\mathcal{H}^{n-1}$-null set

$$\chi_\delta \xrightarrow{\delta \downarrow 0} \chi_{B_{int}} + \frac{1}{2}\chi_{\partial^* B} \quad \mathcal{H}^{n-1}\text{-a.e.}$$

It remains to show that

$$\|D\chi_\delta\|_1 \xrightarrow{\delta \downarrow 0} |D\chi_B|(\Omega)\,.$$

Let $0 < \delta < \frac{\mathrm{dist}_{\partial\Omega}(B)}{2}$ and let $\phi \in C_0^1(\Omega, \mathbb{R}^n)$ be such that $\|\phi\|_C \leq 1$ with $\operatorname{supp} \phi \subset B_\delta$. Then

$$\begin{aligned}\int_\Omega \chi_\delta \operatorname{div} \phi \,\mathrm{d}\mathcal{L}^n &= \int_\Omega (\rho_\delta * \chi_B) \operatorname{div} \phi \,\mathrm{d}\mathcal{L}^n \\ &= \int_\Omega \chi_B (\rho_\delta * \operatorname{div} \phi) \,\mathrm{d}\mathcal{L}^n \\ &= \int_\Omega \chi_B \operatorname{div} (\rho_\delta * \phi) \,\mathrm{d}\mathcal{L}^n \\ &\leq |D\chi_B|(\Omega)\,,\end{aligned}$$

since $|\rho_\delta * \phi| \leq 1$ and $\operatorname{supp}(\rho_\delta * \phi) \subset\subset \operatorname{int}\Omega$.

This implies

$$\|D\chi_\delta\|_1 = |D\chi_\delta|(B_\delta) \leq |D\chi_B|(\Omega).$$

On the other hand, since $\chi_\delta \xrightarrow{L^1} \chi_B$ and the total variation is lower semi-continuous

$$|D\chi_B|(\Omega) \leq \liminf_{\delta\downarrow 0} |D\chi_\delta|(\Omega) = \liminf_{\delta\downarrow 0} |D\chi_\delta|(B_\delta).$$

This yields the statement of the proposition. □

4.2 Vector Fields Having Divergence Measure

This section contains an exposition of a number of useful product rules for vector fields having divergence measure.

The following definition of vector fields having divergence measure is in accordance with Chen [9, p. 402].

Definition 4.9. *Let $U \subset \mathbb{R}^n$ be open. A function $F \in \mathcal{L}^1(U, \mathbb{R}^n, \mathcal{L}^n)$ is called vector field **having divergence measure** if*

$$\sup\left\{\int_U F \cdot D\phi \,\mathrm{d}\mathcal{L}^n \mid \phi \in C_0^1(U), |\phi| \leq 1\right\} < \infty.$$

The spaces of vector fields having divergence measure are defined as follows (cf. [9, p. 402]).

Definition 4.10. *Let $U \subset \mathbb{R}^n$ be open and $p \in [1, \infty]$. The set of all $F \in \mathcal{L}^p(U, \mathbb{R}^n, \mathcal{L}^n)$ having divergence measure is denoted by*

$$\mathcal{DM}^p(U, \mathbb{R}^n).$$

$\mathcal{DM}^p(U, \mathbb{R}^n)$ is a Banach space with the norm

$$\|F\|_{\mathcal{DM}^p} := \sup_{\substack{k\in\mathbb{N}\\ 1\leq k\leq n}} \|F_k\|_p + |\operatorname{div} F|(U) \quad \textit{for } F \in \mathcal{DM}^p(U, \mathbb{R}^n).$$

The following result on the structure of the distributional divergence of vector fields having divergence measure is taken from [31, p. 529].

Proposition 4.11. *Let $U \subset \mathbb{R}^n$ be open and $F \in \mathcal{DM}^1(U, \mathbb{R}^n)$. Then there exists a Radon measure $\operatorname{div} F$ on U such that for every $\phi \in C_0^1(U)$*

$$\int_U F \cdot D\phi \,\mathrm{d}\mathcal{L}^n = -\int_U \phi \,\mathrm{d}\operatorname{div} F.$$

The following important proposition can be found in [11, p. 252]. It states that compactly supported divergence measure fields have zero divergence on sets containing their support.

Proposition 4.12. *Let $U \subset \mathbb{R}^n$ be open and $F \in \mathcal{DM}^1(U, \mathbb{R}^n)$ be such that there exists a compact set $C \subset U$ with*

$$F = 0 \quad \mathcal{L}^n\text{-a.e. on } U \setminus C\,.$$

Then

$$\operatorname{div} F(U) = 0\,.$$

The following result is a specialisation of Proposition 4 from [12, p. 1014]. It states that essentially bounded vector fields having divergence measure can be partially restricted to any bounded set of finite perimeter.

Proposition 4.13. *Let $U \subset \mathbb{R}^n$ be open and $F \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$. Then for every bounded set of finite perimeter $\Omega \in \mathcal{B}(U)$ such that $\operatorname{dist}_\Omega(\partial U) > 0$*

$$F \cdot \chi_\Omega \in \mathcal{DM}^\infty(U, \mathbb{R}^n)\ .$$

The following proposition is taken from Chen [11, p. 250] and Silhavy [32, p. 448]. In the proof of Gauß formulas in the next chapter, it is used to partially integrate compactly supported Lipschitz continuous functions with respect to vector fields having divergence measure.

Proposition 4.14. *Let $U \subset \mathbb{R}^n$ be open, $F \in \mathcal{DM}^1(U, \mathbb{R}^n)$ and let $f \in W^{1,\infty}(U, \mathbb{R})$. Then $F \cdot f \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$ and*

$$\operatorname{div}(F \cdot f) = f \operatorname{div} F + FDf\,.$$

The integrals over the boundaries of δ-neighbourhoods of the normal component of F on $\partial\Omega$ are functions of bounded variation. Note that for sets with sufficiently smooth boundary

$$D\operatorname{dist}_\Omega = \nu^{\Omega_\delta}\,.$$

Proposition 4.15. *Let $U \subset \mathbb{R}^n$ be open and $\Omega \in \mathcal{B}(U)$ be bounded such that $\varepsilon := \operatorname{dist}_\Omega(\partial U) > 0$. Furthermore, let $\mathcal{L}^n(\partial\Omega) = 0$. Then for every $F \in \mathcal{DM}^1(U, \mathbb{R}^n)$ the mapping $h : (0, \varepsilon) \to \mathbb{R}$ defined by*

$$h(\delta) := \int_{\partial\Omega_\delta} F \cdot D\operatorname{dist}_\Omega \, \mathrm{d}\mathcal{H}^{n-1}$$

is in $\mathcal{BV}(0, \varepsilon)$ and thus

$$\lim_{\delta \downarrow 0} \int_{\partial\Omega_\delta} F \cdot D\operatorname{dist}_\Omega \, \mathrm{d}\mathcal{H}^{n-1}$$

exists.

Proof. Let $F \in \mathcal{DM}^1(U, \mathbb{R}^n)$ and $\phi \in C_0^1((0, \varepsilon))$. Then $g := \phi \circ \mathrm{dist}_\Omega(\cdot) \in W^{1,\infty}(U, \mathbb{R})$ and

$$Dg = D\phi(\mathrm{dist}_\Omega) D\mathrm{dist}_\Omega .$$

Using Propositions 4.12 and 4.14 for partial integration

$$\begin{aligned}
\int_{(0,\varepsilon)} D\phi(\delta) \int_{\partial\Omega_\delta} F \cdot D\mathrm{dist}_\Omega \, \mathrm{d}\mathcal{H}^{n-1} \, \mathrm{d}\delta &= \int_{(0,\varepsilon)} \int_{\partial\Omega_\delta} F \cdot D\phi(\mathrm{dist}_\Omega) D\mathrm{dist}_\Omega \, \mathrm{d}\mathcal{H}^{n-1} \, \mathrm{d}\delta \\
&= \int_{(0,\varepsilon)} \int_{\partial\Omega_\delta} F \cdot Dg \, \mathrm{d}\mathcal{H}^{n-1} \, \mathrm{d}\delta \\
&= \int_U F \cdot Dg \, \mathrm{d}\mathcal{L}^n \\
&= -\int_U g \, \mathrm{d}\,\mathrm{div}\, F \\
&\leq \|g\|_C \, |\mathrm{div}\, F| \, (U) \leq c \, \|\phi\|_C \, .
\end{aligned}$$

Thus, $h \in \mathcal{BV}((0, \varepsilon))$ and the statement of the proposition follows with Theorem 3.28 from [2, p. 136]. $\square$

Chapter 5

Gauß Formulas

In this chapter, the main theorems on Gauß formulas are proved. The key to obtain these formulas is the representation of the dual space of $\mathcal{L}^{\infty}(U, \mathbb{R}^n, \mathcal{L}^n)$. The first section covers the case of essentially bounded vector fields. There, the existence of a normal measure ν is proved, and properties thereof are derived. It is shown that these measures yield Gauß formulas in many settings. In particular, this new result together with the product rules from Section 4.2 yields a Gauß formula for essentially bounded functions of bounded variation and essentially bounded vector fields having divergence measure. This is new, compared with the literature. Nevertheless, unbounded vector fields are not integrable with respect to normal measure in the general case. In the second part of this chapter, unbounded vector fields are investigated. A Gauß formula is obtained for bounded open sets with path-connected boundary by extending a result due to Silhavy [32].

5.1 Bounded Vector Fields having Divergence Measure

The following lemma enables the use of the characterisation of the dual of L^{∞} in the following theorem. The key point of this statement is that the dual of a product space is essentially the product of the dual spaces.

Nevertheless, a self-contained proof is given.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$. The dual space of*

$$\mathcal{L}^{\infty}(\Omega, \mathbb{R}^n, \mathcal{L}^n)$$

equipped with the norm

$$\|F\| := \sup_{\substack{k\in\mathbb{N}\\ 1\le k\le n}} \|F_k\|_\infty \quad \textit{for} \quad F \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n, \mathcal{L}^n)$$

is the space

$$(\mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n$$

equipped with the norm

$$\|\nu\| = \sum_{k=1}^n |\nu_k|(\Omega) \ \textit{for}\ \nu \in (\mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n\ .$$

Proof. Let $\nu \in (\mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n$. Then $u^* : \mathcal{L}^\infty(\Omega, \mathbb{R}^n, \mathcal{L}^n) \to \mathbb{R}$ defined by

$$\langle u^*, F\rangle = \sum_{k=1}^n \int_\Omega F_k \,\mathrm{d}\,\nu_k$$

for $F \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n, \mathcal{L}^n)$ is obviously a linear functional on $\mathcal{L}^\infty(\Omega, \mathbb{R}^n, \mathcal{L}^n)$. Furthermore

$$|\langle u^*, F\rangle| \le \sum_{k=1}^n \|F_k\|_\infty |\nu_k|(\Omega) \le \|F\|\,\|\nu\|$$

for $F \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n, \mathcal{L}^n)$, where the norms are defined as in the statement of the proposition.

Now let $u^* \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n, \mathcal{L}^n)^*$. Then for every $k \in \mathbb{N}, 1 \le k \le n$

$$u_k^* : \mathcal{L}^\infty(\Omega, \mathcal{L}^n) \to \mathbb{R} : f \mapsto \langle u^*, fe_k\rangle$$

is a continuous linear functional on $\mathcal{L}^\infty(\Omega, \mathcal{L}^n)$. By Proposition 2.21 there exist $\nu_k \in \mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ such that

$$\langle u_k^*, f\rangle = \int_\Omega f \,\mathrm{d}\,\nu_k$$

for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{L}^n)$. Hence for every $F \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n, \mathcal{L}^n)$

$$\langle u^*, F\rangle = \sum_{k=1}^n \langle u_k^*, F_k\rangle = \sum_{k=1}^n \int_\Omega F_k \,\mathrm{d}\,\nu_k = \int_\Omega F \,\mathrm{d}\,\nu\,,$$

where $\nu = (\nu_1, ..., \nu_n)$.

Now let $\nu \in (\mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n$. Then

$$\begin{aligned}
\sup_{\substack{F \in \mathcal{L}^\infty(\Omega,\mathbb{R}^n,\mathcal{L}^n) \\ \|F\| \le 1}} \left| \int_\Omega F \,\mathrm{d}\,\nu \right| &= \sup_{\substack{F \in \mathcal{L}^\infty(\Omega,\mathbb{R}^n,\mathcal{L}^n) \\ \|F\| \le 1}} \int_\Omega F \,\mathrm{d}\,\nu \\
&= \sup_{\substack{F \in \mathcal{L}^\infty(\Omega,\mathbb{R}^n,\mathcal{L}^n) \\ \|F\| \le 1}} \sum_{k=1}^n \int_\Omega F_k \,\mathrm{d}\,\nu_k \\
&= \sum_{k=1}^n \sup_{\substack{F_k \in \mathcal{L}^\infty(\Omega,\mathcal{L}^n) \\ \|F_k\|_\infty \le 1}} \int_\Omega F_k \,\mathrm{d}\,\nu_k \\
&= \sum_{k=1}^n |\nu_k|\,(\Omega) = \|\nu\| \ .
\end{aligned}$$

This finishes the proof. □

The proof of the upcoming Gauß Theorem relies on the following notion of approximation of the domain Ω. It turns out that this is not only a technical necessity but gives the obtained Gauß formulas a more flexible shape.

Definition 5.2. *Let $U \subset \mathbb{R}^n$ be open and $\Omega \in \mathcal{B}(U)$ with $\mathrm{dist}_{\partial U}(\Omega) > 0$.*

*A sequence $\{\chi_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(U, [0,1])$ of Lipschitz continuous real functions with compact support in U is called **good approximation for** χ_Ω with limit function χ, if*

1. $$\lim_{k\to\infty} \chi_k(x) =: \chi(x) \quad \textit{exists } \mathcal{H}^{n-1}\textit{-a.e. on } U$$

2. $$\chi = 1 \quad \mathcal{H}^{n-1}\textit{-a.e. on } \mathrm{int}\,\Omega$$

3. $$\chi = 0 \quad \mathcal{H}^{n-1}\textit{-a.e. on } \left(\overline{\Omega}\right)^c$$

4. $$\sup_{k\in\mathbb{N}} \|D\chi_k\|_1 < \infty\,.$$

A necessary condition for Ω to allow a good approximation is given in the next proposition.

Proposition 5.3. *Let $U \subset \mathbb{R}^n$ be open and $\Omega \in \mathcal{B}(U)$ be bounded such that $\mathrm{dist}_\Omega(\partial U) > 0$. If there is a good approximation for χ_Ω with $\|\chi - \chi_\Omega\|_1 = 0$. Then Ω is a set of finite perimeter.*

Proof. Let $\{\chi_k\}_{k\in\mathbb{N}}$ be a good approximation for χ_Ω. Since Ω is bounded and every $\mathcal{H}^{n-1}$-null set is a $\mathcal{L}^n$-null set,

$$\chi_k \xrightarrow{L^1} \chi_\Omega \,.$$

Since the total variation is lower semi continuous,

$$|D\chi_\Omega|\,(U) \leq \liminf_{k\to\infty} \|D\chi_k\|_1 < \infty \,.$$

This proves the statement. □

Remark 5.4. In Example 5.10, it is shown that every set of finite perimeter allows a good approximation.

