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Quantified Tauberian Theorems and Applications to Decay of Waves

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Contents

List of Symbols and Notation

We use standard notation as much as we can. In some cases ambiguities might occur and in this case the following list of symbols serves as the reference in our thesis. Also some of our frequently used self-defined symbols occur here.

$\mathbb{R}_+$	$= [0,\infty)$
$\mathbb{C}_+$	$= \{z \in \mathbb{C}; \Re z \geq 0\}$
$\mathbb{N}_j$	$= \{j, j+1, \ldots\}$ for j an integer
$\mathbb{N}$	$= \mathbb{N}_0$
$\exp_j$	$= \exp \circ \ldots \circ \exp$, j-times, $j \in \mathbb{N}_1$
$\log_j$	$= \log \circ \ldots \circ \log$, j-times, $j \in \mathbb{N}_1$
$L_j(s)$	$= \log_j(1+j+s), s \geq 0, j \in \mathbb{N}_1$
Ω_M	$= \{z \in \mathbb{C}; 0 > \Re z > -1/M(\lvert\Im z\rvert)\}$
B_r	Open ball around 0 with radius $r > 0$ in $\mathbb{R}^d$ or $\mathbb{C}$
$\mathrm{R}_0, \mathrm{R}_\alpha$	See Section A.1
$\mathrm{PI}, \mathrm{PD}, \mathrm{BI}, \mathrm{BD}$	See Section A.2
PI_N	See Section A.3
$L^2_{\mathrm{comp}}(\Omega)$	$= \{u \in L^2(\Omega); \operatorname{supp} u \subset \Omega\}, \Omega \subset \mathbb{R}^d$ open
$H^m_{\mathrm{comp}}(\Omega)$	$= \{u \in H^m(\Omega); \operatorname{supp} u \subset \Omega\}, \Omega \subset \mathbb{R}^d$ open, $m \in \mathbb{N}_1$
$H^0_{\mathrm{comp}}(\Omega)$	$= L^2_{\mathrm{comp}}(\Omega), \Omega \subset \mathbb{R}^d$ open
$\nabla H^1(\Omega)$	$= \{\nabla u \in L^2(\Omega)^d; u \in H^1(\Omega)\}, \Omega \subset \mathbb{R}^d$ open
$B^{s,p}_q(\Omega)$	for $\Omega \subseteq \mathbb{R}^d$ open; Besov space; See Appendix C
$H^s(\partial\Omega)$	for $\Omega \subseteq \mathbb{R}^d$ open, Lipschitz; Fractional Sobolev space on closed set; See Appendix C
$\mathcal{L}(X,Y)$	$\{T : X \to Y; T$ bounded linear operator$\}$ where X, Y Banach spaces
$\mathcal{L}(X)$	$= \mathcal{L}(X,X)$

Notation

Generic constants c and C. We use two generic constants $c > 0$ and $C > 0$. Generic means that they may change their value from line to line. The difference between these two constants is that their usage implicitly means that we could always replace c by a smaller constant and C by a larger constant - *if this is necessary*. So one should keep in mind that c is a small number and C a large number.

Small constants $c_1, c_2, \ldots$. By c_j for $j \in \mathbb{N}$ we denote strictly positive real numbers. Usage of these constants implicitly means that all statements in which they occur remain true if one replaces c_j by a smaller number.

Large constants $C_1, C_2, \ldots$. By C_j for $j \in \mathbb{N}$ we denote strictly positive real numbers. Usage of these constants implicitly means that all statements in which they occur remain true if one replaces C_j by a larger number. Small and large constants are not allowed to change their values - unless it is explicitly stated.

Landau notation. Let us denote by ϕ, ϕ_1, ϕ_2 (not necessarily strictly) positive functions and by ψ a complex valued functions defined on $\mathbb{R}\backslash K$ or $\mathbb{R}_+\backslash K$, where K is a compact interval. We define

$$\phi_1(s) \lesssim \phi_2(s) :\Leftrightarrow \exists s_0 > 0, C > 0 \forall \, |s| \geq s_0 : \phi_1(s) \leq C\phi_2(s),$$
$$\phi_1(s) \approx \phi_2(s) :\Leftrightarrow \phi_1(s) \lesssim \phi_2(s) \text{ and } \phi_2(s) \lesssim \phi_1(s).$$

Furthermore we define the following classes (sets) of functions:

$$O(\phi(s)) := \{\psi; |\psi(s)| \lesssim \phi(s)\},$$
$$o(\phi(s)) := \{\psi; \forall \varepsilon > 0 \exists s_\varepsilon > 0 \forall \, |s| \geq s_\varepsilon : |\psi(s)| \leq \varepsilon\phi(s)\}.$$

By abuse of notation we write for example $\psi(s) = O(\phi(s))$ instead of $\psi \in O(\phi(s))$ or $\phi(s) = \phi_1(s) + O(\phi_2(s))$ instead of $|\phi(s) - \phi_1(s)| \lesssim \phi_2(s)$. By $O(s^{-\infty})$ we denote the intersection of all $O(s^{-N})$ for $N \in \mathbb{N}$. We say that a function $\phi : \mathbb{R} \to \mathbb{R}$ decays *rapidly* if for any $n \in \mathbb{N}_0$ there exists a constant C such that $|\phi(t)| \leq C(1+t)^{-n}$.

Asymptotic similarity/equivalence. We say that $\phi_1, \phi_2 : [a, \infty) \to (0, \infty)$ are asymptotically similar if $\phi_1(s) \approx \phi_2(s)$ i.e. $\phi_1(s) \lesssim \phi_2(s)$ and $\phi_2(s) \lesssim \phi_1(s)$. We say that $\phi_1, \phi_2 : [a, \infty) \to (0, \infty)$ are asymptotically equivalent and write $\phi_1(s) \sim \phi_2(s)$ if $\phi_1(s)/\phi_2(s) \to 1, s \to \infty$.

Inverse functions. Given $a \geq 0$ and a continuous non-decreasing function $M : [a, \infty) \to (0, \infty)$ such that $M(s) \to \infty$ as $s \to \infty$, we denote by $M^{-1} : [M(a), \infty) \to [a, \infty)$ its (right-continuous) right-inverse, given by $M^{-1}(s) = \sup\{r \geq a : M(r) \leq s\}$, $s \geq M(a)$. The definition implies that $M(M^{-1}(s)) = s$, $s \geq M(a)$, and $M^{-1}(M(s)) \geq s$, $s \geq a$.

Introduction

This thesis is devoted to the investigation of (semi-uniform) decay rates for C_0-semigroups and applications to the decay of waves. To give the reader an impression of the main contributions of the thesis, in the following we formulate a few mathematical questions to which our results give (partial) answers. We remark that this does not reflect all of our results but certainly the most interesting ones. We further note at this point that Chapter 2 is joint work with Jan Rozendaal and David Seifert.

Quantified Tauberian theorems and decay of C_0-semigroups

In the last decade there has been much activity in the field of *quantified* Tauberian theorems for C_0-semigroups, or more generally for functions of a real variable [34, 6, 19, 8, 12, 35, 10, 18, 7]. See also [44, 45] and references therein for quantified Tauberian theorems on sequences and [26] for Dirichlet series. We refer to [31] and [4, Chapter 4] for a general overview on Tauberian theory.

A milestone in this area of research is without doubt a result of Batty and Duyckaerts [8]. It reads as follows:

Theorem 0.1 (Batty-Duyckaerts [8]). *Let X be a Banach space and let A be the generator of a bounded C_0-semigroup T on X. Suppose that $\sigma(A) \cap i\mathbb{R} = \emptyset$ and that $M : \mathbb{R}_+ \to (0, \infty)$ is a continuous non-decreasing function such that $\|(is - A)^{-1}\| \leq M(|s|)$, $s \in \mathbb{R}$. Then there exists a constant $c > 0$ such that*

$$\|T(t)A^{-1}\| = O\left(\frac{1}{M_{\log}^{-1}(ct)}\right), \quad t \to \infty,$$

where $M_{\log} : \mathbb{R}_+ \to (0, \infty)$ is defined by $M_{\log}(s) = M(s)\log(2 + s + M(s))$, $s \geq 0$. Conversely, suppose that $\left\|T(t)A^{-1}\right\| \leq m(t), t \geq 0$ for a continuous non-increasing function $m : \mathbb{R}_+ \to (0, \infty)$ with $m(t) \to 0, t \to \infty$. Then

$$\left\|(is - A)^{-1}\right\| \leq O\left(m^{-1}\left(\frac{1}{2\,|s| + 2}\right)\right), \quad |s| \to \infty.$$

The proof of the first part of this result is based on the so called *contour method* (using Cauchy's formula) and was inspired by Newman's approach to the prime number theorem [39]. The converse part of the theorem is much simpler to prove but is nevertheless important, as it shows that the first part is sharp up to a *logarithmic loss*. There are generalizations of this theorem allowing for a finite number of spectral points along the imaginary axis but for the sake of simplicity we do not consider them in this introduction.

Theorem 0.1 is a consequence of a more general theorem for functions instead of semigroup orbits [8]. Here, one considers a locally integrable function $f : \mathbb{R}_+ \to X$

and asks for the rate at which the norm of $f(t)$ decays to zero as t tends to infinity. The "regularizing" effect of A^{-1} to the orbit of the semigroup is now replaced by the additional condition $f' \in L^\infty(\mathbb{R}_+; X)$ on the weak derivative of f. We remark that under this assumption the Laplace transform $\hat{f}$ of f is absolutely convergent in $\{z \in \mathbb{C}; \Re z > 0\}$. The condition on the resolvent now translates to a condition on $\hat{f}$. That means we assume that $\hat{f}$ extends analytically across the imaginary axis to a domain

$$\Omega_M = \left\{ z \in \mathbb{C}; 0 > \Re z > -\frac{1}{M(|\Im z|)} \right\}$$

and

$$\left\| \hat{f}(z) \right\| \leq CM(|\Im z|), \quad z \in \Omega_M. \tag{0.1}$$

A proof which is almost identical to the proof of Theorem 0.1 shows

Theorem 0.2 (Batty-Duyckaerts [8]). *Let $f : \mathbb{R}_+ \to X$ be a locally integrable function with $f' \in L^\infty(\mathbb{R}_+; X)$ whose Laplace transform extends analytically to Ω_M and satisfies (0.1). Then*

$$\|f(t)\| = O\left(\frac{1}{M_{\log}^{-1}(ct)} \right), \quad t \to \infty.$$

A result of this type is important for the decay of "perturbed" orbits of semigroups, that is, functions of the form $P_2 T(t) P_1 x$, where P_1, P_2 are bounded operators. This in turn is a natural approach to answer questions on local decay for wave equations on exterior domains. We emphasize at this point that a special case of this theorem, with $M(s) \sim s^\alpha$ for some $\alpha > 0$ and $f(t) = P_2 T(t) P_1 x$ where T is a unitary group, motivated by the wave equation, was already known several years earlier by Popov and Vodev [40]. Observe that the domain Ω_M to which $\hat{f}$ extends analytically is determined by the same function M by which $\hat{f}$ should be bounded. In the situation of Theorem 0.1 this is a natural assumption due to the resolvent equation. However in applications (see e.g. Chapter 3) one sometimes faces the situation that $\hat{f}$ extends to a "relatively large" domain - that means M is "small" - but one only knows a "large" bound on the Laplace transform in the region of analyticity. In this case the results of [8] are not applicable anymore and it would be desirable to answer to following

Question 1. Let $K : \mathbb{R}_+ \to (0, \infty)$ be non-decreasing. Is a version of Theorem 0.2 still valid if one replaces (0.1) by

$$\left\| \hat{f}(z) \right\| \leq K(|\Im z|), \quad z \in \Omega_M,$$

while keeping all other assumptions? For what function $M_{\log K}$ is the decay rate given by

$$\|f(t)\| = O\left(\frac{1}{M_{\log K}^{-1}(ct)} \right), \quad t \to \infty, \tag{0.2}$$

if one chooses $c > 0$ suitably?

After some preliminary work by Borichev and Tomilov [12] a first general answer to this question was given by Batty, Borichev and Tomilov [7]. There the authors allow $K(s) = ((1+s)M(s))^\alpha, s \in \mathbb{R}_+$ for arbitrary $\alpha > 0$ and deduce a decay rate determined by $M_{\log K} = M_{\log}$. The proof is a refinement of the proof of Theorem 0.2. Unfortunately some potential applications of such a result are still not covered by this improved version of Theorem 0.2. In fact, in Chapter 3 we present an example where M is constant, that is Ω_M is a strip, and the best known bound on the Laplace transform is given by $\exp(Cs^\alpha)$ for some $\alpha > 0$. In Chapter 1 we give an answer to Question 1 which covers also this situation. More precisely we allow all non-decreasing functions $K : \mathbb{R}_+ \to (0, \infty)$ which satisfy for some $\varepsilon \in (0, 1)$.

$$K(s) = O(\exp(\exp((sM(s))^{1-\varepsilon}))). \tag{0.3}$$

The decay rate is given by $M_{\log K}(s) = M(s) \log(2+M(s)+K(s))$. The proof of this result uses refined versions of techniques already applied by Chill and Seifert [18]. The main idea of the proof, going back to Ingham and Karamata, is to introduce a splitting $f = [f - \varphi_R * f] + \varphi_R * f$ for a suitably chosen approximate unit φ_R. The first summand can then be estimated in an elementary way and the second one by going to the Fourier space. Heavily using ideas from [12] we show that the decay rate given in (0.2) is optimal up to the choice of c. We also give a partial answer to the therefore natural question on the optimal choice of c.

Other topics of Chapter 1 include the investigation of $\hat{f}$ having s^{-1}-type or log-type singularities at zero. We give more details on that in the introduction of Chapter 1. We only mention that a log-type singularity forces us to give an alternative proof of our version of Theorem 0.2 now based on a refined version of the contour method applied in [8]. Surprisingly, the second proof requires K to satisfy the same constraint (0.3). Based on this observation we conjecture that (0.3) with $\varepsilon = 0$ is not sufficient for a $1/M_{\log K}^{-1}(ct)$ decay rate to hold in general. Unfortunately we have no idea how to prove that.

Another question related to Theorem 0.1 is to ask whether the logarithmic loss in the upper bound for the decay rate can be avoided. It is not difficult to see that the answer to this question - in this generality - is *no*! Consider for example a semigroup on a Hilbert space with a normal generator which has spectral set $\{is - \log(s)^{-1}; s \in [2, \infty)\}$. Clearly, by the spectral theorem, the resolvent of the generator is bounded by a function $M(s) \sim \log(s)$. Moreover, it is not difficult to see that the semiuniform decay rate is given by $e^{-2\sqrt{t}} \sim 1/M_{\log}^{-1}(4t)$. Therefore it is natural to restrict this question to certain classes of functions M.

A first breakthrough concerning this modified question was achieved in a celebrated paper of Borichev and Tomilov [12]. The authors consider polynomial resolvent bounds, that is $M(s) = C(1 + s^\alpha)$ for some $\alpha > 0$. Assuming that M is up to a constant the best upper bound, Theorem 0.1 yields in this case

$$c\left(\frac{1}{t}\right)^{\frac{1}{\alpha}} \leq \left\|T(t)A^{-1}\right\| \leq C\left(\frac{\log(t)}{t}\right)^{\frac{1}{\alpha}}$$

for all $t \geq 2$. In [12] a semigroup on a non-Hilbertian space is constructed for which the upper bound is indeed the precise decay rate. This means that even for the class of functions like $1 + s^\alpha$ the answer is still *no*! Fortunately the situation is different in Hilbert spaces:

THEOREM 0.3 (Borichev-Tomilov [12]). *Let X be a Hilbert space and let A be the generator of a bounded C_0-semigroup T on X. Suppose that $\sigma(A) \cap i\mathbb{R} = \emptyset$ and that $\left\|(is - A)^{-1}\right\| \leq M(|s|) := C(1 + |s|^\alpha)$, $s \in \mathbb{R}$, for some $\alpha > 0$. Then for any*

$c > 0$

$$\left\|T(t)A^{-1}\right\| = O\left(\frac{1}{M^{-1}(ct)}\right)\left(= O\left(\frac{1}{t^{\frac{1}{\alpha}}}\right)\right), \quad t \to \infty. \tag{0.4}$$

We see that in this situation the decay rate is given by $1/M^{-1}(ct)$ and that $c > 0$ can be chosen arbitrarily since it only influences the growth properties of the function $M^{-1}(ct)$ up to a constant - we say M^{-1} and $M^{-1}(c\cdot)$ are *asymptotically similar*. This leads us to the next major question to be answered in this thesis.

QUESTION 2. What is the class $\mathcal{M}$ of (finally) continuous non-decreasing functions M such that the conclusion (0.4) of Theorem 0.3 still holds? For M being a function from this class, are M^{-1} and $M^{-1}(c\cdot)$ asymptotically similar for any $c > 0$?

To the best of our knowledge to this date the only paper addressing this question is [10] (with a preprint on arXiv from 2013). With a heavy use of functional calculus and the theory of regularly varying functions (see e.g. [11]) the authors (Batty, Chill and Tomilov) could show that a subclass of the class of regularly varying functions (see Appendix A for a short introduction to this class of functions) is contained in $\mathcal{M}$. In particular it follows from that paper that the functions given by $s^\alpha/\log(s), s > 2$ for some $\alpha > 0$ are contained in $\mathcal{M}$. However, the very similar functions given by $s^\alpha \log(s), s > 2$ could not be shown to lie in $\mathcal{M}$.

In Chapter 2 we give a complete answer to Question 2. This chapter is based on the preprint [43] which is a joint work with Jan Rozendaal and David Seifert. We prove that $\mathcal{M} = \mathrm{PI}$ which is the class of functions having *positive increase*. The class PI is larger than the class of regularly varying functions but still satisfies the asymptotic similarity condition asked for in Question 2. Moreover, any function having positive increase is bounded from below by s^ε for a suitably chosen $\varepsilon > 0$. In particular $1/M^{-1}$ does not decay at a super-polynomial rate. We refer to Appendix A for more details on functions having positive increase. We mention at this point that our proof of the necessity of the positive increase condition does not rely on certain well constructed semigroups. Indeed the positive increase condition is necessary for every normal semigroup. Actually we prove the necessity for an even wider class of semigroups. We refer to the introduction of Chapter 2 for the details.

Our answer to Question 2 makes the question of polynomial- and sub-polynomial semiuniform decay rates for semigroups on Hilbert spaces a very well studied subject. It also shows the following: if one is interested in *super*-polynomial decay rates and if one has a bounding function M for the resolvent which is sharp (possibly up to a constant), the decay rate is in general not given by (0.4) - instead the rate is strictly slower. However it is not plausible to assume that the decay rate is always not faster than $1/M_{\log}^{-1}(ct)$. This leads us to

QUESTION 3. Let A be the generator of a bounded C_0-semigroup on a Hilbert space satisfying $\sigma(A) \cap i\mathbb{R} = \emptyset$ and $\left\|(is - A)^{-1}\right\| \leq M(|s|)$ for some continuous non-decreasing function M with a sub-polynomial growth. In this situation, what is the optimal (smallest) continuous non-decreasing function M_{opt} such that

$$\left\|T(t)A^{-1}\right\| \leq \frac{1}{M_{\mathrm{opt}}^{-1}(t)}$$

for all $t \in \mathbb{R}_+$?

In this strong formulation we are not able to answer this question. However we can give partial answers to this question which are already far-reaching. First of all, if we restrict to normal semigroups we can indeed give a complete answer. The optimal decay rate in this situation is asymptotically equivalent to $1/M_{\mathrm{qm}}^{-1}(t)$ where

$$M_{\mathrm{qm}}(s) = \sup_{\lambda\in[1,s]} M(\lambda^{-1}s)\log(\lambda), \quad s \geq 1.$$

Clearly, the definition of M_{opt} implies $M_{\mathrm{opt}} \geq M_{\mathrm{qm}}$. This raises the question whether these two functions are equal. Note that equality would mean that normal semigroups always yield the worst decay rate under all semigroups with a given growth behaviour of the resolvent.

To approach an answer to this question, in Chapter 2 we define the notion of *quasi-positive increase (with auxiliary function N)*. The so called auxiliary function $N : \mathbb{R}_+ \to (0,\infty)$ is a non-decreasing function. The definition of quasi-positive increase (see Section A.3) reveals that it is natural to restrict to auxiliary functions with $N(s) = O(\log(s)), s \to \infty$ and that *every* non-decreasing function has quasi-positive increase with auxiliary function $N(s) = \beta\log(2+s)$ where $\beta > 0$ can be chosen arbitrarily. We prove the following

Theorem 0.4. *Let X be a Hilbert space and let A be the generator of a bounded C_0-semigroup T on X with $\sigma(A)\cap i\mathbb{R} = \emptyset$. Let $M, N : \mathbb{R}_+ \to (0,\infty)$ be continuous non-decreasing functions and suppose that $M(s) \to \infty$ as $s \to \infty$, that $N(s) = O(\log s)$ as $s \to \infty$, and that M has quasi-positive increase with auxiliary function N. Suppose further that $\left\|(is - A)^{-1}\right\| \leq M(|s|)$, $s \in \mathbb{R}$. Then there exists a constant $c > 0$ such that*

$$\left\|T(t)A^{-1}\right\| = O\left(\frac{1}{M_N^{-1}(ct)}\right), \quad t \to \infty, \tag{0.5}$$

where $M_N : \mathbb{R}_+ \to (0,\infty)$ is defined by $M_N(s) = M(s)N(s)$, $s \geq 0$. Moreover, in (0.5), for any $\varepsilon > 0$ one can choose $c = be - \varepsilon$ where b is a constant, depending on M and N, arising in the definition of quasi-positive increase.

Note that due to the explicit constant c this result is sharper than Theorem 0.1 even if $N(s) = \beta\log(2+s)$. Even more is true: for a large class of resolvent bounds M we can prove that for the "optimal" choice of the auxiliary function N our result can essentially not be improved, i.e. the conclusion would be false in general (for normal semigroups) if one chooses $c > be$. This is particularly true for the very important special case $M(s) \sim C\log(s)^\alpha$ for certain constants $C, \alpha > 0$.

Applications: decay of waves

This part of the thesis is devoted to the study of three different types of wave equations and their energy decay properties. Two of these wave equations also serve as examples showing the strength of our theoretical results obtained in part one.

Waves on exterior domains. In Chapter 3 we consider

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) = 0 & (t \in (0,\infty), x \in \Omega), \\ u(t,x) = 0 & (t \in (0,\infty), x \in \partial\Omega), \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) & (x \in \Omega). \end{cases}$$

Here $\Omega \subsetneq \mathbb{R}^d$, $d \geq 2$, is a connected open set with bounded complement and non-empty C^∞-boundary. Since this system preserves the (total) energy, energy loss can only occur in spatially bounded regions due to radiation to infinity. Thus it is appropriate to study the so called *local energy*

$$E^\rho(t) = \int_{\Omega \cap B_\rho} |\nabla u(t,x)|^2 + |u_t(t,x)|^2 \, dx.$$

Here the radius $\rho > 0$ has to be chosen large enough so that the obstacle $\mathcal{O}$ is included in the open ball B_ρ around the origin with that radius. In the literature a famous question is the following:

QUESTION 4. Given $m \in \mathbb{N}$, what is the rate

$$p_m(t) = \sup \left\{ E^\rho(t); \|(u_0, u_1)\|_{H^{m+1}_{\mathrm{comp}} \times H^m_{\mathrm{comp}}(\Omega \cap B_\rho)} \leq 1 \right\}$$

at which $E^\rho(t)$ decays uniformly with respect to (normalized) initial data from $H^{m+1} \times H^m$ with compact support in $\Omega \cap B_\rho$?

The well established approach to this question is to formulate the above wave equation in the language of C_0-semigroups. This leads to a unitary group $(T(t))_{t \in \mathbb{R}}$ with generator $\mathcal{A}$ on a suitable Hilbert space $\mathcal{H}$. Question 4 is now essentially equivalent to the decay of $f(t) = PT(t)(1-\mathcal{A})^{-m}P$ for P being a suitable (bounded) multiplication operator. For simplicity we restrict to the case of odd dimensions from now on. In [21] it was shown by Bony and Petkov that whenever $\hat{f}$ extends to a strip to the left of the imaginary axis then $\hat{f}(z) \leq C \exp(C |\Im z|^{d-1})$ for z lying in that strip. By using a Tauberian result due to Popov and Vodev [40] the authors were able to deduce

$$p_m(t) = O\left(\left(\frac{\log(t)}{t} \right)^{\frac{m}{d-1}} \right)$$

from that. Our results from Chapter 1 improve their result and even simplify the argument. That is, in the same situation we can deduce

$$p_m(t) = O\left(\left(\frac{1}{t} \right)^{\frac{m}{d-1}} \right),$$

thus we get rid of the logarithmic factor. This is a taste of our very small contribution to the rather general Question 4.

Damped waves on partially rectangular domains. Let $\Omega \subset \mathbb{R}^2$ be a so called *partially rectangular* domain. That is, there exists a rectangle $R \subseteq \Omega$ such that two opposite sides of R are contained in $\partial\Omega$. Let $a \in L^\infty(\Omega) \backslash \{0\}$ be a (not necessarily strictly) positive function. An example of a partially rectangular domain is of course any rectangle and the in the literature on dynamical billiards well known Bunimovich stadium. We consider the damped wave equation

$$\begin{cases} u_{tt}(t,x,y) - \Delta u(t,x,y) + 2a(x,y)u_t(t,x,y) = 0 & (t \in (0,\infty),\, (x,y) \in \Omega), \\ u(t,x,y) = 0 & (t \in (0,\infty),\, (x,y) \in \partial\Omega), \\ u(0,x,y) = u_0(x,y),\, u_t(0,x,y) = u_1(x,y) & ((x,y) \in \Omega). \end{cases}$$

Without loss of generality we may assume that $R = (0,1)^2$ and that $\{(x,y) \in R; y = 0\}$ and $\{(x,y) \in R; y = 1\}$ correspond to two opposite sides of R which are contained in Ω. Furthermore we assume that a restricted to some neighbourhood of

$\Omega\backslash R$ is bounded from below by a strictly positive constant and that a allows for so called *trapped bouncing rays*. That is, there exists a non-empty interval $I \subset (0,1)$ such that a restricted to $I \times (0,1)$ is zero.

The above wave equation gives rise to a contractive C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. Its generator $\mathcal{A}$ has no spectrum on the imaginary axis. The square of the norm in $\mathcal{H}$ can naturally be interpreted as the energy of the system. Thus the uniform energy decay rate of (normalized) "classical" solutions is up to squaring exactly the rate at which $\left\|T(t)\mathcal{A}^{-1}\right\|$ decays to zero. Burq and Hitrik showed in [15] that

$$c(1+s) \leq \sup_{|\xi|\leq s} \left\|(i\xi - \mathcal{A})^{-1}\right\| \leq C(1+s^2), \quad s > 0.$$

Actually the (easier) proof of the left hand inequality, based on the construction of quasi-modes, was only sketched in [15] and we therefore refer to Anantharaman and Léautaud [3] for more details in case of a torus. In view of Theorems 0.1 and 0.3 this leads to a decay rate estimate of the form

$$\frac{c}{t} \leq \left\|T(t)\mathcal{A}^{-1}\right\| \leq \frac{C}{t^{\frac{1}{2}}}, \quad t \geq 1. \tag{0.6}$$

We remark that an approach via *geometric optics* using the ideas of Ralston [42] also yields the lower bound on $\left\|T(t)\mathcal{A}^{-1}\right\|$. For simplicity let us now restrict to $\Omega = (0,1)^2$. Given $\varepsilon \in (0,1/2]$ it is known that under smoothness assumptions on a (depending on ε) it is possible to show that the decay rate is bounded from above by $Ct^{-1+\varepsilon}$, see e.g. [15, 3]. We also refer to more precise results in a slightly different situation [33, 16, 13]. We observe that the upper bound in the a priori estimate (0.6) receives much attention. However, this is not so for the lower bound. Therefore let us ask

Question 5. Is the lower bound in (0.6) sharp in general?

In the light of the above mentioned results it is natural to assume that discontinuous behaviour of a could yield the slowest possible decay rate. In the appendix of [3] Nonnenmacher considered the situation of a being constant zero on $(0,\sigma)\times(0,1)$ and constant non-zero on $(\sigma,1)\times(0,1)$ for some $\sigma \in (0,1)$. He investigated the spectrum of $\mathcal{A}$ and could prove that there exists a sequence of eigenvalues (z_n) with $\Im z_n \to \infty$ and $-\Re z_n = O((\Im z_n)^{-\frac{3}{2}})$. This yields a lower resolvent estimate

$$\sup_{|\xi|\leq s} \left\|(is - \mathcal{A})^{-1}\right\| \geq c(1+s^{\frac{3}{2}}) \text{ and thus } \left\|T(t)\mathcal{A}^{-1}\right\| \geq \frac{c}{t^{\frac{2}{3}}}, \quad s,t \geq 1. \tag{0.7}$$

It is now an interesting question whether the spectrum reflects the correct behaviour of the resolvent growth. This motivated us to exactly calculate the resolvent growth in Chapter 4. Using Theorem 0.3 we confirm that (0.7) remains true if one reverses the inequality-signs (changing the constants). Our results are published in [48]. Of course this does not answer Question 5 but it leads us to the conjecture that the answer is probably *no*. We furthermore conjecture that the sharp lower bound is given by $ct^{-2/3}$.

Waves subject to viscoelastic boundary damping. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Let us consider a model for the radiation and reflection of sound waves

$$\begin{cases} U_{tt}(t,x) - \Delta U(t,x) = 0 & (t \in \mathbb{R}, x \in \Omega), \\ \partial_n U(t,x) + k * U_t(t,x) = 0 & (t \in \mathbb{R}, x \in \partial\Omega). \end{cases}$$

Here $*$ means the usual convolution with respect to the time variable and $k : \mathbb{R} \to [0,\infty)$ is an integrable function, depending on the time-variable only, which vanishes on $(-\infty, 0)$. We furthermore assume that k is completely monotone, that is there exists a (unique) positive Borel-measure ν such that $k(t) = \int_{[0,\infty)} e^{-ts} d\nu(s), t > 0$. The Laplace transform $\hat{k}$ of k can naturally be interpreted as the *acoustic impedance* of the boundary. One can interpret U_t and $-\nabla U$ as (relative) pressure and fluid velocity. Under these assumptions Desch, Fasangova, Milota and Probst [22] could rephrase this equation as an abstract Cauchy problem $\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t), t > 0, \mathbf{x}(0) = \mathbf{x}_0$ where $\mathcal{A}$ is the generator of a contractive C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. The natural energy of this system is equal to the square of the norm in $\mathcal{H}$. It was also shown in [22] that $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$ and that $\mathcal{A}$ is injective. Since we are interested in decay rates for "classical" solutions a natural question arises. Aiming for results not depending on Ω we ask the following

Question 6. Can one characterize the relation $0 \in \sigma(A)$ in terms of the acoustic impedance only? Can one determine the growth of $\left\|(is - A)^{-1}\right\|$ at infinity in terms of the acoustic impedance, and in case of $0 \in \sigma(A)$, what is the growth rate at zero?

We are not able to answer this question in this generality. However we think our partial results are already far-reaching. In the particular case $\Omega = (0,1)$ we actually can answer this question completely. For arbitrary Ω we prove

$$0 \notin \sigma(A) \Leftrightarrow \exists \varepsilon > 0 : \nu([0,\varepsilon]) = 0 \Leftrightarrow \exists \varepsilon > 0 : k(t) = O(e^{-\varepsilon t}), t \to \infty.$$

Observe that the last equivalence of the preceding line is almost trivial. Moreover, we show

$$0 \in \sigma(A) \Rightarrow \left\|(is - A)^{-1}\right\| \leq C\,|s|^{-1}, \quad |s| \leq 1.$$

The only part of Question 6 we did not answer so far is the part concerned with the growth of the resolvent at infinity. In the 1-dimensional setting we can give a rather precise answer

$$\frac{c}{\Re\hat{k}(is)} \leq \left\|(is - A)^{-1}\right\| \leq \frac{C}{\Re\hat{k}(is)}, \quad |s| \geq 1. \tag{0.8}$$

Note that the function $\Re\hat{k}(i\cdot) = \mathcal{F}k : \mathbb{R} \to (0,\infty)$ is smooth, symmetric and strictly decaying (to zero) on the interval $\mathbb{R}_+$. The proof of this result is based on a rather explicit representation of the resolvent of $\mathcal{A}$. The multi-dimensional case needs a completely new strategy. Under mild additional assumptions on the acoustic impedance and the domain we are able to confirm the upper bound in (0.8). The condition on $\hat{k}$ and Ω involves properties of Laplace-Neumann eigenfunctions recently investigated in [5].

Once we have such a resolvent bound it is desirable to know what kind of functions can arise from $\mathcal{F}k$. One can show that for suitable choices of k (or ν) it is possible to reproduce any regularly varying function with index in $(-2,0)$, at least up to asymptotic equivalence. This shows that our results of Chapter 2 are perfectly adapted to calculate the decay rate for such a kind of equations. Restricting for simplicity to $\Omega = (0, 1)$ and $\nu([0, 1]) = 0$ our results show that

$$\Re\mathcal{F}k \in \mathrm{PD} \Leftrightarrow \left\|T(t)A^{-1}\right\| = O\left(\frac{1}{M^{-1}(t)}\right) \text{ where } M(s) = \mathcal{F}k(s), s \geq 0.$$

That is, we can precisely determine the decay rate for a large class of possible acoustic impedances.

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Part 1

Quantified Tauberian theorems and decay of C_0-semigroups

CHAPTER 1

Decay of vector-valued functions

1.1. Introduction

This Chapter is mainly devoted to a generalization of [8, Theorem 4.1 and Corollary 4.2] which we reproduce here in our terminology.

THEOREM 1.1 (Batty-Duyckaerts [8]). *Let $f : \mathbb{R}_+ \to X$ be a locally integrable function with $f' \in L^\infty(\mathbb{R}_+; X)$ with Laplace transform $\hat{f}$ which extends analytically to Ω_M (defined in (1.2)) and satisfies*

$$\left\| \hat{f}(z) \right\| \leq CM(|\Im z|) \text{ for } z \in \Omega_M. \tag{1.1}$$

Then

$$\|f(t)\| = O\left(\frac{1}{M_{\log}^{-1}(ct)} \right), \quad t \to \infty$$

where $M_{\log}(s) = M(s)\log(2 + s + M(s))$ for $s \geq 0$.

Here and in what follows we set

$$\Omega_M = \left\{ z \in \mathbb{C}; 0 > \Re z > -\frac{1}{M(|\Im z|)} \right\}. \tag{1.2}$$

We present generalizations in two different directions. The first generalization is Theorem 1.3 below. Our theorem is inspired by a paper of Batty, Borichev and Tomilov [7]. We have adapted some ideas from this paper, however our main strategy in the proof follows the “Fourier approach” as in a paper of Chill and Seifert [18]. Although we also consider *L^p-rates of decay* as in [7] our main concern in Theorem 1.3 is to weaken the constraint (1.1) in the sense that we want to decouple (almost) completely the growth bound from the shape of the domain. That is we replace this constraint by

$$\left\| \hat{f}(z) \right\| \leq K(|\Im z|) \text{ for } z \in \Omega_M.$$

For $K : \mathbb{R}_+ \to (0, \infty)$ we allow any non-decreasing function which satisfies

$$K(s) = O(\exp(\exp((sM(s))^{1-\varepsilon}))), s \to \infty \tag{1.3}$$

for some $\varepsilon \in (0, 1)$. This is motivated by applications to the wave equation in exterior domains (see Chapter 3). Now the decay rate is given by $1/M_{\log K}^{-1}(ct)$ with $M_{\log K}(s) = M(s)\log(2 + s + M(s) + K(s))$ for $s \geq 0$. In a special case our results reproduce the results of [7] where K had to be bounded by a polynomial in $sM(s)$. In Sections 1.2.3 and 1.2.4 we show that the decay rate we obtain are in a sense optimal in general. To prove that we slightly generalize an argument from a paper of Borichev and Tomilov [12].

Our second generalization is Theorem 1.31. Our motivation for such a generalization is the wave equation on exterior domains in *even dimensions*. This possible application shows that there is a need for a theorem which allows $\hat{f}$ to have a log-type singularity near zero. This means that in a neighbourhood of $0 \in \mathbb{C}$ there exists an (X-valued) analytic function $\tilde{f}$ such that

$$z \mapsto \hat{f}(z) - \tilde{f}(z)\log(z) \text{ is analytic.}$$

To the best of our knowledge such a theorem is new in this general context. Our work in this direction was inspired by paper of Vodev [51]. Unfortunately we were not able to apply the "Fourier approach" due to - for us - unsurmountable difficulties with the logarithmic singularity at zero. So in contrast to the proof of Theorem 1.3 as in e.g. [8, 7] we now use the contour method for our proof. However our very weak condition (1.3) forces us to very carefully choose the contour along which we integrate. Also the so called *fudge-factor* is heavily influenced by this condition.

Let us finally explain our last main result of this chapter. Before we do so let us first state the following simplified version of a result due to Martinez [35] (see also [18, 10]).

Theorem 1.2. *Let X be a Banach space and let A be the generator of a bounded C_0-semigroup T on X. Suppose that $\sigma(A) \cap i\mathbb{R} = \{0\}$ and that $\left\|(is - A)^{-1}\right\| \leq C(\min\{|s|, 1\})^{-\alpha}$, $s \in \mathbb{R}$ for some $\alpha \geq 1$. Then*

$$\left\|T(t)A(1-A)^{-2}\right\| = O\left(\left(\frac{\log(t)}{t}\right)^{\frac{1}{\alpha}}\right), \quad t \to \infty.$$

The assumption on the resolvent to be bounded at infinity is only made to focus on the essentials in the following. Our results (Theorem 1.26 and 1.38) show that in the special case $\alpha = 1$ the logarithmic loss actually does not occur. This phenomenon seems to be unknown in this context. Analogous results are only known for semigroups on Hilbert spaces and for analytic semigroups.

1.2. No singularity on $i\mathbb{R}$

Given a continuous and non-decreasing function $M : \mathbb{R}_+ \to (0, \infty)$ we denote

$$w_M(t) = \begin{cases} M^{-1}(t) & \text{if } t \geq 1 \\ 1 & \text{else.} \end{cases}$$

The main result of this section is the following theorem, which is a generalization of [10, Theorem 4.1].

Theorem 1.3. *Let $(X, \|\cdot\|)$ be a Banach space, $m \in \mathbb{N}$, and $f : \mathbb{R}_+ \to X$ be a locally integrable function such that its m-th weak derivative $f^{(m)}$ is in $L^p(\mathbb{R}_+; X)$ for some $1 < p \leq \infty$. Assume that there exist continuous and non-decreasing functions $M, K : \mathbb{R}_+ \to (0, \infty)$ satisfying*

(i) $\forall s \geq 0 : K(s) \geq \max\{2, s, M(s)\}$,
(ii) $\exists \varepsilon \in (0, 1) : K(s) = O\left(e^{e^{(sM(s))^{1-\varepsilon}}}\right)$ *as* $s \to \infty$.

such that the Laplace transform $\hat{f}$ of f extends analytically to $\Omega_M \cup \mathbb{C}_+$ and

$$\left\|\hat{f}(z)\right\| \leq K(|\Im z|) \text{ for all } z \in \Omega_M. \tag{1.4}$$

Then there exists a constant $c_1 > 0$ *such that*

$$\left(t \mapsto \left\| w_{M_{\log K}}(c_1 t)^m f(t) \right\| \right) \in L^p(\mathbb{R}_+), \tag{1.5}$$

where $M_{\log K}(s) := M(s)\log(K(s))$.

REMARK 1.4. Note that a function $f \in L^1_{loc}(\mathbb{R}_+; X)$ with $f^{(m)} \in L^p(\mathbb{R}_+; X)$ is polynomially bounded. In fact, $\|f(t)\| \leq C(1+t)^{m-1/p}$ for all $t \geq 0$. In particular the Laplace transform of f is well-defined in the interior of $\mathbb{C}_+$ as an absolutely convergent integral.

REMARK 1.5. One can drop condition (i) on K but then one has to replace $M_{\log K}$ by the function given by $M(s)\log(2 + s + M(s) + K(s))$.

REMARK 1.6. We are not able to prove the theorem for $\varepsilon = 0$ in condition (ii). In Section 1.13 the reader can find a short discussion on a slightly weaker constraint on K.

We prove Theorem 1.3 as a corollary to the following variant which is a generalization of [18, Theorem 2.1(b)]:

THEOREM 1.7. *Let* $(X, \|\cdot\|)$ *be a Banach space,* $m \in \mathbb{N}$*, and* $f : \mathbb{R}_+ \to X$ *be a locally integrable function such that* $f^{(m)} \in L^p(\mathbb{R}_+; X)$ *for some* $1 < p \leq \infty$*. Let* M *and* K *be as in Theorem 1.3. Assume that the Fourier transform* F *of* f *is of class* C^∞ *and its derivatives satisfy*

$$\left\| F^{(j)}(s) \right\| \leq j! K(|s|) M(|s|)^j \text{ for all } j \in \mathbb{N}_0, s \in \mathbb{R}. \tag{1.6}$$

Then there exists a constant $c_1 > 0$ *such that*

$$\left(t \mapsto \left\| w_{M_{\log K}}(c_1 t)^m f(t) \right\| \right) \in L^p(\mathbb{R}_+), \tag{1.7}$$

where $M_{\log K}(s) := M(s)\log(K(s))$.

REMARK 1.8. Note that the Fourier transform of f is well-defined in the sense of tempered distributions since f is polynomially bounded (compare with Remark 1.4).

We show in Lemma 1.14 that the Theorems 1.3 and 1.7 are essentially equivalent. To prove Theorem 1.7 we adapt the proof of [18, Theorem 2.1(b)]. That is - for $m = 1$ - we decompose $f = [f - \phi_R * f] + \phi_R * f = J_1 + J_2$ into two terms with the help of some suitably chosen and scaled convolution kernel $\phi_R(t) = R\phi(Rt)$ with $\int_{\mathbb{R}} \phi(t)dt = 1$. Then we estimate the X-norm of $J_1(t, R)$ and $J_2(t, R)$ in terms of R and t, solely assuming $f' \in L^p$ for the former and the bounds on all derivatives $F^{(j)}$ for the latter. Finally we optimize the sum of these two estimates by choosing $R = M_{\log K}^{-1}(c_1 t)$ for a sufficiently small c_1.

We improve the techniques of [18] in the following way: we estimate $J_1(t, R)$ from above by a Poisson integral $R^{-1}P_{R^{-1}} * \|f'\|(t)$ which makes it possible to apply a fundamental result on Carleson measures. We note that this technique was already applied in [7]. Compared to the proof in [18] we get a better estimate on $J_2(t, R)$ by choosing a better convolution kernel ϕ. Also the Fourier transform ψ of our convolution kernel is a C_c^∞-function which simplifies the proof slightly. Our choice of ψ is based on the Denjoy-Carleman theorem on quasi-analytic functions.

1.2.1. Proof of Theorem 1.7. Without loss of generality we may assume that $f(0) = f'(0) = \ldots = f^{(m-1)}(0) = 0$. If this was not satisfied we could replace f by $f - g$ for some function $g \in C_c^m([0, t_1); X)$ with $g(0) = f(0), \ldots, g^{(m-1)}(0) = f^{(m-1)}(0)$ and $t_1 > 0$ arbitrary. This neither changes the asymptotics of f at infinity nor does it change the growth of F and its derivatives at infinity significantly. To see this note that the Fourier transform G of g satisfies

$$\left\| G^{(j)}(s) \right\| \leq t_1^{j+1} \left\| g \right\|_\infty \text{ for all } j \in \mathbb{N}_0, s \in \mathbb{R}.$$

Now let us extend f by zero on the negative numbers. By our additional assumptions we see that the extended function is $(m-1)$-times continuously differentiable on the whole real line and $f^{(m)} \in L^p(\mathbb{R}; X)$.

Let $\psi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \psi \subseteq [-1, 1]$ and $\psi(0) = 1$ be a function to be fixed later in the proof. Let

$$\phi(t) = \mathcal{F}^{-1}\psi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} \psi(s) ds$$

be its inverse Fourier transform. Note that ϕ is a Schwartz function with $\int \phi dt = \psi(0) = 1$. For $R > 0$ let $\phi_R(t) = R\phi(Rt)$ and $\psi_R(s) = \psi(s/R)$. Let us decompose

$$\begin{aligned} f(t) &= (\delta - \phi_R)^{*m} * f(t) - [(\delta - \phi_R)^{*m} - \delta] * f(t) \\ &= \left[\sum_{j=0}^{m} \binom{m}{j} (-1)^j \phi_R^{*j} * f \right](t) - \left[\sum_{j=1}^{m} \binom{m}{j} (-1)^j \phi_R^{*j} * f \right](t) \\ &=: J_1(t, R) + J_2(t, R). \end{aligned}$$

Here by ϕ^{*j} we denote the j-times convolution of ϕ with itself. We also define $\phi^{*0} = \delta$ (delta function). Note that $(\phi_R)^{*j} = (\phi^{*j})_R$.

1.2.1.1. *Estimation of J_1.* Let us define the Poisson kernel by

$$P_y(t) = \frac{1}{\pi} \cdot \frac{y}{t^2 + y^2}.$$

Recall that by Young's inequality the Poisson kernel acts as a continuous operator on $L^p(\mathbb{R})$ via convolution.

Lemma 1.9. *Let $1 \leq p \leq \infty$ and $m \in \mathbb{N}_1$. Let $f : \mathbb{R} \to X$ be a locally integrable function such that $f^{(m)} \in L^p(\mathbb{R}; X)$. Let ϕ be as above. Then there exists a constant $C > 0$ (only depending on ϕ and m) such that*

$$\left\| (\delta - \phi_R)^{*m} * f(t) \right\| \leq \frac{C}{R^m} P_{\frac{1}{R}} * \left\| f^{(m)} \right\| (t) \tag{1.8}$$

holds for all $t \geq 0$ and $R > 0$.

Remark 1.10. It is clear from the proof that in the statement of the lemma one can replace $P = P_1$ by any positive and integrable kernel bounded from below by $c(1+t)^{-\alpha}$ for some $\alpha > 1$. We then define $P_y(t) = y^{-1}P(y^{-1}t)$. Unfortunately this is not consistent with the definition of ϕ_R, but for the Carleson measure argument below it is more convenient to define P_y as above.

Proof. Let us define two antiderivatives of ϕ

$$\Phi_-(t) = \int_{-\infty}^{t} \phi(\tau) d\tau, \quad \Phi_+(t) = -\int_{t}^{\infty} \phi(\tau) d\tau.$$

Furthermore we define the following auxiliary function

$$\Phi(t) = \begin{cases} \Phi_-(t) & \text{if } t < 0 \\ \Phi_+(t) & \text{if } t \geq 0 \end{cases}. \tag{1.9}$$

We observe that the derivative of Φ is ϕ plus a factor times the delta function at zero. This observation is the reason why we split the integral from the following calculation at 0.

