Line Bundles of Rational Degree Over Perfectoid Space

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Abstract of Dissertation

Line Bundles of Rational Degree Over Perfectoid Space

We begin the story by recalling the notion of degree in topology as given in [Hatcher, 2002, pp 134]. A map of spheres $f : S^n \to S^n$ induces a map in homology $\phi : H_n(S^n) \to H_n(S^n)$ given as a group homomorphism $\mathbb{Z} \to \mathbb{Z}$. This map is simply multiplication by $d \in \mathbb{Z}$ which is called the degree of the map.

If we take direct limit of these maps (1) as in example 3F.3 [Hatcher, 2002, pp 312] by setting $d = p$ a prime

$$\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \cdots \tag{1}$$

we get a moore space $M(\mathbb{Z}[1/p], n)$. We can now talk about maps from $\mathbb{Z}[1/p] \to \mathbb{Z}[1/p]$, and degree d as an element of $\mathbb{Z}[1/p]$. In this thesis we transfer the notion of degree as an element of $\mathbb{Z}[1/p]$ to algebra, analysis and geometry.

We start with non-archimedean analysis and define order and degree for power series with terms of rational degree, and then use these definitions to prove the analogues of theorems in complex analysis: Weierstaß Division, Weierstraß preparation and Maximum principal for the case of rational degrees.

We now move onto algebraic geometry and in chapter 7 we describe vector bundles over perfectoid projective and affine spaces. This is similar to descrip-

tion of vector bundles over $\mathbb{P}^1$ given by Grothendieck. This result requires us to describe and prove the unit elements of rational power series.

In the chapter 8 we define line bundles with degree $d \in \mathbb{Z}[1/p]$ and use it to compute Picard groups and show equivalence of Cartier and Weil divisors on $\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}$.

In the chapter 9 we compute cohomology of line bundles of rational degree and show these are of infinite dimension. The proof uses Čech complex as in Serre's corresponding result of cohomology of line bundles.

Finally, in chapter 10 we describe differential forms on $\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}$ and show that it is also of infinite dimension. We also show that standard Euler sequence holds and also understand why standard Reimann Roch would not hold.

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Chapter 1

Introduction to Rigid Analytic Geometry

The methods of complex-analytic geometry can be used for studying algebraic geometry over $\mathbb{C}$ and for computations of coherent sheaf cohomology. We want similar methods and theory for arithmetic geometry where we use p-adic numbers ($\mathbb{Q}_p$) or non-archimedean fields, but this theory does not have the 'right topology' since the underlying field is totally disconnected and thus we cannot define manifolds the same way as in $\mathbb{C}$. The problem of topology is overcome by using Grothendieck topology instead of regular topology and working with Étale Cohomolgy. This new cohomology theory was used by Grothendieck and Deligne to prove the Weil conjectures.

John Tate in the 60's profitably used Grothedieck topology to get a well behaved coherent sheaf and its cohomology for non-archimedean spaces. This theory introduced by John Tate is called rigid analytic geometry and it has many important applications such as the Langlands correspondence relating automorphic and Galois representations. The central subject of arithmetic geometry is the study of Galois representations which can be associated with Étale Cohomology of a scheme defined over number fields. The most famous example is that of elliptic curves and modular forms. We can associate a Tate Module to an elliptic curve with a Galois action and have an associated Galois Representation . This was used in the proof of Fermat's Last theorem via Shimura-Taniyama conjecture. The perfectoid rings arise from Fontaine's period rings (defined by Jean-Marc Fontaine) which are a collection of commutative rings (over $\mathbb{C}_p$) that are used to classify p-adic Galois representations. Informally, we can think of obtaining perfectoid spaces by attaching all pth power roots, for example, if we start with $\mathbb{Q}_p$ its perfectoid analogue would be completion of $\mathbb{Q}_p(p^{1/p^\infty})$, more formal details are in Chapter 4. These spaces have been used to prove many recent results including the weight-monodromy conjecture. The idea is to construct an equivalence of categories from the realm of algebraic geometry to that of rigid geometry in the sense of Roland Huber. Perhaps, the most interesting part of Perfectoid space is the correspondence between fields of characteristic zero and fields of characteristic p, this correspondence is constructed conceptually via tilting functors or the computational approach of Witt vectors.

In this thesis we lay the foundation for rational degree d as an element of $\mathbb{Z}[1/p]$ by using perfectoid analogue of projective space $\mathbb{P}^n$, and consider power series instead of polynomials. We start the groundwork by proving Weierstraß theorems for perfectoid spaces which are analogues of standard Weierstraß theorems in complex analysis. We then move onto defining sheaves for Projective perfectoid analogue and prove perfectoid analogues of Gorthendieck's classification theorem on $\mathbb{P}^1$, Serre's theorem on Cohomology of line bundles. As intermediate results we also compute Picard groups and define Cartier and Weil divisors for Perfectoid projective space, again these are analogous to their counterparts in Algebraic Geometry.

We start with the definition of Non-Archimedean absolute value.

Definition 1.0.1. Let K be field. A map $|\cdot| : K \to \mathbb{R}_{\geqslant}$ is called a non-Archimedean absolute value if for all $\alpha, \beta \in K$ the following hold.

(i) $|\alpha| = 0$ iff $\alpha = 0$.

(ii) $|\alpha\beta| = |\alpha||\beta|$,

(iii) $|\alpha + \beta| \leqslant \max\{|\alpha|, |\beta|\}$

An absolute value $|\cdot|$ is *trivial* if the only values it takes is $0, 1 \in \mathbb{R}$. We shall assume that absolute value is non trivial.

If we take this absolute value to define the distance function by setting $d(\alpha, \beta) = |\alpha - \beta|$ we can talk about disks/balls around points $a \in K$ with some radius $r \in \mathbb{R}_{>0}$.

$$\text{Open disc: } D^-(a, r) = \{x \in K \ : \ d(x, a) < r\} \tag{1.1}$$

$$\text{Closed disc: } D^+(a, r) = \{x \in K \ : \ d(x, a) \leqslant r\} \tag{1.2}$$

$$\text{Boundary: } \partial D(a, r) = \{x \in K \ : \ d(x, a) = r\} \tag{1.3}$$

The following properties hold

1. The topology of K is totally disconnected.

2. An Open disc is both open and closed.

3. If two discs intersect ($D_1 \cap D_2 \neq \varnothing$) then one disc contains the other ($D_1 \subset D_2$ or $D_2 \subset D_1$).

We define holomorphic (or analytic) functions via convergent power series. This leads to a problem that a function could be locally analytic but might not enjoy global convergence properties, since, non-empty open subsets are not connected, locally analytic functions might not have reasonable global properties. This problem can be resolved via Grothendieck Topology.

1.1 Restricted Power Series

We use lemma 1.1.1 [Bosch, 2014, Lemma 3, p10] as a guiding light to define restricted power series , which states that

Lemma 1.1.1. *If K is complete, the series $\sum_{i \geqslant 0} a_i$ where $a_i \in K$, is convergent iff* $\lim_{i \to \infty} |a_i| = 0$

Let K^{alg} be algebraic closure of a complete field K, the absolute value on K extends uniquely to K^{alg}. We can now define a unit ball in ${K^{\text{alg}}}^n$ given as

$$\mathbb{B}^n(K^{\text{alg}}) = \{(x_1, \ldots, x_n) \in {K^{\text{alg}}}^n : |x_i| \leqslant 1 \text{ for all } i\} \tag{1.4}$$

We can now define power series which converge globally on $\mathbb{B}^n(K^{\text{alg}})$ as given in [Bosch, 2014, Lemma 1, p12].