Now, the Gauß Theorem can be proved using good approximations and the characterisation of the dual of $\mathcal{L}^\infty(U, \mathbb{R}^n, \mathcal{L}^n)$.

Theorem 5.5. *Let $U \subset \mathbb{R}^n$ be open, $\Omega \in \mathcal{B}(U)$ be a bounded set of finite perimeter such that $\operatorname{dist}_\Omega(\partial U) > 0$. Furthermore, let $\{\chi_k\}_{k\in\mathbb{N}}$ be a good approximation with limit χ. Then there exists $\nu \in (\operatorname{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n$ such that for every $k \in \mathbb{N}$, $1 \leq k \leq n$*

$$\operatorname{core}\nu_k \subset \partial\Omega \,.$$

and the Gauß formula

$$\operatorname{div} F\,(\operatorname{int}\Omega) + \int_{\partial\Omega} \chi \,\mathrm{d}\operatorname{div} F = \fint_{\partial\Omega} F \,\mathrm{d}\nu \tag{5.1}$$

holds for every $F \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$. The measure ν is minimal in the norm, i.e. if $\nu' \in (\operatorname{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n$ satisfies (5.1) for every $F \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$, then

$$\|\nu\| \leq \|\nu'\| \,.$$

In addition, for every set of finite perimeter $B \in \mathcal{B}(U)$

$$\nu(B) = -\lim_{k\to\infty} \int_{B\cap\operatorname{supp} D\chi_k} D\chi_k \,\mathrm{d}\mathcal{L}^n = -\int_{\partial^* B\cap\overline{\Omega}} \chi \cdot \nu^B \,\mathrm{d}\mathcal{H}^{n-1}$$

The preceding new Gauß Theorem sets itself apart from the literature by introducing normal measures. In the literature, Gauß formulas for sets of finite perimeter and essentially bounded vector field can be found in the form of functionals on a function space on the boundary (cf. [32, p. 448]) or as functions on the boundary which are obtained by mollification (cf. [13, p. 262f]). The approach chosen here enables a clean separation of geometry and vector field. In plus, it yields the existence of a normal measure which is defined on all Borel subsets.

Definition 5.6. *$\nu \in (\mathrm{ba}\,(U, \mathcal{B}(U), \mathcal{L}^n))^n$ satisfying* (5.1) *for all $F \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$ with some limit χ of a good approximation of χ_Ω that is minimal in the sense of Theorem 5.5 is called* ***normal measure*** *of Ω related to χ.*

Now, the proof of Theorem 5.5 is given.

Proof. Let $\Omega \in \mathcal{B}(U)$ be a bounded set of finite perimeter with $\mathrm{dist}_\Omega(\partial U) > 0$ and let $\{\chi_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(U, [0,1])$ be an associated good approximation with limit function χ.

Now, let $F \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$. Then by the Dominated Convergence Theorem (cf. [20, p. 20])

$$\int_U \chi_k \,\mathrm{d}\,\mathrm{div}\, F \xrightarrow{k\to\infty} \mathrm{div}\, F(\mathrm{int}\,\Omega) + \int_{\partial\Omega} \chi \,\mathrm{d}\,\mathrm{div}\, F\,.$$

Note that $\mathrm{div}\, F <<^w \mathcal{H}^{n-1}$ (cf. [12, p. 1014]). On the other hand,

$$F \cdot \chi_k \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$$

by Proposition 4.14. Furthermore, $F \cdot \chi_k$ is compactly supported in U. Thus by Proposition 4.12 for every $k \in \mathbb{N}$

$$\int_U \chi_k \,\mathrm{d}\,\mathrm{div}\, F = -\int_U D\chi_k \cdot F \,\mathrm{d}\mathcal{L}^n\,.$$

Hence for every $k \in \mathbb{N}$

$$\left|\int_U \chi_k \,\mathrm{d}\,\mathrm{div}\, F\right| \le \|D\chi_k\|_{\mathcal{L}^1} \|F\|_{\infty, \mathrm{supp}\,\chi_k}\,.$$

This implies

$$\left|\mathrm{div}\, F(\mathrm{int}\,\Omega) + \int_{\partial\Omega} \chi \,\mathrm{d}\,\mathrm{div}\, F\right| \le \limsup_{k\to\infty} \|D\chi_k\|_{\mathcal{L}^1} \|F\|_{\infty, \mathrm{supp}\,\chi_k} \le \sup_{k\in\mathbb{N}} \|D\chi_k\|_1 \|F\|_\infty\,.$$

Hence

$$u_0^* : \mathcal{DM}^\infty(U, \mathbb{R}^n) \to \mathbb{R} : F \mapsto \mathrm{div}\, F(\mathrm{int}\,\Omega) + \int_{\partial\Omega} \chi \,\mathrm{d}\,\mathrm{div}\, F$$

is a continuous linear functional on a subspace of $\mathcal{L}^\infty(U, \mathbb{R}^n, \mathcal{L}^n)$. By the Hahn-Banach Theorem [18, p. 63] there exists a continuous linear extension u^* of u_0^* to all of $\mathcal{L}^\infty(U, \mathbb{R}^n, \mathcal{L}^n)$ such that $\|u^*\| = \|u_0^*\|$. In particular, this extension is minimal in the norm. By Proposition 5.1 there exists a $\nu \in (\mathrm{ba}\,(U, \mathcal{B}(U), \mathcal{L}^n))^n$ such that for all $F \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$

$$\mathrm{div}\, F(\mathrm{int}\,\Omega) + \int_{\partial\Omega} \chi \,\mathrm{d}\,\mathrm{div}\, F = \int_U F \,\mathrm{d}\nu\,.$$

and $\|\nu\| = \|u_0^*\|$. Furthermore

$$\sum_{k=1}^{n} |\nu_k|(\Omega) = \|u_0^*\| = \|u^*\| \ .$$

Note that by the Coarea Formula (cf. [20, p. 112]), for a.e. $0 < \delta < \operatorname{dist}_\Omega(\partial U)$ the neighbourhood Ω_δ is a set of finite perimeter. By Proposition 4.13

$$F \cdot \chi_{\Omega_\delta} \in \mathcal{DM}^\infty(U, \mathbb{R}^n) \ .$$

But $F \cdot \chi_{\Omega_\delta}$ and F agree on a neighbourhood of Ω, whence

$$\operatorname{div}(F \cdot \chi_{\Omega_\delta})(\operatorname{int}\Omega) + \int_{\partial\Omega} \chi \, \mathrm{d}\operatorname{div}(F \cdot \chi_{\Omega_\delta}) = \operatorname{div} F(\operatorname{int}\Omega) + \int_{\partial\Omega} \chi \, \mathrm{d}\operatorname{div} F \,.$$

whence

$$\int_{U\setminus\Omega_\delta} F \, \mathrm{d}\nu = 0$$

for every $F \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$ and almost every $0 < \delta < \operatorname{dist}_\Omega(\partial U)$. Thus for almost every such $\delta > 0$ and $F \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$

$$\langle u^*, F \rangle = \int_\Omega F \, \mathrm{d}\nu \lfloor \Omega_\delta \,.$$

This implies

$$\|u_0^*\| \leq \|\nu \lfloor \Omega_\delta\| \leq \|\nu\| = \|u_0^*\|$$

and thus

$$\sum_{k=1}^{n} |\nu_k \lfloor (U \setminus \Omega_\delta)|(\Omega) = \sum_{k=1}^{n} |\nu_k|(U) - |\nu_k \lfloor \Omega_\delta|(\Omega) = \|u_0^*\| - \|u_0^*\| = 0$$

whence

$$|\nu_k|(U \setminus \Omega_\delta) = 0$$

for almost every $0 < \delta < \operatorname{dist}_\Omega(\partial U)$. Since $|\nu_k|$ is monotone, the statement follows for all $0 < \delta < \operatorname{dist}_\Omega(\partial U)$.

Note that by the Coarea formula (cf. [20, p. 112]) $\Omega_{-\delta}$ is a set of finite perimeter for almost every $\delta > 0$. By Proposition 4.13

$$F \cdot \chi_{\Omega_{-\delta}} \in \mathcal{DM}^\infty(U, \mathbb{R}^n) \ .$$

Since Ω is bounded, $F \cdot \chi_{\Omega_{-\delta}}$ is compactly supported in $\operatorname{int}\Omega$ and thus by Proposition 4.12

$$\operatorname{div} F \cdot \chi_{\Omega_{-\delta}}(\operatorname{int}\Omega) + \int_{\partial\Omega} \chi \, \mathrm{d}\operatorname{div}\left(F \cdot \chi_{\Omega_{-\delta}}\right) = 0 \,.$$

This implies

$$\int_\Omega F \,\mathrm{d}\nu\lfloor\Omega_{-\delta} = 0$$

for every such $\delta > 0$ and $F \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$. Hence

$$\|u_0^*\| \leq \|\nu\lfloor U \setminus \Omega_{-\delta}\| \leq \|\nu\| = \|u_0^*\| \;.$$

and analogously to the reasoning for Ω_δ one deduces

$$|\nu_k|\,(\Omega_{-\delta}) = 0$$

for every $k \in \mathbb{N}, 1 \leq k \leq n$.

Now for every $\delta > 0, \delta < \operatorname{dist}_\Omega(\partial U)$ and every $k \in \mathbb{N}, 1 \leq k \leq n$

$$|\nu_k|\,((U \setminus \Omega_\delta) \cup \Omega_{-\delta}) = 0\,.$$

This implies

$$\operatorname{core}\nu_k \subset \partial\Omega$$

for every $k \in \mathbb{N}, 1 \leq k \leq n$. This establishes Equation (5.1).

Now, let $B \in \mathcal{B}(U)$ be a set of finite perimeter. Then for $k \in \mathbb{N}, 1 \leq k \leq n$

$$e_k \cdot \chi_B \in \mathcal{DM}^\infty(U, \mathbb{R}^n) \;.$$

Note that

$$\operatorname{div}(e_k \cdot \chi_B) = \partial_k \chi_B = -(\nu^B)_k \mathcal{H}^{n-1}\lfloor\partial^* B\,.$$

The established Gauß formula yields

$$\begin{aligned}
\int_U \chi_B \,\mathrm{d}\nu_k &= \int_U e_k \cdot \chi_B \,\mathrm{d}\nu \\
&= \operatorname{div}(e_k \cdot \chi_B)(\operatorname{int}\Omega) + \int_{\partial\Omega} \chi \,\mathrm{d}\operatorname{div}(e_k \cdot \chi_B) \\
&= \int_U \chi \,\mathrm{d}\operatorname{div}(e_k \cdot \chi_B) \\
&= \lim_{l\to\infty} \int_U \chi_l \,\mathrm{d}\operatorname{div}(e_k \cdot \chi_B) \\
&= \lim_{l\to\infty} -\int_B e_k \cdot D\chi_l \,\mathrm{d}\mathcal{L}^n
\end{aligned}$$

Since $k \in \mathbb{N}, 1 \leq k \leq n$ was arbitrary

$$\nu(B) = -\lim_{k\to\infty} \int_{B\cap\operatorname{supp} D\chi_k} D\chi_k \,\mathrm{d}\mathcal{L}^n\,.$$

On the other hand, for every set of finite perimeter $B \in \mathcal{B}(U)$ and for $k \in \mathbb{N}$, $1 \leq k \leq n$

$$
\begin{aligned}
\nu_k(B) &= \lim_{l\to\infty} \int_U \chi_l \,\mathrm{d}\,\mathrm{div}\,(e_k \cdot \chi_B) \\
&= \lim_{l\to\infty} \int_U \chi_l \,\mathrm{d}\,\partial_k(\chi_B) \\
&= -\lim_{l\to\infty} \int_{\partial^* B} \chi_l \cdot (\nu^B)_k \,\mathrm{d}\,\mathcal{H}^{n-1} \\
&= -\int_{\partial^* B} \chi \cdot (\nu^B)_k \,\mathrm{d}\,\mathcal{H}^{n-1}\,.
\end{aligned}
$$

Hence

$$
\nu(B) = -\int_{\partial^* B\cap\overline{\Omega}} \chi \cdot \nu^B \,\mathrm{d}\,\mathcal{H}^{n-1}\,.
$$

□

Given a good approximation of χ_Ω, normal measures are uniquely defined on sets of finite perimeter.

Proposition 5.7. *Let $U \subset \mathbb{R}^n$ be open and $\Omega \in \mathcal{B}(U)$ be a bounded set of finite perimeter such that $\operatorname{dist}_\Omega(\partial U) > 0$. Let $\{\chi_k\}_{k\in\mathbb{N}}$ be a good approximation with limit χ. Let $\nu \in (\mathrm{ba}\,(U, \mathcal{B}(U), \mathcal{L}^n))^n$ be an associated normal measure. Then for every set of finite perimeter $B \in \mathcal{B}(U)$ there exists a Lebesgue null set $N \subset \mathbb{R}$*

$$
\nu(B) = \int_{\partial^* B\cap\partial\Omega} -\chi\nu^B \,\mathrm{d}\,\mathcal{H}^{n-1} + \lim_{\substack{\delta\downarrow 0\\ \delta\notin N}} \int_{B_{int}\cap\partial^*\Omega_{-\delta}} \nu^{\Omega_{-\delta}} \,\mathrm{d}\,\mathcal{H}^{n-1}\,.
$$

Proof. Note that B and B_{int} only differ by a $\mathcal{L}^n$-null set (cf. [20, p. 43]). Hence B_{int} is also a set of finite perimeter. W.l.o.g. $B = B_{int}$. The Coarea formula (cf. [20, p. 112]) implies that for a.e. $\delta > 0$ the set

$$
\Omega_\delta \setminus \Omega_{-\delta}
$$

has finite perimeter. Then $(\Omega_\delta \setminus \Omega_{-\delta})_{int}$ is also a set of finite perimeter. By [23, p. 5],

$$
B \cap (\Omega_\delta \setminus \Omega_{-\delta})_{int}
$$

is also a set of finite perimeter. Note that the Coarea Formula also implies that for a.e. $\delta > 0$

$$
\mathcal{H}^{n-1}(\partial_* B \cap \partial(\Omega_\delta \setminus \Omega_{-\delta})) = 0\,.
$$