First, we consider the case $m = 1$.

$$\begin{aligned} [f - \phi_R * f](t) &= \int_{-\infty}^{\infty} (f(t) - f(t-\tau))\phi_R(\tau)d\tau \\ &= \left[\int_{-\infty}^{0} + \int_{0}^{\infty}\right] (f(t) - f(t - \frac{\tau}{R}))\phi(\tau)d\tau \\ &= -\frac{1}{R}\int_{-\infty}^{\infty} f'(t - \frac{\tau}{R})\Phi(\tau)d\tau \\ &= -\frac{1}{R}\Phi_R * f'(t). \end{aligned} \tag{1.10}$$

We need to explain why the partial integration executed from line two to three produces no boundary terms at $-\infty, 0$ and ∞. At zero there are no boundary terms since $(f(t) - f(t - \frac{\tau}{R}))$ vanishes at $\tau = 0$ and the two limits $\lim_{t\to 0\pm} \Phi(t)$ exist. Recall that f is polynomially bounded. Moreover the function Φ decays rapidly at infinity. Thus there are no boundary terms at plus or minus infinity. Finally the last equality together with the fact that Φ decays rapidly implies

$$\begin{aligned} \|[f - \phi_R * f](t)\| &\leq \frac{C}{R}\int_{-\infty}^{\infty} \left\|f'(t - \frac{\tau}{R})\right\| \frac{1}{\tau^2+1}d\tau \\ &\leq \frac{C}{\pi R}\int_{-\infty}^{\infty} \|f'(t-\tau)\| \frac{R^{-1}}{\tau^2 + R^{-2}}d\tau \\ &= \frac{C}{R}P_{\frac{1}{R}} * \|f'\|(t). \end{aligned}$$

Now we consider the case $m \in \mathbb{N}_2$. Let us define recursively $f_{j+1} = f_j - \phi_R * f_j, f_0 = f$ for $j \in \{0, 1, \ldots, m-1\}$. Clearly $f_m = (\delta - \phi_R)^{*m} * f$. We prove now $f_j = (-1/R)^j \Phi_R^{*j} * f^{(j)}$ via induction on j. Observe that for any $j \in \mathbb{N}_1$ the function Φ^{*j} decays rapidly. For $j = 1$ the induction hypothesis is precisely (1.10). Assume that the hypothesis is valid for some $j < m$. Then by (1.10) for f replaced by f_j

$$f_{j+1} = f_j - \phi_R * f_j = -\frac{1}{R}\Phi_R * f_j' = \left(-\frac{1}{R}\right)^{j+1} \Phi_R^{*(j+1)} * f^{(j+1)}.$$

From here we can finish the proof as in the case $m = 1$. □

Since the L^1-norm of the Poisson kernel is 1 (for any $y > 0$) we see from Young's inequality that $\|P_y * g\|_{L^p} \leq \|g\|_{L^p}$ for any $g \in L^p(\mathbb{R})$, $y > 0$. If $p = \infty$ and if we set $R = R(t) = w_{M_{\log K}}(c_1 t)$ we deduce from Lemma 1.9 that, for every $t \geq 0$

$$R(t)^m \left\|(\delta - \phi_{R(t)})^{*m} * f(t)\right\| \leq C_{c_1} < \infty. \tag{1.11}$$

If we compare this with (1.7) we see that this already yields the desired estimate on J_1 in the case $p = \infty$. If $p < \infty$ we need a slightly more involved argument based on a property of Carleson measures.

Let $P*g(t,y) := P_y * g(t)$ and let μ be a Borel measure on the upper half-plane $H = \{(t,y) \in \mathbb{R}^2; y > 0\}$. Now we ask for which measures μ an inequality

$$\|P * g\|_{L^p(H,d\mu)} \le C_p \|g\|_{L^p(\mathbb{R})} \tag{1.12}$$

holds for all $g \in L^p(\mathbb{R})$ with a constant C_p not depending on g? Note that the inequality $\|P_y * g\|_{L^p} \le \|g\|_{L^p}$ is a special case of (1.12) for $C_p = 1$ with μ being the one-dimensional Hausdorff measure of the line $\{(t,y) \in H; t \in \mathbb{R}\} \subset H$. Actually for $1 < p < \infty$ one can characterize the class of all measures μ for which (1.12) holds for all g. These measures are called *Carleson measures* (see [24, Theorem I.5.6.]). Let $\gamma : \mathbb{R} \to (0,\infty)$ be a bounded continuous function with bounded variation. Then the one-dimensional Hausdorff measure restricted to

$$\Gamma = \{(t, \gamma(t)); t \in \mathbb{R}\} \subset H$$

is a Carleson measure. Now let $\gamma(t) = 1/R(t) = 1/M_{\log K}^{-1}(c_1 t)$ for $t > 0$ and $\gamma(t) = M_{\log K}^{-1}(0)$ for $t < 0$. If we set $\mu_{M_{\log K}}$ to be the Carleson measure corresponding to this particular choice of γ then we deduce that for $1 < p < \infty$

$$\left\| P * \left\| f^{(m)} \right\| \right\|_{L^p(H, d\mu_{M_{\log K}})} \le C_p \left\| f^{(m)} \right\|_{L^p(\mathbb{R}_+;X)} < \infty. \tag{1.13}$$

From this together with Lemma 1.9 we deduce

LEMMA 1.11. *Let $c_1 >$ and define $R(t) = M_{\log K}^{-1}(c_1 t)$. Then (i) for $p = \infty$*

$$\sup_{0<t<\infty} R(t)^m \left\| (\delta - \phi_{R(t)})^{*m} * f(t) \right\| \le C \left\| f^{(m)} \right\|_{L^\infty(\mathbb{R}_+;X)},$$

(ii) and for $1 < p < \infty$

$$\int_0^\infty \left\| R(t)^m (\delta - \phi_{R(t)})^{*m} * f(t) \right\|^p dt \le C \left\| f^{(m)} \right\|^p_{L^p(\mathbb{R}_+;X)}.$$

In both cases C does not depend on f.

1.2.1.2. *Estimation of J_2.* The following lemma is only necessary if $p \ne \infty$.

LEMMA 1.12. *There exists a $\delta > 0$ such that $K(M_{\log K}^{-1}(t)) \ge t^\delta$ for all $t \ge M_{\log K}(1)$.*

PROOF. Let $R = M_{\log K}^{-1}(t)$. Since $M_{\log K}^{-1}$ is the right-inverse of $M_{\log K}$ we have

$$t = M_{\log K}(R) = M(R)\log(K(R)) \ge M(R)\log(M(R)).$$

The inverse of the function $x \mapsto x\log(x)$ is asymptotically equivalent to $y \mapsto y/\log(y)$ for large $y > 0$. Hence there exists a $\delta > 0$ such that $M(R) \le \delta^{-1} t/\log(t)$. Thus

$$K(R) = \exp(\log(K(R))) = \exp\left(\frac{t}{M(R)}\right) \ge \exp(\delta \log(t)) = t^\delta.$$

□

At this point in the proof we fix a ψ having one additional property. We assume that the derivatives of ψ satisfy for some $C_1 > 0$

$$\forall j \in \mathbb{N}_0 : \sup_{s \in [-1,1]} \left| \psi^{(j)}(s) \right| \le C_1^{j+1} A_j \text{ with } A_j = (j \log(2+j)^{1+\varepsilon})^j. \tag{1.14}$$

Note that (1.14) cannot be satisfied by any ψ if we would replace A_j by $j!$ since then ψ would be analytic and hence cannot have compact support and $\psi(0) = 1$ at the same time. The Denjoy-Carleman[1] theorem (see e.g. [28, Theorem 1.3.8] or [20]) gives a description of those sequences (A_j) which allow for compactly supported non-zero functions ψ satisfying the inequality in (1.14). In particular, the Denjoy-Carleman theorem implies that our choice of A_j is admissible for the existence of such a ψ. Conversely it implies that there is no $\psi \in C_c^\infty(\mathbb{R})\backslash\{0\}$ which satisfies (1.14) with $\varepsilon = 0$.

Now we proceed with the estimation of $J_2(t,R)$. Therefore we have to estimate $J_{2,j}(t,R) = \phi_R^{*j} * f(t)$ for $j \in \{1,\dots,m\}$. First let us consider $J_{2,1}$. Let $N \in \mathbb{N}_0$. Integration by parts N-times yields

$$J_{2,1}(t,R) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ist}F(s)\psi_R(s)ds = \frac{1}{2\pi}\left(\frac{i}{t}\right)^N \int_{-R}^{R} e^{ist}\left[\sum_{j=0}^{N}\binom{N}{j}F^{(N-j)}R^{-j}(\psi^{(j)})_R\right](s)ds. \tag{1.15}$$

To verify the following calculations recall (1.6) and (1.14). We estimate the integral very roughly from above, by the length of the interval of integration times the supremum of the integrand within this interval. We also use Stirling's formula implying for example that $(cj)^j \le j! \le (Cj)^j$ for appropriate constants $c, C > 0$.

$$\begin{aligned}\|R^m J_{2,1}(t,R)\| &\le Ct^{-N}R^{m+1}\sum_{j=0}^{N}\binom{N}{j}(N-j)!K(R)M(R)^{N-j}\left(\frac{\|\psi^{(j)}\|_\infty^{\frac{1}{j}}}{R}\right)^j\\ &\le C\cdot R^{m+1}K(R)\left(\frac{C_2M(R)N}{et}\right)^N\cdot\sum_{j=0}^{N}\left(\frac{C_3\log(2+N)^{1+\varepsilon}}{RM(R)}\right)^j \qquad (1.16)\\ &=: C\cdot A\cdot B.\end{aligned}$$

The second inequality is valid for sufficiently large $C_2, C_3 > 0$. Now let us set $N = \lfloor t/(C_2M(R))\rfloor$ and $R = M_{\log K}^{-1}(c_1t)$. The constant $c_1 > 0$ will be chosen later. Then the condition (ii) on K implies

$$B \le \sum_{j=0}^{N}\left(\frac{C_3\log((c_1C_2)^{-1}\log(K(R)))^{1+\varepsilon}}{RM(R)}\right)^j \le \sum_{j=0}^{N}\left(\frac{C_4(RM(R))^{1-\varepsilon^2}}{RM(R)}\right)^j \le C.$$

The constant in the last inequality does not depend on t. Moreover,

$$A \le CR^{m+1}K(R)e^{-N} \le CR^{m+1}K(R)e^{-\frac{\log(K(R))}{c_1C_2}} = CR^{m+1}K(R)^{1-\frac{1}{c_1C_2}}.$$

If we choose c_1 sufficiently small, Lemma 1.12 implies that

$$\left\|M_{\log K}^{-1}(c_1t)^m J_{2,1}(t, M_{\log K}^{-1}(c_1t))\right\| \le \begin{cases} C & \text{if } p=\infty,\\ \frac{C}{(1+t)^{2/p}} & \text{if } 1\le p<\infty.\end{cases} \tag{1.17}$$

Clearly (1.15) remains valid if one replaces $J_{2,1}$ by $J_{2,k}$ and ψ by its k-th power ψ^k. It is not difficult to check that ψ^k also satisfies (1.14) if one replaces C_1^{j+1} by

[1] A special version of the Denjoy-Carleman theorem (sufficient for our considerations) reads as follows. Let S be the set of C^∞-functions on $\mathbb{R}$ supported on $[-1,1]$ such that (1.14) holds for a sequence (A_j) such that $(\sqrt[j]{A_j})$ is non-decreasing. Then S contains a non-zero function if and only if $\sum_j 1/\sqrt[j]{A_j} < \infty$.

$C_1^k(kC_1)^j$. Therefore (1.17) remains true after replacing $J_{2,1}$ by J_2. This together with Lemma 1.11 proves Theorem 1.7.

REMARK 1.13. Our proof breaks down if we allow ε to be zero in condition (ii) in Theorem 1.3 (and 1.7). This is essentially due to the fact that by the Denjoy-Carleman theorem a function ψ satisfying (1.14) for $\varepsilon = 0$ is necessarily *quasi-analytic*. This means that $\psi^{(j)}(s_0) = 0$ for a single $s_0 \in \mathbb{R}$ but all $j \in \mathbb{N}$ automatically implies $\psi = 0$. However, one can weaken (ii) slightly by choosing for some given $\varepsilon \in (0,1)$ and $n \in \mathbb{N}_1$

$$A_j = j \cdot L_1(j) \cdot L_2(j) \cdot \ldots \cdot L_n(j) \cdot L_{n+1}(j)^{1+\varepsilon} \text{ with}$$
$$L_k(j) = \underbrace{[\log \circ \ldots \circ \log]}_{k \text{ times}}(1 + k + j).$$

This allows us to replace (ii) by the condition

$$K(s) = O\left(\exp\left(\exp\left(\frac{sM(s)}{L_1(sM(s)) \cdot \ldots \cdot L_{n-1}(sM(s)) \cdot L_n(sM(s))^{1+\varepsilon}}\right)\right)\right).$$

Again choosing $\varepsilon = 0$ is forbidden for any n.

1.2.2. Proof of Theorem 1.3. Lemma 1.14 below implies that Theorem 1.3 and Theorem 1.7 are equivalent. To prepare the formulation of this lemma we introduce some notation. Let $M_1,\ M_2,\ K_1,\ K_2 : \mathbb{R}_+ \to (0,\infty)$ be continuous and non-decreasing functions. For $f : \mathbb{R}_+ \to X$ measurable and polynomially bounded and extended by zero on the negative real numbers we consider two distinct conditions. The first one is

$$\forall z \in \Omega_{M_1} : \left\|\hat{f}(z)\right\| \le K_1(|\Im z|). \tag{1.18}$$

This condition implicitly states that the Laplace transform of f can be extended to Ω_{M_1}. Let F be the Fourier transform of f. The second condition is

$$\forall j \in \mathbb{N}_0, s \in \mathbb{R} : \left\|F^{(j)}(s)\right\| \le j! K_2(|s|) M_2(|s|)^j. \tag{1.19}$$

This condition implicitly states that the Fourier transform is a C^∞-function.

The following lemma relates these conditions to each other under a mild condition on f.

LEMMA 1.14. *Let $f : \mathbb{R}_+ \to X$ be a measurable and polynomially bounded function with $f^{(m)} \in L^p(\mathbb{R}_+; X)$ for some $1 \le p \le \infty$ and $m \in \mathbb{N}_1$. We extend f by zero on the negative real numbers and denote by F its Fourier transform. (a) If F satisfies (1.19) then f satisfies (1.18) with*

$$M_1(s) = (1-\varepsilon)^{-1} M_2(s) \text{ and } K_1(s) = \varepsilon^{-1} K_2(s)$$

for any $\varepsilon \in (0,1)$. (b) If f satisfies (1.18) then F satisfies (1.19) with

$$M_2(s) = M_1\left(s + M_1(s)^{-1}\right) \text{ and}$$

$$K_2(s) = K_1\left(s + M_1(s)^{-1}\right) + C_f \frac{M_1\left(s + M_1(s)^{-1}\right)^{2-\frac{1}{p}}}{(1+s)^m} + C'_f.$$

The constant C_f depends only on $\left\|f^{(m)}\right\|_{L^p}$, the constant C'_f depends only on $\|f(0)\|, \ldots, \left\|f^{(m-1)}(0)\right\|$.

Before proving this lemma we finish the proof of Theorem 1.3. Since f satisfies (1.18) for $M_1 = M$ and $K_1 = K$, Lemma 1.14 implies that (1.19) is true for M_2 and K_2 given as in part (b) of the lemma. In the following we assume $s > 0$ large enough to satisfy $1/M_1(s) \le s$. Note that condition (i) in Theorem 1.3 implies the existence of a (small) constant $c > 0$ such that (for large s)

$$cM_2(s)\log(K_2(s)) \le M(2s)\log(K(2s)).$$

This immediately yields for large t

$$M^{-1}_{\log K}(ct) \le 2(M_2)^{-1}_{K_2}(t).$$

Therefore $(M_2)^{-1}_{K_2}(c_1\cdot)^m f \in L^p$ for some $c_1 > 0$ implies that $M^{-1}_{\log K}(cc_1\cdot)^m f \in L^p$. The proof of Theorem 1.3 is complete.

Proof of Lemma 1.14. Let us begin with the easier part (a). Hadamard's formula shows that (1.19) implies that $\hat{f}$ is analytic in $\Omega_{M_2} \supset \Omega_{M_1}$. Let $z \in \Omega_{M_1}$ and let $s = \Im z$. Then

$$\left\|\hat{f}(z)\right\| = \left\|\sum_{j=0}^{\infty} \frac{1}{j!}\hat{f}^{(j)}(is)(z-is)^j\right\| \le \sum_{j=0}^{\infty} K_2(s)M_2(s)^j\left(\frac{1-\varepsilon}{M_2(s)}\right)^j = \varepsilon^{-1}K_2(s).$$

Let us now prove part (b). Let us fix $s \in \mathbb{R}$, let $r = 1/M_1(|s| + 1/M(|s|))$ and let γ be the positively oriented circle of radius r around is in the complex plane. Note that γ is included in the closure of the union of Ω_{M_1} and $\mathbb{C}_+$. Let γ_+ and γ_- be the intersection of γ with $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. By Cauchy's formula we have

$$\begin{aligned}\hat{f}^{(j)}(is) &= \frac{j!}{2\pi i}\left[\int_{\gamma_-} + \int_{\gamma_+}\right] \frac{\hat{f}(z)}{(z-is)^{j+1}}\left(1 + \frac{(z-is)^2}{r^2}\right)dz \\ &=: j!\left[I_- + I_+\right].\end{aligned}$$

Let us first estimate I_-:

$$\begin{aligned}\|I_-\| &\le \frac{1}{2\pi}\cdot r^{-j-1}\sup_{z\in\gamma_-}\left\|\hat{f}(z)\right\|\cdot\pi r\cdot 2 \\ &\le K_1\left(|s| + M_1(|s|)^{-1}\right)M_1\left(|s| + M_1(|s|)^{-1}\right)^j. \qquad (1.20)\end{aligned}$$

Let us now estimate I_+:

$$\begin{aligned}I_+ &= \frac{1}{2\pi i}\int_{\gamma_+}\frac{\left(1+\frac{(z-is)^2}{r^2}\right)}{(z-is)^{j+1}}\left(\sum_{k=0}^{m-1} z^{-j-1}f^{(k)}(0) + z^{-m}\int_0^\infty e^{-zt}f^{(m)}(t)dt\right)dz \\ &=: \sum_{k=0}^{m-1} I_{+,k} + I_{+,m}\end{aligned}$$

It is an easy exercise to show that the integral of $e^{-rt\cos(\theta)}\cos(\theta)$ over $\theta \in (-\pi/2, \pi/2)$ can be estimated from above by a constant times $((rt)^2+1)^{-1}$. Therefore by Hölder's inequality we get for large $|s|$

$$\begin{aligned}\|I_{+,m}\| &\le \frac{C}{|s|^m r^{j+1}}\int_0^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-rt\cos(\theta)}\cos(\theta)d\theta\left\|f^{(m)}(t)\right\|dt \\ &\le \frac{C}{|s|^m r^{j+2-1/p}}\left\|f^{(m)}\right\|_{L^p}\end{aligned}$$

$$\leq \frac{C \left\| f^{(m)} \right\|_{L^p}}{|s|^m} M_1(|s| + M_1(|s|)^{-1})^{j+2-1/p}.$$

A similar (and easier) estimate is true for the other summands $I_{+,k}$. This together with (1.20) yields the claim. □

1.2.3. Optimality of Theorem 1.3. In this section we show that under the assumptions of Theorem 1.3 and for $p = \infty, m = 1$ one can - up to improvement of the constant c_1 - not get a faster decay rate than the one already given by the theorem. To show this we use almost the same method as in [12]. There the authors showed the optimality in the very particular case that $M(s) = C(1 + s^\alpha)$ and $K(s) = C(1 + s^\beta)$ for $\beta > \alpha/2 > 0$. Theorem 1.15 below contains as a special case [12, Theorem 3.8]. To compare our result with Borichev's and Tomilov's result take also Remark 3.10 from their paper into account.

THEOREM 1.15. *Let $c_1 > 0$ and let $M, K : \mathbb{R}_+ \to [2, \infty)$ be continuous and non-decreasing functions satisfying for some non-decreasing function $N : \mathbb{R}_+ \to [1, \infty)$*

(i) $\lim_{s\to\infty} \frac{M_{\log K}(s)}{\log(2+s)} = \infty$ *and* $\exists \varepsilon > 0, s_0 > 0 \forall s \geq s_0 : K(s) \geq s^\varepsilon$,
(ii) $\exists s_0 > 0 \forall s \geq s_0, s' \geq 0 : M(s + s') \leq N(s')M(s)$.

Then there exists a real number $\gamma \geq 0$, not depending on c_1 and a locally integrable function $f : \mathbb{R}_+ \to \mathbb{C}$ with $f' \in L^\infty(\mathbb{R}_+)$ such that

$$\left|\hat{f}(z)\right| \leq \frac{C}{R} M(|\Im z|)^{\frac{1}{2}} K(|\Im z|)^{\frac{\gamma}{c_1}} \text{ for all } z \in \Omega_M \tag{1.21}$$

and

$$\limsup_{t\to\infty} M_{\log K}^{-1}(c_1 t) |f(t)| \geq c > 0. \tag{1.22}$$

If instead of (ii) we have the stronger assumption that there exists a $\gamma_0 \geq 1$ such that

(ii') $\forall s_1 > 0 \exists s_0 > 0 \forall s \geq s_0, s' \leq s_1 : M(s + s') \leq \gamma_0 M(s)$

and if $\gamma > \gamma_0$ then it is possible to choose f in such a way that (1.21) holds for this choice of γ. If in addition M is unbounded then it is possible to choose f in such a way that (1.21) holds for all $\gamma > \gamma_0$.

REMARK 1.16. Let $\tilde{K} : \mathbb{R}_+ \to (0, \infty)$ be given by $\tilde{K}(s) = M(|\Im z|)^{1/2} K(|\Im z|)^{\gamma/c_1}$, $s \geq 0$. Assume for simplicity that $K(s) \geq \max\{2, s, M(s)\}$ for $s \geq 0$. Then $M_{\log K}$ and $M_{\log \tilde{K}}$ are asymptotically equivalent. After possibly redefining $\tilde{K}$ on a compact interval we can apply Theorem 1.3 to deduce that $M_{\log K}^{-1}(c_2 t) |f(t)| \leq C, t \geq 1$ for certain constants $c_2, C > 0$. This is consistent with (1.22).

REMARK 1.17. Note that condition (i) is only a very mild restriction. In fact, a typical situation where (i) is violated is that M is a constant and K grows at most polynomially. But then Theorem 1.3 implies exponential decay for f. This in turn implies, that the integral which defines $\hat{f}$ is absolutely convergent in a small strip to the left of the imaginary axis. In particular $\hat{f}$ extends analytically to this strip and is bounded there. So our results are trivially optimal in that case.

Before we prove the theorem we need a similar lemma as in [12]. Given a compactly supported measure μ on $\mathbb{C}\backslash\overline{\Omega_M \cup \mathbb{C}_+}$ we use the following notation for

$z \in \Omega_M \cup \mathbb{C}_+$ and $t \geq 0$

$$\mathcal{C}\mu(z) = \int \frac{1}{z-\zeta} d\mu(\zeta),\ \mathcal{L}\mu(t) = \int e^{t\zeta} d\mu(\zeta),\ \mathcal{L}'\mu(t) = \int \zeta e^{t\zeta} d\mu(\zeta).$$

To simplify the notation we extend M and K symmetrically to the negative real axis.

LEMMA 1.18. *Let c_1, M and K be as in Theorem 1.15. There exists a $\delta > 0$ and a $\gamma > 0$, only depending on M and δ, such that for all $\varepsilon > 0$ and $k_0 \in \mathbb{N}_0$ there exists $k \in \mathbb{N}_{k_0}$ and a compactly supported Borel measure μ on $\mathbb{C}\backslash\overline{\Omega_M \cup \mathbb{C}_+}$ such that for all $z \in \Omega_M$ and $t \geq 0$*

$$|\mathcal{C}\mu(z)| \leq \frac{C}{R} M^{\frac{1}{2}} K^{\gamma} 1_{[R-2\delta, R+2\delta]}(\Im z) + \varepsilon, \tag{1.23}$$

$$|\mathcal{L}'\mu(t)| \leq C 1_{[\frac{k}{2\delta}, \frac{2k}{\delta}]}(t) + \varepsilon, \tag{1.24}$$

$$|\mathcal{L}\mu(t)| \leq \frac{C}{R} 1_{[\frac{k}{2\delta}, \frac{2k}{\delta}]}(t) + \frac{\varepsilon}{\max\{R, M^{-1}_{\log K}(c_1 t)\}}, \tag{1.25}$$

$$\left|\mathcal{L}\mu(\frac{k}{\delta})\right| \geq \frac{c}{R}. \tag{1.26}$$

Here $R = M^{-1}_{\log K}(c_1 k/\delta)$. If instead of (ii) we have the stronger assumption that there exists a $\gamma_0 \geq 1$ such that

(ii') $\forall s_1 > 0 \exists s_0 > 0 \forall s \geq s_0, s' \leq s_1 : M(s+s') \leq \gamma_0 M(s)$

and if $\gamma > \gamma_0$ then it is possible to choose f in such a way that (1.23) holds for this choice of γ. If in addition M is unbounded then it is possible to choose f in such a way that (1.23) holds for all $\gamma > \gamma_0$.

REMARK 1.19. For $\Im z = R$ the inequality (1.23) holds also in the reverse direction (for a different value of C). This will be indicated in the proof.

PROOF. Let $\delta > 1/M(0)$ be a real number to be fixed later. Let $k \in \mathbb{N}_{k_0}$ to be fixed later. Let us define

$$w = iR - \delta,\ q = e^{2\pi i/(k+1)},\ \delta A = kl(k)$$

where $l : \mathbb{R}_+ \to (0, \infty)$ is a strictly increasing function such that $l(t) \geq \beta \log(e+t)$ for some $\beta \geq 1$ to be fixed later. By δ_{z_0} we denote the Dirac-measure at $z_0 \in \mathbb{C}$. Let us define

$$\mu = \frac{\tau}{R} \sum_{j=0}^{k} q^j \delta_{w+A^{-1}q^j}.$$

The constant $\tau > 0$ will be chosen later. Before we go on we state a simple lemma which will be frequently applied in the following.

LEMMA 1.20. *Let $n > 0$ be a real number. The function $s \mapsto s^n e^{-s}$ has a unique maximum on $\mathbb{R}_+$. Before this maximum the function is strictly increasing and after that maximum it is strictly decreasing.*

One can prove the lemma by simply taking the derivative of the function.

Part 1: estimation of $\mathcal{L}\mu$. We distinguish the two cases $t \leq A$ and $t > A$.
Case 1: $t \leq A$. We calculate

$$\begin{aligned}\mathcal{L}\mu(t) &= \frac{\tau}{R}\sum_{j=0}^{k} q^j e^{t(w+A^{-1}q^j)} \\ &= \frac{\tau}{R}e^{tw}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{t}{A}\right)^m \sum_{j=0}^{k} q^{(m+1)j} \\ &= \frac{\tau}{R}\cdot e^{tw}\frac{(k+1)t^k}{A^k k!}\cdot\sum_{n=1}^{\infty}\frac{k!}{(n(k+1)-1)!}\left(\frac{t}{A}\right)^{(n-1)(k+1)} \\ &=: \frac{\tau}{R}\cdot I\cdot II.\end{aligned}$$

Clearly II is bounded from below by 1 and bounded from above by a constant which does not depend on k or A. Thus by Stirling's formula we get

$$|\mathcal{L}\mu(t)| \geq c\frac{\tau}{R}\sqrt{k}e^{-\delta t}\left(\frac{e\delta t}{\delta A k}\right)^k.$$

As a function in t we can maximize the right-hand side by setting $\delta t = k$. If we furthermore define

$$\tau = \frac{1}{\sqrt{k}}(\delta A)^k \tag{1.27}$$

we see that (1.26) is proved. Since II is bounded from above we have

$$|\mathcal{L}\mu(t)| \leq C\frac{\tau}{R}\sqrt{k}e^{-\delta t}\left(\frac{e\delta t}{\delta A k}\right)^k. \tag{1.28}$$

Again we maximize the right-hand side by setting $\delta t = k$ and plugging in (1.27). This leads to

$$|\mathcal{L}\mu(t)| \leq C\frac{\tau}{R}\sqrt{k}e^{-k}\left(\frac{e}{\delta A}\right)^k \leq \frac{C}{R}$$

For $t \in [k/2\delta, 2k/\delta]$ this is already what we want to have in (1.25).

Case 1.1: $\delta t \leq k/2$. In this case the maximum in (1.28) with respect to t is attained for $\delta t = k/2$. This yields

$$|\mathcal{L}\mu(t)| \leq C\frac{\tau}{R}\sqrt{k}e^{-\frac{k}{2}}\left(\frac{e}{2\delta A}\right)^k = \frac{C}{R}\left(\frac{e}{4}\right)^{\frac{k}{2}} \leq \frac{\varepsilon}{R}$$

The last inequality holds for sufficiently large k. We proved (1.25) for $\delta t \leq k/2$.

Case 1.2: $2k \leq \delta t \leq \delta A$. Condition (i) from Theorem 1.15 yields $M^{-1}_{\log K}(c_1 t) \leq e^{\delta t/\alpha}$ for any $\alpha > 0$ as long as t is large enough. Thus, if we multiply (1.28) by $M^{-1}_{\log K}(c_1 t)$ we get

$$\begin{aligned}M^{-1}_{\log K}(c_1 t)\,|\mathcal{L}\mu(t)| &\leq C\frac{\tau}{R}\sqrt{k}e^{-(1-\frac{1}{\alpha})\delta t}\left(\frac{e\delta t}{\delta A k}\right)^k \\ &\leq \frac{C}{R}\left(\frac{2}{e^{1-\frac{2}{\alpha}}}\right)^k \leq \varepsilon\end{aligned}$$

for sufficiently large k. From the first to the second line we used that the maximum of the right-hand side of the first line is attained at $\delta t = 2k$ if $\alpha \geq 2$. In the last estimate we used $e^{1-\frac{2}{\alpha}} > 2$ which is true if α is large enough. We proved (1.25) for $2k \le \delta t \le \delta A$.

Case 2: $t > A$. Then we have

$$\begin{aligned}|\mathcal{L}\mu(t)| &\le \frac{\tau}{R}(k+1)e^{-(\delta-A^{-1})t} \\ &\le \frac{C}{R}\sqrt{k}(\delta A)^k e^{-\delta A} e^{-(\delta-A^{-1})(t-A)}\end{aligned}$$

In the following we assume that $\delta - A^{-1} > 0$ which is true for large k.

Case 2.1: $A < t < 2A$. In this case (using again $M^{-1}_{\log K}(c_1 t) \le e^{\delta t/\alpha}$ for large t) we get

$$\begin{aligned}M^{-1}_{\log K}(2c_1 A)\,|\mathcal{L}\mu(t)| &\le \frac{C}{R}\sqrt{k}\left(kl(k)e^{-l(k)}\right)^k e^{\frac{2kl(k)}{\alpha}} \\ &= \frac{C}{R}\sqrt{k}\left(kl(k)e^{-(1-\frac{2}{\alpha})l(k)}\right)^k \le \varepsilon\end{aligned}$$

if we choose $\beta > 1$ and let α satisfy $(1-\frac{2}{\alpha})^{-1} < \beta$ and if k is large enough. We proved (1.25) for $A < t < 2A$.

Case 2.2: $t \ge 2A$. If we use $\sqrt{k}(\delta A)^k e^{-\delta A} \le 1$ for large k we can calculate for an $\alpha > 4$

$$\begin{aligned}M^{-1}_{\log K}(c_1 t)\,|\mathcal{L}\mu(t)| &\le \frac{C}{R}e^{-(1-\frac{1}{kl(k)})(\delta t - \delta A)} e^{\frac{\delta t}{\alpha}} \\ &\le \frac{C}{R}e^{(\frac{1}{\alpha}-\frac{1}{4})\delta t} \le \varepsilon.\end{aligned}$$

This finishes the proof of (1.25).

Part 2: estimation of $\mathcal{C}\mu$. First observe that as long as z is no $(k+1)$-th root of unity we have

$$\sum_{j=0}^{k} \frac{q^j}{z-q^j} = \frac{k+1}{z^{k+1}-1}.$$

Clearly this equation must hold for some k-th order polynomial p if one replaces the term $k+1$ on the right-hand side by $p(z)$. Moreover the left-hand side is invariant under the substitution which replaces z by qz. Thus $p(z) = p(qz)$. But this implies that p is a constant. By plugging in $z = 0$ we see that $p = k+1$.

The observation yields for $z \in \Omega_M$

$$\mathcal{C}\mu(z) = \frac{\tau}{R}\frac{(k+1)A}{(A(z-w))^{k+1}-1}. \tag{1.29}$$

Now it is not difficult to prove (1.23) for $|\Im z - R| > 2\delta$. The latter condition implies $|z-w| > 2\delta$. Thus, using (1.29) we get for $|\Im z - R| > 2\delta$ and k large:

$$|\mathcal{C}\mu(z)| \le C\frac{\tau}{R}kA(2\delta A)^{-k-1} \le \frac{C\sqrt{k}}{\delta R}2^{-k} \le \varepsilon.$$

If we do not have $|\Im z - R| > 2\delta$ we can merely estimate $|z-w| \ge \delta - 1/M(\Im z)$. This yields for $z \in \Omega_M$ with $|\Im z - R| \le 2\delta$ and for all $\gamma_1 > 1$

$$\begin{aligned}|\mathcal{C}\mu(z)| &\le C\frac{\tau}{R}kA\Big(\delta A\big(1-\frac{1}{\delta M(\Im z)}\big)\Big)^{-k-1} \\ &\le \frac{C\sqrt{k}}{\delta R}e^{\gamma_1 \frac{k}{\delta M(\Im z)}}\end{aligned}$$

$$\leq \frac{C\sqrt{k}}{\delta R} e^{\gamma_1 N(2\delta) \frac{k}{\delta M(R)}}$$
$$\leq \frac{C}{\delta R} \sqrt{M_{\log K}(R)} K(R)^{\frac{\gamma_1 N(2\delta)}{c_1}}.$$

From the first to the second line we use the inequality $1 - x \geq e^{-\gamma_1 x}$ which is valid for small $x \geq 0$. If M is bounded we choose δ large enough to make use of this inequality. From the second to the third line we used condition (ii) from Theorem 1.15. Choosing $\gamma = \gamma_1 N(2\delta)$ we get (1.23). Concerning Remark 1.19 a reverse inequality for $\Im z = R$ can be proved analogously but in an even simpler way by using the inequality $1 - x \leq e^{-x}$ which is valid for all $x \geq 0$.

Part 3: estimation of $\mathcal{L}'\mu$. Finally we want to estimate the derivative of $\mathcal{L}\mu$.

Case 1: $t \geq A$. In this case we directly get for large k

$$|\mathcal{L}'\mu(t)| \leq \frac{\tau}{R}(k+1)(R + A^{-1})e^{-(\delta - A^{-1})t}$$
$$\leq C\frac{\sqrt{k}}{R}(\delta A)^k R e^{-\delta A} \leq \varepsilon.$$

Case 2: $t < A$. Let us first get a different representation of $\mathcal{L}\mu$:

$$\mathcal{L}'\mu(t) = \frac{\tau}{R}\sum_{j=0}^{k} q^j (w + A^{-1}q^j) e^{(w + A^{-1}q^j)t}$$
$$= \frac{\tau}{R} e^{tw} \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{t}{A}\right)^m \sum_{j=0}^{k} (wq^{(m+1)j} + A^{-1}q^{(m+2)j})$$
$$= \frac{w}{R}\tau e^{tw} \frac{(k+1)t^k}{A^k k!} \sum_{n=1}^{\infty} \frac{k!}{(n(k+1)-1)!}\left(\frac{t}{A}\right)^{(n-1)(k+1)} \left[1 + \frac{n(k+1)-1}{wt}\right].$$

Note that if $t > t_0 > 0$, the series at the end of the calculation is bounded by a constant which only depends on t_0.

$$|\mathcal{L}'\mu(t)| \leq C\tau\sqrt{k}e^{-\delta t}\left(\frac{e\delta t}{\delta A k}\right)^k \left[1 + \frac{k}{Rt}\right]$$
$$\leq Ce^{-\delta t}\left(\frac{e\delta t}{k}\right)^k \left[1 + \frac{k}{Rt}\right] \tag{1.30}$$

Note that (1.30) as a function in t is increasing for $\delta t < k - 1$ and decreasing for $\delta t > k$. Therefore we see that $|\mathcal{L}'\mu(t)|$ bounded by a constant not depending on t. This shows (1.24) for $k/2\delta \leq t \leq 2k/\delta$.

Case 2.1: $\delta t \leq k/2$. The maximum in (1.30) is then attained for $\delta t = k/2$. This yields

$$|\mathcal{L}'\mu(t)| \leq Ce^{-\frac{k}{2}}\left(\frac{e}{2}\right)^k \leq C\left(\frac{e}{4}\right)^{\frac{k}{2}} \leq \varepsilon$$

if k is large enough.

Case 2.2: $2k \leq \delta t \leq A$. The maximum in (1.30) is then attained for $\delta t = 2k$. This yields

$$|\mathcal{L}'\mu(t)| \leq Ce^{-2k}(2e)^k \leq C\left(\frac{2}{e}\right)^{\frac{k}{2}} \leq \varepsilon$$

if k is large enough. This finishes the proof of Lemma 1.18. □

Proof of Theorem 1.15. For an $\varepsilon_0 > 0$ to be chosen later we define a sequence (ε_n) by $\varepsilon_n = 2^{-n}\varepsilon_0$. There exists a $\delta > 0$, an increasing sequence of natural numbers (k_n) and a sequence of measures (μ_n) according to Lemma 1.18. We may assume that $([R_n - 2\delta, R_n + 2\delta])$ and $([k_n/2\delta, 2k_n/\delta])$ are sequences of pairwise disjoint intervals. Let us define

$$f(t) = \sum_{n=1}^{\infty} \mathcal{L}\mu_n(t) \text{ for } t \geq 0.$$

The series is uniformly convergent because of (1.25). The function f is therefore continuous and since the sequence of derivatives converges uniformly (by (1.24)) we see that f has a bounded weak derivative given by

$$f'(t) = \sum_{n=1}^{\infty} \mathcal{L}'\mu_n(t) \text{ for } t \geq 0.$$

By a similar argument the Laplace transform has the form

$$\hat{f}(z) = \sum_{n=1}^{\infty} \mathcal{C}\mu_n(z) \text{ for } z \in \Omega_M.$$

Here the sum converges uniformly on compact subsets of $\overline{\Omega_M \cup \mathbb{C}_+}$ (by (1.23)). We already know that the derivative of f is bounded. The estimate (1.21) follows immediately from (1.23). It remains to prove (1.22). Let us set $t_n = k_n/\delta$. Then we deduce from (1.25) and (1.26) that

$$\begin{aligned} |f(t_n)| &\geq \frac{c}{R_n} - \varepsilon_0 \sum_{j\neq n} \frac{2^{-j}}{\max\{R_j, M_{\log K}^{-1}(c_1 t_n)\}} \\ &\geq \frac{c}{R_n} - \varepsilon_0 \sum_{j\neq n} \frac{2^{-j}}{R_n} \\ &\geq \frac{c}{R_n} = \frac{c}{M_{\log K}^{-1}(c_1 t_n)}. \end{aligned}$$

In the last line we chose ε_0 small enough. □

Remark 1.21. By the same technique one can also prove the optimality of Theorem 1.3 for $m > 1$. To achieve this one just has to define the measure μ in Lemma 1.18 by $\mu = \tau R^{-m} \sum_{j=0}^{k} q^j \delta_{w+A^{-1}q^j}$.

Remark 1.22. With the help of Remark 1.19 one easily sees that for $\Im z = R_n$ the inequality (1.21) holds also in the reverse direction (for a different constant C).

1.2.4. On the optimality of the constant c_1 in Theorem 1.3. The literature does not seem to pay much attention to the constant c_1 appearing in Theorem 1.3. If we are interested in polynomial decay the constant does not influence the decay rate much. However, if for example $M_{\log K}^{-1}(t) = \exp(t^\alpha)$ for some $\alpha \in (0, 1]$ we immediately see that c_1 influences the decay rate in a crucial way. The aim of this subsection is to give a partial answer concerning the question of the optimality of c_1. Under not too restrictive conditions on M and K we show that Theorem 1.3 is valid for any $c_1 < 1$ and false for $c_1 > 1$. Unfortunately we have to exclude the important special case of exponential decay from our discussion.

THEOREM 1.23. *Let $p = \infty$. (a) In addition to the assumptions in Theorem 1.3 assume that K increases faster than any polynomial and assume that $K(s) \geq c(1+s)^{-m}M(2s)^2$. Then (1.5) holds for all $c_1 < 1$. (b) Let M, K satisfy the assumptions of Theorem 1.3. Assume in addition that for some $\gamma_0 \geq 1$*

$$\forall s_1 > 0 \exists s_0 > 0 \forall s \geq s_0, s' \leq s_1 : M(s+s') \leq \gamma_0 M(s). \tag{1.31}$$

Assume furthermore that K increases faster than any polynomial in $sM(s)$. Let $c_1 > \gamma_0$. Then there exists a locally integrable function $f : \mathbb{R}_+ \to \mathbb{C}$, satisfying the assumptions of Theorem 1.3 such that (1.5) does not hold for this choice of c_1.

REMARK 1.24. It is not difficult to find functions M which satisfy (1.31) for *any* $\gamma_0 > 1$. Take for example M to be a constant, a logarithm or a polynomial. It is also possible to take $M(s) = \exp(s^\alpha)$ for $\alpha \in (0,1)$. On the other hand the example $M(s) = \exp(s)$ does not satisfy this condition for any $\gamma > 1$.

REMARK 1.25. We think that the condition that K increases faster than a polynomial in s is natural in both parts of the theorem. On the other hand we do not know whether the growth condition on K in terms of $M(s)$ or $M(2s)$ is a necessary assumption for the conclusion of Theorem 1.23 to hold. Concerning (a) this condition is only necessary in the proof since we do not know whether Lemma 1.14 is valid for $C_f = 0$. Concerning (b) we need it because of the factor $M(|\Im z|)^{1/2}$ appearing in (1.21).

PROOF. (a) The claim is proved by having a look into the proof of Theorem 1.3. It is not difficult to see that (1.16) is true for any $C_2 > 1$. To get (1.17) one has to choose c_1 in such a way that $K(R)^{\frac{1}{c_1 C_2}-1} \geq cR^{m+1}$. Since K grows super-polynomially in s this means $c_1 < 1/C_2$. Now observe that in the final step of the proof in Section 1.2.2, before the proof of Lemma 1.14, one can choose any $c < 1$. Here we use that $K(s) \geq c(1+s)^{-m}M(2s)^2$. Since C_2 can be chosen arbitrary close to 1 the first assertion is proved.

(b) Let $\gamma_0 < \gamma < c_1$. First observe that the assumptions of Theorem 1.15 (including (ii')) are satisfied (concerning $m > 1$ see also Remark 1.21). Thus there exists a locally integrable function $f : \mathbb{R}_+ \to \mathbb{C}$ such that the conclusion of Theorem 1.15 is satisfied. Since K grows faster than any polynomial of $M(s)$ we can withdraw the factor $M(|\Im z|)^{1/2}$ from (1.21) if we replace γ/c_1 by 1 in this inequality. Now the function satisfies the assumptions of Theorem 1.3 but it fails to satisfy (1.5) for our choice of c_1 by Theorem 1.15. □

1.3. s^{-1}-singularity at zero

THEOREM 1.26. *Let $(X, \|\cdot\|)$ be a Banach space, $m \in \mathbb{N}$, and $f : \mathbb{R}_+ \to X$ be a locally integrable function such that its weak derivative f' and its primitive $t \mapsto f_1(t) = \int_0^t f ds$ are bounded. Assume that there exist continuous and non-decreasing functions $M, K : \mathbb{R}_+ \to (0,\infty)$ satisfying*

(i) $\forall s \geq 0 : K(s) \geq \max\{2, M(s)\}$,

(ii) $\exists \varepsilon \in (0,1) : K(s) = O\left(e^{e^{M(s)^{1-\varepsilon}}}\right)$ *as* $s \to \infty$.

such that the Laplace transform $\hat{f}$ of f extends analytically to $(\Omega_\blacktriangleright \cap \Omega_M) \cup \mathbb{C}_+$ and

$$\left|\hat{f}(z)\right| \leq K(|\Im z|) \text{ for all } z \in (\Omega_\blacktriangleright \cap \Omega_M). \tag{1.32}$$

Here $\Omega_{\blacktriangleright} = \{z \in \mathbb{C}; -c\,|\Im z| < \Re z < 0\}$ *for some* $c > 0$. *Then there exists a constant* $c_1 > 0$ *such that*

$$|f(t)| \le C \max\left\{\frac{1}{t}, \frac{1}{M_{\log K}^{-1}(c_1 t)}\right\} \tag{1.33}$$

for all $t \ge 1$, *where* $M_{\log K}(s) := M(s)\log(K(s))$ *if* M *is unbounded and else we interpret* $M_{\log K}^{-1}(c_1 t)$ *as* ∞.

Remark 1.27. Note that we assume that $\hat{f}$ is bounded in a neighbourhood of zero in $\Omega_{\blacktriangleright}$. The reason why we call this situation "s^{-1}-singularity at zero" will become clear in Section 1.5.3.

Remark 1.28. The contribution to the decay rate of the singularity at zero is the term t^{-1}. No logarithmic loss occurs! It seems to be unknown if one can generalize the above theorem to arbitrary (or at least certain) other types of singularities at zero. For example, let $\alpha > 1$ and define $\Omega_\alpha = \{z \in \mathbb{C}; -c\,|\Im z|^\alpha < \Re z < 0\}$ for some $c > 0$. Then we can pose the following question. If one replaces (1.32) by

$$\left\|\hat{f}(z)\right\| \le K(|\Im z|) \vee C\,|z|^{1-\alpha} \text{ for all } z \in \Omega_\alpha \cap \Omega_M,$$

and (1.33) by

$$\|f(t)\| \le C \max\left\{\frac{1}{t^{\frac{1}{\alpha}}}, \frac{1}{M_{\log K}^{-1}(c_1 t)}\right\},$$

is Theorem 1.26 then still true? We think the answer is "no". However, if one relaxes the conclusion of the theorem to

$$\|f(t)\| \le C \max\left\{\left(\frac{\log(2+t)}{t}\right)^{\frac{1}{\alpha}}, \frac{1}{M_{\log K}^{-1}(c_1 t)}\right\},$$

then one can combine the proof of Theorem 1.3 with techniques (for singularity at zero) from [18] to prove a generalization of the above theorem - with a logarithmic loss. We think that one can perform a similar construction as in Section 1.2.3 to show the optimality of this result. Since the above result suffices for our applications we did no further research in this direction.