Lemma 1.1.2. *A formal power series where* $v = (v_1, \ldots, v_n)$

$$\sum_{i \in \mathbb{N}^n} a_i v^i = \sum_{i \in \mathbb{N}^n} a_{i_1 \ldots i_n} v_1^{i_1} \cdots v_n^{i_n} \in K[[v_1, \ldots, v_n]] \tag{1.5}$$

converges globally on $\mathbb{B}^n(K^{alg})$ *iff* $\lim_{|i| \to \infty} |a_i| = 0$.

1.2 Tate Algebra

The K algebra $T_n = K\langle v_1, \ldots, v_n \rangle$ (by convention $T_0 = K$) of all formal power series converging on the unit ball $\mathbb{B}^n(K^{alg})$ is called Tate Algebra of strictly convergent power series.

$$\sum_{i \in \mathbb{N}^n} a_i v^i \in K[[v_1, \ldots, v_n]], \quad a_i \in K, \quad \lim_{|i| \to \infty} |a_i| = 0 \tag{1.6}$$

1.2.1 Gauß Norm

We can define Gauß norm on the Tate algebra T_n

$$|f| = \max |a_i| \text{ where } f = \sum_i a_i v^i \tag{1.7}$$

The Gauß norm satisfies the following conditions with $c \in K$ and $f, g \in T_n$

$$|f| = 0 \text{ iff } f = 0$$

$$|cf| = |c||f|$$

$$|fg| = |f||g|$$

$$|f + g| = \max(|f|, |g|)$$

This makes T_n a K algebra. The above listed properties can be found on [Bosch, 2014, p13]. In fact T_n is complete with respect to the Gauß norm, thus a Banach K algebra.

1.2.2 Reduction

Since we have an absolute value on K we can define corresponding valuation ring, the maximal ideal and the residue field.

$$\text{Valuation Ring:} \quad R = \{x \in K : |x| \leqslant 1\} \tag{1.8}$$

$$\text{Maximal Ideal:} \quad \mathfrak{m} = \{x \in K : |x| < 1\} \tag{1.9}$$

$$\text{Residue Field:} \quad k = R/\mathfrak{m} \tag{1.10}$$

The R algebra of restricted power series is denoted by $R\langle \upsilon_1, \ldots, \upsilon_n \rangle$ and if $f \in R\langle \upsilon_1, \ldots, \upsilon_n \rangle$ then $|f| \leqslant 1$.

We can extend the natural epimorphism $R \to k$ to the following

$$\pi : R\langle \upsilon_1, \dots \upsilon_n \rangle \to k[\upsilon_1, \dots \upsilon_n] \tag{1.11}$$

$$\sum_i a_i \upsilon^i \mapsto \sum_i \widetilde{a}_i \upsilon^i \tag{1.12}$$

$$f \mapsto \widetilde{f} \tag{1.13}$$

The reduction of f is denoted as $\widetilde{f} = \pi(f)$.

The canonical projection induced the following reduction map which is compatible with evaluations at $\mathbb{B}^n(K^{alg})$.

$$\begin{CD} R\langle \upsilon \rangle @>>> K[\upsilon] \\ @V{\text{eval}}VV @VV{\text{eval}}V \\ \overline{R} @>>> \overline{K} \end{CD}$$

Definition 1.2.1. An element $f \in T_n$ is called υ_n distinguished of order $s \in \mathbb{N}$ if f can be written as $f = \sum_{\nu=0}^{\infty} g_\nu \upsilon^\nu \in T_n \langle \upsilon_n \rangle$ and $g_\nu \in T_{n-1}$ and the following hold

1. g_s is a unit in T_{n-1}.

2. $|g_s| = |f|$.

3. $|g_s| > |g_\nu|$ for $\nu > s$.

Chapter 2

Affinoid Algebra

The elements T_n are functions from $\mathbb{B}^n(K^{alg}) \to K^{alg}$ and we can look at the zero set of T_n

$$V(\mathfrak{a}) = \{x \in \mathbb{B}^n(K^{alg}) \mid f(x) = 0 \text{ for } f \in \mathfrak{a}\} \tag{2.1}$$

Algebras of the type $T_n/\mathfrak{a}$ are called affinoid algebra. Note that ideal $\mathfrak{a}$ in T_n implies that it is closed in T_n.

Definition 2.0.1. Let A be a K algebra, it is called an affinoid K algebra if there is an epimorphism

$$T_n \to A \text{ for some } n \in \mathbb{N} \tag{2.2}$$

2.1 Some properties of Affinoid K Algebra

The following proposition is given on [Bosch, 2014, p 32]

Proposition 2.1.1. *If A is an affinoid K algebra then it has the following properties*

1. *A is Noetherian.*

2. *A is Jacobson*

3. *A satisfies Noether Normalization. In other words we have an injection*

$$T_m \to A \text{ for some } m \in \mathbb{N}.$$

The sup norm for a Tate Algebra T_n coincides with the Gauß norm. Further more

Proposition 2.1.2. *For an affinoid K algebra A the following conditions are equivalent*

1. $|f|_{sup} = 0$
2. *f is nilpotent.*

Definition 2.1.3. The Gauß norm on the Tate Algebra T_n induces **residue norm** on $A = T_n/\mathfrak{a}$. We denote the residue norm by $|\cdot|_\pi$ and is given by

$$|\pi(f)|_\pi = \inf_{a \in \operatorname{Ker} \pi} |f - a|, \tag{2.3}$$

where π is the canonical epimorphism $\pi : T_n \to A$.

Let $f \in T_n$ and $\overline{f} \in A$ then $|\overline{f}|_\pi$ is the infimum of all values $|f|$ varying over all inverse images of $\overline{f}$.

2.1.1 Some Properties of the Residue Norm

Proposition 2.1.4. *let $A : T_n/\mathfrak{a}$ be an affinoid K algebra with the projection map $\pi : T_n \to T_n/\mathfrak{a}$. The residue map $|\cdot|_\pi : A \to \mathbb{R}_{\geqslant 0}$ satifies the following:*

1. *The residue norm $|\cdot|_\pi$ is a K algebra norm and induces quotient topology of T_n on A.*
2. *The map $\pi : T_n \to A$ is continuous and open.*
3. *The affinoid K algebra A is complete under the norm $|\cdot|_\pi$.*
4. *Corresponding to any $\overline{f} \in T_n/\mathfrak{a}$ there exists an inverse image $f \in T_n$ such that $|\overline{f}|_\pi = |f|$.*

Let MaxSpec A denote the set of maximal ideals of A. If $\mathfrak{m} \in$ MaxSpec A then we can define $f(\mathfrak{m})$ as the residue class of f in $A/\mathfrak{m}$, that is $f \mod \mathfrak{m}$. We will often write $x \in$ MaxSpec A and $f(x)$ to mean residue class of f in A/x. The sup norm is given as

$$|f|_{\sup} = \sup_{\mathfrak{m} \in \text{MaxSpec } A} |f(\mathfrak{m})|$$

2.1.2 Some properties of the sup norm

1. $|f^n|_{\sup} = |f|_{\sup}^n$, the sup norm is power multiplicative.

2. If $\phi : B \to A$ is a morphism between two affinoid K algebras. Then

$$|\phi(b)|_{sup} \leqslant |b|_{sup} \text{ for all } b \in B$$

3. Let A be an affinoid K algebra and $f \in A$, with a residue norm $|\cdot|_\pi$ corresponding to the epimorphism $\pi : T_n \to A$. Then

$$|f|_{sup} \leqslant |f|_\pi.$$

   In particular $|f|_{sup}$ is finite.