Using this and [15, p. 199]

$$\partial_*(B \cap (\Omega_\delta \setminus \Omega_{-\delta})_{int})$$

differs from

$$(\partial_* B \cap (\Omega_\delta \setminus \Omega_{-\delta})_{int}) \cup (B \cap \partial_*(\Omega_\delta \setminus \Omega_{-\delta}))$$

only by a $\mathcal{H}^{n-1}$-null set. Since $\Omega_\delta \setminus \Omega_{-\delta}$ has density 1 at points of its measure theoretic interior and the measure theoretic normal is characterised by the halfspace it generates (cf. Remark 4.7, [20, p. 203]), one sees that

$$\nu^{B\cap(\Omega_\delta\setminus\Omega_{-\delta})} = \nu^B \quad \text{on} \quad \partial^* B \cap (\Omega_\delta \setminus \Omega_{-\delta})_{int} \cap \partial^*(B \cap (\Omega \setminus \Omega_{-\delta}))\,.$$

Theorem 5.5 states $\operatorname{core} \nu \subset \partial\Omega$, thus

$$\nu(B) = \nu(B \cap (\Omega_\delta \setminus \Omega_{-\delta})_{int}) = -\int_{\partial^*(B\cap(\Omega_\delta\setminus\Omega_{-\delta})_{int})\cap\overline{\Omega}} \chi \nu^{(B\cap(\Omega_\delta\setminus\Omega_{-\delta})_{int})}\, \mathrm{d}\mathcal{H}^{n-1}\,.$$

for a.e. $\delta > 0$. The integral on the right hand side is for a.e. $\delta > 0$ equal to

$$\int_{\partial^* B\cap\overline{\Omega}\cap(\Omega_\delta\setminus\Omega_{-\delta})_{int}} -\chi\nu^B \,\mathrm{d}\mathcal{H}^{n-1} + \int_{B\cap\overline{\Omega}\cap\partial^*(\Omega_{-\delta})} \chi\nu^{\Omega_{-\delta}}\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

Noting that

$$\int_{\partial^* B\cap\overline{\Omega}\cap A} -\chi\nu^B\,\mathrm{d}\mathcal{H}^{n-1}$$

defines a σ-measure in A and using continuity from above yields

$$\lim_{\delta\downarrow 0}\int_{\partial^* B\cap\overline{\Omega}\cap(\Omega_\delta\setminus\Omega_{-\delta})_{int}} -\chi\nu^B\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial^* B\cap\partial\Omega} -\chi\nu^B\,\mathrm{d}\mathcal{H}^{n-1}$$

On the other hand $\partial^*(\Omega_{-\delta}) \subset \overline{\Omega}$. Hence

$$\partial^*(\Omega_{-\delta}) \cap \overline{\Omega} \cap B = B \cap \partial^*(\Omega_{-\delta})\,.$$

Furthermore $\chi = 1$ on $\operatorname{int}\Omega$. This finishes the proof. □

The following picture illustrates the representation of a normal measure from the preceding proposition.

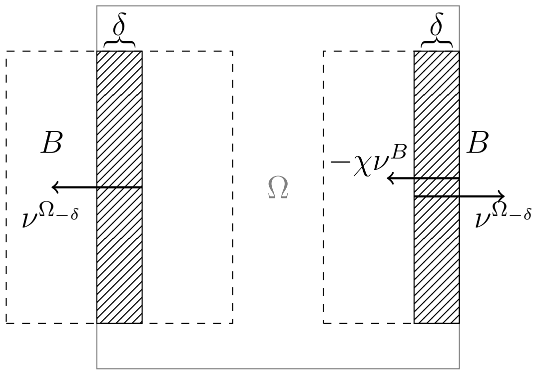

Figure 5.1: Domain of influence for a normal measure and a set of finite perimeter B

The relation of $\mathcal{H}^{n-1}\lfloor\partial^*\Omega$ and $|\nu|$ is treated in the next proposition.

Proposition 5.8. *Let $U \subset \mathbb{R}^n$ be open, $\Omega \in \mathcal{B}(U)$ be a bounded set of finite perimeter such that $\operatorname{dist}_\Omega(\partial U) > 0$. Furthermore, let $\{\chi_k\}_{k\in\mathbb{N}}$ be a good approximation with limit χ and let $\nu \in (\mathrm{ba}\,(U, \mathcal{B}(U), \mathcal{L}^n))^n$ be the associated normal measure.*

If $\|\chi - \chi_\Omega\|_1 = 0$, then for every open set $B \subset U$

$$|\nu|\,(B) \geq (\mathcal{H}^{n-1}\lfloor\partial^*\Omega)(B)\,.$$

Remark 5.9. Note that $\mathcal{L}^n(\partial\Omega) = 0$ implies $\|\chi - \chi_\Omega\|_1 = 0$.

Proof. Let $\phi \in C_0^1\,(U, \mathbb{R}^n)$. Then using the Gauß Theorem from Evans [20, p. 209]

$$\begin{aligned}\int_U \phi\,\mathrm{d}\nu &= \int_{\operatorname{int}\Omega} \operatorname{div}\phi\,\mathrm{d}\mathcal{L}^n + \int_{\partial\Omega} \chi\operatorname{div}\phi\,\mathrm{d}\mathcal{L}^n\\ &= \int_\Omega \operatorname{div}\phi\,\mathrm{d}\mathcal{L}^n\\ &= \int_{\partial^*\Omega} \phi\cdot\nu^\Omega\,\mathrm{d}\mathcal{H}^{n-1}\,.\end{aligned}$$

Hence for every open set $B \subset U$

$$
\begin{aligned}
|\nu|(B) &\geq \sup_{\substack{\phi \in C_0^1(B,\mathbb{R}^n),\\ \|\phi\|_\infty \leq 1}} \int_U \phi \,\mathrm{d}\nu \\
&\geq \sup_{\substack{\phi \in C_0^1(B,\mathbb{R}^n),\\ \|\phi\|_C \leq 1}} \int_{\partial^*\Omega} \phi \cdot \nu^\Omega \,\mathrm{d}\mathcal{H}^{n-1} \\
&= |D\chi_\Omega|(B) \\
&= (\mathcal{H}^{n-1} \lfloor \partial^*\Omega)(B)\,.
\end{aligned}
$$

For the last equality, see e.g. [20, p. 205].

Since $B \in \mathcal{B}(U)$ was arbitrary, this finishes the proof. □

The following example shows that for every set of finite perimeter there exists a canonical normal measure. Hence, Theorem 5.5 is always applicable.

Example 5.10. Canonical normal measure
Let $U \subset \mathbb{R}^n$ be open and $\Omega \in \mathcal{B}(U)$ be a bounded set of finite perimeter such that $\operatorname{dist}_\Omega(\partial U) > 0$. Furthermore, let $\rho \in C_0^\infty(\mathbb{R}^n)$ be the standard mollification kernel (cf. [20, p. 122]). Then by Proposition 4.8.

$$\chi_k(x) := \int_{\mathbb{R}^n} \frac{1}{k^n} \rho\left(k(y-x)\right) \chi_\Omega(x) \,\mathrm{d}y\,.$$

is a good approximation for χ_Ω. The limit function χ satisfies

$$\chi = \chi_{\Omega_{int}} + \frac{1}{2}\chi_{\partial^*\Omega} \ \mathcal{H}^{n-1}\text{-a.e.}\,.$$

Hence, there exists a normal measure $\nu \in (\mathrm{ba}\,(U, \mathcal{B}(U), \mathcal{L}^n))^n$ such that for every $F \in \mathcal{DM}^\infty(U,\mathbb{R}^n)$ the following *Gauß formula* holds

$$\operatorname{div} F(\Omega_{int}) + \frac{1}{2}\operatorname{div} F(\partial^*\Omega) = \fint_{\partial\Omega} F \,\mathrm{d}\nu\,.$$

Furthermore,

$$\operatorname{core} \nu \subset \partial\Omega\,.$$

The divergence on the regular boundary of Ω, weighted with $\frac{1}{2}$, cannot be found in the literature. This is due the fact, that the majority of the texts prohibit the vector fields under consideration from exhibiting such concentrations. The remaining sources treat settings similar to the one of Theorem 5.18 below. The weight $\frac{1}{2}$ appears plausible, when interpreting the divergence as source strength of the field F. At points of the regular boundary,

Ω geometrically resembles a half-space. Then half of the source strength can be seen to flow into the domain and the other half flows outwards.

The next example shows that for many closed sets of finite perimeter a more familiar form of the Gauß Theorem can be derived.

Example 5.11. Outer normal measure
Let $U \subset \mathbb{R}^n$ be open and $\Omega \in \mathcal{B}(\Omega)$ be a bounded, closed set of finite perimeter such that $\delta_0 := \operatorname{dist}_\Omega(\partial U) > 0$. Furthermore, let there be a sequence $\{\delta_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ such that $\lim\limits_{k\to\infty} \delta_k = 0$ and

$$\sup_{k\in\mathbb{N}} \fint_{(0,\delta_k)} \mathcal{H}^{n-1}(\partial\Omega_\delta)\,\mathrm{d}\delta < \infty\,.$$

This is the case if, e.g.

$$\lim_{\delta\downarrow 0} \mathcal{H}^{n-1}(\partial\Omega_\delta) = \mathcal{H}^{n-1}(\partial^*\Omega)\,.$$

For $x \in U$ and $k \in \mathbb{N}$ set

$$\begin{aligned}\chi_k(x) &:= \chi_{\Omega_{\delta_k}}(x)\left(1 - \frac{1}{\delta_k}\chi_{\Omega_{\delta_k}\setminus\Omega}\operatorname{dist}_\Omega(x)\right)\\ &= \max\left\{0, \min\left\{1, 1 - \frac{1}{\delta_k}\operatorname{dist}_\Omega\right\}\right\}\,.\end{aligned}$$

Then $\chi_k \in W^{1,\infty}(U,[0,1])$ is Lipschitz continuous (cf. [14, p. 47]). These functions are called (outer) **Portmanteau functions**. Note that by the Coarea formula for functions of bounded variation (cf. [20, p. 185])

$$\|D\chi_k\|_1 = \int_{(0,1)} \mathcal{H}^{n-1}(\chi_k^{-1}(\delta))\,\mathrm{d}\,\delta = \fint_{(0,\delta_k)} \mathcal{H}^{n-1}(\partial\Omega_\delta)\,\mathrm{d}\delta\,.$$

Hence, the sequence $\{\chi_{\delta_k}\}_{k\in\mathbb{N}}$ is a good approximation for χ_Ω and the limit function is

$$\chi = \chi_\Omega\,.$$

Thus, there exists a normal measure $\nu \in (\operatorname{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n$ such that for every $F \in \mathcal{DM}^\infty(U,\mathbb{R}^n)$ the following *Gauß formula* holds

$$\operatorname{div} F(\Omega) = \fint_{\partial\Omega} F\,\mathrm{d}\nu\,.$$

Open set of finite perimeter can be treated similarly, as the following example shows.

Example 5.12. Inner normal measure
Let $U \subset \mathbb{R}^n$ be open and $\Omega \subset U$ be a bounded, open set of finite perimeter such that $\mathrm{dist}_\Omega(\partial U) > 0$. Furthermore, assume there exists $\{\delta_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ such that $\lim_{k\to\infty} \delta_k = 0$ and

$$\sup_{k\in\mathbb{N}} \fint_{(0,\delta_k)} \mathcal{H}^{n-1}(\partial\Omega_{-\delta})\,\mathrm{d}\delta < \infty\,.$$

For $k \in \mathbb{N}$ and $x \in U$ set

$$\begin{aligned}\chi_k(x) &:= \chi_{\Omega_{-\delta_k}}(x) + \frac{1}{\delta_k}\,\mathrm{dist}_{\Omega^c}(x)\chi_{\Omega\setminus\Omega_{-\delta_k}} \\ &= \min\left\{1, \max\left\{0, \frac{1}{\delta_k}\,\mathrm{dist}_{\Omega^c}\right\}\right\}.\end{aligned}$$

Then $\chi_k \in W^{1,\infty}(U,[0,1])$ is Lipschitz continuous (cf. [14, p. 47]). Then as in Example 5.11, the sequence $\{\chi_{\delta_k}\}_{k\in\mathbb{N}}$ is a good approximation for χ_Ω and the limit function is

$$\chi = \chi_\Omega\,.$$

These functions are called (inner) **Portmanteau functions**. Hence there exists a normal measure $\nu \in (\mathrm{ba}\,(U,\mathcal{B}(U),\mathcal{L}^n))^n$ such that for every vector field $F \in \mathcal{DM}^\infty(U,\mathbb{R}^n)$ the following *Gauß formula* holds

$$\mathrm{div}\,F(\Omega) = \fint_{\partial\Omega} F\,\mathrm{d}\nu\,.$$

The subsequent corollary illustrates the dependence of the integral with respect to normal measure on the good approximation of χ_Ω.

Corollary 5.13. *Let $U \subset \mathbb{R}^n$ and $\Omega \in \mathcal{B}(U)$ be a bounded set of finite perimeter such that $\mathrm{dist}_\Omega(\partial U) > 0$. Then for any two good approximations $\{\chi_k^1\}_{k\in\mathbb{N}}, \{\chi_k^2\}_{k\in\mathbb{N}} \subset W^{1,\infty}(U,[0,1])$ for χ_Ω, associated normal measures $\nu_1,\nu_2 \in (\mathrm{ba}\,(U,\mathcal{B}(U),\mathcal{L}^n))^n$ and any $F \in \mathcal{DM}^\infty(U,\mathbb{R}^n)$*

$$\fint_{\partial\Omega} F\,\mathrm{d}\nu_1 - \nu_2 = \int_{\partial\Omega} \chi_1 - \chi_2\,\mathrm{d}\,\mathrm{div}\,F$$

where χ_1 and χ_2 are the limit functions for $\{\chi_k^1\}_{k\in N}$ and $\{\chi_k^2\}_{k\in\mathbb{N}}$ respectively.

Remark 5.14. In particular, if $|\mathrm{div}\,F|(\partial\Omega) = 0$,

$$\fint_{\partial\Omega} F\,\mathrm{d}\nu$$

is independent of the choice of the good approximation.

Since ν is a bounded measure, all essentially bounded vector fields F are integrable with respect to this measure. This leads to the question, whether $F \in L^1(U,\nu)$ for unbounded vector fields. The next example answers this negatively. The function is similar to the one in [8, p. 100].