1.3.1. Some auxiliary lemmas. First, we prove an analogue of Lemma 1.9. By P we again denote the Poisson kernel.

Lemma 1.29. *Let* $1 \le p \le \infty$ *and* $r > 0$. *Let* $f : \mathbb{R} \to X$ *be a locally integrable function such that its primitive* $t \mapsto f_1(t) = \int_0^t f ds$ *is in* $L^p(\mathbb{R}_+; X)$. *Let* ϕ *be a Schwartz function with integral equal to* 1. *Then there exists a constant* $C > 0$ *(only depending on* ϕ*) such that for all* $t \in \mathbb{R}$

$$\|\phi_r * f(t)\| \le C r P_{r^{-1}} * \|f_1\|(t).$$

Proof. Integrating by parts yields

$$\begin{aligned}\phi_r * f(t) &= \int_{-\infty}^{\infty} f(\tau)\phi(r(t-\tau)) r d\tau \\ &= r\int_{-\infty}^{\infty} f_1(\tau)\phi'(r(t-\tau)) r d\tau.\end{aligned}$$

If we use the inequality $|r\phi'(rt)| \le C P_{r^{-1}}(t)$ we derive the desired estimate. □

Next we prove an analogue of Lemma 1.14. Therefore let $F = \mathcal{F}f$ be the Fourier transform of f.

LEMMA 1.30. *Let $f : \mathbb{R}_+ \to X$ be a locally integrable function such that its primitive $t \mapsto f_1(t) = \int_0^t f ds$ is in $L^p(\mathbb{R}_+; X)$ for some $1 \le p \le \infty$. We extend f by zero on $\mathbb{R}_-$. Let M_1, M_2, K_1, $K_2 : \mathbb{R}_+ \to (0, \infty)$ be continuous and non-decreasing functions. Let $s_0 > 0$ and define*

$$\Omega_{M_1,0} = \left\{ z \in \mathbb{C}; -\frac{1}{M_1(|\Im z|^{-1})} < \Re z < 0 \right\}.$$

(a) If f satisfies

$$\forall j \in \mathbb{N}_0, |s| \le s_0 : \left| F^{(j)}(s) \right| \le j! K_2(|s|^{-1}) M_2(|s|^{-1})^j, \tag{1.34}$$

then it also satisfies

$$\left| \hat{f}(z) \right| \le K_1(|\Im z|^{-1}) \text{ for all } z \in \Omega_{M_1,0} \text{ with } |\Im z| \le s_0 \tag{1.35}$$

with

$$M_1(s) = (1-\varepsilon)^{-1} M_2(s) \text{ and } K_1(s) = \varepsilon^{-1} K_2(s)$$

for any $\varepsilon \in (0,1)$. (b) If f satisfies (1.35) then it also satisfies (1.34) with

$$M_2(s) = M_1\left(s + M_1(s)^{-1}\right) \text{ and}$$

$$K_2(s) = K_1\left(s + M_1(s)^{-1}\right) + C_f \frac{M_1\left(s + M_1(s)^{-1}\right)^{2-\frac{1}{p}}}{1+s}.$$

The constant C_f depends solely on $\|f_1\|_{L^p}$.

SKETCH OF THE PROOF. We omit the easy proof of (a). The proof of (b) is analogous to the proof of Lemma 1.14 part (b). But now use

$$r = \frac{1}{M\left(\left[|s| - M(|s|^{-1})^{-1}\right]^{-1}\right)}$$

as the radius of the circle to integrate over. To estimate I_+ use that for $\Re z > 0$

$$\hat{f}(z) = z \int_0^\infty e^{-zt} f_1(t) dt$$

is valid. ☐

1.3.2. Proof of Theorem 1.26. To emphasize that our proof breaks down if one tries to generalize the theorem as proposed in Remark 1.28 we let $\alpha \ge 1$ be a real number and relax (as in the remark) condition (1.32) to

$$\left\| \hat{f}(z) \right\| \le K(|\Im z|) \vee |z|^{1-\alpha} \text{ for all } z \in \Omega_\alpha \cap \Omega_M.$$

Note that $\alpha = 1$ is the special case in which we are interested in. Without loss of generality $s_0 = 2$. From Lemma 1.30 we deduce that the Fourier transform F of f satisfies

$$\left\| F^{(j)}(s) \right\| \le C^{1+j} j! s^{1-\alpha} s^{-\alpha j}$$

for all $s \in [-2,2]$ and $j \in \mathbb{N}$, where C is independent of s and j. By Lemma 1.14 we may assume that

$$\forall j \in \mathbb{N}_0, |s| \geq 1 : \left\| F^{(j)}(s) \right\| \leq j! K(|s|) M(|s|)^j.$$

Let us fix now a $t \geq 1$. Without loss of generality we may assume that M is unbounded otherwise we can replace it by, for example, $M(s) = \log(2+s)$. We define $R = M^{-1}_{\log K}(c_1 t)$ for a $c_1 > 0$ to be chosen later. Without loss of generality $R \geq 2$. Without loss of generality we assume that $M(R) \geq \log(R)^{\varepsilon}$ and $K(R) \geq R$. If this was not the case we simply replace M (resp. K) by the functions given by $M(s) \vee \log(s)^{\varepsilon}$ (resp. $K(s) \vee s$).

Let us define $\varphi = \mathcal{F}\psi$ where $\psi \in C_c^\infty(\mathbb{R})$ satisfies $0 \leq \psi \leq 1$, $\psi|_{[-1,1]} = 1$ and $\operatorname{supp} \psi \subseteq [-2,2]$. For $R > 0$ we define $\psi_R(s) = \psi(s/R)$ and $\phi_R(t) = R\phi(Rt)$. Moreover, by the Denjoy-Carleman theorem we may assume that

$$\forall j \in \mathbb{N}_0 : \sup_{s \in [-2,2]} \left| \psi^{(j)}(s) \right| \leq C_1^{j+1} A_j \text{ with } A_j = (j \log(2+j)^{1+\varepsilon})^j.$$

Let $0 < r < R < \infty$. We decompose

$$f = [f - \phi_{2R} * f] + [\phi_{2R} * f - \phi * f] + [\phi * f - \phi_r * f] + \phi_r * f. \tag{1.36}$$

By Lemmas 1.9, 1.29 and $\|P_y *\|_{L^\infty \to L^\infty} = 1$ for each $y > 0$ we have

$$\|[f - \phi_{2R} * f](t)\| \leq \frac{C}{R}, \quad \|\phi_r * f(t)\| \leq Cr. \tag{1.37}$$

To estimate $[\phi_{2R} * f - \phi * f](t)$ we follow the estimation of $J_{2,1}$ in the proof of Theorem 1.7. That is, as in (1.15) we integrate

$$[\phi_{2R} * f - \phi * f](t) = \frac{1}{2\pi} \int_{-R}^{R} e^{ist} (\psi_{2R} - \psi) F(s) ds$$

N times by part. Let us define

$$J(t) = \frac{1}{2\pi} \left(\frac{i}{t} \right)^N \int_{1<|s|<2} e^{ist} \left[\sum_{j=1}^{N} \binom{N}{j} F^{(N-j)} \psi^{(j)} \right] (s) ds.$$

Note that the summation starts at $j = 1$. If we choose $N = \lfloor t/(C_1 M(R)) \rfloor$ for a large enough C_1 the estimation of $J_{2,1}$ in the proof of Theorem 1.7 shows

$$\|[\phi_{2R} * f - \phi * f](t) - J(t)\| \leq \frac{C}{M^{-1}_{\log K}(c_1 t)} \tag{1.38}$$

for an appropriate $c_1 > 0$. An analogous argument shows

$$\|J(t)\| \leq C \left(\frac{CN}{t} \right)^N \sum_{j=1}^{N} \log(2+j)^{(1+\varepsilon)j}.$$

Since, by assumption (ii), $\log(K(R)) \leq \log(M(R))^{1-\varepsilon}$ this yields

$$\|J(t)\| \leq \frac{1}{t} \left(\frac{CN \log(2+N)^{1+\varepsilon}}{t} \right)^{N-1} \leq \frac{C}{t} \left(\frac{C\varepsilon^{-1} M(R)^{1-\varepsilon^2}}{M(R)} \right)^{N-1} \leq \frac{1}{t} \tag{1.39}$$

if t (and therefore also R) is large enough.

It remains to estimate $[\phi * f - \phi_r * f](t)$. Integrating two times by parts we get

$$
\begin{aligned}
2\pi(\phi - \phi_r) * f(t) &= \int_{r<|s|<2} e^{ist}(\psi - \psi_r)F(s)ds \\
&= \left(\frac{i}{t}\right)\sum_{j=0}^{2}\binom{2}{j}\int_{r<|s|<2} e^{ist}(\psi^{(2-j)} - r^{j-2}(\psi^{(2-j)})_r)F^{(j)}(s)ds \\
&=: I(t,r) + II(t,r) + III(t,r).
\end{aligned}
$$

Using the estimates on the derivatives of F close to zero we get

$$
\begin{aligned}
\|I(t,r)\| &\le C\int_{r<|s|<2}(1 + r^{-2}1_{r<|s|<2r}(s))s^{1-\alpha}ds \le Cr(r^{-\alpha} + r^{-1-\alpha}), \\
\|II(t,r)\| &\le C\int_{r<|s|<2}(1 + r^{-1}1_{r<|s|<2r}(s))s^{1-2\alpha}ds \le Cr(r^{-2\alpha} + r^{-2\alpha}) \text{ and} \\
\|III(t,r)\| &\le C\int_{r<|s|<2} s^{1-3\alpha}ds \le Crr^{1-\alpha}r^{-2\alpha}.
\end{aligned}
$$

This implies

$$
\|[\phi * f - \phi_r * f](t)\| \le Crr^{1-\alpha}\left(\frac{r^{-\alpha}}{t}\right)^2. \tag{1.40}
$$

Let us plug into (1.36) the estimates (1.37), (1.38), (1.39) and (1.40). This yields

$$
\|f(t)\| \le \frac{C}{t} + \frac{C}{M_{\log K}^{-1}(c_1 t)} + \frac{C}{R}\left[1 + rR\left(1 + r^{1-\alpha}\left(\frac{r^{-\alpha}}{t}\right)^2\right)\right].
$$

Let us use the fact that actually $\alpha = 1$. Let $r = t^{-1}$. Then the preceding estimate implies

$$
\|f(t)\| \le \frac{C}{t} + \frac{C}{M_{\log K}^{-1}(c_1 t)} + \frac{C}{R}\left[1 + \frac{R}{t}\right].
$$

Recalling that $R = M_{\log K}^{-1}(c_1 t)$ we see that this finishes the proof of Theorem 1.26.

1.4. Logarithmic singularity at zero

THEOREM 1.31. *Let $(X, \|\cdot\|)$ be a Banach space and $f : \mathbb{R}_+ \to X$ be a locally integrable function such that its weak derivative $f^{(m)}$ of order $m \in \mathbb{N}_1$ is bounded. Assume that there exist continuous and non-decreasing functions $M, K : \mathbb{R}_+ \to (0, \infty)$ satisfying*

(i) $\forall s \ge 0 : K(s) \ge \max\{2, s, M(s)\}$,
(ii) $\exists \varepsilon \in (0,1) : K(s) = O\left(e^{e^{(sM(s))^{1-\varepsilon}}}\right)$ *as* $s \to \infty$.

Assume that for some $r > 0$ and some analytic function $\tilde{f} : B_r \to X$ the mapping $z \mapsto \hat{f}(z) - \tilde{f}(z)\log(z)$ is analytic on B_r. Assume furthermore that $\hat{f}$ extends analytically to $(\Omega_M \cup \mathbb{C}_+)\backslash\mathbb{R}_-$ and

$$
\left\|\hat{f}(z)\right\| \le K(|\Im z|) \text{ for all } z \in \Omega_M, |\Im z| > \frac{r}{2}. \tag{1.41}
$$

Then there exists a constant $c_1 > 0$ such that for any $k \in \mathbb{N}_1$ there exists another constant $C(k) \geq 0$ such that

$$\left\| f(t) + \tilde{f}_{k-1}\left(\frac{d}{dt}\right) t^{-1} \right\| \leq \max\left\{ \frac{C(k)}{t^{k+1}}, \frac{C}{M_{\log K}^{-1}(c_1 t)^m} \right\} \tag{1.42}$$

for all $t \geq 1$, where $M_{\log K}(s) := M(s)\log(K(s))$ and $\tilde{f}_{k-1}$ is the Taylor polynomial of $\tilde{f}$ up to order $k-1$. More precisely the constant $C(k)$ can be estimated from above by

$$C(k) \leq \sup_{-r<x<0} \left\| \tilde{f}^{(k)}(x) \right\|. \tag{1.43}$$

In particular one can choose $C(k) = 0$ if $\tilde{f}$ is a polynomial of degree at most k.

1.4.1. Proof of Theorem 1.31. For simplicity we assume $m = 1$. At the very end of the proof we briefly explain the modification of the proof which leads to the conclusion of the theorem in case $m > 1$. Let k be a strictly positive natural number to be fixed later. We define the function $\psi : \mathbb{C}\backslash\{-i, +i\} \to \mathbb{C}$ by

$$\psi(z) = c_k \exp\left(-\exp\left(\left(\frac{2}{1+z^2} \right)^k \right) \right), \text{ where } c_k = e^{e^{2^k}}.$$

Let $R > 0$ be a natural number to be chosen later (depending on t). Depending on R and an additional parameter $\delta > 0$ we define now various contours for integration in the complex plane.

$$\begin{aligned}
\gamma_{11} &= \{R(x - i(1 - x^{\frac{1}{k+2}})); \, x \in (0,1)\}, \\
\gamma_{12} &= \{R((1-x) + i(1 - (1-x)^{\frac{1}{k+2}})); \, x \in (0,1)\}, \\
\gamma_{21} &= \{-R(x - i(1 - x^{\frac{1}{k+2}})); \, x \in (0,1)\}, \\
\gamma_{22} &= \{-R((1-x) + i(1 - (1-x)^{\frac{1}{k+2}})); \, x \in (0,1)\}, \\
\gamma_{31} &= \{-R(x - i(1 - x^{\frac{1}{k+2}})); \, x \in (0, (2RM(R))^{-1})\}, \\
\gamma_{32} &= \{-(2M(R))^{-1} + iy; \, y \in (R - R(2RM(R))^{\frac{-1}{k+2}}, \delta)\}, \\
\gamma_{33} &= \{x + i\delta; \, x \in (-(2M(R))^{-1}, 0)\} \cup \{(\delta\cos\varphi, \delta\sin\varphi); \, \varphi \in (\frac{\pi}{2}, -\frac{\pi}{2})\} \\
&\quad \cup \{x - i\delta; \, x \in (0, -(2M(R))^{-1})\} \\
\gamma_{34} &= \{-(2M(R))^{-1} + iy; \, y \in (-\delta, -R + R(2RM(R))^{\frac{-1}{k+2}})\}, \\
\gamma_{35} &= \{-R(x + i(1 - |x|^{\frac{1}{k+2}})); \, x \in (-(2RM(R))^{-1}, 0)\}.
\end{aligned}$$

Since we plan to consider the limit $\delta \downarrow 0$ we may assume that none of the contours intersects another one. If we have to use a parametrization of one of these contours we do it via x, y or φ as indicated in the definitions of the contours. This also determines an orientation of the paths. Moreover, we define $\gamma_1 = \gamma_{11} + \gamma_{12}$, $\gamma_2 = \gamma_{21} + \gamma_{22}$ and $\gamma_3 = \gamma_{31} + \ldots + \gamma_{35}$. Note that $\gamma_1 + \gamma_2$ and $\gamma_1 + \gamma_3$ are closed paths encircling each of the points from the interval (δ, R). Also note that the derivative of the parametrization of any of the above paths approaching $+i$ or $-i$ can be estimated by a constant times Rx^{-1}.

Now let us define the bounded function $g : \mathbb{R} \to X$ via $g(t) = -f'(t)$ for positive t and extend it by 0 for negative arguments. Observe that $\hat{g}(z) = -z\hat{f}(z) + f(0)$.

Without loss of generality we may assume that $f(0) = 0$, otherwise we adjust f appropriately on the interval $[0, 1]$ as in the proof of Theorem 1.7 (Section 1.2.1). Let us define the function h_t on the interior of $\mathbb{C}_+$ by

$$h_t(z) = \hat{g}(z) - \int_0^t e^{-zs} g(s) ds.$$

By assumptions, $\hat{g}$ and h_t extend to analytic functions on $(\Omega_M \cup \mathbb{C}_+) \setminus \mathbb{R}_-$. Observe that $f(t) = h_t(0)$, if we extend $\hat{g}$ by continuity as 0 at 0. Therefore

$$\begin{aligned} f(t) &= \lim_{\lambda \downarrow 0} h_t(\lambda) \psi(R^{-1}\lambda) e^{\lambda t} \\ &= \lim_{\lambda \downarrow 0} \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{\gamma_1 + \gamma_3} \psi(R^{-1}z) h_t(z) e^{zt} \frac{dz}{z - \lambda} \\ &= \lim_{\lambda \downarrow 0} \frac{1}{2\pi i} \int_{\gamma_1} \psi(R^{-1}z) \left(\hat{g}(z) - \int_0^t e^{-zs} g(s) ds \right) e^{zt} \frac{dz}{z - \lambda} \\ &+ \lim_{\lambda \downarrow 0} \frac{1}{2\pi i} \int_{\gamma_2} \psi(R^{-1}z) \left(- \int_0^t e^{-zs} g(s) ds \right) e^{zt} \frac{dz}{z - \lambda} \\ &+ \lim_{\lambda \downarrow 0} \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{\gamma_3} \psi(R^{-1}z) \hat{g}(z) e^{zt} \frac{dz}{z - \lambda} \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Actually at the moment we do not know if the integrals above really exist since ψ has (essential) singularities at $\pm i$. However, the following lemma fixes this problem. It implies that $\psi(R\cdot)$ is actually bounded on all the contours and actually decays fast enough (for our purposes) as z approaches $\pm iR$ along γ_1, γ_2 or γ_3. Thus - in the spirit of Newman [39] - our ψ serves as a fudge factor in our Cauchy integrals.

Lemma 1.32. *Let $\varepsilon \in (0, 1)$ and $k \in \mathbb{N}_1$ with $k > 2\varepsilon^{-1} - 2$. Then*

$$|\psi(z)| \leq C \exp(-\exp(x^{-(1-\varepsilon)}))$$

holds for all $z \in \mathbb{C}$ which can be represented as

$$z = x + i(1 - y) \text{ or } z = x + i(-1 + y)$$

where $y \in (0, 1)$ and $|x| = y^{k+2}$.

Proof. By symmetry of the function ψ it suffices to consider the case where z can be represented as $z = x + i(1 - y)$ for $y \in (0, 1)$ and $|x| = y^{k+2}$. Clearly ψ is bounded if z stays away from i. Thus it suffices to consider the asymptotic behaviour of $\psi(z)$ as y approaches 0. We have $x = o(y)$ as $y \to 0$ and

$$\frac{1}{1 + z^2} = \frac{a - ib}{a^2 + b^2} \text{ with } a = y(2 - y) + x^2 = (2 + o(1))y$$
$$\text{and } b = 2x(1 - y) = (2 + o(1))x.$$

Therefore a short calculation yields

$$\left(\frac{2}{1 + z^2} \right)^k = (1 + o(1)) y^{-k} + i o(1)$$

as $y \downarrow 0$. Here and in the following $o(1)$ replaces *real valued* terms converging to zero as $y \downarrow 0$. The last line in turn implies

$$\Re \exp\left(\left(\frac{2}{1+z^2}\right)^k\right) = e^{(1+o(1))y^{-k}}.$$

This yields the claim. □

1.4.1.1. *Estimation of* I_1. By dominated convergence we can perform the limit $\lambda \downarrow 0$ by simply setting $\lambda = 0$ in the integral. We further split the integral $I_1 = I_{11}+I_{12}$ according to the decomposition of the path $\gamma_1 = \gamma_{11}+\gamma_{12}$. Using $|\dot\gamma_{11}|(x) \leq CRx^{-1}$ we get

$$\begin{aligned} \|I_{11}\| &= \left\| \int_0^1 \int_t^\infty \underbrace{\psi(R^{-1}\gamma_{11}(x))}_{=o(x^\infty) \text{ by Lemma 1.32}} e^{-(s-t)\gamma_{11}(x)} g(s) ds \dot\gamma_{11}(x) \frac{dx}{\gamma_{11}(x)} \right\| \\ &\leq C \int_{-\infty}^\infty \underbrace{\int_0^1 e^{-sRx} x^2 dx}_{\leq C(1+Rs)^{-2}} \|g(t+s)\|\, ds \\ &\leq \frac{C}{R} P_{R^{-1}} * \|g\|(t) \leq \frac{C}{R}. \end{aligned} \tag{1.44}$$

Here we again emphasize the occurrence of the Poisson kernel, defined in Section 1.2.1.1, although we do not need it in this proof since we are not interested in L^p-rates of decay. We have proved that

$$I_1 \leq \frac{C}{R} P_{R^{-1}} * \|g\|(t) \leq \frac{C}{R} \tag{1.45}$$

since the estimation of I_{12} is analogous.

1.4.1.2. *Estimation of* I_2. This is almost the same procedure as in the estimation of I_1. Again we can perform the limit $\lambda \downarrow 0$ by simply setting $\lambda = 0$ in the integral and we split the integral $I_2 = I_{21} + I_{22}$ according to the decomposition of the path $\gamma_2 = \gamma_{21} + \gamma_{22}$.

$$\begin{aligned} \|I_{21}\| &\leq \int_0^1 \int_0^t \underbrace{\left|\psi(R^{-1}\gamma_{21}(x))\right|}_{=o(x^\infty) \text{ by Lemma 1.32}} e^{-(t-s)Rx} \|g(s)\|\, ds\, |\dot\gamma_{21}(x)| \frac{dx}{|\gamma_{21}(x)|} \\ &\leq C \int_{-\infty}^\infty \underbrace{\int_0^1 e^{-sRx} x^2 dx}_{\leq C(1+Rs)^{-2}} \|g(t-s)\|\, ds \\ &\leq \frac{C}{R} P_{R^{-1}} * \|g\|(t) \leq \frac{C}{R}. \end{aligned}$$

Again the estimation of I_{22} is analogous and we have therefore proved

$$I_2 \leq \frac{C}{R} P_{R^{-1}} * \|g\|(t) \leq \frac{C}{R}. \tag{1.46}$$

1.4.1.3. *Estimation of* I_3. We split the integral $I_3 = I_{31} + \ldots + I_{35}$ according to the decomposition of the path $\gamma_3 = \gamma_{31} + \ldots + \gamma_{35}$. It suffices to investigate I_{33}, I_{34} and I_{35} since the estimation of I_{31} and I_{32} is similar to the estimation of I_{35} and I_{34}. By dominated convergence we can perform the limits $\delta \downarrow 0$ and $\lambda \downarrow 0$

by simply setting $\delta = \lambda = 0$ in the integrals I_{35} and I_{34}. The limits in the integral I_{33} are performed later on.

Let us now fix $k = \lceil 4\varepsilon^{-1} - 2 \rceil$, where ε is as in condition (ii) on K, and recall Lemma 1.32. Then we may calculate

$$\begin{aligned}\|I_{35}\| &\leq \int_{-(2RM(R))^{-1}}^{0} \left|\psi(R^{-1}\gamma_{35}(x))\right| e^{Rxt} \|\hat{g}(\gamma_{35}(x))\| \, |\dot{\gamma}_{35}(x)| \frac{dx}{|\gamma_{35}(x)|} \\ &\leq C \int_{-(2RM(R))^{-1}}^{0} \exp(-\exp(x^{-(1-\varepsilon/2)}))RK(R)\frac{dx}{x} \\ &\leq \frac{CK(R)}{M(R)} \exp(-\exp((2RM(R))^{-(1-\varepsilon)})) \leq \frac{C}{R}. \end{aligned} \tag{1.47}$$

In the last inequality we used the condition (ii) on K.

$$\begin{aligned}\|I_{34}\| &\leq \int_{0}^{R-(2RM(R))^{\frac{-1}{k+2}}} \left|\psi(R^{-1}\gamma_{34}(y))\right| \|\hat{g}(\gamma_{34}(y))\| \, e^{-\frac{t}{2M(R)}} \, |\dot{\gamma}_{34}(y)| \frac{dy}{(2M(R))^{-1}} \\ &\leq C \int_0^R RK(R) e^{-\frac{t}{2M(R)}} M(R) dy \\ &\leq CR^2 M(R)K(R)e^{-\frac{t}{2M(R)}}. \end{aligned} \tag{1.48}$$

Before we finally consider the integral I_{33}, let us first summarize what we obtained so far. By (1.45), (1.46), (1.47) and (1.48) we have

$$\|f(t) - I_{33}\| \leq \frac{C}{R}\left(1 + R^3 M(R)K(R)e^{-\frac{t}{2M(R)}}\right). \tag{1.49}$$

Using condition (i) on K the choice $R = M_{\log K}^{-1}(c_1 t)$ for a sufficiently small c_1 yields

$$\|f(t) - I_{33}\| \leq \frac{C}{M_{\log K}^{-1}(c_1 t)}. \tag{1.50}$$

Now let us turn to the estimation of I_{33}. Observe that $\tilde{f}$ satisfies for $-r < x < 0$

$$\tilde{f}(x) = \frac{1}{2\pi i} \cdot \frac{1}{x} \cdot \lim_{\delta \downarrow 0}(\hat{g}(x - i\delta) - \hat{g}(x + i\delta)).$$

Note that, by assumptions on f, the terms to the right of the limit are uniformly bounded. Thus by dominated convergence and a change of variables (we replace x by $-x$ in the parametrization of γ_{33}) we get

$$\begin{aligned} I_{33} &= \int_0^{(2M(R))^{-1}} \psi(-R^{-1}x)e^{-xt}\tilde{f}(-x)dx \\ &= \frac{1}{t}\int_0^{\frac{t}{2M(R)}} e^{-y}\left(1 + O\left((tR)^{-1}y\right)\right)\left(\tilde{f}_{k-1}(-t^{-1}y) + O\left((t^{-1}y)^k\right)\right)dy. \end{aligned}$$

We show now that neglecting the O's and then integrating from 0 to ∞ in the above integral from the last line produces an error of order at most $t^{-k-1} + 1/M_{\log K}^{-1}(c_1 t)$. First we observe, using boundedness of $\tilde{f}$ and then replacing the upper limit of the integral by ∞, that for $t \geq 1$

$$\left\|\frac{1}{t}\int_0^{\frac{t}{2M(R)}} \left(\frac{y}{tR}\right) e^{-y}\tilde{f}(-t^{-1}y)dy\right\| \leq \frac{C}{t^2 R} \leq \frac{C}{M_{\log K}^{-1}(c_1 t)}. \tag{1.51}$$

Using the standard integral representation of the Gamma function we get

$$\frac{1}{t}\int_0^{\frac{t}{2M(R)}} e^{-y}(t^{-1}y)^k dy \leq \frac{k!}{t^{k+1}}. \tag{1.52}$$

By making c_1 smaller, if necessary, we may assume that $K(s) \geq s^{4c_1}$. Thus $t/(2M(R)) = (2c_1)^{-1}\log(K(R)) \geq 2\log(R)$ and we get using the fact that $\tilde{f}_{k-1}$ is polynomially bounded

$$\left\| \frac{1}{t}\int_{\frac{t}{2M(R)}}^{\infty} e^{-y}\tilde{f}(-t^{-1}y)dy \right\| \leq \frac{C}{tM^{-1}_{\log K}(c_1 t)} \leq \frac{C}{M^{-1}_{\log K}(c_1 t)}. \tag{1.53}$$

Let a_j for $j \in \mathbb{N}$ be the j-th Taylor coefficient of $\tilde{f}$, that is $\tilde{f}(z) = \sum_{j=0}^{\infty} a_j z^j$. Then

$$\begin{aligned}\frac{1}{t}\int_0^{\infty} e^{-y}\tilde{f}_{k-1}(-t^{-1}y)dy &= -\sum_{j=0}^{k-1} a_j(-t^{-1})^{j+1}\int_0^{\infty} e^{-y}y^j dy \\ &= -\sum_{j=0}^{k-1} a_j j!(-t^{-1})^{j+1} = \tilde{f}_{k-1}\left(\frac{d}{dt}\right)\frac{1}{t}\end{aligned}$$

We have proved that

$$\left\| I_{33} - \tilde{f}_{k-1}\left(\frac{d}{dt}\right)t^{-1} \right\| \leq \frac{Ck!}{t^{k+1}} + \frac{C}{M^{-1}_{\log K}(c_1 t)}. \tag{1.54}$$

If we combine (1.50) and (1.54) we get the desired decay rate. The upper estimate on $C(k)$ stated in (1.43) follows from

$$\left\|\tilde{f}(x) - \tilde{f}_{k-1}(x)\right\| \leq \frac{1}{k!}\sup_{x<\xi<0}\left\|\tilde{f}^{(k)}(\xi)\right\| \text{ for all } x \in (-r, 0)$$

together with (1.52).

1.4.1.4. *The case $m > 1$.* First, in order to improve (1.50) to

$$\|f(t) - I_{33}\| \leq \frac{C}{M^{-1}_{\log K}(c_1 t)^m},$$

it suffices to improve (1.49) to

$$\|f(t) - I_{33}\| \leq \frac{C}{R^m}\left(1 + R^{m+1}M(R)K(R)e^{-\frac{t}{2M(R)}}\right).$$

We can achieve this if we can replace the final C/R bound in (1.45), (1.46) and (1.47) by a bound C/R^m. For (1.47) this is easy since the estimation above actually shows the better bound $C/R^{m'}$ for any $m' \in \mathbb{N}$. To get the better bound in (1.45) we use that for example $\gamma_{11}(x)$ is bounded from below by a constant times R. Observing that

$$\frac{1}{\gamma_{11}(x)^{m-1}}\left(-\frac{d}{ds}\right)^{m-1} e^{-(s-t)\gamma_{11}(x)} = e^{-(s-t)\gamma_{11}(x)}$$

we see that an integration by parts argument - using that $g^{(m-1)}$ is bounded - yields the desired C/R^m bound. Actually, performing the integration by parts yields boundary terms. However, exactly the same boundary terms with opposite sign occur if we do the same trick for the estimation of I_{21}. So if we estimate directly the sum $I_{11} + I_{21}$ and use that ψ is symmetric we see that the boundary terms cancel out. For $I_{12}+I_{22}$ we do the same trick and get the improved estimate for (1.45) and (1.46).

The proof is complete if we manage to improve (1.51) to

$$\left\|\frac{1}{t}\int_0^{\frac{t}{2M(R)}}\left(1-\psi\left(\frac{y}{tR}\right)\right)e^{-y}\tilde{f}(-t^{-1}y)dy\right\| \le \frac{C}{t^{m+1}R^m} \le \frac{C}{M_{\log K}^{-1}(c_1t)^m}$$

and (1.53) to

$$\left\|\frac{1}{t}\int_{\frac{t}{2M(R)}}^{\infty} e^{-y}\tilde{f}(-t^{-1}y)dy\right\| \le \frac{C}{tM_{\log K}^{-1}(c_1t)^m} \le \frac{C}{M_{\log K}^{-1}(c_1t)^m}.$$

We easily achieve the second goal by choosing c_1 so small that $K(s)\ge s^{4mc_1}$ holds for all $s>0$. We could achieve the first goal if $\psi^{(j)}(0)=0$ for all $j=1,\dots,m$. Our current fudge factor does not satisfy this for $m\ge 2$. However, if we replace it by

$$\psi_m(z)=c_k\exp\left(-\exp\left(\left(\frac{2}{1+z^{4m+2}}\right)^k\right)\right),\ \text{where } c_k=e^{e^{2^k}},$$

then this property is satisfied. One only has to check now that this new fudge factor works as well as the old one in the other parts of the proof. In particular we mention that ψ_m also satisfies Lemma 1.32. The proof of Theorem 1.31 is finished.

1.4.2. A minor relaxation of condition (ii) on K. We cannot prove Theorem 1.31 for $\varepsilon=0$ in condition (ii) on K. However, as in Remark 1.13 on Theorem 1.3 and 1.7 we can relax (ii) slightly by (ii)':

$$K(s)=O\left(\exp\left(\exp\left(\frac{sM(s)}{L_1(sM(s))\cdot\ldots\cdot L_{n-1}(sM(s))\cdot L_n(sM(s))^{1+\varepsilon}}\right)\right)\right)$$

for some $\varepsilon>0$ and $n\in\mathbb{N}_1$. The proof of Theorem 1.31 changes only in the choice of the fudge factor ψ and the contours $\gamma_1,\gamma_2,\gamma_{31}$ and γ_{35}. What we need in the proof is that $\psi(R\cdot)$ is bounded in the domain enclosed by $\gamma_1+\gamma_2$, we have $\psi(z)=O(|\Re z|^\infty)$ if $z\to\pm i$ within this domain and that we can control the absolute value of $\psi(R^{-1}z)K(R)$ for $|\Re z|\le 1/(2RM(R))$. See for example the estimation of I_{35} in (1.47) for the reason why we need the last mentioned control.

To achieve all these things we define (in case $m=1$) the fudge factor by

$$\psi(z)=c_{nk}\exp\left(-\exp_{n+1}\left(\left(\frac{2}{1+z^2}\right)^k\right)\right)$$

for an $k\in\mathbb{N}_1$ to be chosen. The positive real number c_{nk} is chosen in such a way that $\psi(0)=1$. By $\exp_j$ we denote the composition of j exponential functions. Moreover we define $\chi:\mathbb{R}_+\to\mathbb{R}$ by

$$\chi(y)=\frac{y^{k+2}}{\prod_{j=1}^n\exp_j(y^{-k})}.$$

Observe that $(\chi^{-1})'(x)=o(x^{-1})$ as $x\to 0$. In the definition of the contours we replace all occurrences of x or $|x|$ by $\chi^{-1}(x)$ or $\chi^{-1}(|x|)$. To get the desired control on the product of ψ and K above it is crucial to generalize Lemma 1.32 in the following way.

LEMMA 1.33. *Let* $\varepsilon \in (0,1)$ *and* $k \in \mathbb{N}_1$ *with* $k > 2\varepsilon^{-1} - 2$. *Let us denote* $\tilde{L}_{n,\varepsilon}(t) = L_1(t) \cdot \ldots \cdot L_{n-1}(t) \cdot L_n(t)^{1+\varepsilon}$ *for positive real numbers* t. *Then*

$$|\psi(z)| \leq C \exp\left(-\exp\left(\frac{x^{-1}}{\tilde{L}_{n,\varepsilon}(x^{-1})}\right)\right)$$

for all $z \in \mathbb{C}$ *which can be represented as*

$$z = x + i(1-y) \text{ or } z = x + i(-1+y),$$

where $y \in (0,1)$ *and* $|x| = \chi(y)$.

PROOF. Without loss of generality $z = x + i(1-y)$. As in the proof of Lemma 1.32 we get

$$\begin{aligned} \left(\frac{2}{1+z^2}\right)^k &= y^{-k} + iky^{-k-1}O(\chi(y)) \\ \Rightarrow \exp\left(\left(\frac{2}{1+z^2}\right)^k\right) &= \exp(y^{-k}) + iky^{-k-1}\exp(y^{-k})O(\chi(y)) \\ &\vdots \\ \Rightarrow \exp_n\left(\left(\frac{2}{1+z^2}\right)^k\right) &= \exp_n(y^{-k}) + iky^{-k-1}\left(\prod_{j=1}^{n}\exp_j(y^{-k})\right)O(\chi(y)). \end{aligned}$$

This implies

$$|\psi(z)| \leq \exp(-(1+o(1))\exp_{n+1}(y^{-n})).$$

Now let $\delta = 2/k$. From the definition of χ it is not difficult to see that

$$\exp_n(y^{-k}) \geq \frac{x^{-1}}{\tilde{L}_{n,\delta}(x^{-1})}.$$

This finishes the proof. □

We have seen that by the method of two different proofs we arrive at the same condition (ii) (or (ii)') for K in Theorem 1.3 and Theorem 1.31. On this basis we conjecture that both theorems are false if one allows ε to be 0 in any of the constraints (ii) or (ii)'.

1.5. Application: (local) decay of C_0-semigroups

The results of the preceding sections can be applied to calculate local decay rates for C_0-semigroups. To fix some of our notation, let $T = (T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator $A : D(A) \to X$. Except for Section 1.5.1 we naturally restrict our considerations to the case $p = \infty$. A discussion of L^p-rates for semigroups and an application to the wave equation can be found in [7, Section 6].

1.5.1. No singularity on $i\mathbb{R}$. The following is an immediate consequence of Theorem 1.3.

COROLLARY 1.34 (to Theorem 1.3). *Let T be a bounded C_0-semigroup on a Banach space $(X, \|\cdot\|)$ with generator A. Let P_1 and P_2 be two bounded operators on X, let $x \in X$ and let $1 < p \leq \infty$. Let $M, K : \mathbb{R}_+ \to (0, \infty)$ be continuous and non-decreasing functions satisfying*

(i) $\forall s > 1 : K(s) \geq \max\{2, s, M(s)\}$,
(ii) $\exists \varepsilon \in (0,1) : K(s) = O\left(e^{e^{(sM(s))^{1-\varepsilon}}}\right)$ *as* $s \to \infty$.

Let $G(z) = P_2(z - A)^{-1} P_1 x$ for $\Re z > 0$. Assume that G extends analytically to the domain $\Omega_M \cup \mathbb{C}_+$ and satisfies the estimate

$$\|G(z)\| \leq K(|\Im z|) \text{ for } z \in \Omega_M. \tag{1.55}$$

Assume furthermore that $(t \mapsto \|P_2 T(t) P_1 x\|) \in L^p(\mathbb{R}_+)$. Then for all $m \in \mathbb{N}_1$ and $\omega > 0$ we have

$$(t \mapsto w_{M_{\log K}}(t)^m \left\|P_2 T(t)(\omega - A)^{-m} P_1 x\right\|) \in L^p(\mathbb{R}_+)$$

where $M_{\log K}(s) = M(s)\log(K(s))$.

REMARK 1.35. Observe that the condition $(t \mapsto \|P_2 T(t) P_1 x\|) \in L^p(\mathbb{R}_+)$ is trivially satisfied if T is a bounded C_0-semigroup and $p = \infty$. If in this case A is invertible then - as is clear from the proof - one can also take $\omega = 0$. In the case $P_1 = P_2 = 1$ we note that if $p \neq \infty$ and if $(t \mapsto \|T(t)x\|) \in L^p(\mathbb{R}_+)$ is true *for all* $x \in X$ then by Datko's theorem (see e.g. [4, Theorem 5.1.2]) the semigroup is automatically exponentially stable.

REMARK 1.36. In the particular case $P_1 = P_2 = 1$ one typically assumes that the resolvent extends continuously to the imaginary axis and satisfies an estimate $\left\|(is - A)^{-1}\right\| \leq M(|s|)$ for $s \in \mathbb{R}$. This then implies that the resolvent extends analytically to Ω_M and it satisfies (1.55) with K being a multiple of M in a slightly smaller domain. So in this situation our corollary does not improve known results.

However, our main interest in applying this theorem is to consider the case where P_1 and P_2 are not the identity. We think that a typical situation is that M is a slowly increasing function (possibly constant) and K is a (possibly much) faster increasing function. That is, we assume that the perturbed resolvent extends to a relatively large domain to the left of the imaginary axis, but may grow very quickly. We illustrate this philosophy in Chapter 3.

PROOF OF COROLLARY 1.34. Let us define $f(t) = P_2 T(t)(\omega - A)^{-m} P_1 x$. Then we have for $t > 0$ and for $z \in \Omega_M$

$$f^{(m)}(t) = P_2 T(t)[\omega(\omega - A)^{-1} - 1]^m P_1 x \text{ and} \tag{1.56}$$

$$\hat{f}(z) = \sum_{j=0}^{m-1} (\omega - z)^{-(j+1)} P_2 (\omega - A)^{-(m-j)} P_1 x + (\omega - z)^{-m} G(z). \tag{1.57}$$

The second line immediately implies (1.4) up to a constant factor. The first line implies $\left\|f^{(m)}\right\| \in L^p(\mathbb{R}_+)$ since

$$\begin{aligned}\left\|P_2 T(\cdot)(\omega - A)^{-1} P_1 x\right\|_{L^p} &= \left\|\int_0^\infty P_2 e^{-\omega\tau} T(\cdot + \tau) P_1 x d\tau\right\|_{L^p} \\ &\leq \omega^{-1} \left\|P_2 T(\cdot) P_1 x\right\|_{L^p}.\end{aligned}$$

Thus the conclusion of the corollary follows from Theorem 1.3. □

1.5.2. Logarithmic singularity at zero.

COROLLARY 1.37 (to Theorem 1.31). *Let T be a bounded C_0-semigroup on a Banach space $(X, \|\cdot\|)$ with generator A. Let P_1 and P_2 be two bounded operators on X and let $x \in X$. Let $M, K : \mathbb{R}_+ \to (0,\infty)$ be continuous and non-decreasing functions satisfying*

(i) $\forall s > 1 : K(s) \geq \max\{2, s, M(s)\}$,

(ii) $\exists \varepsilon \in (0,1) : K(s) = O\left(e^{e^{(sM(s))^{1-\varepsilon}}}\right)$ *as* $s \to \infty$.

Let $G(z) = P_2(z-A)^{-1}P_1x$ for $\Re z > 0$. Assume that for some $r > 0$ and some analytic function $\tilde{G} : B_r \to X$ the mapping $z \mapsto G(z) - \tilde{G}(z)\log(z)$ is analytic on B_r. For $|z| < r < \omega$ let $\tilde{G}_{m,\omega}(z) = (\omega - z)^{-m}\tilde{G}(z)$ and for $j \in \mathbb{N}$ let $\tilde{G}_{m,\omega,j}$ be its j-th order Taylor expansion. Assume furthermore that G extends analytically to $(\Omega_M \cup \mathbb{C}_+)\backslash\mathbb{R}_-$ and

$$\|G(z)\| \leq K(|\Im z|) \text{ for } z \in \Omega_M, |\Im z| > \frac{r}{2}. \tag{1.58}$$

Then for all $m \in \mathbb{N}_1$ and $\omega > r$ there is a $c_1 > 0$ such that for all $k \in \mathbb{N}_1$ there is another constant $C(k) > 0$ such that for all $t \geq 1$

$$\left\|P_2T(t)(\omega - A)^{-m}P_1x - \tilde{G}_{m,\omega,k-1}\left(\frac{d}{dt}\right)t^{-1}\right\| \leq \max\left\{\frac{C(k)}{t^{k+1}}, \frac{C}{M_{\log K}^{-1}(c_1t)^m}\right\}.$$

Here $M_{\log K}(s) = M(s)\log(K(s))$. More precisely the constant $C(k)$ can be estimated from above by

$$C(k) \leq \sup_{-r<s<0}\left\|\tilde{G}_{m,\omega}^{(k)}(s)\right\|.$$

PROOF. The proof is almost the same as for Corollary 1.34. Note that by (1.57) the Laplace transform $\hat{f}$ has the same singularity of logarithmic type at zero as the function $z \mapsto (\omega - z)^{-m}G(z)$. This explains the definition of $\tilde{G}_{m,\omega}$ in the theorem. □

1.5.3. s^{-1}-singularity at zero.

THEOREM 1.38. *Let T be a bounded C_0-semigroup on a Banach space $(X, \|\cdot\|)$ with generator A. Let $M : [0,\infty) \to (0,\infty)$ be a continuous non-decreasing function. Assume that the resolvent of A extends analytically across the imaginary axis and satisfies*

$$\forall s \in \mathbb{R} : \left\|(is - A)^{-1}\right\| \leq M(|s|) \vee \frac{1}{1 \wedge |s|}.$$

Then there exists a constant $c_1 > 0$ such that for all $t \geq 1$

$$\left\|T(t)A(1-A)^{-2}\right\| \leq C\max\left\{\frac{1}{t}, \frac{1}{M_{log}^{-1}(c_1t)}\right\},$$

where $M_{log}(s) = M(s)\log(2 + s + M(s))$.

PROOF. Let $f(t) = T(t)A(1-A)^{-2}$ for $t \geq 0$. We verify all hypotheses of Theorem 1.26. For $z \in \mathbb{C}$ with strictly positive real part we have

$$\hat{f}(z) = (z-A)^{-1}A(1-A)^{-2} = (z(z-A)^{-1} - 1)(1-A)^{-2}.$$

By the resolvent identity we have for all $s \in \mathbb{R}$ and $0 \leq d \leq D/2$ that

$$\left\|(is - A)^{-1}\right\| \leq D \Rightarrow (is - d - A)^{-1} \text{ exists and } \left\|(is - d - A)^{-1}\right\| \leq 2D.$$

This implies that $\hat{f}$ extends to the left of the imaginary axis and satisfies an estimate there as required in Theorem 1.26 (with M and K replaced by $2M$). Without loss of generality we may assume that $M \geq 1$ to satisfy constraint (i) in Theorem 1.26. Furthermore

$$f'(t) = T(t)A^2(1 - A)^{-2},\ f_1(t) = (T(t) - 1)(1 - A)^{-2}.$$

Thus also the derivative and the primitive of f are bounded, since T is a bounded semigroup. The conclusion now follows from Theorem 1.26. □

CHAPTER 2

Optimal decay for C_0-semigroups on Hilbert spaces

(Joint work with Jan Rozendaal and David Seifert)

2.1. Introduction

One of the most important results establishing decay rates for operator semigroups is the following.

THEOREM 2.1 (Borichev-Tomilov [12]). *Let X be a Hilbert space and let A be the generator of a bounded C_0-semigroup T on X. Suppose that $\sigma(A) \cap i\mathbb{R} = \emptyset$ and that $\left\|(is - A)^{-1}\right\| \leq M(|s|) := C(1 + |s|^{\alpha})$, $s \in \mathbb{R}$ for some $\alpha > 0$. Then for any $c > 0$*

$$\left\|T(t)A^{-1}\right\| = O\left(\frac{1}{M^{-1}(ct)}\right) \left(= O\left(\frac{1}{t^{\frac{1}{\alpha}}}\right)\right), \quad t \to \infty. \tag{2.1}$$

It is well known that for arbitrary M, if M is chosen optimal with respect to the resolvent estimate, the decay rate can never be essentially faster than $1/M^{-1}(ct)$ for some $c > 0$. This is the converse part of Theorem 0.1. The assumption that the function M is a "nice" function (a polynomial) is essential for the above "M-theorem" to be true. In fact considering normal semigroups one can easily see that Theorem 2.1 becomes false if we consider for example $M(s) = \log(2 + s)$ for $s \geq 0$. In this case the "$M_{\log}$-theorem" (Theorem 0.1) is optimal in general (up to the choice of c).