4. Let A be an affinoid K algebra and $f \in A$, with a residue norm $|\cdot|_\pi$. Then we have $|f|_{sup} < 1$ if and only if the sequence $\{|f^n|_\pi\}$ with $n \in \mathbb{N}$ is a zero sequence. We call such f as 'topologically nilpotent' with respect to $|\cdot|_\pi$.

5. (Maximum Principle) Let A be an affinoid K algebra and $f \in A$, there exists a point $x \in \text{MaxSpec } A$ such that $|f(x)| = |f|_{sup}$.

6. Let A, B be affinoid K algebras and there is a morphism $\phi : B \to A$. Then ϕ is continuous with respect to residue norm.

2.2 Localization of Affinoid Algebras

Let A be an affinoid algebra. Assume that $f_0, \ldots, f_r \in A$ without any common zeroes that is $V(f_0 \ldots, f_r) = \varnothing$.

$$A\left\langle \frac{f}{f_j}\right\rangle := A\left\langle \frac{f_0}{f_j},\dots,\frac{f_r}{f_j}\right\rangle$$
$$= \frac{A\left\langle \zeta_1,\dots \zeta_r\right\rangle}{f_i - f_j\zeta_i}, \quad i = 1,\dots,r$$

where f is a tuple $(f_0,\dots,f_r)$

2.3 Affinoid Space

A geometric object $\mathrm{Sp}(A)$ can be associated to every affinoid algebra A.

$$\mathrm{Sp}(A) := \mathrm{MaxSpec}\ (A) := \{\mathfrak{m} : \mathfrak{m} \text{ is a maximal ideal of } A\} \tag{2.4}$$

The set of maximal ideals of A is called an affinoid space.

The crucial result which is useful is the Hilbert Nullstellensatz.

Theorem 2.3.1. *Let $\mathfrak{m} \subset T_n$ be a maximal ideal then $T_n/\mathfrak{m}$ is a finite field extension of K.*

We can extend the valuation on K to the finite field extension $A/\mathfrak{m}$.

2.3.1 Rational Affinoid Subdomain

Rigid geometry is built out of Affinoid spaces. We need to introduce holomorphic functions on an affinoid space. But, the biggest problem we face is the natural topology of the field K which is totally disconnected. Thus, there is a need to provide analytic space with extra topological structure to get a non trivial notion of connectedness. This will give us (1) Analytic continuation and (2) Global expansion of analytic function on polydiscs.

Let $X = \mathrm{Sp}(A)$ then we can define a rational affinoid subdomain of X

$$X_j := X(f_0/f_j, \ldots, f_r/f_j) \tag{2.5}$$

$$:= \{x \in X : |f_i(x)| \leqslant \left|f_j(X)\right| \text{ for } i = 0, \ldots, r\} \tag{2.6}$$

The family $X_0, \ldots, X_r$ is called a *rational covering of* X. We can give a structure ring to X as follows

$$\mathcal{O}_X(X_j) : = A\left\langle \frac{f}{f_j} \right\rangle \tag{2.7}$$

$$= A\left\langle \frac{f_0}{f_j}, \ldots, \frac{f_r}{f_j} \right\rangle \tag{2.8}$$

$$= \frac{A\left\langle \frac{\zeta_0}{\zeta_j}, \ldots, \frac{\zeta_r}{\zeta_j} \right\rangle}{\zeta_j f_i - f_j \zeta_i}, \quad i = 1, \ldots, r \tag{2.9}$$

Chapter 3

Adic Spaces

We will closely follow [Wedhorn, 2012] to describe basic properties of adic spaces.

Definition 3.0.1. Let R be a set endowed with a structure of a topological space such that $(R, +)$ is a topological group R is a ring and the following map is continous

$$A \times A \to A, (a, a') \mapsto aa'. \tag{3.1}$$

We call R a *topological ring.*

Definition 3.0.2. Let R be a topological ring, where the ideal I defines the topology of R, that is $\{I^n\}_{n\in\mathbb{N}}$ forms a basis of neighborhood of 0 in R, and in that case R is called *adic*. The ideal I is called an ideal of definition.

Example 3.0.3. Let $\mathbb{Z}_{(p)}$ be localization of integers at the prime p. Then the ideal of definition is $p\mathbb{Z}_{(p)}$. The ideals $p^n\mathbb{Z}_{(p)}$ form a fundamental system of neighborhoods of zero.

Definition 3.0.4. A ring R is called *Huber Ring* or f *adic ring* is there is an open adic subring $R_0 \subseteq R$ such that the topology of R_0 is defined by a finitely generated ideal I, that is R_0 admits an ideal of definition which is finitely generated. The ring R_0 is called the *ring of definition* of R.

Example 3.0.5. On the field of rational numbers $R = \mathbb{Q}$ we can put a p adic topology with absolute value $|\cdot|_p$. It follows that the ring of definition is $R_0 = \mathbb{Z}_{(p)}$ and the ideal of definition $p\mathbb{Z}_{(p)}$.

Definition 3.0.6. A subset $S \subset R$ is called bounded if for all neighborhood U of zero, there is a neighborhood V of zero such that $S \cdot V \subseteq U$, where $S \cdot V = \{sv : s \in S, v \in V\}$. An element $f \in R$ is power bounded if $\{f^n : n \in \mathbb{N}\}$ is bounded.

$$R^{\circ} := \{f \in R \mid f \text{ is power bounded}\} \tag{3.2}$$

An element $f \in R$ is called *topologically nilpotent* if $\lim_{n\to\infty} f^n = 0$

$$R^{\circ\circ} := \{f \in R \mid f \text{ is topologically nilpotent}\} \tag{3.3}$$

Example 3.0.7. With the rationals $\mathbb{Q}$ we have the ring $\mathbb{Z}_{(p)}$ which corresponds to the subring $\mathbb{Q}^{\circ} = \mathbb{Z}_{(p)}$ of power bounded elements, and $\mathbb{Q}^{\circ\circ} = p\mathbb{Z}_{(p)}$ is the topologically nilpotent elements of $\mathbb{Q}$.

Definition 3.0.8. Let R be a f *adic* or *Huber ring* with a topologically nilpotent unit, then R is called a *Tate Ring.*

Example 3.0.9. $\mathbb{Q}_p$ with the topological nilpotent unit as p, since $p^n \to 0$ in the topology generated by $|\cdot|_p$.

3.1 Valuation

Definition 3.1.1. A valuation on a ring R is a map $|\cdot| : R \to \Gamma \cup \{0\}$, where Γ is a totally ordered group such that

i $|ab| = |a||b|$ for all $a, b \in R$

ii $|a + b| \leqslant \max(|a|, |b|)$ for all $a, b \in A$

iii $|0| = 0$ and $|1| = 1$

The order on $\Gamma \cup \{0\}$ comes from the order on Γ where 0 is the minimum element. $\mathrm{Im}(|\cdot|) \backslash \{0\}$ generates a subgroup of Γ which is called the *value group* of $|\cdot|$.

We also define the *support* of $|\cdot|$ as the set of elements of R that have zero valuation.

$$\mathrm{supp}(|\cdot|) := |\cdot|^{-1}(0) \tag{3.4}$$

The multiplicative property of valuation forces the elements which have a zero valuation (called *support of valuation*) to be a prime ideal inside the ring.

Definition 3.1.2. Let Γ_1 and Γ_2 be ordered groups. Two valuations $|\cdot|_1 : R \to \Gamma_1 \cup \{0\}$ and $|\cdot|_2 : R \to \Gamma_2 \cup \{0\}$ are *equivalent* if there is an isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ and the following diagram commutes.

$$\begin{array}{ccc} \Gamma_1 \cup \{0\} & \xrightarrow{\varphi} & \Gamma_2 \cup \{0\} \\ & {}_{|\cdot|_1}\nwarrow \quad R \quad \nearrow_{|\cdot|_2} & \end{array}$$

Equivalently two valuations $|\cdot|_1$ and $|\cdot|_2$ are equivalent if for all $a, b \in R$ we have $|a|_1 \geqslant |b|_1$ if and only if $|a|_2 \geqslant |b|_2$

We now need the notion of a continuous valuation.