Example 5.15. Let $U := (0,1)^2 \subset \mathbb{R}^2$ and $\Omega := \{(x,y) \in R^2 | x \leq y\} \cap B_{\frac{1}{4}}\left(\frac{1}{2}(1,1)\right)$. Furthermore let $\nu \in (\mathrm{ba}\,(U, \mathcal{B}(U), \mathcal{L}^n))^n$ be a normal measure for Ω and $F \in \mathcal{DM}^1(U,\mathbb{R}^n)$ defined by

$$F(x,y) := |x-y|^{-\frac{1}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for $x \neq y$. Then $\operatorname{div} F = 0$. In order to see that, let $\Delta := \{(x,x) \in U | x \in \mathbb{R}\}$, $1 > \delta > 0$ and note that for $\phi \in C_0^1(U)$

$$\begin{aligned}
\int_U F \cdot D\phi \,\mathrm{d}\mathcal{L}^n &= \int_{U \cap \Delta_\delta} F \cdot D\phi \,\mathrm{d}\mathcal{L}^n + \int_{U \setminus \Delta_\delta} F \cdot D\phi \,\mathrm{d}\mathcal{L}^n \\
&= \int_{U \cap \Delta_\delta} F \cdot D\phi \,\mathrm{d}\mathcal{L}^n - \int_{U \setminus \Delta_\delta} \phi \operatorname{div} F \,\mathrm{d}\mathcal{L}^n - \int_{\partial(U \setminus \Delta_\delta)} \phi F \cdot \nu^{\Delta_\delta} \,\mathrm{d}\mathcal{H}^1
\end{aligned}$$

Since $\operatorname{div} F = 0$ outside of Δ_δ and $F \cdot \nu^{\Delta_\delta} = \pm \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot F = 0$ on U

$$\int_U F \cdot D\phi \,\mathrm{d}\mathcal{L}^n = \int_{U \cap \Delta_\delta} F \cdot D\phi \,\mathrm{d}\mathcal{L}^n \xrightarrow{\delta \downarrow 0} 0\,.$$

Note that for every $c > 0$ with $\frac{1}{c^2} > \delta > 0$

$$\Omega_\delta \cap \{(x,y) \in U \mid |F(x,y)| \geq c\} \supset \Delta_\delta \cap \Omega_\delta$$

Hence for every $F' \in \mathcal{L}^\infty(U, \mathbb{R}^n, \mathcal{L}^n)$

$$\begin{aligned}
&|\nu|\,(\Omega_\delta \cap \{(x,y) \in U \mid |F(x,y) - F'(x,y)| \geq \varepsilon\}) \\
&\quad \geq |\nu|\,(\Omega_\delta \cap \{(x,y) \in U \mid |F(x,y)| \geq \|F'\|_\infty + \varepsilon\}) \\
&\quad \geq |\nu|\,(\Delta_\delta \cap \Omega_\delta) \geq \mathcal{H}^1\left(\Delta \cap B_{\frac{1}{4}}\left(\frac{1}{2}(1,1)\right)\right) = \frac{1}{2} > 0
\end{aligned}$$

for every $0 < \delta < \frac{1}{(\varepsilon + \|F'\|_\infty)^2}$.

Hence, there is no sequence $\{F_k\}_{k\in\mathbb{N}} \subset \mathcal{L}^\infty(U, \mathbb{R}^n, \mathcal{L}^n)$ converging in measure to F. In particular, F cannot be approximated in measure by simple functions.

Remark 5.16. The preceding example indeed works for $U = (0,1)^2$ and every $F \in \mathcal{DM}^1(U, \mathbb{R}^n)$ such that for some $\phi : \mathbb{R} \to \mathbb{R}$ satisfying

1. ϕ is continuously differentiable on $\mathbb{R} \setminus \{0\}$
2. $\lim\limits_{x \to 0} \phi(x) = \infty$
3. $g : U \to \mathbb{R} : (x, y) \mapsto \phi(x - y)$ is integrable on U

it holds
$$F = g \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$
The essential point is that F is tangential to the curve where it is unbounded. Hence, there are many vector fields which cannot even be approximated in measure.

The following example gives a vector field that only blows up at one point and still is not integrable with respect to normal measure. The function is the same as in [9, p. 403].

Example 5.17. Let $n = 2$, $U := B_1(0) \subset \mathbb{R}^2$ and
$$\Omega := B_{\frac{1}{2}}(0) \cap \{(x, y) \in \mathbb{R}^2 | x, y \geq 0\} .$$
Furthermore, let
$$F : U \to \mathbb{R}^2 : x \mapsto \frac{1}{2\pi} \frac{x}{|x|^2} .$$
Then $F \in \mathcal{DM}^1(U, \mathbb{R}^n)$ and $\operatorname{div} F = \delta_0$. Let $\{\chi_k\}_{k \in \mathbb{N}}$ be the canonical good approximation from Example 5.10. Let $\nu \in (\mathrm{ba}\,(U, \mathcal{B}(U), \mathcal{L}^n))^n$ be the normal measure associated with this good approximation. Assume that $F \in L^1(U, \nu)$. Then
$$\int_U |F| \,\mathrm{d}\,|\nu| < \infty .$$
But
$$\int_U |F| \,\mathrm{d}\,|\nu| \geq \frac{1}{2\pi} \int_{\partial\Omega} \frac{1}{|x|} \,\mathrm{d}\,\mathcal{H}^1 \geq \frac{1}{2\pi} \int_{\left(0, \frac{1}{2}\right)} \frac{1}{t} \,\mathrm{d}t = \infty ,$$
a contradiction. Hence $F \notin L^1(U, \nu)$.

Up to now, the Gauss Theorem was given for sets that have a positive distance to the boundary. In order to complement this result, the following theorem states the theorem for the whole set, in the case of $U = \Omega$.

Theorem 5.18. *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ be a bounded open set of finite perimeter. If there exists $\delta_0 > 0$ and $c > 0$ such that for almost every $\delta \in (0, \delta_0)$*

$$\mathcal{H}^{n-1}(\partial\Omega_{-\delta}) \leq c\,,$$

then there exists $\nu \in (\mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n$ such that for every $k \in \mathbb{N}$, $1 \leq k \leq n$

$$\mathrm{core}\,\nu_k \subset \partial\Omega$$

and for all $F \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n)$ the following Gauß formula holds

$$\fint_{\partial\Omega} F \,\mathrm{d}\nu = \mathrm{div}\, F(\Omega)\,.$$

and for every open set $B \subset \mathbb{R}^n$

$$|\nu|\,(B \cap \Omega) \geq (\mathcal{H}^{n-1} \lfloor \partial^*\Omega)(B)\,.$$

Furthermore, ν is minimal in the sense, that if $\nu' \in (\mathrm{ba}\,(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n))^n$ satisfies the equations above, then

$$\|\nu\| \leq \|\nu'\|\ .$$

For every $B \in \mathcal{B}(\Omega)$ having finite perimeter in $\mathbb{R}^n$

$$\nu(B) = -\int_{\partial^* B \cap \Omega} \nu^B \,\mathrm{d}\,\mathcal{H}^{n-1}\,.$$

Remark 5.19. Note that if $\Omega \in \mathcal{B}(\Omega)$ is only supposed to be open and

$$\mathcal{H}^{n-1}(\partial\Omega_{-\delta}) \leq c$$

is required, then Ω is necessarily a set of finite perimeter, due to the total variation being lower semi-continuous.

On the other hand, this condition loosely resembles the definition of Lipschitz deformable boundaries defined in [8, p. 94], but is much more general.

Proof. Let $\{\chi_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\mathbb{R}^n, [0,1])$ be such that

$$\begin{aligned}\chi_k &:= \chi_{\Omega_{-\frac{2}{k}}} + \chi_{\left(\Omega_{-\frac{1}{k}} \setminus \Omega_{-\frac{2}{k}}\right)} (k\,\mathrm{dist}_{\partial\Omega} - 1) \\ &= \min\left\{1, \max\left\{0, k\,\mathrm{dist}_{\partial\Omega} - 1\right\}\right\}\ .\end{aligned}$$

See [14, p. 47] for reference. Then

$$|D\chi_k| = k\chi_{\left(\Omega_{-\frac{1}{k}} \setminus \Omega_{-\frac{2}{k}}\right)}\,.$$

Then the Coarea Formula [20, p. 112] implies

$$\|D\chi_k\|_1 = \int_{\Omega_{-\frac{1}{k}} \setminus \Omega_{-\frac{2}{k}}} k \,\mathrm{d}\mathcal{L}^n = \fint_{(\frac{1}{k},\frac{2}{k})} \mathcal{H}^{n-1}(\partial\Omega_{-\delta})\,\mathrm{d}\delta \le c\,.$$

As in the proof of Theorem 5.5,

$$\lim_{k\to\infty}\int_\Omega F\cdot D\chi_k\,\mathrm{d}\mathcal{L}^n = -\lim_{k\to\infty}\int_\Omega \chi_k\,\mathrm{d}\operatorname{div} F = -\int_\Omega 1\,\mathrm{d}\operatorname{div} F = \operatorname{div} F(\Omega)\,.$$

On the other hand, for every $k \in \mathbb{N}$

$$\left|\int_\Omega F\cdot D\chi_k\,\mathrm{d}\mathcal{L}^n\right| \le \|F\|_\infty \|D\chi_k\|_1 \le \|F\|_\infty \sup_{k\in\mathbb{N}} \|D\chi_k\|_1 \le c\,\|F\|_\infty\,.$$

Hence

$$u_0^* : \mathcal{DM}^\infty(\Omega,\mathbb{R}^n) \to \mathbb{R} : F \mapsto \operatorname{div} F(\Omega)$$

is a continuous linear functional on a subspace of $\mathcal{L}^\infty(\Omega,\mathbb{R}^n,\mathcal{L}^n)$. The Hahn-Banach Theorem (cf. [18, p.63]) implies the existence of a measure $\nu \in (\mathrm{ba}\,(\Omega,\mathcal{B}(\Omega),\mathcal{L}^n))^n$ such that for all $F \in \mathcal{DM}^\infty(U,\mathbb{R}^n)$

$$\operatorname{div} F(\Omega) = \int_\Omega F\,\mathrm{d}\nu\,. \tag{5.2}$$

Furthermore, $\|\nu\| = \|u_0^*\|$, implying minimality in the norm. Now by Proposition 4.13, for almost every $\delta > 0$ and every $F \in \mathcal{DM}^\infty(\Omega,\mathbb{R}^n)$

$$F\cdot\chi_{\Omega_{-\delta}} \in \mathcal{DM}^\infty(\Omega,\mathbb{R}^n)$$

and $F\cdot\chi_{\Omega_{-\delta}}$ has compact support in Ω. By Proposition 4.12

$$\operatorname{div}(F\cdot\chi_{\Omega_{-\delta}})(\Omega) = 0\,.$$

Thus, for every $F \in \mathcal{DM}^\infty(\Omega,\mathbb{R}^n)$

$$\begin{aligned}\int_\Omega F\,\mathrm{d}\nu &= \operatorname{div} F(\Omega)\\ &= \operatorname{div}(F\cdot\chi_{\Omega\setminus\Omega_{-\delta}})(\Omega) + \operatorname{div}(F\cdot\chi_{\Omega_{-\delta}})(\Omega)\\ &= \operatorname{div}(F\cdot\chi_{\Omega\setminus\Omega_{-\delta}})(\Omega)\\ &= \int_\Omega F\,\mathrm{d}\nu\lfloor(\Omega\setminus\Omega_{-\delta})\,.\end{aligned}$$

Thus, $\nu\lfloor(\Omega\setminus\Omega_{-\delta})$ also satisfies Equation (5.2). The minimality of $\|\nu\|$ then implies

$$\|\nu\lfloor\Omega_{-\delta}\| = 0\,.$$

Since $\delta > 0$ can be arbitrarily small

$$\operatorname{core} \nu \subset \partial\Omega .$$

Note that for $B \in \mathcal{B}(\Omega)$ having finite perimeter in $\mathbb{R}^n$

$$e_k \chi_B \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n) \ .$$

In order to see this, compute

$$\operatorname{div}(e_k \cdot \chi_B) = \partial_k \chi_B = -\nu_k^B \mathcal{H}^{n-1} \lfloor \partial^* B .$$

In particular

$$\nu(B) = -\int_{\partial^* B \cap \Omega} \nu^B \,\mathrm{d}\,\mathcal{H}^{n-1} .$$

Now, let $B \subset \mathbb{R}^n$ be open. Then using the Gauß Theorem from Evans (cf. [20, p.209])

$$\begin{aligned}
|\nu|(B \cap \Omega) &\geq \sup_{\substack{\phi \in C_0^1(B,\mathbb{R}^n),\\ \|\phi\|_\infty \leq 1}} \int_\Omega \phi \,\mathrm{d}\,\nu \\
&\geq \sup_{\substack{\phi \in C_0^1(B,\mathbb{R}^n),\\ \|\phi\|_C \leq 1}} \int_\Omega \phi \,\mathrm{d}\,\nu \\
&\overset{(5.2)}{=} \sup_{\substack{\phi \in C_0^1(B,\mathbb{R}^n),\\ \|\phi\|_C \leq 1}} \operatorname{div} \phi(\Omega) \\
&= \sup_{\substack{\phi \in C_0^1(B,\mathbb{R}^n),\\ \|\phi\|_C \leq 1}} \int_{\partial^* \Omega} \phi \cdot \nu^\Omega \,\mathrm{d}\,\mathcal{H}^{n-1} \\
&= |D\chi_\Omega|(B) \\
&= (\mathcal{H}^{n-1} \lfloor \partial^* \Omega)(B) .
\end{aligned}$$

□

Remark 5.20. Theorem 5.18 still holds true for open Ω such that there exists $\{\delta_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ such that $\lim\limits_{k\to\infty} \delta_k = 0$ and

$$\sup_{k\in\mathbb{N}} ⨍_{\left(\frac{\delta_k}{2},\delta_k\right)} \mathcal{H}^{n-1}(\Omega_{-\delta}) \,\mathrm{d}\delta < \infty .$$

The arguments are the same as in Example 5.11 and Example 5.12.

The following proposition is a new Gauß-Green formula for essentially bounded functions of bounded variation and essentially bounded vector fields having divergence measure. In contrast to the literature, where only continuous scalar fields were treated (cf. [32, p. 448], [12, p. 1014]), this is a new quality.

Proposition 5.21. *Let $U \subset \mathbb{R}^n$ be open and $\Omega \in \mathcal{B}(U)$ be a bounded set of finite perimeter such that $\operatorname{dist}_\Omega(\partial U) > 0$. Furthermore, let $\{\chi_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(U,[0,1])$ be a good approximation for χ_Ω and $\nu \in (\operatorname{ba}(U,\mathcal{B}(U),\mathcal{L}^n))^n$ be an associated normal measure.*

Then for every $F \in \mathcal{DM}^\infty(U,\mathbb{R}^n)$ the set function

$$F^\nu : \mathcal{B}(U) \to \mathbb{R} : B \mapsto \int_B F \,\mathrm{d}\nu$$

is an element of $\operatorname{ba}(U,\mathcal{B}(U),\mathcal{L}^n)$ with

$$\operatorname{core} F^\nu \subset \partial\Omega$$

and for every compactly supported $f \in \mathcal{BV}(U) \cap \mathcal{L}^\infty(U,\mathcal{L}^n)$ the following Gauß formula holds

$$\operatorname{div}(f\cdot F)(\operatorname{int}\Omega) + \int_{\partial\Omega} \chi \,\mathrm{d}\operatorname{div}(f\cdot F) = \fint_{\partial\Omega} f \,\mathrm{d}F^\nu = \fint_{\partial\Omega} f\cdot F\,\mathrm{d}\nu\,.$$

*Call F^ν **normal trace of** F **on** $\partial\Omega$.*

Proof. Note that

$$f\cdot F \in \mathcal{DM}^\infty(U,\mathbb{R}^n)\ .$$

See [12, p. 1014] for reference. Hence

$$\operatorname{div}(f\cdot F)(\operatorname{int}\Omega) + \int_{\partial\Omega} \chi \,\mathrm{d}\operatorname{div}(f\cdot F) = \fint_{\partial\Omega} f\cdot F\,\mathrm{d}\nu\,.$$

Note that for every $B \in \mathcal{B}(U)$

$$\left|\int_B F\,\mathrm{d}\nu\right| \leq \|F\|_\infty |\nu|(B)\,,$$

whence

$$F^\nu \in \operatorname{ba}(U,\mathcal{B}(U),\mathcal{L}^n)\ .$$

Since for every $B \in \mathcal{B}(U)$

$$F^\nu(B) = \int_B F\,\mathrm{d}\nu = \int_{B\cap(\Omega_\delta\setminus\Omega_{-\delta})} F\,\mathrm{d}\nu$$

the core of F^ν is a subset of $\partial\Omega$.