It is natural to ask for the class $\mathcal{M}$ of non-decreasing functions $M : \mathbb{R}_+ \to (0, \infty)$ for which Theorem 2.1 remains true. A pioneering work of Batty, Chill and Tomilov gives a first answer to this question [10]. The authors could show that a certain subclass of the class of regularly varying functions belongs to $\mathcal{M}$. The aim of this chapter is to give a precise characterization of $\mathcal{M}$. Our results show that $\mathcal{M} = \mathrm{PI}$ is the class of functions having *positive increase*. Let $M : \mathbb{R}_+ \to (0, \infty)$ be a measurable function. We say that M has *positive increase* [11, Chapter 2.1] and write $M \in \mathrm{PI}$ if

$$\exists \alpha, s_0 > 0, b \in (0, 1] \forall s_0 \leq s \leq R : \frac{M(R)}{M(s)} \geq b \left(\frac{R}{s}\right)^{\alpha}. \tag{2.2}$$

We remark here that if $M \in \mathrm{PI}$ then M^{-1} and $M^{-1}(c\cdot)$ are asymptotically similar for any $c > 0$. We refer the reader to Section A.2 for the required knowledge on positive increase. The necessity of the positive increase condition for an "M-theorem" to be true in general is shown by considering normal semigroups, or more generally by semigroups for which the resolvent growth along the imaginary axis is up to a constant given by the inverse of the distance to the spectrum. Our results remain true in an analogous form if we allow $\sigma(A) \cap i\mathbb{R} = \{0\}$, replacing $T(t)A^{-1}$ by

$T(t)A(1-A)^{-1}$ or $T(t)A(1-A)^{-2}$, depending on whether the resolvent is bounded near infinity or not.

It is an easy consequence from the definition of positive increase that $M \in \mathrm{PI}$ implies $M(s) \gtrsim s^{\varepsilon}$ for some $\varepsilon > 0$. Thus a super-polynomial decay cannot be deduced from an M-theorem. We remark that $T(t)A(1-A)^{-1}$ or $T(t)A(1-A)^{-2}$ cannot decay at a rate faster than $1/t$ if $0 \in \sigma(A)$. Therefore it is natural to assume $\sigma(A) \cap i\mathbb{R} = \emptyset$ if one is interested in faster decay rates. Although the decay rate must now be strictly slower than the one predicted by the conclusion of the M-theorem it is reasonable to still search for an improvement of the $M_{\log}$-theorem. This search lead us to the notion of *quasi-positive increase (with auxiliary function N)*. Let $N : \mathbb{R}_+ \to (0, \infty)$ (the auxiliary function) be a continuous non-decreasing function. Given $a \geq 0$ for a measurable function $M : [a, \infty) \to (0, \infty)$ we write $M \in \mathrm{PI}_N$ and say M has quasi-positive increase (with auxiliary function N) if

$$\exists s_0 \geq a, b \in (0,1] \forall s_0 \leq s \leq R : \frac{M(R)}{M(s)} \geq b \left(\frac{R}{s}\right)^{\frac{1}{N(R)}}. \tag{2.3}$$

The definition of quasi-positive increase implies that that *every* non-decreasing function has quasi-positive increase with auxiliary function $N(s) = \beta \log(2+s), s \geq 0$ where $\beta > 0$ can be chosen arbitrary. We prove that Theorem 2.1 remains valid for $M \in \mathrm{PI}_N$ if (2.1) is replaced by

$$\left\|T(t)A^{-1}\right\| = O\left(\frac{1}{M_N^{-1}(c_1 t)}\right), \quad t \to \infty$$

where $M_N(s) = M(s)N(s)$ for $s \geq 0$. The constant c_1 can be chosen to be equal to $be - \varepsilon$ for any $\varepsilon > 0$. For specific examples for M including $M(s) = \log(s)^{\alpha}$ for any $\alpha \in (0, \infty)$ and $M(s) = \exp(\log(s)^{\alpha})$ for any $\alpha \in (0,1)$ we can show that the decay rates we obtain are optimal up to the ε-loss in the choice of the constant c_1. The proof of the optimality is interesting on its own since we derive a precise formula for the decay of normal semigroups knowing the optimal resolvent bound M (Theorem 2.15). For the required knowledge on quasi-positive increase we refer the reader to Section A.3.

2.2. Sharp ((sub-)polynomial) decay rates under $M \in \mathrm{PI}$

In this section we prove $\mathcal{M} \supseteq \mathrm{PI}$ in the terminology of the introduction. We distinguish three cases: $\sigma(A) \cap i\mathbb{R} = \emptyset$ and $\left\|(is - A)^{-1}\right\|$ possibly unbounded as $|s| \to \infty$ (singularity at infinity), $\sigma(A) \cap i\mathbb{R} = \{0\}$ and $\left\|(is - A)^{-1}\right\|$ bounded for $|s| \geq 1$ (singularity at zero), and finally $\sigma(A) \cap i\mathbb{R} = \{0\}$ and $\left\|(is - A)^{-1}\right\|$ possibly unbounded as $|s| \to \infty$ (singularity at zero and infinity).

2.2.1. Singularity at infinity.

Theorem 2.2. *Let T be a bounded C_0-semigroup on a Hilbert space X with generator A. Suppose that $\sigma(A) \cap i\mathbb{R} = \emptyset$ and assume that there exists a continuous non-decreasing function $M : \mathbb{R}_+ \to (0, \infty)$ having positive increase such that*

$$\forall s \in \mathbb{R} : \left\|(is - A)^{-1}\right\| \leq M(|s|).$$

Then

$$\left\|T(t)A^{-1}\right\| \leq O\left(\frac{1}{M^{-1}(t)}\right), \quad t \to \infty.$$

PROOF. Let $\psi\colon \mathbb{R} \to \mathbb{C}$ be a Schwartz function such that $\psi(0) = \|\psi\|_{L^\infty} = 1$ and $\operatorname{supp}\psi \subseteq [-1,1]$, and let $\phi = \mathcal{F}^{-1}\psi$. For $R > 0$ let $\phi_R(t) = R\phi(Rt)$, $t \in \mathbb{R}$, and $\psi_R = \mathcal{F}\phi_R$, so that $\psi_R(s) = \psi(R^{-1}s)$, $s \in \mathbb{R}$. Note also that $\int_{\mathbb{R}}\phi_R(t)\,\mathrm{d}t = 1$ for all $R > 0$. Now temporarily fix $t > 0$ and, given $n \in \mathbb{N}_0$, let $g_n\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$g_n(s) = \begin{cases} 0, & s < 0, \\ s^n, & 0 \le s \le t, \\ s^n - (s-t)^n, & s > t. \end{cases} \tag{2.4}$$

In particular, $g_0 = \chi_{[0,t]}$. Let $x \in X$ and $n \in \mathbb{N}$ be fixed for now. We define the map $h_n\colon \mathbb{R} \to X$ by $h_n(s) = g_n(s)T(s)A^{-1}x$, $s \in \mathbb{R}$, where the semigroup is extended by zero to the whole of $\mathbb{R}$. Then

$$T(t)A^{-1}x = \frac{n+1}{t^{n+1}}\int_0^t T(t-s)h_n(s)\,\mathrm{d}s. \tag{2.5}$$

Our strategy is to split this integral by writing $h_n = (\delta - \phi_R) * h_n + \phi_R * h_n$, where δ denotes the Dirac mass at zero, and to estimate the resulting two integrals separately by making suitable choices of $R > 0$ and of $n \in \mathbb{N}$.

We begin by introducing the auxiliary function $\Phi\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(s) = \begin{cases} \int_{-\infty}^s \phi(\tau)\,\mathrm{d}\tau, & s < 0, \\ -\int_s^\infty \phi(\tau)\,\mathrm{d}\tau, & s \ge 0, \end{cases}$$

so that $\Phi' = \phi - \delta$ in the sense of distributions. Using the fact that Φ, being a primitive of a Schwartz function, decays rapidly at infinity and that $\int_{\mathbb{R}} \phi_R(s)\,\mathrm{d}s = 1$, a simple calculation using integration by parts yields

$$(\delta - \phi_R) * h_n(s) = -\frac{1}{R}\int_0^\infty \Phi(Rs - \tau)h_n'(R^{-1}\tau)\,\mathrm{d}\tau, \quad s \in \mathbb{R}. \tag{2.6}$$

Now the distributional derivative h_n' of h_n is given by

$$h_n'(s) = ng_{n-1}(s)T(s)A^{-1}x + g_n(s)T(s)x, \quad s \in \mathbb{R},$$

and hence

$$\|h_n'(s)\| \le K(ns^{n-1} + s^n)(\|A^{-1}\| + 1)\|x\|, \quad s \ge 0,$$

where $K = \sup_{t\ge 0}\|T(t)\|$. It follows from (2.6) that

$$\|(\delta - \phi_R) * h_n(s)\| \lesssim \frac{\|x\|}{R}\int_0^\infty |\Phi(Rs - \tau)|\left(n\left(\frac{\tau}{R}\right)^{n-1} + \left(\frac{\tau}{R}\right)^n\right)\mathrm{d}\tau \tag{2.7}$$

for all $s \in \mathbb{R}$, where the implicit constant is independent of R, n, t and x. We now inductively define functions $\Phi_k\colon \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{N}$, by setting $\Phi_1 = |\Phi|$ and

$$\Phi_{k+1}(s) = \begin{cases} \int_{-\infty}^s \Phi_k(\tau)\,\mathrm{d}\tau, & s < 0, \\ -\int_s^\infty \Phi_k(\tau)\,\mathrm{d}\tau, & s \ge 0, \end{cases} \tag{2.8}$$

for $k \ge 1$. Then, for each $k \in \mathbb{N}$, Φ_k vanishes rapidly at infinity and we have $\Phi_{k+1}' = \Phi_k - \langle\Phi_k\rangle\delta$ in the sense of distributions, where $\langle\Phi_k\rangle = \int_{\mathbb{R}}\Phi_k(s)\,\mathrm{d}s$. Hence by a simple inductive argument using integration by parts we see that, for $m \in \mathbb{N}_0$ and $s \ge 0$,

$$\int_0^\infty |\Phi(s-\tau)|\tau^m\,\mathrm{d}\tau = \sum_{k=0}^{m-1}\frac{m!}{(m-k)!}\langle\Phi_{k+1}\rangle s^{m-k} + m!\int_{-\infty}^s \Phi_{m+1}(\tau)\,\mathrm{d}\tau,$$

and therefore

$$\int_0^\infty |\Phi(Rs-\tau)| \left(\frac{\tau}{R}\right)^m \mathrm{d}\tau \le \sum_{k=0}^m \frac{m!}{(m-k)!}\|\Phi_{k+1}\|_{L^1} R^{-k} s^{m-k}.$$

Applying this with $m = n-1$ and $m = n$ in (2.7) we find after a simple calculation that

$$\left\| \frac{n+1}{t^{n+1}} \int_0^t T(t-s)(\delta - \phi_R) * h_n(s)\,\mathrm{d}s \right\| \lesssim \frac{\|x\|}{R}\left(P_n(Rt) + \frac{n+1}{t} P_{n-1}(Rt)\right),$$

where the implicit constant is still independent of R, n, t and x and where, for $m \in \mathbb{N}_0$ and $s \ge 0$,

$$P_m(s) = \sum_{k=0}^m \frac{(m+1)!}{(m+1-k)!} \frac{\|\Phi_{k+1}\|_{L^1}}{s^k}. \tag{2.9}$$

Note that each of the functions P_m, $m \in \mathbb{N}_0$, is non-increasing. In particular, if we assume that $R, t \ge 1$, then

$$\left\| \frac{n+1}{t^{n+1}} \int_0^t T(t-s)(\delta - \phi_R) * h_n(s)\,\mathrm{d}s \right\| \lesssim \frac{\|x\|}{R}, \tag{2.10}$$

where the implicit constant depends on n but is independent of R, t and x.

We now turn to the remaining term in the splitting. Note first that by Hölder's inequality

$$\left\| \frac{n+1}{t^{n+1}} \int_0^t T(t-s)\phi_R * h_n(s)\,\mathrm{d}s \right\| \le K \frac{n+1}{t^{n+1/2}} \|\phi_R * h_n\|_{L^2(\mathbb{R},X)}. \tag{2.11}$$

We now estimate the L^2-norm of $\phi_R * h_n$. Given $\alpha > 0$, define the function $h_{n,\alpha} \in L^1(\mathbb{R})$ by $h_{n,\alpha}(s) = e^{-\alpha s} h_n(s)$, $s \in \mathbb{R}$. Then $h_{n,\alpha}(s) = n!(T_\alpha^{*n} * h_{0,\alpha})(s)$, where $T_\alpha(s) = e^{-\alpha s} T(s)$, $s \in \mathbb{R}$, again after extending the semigroup by zero to the whole of $\mathbb{R}$, and therefore

$$(\mathcal{F} h_{n,\alpha})(s) = n!(is + \alpha - A)^{-n} \widehat{h_0}(is+\alpha), \quad s \in \mathbb{R}. \tag{2.12}$$

Hence by the dominated convergence theorem, given any Schwartz function $\eta\colon \mathbb{R} \to \mathbb{C}$, we have

$$\begin{aligned} \int_{\mathbb{R}} \phi_R * h_n(s)\eta(s)\,\mathrm{d}s &= \lim_{\alpha\to 0+} \int_0^\infty h_{n,\alpha}(s)\zeta_R(s)\,\mathrm{d}s \\ &= \lim_{\alpha\to 0+} \int_{\mathbb{R}} \phi_R * h_{n,\alpha}(s)\eta(s)\,\mathrm{d}s \\ &= \lim_{\alpha\to 0+} \int_{\mathbb{R}} \psi_R(s)(\mathcal{F} h_{n,\alpha})(s)(\mathcal{F}^{-1}\eta)(s)\,\mathrm{d}s, \end{aligned}$$

where $\zeta_R(s) = \int_{\mathbb{R}} \phi_R(\tau - s)\eta(\tau)\,\mathrm{d}\tau$, $s \in \mathbb{R}$. Note that, since $\sigma(A) \cap i\mathbb{R} = \emptyset$, the resolvent of A extends holomorphically across the imaginary axis, and in particular, the resolvent is uniformly bounded in an open neighbourhood of $i \operatorname{supp} \psi_R$. It follows from (2.12) and another application of the dominated convergence theorem that

$$\phi_R * h_n = \mathcal{F}^{-1}(\psi_R m_n \mathcal{F} h),$$

where $m_n(s) = n!(is - A)^{-n} A^{-1}$ and $h(s) = g_0(s)T(s)x$, $s \in \mathbb{R}$. A straightforward estimate using Plancherel's theorem now gives

$$\|\phi_R * h_n\|_{L^2(\mathbb{R},X)} \le \|\psi_R m_n\|_{L^\infty(\mathbb{R},\mathcal{L}(X))} \|h\|_{L^2(\mathbb{R},X)}.$$

Note that $\|h\|_{L^2(\mathbb{R},X)} \leq Kt^{1/2}\|x\|$. Moreover,

$$is(is-A)^{-n}A^{-1}x = (is-A)^{-n+1}A^{-1}x + (is-A)^{-n}x, \quad s \in \mathbb{R},$$

and hence $|s|\|(is-A)^{-n}A^{-1}\| \lesssim M(|s|)^{n-1} + M(|s|)^n$, $s \in \mathbb{R}$. By rescaling M if necessary we may assume that $M(s) \geq 1$ for all $s \geq 0$, and then

$$\|(is-A)^{-n}A^{-1}\| \lesssim \frac{M(|s|)^n}{\max\{s_0, |s|\}}, \quad s \in \mathbb{R},$$

where $s_0 > 0$ is fixed but arbitrary. Now since M is non-decreasing and has positive increase there exist constants $\alpha > 0$ and $c \in (0,1]$ such that

$$\frac{M(R)}{M(|s|)} \geq c\left(\frac{R}{|s|}\right)^{\alpha}, \quad R \geq |s| \geq s_0.$$

We now make a specific choice of n by setting $n = \lceil \alpha^{-1} \rceil$. A simple calculation then gives

$$\|\psi_R m_n\|_{L^\infty(\mathbb{R},\mathcal{L}(X))} \lesssim n! \sup_{|s|\leq R} \frac{M(|s|)^n}{\max\{s_0, |s|\}} \leq \frac{n!}{R}\left(\frac{M(R)}{c}\right)^n.$$

Combining the above estimates in (2.11) we find that

$$\left\|\frac{n+1}{t^{n+1}}\int_0^t T(t-s)\phi_R * h_n(s)\,\mathrm{d}s\right\| \lesssim (n+1)!\frac{\|x\|}{R}\left(\frac{M(R)}{ct}\right)^n, \tag{2.13}$$

where the implicit constant is independent of R, t and x. In fact, the implicit constant would also be independent of n if it were still free to vary, and this will become important in Section 2.4 below. Combining (2.13) with (2.10) in (2.5) gives

$$\|T(t)A^{-1}\| \lesssim \frac{1}{R}\left(1 + \left(\frac{M(R)}{ct}\right)^n\right), \quad R, t \geq 1,$$

where the implicit constant is independent of both R and t. If we now set $R = M^{-1}(ct)$ for $t \geq c^{-1}M(1)$, then the result follows from Lemma A.3. □

2.2.2. Singularity at zero.

THEOREM 2.3. *Let T be a bounded C_0-semigroup on a Hilbert space X with generator A. Suppose that $\sigma(A) \cap i\mathbb{R} = \{0\}$ and assume that there exists a continuous non-decreasing function $M : [1,\infty) \to (0,\infty)$ having positive increase such that for all $s \geq 1$*

$$\sup_{|\xi|\geq s^{-1}} \left\|(i\xi - A)^{-1}\right\| \leq M(|s|).$$

Then

$$\left\|T(t)A(1-A)^{-1}\right\| \leq O\left(\frac{1}{M^{-1}(t)}\right), \quad t \to \infty.$$

REMARK 2.4. Note that $0 \in \sigma(A)$ necessarily implies $M(s) \geq s$ for all $s \geq 1$.

PROOF OF THEOREM 2.3. The proof is similar to that of Theorem 2.2. Let $\psi\colon \mathbb{R} \to \mathbb{C}$ be a Schwartz function such that $\|\psi\|_{L^\infty} = 1$ and $\psi(s) = 1$ for $|s| \leq 1$, and let $\phi = \mathcal{F}^{-1}\psi$. Temporarily fix $x \in X$, $n \in \mathbb{N}$ and $t > 0$, and define the map $h_n\colon \mathbb{R} \to X$ by $h_n(s) = g_n(s)T(s)A(1-A)^{-1}x$, $s \in \mathbb{R}$, where the semigroup is extended by zero to the whole of $\mathbb{R}$ and where g_n is as defined in (2.4). Moreover, let $H_n\colon \mathbb{R}\to X$ be given by $H_n(s)=0$, $s<0$, and

$$H_n(s)=\int_0^s h_n(\tau)\,\mathrm{d}\tau, \quad s\geq 0.$$

A simple calculation shows that

$$\|H_n(s)\|\leq 2Ks^n\|(1-A)^{-1}\|\|x\|, \quad s\geq 0,$$

where $K=\sup_{t\geq 0}\|T(t)\|$. For $r\in(0,1]$ we let $\phi_r(t)=r\phi(rt)$, $t\in\mathbb{R}$, and $\psi_r=\mathcal{F}(\phi_r)$, as in the proof of Theorem 2.2. Integration by parts gives

$$\phi_r * h_n(s)=r\int_0^\infty \phi'(rs-\tau)H_n(r^{-1}\tau)\,\mathrm{d}\tau, \quad s\in\mathbb{R},$$

and hence

$$\|\phi_r * h_n(s)\|\lesssim r\|x\|\int_0^\infty |\phi'(rs-\tau)|\left(\frac{\tau}{r}\right)^n\mathrm{d}\tau, \quad s\in\mathbb{R}.$$

As in the proof of Theorem 2.2 we now introduce functions $\Phi_k\colon\mathbb{R}\to\mathbb{R}$, $k\in\mathbb{N}$, defined as in (2.8) but with $\Phi_1=|\phi'|$. This leads to the estimate

$$\left\|\frac{n+1}{t^{n+1}}\int_0^t T(t-s)\phi_r * h_n(s)\,\mathrm{d}s\right\|\lesssim r\|x\|P_n(rt), \tag{2.14}$$

where the implicit constant is independent of r, n, t and x, and where P_n is as in the proof of Theorem 2.2.

Next we observe that by an argument analogous to that in the proof of Theorem 2.2, and using the assumption that $\sup_{|s|\geq 1}\|(is-A)^{-1}\|<\infty$, we find that

$$(\delta-\phi_r)*h_n=\mathcal{F}^{-1}\big((1-\psi_r)m_n\,\mathcal{F}h\big),$$

where $m_n(s)=n!A(is-A)^{-n}$, $s\in\mathbb{R}\setminus\{0\}$, and $h(s)=g_0(s)T(s)(1-A)^{-1}x$, $s\in\mathbb{R}$. Using the fact that $M(s)\geq s$, $s\geq 1$, it is straightforward to show that $\|A(is-A)^{-n}\|\leq 2|s|M(|s|^{-1})^n$, $0<|s|\leq 1$. Recall that $\|(is-A)^{-1}\|\leq M(1)$, $|s|\geq 1$. Since M is assumed to have positive increase it follows as before that for an appropriate choice of n we have

$$\|(1-\psi_r)m_n\|_{L^\infty(\mathbb{R},\mathcal{L}(X))}\lesssim r\left(\frac{M(r^{-1})}{c}\right)^n,$$

where $c>0$ is a constant. We deduce, upon applying Plancherel's theorem and Hölder's inequality, that

$$\left\|\frac{n+1}{t^{n+1}}\int_0^t T(t-s)(\delta-\phi_r)*h_n(s)\,\mathrm{d}s\right\|\lesssim r\|x\|\left(\frac{M(r^{-1})}{ct}\right)^n,$$

where the implicit constant is independent of r, t and x. Combining this with (2.14) as in the proof of Theorem 2.2 gives

$$\|T(t)A(1-A)^{-1}\|\lesssim r\left(P_n(rt)+\left(\frac{M(r^{-1})}{ct}\right)^n\right),$$

where the implicit constant is independent of both r and t. For $t\geq M(1)$ we now set $r=M^{-1}(ct)^{-1}$. Then in particular $rt\geq c^{-1}$, and since P_n is non-increasing the result follows from Lemma A.3. □

2.2.3. Singularity at zero and infinity.

THEOREM 2.5. *Let T be a bounded C_0-semigroup on a Hilbert space X with generator A. Suppose that $\sigma(A) \cap i\mathbb{R} = \{0\}$ and assume that there exist continuous non-decreasing functions $M_0, M_\infty : [1, \infty) \to (0, \infty)$ such that for all $s \geq 1$*

$$\sup_{1 \geq |\xi| \geq s^{-1}} \left\| (i\xi - A)^{-1} \right\| \leq M_0(s) \text{ and } \sup_{1 \leq |\xi| \leq s} \left\| (i\xi - A)^{-1} \right\| \leq M_\infty(s).$$

Let $M : [1, \infty) \to (0, \infty), M(s) = \max\{M_0(s), M_\infty(s)\}$. Suppose furthermore that M has positive increase. Then

$$\left\| T(t)A(1 - A)^{-2} \right\| \leq O\left(\frac{1}{M^{-1}(t)} \right), \quad t \to \infty.$$

PROOF. The proof follows the same pattern as those of Theorems 2.2 and 2.3, and indeed combines ideas from both proofs. This time the splitting arises from the decomposition

$$\delta = (\delta - \phi_R) + (\phi_R - \phi) + (\phi - \varphi_r) + \varphi_r,$$

where $r \in (0, 1]$, $R > 0$ and the notation is as before, with ϕ being the same as in the proof of Theorem 2.2 and φ being the function arising in the proof of Theorem 2.3. The integrals corresponding to the first two terms of the splitting can now be dealt with as in the proof of Theorem 2.2, the terms arising from the second two as in the proof of Theorem 2.3. □

2.3. Necessity of $M \in \mathrm{PI}$

In some cases one can show that the spectrum determines the resolvent growth along the imaginary axis. This is for example the case if the generator is normal, it is the case for the damped wave equation discussed in Chapter 4 and at least for the 1D case of the wave equation discussed in Chapter 5. We show now that in this situation $M \in \mathrm{PI}$ is necessary for an M-inverse theorem to hold. In particular this shows $\mathcal{M} \subseteq \mathrm{PI}$ in the terminology of the introduction.

2.3.1. Singularity at infinity.

THEOREM 2.6. *Let T be a bounded C_0-semigroup on a Banach space with generator A and $\sigma(A) \cap i\mathbb{R} = \emptyset$. Let $\delta \in (0, 1]$ and $M : \mathbb{R}_+ \to (0, \infty)$ be an increasing unbounded function such that for all $s > 0$*

$$\delta M(s) \leq \sup_{|\xi| \leq s} \operatorname{dist}(i\xi, \sigma(A))^{-1} \leq \sup_{|\xi| \leq s} \left\| (i\xi - A)^{-1} \right\| \leq M(s). \tag{2.15}$$

Assume that there exists a constant $c > 0$ such that

$$\left\| T(t)A^{-1} \right\| = O\left(\frac{1}{M^{-1}(ct)} \right), \quad t \to \infty. \tag{2.16}$$

Then $M \in \mathrm{PI}$.

REMARK 2.7. If A is a normal operator on a Hilbert space, then (2.15) is satisfied for $\delta = 1$ if M is defined by an equality in the rightmost inequality.

PROOF. Consider the function $M_{\mathrm{spec}} : \mathbb{R}_+ \to (0, \infty)$ given by

$$M_{\mathrm{spec}}(s) = \sup_{|\xi| \leq s} \operatorname{dist}(i\xi, \sigma(A))^{-1}, \quad s \geq 0.$$

Then $\delta M(s) \le M_{\mathrm{spec}}(s) \le M(s)$, $s \ge 0$. Recall that the spectral radius of a bounded linear operator is always dominated by the norm of the operator. Hence by (2.16) and the spectral inclusion theorem for the Hille-Phillips functional calculus (see [25, Section 2.7.1] or [27, Theorem 16.3.5]) there exists a constant $C > 0$ such that if $\alpha + i\beta \in \sigma(A)$ then

$$\frac{e^{\alpha t}}{|\alpha + i\beta|} \le \|T(t)A^{-1}\| \le \frac{C}{M^{-1}(ct)}$$

for all sufficiently large t. It follows that

$$-\alpha t \ge \log\left(\frac{M_{\mathrm{spec}}^{-1}(\delta ct)}{C|\alpha + i\beta|}\right) \tag{2.17}$$

whenever $\alpha + i\beta \in \sigma(A)$ and $t > 0$ is sufficiently large. Now given $s \ge 0$ we may find $\xi \in [-s, s]$ and $\alpha + i\beta \in \sigma(A)$ such that $M_{\mathrm{spec}}(s) = |\alpha + i\beta - i\xi|^{-1}$. Note that $-\alpha \le M_{\mathrm{spec}}(s)^{-1}$ and that, for s sufficiently large, we have $|\alpha + i\beta| \le 2s$. In fact, one could replace the factor 2 by $1 + \varepsilon$ for any $\varepsilon > 0$ here. Let $\lambda \ge 1$ and, for s sufficiently large, let $t = (\delta c)^{-1} M_{\mathrm{spec}}(\lambda s)$. Then (2.17) yields

$$\frac{M_{\mathrm{spec}}(\lambda s)}{M_{\mathrm{spec}}(s)} \ge \delta c \log\left(\frac{\lambda}{2C}\right),$$

and replacing δ by δ^2 we see that the same estimate holds with M_{spec} replaced by M. Hence M has positive increase by Lemma A.1, as required. ☐

2.3.2. Singularity at zero.

THEOREM 2.8. *Let T be a bounded C_0-semigroup on a Banach space with generator A and $\sigma(A) \cap i\mathbb{R} = \{0\}$. Let $\delta \in (0, 1]$ and $M : [1, \infty) \to (0, \infty)$ be an increasing unbounded function such that for all $s \ge 1$*

$$\delta M(s) \le \sup_{|\xi| \ge s^{-1}} \operatorname{dist}(i\xi, \sigma(A))^{-1} \le \sup_{|\xi| \ge s^{-1}} \left\|(i\xi - A)^{-1}\right\| \le M(s).$$

Assume that there exists a constant $c > 0$ such that

$$\left\|T(t)A(1 - A)^{-1}\right\| = O\left(\frac{1}{M^{-1}(ct)}\right), \quad t \to \infty.$$

Then $M \in \mathrm{PI}$.

We omit the proof of this theorem since it is almost identical to the proof of Theorem 2.6. Essentially the only thing which changes in the proof is that the role of $|\alpha + i\beta|$, which is large in the proof of Theorem 2.6, is replaced by $|\alpha + i\beta|^{-1}$, which is large in the proof of Theorem 2.8.

2.3.3. Singularity at zero and infinity.

THEOREM 2.9. *Let T be a bounded C_0-semigroup on a Banach space with generator A and $\sigma(A) \cap i\mathbb{R} = \{0\}$. Let $\delta \in (0, 1]$ and $M_0, M_\infty : [1, \infty) \to (0, \infty)$ be non-decreasing functions such that for all $s \ge 1$*

$$\delta M_0(s) \le \sup_{s^{-1} \le |\xi| \le 1} \operatorname{dist}(i\xi, \sigma(A))^{-1} \le \sup_{s^{-1} \le |\xi| \le 1} \left\|(i\xi - A)^{-1}\right\| \le M_0(s),$$

$$\delta M_\infty(s) \le \sup_{1 \le |\xi| \le s} \operatorname{dist}(i\xi, \sigma(A))^{-1} \le \sup_{1 \le |\xi| \le s} \left\|(i\xi - A)^{-1}\right\| \le M_\infty(s).$$

Let $M : [1,\infty) \to (0,\infty), M(s) = \max\{M_0(s), M_\infty(s)\}$. Assume that M is unbounded and that there exists a constant $c > 0$ such that

$$\left\|T(t)A(1+A)^{-2}\right\| = O\left(\frac{1}{M^{-1}(ct)}\right), \quad t \to \infty.$$

Then $M \in \mathrm{PI}$.

We omit also this proof which is essentially a combination of the arguments from the proofs of Theorem 2.6 and Theorem 2.8.

2.4. On super-polynomial decay rates

In this last section of Chapter 2 we want to give a first investigation of the situation when M grows at a sub-polynomial rate. We naturally restrict to the case of a singularity at infinity, i.e. $\sigma(A) \cap i\mathbb{R} = \emptyset$ since otherwise a singularity at zero would force M to increase at least like s for large s. When M grows at a sub-polynomial rate then $M \notin \mathrm{PI}$ and thus our results from the previous sections tell us that the decay rate is strictly slower than $1/M^{-1}(ct)$ for any $c > 0$. It is known that in some cases, also in Hilbert spaces the $M_{\log}$-theorem is optimal with respect to the "form" of the decay rate. In fact, one can construct a bounded normal semigroup with $\sigma(A) \cap i\mathbb{R} = \emptyset$ for which one can choose

$$\log(s) \leq M(s) \leq \log(1+s) \text{ for } s \geq 2$$

and thus $M^{-1}(ct) \sim e^{-ct}$ but

$$\left\|T(t)A^{-1}\right\| = e^{-2\sqrt{t}} \sim \frac{1}{M_{\log}^{-1}(4t)}.$$

See [10, Example 5.2]. This example shows that we need at least some condition on M to get a faster decay rate than the one given by the $M_{\log}$-theorem. Note that even if the decay rate is given by $1/M_{\log}^{-1}(ct)$ for some $c > 0$ the precise rate is heavily influenced by $c > 0$ if M grows at a sub-polynomial rate, as is the case for $M = \log$.

2.4.1. A generalization of Theorem 2.2.

THEOREM 2.10. *Let X be a Hilbert space and let A be the generator of a bounded C_0-semigroup T on X with $\sigma(A) \cap i\mathbb{R} = \emptyset$. Let $M, N : \mathbb{R}_+ \to (0,\infty)$ be continuous non-decreasing functions and suppose that $M(s) \to \infty$ as $s \to \infty$, and that M has quasi-positive increase with auxiliary function N. Suppose further that $\|(is - A)^{-1}\| \leq M(|s|)$, $s \in \mathbb{R}$. Then*

$$\|T(t)A^{-1}\| = O\left(\frac{1}{M_K^{-1}(bet)}\right), \quad t \to \infty, \tag{2.18}$$

where b is as in (2.3) and where $K : \mathbb{R}_+ \to (0,\infty)$ is the function defined by $K(s) = N(s) + 3\log(N(s))/2$, $s \geq 0$. In particular, given any $\varepsilon \in (0,1)$ we have

$$\|T(t)A^{-1}\| = O\left(\frac{1}{M_N^{-1}(be(1-\varepsilon)t))}\right), \quad t \to \infty. \tag{2.19}$$

REMARK 2.11. Note that for a given $\varepsilon \in (0,1)$ the fastest rate in (2.19) is attained by choosing an optimal auxiliary function N (whenever it exists) with respect to M. The same is true for b-minimal auxiliary functions if b is fixed. See Section A.3 for the definition of minimality and optimality.

Remark 2.12. Comparing with Theorem 0.1 one might wonder if it is always possible to find an auxiliary function satisfying $N(s) = O(\log(s)), s \to \infty$. Indeed this is the case, as shown in Section A.3.

Proof. If N is bounded then M has positive increase and the result follows from Theorem 2.2, so we may assume that $N(s) \to \infty$ as $s \to \infty$. Moreover, by Remark 2.12 we may assume that $N(s) = O(\log(s)), s \to \infty$. Let us first prove (2.18). We use the same notation as in the proof of Theorem 2.2 and proceed in exactly the same way except that we now choose $n = \lceil N(R) \rceil$. Note that by Stirling's formula

$$(n+1)! \approx e^{-n} n^{n+\frac{3}{2}} \sim \left(1 + \frac{3\log(n)}{2n}\right)^n e^{-n} n^n$$

as $n \to \infty$. Hence 2.13 implies that if R is sufficiently large and if $t > 0$ then

$$\|T(t)A^{-1}\| \lesssim \frac{1}{R}\left(P_n(Rt) + \frac{n+1}{t}P_{n-1}(Rt) + \left(\frac{M_K(R)}{bet}\right)^n\right), \tag{2.20}$$

where the implicit constant is independent of both R and t. We now set $R = M_K^{-1}(bet)$ for t sufficiently large. Thus 2.18 follows provided the first two terms inside the brackets remain uniformly bounded as $t \to \infty$. By the Denjoy-Carleman theorem [28, Theorem 1.3.8] we may assume that the function ψ in addition to the properties already mentioned satisfies $\|\psi^{(k)}\|_{L^\infty} \leq C_k$, where $C_k = C^k k^{2k}$, $k \in \mathbb{N}_0$, for some constant $C > 0$. Integrating by parts we then find that $|\phi(s)| \lesssim C_k(1+|s|)^{-k}$ for all $k \in \mathbb{N}_0$ and $s \in \mathbb{R}$, and hence $\|\Phi_k\|_{L^1} \lesssim C_{k+2}$ for all $k \in \mathbb{N}_0$. Using (2.9) and estimating crudely we thus find, after adjusting the value of the constant C, that for $t \geq 1$ we have

$$P_n(Rt) \lesssim C_3 + \sum_{k=1}^{\infty} R^{-1}\big(C(N(R)+3)^3\big)^{k+3}, \tag{2.21}$$

where the implicit constant is independent of t and hence of R. Since N grows at most logarithmically, we deduce that $P_n(Rt)$ is uniformly bounded as $t \to \infty$. Moreover, since $N(R) \lesssim t$ we see similarly that the second term in (2.20) remains bounded as t grows large. This completes the proof of (2.18). In order to obtain (2.19) it suffices to observe that $M_K(s) \leq (1-\varepsilon)^{-1} M_N(s)$ for all sufficiently large values of s. □

Example 2.13. Let $\alpha \in (0, \infty)$ and define $M(s) = \log(s)^\alpha$ for $s \geq e$ and $M(s) = 1$ for $s \in [0, e)$. Let us restrict Theorem 2.10 to this particular choice of M. In view of Remark 2.11 it is reasonable to search for an optimal auxiliary function. By Example A.4 an optimal auxiliary function is given by

$$N(s) = \begin{cases} (1+\alpha)^{-1}\log(s) & \text{for } s \geq e, \\ (1+\alpha)^{-1} & \text{for } s \in [0, e). \end{cases}$$

This function is $e^{-1}(1+\alpha^{-1})^\alpha$-minimal. With this optimal choice of N equation (2.19) says that for any $\varepsilon \in (0,1)$

$$\|T(t)A^{-1}\| = O\Big(\exp\big(-(c_\alpha(1-\varepsilon)t)^{\frac{1}{\alpha+1}}\big)\Big), \quad t \to \infty, \tag{2.22}$$

$$\text{where } c_\alpha = (1+\alpha)\left(\frac{1+\alpha}{\alpha}\right)^\alpha.$$

EXAMPLE 2.14. Let $\alpha \in (0,1)$ and define $M(s) = \exp(\log(s)^\alpha)$ for $s \geq 1$ and $M(s) = 1$ for $s \in [0,1)$. By Example A.5 an optimal auxiliary function is given by

$$N(s) = \begin{cases} \alpha^{-1} \log(s)^{1-\alpha} & \text{for } s \geq 1, \\ \alpha^{-1} & \text{for } s \in [0,1). \end{cases}$$

This function is 1-minimal. Clearly M_N increases considerably slower than $M_{\log}$. Therefore Theorem 2.10 yields a faster decay rate then Theorem 0.1 (for any choice of c in the latter theorem).

2.4.2. Sharp decay rates for quasi-multiplication semigroups. Following [10] we say that a C_0-semigroup T with generator A on a Banach space X is a *quasi-multiplication semigroup* if

$$\|T(t)r(A)\| = \sup_{z \in \sigma(A)} |e^{tz} r(z)|, \quad t \geq 0,$$

for every rational function r whose poles lie outside $\sigma(A)$ and which is bounded at infinity. It follows from the spectral theorem that any C_0-semigroup of normal operators is a quasi-multiplication semigroup, but the class also contains multiplication semigroups on non-Hilbertian function spaces. Our next result describes the exact rate of decay for quasi-multiplication semigroups with arbitrary resolvent growth. The proof is an extension of the ideas used in Theorem 2.6; see also [10, Proposition 5.1]. Recall that the spectral bound $\sigma(A)$ of a semigroup generator A is defined as $s(A) = \sup_{z \in \sigma(A)} \Re z$.

THEOREM 2.15. *Let X be a Banach space and let A be the generator of a quasi-multiplication semigroup T on X. Suppose that $s(A) = 0$ but $\sigma(A) \cap i\mathbb{R} = \emptyset$, and let $M \colon \mathbb{R}_+ \to (0,\infty)$ be defined by $M(s) = \sup_{|\xi| \leq s} \|(i\xi - A)^{-1}\|$, $s \geq 0$. Then*

$$\|T(t)A^{-1}\| \sim \frac{1}{M_{\mathrm{qm}}^{-1}(t)}, \quad t \to \infty, \tag{2.23}$$

where $M_{\mathrm{qm}} \colon [1,\infty) \to \mathbb{R}_+$ is defined by

$$M_{\mathrm{qm}}(s) = \max_{1 \leq \lambda \leq s} M(\lambda^{-1} s) \log \lambda, \quad s \geq 1. \tag{2.24}$$

PROOF. Since T is a quasi-multiplication semigroup we have

$$\|T(t)A^{-1}\| = \sup_{z \in \sigma(A)} \frac{e^{t\Re z}}{|z|}, \quad t \geq 0, \tag{2.25}$$

and also $M(s) = \sup_{|r| \leq s} \operatorname{dist}(ir, \sigma(A))^{-1}$, $s \geq 0$. In particular, $M(s) \to \infty$ as $s \to \infty$ since $s(A) = 0$. Now if $z \in \sigma(A)$ then $-\Re z \geq M(|\Im z|)^{-1}$, so

$$\|T(t)A^{-1}\| \leq \sup_{z \in \sigma(A)} \frac{1}{|z|} \exp\left(-\frac{t}{M(|\Im z|)}\right), \quad t \geq 0.$$

Since M is unbounded one may assume, by choosing t to be sufficiently large, that the supremum is unaffected by restricting consideration to points $z \in \sigma(A)$ satisfying $|\Im z| \geq 1$. Thus

$$\|T(t)A^{-1}\| \leq \sup_{s \geq 1} \frac{1}{s} \exp\left(-\frac{t}{M(s)}\right)$$

for all sufficiently large t. Given $t \geq M_{\mathrm{qm}}(1)$ let $R = M_{\mathrm{qm}}^{-1}(t)$. Then for $s \geq R$ we have $s^{-1}\exp(-tM(s)^{-1}) \leq R^{-1}$, while for $1 \leq s \leq R$ the definition of M_{qm} implies that $M_{\text{qm}}(R) \geq M(s)\log(R/s)$ and hence again $s^{-1}\exp(-tM(s)^{-1}) \leq R^{-1}$. Thus $\|T(t)A^{-1}\| \leq 1/M_{\text{qm}}^{-1}(t)$ for all sufficiently large values of t.

Now let $\varepsilon \in (0,1)$ and consider the function $K\colon \mathbb{R}_+ \to (0,\infty)$ defined by

$$K(t) = \frac{1-\varepsilon}{\|T(t)A^{-1}\|}, \quad t \geq 0.$$

Note that, by (2.25), the function K is continuous and strictly increasing. Arguing as in the proof of Theorem 2.6 we see that for sufficiently large values of s we may find $\alpha + i\beta \in \sigma(A)$ such that $-\alpha \leq M(s)^{-1}$ and $|\alpha + i\beta| < s(1-\varepsilon)^{-1}$. It then follows as before from (2.17) with N^{-1} replaced by K, and with the choices $c = \delta = 1$ and $C = 1-\varepsilon$, that there exists a constant $s_0 > 0$ such that $K^{-1}(\lambda s) \geq M(s)\log\lambda$ for all $\lambda \geq 1$ and all $s \geq s_0$. Thus $K^{-1}(s) \geq M(\lambda^{-1}s)\log\lambda$, $1 \leq \lambda \leq s/s_0$, whenever $s \geq s_0$. Using the fact that M is unbounded, it is straightforward to see that

$$M_{\text{qm}}(s) = \max_{1\leq\lambda\leq s/s_0} M(\lambda^{-1}s)\log\lambda$$

and hence $K^{-1}(s) \geq M_{\text{qm}}(s)$ for all sufficiently large values of $s \geq s_0$. Thus when t is sufficiently large we have $M_{\text{qm}}^{-1}(t) \geq K(t)$, and consequently

$$\|T(t)A^{-1}\| \geq \frac{1-\varepsilon}{M_{\text{qm}}^{-1}(t)}.$$

This completes the proof. □

Theorem 2.15 becomes false if we drop the assumption that $s(A) = 0$. For instance, if we let A be the generator of a quasi-multiplication semigroup with spectrum $s(A) = \{i - s : s \geq 1\}$, then $\|T(t)A^{-1}\| = 2^{-1/2}e^{-t}$ but $M_{\text{qm}}^{-1}(t)^{-1} = e^{-t}$, $t \geq 0$. Similarly, if $\sigma(A) = i\mathbb{R} - 1/2$ then $\|T(t)A^{-1}\| = 2e^{-t/2}$ but $M_{\text{qm}}^{-1}(t)^{-1} = e^{-t/2}$, $t \geq 0$.

2.4.3. On optimality of Theorem 2.10. We can use Theorem 2.15 to investigate the quality of the estimates in (2.18) and (2.19).

Example 2.16. Let M be the function from Example 2.13 and let N be the optimal auxiliary function given in that example. It is easy to show that

$$M_{\text{qm}}(s) = \frac{1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\log(s)^{\alpha+1}, \quad s > 1.$$

This shows that for normal semigroups (2.22) is sharp in terms of c_α. In particular (2.19) is sharp up to the ε-loss - in general. Using the finer estimate (2.18) we can even show that for all semigroups which satisfy the hypotheses of Theorem 2.10

$$\|T(t)A^{-1}\| \lesssim \frac{t^{\frac{3}{2(1+\alpha)}}}{M_{\text{qm}}^{-1}(t)}.$$

The question arises if the factor $t^{3/2(1+\alpha)}$ is really necessary here. Unfortunately, by optimality of N this factor cannot be avoided by the direct use of Theorem 2.10. We think it is an interesting question if this factor can be avoided in general or if it is actually necessary.

EXAMPLE 2.17. Let M be the function from Example 2.14 and let N be the optimal auxiliary function given in that example. A tedious but not very difficult calculation yields for $s > 1$ with $L(s) = \log(s)^{-\alpha}$

$$\begin{aligned} M_{\mathrm{qm}}(s) &= \frac{1}{\alpha e}\left(1 - \left(\frac{1-\alpha}{\alpha}\right)^2 L(s)^2 + O\left(L(s)^3\right)\right)\exp(\log(s)^\alpha)\log(s)^{1-\alpha} \\ &= e^{-1}(1+o(1))M_N(s), \quad s \to \infty. \end{aligned}$$

Comparing with Example 2.14, we see that Theorems 2.15 and 2.10 yield the same rate of decay up to the ε-loss in (2.19). Observe that the function L can be replaced by zero without affecting the asymptotic behaviour of M_{qm}^{-1}.

In the *context of these two examples* we observe that Theorem 2.10 is sharp in the sense that (2.19) becomes false (in general) if $be(1-\varepsilon)$ would be replaced by any number strictly larger than be. In Example 2.16 we see that at least for normal semigroups the optimal decay rate is, up to a polynomial factor, given by (2.18) (for N optimal). A similar observation can be made for the function M from Example 2.14, where the correction factor is now a polynomial in $\log(t)$. Unfortunately we are not able to generalize these observations in a systematic way. But we think that it is reasonable to say that our Theorem 2.10 is a "seemingly almost" sharp result. On the other hand we think that there is a need for an improved version of that theorem with a possibly simplified theory behind possibly not relying on quasi-positive increase.

To open up an interesting question for future research we want to formulate an (in our opinion) rather optimistic conjecture. This conjecture can be summarized informally as: *Normal semigroups yield the worst decay rates.*

CONJECTURE 2.18. *Let X be a Hilbert space and let A be the generator of a bounded C_0-semigroup T on X with $\sigma(A) \cap i\mathbb{R} = \emptyset$. Let $M\colon \mathbb{R}_+ \to (0,\infty)$ be a continuous non-decreasing functions such that $M(s) \to \infty$, $s \to \infty$. Suppose $\|(is-A)^{-1}\| \le M(|s|)$, $s \in \mathbb{R}$. Then for some $C_0, t_0 > 0$*

$$\|T(t)A^{-1}\| = \frac{C_0}{M_{\mathrm{qm}}^{-1}(t)} \text{ for } t > t_0. \tag{2.26}$$

The constant C_0 can be chosen to depend only on $\sup_{t\ge 0}\|T(t)\|$.

Note that, restricted to normal semigroups $C_0 = 1$ is the optimal choice of the constant in (2.26) as the proof of Theorem 2.15 shows. We think it would be an interesting question to find also optimal constants (if they exist) for other subclasses of C_0-semigroups. In particular, we ask if (2.26) is satisfied with $C_0 = 1$ for contractive semigroups.