Definition 3.1.3. Let R be a topological ring with valuation $|\cdot|$, then the valuation is called *continuous* if we have a continuous morphism

$$R \to \mathrm{Frac}\left(\frac{R}{\mathrm{supp}|\cdot|}\right) \tag{3.5}$$

The notion of prime ideals and continuous maps arising from $\mathrm{supp}|\cdot|$ is crucial for our purpose. We also need to define some more terms

Definition 3.1.4. Given a valuation $|\cdot|$ we define the following terms

$$\text{Valued Field } K(|\cdot|) := \mathrm{Frac}(R/\mathrm{supp}(|\cdot|)) \tag{3.6}$$

$$\text{Valuation Ring } R(|\cdot|) := \{x \in K \mid |x| \leqslant 1\} \tag{3.7}$$

$$\text{Maximal Ideal } \mathfrak{m}(|\cdot|) := \{x \in K \mid |x| < 1\} \tag{3.8}$$

$$\text{Residue Field } \kappa(|\cdot|) := R(|\cdot|)/\mathfrak{m}(|\cdot|) \tag{3.9}$$

The valuation ring $A(|\cdot|)$ is also written as $\mathcal{O}_K$.

Remark 1. Since the $|r^n| = |r|^n$ for all $r \in K$ and $n \in \mathbb{N}$ (called power multiplicative). We have

$$\text{Valuation Ring } \mathcal{O}_K = K^\circ = \{x \in K \mid |x| \leqslant 1\} \tag{3.10}$$

$$\text{Maximal Ideal } \mathfrak{m}(|\cdot|) = K^{\circ\circ} = \{x \in K \mid |x| < 1\} \tag{3.11}$$

$$\tag{3.12}$$

Let K be a non archimedean field, with open valuation ring K°. Since, K has non trivial valuation there is an element $\pi < 1$. The elements $\{\pi^n\}_{n \in \mathbb{N}}$ form a fundamental system of neighborhood of zero. Also, π is a topologically nilpotent unit of K. Hence, K is a Tate ring.

3.2 Affinoid Rings

We now define affinoid rings and maps between them

Definition 3.2.1. An affinoid ring is a pair (R, R^+), where R is a f adic ring and $R^+ \subseteq R^\circ$ is an open subring of R that is integrally closed in R (the ring of integral elements of R). A morphism φ of affinoid pairs (R, R^+) and (S, S^+) is continuous morphism $\varphi : R \to S$ such that $\varphi : R \to S$ such that $\varphi(R^+) \subseteq S^+$.

We can extend the above definition to a non-archimdean field K and define an affinoid K algebra.

Definition 3.2.2. Let K be a non-archimdean field and $K^+ (\subset K^\circ)$ integrally closed subring, the pair (K, K^+) is called an affinoid field.

An *affinoid* (K, K^+) algebra is a pair of rings (R, R^+) such that

1. R is a K algebra.

2. R is f adic.

3. $R^+ \subseteq R^\circ$ and it is integrally closed in R, and topologically open inside the ring R.

The morphism of (K, K^+) affinoid algebras (R, R^+) and (S, S^+) is continuous morphism $\varphi : R \to S$ such that $\varphi : R \to S$ such that $\varphi(R^+) \subseteq S^+$ and the following diagram commutes

$$\begin{array}{ccc} (R, R^+) & \xrightarrow{\quad \phi \quad} & (S, S^+) \\ & \nwarrow \qquad \nearrow & \\ & (K, K^+) & \end{array}$$

Example 3.2.3. The most relevant example for our case is the Tate Algebra $K\langle T \rangle$. The pair $(K\langle T \rangle, K\langle T \rangle^\circ)$ is a (K, K°) affinoid algebra, where

$$K\langle T \rangle^\circ := \{f \in K\langle T \rangle \mid |f| \leqslant 1\} \tag{3.13}$$

3.3 Adic Spectrum

We can now define the adic spectrum of a pair (R, R^+).

Definition 3.3.1. Let (R, R^+) be affinoid K algebra, then we define

1. adic spectrum as

$$\text{Spa}(R, R+) := \{\text{equivalence classes of continuous valuations } |\cdot| \text{ on } R \text{ such that } |f| \leqslant 1 \text{ for all } f \in R^{\circ}\} \tag{3.14}$$

2. Given a point x of the space $X := \text{Spa}(R, R^{+})$ we can talk about evaluation of a function $f \in R$ at point x as the map

$$f \mapsto |f|_{x} \tag{3.15}$$

3. Topology on $\text{Spa}(R, R^{+})$ is generated by the subsets

$$\{x \in \text{Spa}(R, R^{+}) \text{ such that } |f|_{x} \leqslant |g|_{x} \neq 0, f, g \in R\} \tag{3.16}$$

3.4 Structure Presheaf on Adic Spaces

The basis of topology is given by rational subsets of the form

$$U\left(\frac{T}{S}\right) := \{x \in \text{Spa}(R, R^{+}) | \text{ for all } t \in T \text{ we have } |t|_{x} \leqslant |s|_{x} \neq 0\} \tag{3.17}$$

If $f_1, f_2, \ldots, f_n \in R$ generate the unit ideal and $g \in R$the rational subsets which form the basis of the topology are

$$U\left(\frac{f_1, \ldots, f_n}{g}\right) := \{x \in \text{Spa}(R, R^{+}) \mid |f_i|_{x} \leqslant |g|_{x} i = 1, \ldots, n\} \tag{3.18}$$

These rational subsets are analogous to open sets in Algebraic Geometry which are used to contruct Affine Schemes. These open sets are of the form $D(f) = \text{Spec}(R[1/f])$.

The core idea comes from maximal ideals in $\mathbb{C}$ which are of the form $(x - a), a \in \mathbb{C}$. We form schemes by considering these ideals as a topological space. But, we also can form a valuation ring with uniformizer as $(x - a)$, and look at the topology generated by the valuation. These two ideas are considered together in adic spaces.

Chapter 4

Introduction to Perfectoid Spaces

We begin by a simple observation that the elements of $\mathbb{Z}_p$ (ring of p adic integers) and the elements of $\mathbb{F}_p[[t]]$ look similar

$$a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \ldots, a_i \in \mathbb{F}_p \tag{4.1}$$

$$a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots, a_i \in \mathbb{F}_p \tag{4.2}$$

but the former has characteristic zero, whereas latter has characteristic p. Informally, we have a correspondence $t \leftrightarrow p$. We now get closer to perfection (more precisely we are thinking perfect closure) by adding all p^r-th roots ($r \geqslant 1$) by using an inverse limit with Frobenius as the transition map. Informally, inverse limits are like vectors where adjacent elements are related via Frobenius (our transition maps). Let us record some informal correspondence given as $\leftrightarrow$.

$$(p, p^{1/p}, p^{1/p^2}, \ldots) \leftrightarrow (t, t^{1/p}, t^{1/p^2}, \ldots) \tag{4.3}$$

$$\mathbb{Z}_p[p^{1/p^\infty}]/\langle p\rangle \leftrightarrow \mathbb{F}_p[t^{1/p^\infty}]/\langle t\rangle \tag{4.4}$$

$$\text{Passing to fields } \mathbb{Q}_p(p^{1/p^\infty}) \leftrightarrow \mathbb{F}_p(t^{1/p^\infty}) \tag{4.5}$$

$$\text{Completion p adic } \widehat{\mathbb{Q}_p(p^{1/p^\infty})} \leftrightarrow \mathbb{F}_p((t^{1/p^\infty})) \text{ Completion t adic} \tag{4.6}$$