Let $\varepsilon > 0$. Since $f \in \mathcal{L}^\infty(U, \mathcal{L}^n)$, there exist $m \in \mathbb{N}$, $\{y_k\}_{k=0}^m \subset \mathbb{R}$ and $\{B_k\}_{k=0}^m$ pairwise disjoint, such that

$$\|y_k - f \cdot \chi_{B_k}\|_\infty \leq \varepsilon \text{ and } \bigcup_{k=0}^m B_k = U$$

Set $h := \sum_{k=0}^m y_k \chi_{B_k}$. Then

$$\begin{aligned}
\left| \fint_{\partial\Omega} fF \,\mathrm{d}\nu - \fint_{\partial\Omega} f \,\mathrm{d}F^\nu \right| &\leq \left| \fint_{\partial\Omega} (f-h)F \,\mathrm{d}\nu \right| + \left| \fint_{\partial\Omega} hF \,\mathrm{d}\nu - \fint_{\partial\Omega} h \,\mathrm{d}F^\nu \right| \\
&\quad + \left| \fint_{\partial\Omega} f - h \,\mathrm{d}F^\nu \right| \\
&\leq \varepsilon \|F\|_\infty |\nu|(U) + 0 + \varepsilon |F^\nu|(U).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary

$$\fint_{\partial\Omega} fF \,\mathrm{d}\nu = \fint_{\partial\Omega} f \,\mathrm{d}F^\nu .$$

□

5.2 Unbounded Vector Fields and Open Sets

In the previous section, general Gauß formulas for essentially bounded vector fields having divergence measure were presented. Example 5.15 and 5.17 showed that it is in general not possible to integrate unbounded vector fields with respect to the normal measures obtained. In Proposition 5.21, the measure F^ν was presented as a notion of normal trace.

In the following, this is carried over to the case of unbounded vector fields. Therefore, a result due to Silhavy (cf. [32]) is improved upon. In particular, Silhavy proved that for $F \in \mathcal{DM}^1(U, \mathbb{R}^n)$ there exists a continuous linear functional on $\mathrm{Lip}(\partial\Omega)$, the space of Lipschitz continuous functions on $\partial\Omega$, balancing the volume part of the Gauß formula. The following exposition proves that this functional can be represented by the sum of a Radon measure F^ν and a measure $\mu_F \in (\mathrm{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n$ with core on the boundary. The arguments from Silhavy are retraced, in order to give a self-contained proof of the main theorem.

Throughout this section, for $F \in (\mathcal{L}^\infty(U, \mathcal{L}^n))^n$ and $V \subset U$ open, set

$$\|F\|_{\infty,V} := \operatorname*{ess\,sup}_V |F| .$$

It is essential for the subsequent proofs to be able to compare the Lipschitz constant of a function by the norm of its gradient. The following lemma enables this comparison on balls.

Lemma 5.22. *Let $U \subset \mathbb{R}^n$ be open and $f \in W^{1,\infty}(U, \mathbb{R})$. Then for every $x_0 \in U$ and $0 < \delta < \frac{1}{2}\operatorname{dist}_{x_0}(\partial U)$ with $B_\delta(x_0) \subset U$*

$$\sup_{\substack{x,y \in B_\delta(x_0) \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|} \leq \|Df\|_{\infty, B_{2\delta}(x_0)} .$$

Proof. Let $\varepsilon < \delta$. For $x \in B_\delta(x_0)$ set

$$f_\varepsilon(x) := \int_{\mathbb{R}^n} \rho_\varepsilon(y-x) f(y) \,\mathrm{d}y = \rho_\varepsilon * f(x) ,$$

where ρ_ε is a scaled standard mollification kernel. Then as in Evans [20, p. 123]

$$Df_\varepsilon = \rho_\varepsilon * Df .$$

Note that $f_\varepsilon \to f$ point wise (cf. [20, p. 123]). Hence, for every $x, y \in B_\delta(x_0)$ with $x \neq y$

$$\begin{aligned} |f(x) - f(y)| &= \lim_{\varepsilon \downarrow 0} |f_\varepsilon(x) - f_\varepsilon(y)| \\ &\leq \liminf_{\varepsilon \downarrow 0} \|Df_\varepsilon\|_\infty |x-y| . \end{aligned}$$

Now for every $x \in B_\delta(x_0)$

$$|Df_\varepsilon(x)| \leq \int_{\mathbb{R}^n} |\rho_\varepsilon(y-x)||Df(y)| \,\mathrm{d}\mathcal{L}^n \leq \|Df\|_{\infty, B_{\delta+\varepsilon}(x_0)} .$$

Thus

$$\sup_{\substack{x,y \in B_\delta(x_0) \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|} \leq \liminf_{\varepsilon \downarrow 0} \|Df\|_{\infty, B_{\delta+\varepsilon}(x_0)} \leq \|Df\|_{\infty, B_{2\delta}(x_0)}$$

□

Once the estimate on balls is obtained, it is possible to prove the statement for path-connected sets.

Lemma 5.23. *Let $U \subset \mathbb{R}^n$ be open and $C \subset U$ be compact and path-connected. Furthermore let $0 < \delta < \operatorname{dist}_C(\partial U)$.*

Then there exists $c > 0$ depending only on C and δ such that for every $f \in W^{1,\infty}(U, \mathbb{R})$

$$\sup_{\substack{x,y \in C \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|} \leq c \|Df\|_{\infty, C_\delta} .$$

Proof. First, let $\delta < \frac{1}{6}\operatorname{dist}_C(\partial U)$. Note that

$$C \subset \bigcup_{x \in C} B_\delta\left(x\right) .$$

Since C is compact, there exists $m \in \mathbb{N}$ and $\{x_k\}_{k=0}^m \subset C$ such that

$$C \subset \bigcup_{k=0}^{m} B_\delta\left(x_k\right) .$$

Now, let $x, y \in C$ with $x \neq y$ be such that $|x - y| < \delta$. Then

$$y \in B_\delta\left(x\right) \subset B_{2\delta}\left(x\right) \subset C_{2\delta} \subset U .$$

By Lemma 5.22

$$\frac{|f(x) - f(y)|}{|x - y|} \leq \|Df\|_{\infty, B_{2\delta}(x)} \leq \|Df\|_{\infty, C_{2\delta}} .$$

Now, assume $|x - y| \geq \delta$. Then there exists a continuous $\gamma : [0, 1] \to C$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Let $0 \leq k \leq m$ such that

$$x \in B_\delta\left(x_k\right)$$

and set

$$t_0 := \sup\{t \in [0, 1] \mid \gamma(t) \in B_\delta\left(x_k\right)\} .$$

If $\gamma(t_0) \in B_\delta\left(x_k\right)$, then $t_0 = 1$ and $y \in B_\delta\left(x_k\right)$. Hence

$$\frac{|f(x) - f(x)|}{|x - y|} \leq \|Df\|_{\infty, B_{2\delta}(x_k)} \leq \|Df\|_{\infty, C_{2\delta}} .$$

Otherwise, $\gamma(t_0) \notin B_\delta\left(x_k\right)$. But then there exists $0 \leq l \leq m$ and $l \neq k$ such that

$$\gamma(t_0) \in B_\delta\left(x_l\right)$$

and

$$\gamma(t) \notin B_\delta\left(x_k\right) \text{ for all } t \geq t_0 .$$

Set

$$\gamma^0(t) = \begin{cases} x + \frac{t}{t_0}(\gamma(t_0) - x) & \text{for } t \leq t_0 \\ \gamma(t) & \text{otherwise.} \end{cases}$$

In essence, γ^0 is a shortcut in $\overline{B_\delta\left(x_k\right)}$ to the last point where γ is in $\overline{B_\delta\left(x_k\right)}$.

Repeating the steps above, induction yields a continuous path $\overline{\gamma} : [0,1] \to \overline{C_\delta}$, $0 \le m' \le m$ and $\{t_l\}_{l=1}^{m'} \subset [0,1]$ such that

$$|\overline{\gamma}(t_l) - \overline{\gamma}(t_{l+1})| \le 2\delta \text{ for } l = 0, ..., m' - 1$$

and

$$|x - \overline{\gamma}(t_0)| \le 2\delta \text{ and } |y - \overline{\gamma}(t_{m'})| \le 2\delta\,.$$

Using Lemma 5.22 again for balls of radius 3δ

$$\begin{aligned}
|f(x) - f(y)| \le & |f(x) - f(\overline{\gamma}(t_0)| + \sum_{l=0}^{m'-1} |f(\overline{\gamma}(t_l)) - f(\overline{\gamma}(t_{l+1}))| + |f(\overline{\gamma}_{m'}) - f(y)| \\
\le & \|Df\|_{\infty, B_{6\delta}(x)} |x - \overline{\gamma}(t_0)| + \sum_{l=0}^{m'-1} \|Df\|_{\infty, B_{6\delta}(\overline{\gamma}(t_l))} |\overline{\gamma}(t_l) - \overline{\gamma}(t_{l+1})| \\
& + \|Df\|_{\infty, B_{6\delta}(y)} |y - \overline{\gamma}(t_{m'})| \\
\le & \|Df\|_{\infty, C_{6\delta}} 2\delta(m' + 2) \\
\le & 2(m+2) \|Df\|_{\infty, C_{6\delta}} |x - y|\,.
\end{aligned}$$

Since $x, y \in C$ were arbitrary

$$\sup_{\substack{x,y \in C \\ x \neq x}} \frac{|f(x) - f(y)|}{|x - y|} \le 2(m+2) \|Df\|_{\infty, C_{6\delta}}\,.$$

Note that m only depends on C and δ. Finally, for $0 < \delta < \operatorname{dist}_C(\partial U)$ set

$$\overline{\delta} := \frac{1}{6}\delta\,.$$

Then the inequality above yields

$$\sup_{\substack{x,y \in C \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|} \le 2(m+2) \|Df\|_{\infty, C_\delta}\,.$$

This finishes the proof. □

Remark 5.24. The requirement that C is path-connected cannot be dropped. In order to see this, let $U := \mathbb{R}^2$ and

$$C := [-1, 1] \times \{-2, 2\}\,.$$

Let $f \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R})$ be a Lipschitz continuous function such that

$$f := \begin{cases} 1 & \text{on } [-2,2] \times [1,3] \\ 0 & \text{on } [-2,2] \times [-3,-1]\,. \end{cases}$$

Then for $0 < \delta < 1$

$$\|Df\|_{\infty, C_\delta} = 0$$

but

$$\sup_{\substack{x,y \in C \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|} > 0 \,.$$

Since the trace operator of Silhavy [32] is defined on the space of Lipschitz continuous functions, this space needs to be introduced now.

Definition 5.25. *Let $\Omega \subset \mathbb{R}^n$. Let*

$$\operatorname{Lip}(\Omega)$$

denote the set of all Lipschitz continuous functions on Ω. For $f \in \operatorname{Lip}(\Omega)$ set

$$\|f\|_{\mathrm{Lip}} := \|f\|_C + \sup_{\substack{x,y \in \Omega, \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|} \,.$$

The following result is a slight variation of Lemma 3.2 in Silhavy [32, p. 451]. It states that the Gauß formula yields zero, if the scalar field is zero on the boundary.

Proposition 5.26. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $F \in \mathcal{DM}^1(\Omega, \mathbb{R}^n)$ and $f \in \operatorname{Lip}(\overline{\Omega})$ be such that*

$$f|_{\partial\Omega} = 0 \,.$$

Then

$$\int_\Omega F \cdot Df \,\mathrm{d}\mathcal{L}^n + \int_\Omega f \,\mathrm{d}\operatorname{div} F = 0 \,.$$

Proof. First, suppose that

$$\operatorname{supp} f \subset\subset \Omega \,.$$

By Proposition 4.12 and Proposition 4.14

$$\int_\Omega 1 \,\mathrm{d}\operatorname{div}(F \cdot f) = \int_\Omega f \,\mathrm{d}\operatorname{div} F + \int_\Omega F \cdot Df \,\mathrm{d}\mathcal{L}^n = 0 \,.$$

For the general case, let

$$\begin{aligned} \chi_k &:= \chi_{\Omega_{-\frac{2}{k}}} + (k \operatorname{dist}_{\partial\Omega} - 1)\, \chi_{\Omega_{-\frac{1}{k}} \setminus \Omega_{-\frac{2}{k}}} \\ &= \min\{1, \max\{0, k \operatorname{dist}_{\partial\Omega} - 1\}\} \in \operatorname{Lip}(\overline{\Omega}) \,. \end{aligned}$$

Then $f \cdot \chi_k \in \mathrm{Lip}(\overline{\Omega})$ (cf. [14, p. 48]). In order to estimate the norm independently of $k \in \mathbb{N}$, let $x, y \in \Omega$. If $x, y \in \Omega_{-\frac{2}{k}}$ then

$$|f(x)\chi_k(x) - f(y)\chi_k(y)| = |f(x) - f(y)| \leq \|f\|_{\mathrm{Lip}} |x - y| \, .$$

Otherwise, w.l.o.g. $x \in \Omega \setminus \Omega_{-\frac{2}{k}}$ and

$$\begin{aligned} |f(x) \cdot \chi_k(x) - f(y)\chi_k(y)| &\leq |f(x)||\chi_k(x) - \chi_k(y)| + |\chi_k(y)||f(x) - f(y)| \\ &\leq \|f\|_{C\left(\Omega\setminus\Omega_{-\frac{2}{k}}\right)} |\chi_k(x) - \chi_k(y)| + |f(x) - f(y)| \\ &\leq \left(\sup_{0 \leq \mathrm{dist}_{\partial\Omega}(x) \leq \frac{2}{k}} |f(x)| k + \|f\|_{\mathrm{Lip}} \right) |x - y| \, . \end{aligned}$$

Since f vanishes on $\partial\Omega$

$$\sup_{0 \leq \mathrm{dist}_{\partial\Omega}(x) \leq \frac{2}{k}} |f(x) - 0| \leq \|f\|_{\mathrm{Lip}} \frac{2}{k} \, ,$$

whence

$$\|f \cdot \chi_k\|_{\mathrm{Lip}} \leq 3 \|f\|_{\mathrm{Lip}} \, .$$

Furthermore, for every $k \in \mathbb{N}$

$$\mathrm{supp}\, f \cdot \chi_k \subset\subset \Omega \, .$$

Hence, for every $k \in \mathbb{N}$

$$\int_\Omega F \cdot D(f \cdot \chi_k) \, \mathrm{d}\mathcal{L}^n + \int_\Omega f \cdot \chi_k \, \mathrm{d}\, \mathrm{div}\, F = 0 \, .$$

First note that

$$\int_\Omega f \cdot \chi_k \, \mathrm{d}\, \mathrm{div}\, f \xrightarrow{k \to \infty} \int_\Omega f \, \mathrm{d}\, \mathrm{div}\, F$$

by the Dominated Convergence Theorem (cf. [20, p. 20]).