Part 2

Applications: decay of waves

CHAPTER 3

Local decay for waves in exterior domains

3.1. Introduction

Let $\Omega \subsetneq \mathbb{R}^d$ be a connected open set with bounded complement and non-empty C^∞-boundary. The dimension d is assumed to be at least 2. We consider the wave equation on this domain:

$$\left\{ \begin{array}{ll} u_{tt}(t,x) - \Delta u(t,x) = 0 & (t \in (0,\infty), x \in \Omega), \\ u(t,x) = 0 & (t \in (0,\infty), x \in \partial\Omega), \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) & (x \in \Omega). \end{array} \right. \tag{3.1}$$

Let us fix a radius $\rho > 0$ such that the obstacle $\mathcal{O} = \mathbb{R}^d \backslash \Omega$ is included in the open ball B_ρ of radius ρ and center 0. We define a *state* (at time t) of the system by $\mathbf{x}(t) := (u,v)(t) := (u(t), u_t(t))$. We define the local energy of a state by

$$E^{\mathrm{loc}}(\mathbf{x}) = \int_{\Omega \cap B_\rho} |\nabla u|^2 + |v|^2 \, dx. \tag{3.2}$$

Clearly, equation (3.2) is well defined for all $u \in C_c^\infty(\Omega)$ and $v \in L^2(\Omega)$. Therefore, it is also well defined on the *energy space*

$$\mathcal{H} = H_D^1(\Omega) \times L^2(\Omega),$$

where $H_D^1(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the norm $u \mapsto (\int_\Omega |\nabla u|^2)^{1/2}$. We remark at this point that for any compactly supported C^∞-function $\chi : \mathbb{R}^d \to \mathbb{C}$ the corresponding multiplication operator $f \mapsto \chi f$ is continuous from $H_D^1(\Omega)$ to $H_D^1(\Omega)$ and $L^2(\Omega)$. This is not completely obvious since $H_D^1(\Omega)$ is not a subspace of $L^2(\Omega)$ and actually the statement would be false if $\partial\Omega = \emptyset$. Fortunately we have assumed $\partial\Omega \neq \emptyset, \partial\Omega \in C^\infty$ and therefore the statement follows from the Poincaré-Steklov inequality applied to the open set $\Omega \cap B_r$ where the radius $r > 0$ is chosen so large that $\Omega \cap B_r \neq \emptyset$ is connected and the support of χ is contained in B_r.

Let $m \in \mathbb{N}_0$. We are interested in the uniform decay rate of the local energy with respect to sufficiently smooth initial data, compactly supported in the ball of radius ρ:

$$p_m(t) := \sup \left\{ \left(\frac{E^{\mathrm{loc}}(\mathbf{x}(t))}{\|\mathbf{x}_0\|^2_{H^{m+1} \times H^m}} \right)^{\frac{1}{2}} ; \mathbf{x}_0 \in H^{m+1}_{\mathrm{comp}} \times H^m_{\mathrm{comp}}(\Omega \cap B_\rho) \right\}. \tag{3.3}$$

Here, by $H^m_{\mathrm{comp}}(\Omega \cap B_\rho)$ we denote all square-integrable functions, compactly supported on $\Omega \cap B_\rho$ for which all weak derivatives up to order m are square-integrable too. We also write $L^2_{\mathrm{comp}} = H^0_{\mathrm{comp}}$. It is well known that p_0 either does not decay to zero, or decays exponentially for d odd and like t^{-d} for d even. Moreover, the decay can be characterized by boundedness of the local resolvent of $\mathcal{A}$ on the imaginary axis. We refer to [51] and references therein for these facts.

3.2. The associated unitary C_0-group, its generator and basic properties of the truncated outgoing resolvent

The wave equation (3.1) on the energy space $\mathcal{H}$ can be reformulated in the language of C_0-semigroups. Therefore, as above, we set $\mathbf{x}(t) = (u(t), u_t(t))$, $\mathbf{x}_0 = (u_0, u_1)$ and write

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t), \\ \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{H} \end{cases} \text{where } \mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \tag{3.4}$$

$$\text{with } D(\mathcal{A}) = D_\Delta \times (\underbrace{H_D^1 \cap L^2}_{H_0^1})(\Omega).$$

Here $D_\Delta = \{u \in H_D^1(\Omega); \Delta u \in L^2(\Omega)\}$, where Δ denotes the Laplace operator in the sense of distributions. It can be proved that the wave operator $\mathcal{A}$ is skew-adjoint (see e.g. [32, Theorem V.1.2]). Therefore the following theorem follows by Stone's theorem (see e.g. [32, Appendix 1, Theorem 2]).

THEOREM 3.1. *The wave operator $\mathcal{A}$ generates a unitary C_0-group on $\mathcal{H}$.*

In the following we investigate the resolvent of $\mathcal{A}$ to get decay rates p_m for the local energy. In the literature on local energy decay it is common to investigate the *outgoing resolvent* of the stationary wave equation. For $\Re z > 0$ and $f \in L^2(\Omega)$ the outgoing resolvent is defined as the Laplace transform

$$R(z)f = \int_0^\infty e^{-zt} u(t) dt$$

where u is the first component of the solution to (3.4) for $\mathbf{x}_0 = (0, f) \in \mathcal{H}$. Taking the Laplace transform of (3.4) it is not difficult to show that $w = R(z)f$ for $\Re z > 0$ and $f \in L^2(\Omega)$ is the unique distributional solution in $L^2(\Omega)$ to the stationary wave equation

$$\begin{cases} z^2 w(x) - \Delta w(x) = f(x) & (x \in \Omega), \\ w(x) = 0 & (x \in \partial\Omega). \end{cases} \tag{3.5}$$

That is, $R(z) = (z^2 - \Delta_0)^{-1}$ where by Δ_0 we denote the Dirichlet-Laplace operator with domain $D(\Delta_0) = \{u \in H_0^1(\Omega); \Delta u \in L^2(\Omega)\}$. We emphasize that $D(\Delta_0) \neq D_\Delta$. There is an important relation between R and the resolvent of $\mathcal{A}$: For $\Re z > 0$ we have

$$(z - \mathcal{A})^{-1} = \begin{pmatrix} zR(z) & R(z) \\ z^2 R(z) - 1 & zR(z) \end{pmatrix}. \tag{3.6}$$

Let us fix a cut-off function $\chi \in C_c^\infty(\mathbb{R}^d)$ with $0 \leq \chi \leq 1$ such that $\chi = 1$ on a neighbourhood of $\mathcal{O}$. We define the truncated resolvent by $R_\chi(z) = \chi R(z)\chi$, where we consider χ as a multiplication operator on $L^2(\Omega)$. From the definition we see that the outgoing truncated resolvent is an analytic function in the interior of $\mathbb{C}_+$. The next proposition illuminates its behaviour on the other half of the complex plane.

PROPOSITION 3.2. *(i)[14, Appendix B] The truncated outgoing resolvent R_χ extends analytically to a neighbourhood of $i\mathbb{R}\backslash\{0\}$. Moreover, for any open sector $S \supseteq \mathbb{R}_-$ with vertex at 0 the operator $R_\chi(z) : L^2(\Omega) \to L^2(\Omega)$ is uniformly bounded for z in a small neighbourhood of 0 outside the sector S. (ii)[32, Corollary V.3.3* *together with Remark V.4.3] If the dimension $d \geq 3$ is odd, R_χ extends meromorphically to $\mathbb{C}$. (iii)[51, Proposition 3.1] If the dimension $d \geq 2$ is even, then R_χ extends meromorphically to $\mathbb{C}\backslash\mathbb{R}_-$ and there exists a rank one operator R_0 such that*

$$z \mapsto R_\chi(z) - R_0 z^{d-2}\log(z) \text{ is analytic}$$

in a neighbourhood of 0.

Since the spectrum of Δ_0 is $(-\infty, 0]$ the (maximal) domain of analyticity of the operator R is the interior of $\mathbb{C}_+$. In particular, R does not extend across the imaginary axis if we consider it as an operator on $L^2(\Omega)$. However, if we consider it as an operator $R(z) : L^2_{\text{comp}}(\overline{\Omega}) \to L^2_{\text{loc}}(\Omega)$, then the above proposition says that this operator does extend across the imaginary axis. Moreover, if $f \in L^2_{\text{comp}}(\overline{\Omega})$ and $z \in \mathbb{C}$ is such that $R(z)$ is defined, the function $w = R(z)f \in L^2_{\text{loc}}(\Omega)$ is a solution to (3.5). For $\Re z < 0$ the function w thus defined is not necessarily in $L^2(\Omega)$ and in particular it need not be the unique L^2-solution of (3.5). In other words, $R_\chi(z) \neq \chi(z^2 - \Delta_0)^{-1}\chi$ if $\Re z < 0$.

Let us define the analytic function $G_\chi : \mathbb{C}_+\backslash i\mathbb{R} \to \mathcal{L}(L^2(\Omega))$ by

$$G_\chi(z) = \chi(z - \mathcal{A})^{-1}\chi.$$

Here, we consider χ as an operator on $\mathcal{H}$ acting as $\chi(u_0, u_1) = (\chi u_0, \chi u_1)$. In case $d \geq 3$ is odd, by Proposition 3.2 together with (3.6), we immediately see that G extends to a meromorphic function on $\mathbb{C}$ which has no poles on $i\mathbb{R}$. If $d \geq 2$ is even, then G_χ extends to a meromorphic function on $\mathbb{C}_+\backslash\mathbb{R}_-$. Moreover, by Proposition 3.2(iii) together with (3.6) there exists a finite rank operator P_0 such that

$$z \mapsto G_\chi(z) - P_0 z^{d-1}\log(z) \text{ is analytic} \tag{3.7}$$

in a small ball around 0. Since the spectrum of $\mathcal{A}$ is the entire imaginary axis (this follows from $\sigma(\Delta_0) = (-\infty, 0]$) the equality $G_\chi(z) = \chi(z - \mathcal{A})^{-1}\chi$ *does not hold* for $\Re z < 0$ in general.

The following proposition seems to be well-known. Unfortunately we could not find a complete proof in the literature. Therefore we give a proof in Section B.2.

Proposition 3.3. *Let $\delta > 0$ and let $\tilde{\chi}$ be defined as χ but with $\tilde{\chi} = 1$ on a neighbourhood of the support of χ. Let z with $-\delta < \Re z < 0$ be no pole of R_χ, then*

$$\|G_\chi(z)\| \leq C\left((1 \vee |z|)^{-1} + |z|\,\|R_{\tilde{\chi}}(z)\|_{L^2\to L^2}\right)$$

with a constant $C > 0$ independent of z. The reverse inequality - with a different constant, ignoring the first summand on the right hand side and $\tilde{\chi}$ replaced by χ - is also true.

3.3. Decay of the local energy

It can happen that a whole strip $\{z \in \mathbb{C}; -\delta < \Re z < 0\}$ is free of poles of G_χ - see for instance [29]. In [21] the impact of the presence of such a strip on local energy decay was studied. There it was shown in a first step that such a strip implies that the norm of G_χ can be estimated by $C\exp(C\,|\Im(z)|^\alpha)$ for large z on this strip, and for some $\alpha > 0$. Indeed $\alpha = d - 1$ in this article but it was not shown that this is optimal. In a second step the authors showed that this implies a bound of the form $(1 + |\Im z|)^\alpha$ on G_χ for large arguments in a region of the form $\{z \in \mathbb{C}; -C(1 + |\Im z|)^{-\alpha} < \Re z < 0\}$. This step is rather abstract and relies only on the fact that G_χ is a truncated resolvent of a bounded C_0-semigroup. Finally, in a third step they applied a Tauberian theorem (more precisely, [40, Proposition 1.4]) to get, for d odd, a $(\log(t)/t)^{m/\alpha}$ decay rate for the local energy. If d is even one gets a $t^{-d} \vee (\log(t)/t)^{m/\alpha}$ decay rate.

In the following we get rid of the logarithmic term, and simplify the proof compared to [21], by using a single application of Corollary 1.34 to the local resolvent on a strip. To present a more general result we consider the following conditions.

(a) There is a continuous and non-decreasing function $M : \mathbb{R}_+ \to (0, \infty)$ such that R_χ has no poles in Ω_M.
(b) There is a real number $r > 0$ and a continuous and non-decreasing function $K : \mathbb{R}_+ \to [2, \infty)$ satisfying $K(s) \geq c \max\{s, M(s)\}$ for any $s \geq 0$ such that
$$|\Im z| \, \|R_\chi(z)\|_{L^2 \to L^2} \leq CK(|\Im z|)$$
for all $z \in \Omega_M$ with $|\Im z| \geq r/2$.
(c) If d is even we assume furthermore that the number r from condition (b) is chosen so small that (3.7) is true for all z in a ball of radius r around 0.

Under these assumptions we can prove

Theorem 3.4. *Let $m \in \mathbb{N}_1$ and assume that the conditions (a-c) above are satisfied. (i) If $d \geq 3$ is odd, then*
$$p_m(t) \leq \frac{C}{M_{\log K}^{-1}(c_1 t)^m} \text{ for every } t \geq 1$$
and for a sufficiently small constant $c_1 > 0$. (ii) If $d \geq 2$ is even then
$$p_m(t) \leq C \max\left\{\frac{1}{t^d}, \frac{1}{M_{\log K}^{-1}(c_1 t)^m}\right\} \text{ for every } t \geq 1$$
and for a sufficiently small constant $c_1 > 0$. Here, $M_{\log K}(s) = M(s)\log(K(s))$ for $s \geq 0$.

Proof. (i). For $\Re z > 0$, let $G_\chi(z) = \chi(z - \mathcal{A})^{-1}\chi$. Assumptions (a) and (b) together with Proposition 3.3 imply that G_χ extends analytically to $\Omega_M \cup \overline{\mathbb{C}_+}$ and satisfies
$$\|G_\chi(z)\| \leq CK(|\Im z|) \text{ for } z \in \Omega_M.$$
Thus, by Corollary 1.34, for every $\mathbf{x}_0 \in \mathcal{H}$
$$\left\|\chi e^{t\mathcal{A}}(1 - \mathcal{A})^{-m}\chi \mathbf{x}_0\right\| \leq \frac{C}{M_{\log K}^{-1}(c_1 t)^m} \|\mathbf{x}_0\|. \tag{3.8}$$
By the closed graph theorem the constant C does not depend on $\mathbf{x}_0$. For simplicity we assume $m = 1$ in the following. The general case can be treated in almost the same way.

Let $\chi_1 \in C_c^\infty(\mathbb{R}^d)$ be a function such that $0 \leq \chi_1 \leq 1$ and $\chi_1 = 1$ on $\operatorname{supp} \chi$. Of course, Propositions 3.2 and 3.3 remain valid if one replaces χ by χ_1. Note that the commutator $[\chi, 1 - \mathcal{A}]$ is a bounded operator on $\mathcal{H}$. Let $\mathbf{x}_1 = (1 - \mathcal{A})^{-1}\mathbf{x}_0 \in D(\mathcal{A})$. By Corollary 1.34,
$$\left\|\chi e^{t\mathcal{A}}\chi \mathbf{x}_1\right\| \leq \left\|\chi e^{t\mathcal{A}}(1 - \mathcal{A})^{-1}\chi \mathbf{x}_0\right\| + \left\|\chi(\chi_1 e^{t\mathcal{A}}(1 - \mathcal{A})^{-1}\chi_1)[\chi, (1 - \mathcal{A})]\mathbf{x}_1\right\|$$

$$\leq \frac{C}{M_{\log K}^{-1}(c_1 t)}(\|\mathbf{x}_0\| + \|\mathbf{x}_1\|)$$
$$\leq \frac{C}{M_{\log K}^{-1}(c_1 t)} \|\mathbf{x}_1\|_{D(\mathcal{A})} .$$

Without loss of generality we may assume that $\chi = 1$ on B_ρ. Observe that the norm of elements of $D(\mathcal{A})$, supported in $\overline{\Omega} \cap B_r$, is equivalent to the norm in the space $H^2 \times H^1(\Omega)$. This follows from maximal regularity of the Dirichlet-Laplace operator on the bounded and smooth domain $\Omega \cap B_\rho$. Thus the last inequality (restricted to those $\mathbf{x}_1$ with support in B_ρ) implies the conclusion of the theorem.

(ii) The proof of the second assertion is analogous and uses Corollary 1.37 instead of 1.34. □

Let us go back to the situation described at the beginning of Section 3.3. We assume for simplicity of presentation that d is odd. We see that we can apply the above theorem with $M = \delta$ for some $\delta > 0$ and $K(s) = C \exp(C\, |\Im(z)|^{\alpha})$. Thus we get

$$p_m(t) \leq \frac{C}{t^{\frac{m}{\alpha}}} \text{ for } t \geq 1.$$

So our approach helped to remove the logarithmic loss in this situation.

CHAPTER 4

Waves on a square with constant damping on a strip

4.1. Introduction

Let $\square = (0,1)^2$ be the unit square. We parametrize it by Cartesian coordinates x and y. Let a - the damping - be a function on $\square$ which depends only on x such that $a(x) = a_0 > 0$ for $x < \sigma$ and $a(x) = 0$ for $x > \sigma$ where σ is some fixed number from the interval $(0,1)$. We consider the damped wave equation:

$$\begin{cases} u_{tt}(t,x,y) - \Delta u(t,x,y) + 2a(x)u_t(t,x,y) = 0 & (t \in (0,\infty),\ (x,y) \in \square), \\ u(t,x,y) = 0 & (t \in (0,\infty),\ (x,y) \in \partial\square), \\ u(0,x,y) = u_0(x,y),\ u_t(0,x,y) = u_1(x,y) & ((x,y) \in \square). \end{cases}$$

We are interested in the energy

$$E(t,\mathbf{x}_0) = \frac{1}{2}\int\int |\nabla u(t,x,y)|^2 + |u_t(t,x,y)|^2\, dxdy$$

of a wave at time t with initial data $\mathbf{x}_0 = (u_0, u_1)$. Let $D = (H^2 \cap H_0^1) \times H_0^1(\square)$ denote the set of classical initial data. In this chapter we aim to prove

THEOREM 4.1. *Let $\square$, a and $E(t,\mathbf{x}_0)$ be as above. Then* $\sup E(t,\mathbf{x}_0)^{1/2} \approx t^{-2/3}$ *where the supremum is taken over initial data satisfying* $\|\mathbf{x}_0\|_D = 1$.

In Section 4.3 we show that this theorem is equivalent to Theorem 4.3 below. Section 4.2 is devoted to the proof of Theorem 4.3.

REMARK 4.2. The proof of Theorem 4.1 shows that a higher dimensional analogue is also true. That is, one can replace $y \in \mathbb{R}$ by $y \in \mathbb{R}^{d-1}$ for any natural number $d \geq 2$. The exact decay rate remains the same for all d.

We want to acknowledge that our work on this topic was partly motivated and influenced by a lecture series given by Matthieu Léautaud at the conference "Modern Applications of Operator Theory" in Będlewo (2016) and a paper of Batty, Paunonen and Seifert [9].

4.1.1. The semigroup approach. If we set $\mathbf{x}(t) = (u(t), u_t(t))$ and $\mathbf{x}_0 = (u_0, u_1)$ we may formulate the damped wave equation as an abstract Cauchy problem on the Hilbert space $\mathcal{H} = H_0^1 \times L^2(\square)$:

$$\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t),\ \mathbf{x}(0) = \mathbf{x}_0\,, \text{ where } \mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta & -2a(x) \end{pmatrix}. \tag{4.1}$$

The domain of $\mathcal{A}$ is $D(\mathcal{A}) = (H^2 \cap H_0^1) \times H_0^1(\square)$. The operator $\mathcal{A}$ is dissipative on $\mathcal{H}$ (we equip $H_0^1(\square)$ with the gradient norm), that is

$$\forall (u,v) \in D(\mathcal{A}) : \Re\langle \mathcal{A}(u,v), (u,v)\rangle_{\mathcal{H}} = -2 \int_{\square} a\,|v|^2 \leq 0.$$

Note that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is invertible since $\mathcal{A}(u,v) = (f,g)$ is equivalent to $\Delta u = g - 2af$, $v = f$ and since the Dirichlet-Laplace operator $-\Delta : H^2 \cap H_0^1(\square) \to L^2(\square)$ is invertible. By the Lumer-Phillips theorem (see for example [4, Theorem 3.4.5]) the operator $\mathcal{A}$ generates a C_0-semigroup of contractions. In particular (4.1) is well-posed, i.e. for any $\mathbf{x}_0 \in \mathcal{H}$ there exists a unique *mild* solution $\mathbf{x} \in C([0,\infty);\mathcal{H})$ to (4.1); see for example [4, Chapter 3.1] for the definition of mild solutions. As in [4, Chapter 3.1] we call $\mathbf{x}$ a *classical* solution of (4.1) if it is a mild solution and if in addition $\mathbf{x} \in C^1([0,\infty);\mathcal{H}) \cap C([0,\infty);D(\mathcal{A}))$. By [4, Proposition 3.1.9. h)] a mild solution with initial value in $D(\mathcal{A})$ is already a classical solution.

Note that the inclusion $D(\mathcal{A}) \hookrightarrow \mathcal{H}$ is compact by the Rellich-Kondrachov theorem. Thus the spectrum of $\mathcal{A}$ contains only eigenvalues of finite multiplicity.

4.1.2. Classification of the main result. Our situation is a very particular instance of the so called *partially rectangular* situation. A bounded domain Ω is called *partially rectangular* if its boundary $\partial\Omega$ is piecewise C^∞ and if Ω contains an open rectangle R such that two opposite sides of R are contained in $\partial\Omega$. We call these two opposite sides *horizontal*. One can decompose $\overline{\Omega} = \overline{R} \cup \overline{W}$, where W is an open set which is disjoint to R. In our particular situation we can W choose to be empty. Furthermore it is assumed, that $a > 0$ on $\overline{W}$ and $a = 0$ on S, where $S \subseteq R$ is an open rectangle with two sides contained in the horizontal sides of R. To avoid the discussion of null-sets we assume for simplicity that either a is continuous up to the boundary or it is as in the beginning of the introduction of this chapter.

Under these constraints one can show that the energy of classical solutions can never decay uniformly faster than $1/t^2$, i.e.

$$\sup_{\mathbf{x}_0 \in D(A)} E(t,\mathbf{x}_0)^{\frac{1}{2}} \gtrsim \frac{1}{t}. \tag{4.2}$$

This result seems to be well-known. Unfortunately we do not know an original reference to this bound on the energy. A short modern proof using [8, Proposition 1.3] can be found in [3]. But there is also a *geometric optics* proof using quantified versions of the techniques of [42]. Unfortunately the latter approach seems to be never published anywhere.

On the other hand, if we assume that the damping does not vanish completely in R (this is an additional assumption only if W is empty), then

$$\forall \mathbf{x}_0 \in D(\mathcal{A}) : E(t,\mathbf{x}_0)^{\frac{1}{2}} \lesssim \frac{1}{t^{\frac{1}{2}}}. \tag{4.3}$$

This is a corollary of one of the main results in [3]. There, the authors showed that *stability at rate* $t^{-1/2}$ for an *abstract* damped wave equation is equivalent to an observability condition for a related Schrödinger equation. Earlier contributions towards (4.3) were given by [15] and [34].

Having the two bounds (4.2) and (4.3) at hand a natural question arises: are these bounds sharp? Concerning the fast decay rates related to (4.2) this is partly answered by [15] and [3]. Essentially the authors showed that if the damping function is smooth enough than one can get a decay rate as close to t^{-1} as one wishes. Unfortunately they could not *characterize* the *exact* decay rate in terms of properties of a. A breakthrough into this direction was achieved in [33] (see also [16, 13]) in a slightly different situation (there S degenerates to a line).

To the best of our knowledge it is completely unknown if the slowest possible rate $t^{-1/2}$ is attained. To us the only known result in this direction is due to Nonnenmacher: if we are in the very particular situation of a damped wave equation on a square with constant damping on a strip, parallel to one of the sides of the square, then

$$\sup_{\mathbf{x}_0 \in D(A)} E(t, \mathbf{x}_0)^{\frac{1}{2}} \gtrsim \frac{1}{t^{\frac{2}{3}}};$$

see [3, Appendix B]. So this situation is a candidate for the slow decay rate. In this chapter we show that Nonnenmacher's bound is actually equal to the exact decay rate.

This of course raises a new question. Is it possible to find a non-vanishing bounded damping in a partially rectangular domain, satisfying the constraints specified above, but discarding the continuity assumptions, such that the exact decay rate for $E(t, \mathbf{x}_0)^{\frac{1}{2}}$ is strictly slower than $t^{-2/3}$? We think this is an interesting question for future research.

4.1.3. From waves to stationary waves. Let $f \in L^2(\Box)$. We consider the stationary damped wave equation with Dirichlet boundary conditions

$$\begin{cases} P(s)u(x,y) = (-\Delta - s^2 + 2isa(x))u(x,y) = f(x,y) & \text{in } \Box \\ u(x,y) = 0 & \text{on } \partial\Box \end{cases} \tag{4.4}$$

As already said above, to prove Theorem 4.1 is essentially to show

THEOREM 4.3. *The operator $P(s) : H^2 \cap H_0^1(\Box) \to L^2(\Box)$ from (4.4) is invertible for every $s \in \mathbb{R}$. Moreover*

$$\left\| P(s)^{-1} \right\|_{L^2 \to L^2} \approx 1 + |s|^{\frac{1}{2}}.$$

Actually we only prove the $\lesssim$-inequality since the reverse inequality is a consequence of Proposition 4.6. This proposition, due to Nonnenmacher, shows that there are eigenvalues of $\mathcal{A}$ approaching the imaginary axis fast enough to get the desired lower bound on the stationary resolvent. Since it is well-known we also do not prove the invertability of $P(s)$. The (simple) standard proof is based on testing the homogeneous stationary wave equation with $\overline{u}$. From considering real and imaginary part of the resulting expression one easily checks $u = 0$ by a *unique continuation principle.*

4.2. A sharp resolvent estimate

Here is the plan for the proof of Theorem 4.3: First, we separate the y-dependence of the stationary wave equation from the problem. As a result we are dealing with a family of one dimensional problems which are parametrized by the vertical wave number $n \in \mathbb{N}$. Then we derive explicit solution formulas for the separated problems. These formulas allow us to estimate the solutions of the separated problems by their right-hand side with a constant essentially depending *explicitly* on s and n. In the final step we introduce appropriate regimes for s relative to n which allow us to drop the n-dependence of the constant by a (short) case study.

Because of the symmetry of (4.4) we have $\left\|P(-s)^{-1}\right\|_{L^2\to L^2} = \left\|P(s)^{-1}\right\|_{L^2\to L^2}$. Therefore in the following we always assume s to be *positive*.

4.2.1. Separation of variables. First recall that the functions $s_n(y) = \sqrt{2}\sin(n\pi y)$ for $n \in \{1, 2, \dots\}$ form a complete orthonormal system of $L^2(0,1)$. Thus considering u and f satisfying (4.4) we may write

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x)s_n(y) \text{ and } f(x,y) = \sum_{n=1}^{\infty} f_n(x)s_n(y). \tag{4.5}$$

In terms of this separation of variables the stationary wave equation is equivalent to the one dimensional problem $P_n(s)u_n = f_n$ where

$$P_n(s) = -\partial_x^2 - k_n^2 + 2isa(x), \text{ and} \tag{4.6}$$
$$k_n^2 = s^2 - (n\pi)^2.$$

Note that k_n might be an imaginary number. In a few lines we see that only the real case is important. In that case we choose $k_n \geq 0$. But first we prove the following simple

LEMMA 4.4. *Let $\phi : \mathbb{R}_+ \to (0,\infty)$. Then the estimate $\left\|P_n(s)^{-1}\right\|_{L^2\to L^2} \lesssim \phi(|s|)$ uniformly in n is equivalent to the estimate $\left\|P(s)^{-1}\right\|_{L^2\to L^2} \lesssim \phi(|s|)$.*

PROOF. Let $P(s)u = f$ and expand u and f as in (4.5). Then the implication from the left to the right is a consequence of the following chain of equations and inequalities:

$$\|u\|_{L^2}^2 = \sum_{n=1}^{\infty} \|u_n\|_{L^2}^2 \lesssim \phi(|s|)^2 \sum_{n=1}^{\infty} \|f_n\|_{L^2}^2 = \phi(|s|)^2 \|f\|_{L^2}^2 .$$

The reverse implication follows from looking at $f(x,y) = f_n(x)s_n(y)$ and $u(x,y) = u_n(x)s_n(y)$. ☐

So below we are concerned with the separated stationary wave equation

$$\begin{cases} P_n(s)u_n(x) = f_n(x) & \text{for } x \in (0,1) \\ u_n(0) = u_n(1) = 0 \end{cases} \tag{4.7}$$

where $P_n(s)$ is defined in (4.6). In view of Lemma 4.4 we are left to show $\|u_n\|_{L^2} \lesssim s^{1/2}\|f_n\|_{L^2}$ uniformly in n in order to prove Theorem 4.3. It turns out that such an estimate is easy to prove if k_n is imaginary.

LEMMA 4.5. *There exists a constant $c > 0$ such that $\left\|P_n(s)^{-1}\right\|_{L^2\to H_0^1} \lesssim 1$ holds uniformly in n whenever $s^2 \leq (n\pi)^2 + c$.*

Note that $P_n(s)^{-1}$ is considered as an operator mapping to $H_0^1(0,1)$. But it does not really matter since we will only use this estimate after replacing H_0^1 by L^2.

PROOF. Testing equation (4.7) by $\overline{u}_n$ and taking the real part leads to

$$\int_0^1 |u_n'|^2 - c\int_0^1 |u_n|^2 \leq \int_0^1 |f_n u_n| .$$

Recall that $\|v'\|_{L^2}^2 \geq \pi^2 \|v\|_{L^2}^2$ for all $v \in H_0^1(0,1)$ since π^2 is the lowest eigenvalue of the Dirichlet-Laplacian on the unit interval. Thus the conclusion of the Lemma holds for all $c < \pi^2$. ☐

This lemma allows us to assume

$$k_n = \sqrt{s^2 - (n\pi)^2} > c \tag{4.8}$$

for some universal constant $c > 0$ not depending on neither s nor n.

4.2.2. Explicit formula for $P_n(s)^{-1}$. From now on we consider (4.7) under the constraint (4.8). To avoid cumbersome notation we drop the subscript n from k_n, i.e. we write k instead from now on. Next let $v = u_n|_{[0,\sigma]}, g = f_n|_{(0,\sigma)}$ and $w = u_n|_{[\sigma,1]}, h = f_n|_{(\sigma,1)}$. We may write (4.7) as a coupled system consisting of a wave equation with constant damping and an undamped wave equation:

$$\begin{cases} (-\partial_x^2 - k^2 + 2isa_0)v(x) = g(x) & \text{for } x \in (0,\sigma), \\ (-\partial_x^2 - k^2)w(x) = h(x) & \text{for } x \in (\sigma,1), \\ v(0) = w(1) = 0, \\ v(\sigma) = w(\sigma), v'(\sigma) = w'(\sigma). \end{cases} \tag{4.9}$$

4.2.2.1. *Solution of the homogeneous equation.* The following ansatz satisfies the first three lines of (4.9) with $g, h = 0$:

$$v_0(x) = \frac{1}{k'}\sin(k'x), \quad w_0(x) = \frac{1}{k}\sin(k(1-x)), \tag{4.10}$$

where k' is the solution of $k'^2 = k^2 - 2isa_0$ which has negative imaginary part.

4.2.2.2. *Solution of the inhomogeneous equation.* The following ansatz satisfies the first three lines of (4.9):

$$v_g(x) = -\frac{1}{k'}\int_0^x \sin(k'(x-y))g(y)dy,\ w_h(x) = -\frac{1}{k}\int_x^1 \sin(k(y-x))h(y)dy. \tag{4.11}$$

This is simply the variation of constants (or Duhamel's) formula. It is useful to know the derivatives of these particular solutions:

$$v_g'(x) = -\int_0^x \cos(k'(x-y))g(y)dy,\ w_h'(x) = +\int_x^1 \cos(k(y-x))h(y)dy. \tag{4.12}$$

4.2.2.3. *General solution.* The general solution of the first three lines of (4.7) has the form

$$v = av_0 + v_g, \quad w = bw_0 + w_h. \tag{4.13}$$

Our task is to find the coefficients $a = a(s,n)$ and $b = b(s,n)$. Therefore we have to analyze the coupling condition in line four of (4.9). A short calculation shows that it is equivalent to

$$\underbrace{\begin{pmatrix} v_0 & -w_0 \\ v_0' & -w_0' \end{pmatrix}\Big|_{x=\sigma}}_{=:M(s,n)} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} w_h - v_g \\ w_h' - v_g' \end{pmatrix}\Big|_{x=\sigma}.$$

From the preceding equation we easily deduce

$$a = \frac{1}{\det M}\left[w_0'(v_g - w_h) - w_0(v_g' - w_h)\right]_{x=\sigma}, \tag{4.14}$$

$$b = \frac{1}{\det M}\left[v_0'(v_g - w_h) - v_0(v_g' - w_h)\right]_{x=\sigma}. \tag{4.15}$$

Moreover,

$$\det M = \frac{1}{k'}\sin(k'\sigma)\cos(k(1-\sigma)) + \frac{1}{k}\cos(k'\sigma)\sin(k(1-\sigma)). \tag{4.16}$$

4.2.3. Proving a general estimate $\|u_n\|_{L^2} \leq C(k,k',M)\|f_n\|_{L^2}$. For this inequality we will derive an *explicit* formula for C in terms of k, k' and M. In the next subsection we identify the qualitatively different regimes in which s can live. By *regime* we mean a relation which says how big s - the full momentum - is compared to $n\pi$ - the momentum in y-direction. For each of these regimes we then easily translate the *explicit* k, k', M dependence of C to a an *explicit* dependence on s.

4.2.3.1. *Elementary estimates for w_0 and w_h.* Directly from the definition of w_0 (see (4.10)) we deduce

$$\|w_0\|_\infty \leq \frac{1}{k},\ \|w_0'\|_\infty \leq 1 \text{ and } \|w_0\|_2 \leq \frac{\sqrt{1-\sigma}}{k}. \tag{4.17}$$

In the same manner for w_h from (4.11) and (4.12) we deduce:

$$\|w_h\|_\infty \leq \frac{\sqrt{1-\sigma}}{k}\|h\|_2,\ \|w_h'\|_\infty \leq \sqrt{1-\sigma}\,\|h\|_2 \text{ and } \|w_h\|_2 \leq \frac{1-\sigma}{k}\|h\|_2. \tag{4.18}$$

4.2.3.2. *Estimating w.* Recall from (4.13) that $w = bw_0 + w_h$. Recall the formula (4.15) for b. Note that

$$(v_0'v_g - v_0v_g')(\sigma) = \frac{1}{k'}\int_0^\sigma \sin(k'y)g(y)dy.$$

Thus it seems to be natural to decompose

$$\begin{aligned} b &= \frac{1}{\det M}\left[(v_0w_h' - v_0'w_h) + (v_0'v_g - v_0v_g')\right]_{x=\sigma} \\ &=: b_1 + b_2. \end{aligned}$$

This leads to the decomposition of $w = b_1w_0 + b_2w_0 + w_h$ into three parts. With the help of (4.17) and (4.18) each part can easily be estimated as follows:

$$\begin{aligned} \|b_1w_0\|_2 &\lesssim \frac{e^{|\Im k'|\sigma}}{|k'\det M|}\left(\frac{1}{k} + \frac{|k'|}{k^2}\right)\|h\|_2, \\ \|b_2w_0\|_2 &\lesssim \frac{e^{|\Im k'|\sigma}}{|k'\det M|}\frac{1}{k}\|g\|_2,\ \|w_h\|_2 \lesssim \frac{1}{k}\|h\|_2. \end{aligned} \tag{4.19}$$

We could now add all three single estimates to get the desired estimate on w but we wait until we have done the same thing for v.

4.2.3.3. *Estimating v.* Recall from (4.13) that $v = av_0 + v_h$. Recall the formula (4.14) for a. Note that

$$\begin{aligned} (w_0w_h' - w_0'w_h)(\sigma) &= \frac{1}{k}\int_\sigma^1 \sin(k(1-y))h(y)dy \text{ and} \\ v_g &= \frac{(-w_0'v_0 + w_0v_0')(\sigma)}{\det M}v_g =: v_{g,2} + v_{g,3}. \end{aligned}$$

Thus it seems to be natural to decompose

$$\begin{aligned} a &= \frac{1}{\det M}\left[(w_0w_h' - w_0'w_h) + w_0'v_g - w_0v_g'\right]_{x=\sigma} \\ &=: a_1 + a_2 + a_3. \end{aligned}$$

This in turn leads to a decomposition of $v = a_1v_0 + (a_2v_0 + v_{g,2}) + (a_3v_0 + v_{g,3})$ into three parts. Essentially it remains to find a good representation of the second and the third part of v. First let us write

$$a_2 v_0 + v_{g,2} = \frac{w_0'(\sigma)}{k' \det M} \underbrace{\left(v_g(\sigma)\sin(k'x) - k' v_0(\sigma) v_g(x)\right)}_{=:I(x)},$$

$$a_3 v_0 + v_{g,3} = \frac{w_0(\sigma)}{k' \det M} \underbrace{\left(-v_g'(\sigma)\sin(k'x) + k' v_0'(\sigma) v_g(x)\right)}_{=:II(x)}.$$

Simple calculations yield

$$\begin{aligned} -2I(x) = &\int_0^\sigma \cos(k'(\sigma - x - y))g(y)dy - \int_0^x \cos(k'(\sigma - x + y))g(y)dy \\ &- \int_x^\sigma \cos(k'(\sigma + x - y))g(y)dy, \end{aligned}$$

and

$$\begin{aligned} 2II(x) = &\int_x^\sigma \sin(k'(\sigma + x - y))g(y)dy - \int_0^x \sin(k'(\sigma - x + y))g(y)dy \\ &- \int_0^\sigma \sin(k'(\sigma - x + y))g(y)dy. \end{aligned}$$

Using this and again the elementary estimates (4.17) and (4.18) for w_0 and w_h we deduce

$$\begin{aligned} \|a_3 v_0 + v_{g,3}\|_2 &\lesssim \frac{e^{|\Im k'|\sigma}}{|k' \det M|} \frac{1}{k} \|g\|_2 , \\ \|a_2 v_0 + v_{g,2}\|_2 \lesssim \frac{e^{|\Im k'|\sigma}}{|k' \det M|} \|g\|_2 , \ \|a_1 v_0\|_2 &\lesssim \frac{e^{|\Im k'|\sigma}}{|k' \det M|} \frac{1}{k} \|h\|_2 . \end{aligned} \tag{4.20}$$

4.2.3.4. *Conclusion.* Putting (4.19) and (4.20) together we get the desired inequality

$$\|u_n\|_{L^2} \lesssim \left[\frac{e^{|\Im k'|\sigma}}{|k' \det M|} \left(1 + \frac{|k'|}{k^2}\right) + \frac{1}{k} \right] \|f_n\|_{L^2} . \tag{4.21}$$

4.2.4. Regimes where s can live. Keeping (4.21) in mind, our task is now to find asymptotic dependencies of k and k' on s and a lower bound for $|k' \det M|$. A priori there is no unique asymptotic behavior of $k = \sqrt{s^2 - (n\pi)^2}$ as s tends to infinity because of k's dependence on n. To overcome this difficulty we introduce the following four *regimes*:

(i) $c \le k \le cs^{\frac{1}{2}}$, (ii) $cs^{\frac{1}{2}} \le k \le Cs^{\frac{1}{2}}$, (iii) $Cs^{\frac{1}{2}} \le k \le cs$, (iv) $cs \le k < s$.

Recall that c (resp. C) means a small (resp. large) constant. Both constants may be different in each regime. But by convention made for the symbols c and C we may assume that consecutive regimes overlap.

Since we want to investigate the asymptotics $s \to \infty$ we always may assume $s > s_0$ for some sufficiently large number $s_0 > 0$.

4.2.4.1. *Regime (i): $c \le k \le cs^{\frac{1}{2}}$.* For sufficiently small c the first order Taylor expansion of the square root at 1 gives a good approximation of

$$k' = \sqrt{2a_0} s^{\frac{1}{2}} e^{-\frac{i\pi}{4}} \left(1 + \frac{ik^2}{a_0 s} + O(k^4 s^{-2})\right).$$

In particular $\Im k' = -\sqrt{a_0} s^{\frac{1}{2}}(1 + O(k^2 s^{-1}))$ tends with a polynomial rate to minus infinity as s tends to infinity. Therefore $\cot(k'\sigma) = i + O(s^{-\infty})$. Together with (4.16) this gives us the following useful formula for

$$\det M = \frac{\sin(k'\sigma)}{k'} \left[\cos(k(1-\sigma)) + \frac{k'}{k}(i + O(s^{-\infty}))\sin(k(1-\sigma))\right]. \tag{4.22}$$

It is not difficult to see that the term within the brackets is bounded away from zero. Thus $|k' \det M| \gtrsim \exp(|\Im k'|\,\sigma)$. From (4.21) now follows (recall also (4.8))

$$\|u_n\|_{L^2} \lesssim \left(1 + \frac{|k'|}{k^2}\right) \|f_n\|_{L^2} \lesssim s^{\frac{1}{2}} \|f_n\|_{L^2} \text{ uniformly in } n.$$

4.2.4.2. *Regime (ii): $cs^{\frac{1}{2}} \le k \le Cs^{\frac{1}{2}}$.* Because of $k'^2 = k^2 - 2isa_0$ we see that both $\Re k'$ and $-\Im k'$ are of order $s^{\frac{1}{2}}$. Therefore (4.22) is valid also in this regime. Again the term within the brackets is bounded away from zero. Thus $|k' \det M| \gtrsim \exp(|\Im k'|\,\sigma)$ and (4.21) imply

$$\|u_n\|_{L^2} \lesssim \|f_n\|_{L^2} \text{ uniformly in } n.$$

4.2.4.3. *Regime (iii): $Cs^{\frac{1}{2}} \le k \le cs$.* Using first order Taylor expansion for the square root at 1 gives

$$k' = k\left(1 - ia_0 sk^{-2} + O(s^2 k^{-4})\right).$$

In particular, if we choose C big enough we can assume the ratio k'/k to be as close to 1 as we wish. Similarly, if we choose c small enough we may assume $-\Im k'$ to be as large as we want. Therefore we may assume $\cot(k'\sigma)$ to be as close to i as we wish. This means that the following variant of (4.22) is true for this regime

$$\det M = \frac{\sin(k'\sigma)}{k'} \left[\cos(k(1-\sigma)) + (i + \varepsilon)\sin(k(1-\sigma))\right],$$

where $\varepsilon \in \mathbb{C}$ is some error term with a magnitude as small as we wish. If we choose c and C such that $|\varepsilon| \le 1/2$ we see that the term within the brackets is bounded away from zero. Thus $|k' \det M| \gtrsim \exp(|\Im k'|\,\sigma)$ and (4.21) imply

$$\|u_n\|_{L^2} \lesssim \|f_n\|_{L^2} \text{ uniformly in } n.$$

4.2.4.4. *Regime (iv): $cs \le k < s$.* As in the previous regime

$$k' = k\left(1 - ia_0 sk^{-2} + O(s^{-2})\right).$$

In particular $k'/k = 1 + O(s^{-1}) \to 1$ and $\Im k' = -a_0 sk^{-1} + O(s^{-1})$ is bounded away from $0, +\infty$ and $-\infty$. Thus

$$\begin{aligned}\det M &= \frac{1}{k'}\left[\sin(k'\sigma)\cos(k(1-\sigma) + \cos(k'\sigma)\sin(k(1-\sigma)))\right] + O(s^{-2}) \\ &= \frac{\sin(k + (k'-k)\sigma)}{k'} + O(s^{-2}).\end{aligned}$$

This implies that $|k' \det M| \approx 1$. Thus from (4.21) we deduce

$$\|u_n\|_{L^2} \lesssim \|f_n\|_{L^2} \text{ uniformly in } n.$$

4.2.5. Conclusion. Let u_n solve $P_n(s)u_n(x) = f_n(x)$, where $P_n(s)$ is defined in (4.6). Section 4.2.4 together with Lemma 4.5 shows that the estimate $\|u_n\|_{L^2} \lesssim s^{1/2} \|f_n\|_{L^2}$ holds uniformly for any n. Therefore, Lemma 4.4 implies Theorem 4.3.

4.3. Sharp $t^{-\frac{4}{3}}$-decay rate for the energy

Now we prove Theorem 4.1. Therefore, recall the definition of the energy E and the damped wave operator $\mathcal{A}$ from Section 4.1. Then Theorem 0.3 (or Theorem 2.2) together with the converse part of Theorem 0.1 restricted to our situation says in particular that for any $\alpha > 0$

$$\sup_{\|\mathbf{x}_0\|_{D(\mathcal{A})}=1} E(t, \mathbf{x}_0)^{\frac{1}{2}} \approx t^{-\frac{1}{\alpha}} \Leftrightarrow \left\|(is - \mathcal{A})^{-1}\right\| \approx s^{\alpha}. \tag{4.23}$$

From Proposition B.1 (see also [3, Proposition 2.4 and Lemma 4.6]) we get

$$\left\|(is - \mathcal{A})^{-1}\right\| \approx s^{\alpha} \Leftrightarrow \left\|P(s)^{-1}\right\|_{L^2 \to L^2} \approx s^{\alpha-1}. \tag{4.24}$$

In the appendix of [3] Stéphane Nonnenmacher proved

Proposition 4.6 (Nonnenmacher, 2014). *The spectrum of $\mathcal{A}$ contains an infinite sequence (z_j) with $\Im z_j \to \infty$ such that $(\Im z_j)^{-3/2} \lesssim \Re z_j < 0$.*

Actually he proved this theorem under periodic boundary conditions, but the proof applies also to Dirichlet or Neumann boundary conditions. Note that Proposition 4.6 together with (4.24) establishes the '$\gtrsim$'-inequality of Theorem 4.3.

Using (4.23) and (4.24) together with Theorem 4.3 yields Theorem 4.1.

CHAPTER 5

A viscoelastic boundary damping model

5.1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and $k : \mathbb{R} \to [0, \infty)$ be an integrable function, depending on the time-variable only and vanishing on $(-\infty, 0)$. We consider a model for the reflection of sound on a wall (see e.g. [41]):

$$\begin{cases} U_{tt}(t,x) - \Delta U(t,x) = 0 & (t \in \mathbb{R}, x \in \Omega), \\ \partial_n U(t,x) + k * U_t(t,x) = 0 & (t \in \mathbb{R}, x \in \partial\Omega). \end{cases} \tag{5.1}$$

The function U is called the *velocity potential*. One can derive the acoustic pressure $p(t,x) = U_t(t,x)$ and fluid velocity $v(t,x) = -\nabla U(t,x)$ from U. The second formula gives the velocity potential its name. Extending k by 0 for negative arguments the convolution with U_t is given by the usual formula $k * U_t(t,x) = \int_0^\infty k(r) U_t(t-r,x) dr$. Here n is the outward normal vector of $\partial\Omega$, which exists almost everywhere for Lipschitz domains. Furthermore ∂_n denotes the normal derivative on the boundary.