We follow [Bhatt, 2014] and consider field of p adic numbers $\mathbb{Q}_p$ and the field of Laurent series over $\mathbb{F}_p$ given as $\mathbb{F}_p((t))$. In both of the fields we can represent elements as Laurent series, in particular for $\mathbb{Q}_p$ we replace t with p. But, $\mathbb{Q}_p$ has characteristic zero, whereas $\mathbb{F}_p((t))$ has characteristic p.

$$\mathbb{Q}_p(p^{\frac{1}{p^\infty}}) := \bigcup_n \mathbb{Q}_p\left(p^{\frac{1}{p^n}}\right) \quad \text{and} \quad \mathbb{F}_p((t^{1/p^\infty})) = \bigcup_n \mathbb{F}_p\left(\left(t^{\frac{1}{p^n}}\right)\right) \tag{4.7}$$

A classical theorem of Fontaine and Wintenberger states that the absolute Galois Groups of $\mathbb{Q}_p(p^{\frac{1}{p^\infty}})$ and $\mathbb{F}_p((t))$ are isomorphic.

The key observation is the following correspondence

$$\mathbb{Z}_p\left[p^{\frac{1}{p^\infty}}\right]/\langle p\rangle \simeq \mathbb{F}_p\left[t^{\frac{1}{p^\infty}}\right]/\langle t\rangle \tag{4.8}$$

$$p^{\frac{1}{p^\infty}} \mapsto t^{\frac{1}{p^\infty}} \tag{4.9}$$

4.1 Perfectoid Field

Let K be a complete non-archimedean field of characteristic 0 with a residue field of characteristic p (called mixed characteristic $(0, p)$), which is equipped with a non-discrete valuation of rank 1. Let $K^\circ \subset K$ denote the subring formed by elements of norm $\leqslant 1$. We call K perfectoid if the Frobenius map

$$K^\circ/p \to K^\circ/p, x \mapsto x^p \tag{4.10}$$

is surjective. The Frobenius map is a homeomorphism on topological spaces, as well as on etale topoi.

More formally we state the definition from [Scholze, 2012]

Definition 4.1.1. A perfectoid field is a complete topological field K whose topology is induced by a nondiscrete valuation of rank 1, such that the Frobenius Φ is surjective on K°/p

Example 4.1.2. The basic examples are completions of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\mu_{p^\infty})$. A non example is the field $\mathbb{Q}_p$ since its underlying value group $\mathbb{Z}$ is discrete.

We now define tilting which is given as an inverse limit

$$K^\flat = \varprojlim_{x \mapsto x^p} K \tag{4.11}$$

Elements of $K^\flat$ are sequences given as

$$(x_0, x_1, x_2, \ldots) \text{ such that } x_n^p = x_{n-1} \tag{4.12}$$

The examples include $X \mapsto (X, X^{1/p}, X^{1/p^2}, \ldots)$ or $X \mapsto (X^2, X^{2/p}, X^{2/p^2}, \ldots)$. In general for the multiplicative (but non-additive) map $K^\flat \to K, x \mapsto x^\sharp$ is simply the projection on the first factor in most cases. A more precise definition requires introduction of Witt vectors.

The big result linking these K and its tilt $K^\flat$ is the following

Theorem 4.1.3. *The absolute Galois groups of K and $K^\flat$ are canonically isomorphic.*

We can explain the above theorem a bit more using an example given in [Scholze, 2013]. Let $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$, the above theorem states that we have a natural equivalence between the category of finite extensions L/K and category of finite extensions $M/K^\flat$. For example say M is obtained by adjoining a root of $X^2 - 7tX + X^5$, the basic idea is $t \mapsto p$ (as seen in introduction to this chapter). But for $p = 3$ we have $7 \equiv 1 \mod 3$, therefore, $X^2 - 7tX + t^5 = X^2 - tX + t^5$ but the two polynomials are not the same if t is replaced by p

$$X^2 - 7pX + p^5 \neq X^2 - pX + p^5$$

and hence Galois groups are different. We can resolve this issue by defining M as the splitting field of $X^2 - 7t^{1/p^n}X + t^{5/p^n}$ where $n \in \mathbb{Z}_{\geqslant 0}$ (this makes sense as $K^\flat$ is perfect and we still have a quadratic polynomial). Let L_n denote the splitting field of $X^2 - 7p^{1/p^n}X + p^{5/p^n}$ as n varies. Letting $n \to \infty$ the fields L_n stabilize and we get the desired field L. Further details can be obtained from [Scholze, 2013].

A theorem of Scholze states that

Theorem 4.1.4. *The categories of perfectoid K-spaces and perfectoid* $K^\flat$ *spaces are canonically identified; this identification preserves the étale topology.*

We now focus on Example 2 [Bhatt, 2014]

Consider $A' = K^\circ[\widehat{X^{1/p^\infty}}]$ a p adically complete K° algebra. Set

$$A := A'\left[\frac{1}{p}\right] \tag{4.13}$$

It can be shown that A is perfectoid K algebra and it is denoted by $K\left\langle X^{1/p^\infty}\right\rangle$ and $A^\flat := K^\flat\left\langle X^{1/p^\infty}\right\rangle$.

We can the standard gluing maps to obtain the Projective perfectoid given as $K\left\langle X^{\pm/p^\infty}\right\rangle$. These are the perfectoid spaces we study in this Thesis.

4.2 Untilts of a perfectoid field

The primary reference for this section will be [Kedlaya, 2017, Weinstein, 2017].

Definition 4.2.1. An untilt of K is a pair $(K^\sharp, \iota)$, where $\iota :\to K^{\sharp,\flat}$ is an isomorphism, and $K^\sharp$ is a perfectoid field.

Let $W(K^\circ)$ denote the ring of Witt vectors associated with K°. Then

$$K^\circ \xrightarrow{\iota} K^{\sharp\flat\circ} \xrightarrow{\sharp} K^{\sharp\circ} \text{ induces the map} \tag{4.14}$$

$$\theta_{K^\sharp} : W(K^\circ) \to K^{\sharp\circ} \tag{4.15}$$

$$\sum_{n=0}^{\infty} [a_n]p^n \mapsto \sum_{n=0}^{\infty} a_n^\sharp p^n \tag{4.16}$$

In the above mapping $\operatorname{Ker}\theta_{K^\sharp}$ is a primitive ideal of degree 1. We have the following theorem from [Weinstein, 2017, pp18]

Theorem 4.2.2. *Let* I *be the primitive ideal (of degree 1) of* $W(K^\circ)$*. Then we have a bijection*

(4.17) *Primitive ideals of degree 1* $\simeq$ *Isomorphism classes of untilts of* K

$$I \mapsto (W(K^\circ)/I)\left[\frac{1}{p}\right] \tag{4.18}$$

We now define Witt vectors concretely as in the book of [Greenberg and Serre, 2013, pp 40].