On the other hand, since $\|D(f \cdot \chi_k)\|_\infty \leq \|f \cdot \chi_k\|_{\mathrm{Lip}}$ is bounded independently of $k \in \mathbb{N}$ the Dominated Convergence Theorem also yields

$$\int_\Omega F \cdot D(f \cdot \chi_k) \, \mathrm{d}\mathcal{L}^n \xrightarrow{k \to \infty} \int_\Omega F \cdot Df \, \mathrm{d}\mathcal{L}^n \, .$$

Hence

$$\int_\Omega F \cdot Df \, \mathrm{d}\mathcal{L}^n + \int_\Omega f \, \mathrm{d}\, \mathrm{div}\, F \xleftarrow{k \to \infty} \int_\Omega F \cdot D(f \cdot \chi_k) \, \mathrm{d}\mathcal{L}^n + \int_\Omega f \cdot \chi_k \, \mathrm{d}\, \mathrm{div}\, F = 0 \, .$$

$\square$

The following proposition is a specialised version of Theorem 2.3 in Silhavy [32, p. 448]. It states that the volume part of a Gauß formula only depends on the boundary values of the Lipschitz continuous scalar function.

Proposition 5.27. *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded and* $F \in \mathcal{DM}^1(\Omega, \mathbb{R}^n)$. *Then there exists a continuous linear functional*

$$\mathcal{NT}_F(\Omega) : \mathrm{Lip}(\partial\Omega) \to \mathbb{R}$$

such that for every $f \in \mathrm{Lip}(\overline{\Omega})$

$$\mathcal{NT}_F(\Omega)(f|_{\partial\Omega}) = \int_\Omega f \,\mathrm{d}\,\mathrm{div}\, F + \int_\Omega F \cdot Df \,\mathrm{d}\mathcal{L}^n \,.$$

Furthermore

$$\|\mathcal{NT}_F(\Omega)\| \leq \|F\|_{\mathcal{DM}^1} \,.$$

Proof. The proof follows the same lines as the one in [32, p. 452]. Let $f \in \mathrm{Lip}(\partial\Omega)$ and $f_1, f_2 \in \mathrm{Lip}(\mathbb{R}^n)$ be extensions of f to all of $\mathbb{R}^n$ (cf. [21, p. 201]). Note that $(f_1 - f_2)|_{\partial\Omega} = 0$. Then by Proposition 5.26

$$\int_\Omega f_1 \,\mathrm{d}\,\mathrm{div}\, F + \int_\Omega F \cdot Df_1 \,\mathrm{d}\mathcal{L}^n = \int_\Omega f_2 \,\mathrm{d}\,\mathrm{div}\, F + \int_\Omega F \cdot Df_2 \,\mathrm{d}\mathcal{L}^n \,.$$

For $f \in \mathrm{Lip}(\partial\Omega)$ and any extension $\overline{f} \in \mathrm{Lip}(\mathbb{R}^n)$ of f define

$$\mathcal{NT}_F(\Omega)(f) := \int_\Omega \overline{f} \,\mathrm{d}\,\mathrm{div}\, F + \int_\Omega F \cdot D\overline{f} \,\mathrm{d}\mathcal{L}^n \,.$$

Then $\mathcal{NT}_F(\Omega) : \mathrm{Lip}(\partial\Omega) \to \mathbb{R}$ is well-defined and a linear functional. For $f \in \mathrm{Lip}(\partial\Omega)$ there exists an extension $\overline{f} \in \mathrm{Lip}(\mathbb{R}^n)$ such that

$$\left\|\overline{f}\right\|_{\mathrm{Lip}} = \|f\|_{\mathrm{Lip}} \,.$$

See Silhavy [32, p. 452] and Federer [21, p. 201] for reference. With this extension

$$\begin{aligned}
|\mathcal{NT}_F(\Omega)(f)| &\leq |\mathrm{div}\, F|\,(\Omega) \left\|\overline{f}\right\|_C + \|F\|_1 \left\|D\overline{f}\right\|_\infty \\
&\leq \|F\|_{\mathcal{DM}^1} \left\|\overline{f}\right\|_{\mathrm{Lip}} \\
&= \|F\|_{\mathcal{DM}^1} \|f\|_{\mathrm{Lip}} \,.
\end{aligned}$$

□

Up to now, the arguments from Silhavy [32] were retraced. Now, the representation of $\mathcal{NT}_F(\Omega)$ by the sum of a Radon measure and a measure $\mu_F \in (\mathrm{ba}\,(U, \mathcal{B}(U), \mathcal{L}^n))^n$ is proved. This result is new because it gives the abstract functionals found in the literature a concrete representation as integral functionals.

Theorem 5.28. *Gauß Theorem*
Let $U \subset \mathbb{R}^n$ be open, $\Omega \subset U$ be open with $\overline{\Omega} \subset U$ compact and $\partial\Omega$ path-connected. Furthermore, let $F \in \mathcal{DM}^1(U, \mathbb{R}^n)$.

Then there exists a Radon measure F^ν on $\partial\Omega$ and $\mu_F \in (\mathrm{ba}\,(U, \mathcal{B}(U), \mathcal{L}^n))^n$ with

$$\operatorname{core} \mu_F \subset \partial\Omega$$

such that for all $f \in W^{1,\infty}(U, \mathbb{R})$ the following Gauß-Green formula holds

$$\int_{\partial\Omega} f \,\mathrm{d}F^\nu + ⨍_{\partial\Omega} Df \,\mathrm{d}\mu_F = \int_\Omega f \,\mathrm{d}\operatorname{div} F + \int_\Omega F \cdot Df \,\mathrm{d}\mathcal{L}^n .$$

Note that the existence of the measures in the above theorem is trivial, neglecting the core and the support, $\mu_F = F\mathcal{L}^n$ and $F^\nu = \operatorname{div} F$ would be viable choices. The difficulty lies in the localisation of $\operatorname{core} \mu_F \subset \partial\Omega$ and the support of F^ν.

Proof. By Proposition 5.27 there exists a continuous linear functional $\mathcal{NT}_F(\Omega)$ on $\mathrm{Lip}(\partial\Omega)$ such that for every $f \in W^{1,\infty}(U, \mathbb{R})$

$$\mathcal{NT}_F(\Omega)(f|_{\partial\Omega}) = \int_\Omega f \,\mathrm{d}\operatorname{div} F + \int_\Omega F \cdot Df \,\mathrm{d}\mathcal{L}^n .$$

Let $0 < \delta < \operatorname{dist}_\Omega(\partial U)$. Note that by [20, p. 131f] every $f \in W^{1,\infty}(U, [0,1])$ is locally Lipschitz continuous. Since $\overline{\Omega}$ is compact and path-connected, $f \in \mathrm{Lip}(\overline{\Omega})$ (cf. Lemma 5.23). Then by Lemma 5.23 for every $f \in W^{1,\infty}(U, \mathbb{R})$

$$\|f|_{\partial\Omega}\|_{\mathrm{Lip}} \le \|f|_{\partial\Omega}\|_C + c\,\|Df\|_{\infty,(\partial\Omega)_\delta} \tag{5.3}$$

with $c > 0$ depending only on δ and $\partial\Omega$. Note that

$$\iota : W^{1,\infty}(U, \mathbb{R}) \to C_0(\partial\Omega) \times \mathcal{L}^\infty((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n) \text{ with } \iota(f) = \left(f|_{\partial\Omega}, Df|_{(\partial\Omega)_\delta}\right)$$

is continuous and linear. Set

$$X_0 := \iota(W^{1,\infty}(U, \mathbb{R}))$$

and define

$$u_0^* : X_0 \to \mathbb{R} \text{ with } \left\langle u_0^*, (f|_{\partial\Omega}, Df|_{(\partial\Omega)_\delta})\right\rangle = \mathcal{NT}_F(\Omega)(f|_{\partial\Omega}) .$$

Then u_0^* is a continuous linear functional on the linear space $X_0 \subset C_0(\partial\Omega) \times \mathcal{L}^\infty((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n)$ by equation (5.3).

By the Hahn-Banach Theorem (cf. [18, p. 63]) there exists a continuous linear extension u^* of u_0^* to all of $C_0(\Omega) \times \mathcal{L}^\infty((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n)$ with $\|u^*\| = \|u_0^*\|$. Note that the dual of a product space can be identified with the product of the dual spaces. Hence, as in Proposition 5.1, there exist a Radon measure F^ν on $\partial\Omega$ and a measure $\mu \in (\mathrm{ba}((\partial\Omega)_\delta, \mathcal{B}((\partial\Omega)_\delta), \mathcal{L}^n))^n$ such that for all $f \in W^{1,\infty}(U, \mathbb{R})$

$$\mathcal{NT}_F(\Omega)(f|_{\partial\Omega}) = \left\langle u^*, (f|_{\partial\Omega}, Df|_{(\partial\Omega)_\delta})\right\rangle = \int_{\partial\Omega} f \,\mathrm{d}F^\nu + \int_{(\partial\Omega)_\delta} Df \,\mathrm{d}\mu\,. \quad (5.4)$$

This proves that there is $\mu \in (\mathrm{ba}(U, \mathcal{B}(U), \mathcal{L}^n))^n$ with $\mathrm{core}\,\mu \subset (\partial\Omega)_\delta$ such that the above equation is satisfied. It remains to show that there exists μ_F with $\mathrm{core}\,\mu_F \subset \partial\Omega$ satisfying the same equation. Now, let

$$\begin{aligned} X_1 := \{\tilde{F} \in \mathcal{L}^\infty((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n) \mid & \exists f \in W^{1,\infty}(U, \mathbb{R}) \\ & \exists F \in \mathcal{L}^\infty((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n) : \\ & F = 0 \text{ on } (\partial\Omega)_{\tilde{\delta}} \text{ for some } 0 < \tilde{\delta} < \delta \\ & \tilde{F} = Df + F\} \end{aligned}$$

Then $u_1^* : X_1 \to \mathbb{R}$ with

$$u_1^*(\tilde{F}) := \int_{(\partial\Omega)_\delta} Df \,\mathrm{d}\mu$$

defines a linear functional on X_1 with

$$\|u_1^*\| \le \|\mu\|\ .$$

First, it is shown that the definition is independent of the decomposition of $\tilde{F}$. Therefore, let $\tilde{F} \in X_1$ and $f_1, f_2 \in W^{1,\infty}(U, \mathbb{R})$, $F_1, F_2 \in \mathcal{L}^\infty((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n)$ be such that for some $0 < \tilde{\delta} < \delta$

$$F_1 = F_2 = 0 \text{ on } (\partial\Omega)_{\tilde{\delta}}$$

and

$$\tilde{F} = Df_1 + F_1 = Df_2 + F_2\,.$$

Then

$$Df_1 = Df_2 \text{ on } (\partial\Omega)_{\tilde{\delta}}\,.$$

Since $\partial\Omega$ is path-connected, $(\partial\Omega)_{\tilde{\delta}}$ is path-connected. Hence $f_1 - f_2$ is constant on $(\partial\Omega)_{\tilde{\delta}}$. Note that for $\tilde{c} \in \mathbb{R}$

$$\mathcal{NT}_F(\Omega)(\tilde{c}) = \int_{\partial\Omega} \tilde{c}\, \mathrm{d}\, F^\nu + \int_{(\partial\Omega)_\delta} 0\, \mathrm{d}\mu = \int_{\partial\Omega} \tilde{c}\, \mathrm{d}\, F^\nu .$$

Hence Equation (5.4) yields

$$\int_{(\partial\Omega)_\delta} D(f_1 - f_2)\, \mathrm{d}\mu = \mathcal{NT}_F(\Omega)((f_1 - f_2)|_{\partial\Omega}) - \int_{\partial\Omega} f_1 - f_2\, \mathrm{d}\, F^\nu = 0 .$$

This shows that u_1^* is well-defined.

Since $X_1 \subset \mathcal{L}^\infty\left((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n\right)$ is a linear subspace, the Hahn-Banach Theorem (cf. [18, p. 63]) yields an extension of u_1^* to all of $\mathcal{L}^\infty\left((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n\right)$ and by Proposition 5.1 a measure $\mu_F \in \left(\mathrm{ba}\left((\partial\Omega)_\delta, \mathcal{B}((\partial\Omega)_\delta), \mathcal{L}^n\right)\right)^n$ with

$$\left\langle u_1^*, \tilde{F} \right\rangle = \int_{(\partial\Omega)_\delta} \tilde{F}\, \mathrm{d}\mu_F .$$

By definition,

$$\int_{(\partial\Omega)_\delta} F\, \mathrm{d}\mu_F = \langle u_1^*, 0 + F\rangle = 0$$

for $F \in \mathcal{L}^\infty\left((\partial\Omega)_\delta, \mathbb{R}^n, \mathcal{L}^n\right)$ with $F = 0$ on $(\partial\Omega)_{\tilde{\delta}}$ for some $0 < \tilde{\delta} < \delta$. Hence,

$$\mathrm{core}\, \mu_F \subset \partial\Omega .$$

Since for every $f \in W^{1,\infty}(U, \mathbb{R})$

$$\int_{(\partial\Omega)_\delta} Df\, \mathrm{d}\mu = \langle u_1^*, Df + 0\rangle = \int_{(\partial\Omega)_\delta} Df\, \mathrm{d}\mu_F$$

by definition, the statement of the theorem follows. $\square$

Remark 5.29. Note that the measure μ_F is a direct result of the analysis. In regular settings, this measure is expected to be zero (see also Example 5.30). For $F \in \mathcal{DM}^\infty(U, \mathbb{R}^n)$ and open $\Omega \in \mathcal{B}(U)$ having finite perimeter such that the inner normal measure exists (see Example 5.12), Proposition 5.21 and Proposition 3.23 yield the existence of a Radon measure F^ν on $\partial\Omega$ such that for all compactly supported continuous functions $f \in \mathcal{BV}(U)$

$$\mathrm{div}\,(f \cdot F)(\Omega) = \int_{\partial\Omega} f\, \mathrm{d}\, F^\nu .$$

In particular, $\mu_F = 0$.