We assume that $k \in L^1(0,\infty)$ is a completely monotone function. That is, there exists a positive Radon measure ν on $[0,\infty)$ such that $k(t) = \int_{[0,\infty)} e^{-\tau t} d\nu(\tau)$. We note here that the integrability assumption on k is easily checked to be equivalent to

$$\nu(\{0\}) = 0 \text{ and } \int_0^\infty \tau^{-1} d\nu(\tau) < \infty. \tag{5.2}$$

Let $e_\tau(t) = e^{-\tau t} 1_{[0,\infty)}(t)$ and

$$\psi(t,\tau,x) = e_\tau * U_t(t,x) \quad (t \in \mathbb{R}, \tau \geq 0, x \in \partial\Omega).$$

By defining $p = U_t$, $v = -\nabla U$ and ψ as above one can rephrase (5.1) in an equivalent way as

$$\begin{cases} p_t(t,x) + \operatorname{div} v(t,x) = 0 & (t > 0, x \in \Omega), \\ v_t(t,x) + \nabla p(t,x) = 0 & (t > 0, x \in \Omega), \\ [\psi_t + \tau\psi - p](t,\tau,x) = 0 & (t > 0, \tau > 0, x \in \partial\Omega), \\ \left[-v \cdot n + \int_0^\infty \psi(\tau) d\nu(\tau)\right](t,x) = 0 & (t > 0, x \in \partial\Omega), \end{cases} \tag{5.3}$$

Note that we restrict here to positive times. This is to arrive at an abstract Cauchy problem. The initial state is described by the triplet $\mathbf{x}_0 = (p_0, v_0, \psi_0)$ consisting of p, v and ψ evaluated at time $t = 0$. It is important to observe that p_0 and v_0 cannot fully describe the system's state at $t = 0$ since there are memory effects at the boundary. The missing data from the past is stored in the auxiliary function ψ.

Let us define the energy of the system to be the sum of potential, kinetic and boundary energy:

$$E(\mathbf{x}_0) = \int_\Omega |p_0(x)|^2 + |v_0(x)|^2 \, dx + \int_0^\infty \int_{\partial\Omega} |\psi_0(\tau, x)|^2 \, dS(x) d\nu(\tau).$$

Furthermore we introduce the homogeneous first order energy by

$$E_1^{hom}(\mathbf{x}_0) = \int_\Omega |\nabla p_0|^2 + |\operatorname{div} v_0|^2 \, dx + \int_0^\infty \int_{\partial\Omega} |\tau\psi_0 - p_0|^2 \, dS d\nu(\tau).$$

The first order energy is defined by $E_1 = E + E_1^{hom}$. Let us define the (zeroth order) energy space, and the first order energy space by

$$\mathcal{H} = \mathcal{H}_0 = L^2(\Omega) \times \nabla H^1(\Omega) \times L^2_\nu((0,\infty)_\tau; L^2(\partial\Omega)), \tag{5.4}$$

$$\mathcal{H}_1 = \{\mathbf{x}_0 \in \mathcal{H} : E_1(\mathbf{x}_0) < \infty \text{ and } \left[-v_0 \cdot n|_{\partial\Omega} + \int_0^\infty \psi_0(\tau) d\nu(\tau)\right] = 0\}. \tag{5.5}$$

Here $\nabla H^1(\Omega)$ is the space of vector fields $v \in (L^2(\Omega))^d$ for which there exists a function (potential) $U \in H^1(\Omega)$ such that $v = -\nabla U$. We note that the space of gradient fields $\nabla H^1(\Omega)$ is a closed subspace of $(L^2(\Omega))^d$ since Ω satisfies the Poincaré inequality[1]. To make the boundary condition, appearing in the definition of $\mathcal{H}_1$, meaningful we use that the trace operator $\Gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega), u \mapsto u|_{\partial\Omega}$ is continuous and has a continuous right inverse. Therefore we see that $v \cdot n|_{\partial\Omega}$ is well defined as an element of $H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))^*$ for vector fields $v \in (L^2(\Omega))^d$ with $\operatorname{div} v \in L^2(\Omega)$ by the relation

$$\langle v \cdot n, \Gamma u \rangle_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\partial\Omega)} = \int_\Omega \operatorname{div} v \overline{u} + \int_\Omega v \cdot \nabla \overline{u} \tag{5.6}$$

for all $u \in H^1(\Omega)$. Also note that $E_1(\mathbf{x}_0) < \infty$ implies $\psi_0 \in L^1_\nu$ since $\psi_0(\tau) = \frac{\psi_0(\tau)}{1+\tau} + \frac{\Gamma p}{1+\tau} + \frac{\tau\psi_0(\tau) - \Gamma p}{1+\tau}$ and $(\tau \mapsto \frac{1}{1+\tau}) \in L^1_\nu \cap L^2_\nu(0,\infty)$ by (5.2).[2] The quadratic forms E and E_1 turn $\mathcal{H}$ and $\mathcal{H}_1$ into Hilbert spaces, respectively.

An initial state $\mathbf{x}_0$ is called *classical* if its first order energy is finite and the boundary condition is satisfied (i.e. $\mathbf{x}_0 \in \mathcal{H}_1$). We say that $\mathbf{x} \in C^1([0,\infty); \mathcal{H}) \cap C([0,\infty); \mathcal{H}_1)$ is a *(classical) solution* of (5.3) if it satisfies the first two lines in the sense of distributions and the last two lines in the trace sense, i.e. with $v \cdot n$ defined by (5.6) and p replaced by Γp. From Theorem 5.1 below plus basics from the theory of C_0-semigroups it follows that the initial value problem corresponding to (5.3) is well-posed in the sense that for all classical initial data $\mathbf{x}_0 \in \mathcal{H}_1$ there is a unique solution $\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$ and the mapping $\mathcal{H}_j \ni \mathbf{x}_0 \mapsto \mathbf{x} \in C([0,\infty); \mathcal{H}_j)$ is continuous for $j \in \{0, 1\}$. For a solution $\mathbf{x}$ with $\mathbf{x}_0 = \mathbf{x}(0)$ we also write e.g. $E(t, \mathbf{x}_0)$ instead of $E(\mathbf{x}(t))$. Note that $E_1^{hom}(\mathbf{x}(t)) = E(\dot{\mathbf{x}}(t))$ - this justifies the adjective "homogeneous" for the quadratic form E_1^{hom}.

Our aim is to find the optimal decay rate of the energy, uniformly with respect to classical initial states. This means that we want to find the smallest possible non-increasing function $N : [0,\infty) \to [0,\infty)$ such that

$$E(t, \mathbf{x}_0) \le N(t)^2 E_1(\mathbf{x}_0)$$

[1] Poincaré inequality: if Ω is a bounded Lipschitz domain then there exists a $C > 0$ such that for all $p \in H^1(\Omega)$ with $\int_\Omega p = 0$ we have $\int_\Omega |p|^2 \le C \int_\Omega |\nabla p|^2$.

[2] Here and in the following we abbreviate $L^p_\nu((0,\infty)_\tau; L^2(\partial\Omega))$ simply by L^p_ν for $p \in \{1, 2\}$.

for all $\mathbf{x}_0 \in \mathcal{H}_1$. By Part 1 of this thesis this is essentially equivalent to estimating the resolvent of the wave equation's generator $\mathcal{A}$ (defined in Section 5.2 below) along the imaginary axis near infinity and near zero.

The two main results of this chapter are Theorem 5.4 and 5.9. The Sections 5.3 and 5.4 are devoted to the proofs. We illustrate the application of our main results to energy decay by several examples in Section 5.5. Our first main result (Theorem 5.4) implies in particular that the task of estimating the resolvent of the complicated 3×3-matrix operator $\mathcal{A}$ is equivalent to estimating the resolvent of the corresponding (and much simpler) stationary operator. Our second main result (Theorem 5.9) thus determines an upper resolvent estimate of $\mathcal{A}$ at infinity. Unfortunately we need *additional assumptions* on the acoustic impedance (see (5.21)). However in our separate treatment of the case $\Omega = (0, 1)$ in Section 5.6 we see that in this case actually no additional assumptions are required for the conclusion of Theorem 5.9 to hold. Even more is true, the given upper bound on the resolvent is also optimal in the 1D setting. This and observations from the examples lead us to three questions and corresponding conjectures formulated in Section 5.8.

In Section 5.2 we recall the semigroup approach of Desch, Fašangová, Milota and Probst [22]. For convenience of the reader we recall some basic and some not so basic facts from the literature concerning the trace operator, fractional Sobolev spaces and Besov spaces in Appendix C. For the reader interested in the physical background of equation (5.1) we recommend [38].

5.2. The semigroup approach

We reformulate (5.3) as an abstract Cauchy problem in a Hilbert space:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t), \\ \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{H}. \end{cases} \tag{5.7}$$

Following the approach of [22] we define the energy/state space $\mathcal{H}$ as in (5.4) and write $\mathbf{x} = (p, v, \psi)$ for its elements (the states). Again let $\Gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega), u \mapsto u|_{\partial\Omega}$ be the trace operator on Ω. By abuse of notation let τ denote the multiplication operator on $L^2_\nu(0, \infty)$ mapping $\psi(\tau)$ to $\tau\psi(\tau)$. We define the wave operator by

$$\mathcal{A} = \begin{pmatrix} 0 & -\operatorname{div} & 0 \\ -\nabla & 0 & 0 \\ \Gamma & 0 & -\tau \end{pmatrix} \text{ with } D(\mathcal{A}) = \mathcal{H}_1.$$

Note that $E_1(\mathbf{x}_0) = \|\mathbf{x}_0\|^2_{D(\mathcal{A})} = \|\mathbf{x}_0\|^2_{\mathcal{H}} + \|\mathcal{A}\mathbf{x}_0\|^2_{\mathcal{H}_1}$ for all $\mathbf{x}_0 \in D(\mathcal{A})$.

THEOREM 5.1 ([22]). *The Cauchy problem (5.7) is well posed. More precisely, $\mathcal{A}$ is the generator of a C_0-semigroup of contractions in $\mathcal{H}$.*

Taking formal Laplace transform of the wave equation (5.1) yields

$$\begin{cases} z^2 u(x) - \Delta u(x) = f & (x \in \Omega), \\ \partial_n u(x) + z\hat{k}(z)u(x) = g & (x \in \partial\Omega). \end{cases} \tag{5.8}$$

Here z is a complex number and formally $u = \hat{U}(z) = \int_0^\infty e^{-zt} U(t) dt$, $f = zU(0) + U_t(0)$ and $g = \hat{k}(z)U(0)|_{\partial\Omega}$. A way to give (5.8) a precise meaning is via the form method. Thus for $z \in \mathbb{C}\backslash(-\infty, 0)$ let us define the bounded sesquilinear form $a_z : H^1 \times H^1(\Omega) \to \mathbb{C}$ by

$$a_z(p, u) = z^2 \int_\Omega p\overline{u} + \int_\Omega \nabla p \cdot \nabla \overline{u} + z\hat{k}(z) \int_{\partial\Omega} \Gamma p \Gamma \overline{u} dS.$$

If we replace the right-hand side f, g by $F \in H^1(\Omega)^*$ (dual space of H^1), given by $\langle F, \eta\rangle = \int_\Omega f\overline{\eta} + \int_{\partial\Omega} g\Gamma\overline{\eta} dS$, then a functional analytic realization of (5.8) is given by

$$\forall \eta \in H^1(\Omega) : \ a_z(u, \eta) = \langle F, \eta\rangle_{(H^1)^*, H^1(\Omega)} . \tag{5.9}$$

For all $z \in \mathbb{C}\backslash(-\infty, 0)$ for which (5.9) has for all $F \in H^1(\Omega)^*$ a unique solution $u \in H^1(\Omega)$ we define the stationary resolvent operator $R(z) : H^1(\Omega)^* \to H^1(\Omega), F \mapsto u$.

Theorem 5.2 ([22]). *The spectrum of the wave operator satisfies*

$$\begin{aligned} \sigma(\mathcal{A})\backslash(-\infty, 0] &= \{z \in \mathbb{C}\backslash(-\infty, 0] : R(z) \text{ does not exist.}\} \\ &\subseteq \{z \in \mathbb{C} : \Re z < 0\}. \end{aligned}$$

Furthermore all spectral points in $\mathbb{C}\backslash(-\infty, 0]$ are eigenvalues.

Following the proof of the preceding theorem given in [22] one sees that for $s \in \mathbb{C}\backslash i[0, \infty)$

$$(is - \mathcal{A})(p, v, \psi) = (q, w, \varphi) \in \mathcal{H} \tag{5.10}$$

is equivalent to

$$\begin{aligned} &\forall u \in H^1(\Omega) : a_{is}(p, u) = \langle F, u\rangle_{(H^1)^*, H^1(\Omega)} \\ &\text{and } v = \frac{w + \nabla p}{is}, \ \psi(\tau) = \frac{\Gamma p + \varphi(\tau)}{is + \tau}, \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} \langle F, u\rangle &= is \int_\Omega q\overline{u} - \int_\Omega w \cdot \nabla \overline{u} - is \int_{\partial\Omega} \left[\int_0^\infty \frac{\varphi(\tau)}{is + \tau} d\nu(\tau)\right] \Gamma u \, dS \\ &=: \langle F_1, u\rangle + \langle F_2, u\rangle + \langle F_3, u\rangle . \end{aligned} \tag{5.12}$$

Observe that the adjoint operator of $R(z)$ is given by $R(z)^* = R(\overline{z})$ for all $z \in \mathbb{C}\backslash(-\infty, 0)$ for which $R(z)$ is defined. Finally, we mention:

Theorem 5.3 ([22]). *The wave operator $\mathcal{A}$ is injective.*

In the next section we characterize all kernels k for which $\mathcal{A}$ is invertible.

5.3. $(is - \mathcal{A})^{-1}$ at zero and relation to $R(is)$

In this section we prove the first main result of this chapter.

Theorem 5.4. *The following holds:*

(i) *Let $M : (0, \infty) \to [1, \infty)$ be a non-decreasing function. Then*

$$\begin{aligned} &\left[\exists s_1 > 0 \forall |s| \geq s_1 : \left\|(is - \mathcal{A})^{-1}\right\| \leq CM(|s|)\right] \\ \Leftrightarrow &\left[\exists s_2 > 0 \forall |s| \geq s_2 : \|R(is)\|_{L^2 \to L^2} \leq C |s|^{-1} M(|s|)\right]. \end{aligned}$$

(ii) $\exists s_3 > 0 \forall |s| \leq s_3 : \left\|(is - \mathcal{A})^{-1}\right\| \leq C |s|^{-1}$.

(iii) *$\mathcal{A}$ is invertible iff $(\tau \mapsto \tau^{-1}) \in L^\infty_\nu$, i.e. $\exists \varepsilon > 0 : \nu|_{(0,\varepsilon)} = 0$.*

If $\mathcal{A}$ is not invertible we deduce from Theorem 5.3 that $\mathcal{A}$ cannot be surjective in this case. In Section 5.3.4 we characterize the range of $\mathcal{A}$.

5.3.1. Singularity at ∞. In this subsection we prove Theorem 5.4 (i). Therefore, let us first define the auxiliary spaces X^θ by the real interpolation method:

$$X^\theta = \begin{cases} L^2(\Omega) \text{ resp. } H^1(\Omega) & \text{if } \theta = 0 \text{ resp. } 1, \\ (L^2(\Omega), H^1(\Omega))_{\theta,1} & \text{if } \theta \in (0,1), \\ (X^\theta)^* & \text{if } \theta \in [-1, 0). \end{cases}$$

For $\theta \in (0,1)$ the space X^θ coincides with the Besov space $B_1^{\theta,2}(\Omega)$.

Let us explain why we use the Besov spaces X^θ instead of the Bessel potential spaces $H^\theta(\Omega)$. The reason is that while the trace operator $\Gamma : H^\theta(\Omega) \to H^{\theta-1/2}(\partial\Omega)$ is continuous for $\theta \in (1/2, 1]$ this is no longer true for $\theta = 1/2$ (with the convention $H^0 = L^2$). On the other hand $\Gamma : X^{1/2} \to L^2(\partial\Omega)$ *is* continuous (see Proposition C.2). A corollary of this fact is that for some $C > 0$

$$\forall u \in H^1(\Omega) : \|\Gamma u\|^2_{L^2(\partial\Omega)} \le C \|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} . \tag{5.13}$$

Actually, by Lemma C.4, the preceding trace inequality is equivalent to the continuity of the trace operator $\Gamma : X^{1/2} \to L^2(\partial\Omega)$.

Let us prove the following extrapolation result.

PROPOSITION 5.5. *Let $M : (1, \infty) \to [1, \infty)$ be a non-decreasing function. If*

$$\|R(is)\|_{X^{-a} \to X^b} = O(|s|^{a+b-1} M(|s|)) \text{ as } |s| \to \infty \tag{5.14}$$

is true for $a = b = 0$, then it is also true for all $a, b \in [0, 1]$.

PROOF. Throughout the proof we may assume $|s|$ to be sufficiently large. Assume that (5.14) is true for $a = b = 0$. Let $f \in L^2(\Omega)$ and $p = R(is)f$, i.e.

$$\forall u \in H^1(\Omega) : \; a_{is}(p, u) = \int_\Omega f\overline{u}.$$

Because of (5.13) and the uniform boundedness of $\hat{k}(is)$ there are constants $c, C > 0$ such that $\Re a_{is}(p,p) \ge c \|p\|^2_{H^1} - Cs^2 \|p\|^2_{L^2}$. This helps us to estimate

$$\begin{aligned} c \|p\|^2_{H^1} &\le \Re a_{is}(p,p) + Cs^2 \|p\|^2_{L^2} \\ &\le \|f\|_{L^2} \|p\|_{L^2} + Cs^2 \|p\|^2_{L^2} \\ &\le s^{-2} \|f\|^2_{L^2} + Cs^2 \|p\|^2_{L^2} \\ &\le CM(|s|)^2 \|f\|^2_{L^2} . \end{aligned}$$

In other words, (5.14) is true for $a = 0, b = 1$. By duality (recall $R(z)^* = R(\overline{z})$) it is also true for $a = -1, b = 0$. Almost the same calculation as above but now with the help of (5.14) for the now known case $a = -1, b = 0$ shows that (5.14) is also true for $a = -1, b = 1$.

It remains to interpolate. First interpolate between the parameters ($a = 0, b = 1$) and ($a = 1, b = 1$) to get (5.14) for $a \in [0, 1], b = 1$. Then interpolate between the parameters ($a = 0, b = 0$) and ($a = 1, b = 0$) to get (5.14) for $a \in [0, 1], b = 0$. One last interpolation gives us the desired result. ☐

Let us proceed with the proof of Theorem 5.4(i). The implication "⇒" follows immediately from the equivalence of (5.10) and (5.11) with $w, \varphi = 0$. Therefore we have to show $\|\mathbf{x}\|_{\mathcal{H}} \le CM(|s|)\|\mathbf{y}\|_{\mathcal{H}}$, for all large $|s|$ and for all $\mathbf{x} = (p, v, \psi) \in D(\mathcal{A}), \mathbf{y} = (q, w, \varphi) \in \mathcal{H}$ satisfying (5.10), where C does not depend on s and $\mathbf{y}$.

Let F_j for $j \in \{1, 2, 3\}$ be defined by (5.12) and let p_j satisfy

$$\forall u \in H^1(\Omega) : a_{is}(p_j, u) = \langle F_j, u\rangle_{(H^1)^*, H^1(\Omega)} .$$

Case $j = 1$. It is clear that $\|F_1\|_{L^2} = |s|\,\|q\|_{L^2}$. By Proposition 5.5 we have $\|p_1\|_{X^b} = O(|s|^b M(|s|))\,\|q\|_{L^2}$ for all $b \in [0, 1]$. *Case* $j = 2$. It is clear that $\|F_2\|_{X^{-1}} \le \|w\|_{L^2}$. By Proposition 5.5 we have $\|p_2\|_{X^b} = O(|s|^b M(|s|))\,\|w\|_{L^2}$ for all $b \in [0, 1]$. *Case* $j = 3$. By the continuity of the trace $\Gamma : X^{1/2} \to L^2(\partial\Omega)$, Hölder's inequality and (5.2) we have

$$\begin{aligned}\|F_3\|_{X^{-\frac{1}{2}}} &\le C\,|s| \left\| \int_0^\infty \frac{\varphi(\tau)}{is+\tau} d\nu(\tau) \right\|_{L^2(\partial\Omega)} \\ &\le C\,|s|^{\frac{1}{2}}\,\|\varphi\|_{L^2_\nu} .\end{aligned}$$

Again by Proposition 5.5 this yields $\|p_3\|_{X^b} = O(|s|^b M(|s|))\,\|\varphi\|_{L^2_\nu}$ for all $b \in [0, 1]$. Overall we derived the estimate $\|p\|_{X^b} = O(|s|^b M(|s|))\,\|\mathbf{y}\|_{\mathcal{H}}$ for all $b \in [0, 1]$. Finally, this together with (5.11) implies

$$\begin{aligned}\|v\|_{L^2} &\le C\,|s|^{-1}\,(\|w\|_{L^2} + \|p\|_{H^1}) \\ &\le CM(|s|)\,\|\mathbf{y}\|_{\mathcal{H}}\end{aligned}$$

and

$$\begin{aligned}\|\psi\|_{L^2_\nu} &\le |s|^{-1}\,\|\varphi\|_{L^2_\nu} + \|\Gamma p\|_{L^2} \left(\int_0^\infty \frac{1}{|is+\tau|^2} d\nu(\tau) \right)^{\frac{1}{2}} \\ &\le |s|^{-1}\,\|\varphi\|_{L^2_\nu} + C\,|s|^{-\frac{1}{2}}\,\|p\|_{X^{\frac{1}{2}}} \\ &\le CM(|s|)\,\|\mathbf{y}\|_{\mathcal{H}} .\end{aligned}$$

This concludes the proof of Theorem 5.4 part (i).

5.3.2. Singularity at 0. Now we prove Theorem 5.4 (ii). For $s \ne 0$ we equip the Sobolev space $H^1(\Omega)$ with the equivalent norm $\|u\|_{H^1_s}^2 := \|u\|_{L^2}^2 + \left\|s^{-1}\nabla u\right\|_{L^2}^2$. In what follows we are interested in the asymptotics $s \to 0$ while $s \ne 0$. As in the preceding subsection we introduce some auxiliary spaces by the real interpolation method

$$X_s^\theta = \begin{cases} L^2(\Omega) \text{ resp. } H_s^1(\Omega) & \text{if } \theta = 0 \text{ resp. } 1, \\ (L^2(\Omega), H_s^1(\Omega))_{\theta,1} & \text{if } \theta \in (0, 1), \\ (X_s^\theta)^* & \text{if } \theta \in [-1, 0). \end{cases}$$

We prove an analog of Proposition 5.5 - but without the unknown function M.

PROPOSITION 5.6. *Let* $a, b \in [0, 1]$ *and* $\theta_+ = \max\{a + b - 1, 0\}$. *Then*

$$\|R(is)\|_{X_s^{-a} \to X_s^b} = O(|s|^{-1-\theta_+}) \text{ as } s \to 0. \tag{5.15}$$

Before we can prove this proposition we show

LEMMA 5.7. *There is a constant $C(\Omega)$ solely depending on the dimension and the volume of Ω such that for all $u \in H^1(\Omega)$*

$$\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} |u|^2 \, dS \geq C(\Omega) \int_\Omega |u|^2 .$$

PROOF. For the dimension $d = 1$ this is an easy exercise for the reader. For $d \geq 2$ we recall the isoperimetric inequality of Maz'ya [36, Chapter 5.6] which is valid for all functions $v \in W^{1,1}(\Omega)$:

$$\int_\Omega |\nabla v| + \int_{\partial\Omega} |v| \, dS \geq \frac{d\sqrt{\pi}}{\Gamma(1+\frac{d}{2})^{\frac{1}{d}}} \left(\int_\Omega |v|^{\frac{d}{d-1}} \right)^{\frac{d-1}{d}} .$$

The right-hand side can easily be estimated from below by a constant times the $L^1(\Omega)$-norm of v since Ω is bounded. The conclusion now follows by plugging in $v = u^2$. □

PROOF OF PROPOSITION 5.6. Because of (5.13) and the continuity of $\mathbb{R} \ni s \mapsto \hat{k}(is)$ at zero we have for all $u \in H^1(\Omega)$

$$a_{is}(u,u) = \int_\Omega |\nabla u|^2 + is\hat{k}(0) \int_{\partial\Omega} |u|^2 \, dS + o(1) \|\nabla u\|_{L^2}^2 + O(s^2) \|u\|_{L^2}^2 .$$

Thus for sufficiently small $|s|$ we deduce from Lemma 5.7 and the fact $\hat{k}(0) > 0$ that for all solutions $p \in H^1(\Omega)$ of the stationary wave equation (5.11) with $F = f \in L^2(\Omega)$ the following estimate holds:

$$\begin{aligned} |s| \, \|p\|_{L^2}^2 &\leq C \, |a_{is}(p,p)| = C \, |\langle f, p \rangle| \\ &\leq C \, |s|^{-1} \, \|f\|_{L^2}^2 + \frac{|s|}{2} \, \|p\|_{L^2}^2 . \end{aligned}$$

This shows (5.15) in the case $a = b = 0$.

Let us define the semi-linear functional

$$G_s(u) = -s \int_\Omega \overline{u} + i\hat{k}(is) \int_{\partial\Omega} \overline{u} dS$$

for $u \in H^1(\Omega)$. Observe that $G_s(1) \to i\hat{k}(0) \, |\partial\Omega| \neq 0$ as s tends to 0. It is easy to see from Poincaré's inequality (recall that Ω has Lipschitz boundary) that the expression $\|\nabla u\|_{L^2} + |G_s(u)|$ defines a norm on $H^1(\Omega)$ which is equivalent to the usual one - uniformly for small $|s|$. In particular $p \mapsto \|\nabla p\|_{L^2}$ is an equivalent norm on the kernel of G_s.

Remember that p is the solution of (5.11) for $F = f \in L^2(\Omega)$. We decompose $p = p_0 + p_G$ with $p_G = G_s(p) = \text{const.} \in L^2(\Omega)$ and $G_s(p_0) = 0$. Then

$$a_{is}(p, p_0) = a_{is}(p_0, p_0) = (1 + O(|s|)) \int_\Omega |\nabla p_0|^2 .$$

This implies

$$\|\nabla p_0\|_{L^2}^2 \leq C \, |a_{is}(p, p_0)| \leq C \, |\langle f, p_0 \rangle| \leq C \, \|f\|_{L^2} \, \|\nabla p_0\| .$$

This in combination with (5.15) for $a = b = 0$ implies $\|p\|_{H_s^1} \leq C \, |s|^{-1} \, \|f\|_{L^2}$ which is (5.15) for the parameters $a = 0, b = 1$. By duality (recall $R(z)^* = R(\overline{z})$), equation (5.15) is also true for $a = 1, b = 0$. A similar calculation as above with f replaced by $F \in H^1(\Omega)^*$ and (5.15) for $a = 1, b = 0$ shows (5.15) for $a = 1, b = 1$.

What remains to do is some interpolation. It is important to interpolate in the right order. First, one has to show

$$\|R(is)\|_{X_s^0 \to X_s^{b_1}}, \|R(is)\|_{X_s^{a_1} \to X_s^0} = O(|s|^{-1})$$

for $a_1, b_1 \in [0,1]$. This can be done via interpolation between $(a=0, b=0)$ and $(a=0, b=1)$ for the first estimate and between $(a=0, b=0)$ and $(a=1, b=0)$ for the second estimate. Choosing a_1 and b_1 appropriately, the preceding estimates imply (5.15) in the case $a+b \leq 1$. Interpolation between the preceding case and $a=1, b=1$ yields the remaining part of the proposition. □

Let us proceed with the proof of Theorem 5.4(ii) in a similar fashion as for part (i). We have to show $\|x\|_{\mathcal{H}} \leq C|s|^{-1}\|y\|_{\mathcal{H}}$ for all small $|s|$ and for all $x = (p, v, \psi) \in D(A), y = (q, w, \varphi) \in \mathcal{H}$ satisfying (5.10) where C does not depend on s and y. Let F_j for $j \in \{1,2,3\}$ be defined by (5.12) and let p_j satisfy

$$\forall u \in H^1(\Omega) : a_{is}(p_j, u) = \langle F_j, u\rangle_{(H^1)^*, H^1(\Omega)}$$

Case $j=1$. It is clear that $\|F_1\|_{L^2} = |s|\,\|q\|_{L^2}$. By Proposition 5.6 we have $\|p_1\|_{X_s^b} = O(1)\|q\|_{L^2}$ for all $b \in [0,1]$. *Case* $j=2$. It is clear that $\|F_2\|_{X_s^{-1}} \leq |s|\,\|w\|_{L^2}$. By Proposition 5.6 we have $\|p_2\|_{X_s^b} = O(|s|^{-b})\|w\|_{L^2}$ for all $b \in [0,1]$. *Case* $j=3$. By the continuity of the trace $\Gamma : X^{1/2} \to L^2(\partial\Omega)$ and by Hölder's inequality we have for all $|s| \leq 1$

$$\begin{aligned}\|F_3\|_{X_s^{-\frac{1}{2}}} \leq \|F_3\|_{X^{-\frac{1}{2}}} &\leq C|s|\left\|\int_0^\infty \frac{\varphi(\tau)}{is+\tau}d\nu(\tau)\right\|_{L^2(\partial\Omega)} \\ &\leq C|s|^{\frac{1}{2}}\|\varphi\|_{L^2_\nu}.\end{aligned}$$

By Proposition 5.6 this yields $\|p_3\|_{X_s^b} = O(|s|^{-\frac{1}{2}-(b-\frac{1}{2})_+})\|\varphi\|_{L^2_\nu}$ for all $b \in [0,1]$. Overall we derived the estimate $\|p\|_{X_s^b} = O(|s|^{-\frac{1}{2}-(b-\frac{1}{2})_+})\|\mathbf{y}\|_{\mathcal{H}}$ for all $b \in [0,1]$. Finally, this together with (5.11) implies

$$\begin{aligned}\|v\|_{L^2} &\leq C|s|^{-1}(\|w\|_{L^2} + \|\nabla p\|_{L^2}) \\ &\leq C|s|^{-1}\|w\|_{L^2} + C\|p\|_{H^1_s} \\ &\leq C|s|^{-1}\|\mathbf{y}\|_{\mathcal{H}}\end{aligned}$$

and because of $\|p\|_{X^{\frac{1}{2}}} \leq \|p\|_{X_s^{\frac{1}{2}}}$ for $|s| \leq 1$

$$\begin{aligned}\|\psi\|_{L^2_\nu} &\leq |s|^{-1}\|\varphi\|_{L^2_\nu} + \|\Gamma p\|_{L^2}\left(\int_0^\infty \frac{1}{|is+\tau|^2}d\nu(\tau)\right)^{\frac{1}{2}} \\ &\leq |s|^{-1}\|\varphi\|_{L^2_\nu} + C|s|^{-\frac{1}{2}}\|p\|_{X_s^{\frac{1}{2}}} \\ &\leq C|s|^{-1}\|\mathbf{y}\|_{\mathcal{H}}.\end{aligned}$$

This concludes the proof of Theorem 5.4 part (ii).

5.3.3. Spectrum at 0. Let us prove part (iii) of Theorem 5.4.

"⇒". Let us first assume that $\mathbf{y} = (q, w, \varphi) \in \mathcal{H}$ and $\mathbf{x} = (p, v, \psi) \in D(\mathcal{A})$ satisfy (5.10) for $s = 0$. There is a function $u \in H^1(\Omega)$ such that $w = \nabla u$. We may assume $\int_\Omega u = 0$ to make u unique. Then (5.10) for $s = 0$ is

$$\begin{cases} \operatorname{div} v(x) = q(x) & (x \in \Omega), \\ \nabla p(x) = w(x) = \nabla u(x) & (x \in \Omega), \\ \tau\psi(\tau, x) - p(x) = \varphi(\tau, x) & (\tau > 0, x \in \partial\Omega), \\ -v \cdot n(x) + \int_0^\infty \psi(\tau, x) d\nu(\tau) = 0 & (x \in \partial\Omega). \end{cases} \tag{5.16}$$

From the second line we see that necessarily $p = u + \alpha$ for some complex number α. We have

$$\psi = \frac{\varphi + \Gamma u + \alpha}{\tau} \in (L^1_\nu \cap L^2_\nu)(0, \infty; L^2(\partial\Omega)). \tag{5.17}$$

The L^1_ν-inclusion follows by the definition of $D(\mathcal{A})$ as explained in the paragraph following (5.5). Let us now specialize to the situation $q, w = 0$ and $\|\varphi\|_{L^2_\nu} \le 1$. Then $u = 0$. By the existence of $\mathcal{A}^{-1}$ there must be a uniform bound $|\alpha| \le C$ where the constant does not depend on φ. Because of this, (5.17) and $\int_0^\infty \tau^{-1} d\nu(\tau) < \infty$ we deduce a bound $\left\|\tau^{-1}\varphi\right\|_{L^1_\nu} = \|\psi\|_{L^1_\nu} + C \le C$ where C does not depend on φ. Since this is true for all $\varphi \in L^2_\nu(0, \infty; L^2(\partial\Omega))$ we deduce that the function $(0, \infty) \ni \tau \mapsto \tau^{-1}$ is in $L^2_\nu(0, \infty)$. If we use this in the L^2_ν-inclusion in (5.17) we see that $\left\|\tau^{-1}\varphi\right\|_{L^2_\nu} = \|\psi\|_{L^2_\nu} + C \le C$ where C does not depend on φ. Thus τ^{-1} is an L^2_ν-multiplier and thus it must be bounded with respect to the measure ν.

"⇐". Assume now that $\nu|_{(0,\varepsilon)} = 0$ for some $\varepsilon > 0$. Given $\mathbf{y} = (q, w, \varphi) \in \mathcal{H}$ we show that there is a unique solution $\mathbf{x} = (p, v, \psi) \in D(\mathcal{A})$ of (5.16). From the second line of (5.16) we see that necessarily $p = u + \alpha$ for some complex number α and u as in the first part of the proof. The definition of $\mathcal{H}$ forces the necessity of the ansatz $v = -\nabla U$ for some function $U \in H^1(\Omega)$ with $\int_\Omega U = 0$ for uniqueness purposes. It remains to uniquely determine α and U since then ψ is uniquely given by (5.17). Let $h = -\int_0^\infty \psi d\nu \in L^2(\partial\Omega)$. Then the first and the last line of (5.16) are equivalent to

$$\begin{cases} -\Delta U(x) = q(x) & (x \in \Omega), \\ \partial_n U(x) = h(x) & (x \in \partial\Omega). \end{cases}$$

By the Poincaré inequality this equation has a solution U - which is unique under the constraint $\int_\Omega U = 0$ - if and only if

$$\begin{aligned} 0 &= \int_\Omega q + \int_{\partial\Omega} h dS \\ &= \int_\Omega q - \int_{\partial\Omega} \left(\hat{k}(0)\Gamma u + \int_\varepsilon^\infty \frac{\varphi(\tau)}{\tau} d\nu(\tau) \right) dS - \alpha\, |\partial\Omega|\, \hat{k}(0). \end{aligned} \tag{5.18}$$

In the second equality we also used (5.17). Since $\hat{k}(0) \neq 0$ this determines α and thus also U uniquely. This completes the proof.

5.3.4. The range of $\mathcal{A}$. In the case that $\mathcal{A}$ is not invertible (i.e. $(\tau \mapsto \tau^{-1}) \notin L^\infty_\nu$) in spite of Theorem 2.5 and [35, Proposition 3.1] it is important to know the range $R(\mathcal{A})$ of $\mathcal{A}$. To characterize the range we have to distinguish two cases: (i) $(\tau \mapsto \tau^{-1}) \in L^2_\nu$ and (ii) $(\tau \mapsto \tau^{-1}) \notin L^2_\nu$. In case (ii) for a given $\varphi \in L^2_\nu(0,\infty; L^2(\partial\Omega))$ there might exist no $p \in H^1(\Omega)$ such that

$$\left(\tau \mapsto \frac{\varphi(\tau) + \Gamma p}{\tau}\right) \in L^2_\nu(0,\infty; L^2(\partial\Omega)).$$

In the case that p exists, its boundary value Γp is uniquely determined and the function $(\tau \mapsto \varphi(\tau)/\tau)$ is integrable with respect to ν. Therefore we can define the complex number

$$m_{\varphi,p} = \int_{\partial\Omega}\int_0^\infty \frac{\varphi(\tau) + \Gamma p}{\tau} d\nu(\tau) dS. \tag{5.19}$$

Equipped with this notation we can now formulate:

THEOREM 5.8. *Assume that $\mathcal{A}$ is not invertible (i.e. $(\tau \mapsto \tau^{-1}) \notin L^\infty_\nu$). (i) If $(\tau \mapsto \tau^{-1}) \in L^2_\nu$, then*

$$R(\mathcal{A}) = \left\{ (q,w,\varphi) \in \mathcal{H}; \int_0^\infty \left\| \frac{\varphi(\tau)}{\tau} \right\|^2_{L^2(\partial\Omega)} d\nu(\tau) < \infty \right\}.$$

(ii) If $(\tau \mapsto \tau^{-1}) \notin L^2_\nu$, then

$$R(\mathcal{A}) = \left\{ (q,w,\varphi) \in \mathcal{H}; \exists p \in H^1(\Omega) : w = \nabla p, \int_\Omega q = m_{\varphi,p} \text{ and } \int_0^\infty \left\| \frac{\varphi(\tau) + \Gamma p}{\tau} \right\|^2_{L^2(\partial\Omega)} d\nu(\tau) < \infty \right\}$$

where $m_{\varphi,p}$ is given by (5.19). If (q,w,φ) is in the image of $\mathcal{A}$ then p is unique. In fact it is the first component of the pre-image of (q,w,φ).

PROOF. Let $\mathbf{y} = (q,w,\varphi) \in \mathcal{H}$. Clearly $\mathbf{y} \in R(-\mathcal{A})$ if and only if we can find $\mathbf{x} = (p,v,\psi) \in \mathcal{H}_1$ such that $-\mathcal{A}\mathbf{x} = \mathbf{y}$. Let $u \in H^1(\Omega)$ be such that $\nabla u = w$ and $\int_\Omega u = 0$. As in the proof of Theorem 5.4(iii) we see that necessarily $p = u + \alpha$ for some complex number α and

$$\frac{\varphi + \Gamma p}{\tau} = \psi \in L^2_\nu(0,\infty; L^2(\partial\Omega)). \tag{5.20}$$

Let us assume that case (i) is valid. Then ψ thus defined is in L^2_ν if and only if $(\tau \mapsto \varphi(\tau)/\tau)$ is square integrable with respect to ν. Now one can proceed as in the "$\Leftarrow$"-part of the proof of Theorem 5.4(iii) to find the unique p and v such that $-\mathcal{A}\mathbf{x} = \mathbf{y}$.

Let us now assume that case (ii) is valid. By (5.20) it is clear that the existence of p as in the definition of $R(\mathcal{A})$ is necessary. From the fact that $(\tau \mapsto \tau^{-1})$ is not square integrable we see that Γp is uniquely defined. Now we can again proceed as in the "$\Leftarrow$"-part of the proof of Theorem 5.4(iii) to find the unique p and v such that $-\mathcal{A}\mathbf{x} = \mathbf{y}$. The condition $\int_\Omega q = m_{\varphi,p}$ on $\mathbf{y}$ comes from (5.18), where we have to replace $\hat{k}(0)\Gamma u + \int_\varepsilon^\infty \frac{\varphi(\tau)}{\tau} d\nu(\tau) + \alpha\hat{k}(0)$ by $\int_0^\infty \frac{\varphi(\tau)+\Gamma p}{\tau} d\nu(\tau)$ in our situation. □

5.4. $(is - \mathcal{A})^{-1}$ at infinity

We are seeking for a non-decreasing function $M : [1,\infty) \to [1,\infty)$ such that for some constant $C > 0$

$$\left\|(is - \mathcal{A})^{-1}\right\| \le CM(|s|) \quad (|s| \ge 1).$$

In this section we show that the function $M(s) = 1/\Re\hat{k}(is)$ is an upper bound (up to a constant) for the norm of $(is - \mathcal{A})^{-1}$ when $|s|$ is large and if some additional assumptions on the acoustic impedance $\hat{k}$ and the domain are satisfied.

More precisely we assume that the acoustic impedance satisfies

$$\left[\left|\hat{k}\right| \frac{\left|\hat{k}\right|^2}{(\Re\hat{k})^2}\right](is) = o\left(\frac{1}{L(s)}\right) \text{ as } s \to \infty, \tag{5.21}$$
$$\text{where } L(s) = s^\alpha(1 + \log(s)) \text{ for } s \geq 1.$$

The real number $\alpha \in [0, 1)$ is a domain dependent constant which will be defined below. Note that for $\alpha \geq 1$ there cannot be any integrable completely monotone function which satisfies this condition.

Let (u_j) be the sequence of normalized eigenfunctions of the Neumann-Laplacian with respect to the corresponding (non-negative) frequencies (λ_j). That is

$$\begin{cases} \lambda_j^2 u_j(x) + \Delta u(x) = 0 & (x \in \Omega), \\ \partial_n u_j(x) = 0 & (x \in \partial\Omega), \\ \|u_j\|_{L^2(\Omega)} = 1. \end{cases} \tag{5.22}$$

The eigenfrequencies are counted with multiplicity and we may order them so that $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$. We call a function $p \in L^2(\Omega)$ a spectral cluster of width $\delta > 0$ whenever $\sup\{|\lambda_j - \lambda_i|\,; a_j, a_i \neq 0\} \leq \delta$ where $p = \sum a_j u_j$ is the expansion of p into eigenfunctions. We define the (mean) frequency $\lambda(p) \geq 0$ of p by $\lambda(p)^2 = \sum |(a_j/\|p\|_{L^2})|^2 \lambda_j^2$. We assume that the domain has the property that for sufficiently small $\delta > 0$ there are constants $c, C > 0$ such that for any spectral cluster p of width δ the following estimate is true

$$c\|p\|_{L^2(\Omega)}^2 \leq \int_{\partial\Omega} |\Gamma p|^2 \, dS \leq C\lambda(p)^\alpha \|p\|_{L^2(\Omega)}^2. \tag{5.23}$$

We call the left inequality the *lower estimate* and the right inequality the *upper estimate*. Note that the upper estimate is trivially satisfied for $\alpha = 1$ by applying the trace inequality from Lemma C.3. It is indeed reasonable to assume that this estimate holds for some α strictly smaller than 1. For example if the boundary of $\partial\Omega$ is of class C^∞ then both estimates hold with $\alpha = 2/3$; see [5] for this result. For Ω being an interval one can choose $\alpha = 0$ and for a square $\alpha = 1/2$ is optimal.

This section is devoted to the proof of our second main result:

THEOREM 5.9. *Assume that (5.21) is satisfied, where $\alpha \in [0, 1)$ is such that (5.23) holds for all spectral clusters p of sufficiently small width $\delta > 0$. Then there is a constant $C > 0$ such that*

$$\|R(s)\|_{L^2 \to L^2} \leq \frac{C}{s\Re\hat{k}(is)}$$

for all $s \geq 1$.

Compare this result to Theorem 5.4 to obtain that the norm of $\left\|(is - \mathcal{A})^{-1}\right\|$ is bounded by $\frac{C}{\Re\hat{k}(is)}$ under the constraints of the preceding theorem.

5.4.1. Some auxiliary definitions. We fix a $\delta > 0$ such that (5.23) is true for any spectral cluster of width 3δ. For $p, q \in H^1(\Omega)$ we define the Neumann form by

$$a_z^N(p,q) = z^2 \int_\Omega p\overline{q} + \int_\Omega \nabla p \cdot \nabla \overline{q}.$$

We cover $[0,\infty)$ by disjoint intervals $I_k = [k\lambda, (k+1)\lambda)$ for $k = 0,1,2,\dots$ such that

(i) $\lambda \in [2\delta, 3\delta]$,
(ii) $\exists k_c \in \mathbb{N} : I_{k_c} \supset (s-\delta, s+\delta)$.

The covering depends on $s \geq 1$ but this does not matter for our considerations. With the help of this partition we can uniquely expand every function $p \in L^2(\Omega)$ in terms of spectral clusters in the following way:

$$p = \sum_{k=0}^{\infty} c_k p_k \text{ where } p_k = \sum_{\lambda_j \in I_k} a_j u_j,\ \|p_k\|_{L^2(\Omega)} = 1.$$

Let $s_k(p) \in I_k$ be such that

$$s_k^2(p) = \int_\Omega |\nabla p_k|^2 .$$

Let $p^0_{+(-)} = \sum_{k>(<)k_c} c_k p_k$ and $p^0 = p^0_- + p^0_+$. Let $p_c = c_{k_c} p_{k_c}$. Obviously $p = p^0 + p_c$. Define

$$p_+ = \begin{cases} p^0_+ + p_c & \text{if } a_{is}^N(p_c) \geq 0, \\ p^0_+ & \text{else,} \end{cases}$$

and let p_- be given by $p = p_+ + p_-$. Finally let $\tilde{p} = p_+ - p_-$.

5.4.2. Some auxiliary lemmas. For the remaining part of Section 5.4 we use the notation introduced in Subsection 5.4.1 and we assume that $|s| \geq 1$.

LEMMA 5.10. *For all $p \in H^1(\Omega)$ we have $a_{is}^N(p,\tilde{p}) \geq |s|\,\delta \left\|p^0\right\|^2_{L^2(\Omega)}$.*

PROOF.

$$a_{is}^N(p,\tilde{p}) \geq a_{is}^N(p^0,(\tilde{p})^0) = \sum_{k\neq k_c} \left|s^2 - s_k^2\right| |c_k|^2 \geq s\delta \sum_{k\neq k_c} |c_k|^2 = s\delta \left\|p^0\right\|^2_{L^2(\Omega)} .$$

□

A little bit more involved is the proof of the next lemma.