4.3 Witt Vectors

Definition 4.3.1. Let p be a prime number and $(X_0, X_1, \ldots, X_n, \ldots)$ be an infinite sequence of indeterminates. For $n \geqslant 0$ we define the n-th *Witt Poly-*

nomial as

$$W_n = \sum_{i=0}^{n} p^i X_i^{p^{n-i}} = X_0^{p^n} + pX_1^{p^{n-1}} + \ldots + p^n X_n \tag{4.19}$$

The first three Witt polynomials are

$$W_0 = X_0 \tag{4.20}$$

$$W_1 = X_0^p + pX_1 \tag{4.21}$$

$$W_2 = X_0^{p^2} + pX_1^p + p^2 X_2 \tag{4.22}$$

Let $(Y_0, Y_1, \ldots, Y_n, \ldots)$ be another sequence of indeterminates, then we have the following Theorem 6 on page 40 of [Greenberg and Serre, 2013]

Theorem 4.3.2. *For every* $\Phi \in \mathbb{Z}[X, Y]$ *there exists a unique sequence* $(\phi_0, \ldots, \phi_n, \ldots)$ *of elements of* $\mathbb{Z}[X_0, \ldots, X_n, \ldots; Y_0, \ldots, Y_n]$ *such that:*

$$W_n(\phi_0, \ldots, \phi_n, \ldots) = \Phi(W_n(X_0, \ldots), W_n(Y_0, \ldots)) \tag{4.23}$$

Using the above theorem we can define addition S_i and multiplication P_i recursively via polynomials ϕ_i.

$$S_i \text{ associated with } \Phi(X, Y) = X + Y \tag{4.24}$$

$$P_i \text{ associated with } \Phi(X, Y) = X \cdot Y \tag{4.25}$$

(4.26)

Let R be an arbitrary commutative ring, and let $A = (a_0, a_1, \dots,)$ and $B = (b_0, b_1, \dots,)$ be elements of $R^{\mathbb{N}}$ (Witt vectors with coefficients in R), set:

(4.27) Addition $A + B = (S_0(A, B), \dots, S_n(A, B), \dots)$

(4.28) Multiplication $A \cdot B = (P_0(A, B), \dots, P_n(A, B), \dots)$

Using the theorem 4.3.2 and setting $S_0(A, B) = \phi_0, S_1(A, B) = \phi_1, \dots, S_i = \phi_i \dots$, we show explicit formulae for adding Witt vectors.

$$
\begin{aligned}
W_0(\phi_0, \dots) &= \Phi(W_0(A), W_0(B)) \\
&= W_0(A) + W_0(B) \\
S_0(A, B) = \phi_0 &= a_0 + b_0 \\
W_1(\phi_0, \phi_1, \dots) &= \Phi(W_1(A), W_1(B)) \\
&= W_1(A) + W_1(B) \\
\phi_0^p + p\phi_1 &= a_0^p + pa_1 + b_0^p + pb_1 \\
S_1(A, B) = \phi_1 &= a_1 + b_1 + \frac{a_0^p + b_0^p - (a_0 + b_0)^p}{p}
\end{aligned}
$$

Similarly product can be defined by setting $P_i = \phi_i$.

$$
\begin{aligned}
W_0(\phi_0, \ldots) &= \Phi(W_0(A), W_0(B)) \\
&= W_0(A) \cdot W_0(B) \\
P_0(A, B) = \phi_0 &= a_0 b_0 \\
W_1(\phi_0, \phi_1, \ldots) &= \Phi(W_1(A), W_1(B)) \\
&= W_1(A) \cdot W_1(B) \\
\phi_0^p + p\phi_1 &= (a_0^p + pa_1)(b_0^p + pb_1) \\
p\phi_1 &= (a_0^p + pa_1)(b_0^p + pb_1) - a_0^p b_0^p \\
P_1(A, B) = \phi_1 &= b_0^p a_1 + a_0^p b_1 + pa_1 b_1
\end{aligned}
$$

Example 4.3.3. The Witt ring of the finite field of order p ($W(\mathbb{F}_p)$) is the ring of p-adic integers $\mathbb{Z}_p$.

We now follow [Kedlaya and Liu, 2015]

Definition 4.3.4. A *strict* p *ring* is p adically complete and p torsion free ring S for which $S/(p)$ is perfect.

Given a strict p ring S and a padically complete ring U with a ring morphism $\bar{t} : S/(p) \to U/(p)$, we can lift elements of $S/(p)$ to U uniquely via map $t : S/(p) \to U$. For any $\bar{x} \in S/(p)$ we can talk about lifting $\bar{t}(x^{-p^n})$ to $y \in U$, with $t(\bar{x}) \equiv y^{p^n} \mod (p^{n+1})$. We have a multiplicative section $[\cdot] : S/(p) \to S$ (called Teichmuller map); coming from the projection map $S \to S/(p)$. Each $x \in S$ with $\bar{x}_n \in S/(p)$ can be written uniquely as $\sum_{n=0}^{\infty} p^n [\bar{x}_n]$.

We now state the lemma from [Kedlaya and Liu, 2015, pp. 72-73]

Lemma 4.3.5. *Let* S *be a strict* p*-ring, let* U *be a* p*-adically complete ring, and let* $\pi : U \to U/(p)$ *be the natural projection. Let* $\bar{t} : S/(p) \to U/(p)$ *be a ring homomorphism, and lift* $\bar{t}$ *to a multiplicative map* $t : S/(p) \to U$*. The formulae*

$$T\left(\sum_{n=0}^{\infty} p^n [\bar{x}_n]\right) = \sum_{n=0}^{\infty} p^n t(\bar{x}_n) \tag{4.29}$$

defines a unique homomorphism $T : S \to U$ *such that* $T \circ [\cdot] = t$*.*

The above lemma is used as an input in the following theorem.

Theorem 4.3.6. *The functor* $S \rightsquigarrow S/(p)$ *is an equivalence of categories between strict* p*-rings and perfect* $\mathbb{F}_p$*-algebras.*

Chapter 5

Grothendieck Topology

The topology that is induced by a non-archimedean valuation is always disconnected. Hence, we need to introduce Grothendieck Topology to counteract the property of disconnectedness.

5.1 Introduction

We want to put a topology on a category, and thus require a notion of intersection and covering. Intersection can be interpreted as a fiber product (or pullback) of a category. More concretely, if we are given two open sets U, V of a topological space X we want a category theoretic notion of intersection $U \cap V$ on the topological space X. The inclusion maps $U \subset X$ becomes a morphism $U \to X$ and the intersection becomes fibered product $U \times_X V$. In other words, if there are two morphism $f : U \to X$ and $g : V \to X$, the fibre product is $U \times_X V$ and it comes equipped with two morphisms $U \times_X V \to U$ and $U \times_X V \to V$.

Let X be a topological space with topology $\mathscr{T}$. We can make $\mathscr{T}$ into a category which has objects as open sets of X and morphisms as inclusion maps. Let U, V be objects of $\mathscr{T}$ (or open sets of X). We can define morphisms as

$$\text{Hom}(U, V) = \begin{cases} \varnothing & \text{if } U \not\subseteq V \\ \{U \to V\} & \text{if } U \subseteq V \end{cases} \tag{5.1}$$

The space X is the final object of the category $\mathscr{T}$.

We also need the notion of covering of a set U in some topological space. If $\{U_i\}_{i \in I}$ forms a covering of U, we can consider covering more abstractly as a set of morphisms $\mathcal{U} = \{\phi_i : U_i \to U\}_{i \in I}$. We follow the definition from ([Stacks Project Authors, 2016, Tag 03NG],[Stacks Project Authors, 2016, Tag 03NH])

Definition 5.1.1. Let $\mathscr{C}$ be a category. The following data gives a *a family of morphisms with a fixed target* $\mathcal{U} = \{\phi_i : U_i \to U\}_{i \in I}$

1. an object $U \in \mathscr{C}$.
2. Index set I (which could be empty)
3. for all $i \in I$, there is a morphism ϕ_i with target U given as $\phi_i : U_i \to U$

If U is an object of the category, we can give a set of coverings to all objects of the category.