For the general case note that for $k \in \mathbb{N}$

$$\chi_k := \chi_{(\partial\Omega)_{\frac{1}{k}}}(1 - k\,\mathrm{dist}_{\partial\Omega}) \in W^{1,\infty}(U, \mathbb{R}) \,.$$

Since $\chi_k = 1$ on $\partial\Omega$, Proposition 5.26 yields

$$\int_\Omega f\chi_k \,\mathrm{d}\,\mathrm{div}\, F + \int_\Omega F \cdot D(f \cdot \chi_k) \,\mathrm{d}\mathcal{L}^n = \int_\Omega f \,\mathrm{d}\,\mathrm{div}\, F + \int_\Omega F \cdot Df \,\mathrm{d}\mathcal{L}^n \,.$$

Using Dominated Convergence (cf. [20, p. 20]) yields

$$\begin{aligned} \int_\Omega f\chi_k \,\mathrm{d}\,\mathrm{div}\, F &\xrightarrow{k\to\infty} 0 \\ \int_\Omega F \cdot D(f\chi_k) \,\mathrm{d}\mathcal{L}^n &= \int_\Omega \chi_k F Df + fFD\chi_k \,\mathrm{d}\mathcal{L}^n \\ &\xrightarrow{k\to\infty} 0 + \lim_{k\to\infty} \int_\Omega fFD\chi_k \,\mathrm{d}\mathcal{L}^n \,, \end{aligned}$$

where the last limit exists because the other addends tend to zero and their sum is constant. Hence

$$\lim_{k\to\infty} \int_\Omega fFD\chi_k \,\mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} f \,\mathrm{d}F^\nu + \fint_{\partial\Omega} Df \,\mathrm{d}\mu_F \,.$$

Note that the left-hand side is essentially the same as in Schuricht [31, p. 534] (cf. [32, p. 449]).

The following example computes the Radon measure F^ν for a concrete vector field F.

Example 5.30. Revisiting a rotated version of Example 5.15, let $U := B_2(0)$, $\Omega := (0,1)^2$ and $F \in \mathcal{DM}^1(U, \mathbb{R}^n)$ be defined via

$$F(x,y) := \frac{1}{\sqrt{|x|}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,.$$

Then $\mathrm{div}\, F = 0$. For every $k \in \mathbb{N}$ define

$$\chi_k := \max\{1 - k\,\mathrm{dist}_{\partial\Omega}, 0\} \in W^{1,\infty}(U, [0,1]) \,.$$

Then for every $k \in \mathbb{N}$ with $k > 2$ and every $f \in W^{1,\infty}(U, \mathbb{R})$

$$\int_\Omega f \,\mathrm{d}\,\mathrm{div}\, F + \int_\Omega F \cdot Df \,\mathrm{d}\mathcal{L}^n = \int_\Omega f\chi_k \,\mathrm{d}\,\mathrm{div}\, F + \int_\Omega F \cdot D(f \cdot \chi_k) \,\mathrm{d}\mathcal{L}^n$$

by Proposition 5.26. Since $\operatorname{div} F = 0$, this is equal to

$$\int_{\Omega\setminus\Omega_{-\frac{1}{k}}} F\cdot\left(Df - k\,\mathrm{dist}_{\partial\Omega}\,Df - kfD\mathrm{dist}_{\partial\Omega}\right)\mathrm{d}\mathcal{L}^n\,. \tag{5.5}$$

This integral can be computed by partitioning $\Omega\setminus\Omega_{-\frac{1}{k}}$ into the sets

$$\begin{aligned}
Q_1 &:= \left(0,\frac{1}{k}\right)^2, Q_2 := \left(1-\frac{1}{k},1\right)^2\\
Q_3 &:= \left(0,\frac{1}{k}\right)\times\left(1-\frac{1}{k},1\right), Q_4 := \left(1-\frac{1}{k},1\right)\times\left(0,\frac{1}{k}\right)\\
Q_5 &:= \left(0,\frac{1}{k}\right)\times\left(\frac{1}{k},1-\frac{1}{k}\right), Q_6 := \left(1-\frac{1}{k},1\right)\times\left(\frac{1}{k},1-\frac{1}{k}\right)\\
Q_7 &:= \left(\frac{1}{k},1-\frac{1}{k}\right)\times\left(0,\frac{1}{k}\right), Q_8 := \left(\frac{1}{k},1-\frac{1}{k}\right)\times\left(1-\frac{1}{k},1\right).
\end{aligned}$$

Here, the computation is only carried out for Q_7, the other parts are evaluated by similar elementary computations. Note that

$$\mathrm{dist}_{\partial\Omega}(x,y) = y \text{ on } Q_7\,.$$

Hence 5.5 on Q_7 is equal to

$$\int_{\left(\frac{1}{k},1-\frac{1}{k}\right)}\int_{\left(0,\frac{1}{k}\right)}\frac{\partial_2 f - ky\partial_2 f - kf}{\sqrt{|x|}}\,\mathrm{d}y\,\mathrm{d}x\,.$$

Note that $\partial_2 f$ is bounded, whence $\frac{\partial_2 f}{\sqrt{|x|}}\in\mathcal{L}^1\left(U,\mathcal{L}^n\right)$ and

$$\int_{\left(\frac{1}{k},1-\frac{1}{k}\right)\times\left(0,\frac{1}{k}\right)}\frac{\partial_2 f}{\sqrt{|x|}}\,\mathrm{d}\mathcal{L}^n \xrightarrow{k\to\infty} 0\,.$$

Furthermore

$$\begin{aligned}
\left|\int_{\left(\frac{1}{k},1-\frac{1}{k}\right)\times\left(0,\frac{1}{k}\right)}\frac{-ky\partial_2 f}{\sqrt{|x|}}\,\mathrm{d}\mathcal{L}^n\right| &= \left|⨍_{\left(0,\frac{1}{k}\right)} y\int_{\left(\frac{1}{k},1-\frac{1}{k}\right)}\frac{\partial_2 f}{\sqrt{|x|}}\,\mathrm{d}x\,\mathrm{d}y\right|\\
&\le \frac{1}{k}\,\|Df\|_\infty\int_{(0,1)}\frac{1}{\sqrt{|x|}}\,\mathrm{d}x\\
&\xrightarrow{k\to\infty} 0\,.
\end{aligned}$$

Finally

$$\int_{\left(\frac{1}{k},1-\frac{1}{k}\right)\times\left(0,\frac{1}{k}\right)} \frac{-kf}{\sqrt{|x|}} \,\mathrm{d}\mathcal{L}^n = -\fint_{\left(0,\frac{1}{k}\right)} \int_{\left(\frac{1}{k},1-\frac{1}{k}\right)} \frac{f}{\sqrt{|x|}} \,\mathrm{d}x \,\mathrm{d}y$$
$$\xrightarrow{k\to\infty} -\int_{(0,1)} \frac{f(x,0)}{\sqrt{x}} \,\mathrm{d}x \,.$$

Hence

$$\int_{Q_7} F \cdot D(f \cdot \chi_k) \,\mathrm{d}\mathcal{L}^n \xrightarrow{k\to\infty} -\int_{(0,1)} \frac{f(x,0)}{\sqrt{x}} \,\mathrm{d}x = -\int_{(0,1)\times\{0\}} \frac{f}{\sqrt{x}} \,\mathrm{d}\mathcal{H}^{n-1} \,.$$

Computing the remaining integrals in a similar way yields

$$F^\nu = \frac{1}{\sqrt{x}} \mathcal{H}^{n-1} \lfloor (0,1) \times \{1\} - \frac{1}{\sqrt{x}} \mathcal{H}^{n-1} \lfloor (0,1) \times \{0\}$$

and

$$\mu_F = 0 \,.$$

The preceding example illustrates that μ_F can vanish even for vector fields that are unbounded near an $n-1$-dimensional manifold. The function in the following example is the same as in [32, p. 449f].

Example 5.31. Let $n = 2$ and $U = B_2(0) \subset \mathbb{R}^2$. Furthermore, let $\Omega = (0,1)^2$ and $F \in \mathcal{DM}^1(U, \mathbb{R}^n)$ be defined via

$$F(x,y) := \frac{1}{x^2+y^2} \begin{pmatrix} y \\ -x \end{pmatrix} .$$

Note that $\operatorname{div} F$ is the zero measure. In order to see this, let $\phi \in C_0^1(U)$. Then

$$\int_U F \cdot D\phi \,\mathrm{d}\mathcal{L}^n = \lim_{\delta\downarrow 0} \int_{U\setminus B_\delta(0)} F \cdot D\phi \,\mathrm{d}\mathcal{L}^n$$
$$= \lim_{\delta\downarrow 0} \int_{\partial B_\delta(0)} \phi F \cdot \nu \,\mathrm{d}\mathcal{H}^{n-1} - \int_{U\setminus B_\delta(0)} \phi \operatorname{div} F \,\mathrm{d}\mathcal{L}^n \,.$$

But $F \cdot \nu = 0$ on $\partial B_\delta(0)$ and $\operatorname{div} F = 0$ on $U \setminus B_\delta(0)$.

Now, set

$$f_k := \chi_{\left(\frac{1}{k},\infty\right)\times\mathbb{R}} + \chi_{\left(0,\frac{1}{k}\right)\times\mathbb{R}} k \operatorname{dist}_{\{0\}\times\mathbb{R}} \in W^{1,\infty}(U,[0,1]) \,.$$

Then

$$Df_k = \chi_{\left(0,\frac{1}{k}\right)\times\mathbb{R}} k e_1 \,.$$

By Theorem 5.28, there exists a Radon measure F^ν on $\partial\Omega$ and $\mu_F \in (\mathrm{ba}\,(U, \mathcal{B}(U), \mathcal{L}^n))^n$ with $\operatorname{core}\mu_F \subset \partial\Omega$ such that for $k \in \mathbb{N}$

$$\int_\Omega F \cdot Df_k \,\mathrm{d}\mathcal{L}^n + \int_\Omega f_k \,\mathrm{d}\operatorname{div} F = \fint_{\partial\Omega} Df_k \,\mathrm{d}\mu_F + \int_{\partial\Omega} f_k \,\mathrm{d}F^\nu\,.$$

But $f_k = 0$ on $\partial\Omega$ and $\operatorname{div} F = 0$. Hence

$$\int_\Omega F \cdot Df_k \,\mathrm{d}\mathcal{L}^n = \fint_{\partial\Omega} Df_k \,\mathrm{d}\mu_F\,.$$

Furthermore

$$\begin{aligned}\int_\Omega F \cdot Df_k \,\mathrm{d}\mathcal{L}^n &= \fint_{\left(0,\frac{1}{k}\right)} \int_{(0,1)} \frac{y}{x^2+y^2} \,\mathrm{d}y \,\mathrm{d}x \\ &= \fint_{\left(0,\frac{1}{k}\right)} \left[\frac{1}{2}\ln(x^2+y^2)\right]_0^1 \mathrm{d}x \\ &= \fint_{\left(0,\frac{1}{k}\right)} \frac{1}{2}\ln\left(\frac{1}{x^2}+1\right) \mathrm{d}x \\ &\geq \frac{1}{2}\ln(k^2+1) \xrightarrow{k\to\infty} \infty\,.\end{aligned}$$

The example above shows that μ_F can actually be non-zero, if the concentrations of the vector field F are sufficiently large near $\partial\Omega$. Thus, μ_F is indeed necessary for the characterisation of the Gauß-Green formula.

Example 5.32. Revisiting Example 5.17, let $n = 2$ and $U := B_2(0) \subset \mathbb{R}^n$. Furthermore, let $\Omega := (0,1) \times (-1,1)$ and $F \in \mathcal{DM}^1(U, \mathbb{R}^n)$ be defined by

$$F(x,y) := \frac{1}{2\pi}\frac{1}{x^2+y^2}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Recall that $\operatorname{div} F = \delta_0$. For $k \in \mathbb{N}$ let $f_k \in W^{1,\infty}(U, [0,1])$ be defined by

$$f_k := \chi_{\left(\frac{1}{k},\infty\right)\times\mathbb{R}} + k \operatorname{dist}_{\{0\}\times\mathbb{R}} \chi_{\left(0,\frac{1}{k}\right)\times\mathbb{R}}\,.$$

Then

$$\begin{aligned}\int_\Omega F \cdot Df_k \,\mathrm{d}\mathcal{L}^n &= \fint_{\left(0,\frac{1}{k}\right)} \int_{(-1,1)} \frac{x}{2\pi(x^2+y^2)} \,\mathrm{d}y \,\mathrm{d}x \\ &= \frac{1}{2\pi}\fint_{\left(0,\frac{1}{k}\right)} \left[\arctan\frac{y}{x}\right]_{-1}^1 \mathrm{d}x \\ &= \frac{1}{2\pi}\fint_{\left(0,\frac{1}{k}\right)} 2\arctan\frac{1}{x} \,\mathrm{d}x \\ &\xrightarrow{k\to\infty} \frac{1}{2}\,.\end{aligned}$$

In contrast to the previous example, this example shows a vector field with a strongly concentrated divergence in zero, yet μ_F seems to be zero. Indeed, in [32, p. 449] Silhavy shows that the normal trace can be represented by a Radon measure, if

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{\Omega \setminus \Omega_{-\delta}} |F \cdot D\mathrm{dist}_{\partial\Omega}| \, \mathrm{d}\mathcal{L}^n < \infty \, .$$

This holds true in the last example and thus $\mu_F = 0$.

Chapter 6

Conclusion

In this thesis, pure measures and their application to Gauß formulas are investigated. The characterisation of the dual of the essentially bounded functions is improved by decomposing the weakly absolutely continuous measures into pure parts and a σ-measure with a Lebesgue density. Moreover, a new large class of pure measures on $\mathbb{R}^n$ is identified. This class of so-called density measures is comprised of measures that concentrate on Lebesgue null sets and are often explicitly computable, in contrast to the examples given in the literature. In plus, they have a natural connection to singular Radon measures, i.e. every pure measure can be represented by a Radon measure on its core and every Radon measure can be extended to a measure on the entire domain.

This connection motivates the use of pure measures for Gauß formulas. It turns out that in the case of essentially bounded vector fields on sets of finite perimeter there exist normal measures. In general, these measures are pure and are explicitly computable on sets of finite perimeter. Together with the product formulas for vector fields having divergence measure, they yield not one, but many Gauß formulas for essentially bounded functions and vector fields. These new Gauß formulas separate the geometry and the vector field very well and enable the use of integration theory for the normal trace. In the case of unbounded vector fields, integrability with respect to normal measures cannot be assured. This is shown in concrete examples.