LEMMA 5.11. *There is a constant $C > 0$ (depending on δ and α) such that for all $p \in H^1(\Omega)$*

$$\int_{\partial\Omega} \left|\Gamma p^0\right|^2 dS \leq C\,|s|^\alpha\,(1+\log(|s|))\frac{a_{is}^N(p,\tilde{p})}{|s|}$$

PROOF. Since $a_{is}^N(p,\tilde{p}) \geq a_{is}^N(p^0,(\tilde{p})^0)$ we may assume that $p_c = 0$. Because of

$$\int_{\partial\Omega} |\Gamma p|^2\, dS \leq 2\int_{\partial\Omega} |\Gamma p_-|^2 + |\Gamma p_+|^2\, dS$$
$$\text{and } a_{is}^N(p,\tilde{p}) = a_{is}^N(p_+) - a_{is}^N(p_-)$$

we may assume without loss of generality that either $p = p_+$ or $p = p_-$. We give the proof in detail for the case $p = p_+$. The case $p = p_-$ is analogous and therefore we omit it.

$$
\begin{aligned}
\int_{\partial\Omega} |\Gamma p_+|^2 \, dS &= \Big\| \sum_{k>k_c} c_k \Gamma p_k \Big\|^2_{L^2(\partial\Omega)} \\
&\le \Big(\sum_{k>k_c} |c_k| \, \|\Gamma p_k\|_{L^2(\partial\Omega)} \Big)^2 \\
&\le \Big(C\delta^{\frac{\alpha}{2}} \sum_{k>k_c} |c_k| \, k^{\frac{\alpha}{2}} \Big)^2 \\
&\le C\delta^\alpha \underbrace{\Big(\sum_{k>k_c} |c_k|^2 \, (s_k^2 - s^2) \Big)}_{a_{is}^N(p_+)} \underbrace{\Big(\sum_{k>k_c} \frac{k^\alpha}{s_k^2 - s^2} \Big)}_{=:J} .
\end{aligned}
$$

In the first line we used the continuity of the trace operator $\Gamma : H^1(\Omega) \to L^2(\partial\Omega)$. From the second to the third line we used the upper estimate (5.23) together with $s_k \in I_k = \lambda[k, k+1)$ with $\lambda \in [2\delta, 3\delta]$. It remains to estimate J. It is a well known trick to estimate sums of positive and non-increasing summands by corresponding integrals.

$$
\begin{aligned}
J = \sum_{k>k_c} \frac{k^\alpha}{s_k^2 - s^2} &\le \sum_{k>k_c} \frac{k^\alpha}{\lambda^2 k^2 - s^2} \\
&\le \frac{(k_c+1)^\alpha}{\lambda^2 (k_c+1)^2 - s^2} + \int_{k_c+1}^\infty \frac{x^\alpha}{\lambda^2 x^2 - s^2} dx \\
&=: J_1 + J_2.
\end{aligned}
$$

It is not difficult to see that J_1 can be estimated by a constant times $\delta^{-1-\alpha} s^{\alpha-1}$. For J_2 we substitute $y = \lambda x / s$ and use that $\lambda(k_c + 1) \ge 1 + \delta$. This yields

$$
\begin{aligned}
J_2 &\le C\delta^{-1-\alpha} s^{\alpha-1} \int_{1+\frac{\delta}{s}}^\infty \frac{y^\alpha}{y^2 - 1} dy \\
&\le C\delta^{-1-\alpha} s^{\alpha-1} \left(\int_{1+\frac{\delta}{s}}^2 \frac{1}{y-1} dy + \int_2^\infty \frac{1}{y^{2-\alpha}} dy \right) \\
&\le C\delta^{-1-\alpha} s^{\alpha-1} (\log(\frac{s}{\delta}) + 1).
\end{aligned}
$$

This conclues the proof. □

5.4.3. Proof of Theorem 5.9. Let $p \in H^1(\Omega)$ and $|s| \ge 1$. We have to verify

$$
\sup\{|a_{is}(p,u)| \, ; u \in H^1(\Omega), \, \|u\|_{L^2(\Omega)} \le 1\} \ge c \, |s| \, \Re\hat{k}(is) \, \|p\|_{L^2(\Omega)}
$$

for some constant $c > 0$ independent of p and s. In the following we assume that $a_{is}^N(p_c) \ge 0$. This implies that $p_+ = (p^0)_+ + p_c$ and $p_- = (p^0)_-$. The case $a_{is}^N(p_c) < 0$ can be treated similarly and we therefore omit it. First we prove an auxiliary estimate with the help of Lemma 5.11:

$$\begin{aligned}\int_{\partial\Omega} |\Gamma p_+|^2 + |\Gamma p_-|^2\, dS &= \int_{\partial\Omega} \left|\Gamma p_+^0\right|^2 + \left|\Gamma p_-^0\right|^2 + |\Gamma p_c|^2 + 2\Re(\Gamma p_+^0 \overline{\Gamma p_c}) dS \\ &\le \int_{\partial\Omega} 2\left|\Gamma p_+^0\right|^2 + \left|\Gamma p_-^0\right|^2 + 2\,|\Gamma p_c|^2\, dS \\ &\le C L(s) \frac{a_{is}^N(p,\tilde p)}{|s|} + 2\int_{\partial\Omega} |\Gamma p_c|^2\, dS. \qquad (5.24)\end{aligned}$$

Let us define

$$L_1(s) = \left(\frac{\left|\hat k(is)\right|}{\Re \hat k(is)}\right)^2 L(s) \ge L(s).$$

Our assumption (5.21) on k is equivalent to $|\hat k|(is) = o(1/L_1(s))$ as $|s| \to \infty$. Now we come to the final part of the proof which consists of distinguishing two cases. Essentially the first case means that p is roughly the same as p^0 and the second case means that p is roughly the same as p_c. We fix a constant $\varepsilon_1 \in (0,1)$ to be chosen later. The choice of ε_1 does not depend on s.

Case 1: $L_1(s) a_{is}^N(p,\tilde p) \ge \varepsilon_1 |s| \int_{\partial\Omega} |\Gamma p_c|^2\, dS$. We first show that in this case the Neumann form dominates the form a_{is} for $|s|$ big enough in the following sense:

$$\begin{aligned}\left|a_{is}(p,\tilde p) - a_{is}^N(p,\tilde p)\right| &= \left| s\hat k(is) \int_{\partial\Omega} (\Gamma p_+ + \Gamma p_-)\overline{(\Gamma p_+ - \Gamma p_-)} dS \right| \\ &\le 2\left|s\hat k(is)\right| \int_{\partial\Omega} |\Gamma p_+|^2 + |\Gamma p_-|^2\, dS \\ &\le C\left|s\hat k(is)\right| \varepsilon_1^{-1} L_1(s) \frac{a_{is}^N(p,\tilde p)}{|s|} \\ &\le \frac{1}{2} a_{is}^N(p,\tilde p).\end{aligned}$$

From the second to the third line we used the assumption of case 1 and (5.24). By (5.21) the last line is valid for all $s \ge s_0$, where s_0 is sufficiently large depending on how small ε_1 is. Therefore we have

$$\begin{aligned}|a_{is}(p,\tilde p)| &\ge \left(\frac14 + \frac14\right) a_{is}^N(p,\tilde p) \\ &\ge \frac{s\delta}{4} \left\|p^0\right\|_{L^2(\Omega)}^2 + \frac{\varepsilon_1 |s|}{4L_1(s)} \int_{\partial\Omega} |\Gamma p_c|^2\, dS \\ &\ge \frac{c\varepsilon_1 |s|}{L_1(s)} \left(\left\|p^0\right\|_{L^2(\Omega)}^2 + \|p_c\|_{L^2(\Omega)}^2 \right) \\ &\ge c\varepsilon_1 |s| \Re\hat k(is) \|p\|_{L^2(\Omega)}^2 .\end{aligned}$$

From the second to the third line we used the lower estimate (5.23) and in the last step we used our assumptions on the acoustic impedance (5.21). The theorem is proved for case 1.

Case 2: $L_1(s) a_{is}^N(p,\tilde p) < \varepsilon_1 |s| \int_{\partial\Omega} |\Gamma p_c|^2\, dS$. By Lemma 5.10 and since $\lim_{|s|\to\infty} L_1(s) = \infty$ this yields

$$\int_{\partial\Omega} |\Gamma p_c|^2\, dS \ge \left\|p^0\right\|_{L^2(\Omega)}^2 \qquad (5.25)$$

for all $|s| \geq s_1$ with an $s_1 > 0$ not depending on ε_1. We show now that in case 2 the form a_{is} is dominated by the contribution from the boundary. By Lemma 5.11 we have

$$\begin{aligned}\left|\int_{\partial\Omega} p^0 \overline{p_c} dS\right| &\leq \left(CL(s)\frac{a_{is}^N(p,\tilde{p})}{|s|}\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} |p_c|^2 \, dS\right)^{\frac{1}{2}} \\ &\leq C\sqrt{\varepsilon_1}\left(\frac{L(s)}{L_1(s)}\right)^{\frac{1}{2}} \int_{\partial\Omega} |p_c|^2 \, dS \\ &\leq \frac{\Re\hat{k}(is)}{2\left|\hat{k}(is)\right|} \int_{\partial\Omega} |p_c|^2 \, dS.\end{aligned}$$

In the last step we choose ε_1 so small that $C\sqrt{\varepsilon_1} \leq 1/2$. Finally from this, (5.25) and the lower estimate (5.23) we deduce that

$$\begin{aligned}\Im a_{is}(p,p_c) &\geq \frac{1}{2}|s|\,\Re\hat{k}(is)\int_{\partial\Omega}|p_c|^2\,dS \\ &\geq c\,|s|\,\Re\hat{k}(is)(\|p_c\|^2_{L^2(\Omega)} + \|p^0\|^2_{L^2(\Omega)}) \\ &= c\,|s|\,\Re\hat{k}(is)\,\|p\|^2_{L^2(\Omega)}\end{aligned}$$

which yields the claimed result.

5.5. Examples: sharp decay rates for $\hat{k}$ satisfying a power law

To illustrate Theorem 5.4 and Theorem 5.9, we consider special *standard kernels* $k = k_{\beta,\varepsilon}$ (with $\varepsilon > 0$ and $0 < \beta < 1$) introduced below. These standard kernels have the property that $\Re\hat{k}(is) \approx |\hat{k}(is)| \approx |s|^{\beta-1}$ for large $|s|$. This makes it easy to check whether (5.21) is satisfied or not. We take a closer look at Ω being a square or a disk. In the case of the disk we show the optimality of the resolvent estimate, that is we show that $\left\|(is - \mathcal{A})^{-1}\right\|$ is not only bounded from above by a constant times $1/\Re\hat{k}(is)$ but also from below. The standard kernels are designed in such a way that $\mathcal{A}$ is invertible (i.e. $(\tau \mapsto \tau^{-1}) \in L^2_\nu$; see Theorem 5.4). We have assumed this for the simplicity of exposition. However, in Subsection 5.5.5 we briefly show that our results yield (sharp) decay rates also in the presence of a singularity at zero. The case $\Omega = (0,1)$ is treated separately in Sections 5.6 and 5.7.

5.5.1. Properties of the standard kernels.

For $\varepsilon > 0$ and $0 < \beta < 1$ let

$$k_{\beta,\varepsilon}(t) = e^{-\varepsilon t} t^{-(1-\beta)} \text{ for } t > 0.$$

To keep the notation short we fix ε and β now and write k instead of $k_{\beta,\varepsilon}$ throughout this section. Obviously $k \in L^1(0,\infty)$ and for all $n \in \mathbb{N}_0$ we have $(-1)^n d^n k/dt^n(t) > 0$. The latter property is a characterization of completely monotone functions. Thus the kernel k is admissible in the sense that the semigroup from Section 5.2 is defined.

Let Γ denote the Gamma function. Taking Laplace transform yields for $z > -\varepsilon$

$$\hat{k}(z) = \int_0^\infty e^{-(\varepsilon+z)t} t^{-(1-\beta)} dt = \frac{1}{(\varepsilon+z)^\beta}\int_0^\infty s^{-(1-\beta)} e^{-s} ds = \frac{\Gamma(\beta)}{(\varepsilon+z)^\beta}.$$

By analyticity the equality between the left end and the right end of this chain of equations extends to $\mathbb{C}\backslash(-\infty,-\varepsilon]$.

For $s \in \mathbb{R}$, let $\varphi(s) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ be the argument of $\varepsilon - is$. Note that $\varphi(s) \to \mp\frac{\pi}{2}$ as $s \to \pm\infty$. Then we have

$$\hat{k}(is) = \Gamma(\beta) \left| \frac{\varepsilon - is}{\varepsilon^2 + s^2} \right|^\beta (\cos(\beta\varphi(s)) + i \sin(\beta\varphi(s))) .$$

In particular

$$\Re\hat{k}(is) \approx \left| \Im\hat{k}(is) \right| \approx \frac{1}{|s|^\beta} \text{ for } |s| \geq 1.$$

Here by $\approx$ we mean that the left-hand side is up to a constant, which does not depend on s, an upper bound for the right-hand side and vice versa. The first $\approx$-relation implies that the condition (5.21) is equivalent to the simpler estimate $\Re\hat{k}(is) = o(1/L(s))$ as $|s|$ tends to infinity. More precisely we have

$$(5.21) \Leftrightarrow \beta > \alpha. \tag{5.26}$$

It is well known that for $z > 0$ and $\beta \in (0, 1)$

$$z^{-\beta} = \frac{\sin(\pi\beta)}{\pi} \int_0^\infty \frac{1}{\tau + z} \frac{d\tau}{\tau^\beta}.$$

Thus

$$\hat{k}(z) = \frac{\sin(\beta\pi)}{\pi\Gamma(\beta)} \int_\varepsilon^\infty \frac{1}{\tau + z} \frac{d\tau}{(\tau - \varepsilon)^\beta}.$$

In the notation of Section 5.1 this means

$$d\nu(\tau) = \frac{\sin(\beta\pi)}{\pi\Gamma(\beta)} \cdot \frac{1_{[\varepsilon,\infty)}}{(\tau - \varepsilon)^\beta} d\tau.$$

By Theorem 5.4 (iii) we see that $\mathcal{A}$ is invertible.

5.5.2. Smooth domains. Let us suppose that Ω has a C^∞ boundary and let $k = k_{\beta,\varepsilon}$ for some $\varepsilon > 0$ and $0 < \beta < 1$. By [5] we know that (5.23) is satisfied for $\alpha = 2/3$. Thus by (5.26) and Theorem 5.9 we have

$$\beta > \frac{2}{3} \quad \Longrightarrow \quad \forall s \in \mathbb{R} : \left\| (is - \mathcal{A})^{-1} \right\| \leq C(1 + |s|)^\beta. \tag{5.27}$$

By Theorem 0.3 or 2.2 this implies

PROPOSITION 5.12. *Let $\partial\Omega$ be of class C^∞ and $k = k_{\beta,\varepsilon}$. If $\beta > 2/3$ then, for all $t > 0$ and $\mathbf{x}_0 \in \mathcal{H}_1$,*

$$E(t, \mathbf{x}_0) \leq Ct^{-\frac{2}{\beta}} E_1(\mathbf{x}_0).$$

5.5.3. The disk. Let $\Omega = D$ be the unit disk in $\mathbb{R}^2$. The smallest possible choice of α in (5.23) is indeed 2/3. The simple proof is based on a *Rellich-type identity*, see for instance [5, page 5]. So the circle already realizes the "worsed case scenario" with respect to the upper bounds for Neumann eigenfunctions. Thus in Proposition 5.12 we cannot replace the condition $\beta > 2/3$ by a weaker one. Instead we show the optimality of the upper bound for the energy decay. Therefore we investigate the spectrum of $\mathcal{A}$.

LEMMA 5.13. *Let $\Omega = D$ and $k = k_{\beta,\varepsilon}$. Then there exists a sequence (z_n) in the spectrum of $\mathcal{A}$ such that $(\Im z_n)$ is positive and increasing and such that there exists a constant $C > 0$ such that, for every n,*

$$\frac{C}{(\Im z_n)^\beta} \leq \Re z_n < 0.$$

As a corollary we have

$$\forall s > 0 : \sup_{|\sigma| \leq s} \left\| (i\sigma - \mathcal{A})^{-1} \right\| \geq C(1+s)^\beta.$$

By the converse part of Theorem 0.1 together with Theorem 0.3 or 2.2 this implies

PROPOSITION 5.14. *Let $\Omega = D$ and $k = k_{\beta,\varepsilon}$. If $\beta > 2/3$ then we have for all $t \geq 1$ that*

$$ct^{-\frac{2}{\beta}} \leq \sup_{E_1(\mathbf{x}_0) \leq 1} E(t, \mathbf{x}_0) \leq Ct^{-\frac{2}{\beta}}.$$

If $\beta \in (0, 1)$ is arbitrary the left inequality remains valid.

PROOF OF LEMMA 5.13. Except for the rate of convergence of (z_n) towards the imaginary axis the content of our lemma is included in [23, Theorem 5.2]. Therefore we only sketch the existence of a sequence (z_n) with imaginary part tending to infinity and real part tending to zero.

First recall that an eigenvalue is a complex number z_n such that (5.9) with $F = 0$ and $z = z_n$ has a non-zero solution u. After a transformation to polar coordinates, by a separation of variables argument one can show that the existence of u is equivalent to the existence of a non-zero solution v of

$$\begin{cases} v''(r) + \frac{1}{r}v'(r) - (\frac{l^2}{r^2} + z^2)v(r) = 0 \quad (0 < r < 1), \\ v'(1) + z\hat{k}(z)v(1) = 0, \\ v(0+) \text{ is finite,} \end{cases}$$

for some $l \in \mathbb{N}_0$. The first and the third line forces $v(r)$ to be proportional to $J_l(izr)$, where J_l is the l-th order Bessel function of the first kind (see e.g. [1, Chapter 9]). Therefore the second line implies

$$\frac{J_l'(iz)}{J_l(iz)} = i\hat{k}(z). \tag{5.28}$$

We have seen that a complex number $z_n \notin (-\infty, 0]$ is an eigenvalue of the wave operator if and only if it is a zero of (5.28) for some l. Let us fix l now. Following the approach of [23] one can prove the existence of a sequence of zeros $(z_n) = (is_n - \xi_n)$ with $s_n = n\pi + (2l+1)\pi/4$, $\Re\xi_n > 0$ and ξ_n tending to zero, by a Rouché argument.

It remains to prove that $\xi_n = O((\Im z_n)^{-(1-\beta)})$. By [1, Formula 9.2.1] the following asymptotic formula holds if z tends to infinity while $\Re z$ stays bounded (and l is fixed):

$$J_l(iz) = \sqrt{\frac{2}{\pi z}} \cos\left(iz - \frac{(2l+1)}{4}\pi\right) + O(|z|^{-1}). \tag{5.29}$$

A naive way to get the corresponding asympotic formula for J' and J'' would be to take derivatives of the cosine. In fact this yields the correct leading term. The error term is again $O(|z|^{-1})$ in both cases. For the first derivative this is

[1, Formula 9.2.11]. The formula for the second derivative then follows from the ordinary differential equation satisfied by J_l.

Thus by a Taylor expansion of (5.28) we get:

$$0 + i\xi_n + O(|\xi_n|^2 + n^{-1}) = i\hat{k}(is_n) - i\xi_n \hat{k}'(is_n) + O(|\xi_n|^2 + n^{-1}).$$

This implies

$$\begin{aligned} \xi_n &= (1 + o(1))\hat{k}(is_n) \\ &= (1 + o(1))\frac{\Gamma(\beta)}{s_n^\beta}\left(\cos(\beta\varphi(s_n)) + i\sin(\beta\varphi(s_n))\right). \end{aligned} \tag{5.30}$$

Here $\varphi(s)$ is the argument of $\varepsilon - is$ (see Section 5.5.1). □

Note that in the undamped case $k = 0$ we have $z_n^0 = s_n + O(s_n^{-1})$ by [1, Formula 9.5.12] for the eigenvalues z_n^0. Here again $s_n = n\pi + (2l+1)\pi/4$ and l is fixed. Thus (5.30) implies that $z_n = z_n^0 - (1 + o(1))\hat{k}(is_n)$.

5.5.4. The square. Let $\Omega = Q = (0, \pi)^2$ be a square. In terms of upper bounds for boundary values of spectral clusters the square behaves slightly better than the disk. It seems to be reasonable to believe that this is due to the fact that the square has no *whispering gallery modes*.

LEMMA 5.15. *Let $\Omega = Q$, $k = k_{\beta,\varepsilon}$ and $\delta > 0$. If δ is sufficiently small then for each $L^2(Q)$-normalized spectral cluster p of width δ of the Neumann-Laplace operator*

$$c \le \int_{\partial\Omega} |\Gamma p|^2 dS \le Cs(p)^{\frac{1}{2}}.$$

The constants $c, C > 0$ do not depend on p. Furthermore the exponent $\alpha(Q) = 1/2$ is optimal, i.e. one cannot replace it by a smaller one.

The optimality assertion of Lemma 5.15 may be somewhat surprising. If p was restricted to be a (pure) eigenfunction of the Neumann-Laplace operator the optimal exponent would be $\alpha = 0$. This is a direct consequence of the explicit formula available for the eigenfunctions. However, it will be clear from the proof why spectral clusters behave differently.

As in the preceding examples the lemma implies

PROPOSITION 5.16. *Let $\Omega = Q$, $k = k_{\beta,\varepsilon}$. If $\beta > 1/2$ then, for all $t > 0$ and $\mathbf{x}_0 \in \mathcal{H}_1$,*

$$E(t, \mathbf{x}_0) \le Ct^{-\frac{2}{\beta}} E_1(\mathbf{x}_0).$$

PROOF OF LEMMA 5.15. The explicit form of the normalized Neumann eigenfunctions $u_{m,n}$ and its eigenfrequencies $\lambda_{m,n} \ge 0$ is

$$u_{m,n}(x, y) = 2\cos(mx)\cos(ny),\ \lambda_{m,n}^2 = m^2 + n^2.$$

Let $p = \sum_{m,n} a_{n,m} u_{n,m}$ be a normalized spectral cluster of width δ. We choose $s \ge 0$ such that the set of indices (m, n) with $a_{m,n} \ne 0$ is included in I which is given by

$$\begin{aligned} I &= \{(m, n) \in \mathbb{N}_0^2; s^2 \le m^2 + n^2 \le (s + \delta)^2\}, \\ I_1 &= \{(m, n) \in I; m \ge n\}. \end{aligned}$$

Without loss of generality we may assume that $\sum_{I_1} |a_{m,n}|^2 \geq 1/2$. We first prove the lower bound:

$$\begin{aligned}\int_{\partial\Omega} |\Gamma p|^2 dS &= \sum_n \left\| \sum_m a_{m,n} \Gamma u_{m,n} \right\|^2_{L^2(\partial\Omega)} \\ &\geq 16\pi \sum_{I_1} |a_{m,n}|^2 \\ &\geq 8\pi \|p\|^2_{L^2(\Omega)} .\end{aligned}$$

In the first line we use the orthogonality relation for the cosine functions with respect to the y variable. In the second line we use $\|u_{m,n}\|_{L^2(\partial\Omega)} = 4\sqrt{\pi}$ and the fact that the partial sum over m in the preceding step includes only one member if δ is small and if the index set is restriced to I_1.

Let N_n be the number of non-zero summands with respect to the inner sum in line one. It is not difficult to see that $N_n \leq C\sqrt{s}$ for a constant independent of n and s. Therefore we have

$$\begin{aligned}\int_{\partial\Omega} |\Gamma p|^2 dS &= \sum_n \left\| \sum_m a_{m,n} \Gamma u_{m,n} \right\|^2_{L^2(\partial\Omega)} \\ &= \sum_n N_n^2 \left\| \frac{1}{N_n} \sum_m a_{m,n} \Gamma u_{m,n} \right\|^2_{L^2(\partial\Omega)} \\ &\leq C \sum_{m,n} N_n |a_{m,n}|^2 \\ &\leq C s^{\frac{1}{2}} \|p\|^2_{L^2(\Omega)} .\end{aligned}$$

It remains to prove optimality of the exponent $\alpha = 1/2$. For $n_1 \in \mathbb{N}$ we consider a special spectral cluster p_1 of the form

$$p_1 = 2 \sum_{m=0}^{N-1} a_m \cos(mx) \cos(n_1 y)$$
$$\text{where } N = N(n_1) = \lceil \varepsilon \sqrt{n_1} \rceil .$$

If $\varepsilon > 0$ is sufficiently small and n_1 large enough we see that p_1 is a spectral cluster of width δ. If we set $a_m = 1/\sqrt{N}$ we see that the $L^2(\Omega)$-norm of p_1 is 1 and

$$\int_{\{0\}\times(0,1)} |\Gamma p_1|^2 \, dS = \left| \sum_{m=1}^{N} a_m \right|^2 = N(n_1) \geq \varepsilon\sqrt{n_1}.$$

This finishes the proof since $s(p_1) \in [n_1, n_1 + \delta]$. □

5.5.5. Decay in the presence of a singularity at zero. So far in this section we have excluded the case when $\mathcal{A}$ has a singularity at zero. The purpose of this subsection is to show that getting decay rates in this case is not more difficult than in the case where there is no singularity at zero. As in the previous subsection we simplify our presentation by considering a special family $(\hat{k}'_{\alpha,\beta})_{\alpha,\beta}$ of acoustic impedances given by the measures

$$d\nu'_{\alpha,\beta} = \tau^\alpha d\tau|_{(0,1)} + (\tau - 1)^{-\beta} d\tau|_{(1,\infty)} \quad (\alpha \in (0,\infty), \beta \in (0,1)).$$

Obviously $(\tau \mapsto \tau^{-1})$ is integrable with respect to $\nu'_{\alpha,\beta}$ (thus $k'_{\alpha,\beta}$ is integrable) but it is not bounded with respect to that measure. Observe that $\alpha > 1$ implies that $(\tau \mapsto \tau^{-1})$ is square integrable with respect to ν'. In the following we assume for simplicity that $\alpha > 1$. The reason is that by Theorem 5.8 the range of $\mathcal{A}$ has a simpler representation in this case.

LEMMA 5.17. *Let $\alpha \in (1,\infty), \beta \in (0,1)$. Then $(\tau \mapsto \tau^{-1})$ is integrable, square integrable but unbounded with respect to $\nu'_{\alpha,\beta}$. Moreover*

$$\hat{k}'_{\alpha,\beta}(z) = \frac{\pi}{\sin(\beta\pi)}(1+z)^{-\beta} + O(|z|^{-1})$$

as z tends to infinity avoiding $\mathbb{R}_-$.

PROOF. We only have to prove the last statement. We calculate

$$\hat{k}'(z) = \int_0^1 \frac{\tau^\alpha}{z+\tau}d\tau + \int_1^\infty \frac{1}{z+\tau}\frac{d\tau}{(\tau-1)^\beta} =: I + II.$$

It is easy to see that the modulus of I is bounded by $(|z|-1)^{-1}$ for all z with $|z| > 1$. With regard to II we see that the well known identity

$$z^{-\beta} = \frac{\sin(\beta\pi)}{\pi}\int_0^\infty \frac{1}{z+\tau}\frac{d\tau}{\tau^\beta}$$

finishes the proof. □

PROPOSITION 5.18. *Let $\alpha \in (0,\infty), \beta \in (2/3,1)$ and $k = k'_{\alpha,\beta}$. Let $\partial\Omega$ be a C^∞-manifold. Then*

$$\left\|(is-\mathcal{A})^{-1}\right\| = O(|s|^\beta)$$

as $|s| > 1$ tends to infinity.

PROOF. This is an immediate consequence of Lemma 5.17 together with Theorem 5.4(i) and Theorem 5.9. □

We are now in the position to prove an *optimal* decay estimate.

PROPOSITION 5.19. *Let $\alpha \in (1,\infty), \beta \in (2/3,1)$ and $k = k_{\alpha,\beta}$. Let $\partial\Omega$ be a C^∞-manifold. Then*

$$E(t,\mathbf{x}_0) \le \frac{C}{1+t^2}\left[E_1(\mathbf{x}_0) + \int_0^\infty \|\psi_0(\tau)\|^2_{L^2(\partial\Omega)}\frac{d\nu(\tau)}{\tau^2}\right]$$

holds for all $t \ge 0$ and for all $\mathbf{x}_0 = (p_0, v_0, \psi_0) \in \mathcal{H}$ for which the right-hand side is finite. The constant $C > 0$ does not depend on $\mathbf{x}_0$ or t. Moreover this estimate is sharp in the sense that it would be invalid if one replaces $C/(1+t^2)$ by $o(1/(1+t^2))$ as t tends to infinity.

PROOF. Proposition 5.18, Theorem 5.4(ii) together with Theorem 2.5 yield

$$\left\|e^{t\mathcal{A}}\mathbf{x}_0\right\| \le \frac{C}{1+t}\|\mathbf{x}_0\|_{D(\mathcal{A})\cap R(\mathcal{A})} \text{ for all } \mathbf{x}_0 \in D(\mathcal{A}) \cap R(\mathcal{A}).$$

We know that the norm of $D(\mathcal{A})$ is (equivalent to) the square root of the first order energy E_1. By Theorem 5.8 the norm on $R(\mathcal{A})$ is given by

$$\|\mathbf{x}_0\|^2_{R(\mathcal{A})} = E(\mathbf{x}_0) + \int_0^\infty \|\psi_0(\tau)\|^2_{L^2(\partial\Omega)}\frac{d\nu(\tau)}{\tau^2}.$$

This gives the desired estimate. The sharpness of this estimate follows from [10, Theorem 6.9 and the remarks in Section 8]. □

5.6. Sharp resolvent bounds for the 1D case

Throughout this section $\Omega = (0,1)$ and k is a completely monotone, integrable function. We aim to show that in the 1D setting the conclusion of Theorem 5.9 remains true without any further hypothesis - like (5.21) - on the acoustic impedance. Even more can be done - we prove that the upper estimate is optimal. More precisely we prove

THEOREM 5.20. *Let $\Omega = (0,1)$. Then there are constants $c, C > 0$ such that for all $s > 1$ we have*

$$\frac{c}{\Re \hat{k}(is)} \leq \sup_{1 \leq |\sigma| \leq s} \left\| (i\sigma - \mathcal{A})^{-1} \right\| \leq \frac{C}{\Re \hat{k}(is)}.$$

We prove the lower bound by investigating the spectrum of $-\mathcal{A}$ which is close to the imaginary axis (Subsection 5.6.1). Furthermore we give a more or less concrete formula for the stationary resolvent operator $R(is)$ which allows to prove the upper bound (Subsection 5.6.2). Section 5.7 contains implications of Theorem 5.20 for the decay rates of the energy of the wave equation.

5.6.1. The spectrum. The spectrum of $\mathcal{A}$ satisfies a characteristic equation which is implicitly contained in [23]. For convenience of the reader we give a complete proof.

PROPOSITION 5.21. *A number $z \in \mathbb{C} \backslash (-\infty, 0]$ is in the spectrum of $\mathcal{A}$, and hence an eigenvalue, if and only if it satisfies*

$$\left(\hat{k}(z) - i \tan\left(\frac{iz}{2}\right)\right) \cdot \left(\hat{k}(z) + i \cot\left(\frac{iz}{2}\right)\right) = 0 \tag{5.31}$$

PROOF. By Theorem 5.2 together with the equivalence between (5.10) and (5.11) we see that z is a spectral point if and only if there is a non-zero function p solving

$$\begin{cases} z^2 p(x) - p''(x) = 0 & (x \in (0,1)), \\ -p'(0) + z\hat{k}(z)p(0) = 0, \\ p'(1) + z\hat{k}(z)p(1) = 0. \end{cases}$$

Up to a scalar factor the first two lines are equivalent to the following ansatz

$$p(x) = \cos(izx) - i\hat{k}(z)\sin(izx).$$

Plugging this into the third line yields that z is an eigenvalue if and only if

$$\left(\hat{k}(z)^2 + 2i\hat{k}(z)\cot(iz) + 1\right) z \sin(iz) = 0. \tag{5.32}$$

Note that the zeros of the sine function do not lead to an eigenvalue since the cotangent function has a singularity at the same point. Actually we already know from the situation of general domains that an eigenvalue which is neither zero nor a negative number must have negative real-part. Thus we may simplify (5.32) by dividing by $z \sin(iz)$. The claim now follows from the formula $\cot(\zeta) - \tan(\zeta) = 2\cot(2\zeta)$ which is valid for all complex numbers ζ. □

Let $H, R > 0$. The reader may consider H and R as large numbers. We are interested in the part of the spectrum of $-\mathcal{A}$ contained in the strip

$$\mathcal{U}_H^R = \{z \in \mathbb{C}; |\Im z| > R \text{ and } -H < \Re z < 0\}.$$

PROPOSITION 5.22. *Let $H > 0$. Then for $R > 0$ large enough there exists a natural number $n_0 > 0$ such that the part of the spectrum of $\mathcal{A}$ which is contained in $\mathcal{U}_H^R$ is given by a doubly infinite sequence $(z_n)_{n=\pm n_0}^{\infty}$ with $z_{-n} = \overline{z_n}$ for all n and*

$$\Im z_n = \pi n - \left[(2 + O(|\hat{k}|))\Im \hat{k}\right](i\pi n),$$
$$\Re z_n = -\left[(2 + O(|\hat{k}|))\Re \hat{k}\right](i\pi n).$$

As a consequence the lower bound in Theorem 5.20 is proved.

Note that the two asymptotic formulas given by the proposition imply $z_n = (2+o(1))\hat{k}(in\pi)$ for n tending to plus or minus infinity. This formula can be proved by the same Taylor expansion argument as in the proof of Lemma 5.13. See also the remark after the proof of the mentioned lemma. But this is not enough in order to prove the lower bound in Theorem 5.20 since it might happen that the real part of $\hat{k}(is)$ tends much faster to zero than its imaginary part! This explains the more elaborate Taylor expansion technique in the proof below.

PROOF OF PROPOSITION 5.22. We are searching for the solutions $z \in \mathcal{U}_H^R$ of the characteristic equation (5.31). For simplicity we only consider the solutions of

$$z \in \mathcal{U}_H^R \text{ and } F(z) := \hat{k}(z) - i \tan\left(\frac{iz}{2}\right) = 0.$$

We apply a Rouché argument to show that the zeros of this equation are close to the zeros $is_{2n} = 2n\pi i$ of the tangens-type function on the right-hand side. Let (ε_{2n}) be a null-sequence of positive real numbers, smaller than H, to be fixed later. Let B_{2n} be the open ball of radius ε_{2n} around the center is_{2n}. For $r > 0$ let

$$\mathcal{V}_H^R(r) = \{z \in \mathbb{C}; R < \Im z < R + r \text{ and } -H < \Re z < H\}. \tag{5.33}$$

Take $K(r)$ to be the boundary of the set $\mathcal{V}_H^R(r)\backslash(\bigcup_n B_{2n})$. Since $\hat{k}(z)$ tends to zero as z tends to infinity with bounded real part we can choose R so large and (ε_{2n}) so slowly decreasing such that $|\hat{k}(z)| < |i\tan(iz/2)|$ for $z \in K(r)$. Thus Rouché's theorem for meromorphic functions says that for F and for $(z \mapsto i\tan(iz/2))$ restricted to $\mathcal{V}_H^R(r)$ the number of zeros minus the number of poles (counted with multiplicity) is the same for all $r > 0$. The poles of F are actually the same as for for the tangens type function. Thus it is proved that for large enough $n_0 \in \mathbb{N}$ the zeros of F from $\mathcal{U}_H^R$ for $R = (2n_0 - 1)\pi$ are simple and contained in the balls B_{2n} for $|n| \geq n_0$. Note that we used that we already know that zeros of the characteristic equation must have negative real part.

We have verified that all zeros z_{2n} of $F|_{\mathcal{U}_H^R}$ are given by the following ansatz:

$$z_{2n} = is_{2n} - \xi_{2n} \text{ with } \Re\xi_{2n} > 0 \text{ and } \xi_{2n} = o(1).$$

In the remaining part of the proof we want to simplify the notation by dropping the indices from z, s and ξ. We also write $\hat{k}$ instead of $\hat{k}(z)$. It is not difficult to verify that $F(z) = 0$ is equivalent to

$$e^z = \frac{1 - \hat{k}}{1 + \hat{k}} = \frac{(1 - i\Im\hat{k}) - \Re\hat{k}}{(1 + i\Im\hat{k}) + \Re\hat{k}}.$$

Let $\alpha = \arg(1 + i\Im\hat{k})$ be the argument of $1 + i\Im\hat{k}$ and $L = (1 + (\Im\hat{k})^2)^{1/2}$. Then

$$\begin{aligned}&\arg(1 \pm \hat{k}) = \pm\alpha(1 + O(\Re\hat{k})) = \pm(1 + O(|\hat{k}|))\Im\hat{k},\\ &\text{thus } \Im\xi = 2(1 + O(|\hat{k}|))\Im\hat{k}.\end{aligned}$$

This yields the first asymptotic formula claimed in the proposition. The second asympotic formula is a direct consequence of

$$e^{-\Re\xi} = \frac{L - (1 + O(|\alpha|^2))\Re\hat{k}}{L + (1 + O(|\alpha|^2))\Re\hat{k}} = 1 - \frac{2}{L}(1 + O(|\Im\hat{k}|^2))\Re\hat{k} + O((\Re\hat{k})^2).$$

□

5.6.2. Upper resolvent estimate. We prove the upper estimate stated in Theorem 5.20. By Theorem 5.4 part (i) it suffices to show

PROPOSITION 5.23. *For all* $|s| \geq 1$ *we have* $\|R(is)\|_{L^2\to L^2} \leq C(|s|\,\Re\hat{k}(is))^{-1}$.

PROOF. For some $f \in L^2(0,1)$ let p be the solution of

$$\begin{cases} -s^2p(x) - p''(x) = f & (x \in (0,1)),\\ -p'(0) + is\hat{k}(is)p(0) = 0, \\ p'(1) + is\hat{k}(is)p(1) = 0. \end{cases} \tag{5.34}$$

Let us define two auxiliary functions p_f and p^0 by

$$p_f(x) = -\frac{1}{s}\int_0^x \sin(s(x-y))f(y)dy \text{ and } p^0(x) = \cos(sx) + i\hat{k}(is)\sin(sx).$$

It is easy to see that $p = ap^0 + p_f$ with $a \in \mathbb{C}$ is the only possible ansatz which satisfies the first two lines in (5.34). The parameter a is uniquely defined by the condition from the third line. A short calculation yields that this condition is equivalent to

$$as \cdot \underbrace{\left(\hat{k}(is) + i\tan\left(\frac{s}{2}\right)\right)\left(\hat{k}(is) - i\cot\left(\frac{s}{2}\right)\right)}_{=:D(s)} \cdot \sin(s) = -p_f'(1) - is\hat{k}(is)p_f(1).$$

Note that the singularities of D cancel the zeros of the sine function. Thus we have an explicit formula for a in terms of f. Further note that the absolute values of $sp_f(1)$ and $p_f'(1)$ can be estimated from above by a constant times $\|f\|_{L^2}$. Thus

$$|a| \leq \frac{C}{|s|} \cdot \frac{1}{|D(s)\sin(s)|} \cdot \|f\|_{L^2(0,1)}.$$

By the presence of the tangent and contangent type function the factor $D(s)\sin(s)$ can only be small in a neighbourhood of $s = 2n\pi$ or $s = (2n+1)\pi$. But in this case the real part of $\hat{k}$ prevents D from getting too small. We thus have an estimate $|D(s)\sin(s)| \geq c\Re\hat{k}(is)$ for $|s| \geq 1$ which in turn gives an upper bound on $|a|$. Since the L^2-norm of p^0 can be estimated from above by a constant the proof is finished. □

5.7. Examples: sharp decay rates under $\Re\hat{k}(i\cdot) \in \mathrm{PD}$

For simplicity of exposition we assume that $\Omega = (0,1)$ throughout this section. Let us summarize what we found out in Section 5.6.

THEOREM 5.24. *Let $\Omega = (0,1)$ and k be an integrable completely monotone function. Then there is $c, C > 0$ such that for all $s \geq 1$*

$$\frac{c}{\Re\hat{k}(is)} \leq \sup_{1\leq|\sigma|\leq s} \left\|(i\sigma - \mathcal{A})^{-1}\right\| \leq \frac{C}{\Re\hat{k}(is)}.$$

Moreover, $\mathcal{A}$ is injective and has no singularity at zero iff $\nu|_{[0,\varepsilon)} = 0$ for some $\varepsilon > 0$. If this condition is violated the singularity is of the weakest possible type: s^{-1}.

Again to simplify the presentation we assume in the following $\nu|_{[0,\varepsilon)} = 0$ for some $\varepsilon > 0$. This is to avoid a singularity of $\mathcal{A}$ at zero. An immediate consequence of Theorem 5.24, 0.1 (or 1.3) is

THEOREM 5.25. *Assume that $\nu|_{[0,\varepsilon)} = 0$ for some $\varepsilon > 0$. Then there are constants $c, C > 0$ such that for all $t \geq 1$*

$$\frac{c}{M^{-1}(Ct)} \leq \sup_{E_1(\mathbf{x}_0)\leq 1} E(t, \mathbf{x}_0)^{\frac{1}{2}} \leq \frac{C}{M_{\log}^{-1}(ct)}$$

where the strictly increasing function $M : \mathbb{R}_+ \to (0,\infty)$ is given by $M(s) = (\Re\hat{k}(is))^{-1}$.

A recipe how to adapt the formulation of the above theorem in case of a non-invertible $\mathcal{A}$ was given in Section 5.5.5. A disadvantage of Theorem 5.25 is that, although we know (thanks to Theorem 5.24) precisely the exact growth rate of the resolvent along the imaginary axis, it does not exactly determine the decay rate. There is a "logarithmic gap" between the upper and the lower bound. The (main) purpose of this section is to find weak conditions on $\hat{k}$ which allow to replace $M_{\log}$ by M. Actually the results of Chapter 2 allow us to *characterize* those acoustic impedances $\hat{k}$ for which this is possible.

5.7.1. Sharp decay rates under $\Re\hat{k}(i\cdot) \in \mathrm{PD}$. Clearly we can replace $M_{\log}$ by M in Theorem 5.25 if we assume that k is a standard kernel as discussed in Section 5.5. This follows from Theorem 0.3 and 5.24. Proposition 5.26 shows that $\Re\hat{k}(i\cdot)$ can be chosen in such a way that it is asymptotically equivalent to any prescribed regularly varying function with index in $(-2, 0)$. In this situation Theorem 0.3 is not applicable anymore. However, Theorem 2.2 is still applicable. We give a precise statement right after the next Proposition.

PROPOSITION 5.26. *Let $\alpha \in (0,2)$ and $\ell : \mathbb{R}_+ \to (0,\infty)$ be a slowly varying function. Then one can choose ν in such a way that $\nu|_{[0,1)} = 0$, $(\tau \mapsto \tau^{-1}) \in L^1_\nu$ and*

$$\Re\hat{k}(is)^{-1} \sim s^\alpha \ell(s)$$

as $s \to \infty$.

PROOF. Let us define the measure ν by the following Lebesgue-density $u : \mathbb{R}_+ \to [0,\infty)$:

$$u(t) = \begin{cases} 0 & \text{for } t < 1 \\ \frac{(2-\alpha)}{\Gamma(\frac{\alpha}{2})\Gamma(2-\frac{\alpha}{2})} t^{-\alpha} l(t)^{-1} & \text{for } t \geq 1 \end{cases}.$$

The Laplace-transform $\hat{k}$ of k is given as a Stieltjes-transform of the measure ν. For $\Re z > 0$ we have

$$\hat{k}(z) = \int_{[0,\infty)} \frac{1}{z+\tau} d\nu(\tau) = \int_1^\infty \frac{1}{z+\tau} u(\tau) d\tau.$$

By a change of variables under the integral sign it is not difficult to see that the real part of $\hat{k}$ is a composition of the square-function with another Stieltjes-transform.

$$\Re\hat{k}(is) = \int_1^\infty \frac{\tau}{s^2+\tau} u(\tau) d\tau = \int_1^\infty \frac{1}{s^2+t} \underbrace{\frac{1}{2} u(\sqrt{t})}_{=:v(t)} dt.$$

From [11, Theorem 1.5.8] we deduce

$$V(t) := \int_0^t v(\tau) d\tau \sim \frac{1}{\Gamma(\frac{\alpha}{2})\Gamma(2-\frac{\alpha}{2})} t^{\frac{2-\alpha}{2}} \ell(t^{\frac{1}{2}})^{-1}.$$

Thus [11, Theorem 1.7.4] yields

$$\Re\hat{k}(is) = \int_1^\infty \frac{1}{s^2+t} dV(t) \sim s^{-\alpha} l(s)^{-1}.$$

This finishes the proof. □

Now we can formulate a nice characterization of those $\hat{k}$ for which we get rid of the logarithmic loss in Theorem 5.25.

THEOREM 5.27. *Let $\Omega = (0,1)$ and k be an integrable completely monotone function. Let the strictly increasing function $M : \mathbb{R}_+ \to (0,\infty)$ be given by $M(s) = (\Re\hat{k}(is))^{-1}$. Then*

$$\forall t \geq 1 : \sup_{E_1(\mathbf{x}_0)\leq 1} E(t, \mathbf{x}_0)^{\frac{1}{2}} \leq \frac{C}{M^{-1}(ct)} \tag{5.35}$$

holds for some $c, C > 0$ if and only if $\Re\hat{k}(i\cdot) \in \mathrm{PD}$. Moreover, in case $\Re\hat{k}(i\cdot) \in \mathrm{PD}$ we have for all $c > 0$ that $M^{-1}(c(\cdot)) \approx M^{-1}$ and in particular (5.35) holds for any $c > 0$ if one adjusts C appropriately.

PROOF. First assume $\Re\hat{k}(i\cdot) \in \mathrm{PD}$. Then $M \in \mathrm{PI}$. Therefore (5.35) follows from Theorem 2.2. The fact that $c > 0$ can be chosen arbitrary if one adjusts $C > 0$ appropriately follows from Lemma A.3.

Let us now assume that (5.35) holds. Proposition 5.22 tells us that the spectrum of the 1D wave equation already determines the growth of the resolvent along the imaginary axis. Therefore Proposition 2.6 yields $M \in \mathrm{PI}$ and thus $\Re\hat{k}(i\cdot) \in \mathrm{PD}$. □

5.8. Open questions

For a complete treatment of resolvent estimates for wave equations like (5.1) it would be desirable to answer at least the following two questions.

Question 1. Is the upper bound on $\|(is - \mathcal{A})^{-1}\|$, given by Theorem 5.9, optimal?

Question 2. Can one discard the additional assumption (5.21) on $\hat{k}$ without changing the conclusion of Theorem 5.9?

A strategy to positively answer question 1 is to show that there exists a sequence of eigenvalues of $-\mathcal{A}$ which tend to infinity and approach the imaginary axis sufficiently fast. We have seen that this strategy works at least for $\Omega = (0,1)$

and $\Omega = D$ (see Section 5.6 and Subsection 5.5.3). For the disk we restricted to kernels $k = k_{\beta,\varepsilon}$. However, with the more elaborate Taylor argument which proved Proposition 5.22 one can discard this restriction from Lemma 5.13. We believe that there is a general argument for any bounded Lipschitz domain Ω yielding the existence of such a sequence of eigenvalues.

By our investigations in Section 5.6 we already have a positive answer for question 2 in the 1D setting. Moreover, if $\Omega = D$ is the disk we already know from the spectrum that a non-decreasing function M with $M(s) = o((\Re\hat{k}(is))^{-1})$ can never be an upper bound for $\left\|(is - \mathcal{A})^{-1}\right\|$ for all large $|s|$. We think that the answer to question 2 is either "yes", or if "no" then the upper bound solely depends on $\Re\hat{k}$ and the infimum of all α making the upper estimate in (5.23) true for all spectral cluster p.

Concerning the application of resolvent estimates to energy decay there is also a third question. Let us assume for a moment that the answers to questions 1 and 2 were positive. Then Theorem 5.25 was true for any Ω. In general it is not possible to replace $M_{\log}$ by M in Theorem 0.1. However, does our particular situation allow for a smaller upper bound? Motivated by the results of Chapter 2 we ask

Question 3. Is Theorem 5.25 true for all bounded Lipschitz domains Ω - even with $M_{\log}^{-1}(ct)$ and $M^{-1}(Ct)$ replaced by $M_{\mathrm{qm}}^{-1}(t)$? Here M_{qm} is as defined in Theorem 2.15.