Definition 5.1.2. A *site* (or *topology*) consists of two things a category $\mathscr{C}$ and a set $\text{Cov}(\mathscr{C})$ called covering which consists of families of morphism with a fixed target.

1. (isomorphism) if $\phi : V \to U$ is an isomorphism in $\mathscr{C}$, the ϕ is a covering.

2. (localilty) Let $\{\phi_i : U_i \to U\}_{i \in I}$ be a covering of object U and furthermore each U_i has a covering $\{\psi_{ij} : U_{ij} \to U_i\}_{j \in I_i}$ then we get a covering given as

$$\{\phi_i \circ \psi_{ij} : U_{ij} \to U\}_{(i,j)} \text{ where } (i,j) \in \prod_{i \in I} i \times I_i \tag{5.2}$$

3. (base change) if $\{U_i \to U\}_{i \in I}$ is a covering and $V \to U$ is a morphism. Then we can form a covering for V.

   (a) the fibre product $U_i \times_U V$ exists

   (b) there is a covering $\{U_i \times_U V \to V\}_{i \in I}$

A site is also called a category with Grothendieck Topology. We will always assume that collection of coverings of a site is a set, and collection of objects of the category $\mathscr{C}$ underlying the site is a also a set (that is $\mathscr{C}$ is a small category).

Examples We now give some examples of Sites.

1. Every category can be turned into a Site in a canonical way. We just need the notion of coverings which is simply the identity map $\{\mathbb{1}_U : U_i \to U\}_{i \in I}$ such that I is countable.

2. For a topological space X the associated site is denoted as X_{Zar} and is defined as follows:

(a) Open sets of X are the objects of X_{Zar}.

(b) Inclusion maps become morphisms.

(c) Usual topological coverings, become coverings in X_{Zar}.

For open sets $U, V \subset W \subset X$ the fiber product $U \times_W V = U \cap V$ exists.

3. Category of G-sets can be endowed with a site $\mathscr{T}_G$.

(a) Objects are sets with left G action.

(b) Morphisms are G equivariant maps.

(c) Covering maps are families $\{\phi_i : U_i \to U\}_{i\in I}$ which satisfies $\bigcup_{i\in I} \phi_i(U_i) = U$

Definition 5.1.3. A *presheaf* of sets on a site with underlying category $\mathscr{C}$ is a contravariant functor from $\mathscr{C}$ to the category of sets. The presheaf $\mathscr{F}$ of sets is called a separated presheaf on the site $\mathscr{C}$ if for all coverings $\{\varphi_i : U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$ the map

$$\mathscr{F}(U) \longrightarrow \prod_{i\in I} \mathscr{F}(U_i) \tag{5.3}$$

is injective. Here the map is $s \mapsto (s|_{U_i})_{i\in I}$.

A *sheaf* on the site is a presheaf $\mathscr{F}$ which satisfies the equalizer condition

$$\mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{(i,j)\in I\times I} \mathscr{F}(U_i \times_U U_j) \tag{5.4}$$

The first map in the equalizer above is $s \mapsto (s|_{U_i})_{i\in I}$ and the two maps on the right are

$$(s_i)_{i\in I} \mapsto (s_i|_{U_i \times_U U_j})$$

$$(s_i)_{i\in I} \mapsto (s_j|_{U_i \times_U U_j})$$

Definition 5.1.4. A *Grothendieck Topos* is a category of sheaves of sets on a site.

5.2 Strong Grothendieck Topology

In order to work with rigid analytic spaces we need strong Grothendieck Topology.

Definition 5.2.1. The strong Grothendieck Topology on an affinoid space $X = (\mathrm{Sp}A)$ and underlying category $\mathscr{C}$ (in the sense of Site) is given as:

1. $\varnothing, X \in$ category $\mathscr{C}$.

2. (admissible open) Let U be a subset of X. U is called *admissible open* if there is a covering $U = \bigcup_i U_i$ (this covering is not necessarily finite) by affinoid subdomains U_i of X such that for all morphisms of affinoid spaces

$$\phi : Y \to X \text{ satisfying } \phi(Y) \subset U \tag{5.5}$$

the covering $\{\phi^{-1}(U_i)\}_{i\in I}$ of Y admits a refinement by a finite covering with affinoid subdomains of Y.

3. (admissible covering) Let V be an admissible open set of X and $V = \cup_i V_i$ is covering of V where each V_i is admissible open. Such a covering is called *admissible* if for each morphism of affinoid spaces

$$\phi : Y \to X \text{ satisfying } \phi(Y) \subset U \tag{5.6}$$

the covering $\{\phi^{-1}(U_i)\}_{i \in I}$ of Y admits a refinement by a finite covering with affinoid subdomains of Y.

We often refer to Grothendieck Topology as G topology.

Definition 5.2.2. A *rigid analytic variety* over K is a locally G-ringed space $(X, \mathcal{O}_X)$ which satisfies the following properties

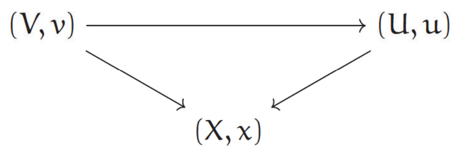
1. $X, \varnothing$ are admissible open sets.
2. X has admissible covering $\{U_i\}_{i \in I}$ such that $(U_i \mathcal{O}_{U_i})$ is an affinoid variety for all $i \in I$.

5.3 Sheaves on Rigid Analytic Spaces

We are primarily concerned with affinoid spaces $X = \text{Sp } A$ and the functor $\mathcal{O}_X$ representing the structure sheaf. The space X has a finite covering by affinoid subdomains $\{X_0 \ldots, X_n\}$ and we have the corresponding structure ring $\mathcal{O}_X(X_j)$ for each subdomain X_j in the rational covering.

5.4 Stalk

The stalk is obtained by putting an order on the admissible sets $\{U_i\}$. The following commutative diagram gives an order $(V, v) \leqslant (U, u)$ over space (X, x). The points $u \in U, v \in V$ get mapped to $x \in X$.



The stalk is given as

$$\mathcal{O}_{X,\bar{x}} = \varinjlim_{(U,u)} \Gamma(U, \mathcal{O}_U) \tag{5.7}$$

It is important to note that the notion of stalk is very different from topology. It is possible that a non-zero section $s \in \mathcal{O}(X)$ has a zero stalk $s_x \in \mathcal{O}_x$ for every $x \in X$. This does not violate the gluability axiom of sheaves because it is possible that $\{U_x\}_x$ does not form an admissible cover of X.

5.5 Cohomology

Given a presheaf $\mathscr{F}$ on the site $\mathscr{C}$ with covering $\mathcal{U} = \{U_i \to U\} \in \mathrm{Cov}(\mathscr{C})$ we can define the zeroth cohomology group as

(5.8)

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i) \text{ such that } s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \right\}.$$

The fundamental observation ([Stacks Project Authors, 2016, Tag 03NG]) is that category of abelian sheaves on a site is an abelian category and therefore has enough injectives. This helps us define cohomology as the right derived functor of the sections functor $\mathscr{F}$. Let $\mathscr{F} \to \mathcal{I}^\bullet$ be an injective resolution where $\mathscr{F} \in \mathrm{Ab}(\mathscr{C})$, and let $U \in \mathrm{Ob}(\mathscr{C})$ then

$$H^p(U, \mathcal{F}) := R^p\Gamma(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet)) \tag{5.9}$$

Since $\mathrm{Ab}(\mathscr{C})$ is an abelian category the global sections functor $\Gamma(U, -)$ is left exact.