Nevertheless, Gauß formulas with the normal trace being a continuous linear functional on the space of Lipschitz continuous functions on the boundary were proved by Silhavy [32] and Torres [13], among others. In this thesis, the result from Silhavy is improved by proving that his normal trace functional can be represented as a sum of a Radon measure and a measure which is pure if the boundary of the domain of integration has no volume. This gives the normal trace the shape of an integral over the boundary. Interestingly, the new measure μ_F that emerges from the analysis acts on the values of the gradient on the boundary.

Future research might clarify the structure of the extremal points of the set of all density measures. They are of interest because any bounded measure can essentially be approximated by convex combinations of these directionally concentrated measures.

In this thesis, the application of pure measures to normal traces and Gauß formulas is emphasised. It is to be expected that traces of Sobolev functions and functions of bounded variation can be treated in a similar manner. In addition, density measures could be used to establish another approach to set valued gradients. The approach sketched in this exposition shows that at least basic rules of differential calculus are readily available.

The fine structure of the measures representing the normal trace functional due to Silhavy also poses an interesting problem. Since the test functions available are only Lipschitz continuous and point wise convergence does not imply convergence in measure, their values are not readily available for arbitrary sets of finite perimeter.

Another venue of research is the application to Continuum Mechanics. The literature on edge contact forces contains models where terms similar to the measure μ_F appear (cf. [17, p. 44], [16, p. 96]). If μ_F turned out to be non-zero in these models, the structure of forces considered would be justified analytically.

Pure measures turn out to be well suited for the representation of traces and normal traces in particular. This can be explained by the fact that they represent the dual space of essentially bounded functions. All in all, pure measures seem to be a good tool to describe phenomena where quantities defined on a volume concentrate on low-dimensional sets.

Bibliography

[1] A. D. Alexandroff. Additive set functions in abstract spaces II. *Recueil Mathématique [Mat. Sbornik]*, 9(51)(3):563–628, 1941.

[2] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford University Press, New York, 2000.

[3] Gabriele Anzellotti. Pairings between measures and bounded functions and compensated compactness. *Annali di Matematica Pura ed Applicata*, 135(1):293–318, December 1983.

[4] Garret Birkhoff. *Lattice Theory*, volume 25 of *Colloquium publications*. American Mathematical Society, Providence, 1993.

[5] S. Bochner. Remark on the theorem of Green. *Duke Math. J.*, 3(2):334–338, June 1937.

[6] S. Bochner. Additive Set Functions on Groups. *Annals of Mathematics*, 40(4):769–799, 1939.

[7] S. Bochner. Finitely Additive Integral. *Annals of Mathematics*, 41(3):495–504, 1940.

[8] Gui-Qiang Chen and Hermano Frid. Divergence-Measure Fields and Hyperbolic Conservation Laws. *Archive for Rational Mechanics and Analysis*, 147(2):89–118, 1999.

[9] Gui-Qiang Chen and Hermano Frid. On the theory of divergence-measure fields and its applications. *Boletim da Sociedade Brasileira de Matematica*, 32(3):401, November 2001.

[10] Gui-Qiang Chen and Hermano Frid. Extended Divergence-Measure Fields and the Euler Equations for Gas Dynamics. *Communications in Mathematical Physics*, 236(2):251–280, May 2003.

[11] Gui-Qiang Chen and Monica Torres. Divergence-Measure Fields, Sets of Finite Perimeter, and Conservation Laws. *Archive for Rational Mechanics and Analysis*, 175(2):245–267, February 2005.

[12] Gui-Qiang Chen and Monica Torres. On the structure of solutions of nonlinear hyperbolic systems of conservation laws. *Communications on Pure and Applied Analysis*, 10(4):1011–1036, 2011.

[13] Gui-Qiang Chen, William P. Ziemer, and Monica Torres. Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws. *Communications on Pure and Applied Mathematics*, 62(2):242–304, February 2009.

[14] Franke H. Clarke. *Optimization and Nonsmooth Analysis.* Classics in Applied Mathematics. Wiley, 1984.

[15] Marco Degiovanni, Alfredo Marzocchi, and Alessandro Musesti. Cauchy Fluxes Associated with Tensor Fields Having Divergence Measure. *Archive for Rational Mechanics and Analysis*, 147(3):197–223, August 1999.

[16] Marco Degiovanni, Alfredo Marzocchi, and Alessandro Musesti. Edge-force densities and second-order powers. *Annali di Matematica Pura ed Applicata*, 185(1):81–103, February 2006.

[17] F. Dell'Isola and P. Seppecher. Edge Contact Forces and Quasi-Balanced Power. *Meccanica*, 32(1):33–52, February 1997.

[18] Nelson Dunford and Jacob T. Schwartz. *Linear Operators, General Theory - Part 1.* Wiley Classics Library. Wiley Interscience, 1988.

[19] Yuli Eidelman, Vitali Milman, and Antonis Tsolomitis. *Functional Analysis - An Introduction*, volume 66 of *Graduate Studies in Mathematics.* American Mathematical Society, 2004.

[20] Lawrence C. Evans and R. Gariepy. *Measure Theory and Fine Properties of Functions.* CRC Press, 1992.

[21] Herbert Federer. *Geometric Measure Theory.* Springer, Berlin, reprint of 1969 ed. edition, 1996.

[22] Carl Friedrich Gauß. Theoria Attractionis Corporum Sphaeroidicorum Ellipticorum Homogeneorum Methodo nova tractata. In *Werke*, volume 5, pages 1–22. 2 edition, 1813.

[23] Morton E. Gurtin, William O. Williams, and William P. Ziemer. Geometric measure theory and the axioms of continuum thermodynamics. *Archive for Rational Mechanics and Analysis*, 92(1):1–22, March 1986.

[24] Paul R. Halmos. *Measure Theory*, volume 18 of *Graduate Texts in Mathematics*. Springer Berlin Heidelberg, 1974.

[25] Heinrich Hertz. Über die Berührung fester elastischer Körper. *Journal für die reine und angewandte Mathematik*, 92:156–171, 1882.

[26] Victor J. Katz. The history of Stokes' theorem. *Mathematics Magazine*, 52(3):146–156, 1979.

[27] A. Kolmogoroff. Untersuchungen über den Integralbegriff. *Mathematische Annalen*, 103(1):654–696, 1930.

[28] Solomon Leader. The theory of Lp-spaces for finitely additive set functions. *Annals of Mathematics. Second Series*, 58:528–543, 1953.

[29] Gert. K. Pedersen. *Analysis Now*, volume 118 of *Graduate Texts in Mathematics*. Springer New York, 1989.

[30] K. P. S. Bhaskara Rao and M. Bhaskara Rao. *Theory of Charges - A Study of Finitely Additive Measures*. Pure and Applied Mathematics - A Series of Monographs and Textbooks. Academic Press, London, 1983.

[31] Friedemann Schuricht. A New Mathematical Foundation for Contact Interactions in Continuum Physics. *Archive for Rational Mechanics and Analysis*, 184(3):495–551, June 2007.

[32] Miroslav Silhavy. The Divergence Theorem for Divergence Measure Vectorfields on Sets with Fractal Boundaries. *Mathematics and Mechanics of Solids*, 14(5):445–455, 2009.

[33] Kôsaku Yosida and Edwin Hewitt. Finitely Additive Measures. In *Bulletin of the American Mathematical Society*, volume 57, pages 46–66, Providence, 1951. American Mathematical Society.

[34] Eberhard Zeidler. *Nonlinear Functional Analysis and Its Applications*, volume III. Springer Berlin Heidelberg, 1986.

[35] Eberhard Zeidler. *Nonlinear Functional Analysis and Its Applications*, volume I. Springer Berlin Heidelberg, 1986.

List of Symbols

Functions and Function Spaces

X	a normed space
$\mathcal{BV}(\Omega)$	space of functions of bounded variation, page 55
$\lvert Df\rvert(\Omega)$	total variation of a function of bounded variation, page 55
$C_0(\Omega)$	continuous functions with compact support
$C(\overline{\Omega})$	continuous functions on Ω, with continuous extension to $\overline{\Omega}$
$C^k(\Omega)$	k times differentiable functions with bounded derivatives
u^*	an element of X^*
Df	the (weak) derivative of F
$\langle\cdot,\cdot\rangle$	dual pairing
$\operatorname{div} F$	divergence measure of F, page 59
$\mathcal{DM}^p(U,\mathbb{R}^n)$	$\mathcal{L}^p$ vector fields having divergence measure, page 59
X^*	dual space of the Banach space X
p'	$\frac{1}{p'}+\frac{1}{p}=1$, the Hölder conjugate
χ_B	$\chi_B=1$ on B, zero otherwise
$\mathrm{Lip}(\Omega)$	Lipschitz continuous functions on Ω, page 85
$L^p(\Omega,\mathcal{A},\mu)$	p-integrable functions w.r.t to algebra $\mathcal{A}$ and measure μ, page 28

$\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$	Equivalence classes in $L^p(\Omega, \mathcal{A}, \mu)$, page 28
$L^p(\Omega, \mu)$	$L^p(\Omega, \mathcal{A}, \mu)$ with $\mathcal{A} = \mathcal{B}(\Omega)$, page 28
$\mathcal{L}^p(\Omega, \mu)$	$\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ with $\mathcal{A} = \mathcal{B}(\Omega)$, page 28
χ_k	good approximation of the characteristic function, page 64
μ_F	measure from Gauß Theorem, page 88
F^ν	Radon measure from Gauß Theorem, page 88
$\text{supp}(\phi)$	closure of the set where $\phi \neq 0$
$f\|_A$	restriction of f to A
$C_0^\infty(\Omega)$	smooth functions with compact support
h	simple function, page 24
$W^{k,p}(U, \mathbb{R})$	Sobolev space of order k and exponent p
$\mathscr{L}^p(\Omega, \Sigma, \mu)$	measures λ, $\lambda << \mu$, $\|\lambda\|_p < \infty$, page 33

Integrals

$f_1 = f_2$ μ-a.e.	$f_1 = f_2$ a.e., page 25
$f_k \xrightarrow{\mu} f$	f_k converge in measure to f, page 25
$\int_\Omega f \, \mathrm{d}\mu$	integral of f w.r.t. measure μ, page 28
$⨍_B f \, \mathrm{d}\mu$	$\frac{1}{\mu(B)} \int_B f \, \mathrm{d}\mu$
$\int_A^{\mathcal{R}} \zeta$	refinement integral of ζ, page 32
$⨍_C f \, \mathrm{d}\mu$	$\int_{C_\delta} f \, \mathrm{d}\mu$, where $C = \operatorname{core} \mu$, page 30

Lattices

$\|l\|$	$l^+ + l^-$, page 6
l^+	$l \vee 0$, page 6
l^-	$l \wedge 0$, page 6
$L^\perp$	orthogonal complement, page 7

$l_1 \perp l_2$	$\lvert l_1\rvert \wedge \lvert l_2\rvert = 0$, page 6
$l_1 \vee l_2$	$\sup\{l_1, l_2\}$, page 6
$l_1 \wedge l_2$	$\inf\{l_1, l_2\}$, page 6

Measures Theory

$\mathcal{A}$	algebra, page 10
$\alpha(n)$	volume of the n-dimensional unit ball
$\mathrm{ba}(\Omega, \mathcal{A})$	space of bounded measures on Ω, page 10
$\mathrm{ba}\,(\Omega, \mathcal{A}, \lambda)$	bounded and weakly absolutely continuous measures w.r.t. λ, page 14
$\mathrm{ca}(\Omega, \mathcal{A})$	space of bounded σ-measures on Ω, page 10
$\mathrm{core}\,\mu$	core of μ, page 17
$\mathrm{dens}_C(B)$	$\lim\limits_{\delta\downarrow 0} \frac{\mathcal{L}^n(B\cap C_\delta\cap\Omega)}{\mathcal{L}^n(C_\delta\cap\Omega)}$, page 17
$\mathrm{Dens}\,(C)$	density measures for C, page 37
$\mathcal{H}^d$	d-dimensional Hausdorff measure
$\mathcal{L}^n$	n-dimensional Lebesgue measure
$\mu << \lambda$	μ is absolutely continuous w.r.t. λ, page 13
$\mu \leq \lambda$	$\mu(A) \leq \lambda(A)$ for all $A \in \mathcal{A}$, page 11
$\mu <<^w \lambda$	μ is weakly absolutely continuous w.r.t. λ, page 14
μ, λ	measures, page 9
μ_c	σ-additive part of μ, page 12
μ_p	pure part of μ, page 12
μ^-	negative part of μ, page 11
ν^B	(measure theoretic) normal to B
μ^*	outer measure for μ, page 24
μ^+	positive part of μ, page 11

$\mu\lfloor A$	restriction of μ to A, page 11
Σ	σ-algebra, page 10
σ	σ-measure, page 10
$\lvert\mu\rvert$	total variation of μ, page 11

Norms

$\lVert f\rVert_{\mathrm{Lip}}$	$\lVert f\rVert_C + \sup\limits_{\substack{x,y\in\Omega,\\ x\neq y}} \frac{\lvert f(x)-f(y)\rvert}{\lvert x-y\rvert}$, page 85
$\lVert F\rVert_{\infty,V}$	$\operatorname*{ess\,sup}\limits_{V} \lvert F\rvert$, page 81
$\lVert f\rVert_C$	$\sup \lvert f(x)\rvert$
$\lVert f\rVert_p$	L^p-norm of f, page 29
$\lVert\lambda\rVert_p$	p-norm of the measure λ, page 33
$\lVert u^*\rVert$	$\sup\limits_{\lVert x\rVert\leq 1} \langle u^*, x\rangle$
$\lVert\mu\rVert$	$\lvert\mu\rvert(\Omega)$, page 12

Sets

A^c	set theoretic complement of A
$B_\delta(x)$	ball centred at x with radius $\delta > 0$
$\partial\Omega$	topological boundary of Ω
$\mathcal{B}(\Omega)$	Borel subsets of Ω, i.e. σ-algebra generated by relatively open sets
$\overline{\Omega}$	topological closure of Ω
dist_Ω	signed distance function, page 37
Ω_δ	$\{x \in U \mid \mathrm{dist}_\Omega(x) < \delta\}$, page 37
Ω	a subset of $\mathbb{R}^n$, page 9
$\mathrm{Graph}(f)$	set of points $(x, f(x))$
$\mathrm{int}\,\Omega$	topological interior of Ω

$\mathcal{I}$	an (index) set
$\partial_* B$	measure theoretic boundary of B, page 56
B_{ext}	measure theoretic exterior of B, page 56
B_{int}	measure theoretic interior of B, page 56
$\mathbb{N}$	natural numbers
$\mathcal{P}^{\mathcal{A}}(A)$	Partitions of A by sets in $\mathcal{A}$, page 32
2^{Ω}	class of subsets of Ω
$\mathcal{R}(h)$	$\{y \in \mathbb{R} \mid \exists x \in \Omega : h(x) = y\}$, page 24
$\partial^* B$	reduced boundary of B, page 57
$A \subset B$	$\forall x \in A : x \in B$
n	$n \in \mathbb{N}, n > 0$, page 9

Glossary

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Datum, Unterschrift