Appendix

APPENDIX A

Regular variation

A.1. Regularly and slowly varying functions (R_α and R_0)

Given $a \geq 0$ let $M : [a, \infty) \to (0, \infty)$ be a measurable function. We say that M is *regularly varying of index* $\alpha \in \mathbb{R}$ [11, Chapter 1] and write $M \in \mathrm{R}_\alpha$ if

$$\forall \lambda \in (0, \infty) : \lim_{s \to \infty} \frac{M(\lambda s)}{M(s)} = \lambda^\alpha.$$

One can show that a function is regularly varying of some index if the above limit merely exists for a large enough set of $\lambda > 0$. "Large enough" means that if one constructs the closure of the set under multiplication and inversion one gets $(0, \infty)$. Functions in R_0 are said to be *slowly varying*. It is easy to see that for each function $M \in \mathrm{R}_\alpha$ there exists a slowly varying function ℓ such that

$$M(s) = s^\alpha \ell(s) \text{ for } s \geq a.$$

By Karamata's Theorem [11, Theorem 1.3.1] every slowly varying function ℓ can be represented as

$$\ell(s) = c(s) \exp\left(\int_a^s \varepsilon(\tau) \frac{d\tau}{\tau}\right) \text{ for } s \geq a. \tag{A.1}$$

Here $a \geq 0$ is a real number, $\varepsilon : [a, \infty) \to \mathbb{R}$ is a locally integrable function with $\varepsilon(\tau) \to 0$ as $\tau \to \infty$ and $c : [a, \infty) \to (0, \infty)$ is measurable with $c(\tau) \to c_0 > 0$ as $\tau \to \infty$. In case $a = 0$ we furthermore assume that $\tau \mapsto \varepsilon(\tau)/\tau$ is integrable on $[0, 1]$. We call ℓ *normalized* if one can choose c to be constant. From the representation (A.1) one can deduce that each regularly varying function of strictly positive (negative) index is asymptotically equivalent to a smooth and finally strictly increasing (decreasing) regularly varying function of the same index.

A.2. The classes PI, PD, BI and BD

Given $a \geq 0$ let $M : [a, \infty) \to (0, \infty)$ be a measurable function. We say that M has *positive increase* [11, Chapter 2.1] and write $M \in \mathrm{PI}$ if

$$\exists \alpha > 0, s_0 \geq a, b \in (0, 1] \forall s \geq s_0, \lambda \geq 1 : \frac{M(\lambda s)}{M(s)} \geq b \lambda^\alpha. \tag{A.2}$$

If the function M is non-decreasing, then it is easy to see that if (A.2) holds for some $s_0 > a$, then it also holds for any $s_0 > a$ with the same choice of α and possibly a different choice of b for each s_0. Similarly we say that M has *positive decrease* and write $M \in \mathrm{PD}$ if

$$\exists \alpha > 0, s_0 \geq a, B \in [1, \infty) \forall s \geq s_0, \lambda \geq 1 : \frac{M(\lambda s)}{M(s)} \leq B \lambda^{-\alpha}.$$

Clearly, if $M \in \mathrm{PD}$ then $M(s) = O(s^{-\alpha}), s \to \infty$ for some $\alpha > 0$. Moreover, M has positive increase if and only if $1/M$ has positive decrease.

There are several equivalent characterizations of functions of positive increase/decrease [11, Chapter 2]. For convenience of the reader we give a useful characterization of positive increase/decrease for non-decreasing/increasing functions.

Lemma A.1. *Let $a \geq 0$. If $M\colon [a,\infty) \to (0,\infty)$ is non-decreasing, then M has positive increase if and only if*

$$\exists \lambda > 1 : \liminf_{s\to\infty} \frac{M(\lambda s)}{M(s)} > 1. \tag{A.3}$$

Similarly, if $M\colon [a,\infty) \to (0,\infty)$ is non-increasing, then M has positive decrease if and only if

$$\exists \lambda > 1 : \limsup_{s\to\infty} \frac{M(\lambda s)}{M(s)} < 1.$$

Proof. It is clear that if M has positive increase then (A.3) holds for all sufficiently large $\lambda > 1$, even without the monotonicity assumption. Suppose therefore that (A.3) holds. We fix an $s_0 > a$ and consider the function $m\colon [1,\infty) \to [1,\infty)$ defined by

$$m(\lambda) = \inf_{s\geq s_0} \frac{M(\lambda s)}{M(s)} \text{ for } \lambda \geq 1.$$

Then m is non-decreasing and, for $\lambda, \mu \geq 1$, we have

$$m(\lambda\mu) = \inf_{s\geq s_0} \frac{M(\lambda\mu s)}{M(\mu s)} \frac{M(\mu s)}{M(s)} \geq \inf_{t\geq \mu s_0} \frac{M(\lambda t)}{M(t)} \cdot \inf_{s\geq s_0} \frac{M(\mu s)}{M(s)} \geq m(\lambda)m(\mu).$$

Thus m is super-multiplicative, and using (A.3) we also see that there exists $\lambda_0 > 1$ such that $m(\lambda_0) > 1$. Now given $\lambda \geq 1$ there exist unique $n \in \mathbb{N}_0$ and $\theta \in [0,1)$ such that $\lambda = \lambda_0^{n+\theta}$. Let $b = m(\lambda_0)^{-1}$ and $\alpha = \log m(\lambda_0)/\log \lambda_0$. Then $b \in (0,1]$, $\alpha > 0$ and by super-multiplicativity of m we have

$$m(\lambda) \geq m(\lambda_0^n) \geq m(\lambda_0)^n \geq b m(\lambda_0)^{n+\theta} = b\lambda^\alpha,$$

giving the first result. The second statement follows by applying the first part to the function $1/M$. □

Given $a \geq 0$ let $M : [a,\infty) \to (0,\infty)$ be a measurable function. We say that M has *bounded increase* [11, Chapter 2.1] and write $M \in \mathrm{BI}$ if

$$\exists \alpha > 0, s_0 \geq a, B \in [1,\infty) \forall s \geq s_0, \lambda \geq 1 : \frac{M(\lambda s)}{M(s)} \leq B\lambda^\alpha.$$

Similarly we say that M has *bounded decrease* and write $M \in \mathrm{BD}$ if

$$\exists \alpha > 0, s_0 \geq a, b \in (0,1] \forall s \geq s_0, \lambda \geq 1 : \frac{M(\lambda s)}{M(s)} \geq b\lambda^{-\alpha}. \tag{A.4}$$

Obviously $M \in \mathrm{BI}$ if and only if $1/M \in \mathrm{BD}$. An analogous argument as in the proof of Lemma A.1 shows that

Lemma A.2. *Let $a \geq 0$ and let $M_1, M_2 : [a,\infty) \to \infty$ be non-increasing and non-decreasing, respectively. Then*

$$M_1 \in \mathrm{BI} \Leftrightarrow \exists \lambda > 1 : \limsup_{s\to\infty} \frac{M_1(\lambda s)}{M_1(s)} < \infty \textit{ and}$$

$$M_2 \in \mathrm{BD} \Leftrightarrow \exists \lambda > 1 : \liminf_{s\to\infty} \frac{M_2(\lambda s)}{M_2(s)} > 0.$$

The following inclusions are easy to see:

$$\forall \alpha \in \mathbb{R} : \mathrm{R}_\alpha \subset \mathrm{BI} \cap \mathrm{BD},$$
$$\forall \alpha > 0 : \mathrm{R}_\alpha \subset \mathrm{PI},\ \mathrm{R}_{-\alpha} \subset \mathrm{PD}.$$

Moreover there is no slowly varying function which has positive increase or decrease. We conclude this section with a useful lemma.

LEMMA A.3. *Let $a \geq 0$ and suppose that $M\colon [a,\infty) \to (0,\infty)$ is a continuous non-decreasing function which has positive increase. Then for every $c > 0$ we have $M^{-1}(t) \approx M^{-1}(ct)$ as $t \to \infty$.*

PROOF. The fact that M has positive increase implies that there exist constants $\alpha > 0$, $b \in (0,1]$ and $s_0 \geq a$ such that

$$\frac{M(\sigma)}{M(s)} \geq b\left(\frac{\sigma}{s}\right)^\alpha \text{ for } \sigma \geq s \geq s_0. \tag{A.5}$$

Let $t \geq M(s_0)$ and $\lambda \geq 1$. Setting $\sigma = M^{-1}(\lambda t)$ and $s = M^{-1}(t)$ in (A.5) we see that

$$\frac{M^{-1}(\lambda t)}{M^{-1}(t)} \leq b^{-1/\alpha}\lambda^{1/\alpha}.$$

Thus according to A.4 the function M^{-1} has bounded increase, which implies the desired result since M^{-1} is non-decreasing. □

A.3. The class PI_N

Let $a \geq 0$ be fixed throughout this section. Given a measurable function $M : [a,\infty) \to (0,\infty)$ we say that M has *quasi-positive increase (with auxiliary function N)* if there exists an $s_0 \geq a$ and a continuous non-decreasing function $N : [s_0,\infty) \to (0,\infty)$ such that

$$\exists b \in (0,1] \forall s \geq s_0, \lambda \geq 1 : \frac{M(\lambda s)}{M(s)} \geq b\lambda^{1/N(\lambda s)}. \tag{A.6}$$

Notice in particular that a measurable function $M : [a,\infty) \to (0,\infty)$ has positive increase if and only if it has quasi-positive increase and admits a bounded auxiliary function.

Given $(b, s_0) \in (0,1] \times [a,\infty)$ a continuous non-decreasing function $N : [a,\infty) \to (0,\infty)$ is called a (b,s_0)-*admissible* auxiliary function (with respect to M) if (A.6) holds for these choices of b and s_0. We call N admissible (b-admissible) if it is (b,s_0)-admissible for some choice of (b,s_0). A b-admissible auxiliary function N is called *b-minimal* if for any b-admissible auxiliary function N_1 there exists $s_1 \geq a$ such that $N(s) \leq N_1(s)$ for all $s \geq s_1$. An auxiliary function N is called *optimal* if it is b-minimal for some $b \in (0,1]$ and if for any $b_1 \in (0,1]$ and any b_1-admissible auxiliary function N_1 there exists $s_1 \geq a$ such that $b^{-1}N(s) \leq b_1^{-1}N_1(s)$ for all $s \geq s_1$. We refer to Remark 2.11 for the purpose of these definitions. Note that N is (b,s_0)-admissible if and only if

$$N(s) \geq \sup_{1<\lambda\leq \frac{s}{s_0}} \frac{\log(\lambda)}{\log\left(\frac{M(s)}{M(\lambda^{-1}s)}\right) + \log(b^{-1})} \text{ for } s > s_0. \tag{A.7}$$

This formula also implies, in case M is a normalized slowly varying function where ε in its Karamata representation (A.1) is positive, continuous and non-decreasing, that $N(s) = \varepsilon(s)^{-1}, s \geq a$ defines a 1-minimal auxiliary function. Observe that (A.7) implies that a (b, s_0)-admissible auxiliary function with $N(s) = O(\log(s)), s \to \infty$ can always be found if $b \neq 1$ and M is non-decreasing. This is not true in general for $b = 1$ as examples with non-decreasing slowly varying functions show (take e.g. $\varepsilon(s) = \log(s)^{-1} \log(\log(s))^{-1}, s \geq e^e$).

Before going to the examples we remind the reader that, by definition, (b-)minimal and optimal auxiliary functions are essentially unique (if they exist) in the following sense: whenever there exist two b-minimal (or optimal) auxiliary functions then for suitable $s_1 \geq a$ they must coincide on the interval $[s_1, \infty)$.

Example A.4. For $\alpha \in (0, \infty)$ let us consider the function given by $M(s) = \log(s)^\alpha, s \geq e$ and $M(s) = 1, s \in [0, e)$. Given $b \in (0, 1]$ we want to find a b-minimal auxiliary function N of M. By using (A.7) and the substitution $\theta = \log(\lambda)/\log(s)$ we get that for $s_0 \geq e$ any (b, s_0)-admissible auxiliary function N satisfies

$$N(s) \geq \sup\left\{\frac{\theta}{\log\left(\frac{b^{-1}}{(1-\theta)^\alpha}\right)}; \theta \in \left(0, 1 - \frac{\log(s_0)}{\log(s)}\right]\right\} \cdot \log(s) \text{ for } s > s_0. \tag{A.8}$$

Note that for large s the supremum in (A.8) is attained for $0 < \theta < 1 - \log(s_0)/\log(s)$. In case $b = 1$ the supremum is "attained" in the limit $\theta \downarrow 0$. Hence, a b-minimal auxiliary function N is given by

$$N(s) = \beta \log(s) \text{ for } s \geq e \text{ where } \beta = \sup\left\{\frac{\theta}{\log\left(\frac{b^{-1}}{(1-\theta)^\alpha}\right)}; \theta \in (0, 1]\right\}. \tag{A.9}$$

For arguments $s \in [0, e)$ we may extend N by the value β. We proved that for the logarithm raised to some power all (b-)minimal auxiliary functions are essentially again the logarithm - up to a scaling factor which only depends on α and the value of b.

Let us now find an optimal auxiliary function N of M. We already know from the above reasoning that necessarily $N(s) = \beta \log(s), s \geq s_1$ for some $\beta > 0$ and some $s_1 > e$ since an optimal auxiliary function is also minimal. Given $\beta > 0$ let $b_\beta \in (0, 1]$ be the supremum of all $b \in (0, 1]$ such that a b-minimal auxiliary function exists for M. We aim to find the maximal possible value for $\beta^{-1} b_\beta$. A short calculation shows that

$$b_\beta = \begin{cases} (\alpha\beta)^{-\alpha} e^{\alpha - \beta^{-1}} & \text{for } \beta < \alpha^{-1}, \\ 1 & \text{for } \beta \geq \alpha^{-1}. \end{cases}$$

Moreover, one can show that b_β-minimal functions actually exist. Another short calculation shows that $\beta^{-1} b_\beta$ gets maximal for $\beta = (1 + \alpha)^{-1}$. We proved that

$$N(s) = \begin{cases} (1 + \alpha)^{-1} \log(s) & \text{for } s \geq e, \\ (1 + \alpha)^{-1} & \text{for } s \in [0, e), \end{cases}$$

defines an optimal auxiliary function for M which is $e^{-1}(1 + \alpha^{-1})^\alpha$-minimal.

Example A.5. For $\alpha \in (0, 1)$ let us consider the function given by $M(s) = \exp(\log(s)^\alpha), s \geq 1$ and $M(s) = 1, s \in [0, 1)$. Again, by using (A.7) and the substitution $\theta = \log(\lambda)/\log(s)$ we get that for $s_0 \geq 1$ any (b, s_0)-admissible auxiliary function N satisfies for $s > s_0$

$$N(s) \geq \sup\left\{\frac{\theta}{1-(1-\theta)^\alpha + \log(b^{-1})\log(s)^{-\alpha}}; \theta \in \left(0, 1 - \frac{\log(s_0)}{\log(s)}\right]\right\} \cdot \log(s)^{1-\alpha}.$$

Observe that the supremum is attained for $0 < \theta < 1 - \log(s_0)/\log(s)$ if s is large and $b \neq 1$. If $b = 1$ the supremum is "attained" in the limit $\theta \downarrow 0$. From this formula it is easily seen that any optimal auxiliary function is necessarily 1-minimal. We proved that an optimal auxiliary function is given by

$$N(s) = \begin{cases} \alpha^{-1}\log(s)^{1-\alpha} & \text{for } s \geq 1, \\ \alpha^{-1} & \text{for } s \in [0,1). \end{cases}$$

APPENDIX B

Basic resolvent estimates for the wave equation

B.1. The damped wave equation on bounded domains

On an open, bounded and connected subset $\Omega \subset \mathbb{R}^d$, with $d \geq 1$, we consider the damped wave equation

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + 2a(x)u_t(t,x) = 0 & (t \in (0,\infty), x \in \Omega), \\ u(t,\cdot)|_{\partial\Omega} = 0 & (t \in (0,\infty)), \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) & (x \in \Omega). \end{cases} \tag{B.1}$$

with a positive damping function $a \in L^\infty(\Omega)$. If we set $\mathbf{x}(t) = (u(t), u_t(t))$ and $\mathbf{x}_0 = (u_0, u_1)$ we can formulate the wave equation as an abstract Cauchy problem

$$\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t),\ \mathbf{x}(0) = \mathbf{x}_0 \ \text{ where } \mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta & -2a(x) \end{pmatrix}, \tag{B.2}$$
$$\text{and } D(\mathcal{A}) = \{u \in H_0^1(\Omega); \Delta u \in L^2(\Omega)\} \times H_0^1(\Omega).$$

on the Hilbert space $\mathcal{H} = H_0^1 \times L^2(\Omega)$. Note that $\partial\Omega \neq \emptyset$. Thus the *energy* $E(\mathbf{x}_0) = \int_\Omega |\nabla u_0(x)|^2 + |u_1(x)|^2 \, dx$ defines a norm on the *energy space* $\mathcal{H}$. Everything what was said in Section 4.1.1 for the special case $\Omega = [0,1]^2$ remains true in the general case. In particular the resolvent mapping $z \mapsto (z - \mathcal{A})^{-1}$ is a meromorphic function on $\mathbb{C}$ with poles of finite order which are located in the strip $[-2\|a\|_\infty, 0) + i\mathbb{R}$.

B.1.1. Basic resolvent estimates. The inhomogeneous eigenvalue equation for the damped wave equation is $(is - \mathcal{A})(u,v) = (f,g)$. Here, s is in the strip $\mathbb{R} + i(0, 2\|a\|_\infty]$. This vector-valued equation is equivalent to the stationary equation:

$$-\Delta u - s^2 u + 2isa(x)u = h := g + (is + 2a(x))f. \tag{B.3}$$

The function v is then simply equal to $isu - f$. If is is not a pole of $(is - \mathcal{A})^{-1}$, then we can define $R(s) := (-\Delta - s^2 + 2isa(x))^{-1}$. This is an operator from $L^2(\Omega)$ to $H_0^1(\Omega)$. Now we can express the resolvent of the damped wave operator in terms of the resolvent $R(s)$ of the stationary wave equation:

$$(is - \mathcal{A})^{-1} = \begin{pmatrix} R(s)(is + 2a(x)) & R(s) \\ R(s)(-s^2 + 2isa(x)) - 1 & isR(s) \end{pmatrix}. \tag{B.4}$$

Proposition B.1. *Let $\mathcal{A}$ and R be as above. Then for real s with large modulus*

$$\left\|(is - \mathcal{A})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \approx \|R(s)\|_{L^2 \to H_0^1} \approx |s| \, \|R(s)\|_{L^2 \to L^2}.$$

Only in the case of no damping ($a = 0$) it can happen that is is an eigenvalue. But then we formally set the norms appearing in Proposition B.1 to be equal to ∞.

The next proposition deals with estimates of the form

$$\|u\|_{L^2}^2 \lesssim \left(\frac{M(s)}{|s|}\right)^2 \|h\|_{L^2}^2 + M(s) \int_\Omega a\,|u|^2 \tag{B.5}$$

and

$$\|u\|_{H_0^1}^2 \lesssim M(s)^2 \|h\|_{L^2}^2 + |s|^2 M(s) \int_\Omega a\,|u|^2 \tag{B.6}$$

for solutions u of (B.3) with $h \in L^2(\Omega)$. The function M is defined for large positive values of s, is bounded from below by a strictly positive number and is extended symmetrically for negative s.

PROPOSITION B.2. *Let M be as above. Then*

$$\left\|(is-\mathcal{A})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \lesssim M(s) \Leftrightarrow \forall u,h : (B.5) \Leftrightarrow \forall u,h : (B.6).$$

Here "$\forall u,h$" means "for all $h \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ which satisfy (B.3)".

The third proposition deals with estimates of the form (B.6), locally where the damping is non vanishing.

PROPOSITION B.3. *Suppose $\omega \subseteq \Omega$ is an open subset such that there exists another open subset $\omega' \subseteq \Omega$ containing the closure of ω such that a restricted to ω' is bounded from below by a strictly positive number. Then*

$$\|u\|_{H^1(\omega)}^2 \lesssim \frac{1}{|s|^2}\|h\|_{L^2}^2 + |s|^2 \int_\Omega a\,|u|^2$$

for every solution u of (B.3) with $h \in L^2(\Omega)$.

In view of Propositions B.2 and B.5 below, the aforementioned proposition shows that in regions where the damping acts, the strongest possible estimates ($M(s) = 1$) for $(is-\mathcal{A})^{-1}$ are valid. Thus only estimates on undamped regions of Ω are of interest.

B.1.2. Proof of Proposition B.1. We give the proof as a sequence of lemmas. First, we prove the easy direction of the inequalities.

LEMMA B.4. *For real s*

$$\|R(s)\|_{L^2\to H_0^1}^2 + |s|^2 \|R(s)\|_{L^2\to L^2}^2 \le \left\|(is-\mathcal{A})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}}^2.$$

Here we equip $H_0^1(\Omega)$ with the norm $(\int |\nabla u|^2)^{\frac{1}{2}}$.

PROOF. This is a direct consequence of (B.4), which implies

$$(is-\mathcal{A})^{-1}(0,g) = (R(s)g, isR(s)g)$$

for all $g \in L^2(\Omega)$, and the definition of the energy norm in $\mathcal{H}$. □

The next proposition is interesting for its own sake. It is a consequence of Weyl's law for the eigenvalues of the Dirichlet Laplacian.

PROPOSITION B.5. *For real s of large modulus*

$$\left\|(is-\mathcal{A})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \gtrsim 1.$$

Note that an estimate of the form $\left\|(is-\mathcal{A})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \gtrsim \frac{1}{|s|}$ is valid for any densely defined closed operator $\mathcal{A}$. This can be proved by contradiction using the resolvent identity. However this estimate seems to be slightly too weak for our purposes as we will see for example at the end of the proof of Lemma B.6.

PROOF OF PROPOSITION B.5. Only for this proof we introduce the notation $\mathcal{A}_a := \mathcal{A}$ to make the dependence of $\mathcal{A}$ on the damping a visible. Let us first investigate the case $a = 0$. For positive s let $N(s)$ be the number of linearly independent smooth functions u which satisfy

$$\begin{cases} -\Delta u(x) - \sigma^2 u(x) = 0 \quad (x \in \Omega) \\ u|_{\partial\Omega} = 0 \end{cases}$$

for some $0 < \sigma \leq s$. By Weyl's law this number satisfies

$$N(s) = \frac{\omega_d}{(2\pi)^d} \mathrm{vol}(\Omega) s^d + O(s^{d-1}).$$

The number ω_d is the volume of the unit ball in $\mathbb{R}^d$. Let $(\delta_s)_{s>0}$ be an increasing family of positive numbers which tend to infinity but satisfy $\delta_s = o(s)$. Then

$$N(s + \delta_s) - N(s) \gtrsim \delta_s s^{d-1}.$$

In particular $\mathcal{A}_0$ has many eigenvalues in the interval $i[s, s+\delta_s]$ if s is sufficiently large. Now - hoping to get a contradiction - suppose that the assertion of the lemma was not true. This means that there is a sequence (s_n) tending to infinity such that $\left\|(is_n - \mathcal{A}_0)^{-1}\right\|_{\mathcal{H}\to\mathcal{H}}$ tends to zero. Without loss of generality we may assume that

$$\left\|(is_n - \mathcal{A}_0)^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} = o(\delta_{s_n}^{-1}).$$

Then a Neumann series argument shows that $\mathcal{A}_0$ cannot have any eigenvalues in the interval $i[s_n, s_n + \delta_{s_n}]$ for large n - contradiction!

We treat the case $a \neq 0$ by a perturbation argument.

$$\mathcal{A}_a = \mathcal{A}_0 + B \text{ where } B = \begin{pmatrix} 0 & 0 \\ 0 & -2a(x) \end{pmatrix}.$$

Again suppose that there exists a sequence (s_n) tending to infinity such that $\left\|(is_n - \mathcal{A}_a)^{-1}\right\|_{\mathcal{H}\to\mathcal{H}}$ tends to zero. Then a Neumann series argument gives

$$(is_n - \mathcal{A}_0)^{-1} = (1 - (is_n - \mathcal{A}_a)^{-1}B)^{-1}(is_n - \mathcal{A}_a)^{-1} \to 0.$$

This is a contradiction to the first part of the proof. □

LEMMA B.6. *For real s of large modulus*

$$\left\|(is - \mathcal{A})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \lesssim \|R(s)\|_{L^2\to H_0^1} + |s| \, \|R(s)\|_{L^2\to L^2}.$$

PROOF. The proof is done if we can estimate the components of the first row of the matrix in (B.4) against the components of the second row. The estimation of the second component of the first row of (B.4) is straightforward:

$$\begin{aligned} &\left\|R(s)(-s^2 + 2isa(x)) - 1\right\|_{H_0^1\to L^2} \\ &= \|R(s)\Delta\|_{H_0^1\to L^2} \lesssim \|R(s)\|_{H^{-1}\to L^2} = \|R(s)\|_{L^2\to H_0^1}. \end{aligned} \tag{B.7}$$

For the last equality we used that $R(-s)$ is the adjoint of $R(s)$ and $R(-s)g = \overline{R(s)\overline{g}}$ for all $g \in L^2(\Omega)$. Next observe

$$R(s)(is + 2a(x)) = \frac{1}{is} R(s)(-s^2 + 2isa(x)) = \frac{1}{is}(1 + R(s)\Delta). \tag{B.8}$$

Therefore let us consider $u, f \in H_0^1(\Omega)$ such that the stationary wave equation (B.3) is satisfied with h replaced by Δf. Then testing the stationary wave equation against u implies

$$\|\nabla u\|_{L^2}^2 - s^2 \|u\|_{L^2}^2 + 2is \int a\, |u|^2 = \langle \Delta f, u\rangle_{H^{-1},H_0^1}\,.$$

This in turn implies

$$\begin{aligned}\|u\|_{H_0^1}^2 &\leq s^2 \|u\|_{L^2}^2 + \|\Delta f\|_{H^{-1}} \|u\|_{H_0^1}\\ &\leq C(s^2 \|u\|_{L^2}^2 + \|f\|_{H_0^1}^2) + \frac{1}{2} \|u\|_{H_0^1}^2\,.\end{aligned}$$

This together with (B.7) yields $\|R(s)\Delta\|_{H_0^1 \to H_0^1} \lesssim |s| \, \|R(s)\|_{H^{-1}\to L^2} + 1$. By (B.8) the last estimate can be used to estimate the upper left entry in the matrix given in (B.4). Summing up our calculations we get

$$\left\|(is - \mathcal{A})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \lesssim \frac{1}{|s|} + \|R(s)\|_{L^2\to H_0^1} + |s| \, \|R(s)\|_{L^2\to L^2}\,.$$

By Proposition B.5 we can absorb the term $1/|s|$ on the right-hand side into the left-hand side. □

From the next lemma we conclude the validity of Proposition B.1.

Lemma B.7. *For real s of large modulus*

$$\|R(s)\|_{L^2\to H_0^1} \approx |s| \, \|R(s)\|_{L^2\to L^2}\,.$$

Proof. Let $h \in L^2(\Omega)$, $s \in \mathbb{R}$ and $u \in H_0^1(\Omega)$ be a solution of the stationary wave equation (B.3). Testing this equation with u and taking the real part yields

$$\begin{aligned}\|\nabla u\|_{L^2}^2 &\leq \|h\|_{L^2} \|u\|_{L^2} + s^2 \|u\|_{L^2}^2\\ &\lesssim \frac{1}{s^2} \|h\|_{L^2}^2 + s^2 \|u\|_{L^2}^2\,.\end{aligned}$$

Therefore

$$\|R(s)\|_{L^2\to H_0^1} \lesssim \frac{1}{|s|} + |s| \, \|R(s)\|_{L^2\to L^2}\,. \tag{B.9}$$

Similarly we get

$$\begin{aligned}s^2 \|u\|_{L^2}^2 &\leq \|h\|_{L^2} \|u\|_{L^2} + \|\nabla u\|_{L^2}^2\\ &\leq \frac{C}{s^2} \|h\|_{L^2}^2 + \frac{s^2}{2} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\,,\end{aligned}$$

which implies

$$|s| \, \|R(s)\|_{L^2\to L^2} \lesssim \frac{1}{|s|} + \|R(s)\|_{L^2\to H_0^1}\,. \tag{B.10}$$

It remains to explain why we can drop the term $1/|s|$ in the inequalities (B.9) and (B.10). If we could not drop this term in either (B.9) or (B.10), it would be possible to find a sequence (s_n) of positive numbers which tend to infinity such that *both* $|s_n| \, \|R(s_n)\|_{L^2\to L^2}$ and $\|R(s_n)\|_{L^2\to H_0^1}$ could be estimated by $1/|s_n|$. But this contradicts Proposition B.5 and Lemma B.6. □

B.1.3. Proof of Proposition B.2. It is trivial that $\left\|(is-\mathcal{A})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \lesssim M(s)$ implies (B.5) and (B.6). Let $h \in L^2(\Omega)$ and let $u \in H_0^1(\Omega)$ be a solution of the stationary wave equation. We have to show that (B.5) and (B.6) both imply $\left\|(is-\mathcal{A})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \lesssim M(s)$.

(i) Let (B.5) be true. Testing the stationary wave equation with u and taking the imaginary part yields

$$\begin{aligned} |s| \int a\,|u|^2 &\le \|h\|_{L^2} \|u\|_{L^2} \\ &\le C\frac{M(s)}{\varepsilon\,|s|} \|h\|_{L^2}^2 + \frac{\varepsilon\,|s|}{M(s)} \|u\|_{L^2}^2 \end{aligned}$$

for any $\varepsilon > 0$. If we choose ε small enough, the aforementioned statement together with (B.5) implies $\|u\|_{L^2} \lesssim M(s)\|h\|_{L^2}$. This gives the assertion by Proposition B.1.

(ii) Let (B.6) be true. Testing the stationary wave equation with u and taking the real part yields $|s|\,\|u\|_{L^2} \lesssim \|\nabla u\|_{L^2} + \|h\|_{L^2}/|s|$. Thus for all $\varepsilon > 0$

$$\begin{aligned} s^2 \int a\,|u|^2 &\le |s|\,\|h\|_{L^2} \|u\|_{L^2} \\ &\lesssim \|h\|_{L^2} \left(\frac{1}{|s|}\|h\|_{L^2} + \|u\|_{H_0^1}\right) \\ &\lesssim \frac{1}{|s|}\|h\|_{L^2}^2 + \frac{M(s)}{\varepsilon}\|h\|_{L^2}^2 + \frac{\varepsilon}{M(s)}\|u\|_{H_0^1}^2 . \end{aligned}$$

Using this estimate in (B.6) with ε small yields together with Proposition B.1 the claim.

B.1.4. Proof of Proposition B.3. Choose a smooth cut-off function $0 \le \eta \le 1$ which is equal to 1 in ω such that its support is in a region where the damping a is bounded from below by a strictly positive number. Let us test equation (B.3) with $u\eta^2$. Then

$$\int_\Omega \left(|\nabla u|^2 \eta^2 + 2\nabla u \cdot \nabla\eta\bar{u}\eta - s^2 |u|^2 \eta^2 + 2isa\,|u|^2 \eta^2\right) = \int_\Omega h\bar{u}\eta^2.$$

By using $\nabla u \cdot \nabla\eta\bar{u}\eta \ge -\frac{1}{2}|\nabla u|^2\eta^2 - \frac{1}{2}|\nabla\eta|^2|u|^2$ and $\left|h\bar{u}\eta^2\right| \le \frac{1}{2|s|^2}|h|^2 + \frac{|s|^2}{2}|u|^2\eta^2$ we end up with

$$\begin{aligned} \int_\omega |\nabla u|^2 &\le \int_\Omega |\nabla u|^2 \eta^2 \\ &\lesssim \frac{1}{|s|^2}\|h\|_{L^2}^2 + |s|^2 \int_\Omega |u|^2 \eta^2 \\ &\lesssim \frac{1}{|s|^2}\|h\|_{L^2}^2 + |s|^2 \int_\Omega a\,|u|^2 . \end{aligned}$$

The corresponding estimate for ∇u replaced by u is trivial.

B.2. The (undamped) wave equation on exterior domains

We are in the situation of Chapter 3. We aim to prove Proposition 3.3 which we repeat here for convenience of the reader.

PROPOSITION B.8. *Let $\delta > 0$ and let $\tilde{\chi}$ be defined as χ but with $\tilde{\chi} = 1$ on a neighbourhood of the support of χ. Let z with $-\delta < \Re z < 0$ be no pole of R_χ, then*

$$\|G_\chi(z)\| \leq C\left((1 \vee |z|)^{-1} + |z| \, \|R_{\tilde{\chi}}(z)\|_{L^2 \to L^2}\right)$$

with a constant $C > 0$ independent of z. The reverse inequality - with a different constant, ignoring the first summand on the right hand side and $\tilde{\chi}$ replaced by χ - is also true.

PROOF. From (3.6) we deduce that

$$G_\chi(z) = \begin{pmatrix} zR_\chi(z) & R_\chi(z) \\ z^2 R_\chi(z) - \chi^2 & zR_\chi(z) \end{pmatrix}, \; z^2 R_\chi(z) - \chi^2 = \chi R(z) \Delta \chi. \tag{B.11}$$

Therefore the last statement of the proposition follows directly from

$$G_\chi(z)(0, g) = (R_\chi(z)g, zR_\chi(z)g) \, .$$

To prove the inequality displayed in the proposition we assume without loss of generality that $|z| \geq 1$. Furthermore we let χ_1 be a function satisfying the same constraints as $\tilde{\chi}$ but with support contained in the interior of the set where $\tilde{\chi}$ is equal to 1. Let $H_D^{-1}(\Omega)$ be the dual space of $H_D^1(\Omega)$. Clearly $\Delta : H_D^1(\Omega) \to H_D^{-1}(\Omega)$ is continuous. Furthermore the commutator $[\Delta, \chi] : H_D^1(\Omega) \to L^2(\Omega)$ is continuous too. This is not completely obvious since $[\Delta, \chi] = \nabla\chi \cdot \nabla + (\Delta\chi)$ has a zeroth order term. Fortunately, $\Delta\chi$ is compactly supported, $\partial\Omega \neq \emptyset, \partial\Omega \in C^\infty$ and therefore $\Delta\chi$ acts as a bounded operator on $H_D^1(\Omega)$ by the Poincaré-Steklov inequality for bounded domains. By the same reasoning we have already seen in Chapter 3 that χ acts as a bounded operator on $H_D^1(\Omega)$. Before coming to the first estimates let us finally note that for all $z \in \mathbb{C} \backslash \mathbb{R}_-$ and $g \in L^2(\Omega)$ we have

$$R_\chi(z)^* g = R_\chi(\overline{z})g = \overline{R_\chi(z)\overline{g}}. \tag{B.12}$$

Here the bars mean the complex conjugate and * means the L^2-adjoint of an operator. If z is a pole of R_χ this equality simply means that $\overline{z}$ is a pole too.

Our goal is to verify the following estimates:

$$\|zR_\chi(z)\|_{H_D^1 \to H_D^1} \lesssim \frac{1}{|z|} + |z| \, \|R_{\tilde{\chi}}(z)\|_{L^2 \to L^2} \, , \tag{B.13}$$

$$\|\chi R(z) \Delta \chi\|_{H_D^1 \to L^2} \lesssim \frac{1}{|z|} + |z| \, \|R_{\chi_1}(z)\|_{L^2 \to L^2} \, , \tag{B.14}$$

$$\|R_\chi(z)\|_{L^2 \to H_D^1} \lesssim \frac{1}{|z|} + |z| \, \|R_{\chi_1}(z)\|_{L^2 \to L^2} \, . \tag{B.15}$$

By (B.11) this implies the conclusion of the proposition.

Step 1. Estimation of $\|R_\chi(z)\|_{L^2 \to H_D^1}$. Let $f \in L^2(\Omega)$ and $u = R(z)\chi f$. Then, by Proposition 3.2, the L^2_{loc}-function u is a distributional solution of

$$\begin{cases} z^2 u(x) - \Delta u(x) = \chi(x) f(x) & (x \in \Omega), \\ u(x) = 0 & (x \in \partial\Omega). \end{cases} \tag{B.16}$$

Testing the equation with $\chi\overline{u}$ leads after a short calculation, using integration by parts, to

$$\|\chi \nabla u\|_{L^2}^2 \lesssim \frac{1}{|z|^2} \|\chi f\|_{L^2}^2 + |z|^2 \, \|(\nabla\chi)u\|_{L^2}^2 \, .$$

This implies (B.15).

Step 2. Estimation of $\|\chi R(z)\Delta\chi\|_{H^1_D\to L^2}$.

$$\begin{aligned}\|\chi R(z)\Delta\chi\|_{H^1_D\to L^2} &= \|R_\chi(z)\Delta + \chi R(z)[\Delta,\chi]\|_{H^1_D\to L^2}\\ &\lesssim \|R_\chi(z)\|_{H^{-1}_D\to L^2} + \|R_{\chi_1}(z)\|_{L^2\to L^2}\\ &\lesssim \frac{1}{|z|} + |z|\,\|R_{\chi_1}(z)\|_{L^2\to L^2}\,.\end{aligned}$$

From the second to the third line we used a duality argument (using (B.12)) together with (B.15). We have proved (B.14).

Step 3. Estimation of $\|zR_\chi(z)\|_{H^1_D\to H^1_D}$. First we observe that by (B.15)

$$\begin{aligned}\left\|z^2R_\chi(z)\right\|_{H^1_D\to H^1_D} &= \|1 + R_\chi(z)\Delta + \chi R(z)[\Delta,\chi]\|_{H^1_D\to H^1_D}\\ &\le 1 + \|R_\chi(z)\Delta\|_{H^1_D\to H^1_D} + \|R_{\chi_1}(z)\|_{L^2\to H^1_D}\\ &\lesssim 1 + \|R_\chi(z)\|_{H^{-1}_D\to H^1_D} + |z|\,\|R_{\chi_1}(z)\|_{L^2\to L^2}\,.\end{aligned}$$

It remains to estimate the middle term in the last line. Let $f \in H^{-1}_D(\Omega)$ and let $u \in H^1_D(\Omega)$ be the solution of (B.16) given by $R(z)\chi f$. Testing the equation with $\chi\overline{u}$ leads after a short calculation to

$$\|\chi\nabla u\|^2_{L^2} \lesssim \|\chi f\|^2_{H^{-1}_D} + |z|^2\,\|(\nabla\chi)u\|^2_{L^2}\,.$$

This implies together with a duality argument (using (B.12)) and (B.15)

$$\|R_\chi(z)\|_{H^{-1}_D\to H^1_D} \lesssim 1 + |z|\,\|R_{\chi_1}(z)\|_{H^{-1}_D\to L^2} \lesssim 1 + |z|^2\,\|R_{\tilde\chi}(z)\|_{L^2\to L^2}\,.$$

But now this in turn implies (B.13). The proof is finished. □

APPENDIX C

Besov spaces: a borderline case for the trace theorem

In this thesis we work with fractional Sobolev spaces, Besov spaces and the trace operator acting on them. Note also that we work with the space $H^s(\partial\Omega)$ which is not only a fractional Sobolev space but also is a function space on a closed subset of $\mathbb{R}^d$ which has empty interior. In this appendix we aim at providing some results from the literature about Sobolev/Besov spaces and their relation to interpolation spaces which is necessary to follow the arguments from Chapter 5.

Of exceptional importance for the proof of Theorem 5.4 (i) and (ii) is the validity of the borderline trace theorem - Proposition C.2. This borderline case seems to be well-known to the experts - also for Lipschitz domains - but unfortunately we were not able to find it in the literature except in [50, Theorem 18.6]. The proof given there is not in our spirit since Besov spaces are not defined as interpolation spaces there. Therefore we give a simple direct proof via the characterization of Besov spaces as interpolation spaces which is true if Ω has the so called extension property (which in turn is satisfied if Ω is a Lipschitz domain).

C.1. Fractional Sobolev- and Besov spaces

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Here by *Lipschitz* we mean that locally near any boundary point and in an appropriate coordinate system one can describe Ω as the set of points which are above the graph of some Lipschitz continuous function from $\mathbb{R}^{d-1}$ into $\mathbb{R}$.

Let $1 \leq p \leq \infty$. We assume the reader to be familiar with the usual *Sobolev space* $W^{1,p}(\Omega)$ which consists of all functions $u \in L^p(\Omega)$ for which all distributional derivatives $\partial_j u$ are in $L^p(\Omega)$. There are different methods of defining Besov spaces. For our purposes it is most convenient to define the *Besov spaces* for $0 < s < 1$ and $1 \leq q \leq \infty$ as real interpolation spaces:

$$B_q^{s,p}(\Omega) = (L^p(\Omega), W^{1,p}(\Omega))_{s,q}. \tag{C.1}$$

Another approach is to define $B_q^{s,p}(\mathbb{R}^d)$ for example via interpolation and then to define the Besov space on Ω as restrictions to Ω of Besov function on $\mathbb{R}^d$. In general these approaches are not equivalent but if Ω satisfies the extension property they are equivalent [49, Chapter 34]. In our setting ($0 < s < 1$) we say that Ω satisfies the *extension property* if there is a linear and continuous operator $\mathrm{Ext} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ such that $(\mathrm{Ext}\, u)|_\Omega = u$ for each u from $W^{1,p}(\Omega)$. The extension property is fulfilled if Ω is bounded and has a Lipschitz boundary. In the following we always assume that this extension property is fulfilled - otherwise some statements from below are not valid.

The *Sobolev-Slobodeckij spaces* are defined as special Besov spaces $W^{s,p}(\Omega) = B_p^{s,p}(\Omega)$. It is common to write H^s instead of $W^{s,2}$ in the Hilbert space setting. For $0 \le s \le 1$ it is also possible to define the scale of *fractional Sobolev spaces* (also known as *Bessel potential spaces*) $H^{s,p}(\Omega)$ via Fourier methods for the special case $\Omega = \mathbb{R}^d$ and via restriction for the general case. These spaces form a scale of complex interpolation spaces. In general the fractional Sobolev spaces differ from the Sobolev-Slobodeckij spaces but coincide in the case $p = 2$ (see [2, Chapter 7.67]). Note that in the book [2] the letter W stands for the fractional Sobolev spaces. We also have $H^{1,p}(\Omega) = W^{1,p}(\Omega)$ for $1 < p < \infty$ - which is Calderón's Theorem (see [30, page 7]).

We mention that for all $0 < s_1 \le s < 1$ and $q, q_1 \in [1, \infty]$ with the restriction $q \le q_1$ if $s_1 = s$:

$$B_q^{s,p}(\Omega) \hookrightarrow B_{q_1}^{s_1,p}(\Omega).$$

This is a direct consequence of a general result about the real interpolation method (see e.g. [49, Lemma 22.2]).

It is possible to define the Besov space $B_q^{s,p}(A)$ on a general class of closed subsets A of $\mathbb{R}^d$ - the so called *d-sets*. For Ω having a Lipschitz boundary its boundary $\partial\Omega$ is such a set, since it is a $(d-1)$-dimensional manifold topologically. The required background is included in [30, Chapter V]. Again we write $H^s(A) = B_2^{s,2}(A)$ in the Hilbert space setting.

C.2. Traces for functions with $1/p$ or more derivatives

Throughout this subsection $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with Lipschitz boundary and we let $1 < p < \infty$. For $1/p < s < 1$ the following theorem is a special case of [30, Chapter VI, Theorem 1-3]. For $s = 1$ it is a special case of [30, Chapter VII, Theorem 1-3], keeping in mind that by Calderón's Theorem the Bessel potential spaces are the ordinary Sobolev spaces for positive integer orders s.

THEOREM C.1. *Let $1/p < s < 1$. Then the trace operator $\Gamma : C(\overline{\Omega}) \to C(\partial\Omega), u \mapsto u|_{\partial\Omega}$ extends continuously to an operator*

$$\Gamma : B_q^{s,p}(\Omega) \to B_q^{s-\frac{1}{p},p}(\partial\Omega).$$

Furthermore Γ has a continuous right inverse:

$$\mathrm{Ext} : B_q^{s-\frac{1}{p},p}(\partial\Omega) \to B_q^{s,p}(\Omega), \quad \Gamma \circ \mathrm{Ext} = \mathrm{id}_{B_q^{s-\frac{1}{p},p}(\partial\Omega)}.$$

The theorem remains valid for $s = 1$, $q = p$ if one replaces $B_q^{s,p}(\Omega)$ by $W^{1,p}(\Omega)$.

Unfortunately this theorem is false for any $1 \le q \le \infty$ in the borderline case $s = 1/p$ if one replaces the target space of Γ by $L^p(\partial\Omega)$. But for our purposes it is sufficient that a weakened version remains valid.

PROPOSITION C.2. *The trace operator $\Gamma : B_1^{\frac{1}{p},p}(\Omega) \to L^p(\partial\Omega)$ is continuous.*

Actually the trace operator is indeed surjective (but we do not need this property in this thesis) and a more general version is proved in [50, Section 18.6]. However there is no *linear* extension operator from $L^p(\partial\Omega)$ back to the Besov space (see [50] and references therein).

We indicate a simple direct proof of Proposition C.2. It is based on two lemmas which have very simple proofs on their own. The first one is

LEMMA C.3. *There exists a constant $C > 0$ such that for every C^∞ function u with compact support in $\mathbb{R}^d$*

$$\|\Gamma u\|_{L^p(\partial\Omega)} \leq C \|u\|_{L^p(\Omega)}^{1-\frac{1}{p}} \|u\|_{W^{1,p}(\Omega)}^{\frac{1}{p}}.$$

The straightforward proof can be found in [49, Lemma 13.1]. For a different proof in the case $p = 2$ we refer to [37]. The second ingredient to the proof of Proposition C.2 is [49, Lemma 25.3] which we recall here for the convenience of the reader.

LEMMA C.4. *Let (X_0, X_1) be an interpolation couple, Y a Banach space and let $0 < \theta < 1$. Then a linear mapping $L : X_0 \cap X_1 \to Y$ extends to a continuous operator $L : (X_0, X_1)_{\theta,1} \to Y$ if and only if there exists a $C > 0$ such that for all $u \in X_0 \cap X_1$ we have $\|Lu\|_Y \leq C \|u\|_{X_0}^{1-\theta} \|u\|_{X_1}^{\theta}$.*

PROOF OF PROPOSITION C.2. Apply the if-part of Lemma C.4 to $X_0 = L^p(\Omega)$, $X_1 = W^{1,p}(\Omega)$, $Y = L^p(\partial\Omega)$, $L = \Gamma$ and $\theta = s$. Use Lemma C.3 to verify the converse. □

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit zum Thema *Quantified Tauberian Theorems and Applications to Decay of Waves* unter Betreuung von Prof. Dr. Ralph Chill (Betreuer)an der TU Dresden ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt. Ich erkenne die Promotionsordnung in der Version vom 23.02.2011 an.

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