Chapter 6

Some properties of $K\left\langle \upsilon^{1/p^{\infty}} \right\rangle$

6.1 Notation

We start by defining the following $K\left\langle \upsilon^{\pm 1/p^{\infty}} \right\rangle$K-Algebras. Let $a, b \in \mathbb{N} \cup \{0\}$ and $i = a/p^{b}$,then we define

(6.1)

$$K\left\langle \upsilon^{1/p^{\infty}} \right\rangle := \sum_{a,b} c_{(a,b)} \upsilon^{i}, \quad c_{(a,b)} \in K, \quad \lim_{a+b \to \infty} |c_{(a,b)}| \to 0$$

(6.2)

$$K\left\langle \upsilon^{-1/p^{\infty}} \right\rangle := \sum_{a,b} c_{(a,b)} \frac{1}{\upsilon^{i}}, \quad c_{(a,b)} \in K, \quad \lim_{a+b \to \infty} |c_{(a,b)}| \to 0$$

(6.3)

$$K\left\langle \upsilon^{\pm 1/p^{\infty}} \right\rangle := \text{ Generated by } \alpha, \beta \text{ where } \alpha \in K\left\langle \upsilon^{1/p^{\infty}} \right\rangle \text{ and } \beta \in K\left\langle \upsilon^{-1/p^{\infty}} \right\rangle$$

For the ease of notation we will write c_i (or even a_i) in place of $c_{(a,b)}$.

It is possible to put an order on the objects defined above. We give one such order below.

6.1.1 Order and Grading

Polynomials come equipped with standard grading, but here we are working with power series with degree of individual terms of the form $a/p^b \in \mathbb{Q}$ where $a, b \in \mathbb{Z}$ and p a prime. We have to fix a convention for expressing terms as summation, and we make sure that there are finitely many terms in each grading. First we grade $K[\upsilon, \upsilon^{1/p}, \upsilon^{1/p^2}, \ldots]$.

Consider antidiagonal in the first quadrant, it consists of terms (a, b) with $a, b \in \mathbb{N} \cup \{0\}$. The sum of the terms is fixed say $k \in \mathbb{N} \cup \{0\}$. For example, corresponding to $k = 3$ we have the following tuples $(0, 3), (1, 2), (2, 1), (3, 0)$ as (a, b), and every antidiagonal has a fixed number of terms in the first quadrant. We will use this as a model for grading. The term (a, b) will correspond to υ^{a/p^b}. The terms on the x axis of the form $(a, 0)$ give us the grading on the polynomial in υ, and as we go to higher and higher antidiagonal we keep recovering higher powers of $1/p$. The vertical line $x = 1$ gives us just the powers of υ in $1/p$. As the reader would have noticed, the notation follows the proof of countability of rationals, skipping any duplicate terms.

Our polynomials are finite sums of the form

$$\sum_{a+b=i} a_i \upsilon^{a/p^b}, \quad a, b, i \in \mathbb{N} \cup \{0\}, a_i \in K \tag{6.4}$$

(there is no relation between a_i and a) and can be clearly extended to power series by making the sum infinite, we denote power series by $K\langle\upsilon\rangle$. In case of power series we also add an extra condition that $|a_i| \to 0$ as $i \to \infty$. Laurent polynomials can be added by duplicating the above sum (we will still have finitely many terms in the antidiagonal and thus grading).

6.2 Order and continuous automorphism

We want to give a unit of $K\left\langle\upsilon^{1/p^\infty}\right\rangle$ as degree zero, and since a_i in the trail end of series $f \in K\left\langle\upsilon^{1/p^\infty}\right\rangle$ go to zero, we can always find a dominant term somewhere in the tail and rewrite as the series as

$$g + \upsilon^{n/p^b} \cdot \left\{\text{unit of } K\left\langle\upsilon^{1/p^\infty}\right\rangle\right\} \tag{6.5}$$

We want to think of elements of $K\left\langle\upsilon^{1/p^\infty}\right\rangle$ as polynomials with the series part lying in some unit, this will also give us finite number of zeros for elements of $K\left\langle\upsilon^{1/p^\infty}\right\rangle$.

We can formalize the above notion in terms of distinguished restricted power series.

$$K_n := K\left\langle\upsilon_1^{1/p^\infty}, \ldots, \upsilon_n^{1/p^\infty}\right\rangle \tag{6.6}$$

Definition 6.2.1. Let $f \in K_n$ be a restricted power series with $f = \sum_{\nu=0}^{\infty} g_\nu \upsilon_n^\nu \in K_{n-1}\left\langle\upsilon_n^{1/p^\infty}\right\rangle$ is called υ_n distinguished of order $s \in \mathbb{Q}$ if the following hold

1. g_s is a unit in K_{n-1}.

2. $|g_s| = |f|$ and $|g_s| > |g_\nu|$ for $\nu > s$.

Notice that our definition satisfies our requirement as in (6.5).

Here we will follow [Bosch, 2014, Lemma 7,p 16] and apply the result to our case with minimal changes. The only thing we have to notice that we have fractional powers in addition to integer powers. We will split the fractional power into an integer part and a fractional part (which would be less than one). When we apply the automorphism in the lemma below we deliberately avoid any changes to the fractional part.

Lemma 6.2.2. *If we are given finitely many* $f_1, \dots, f_r \in K_n \backslash \{0\}$*, there is a continuous automorphism*

$$\sigma : K_n \to K_n, \quad \upsilon_i^m \mapsto \begin{cases} (\upsilon_i)^m & \text{for } i < n, m \in \mathbb{Q} \backslash \mathbb{Z} \cap (0,1) \\ (\upsilon_i + \upsilon_n^{\alpha_i})^m & \text{for } i < n, m \in \mathbb{Z} \\ (\upsilon_n)^m & \text{for } i = n \end{cases} \tag{6.7}$$

with suitable exponents $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{N}$ *such that the elements* $\sigma(f_1), \dots, \sigma(f_r)$ *are* υ_n *distinguished. Furthermore,* $|\sigma(f)| = |f|$ *for all* $f \in K_n$.

6.3 Weierstraß Division

This theorem also appears in the thesis of [Das, 2016] following the proof of Tate Algebras from [Fresnel and van der Put, 2012], here we translate from [Bosch, 2014].

Theorem 6.3.1. *Let $g \in K_n$ be υ_n distinguished of some order s. Then, for any $f \in K_n$, there is a unique series $q \in K_n$ and a unique polynomial $r \in K_{n-1}[\upsilon_n^{1/p^\infty}]$ of degree $r < s$ satisfying*

$$f = qg + r \tag{6.8}$$

Furthermore, $|f| = \max(|q||g|, |r|)$.

Proof. WLOG we assume $|g| = 1$. We consider the equation $f = qg + r$ which immediately implies $|f| \leqslant \max(|q||g|, |r|)$. If $|f|$ is strictly smaller that the right hand side and we could assume that $\max(|q||g|, |r|) = 1$. Then we would have $\overline{qg} + \overline{r} = 0$ with $\overline{q} \neq 0 \neq \overline{r}$ and this contradicts the division algorithm in $k[\upsilon_1^{1/p^\infty}, \ldots, \upsilon_n^{1/p^\infty}]$. Hence, we must have $|f| = \max(|q||g|, |r|)$ (and uniqueness follows). Now we show existence. Let $g = \sum_{\nu=0}^{\infty} g_\nu \upsilon_n^\nu$ where $g_\nu \in K_n$ and g_s is a unit with $|g_\nu| < |g_s| = 1$ with $s < \nu$. Here we are indexing with natural numbers, but we can put these natural numbers in correspondence with rationals of the form a/p^b. Let $\max_{\nu>s} g_\nu = \epsilon < 1$. We will start with a weaker version.

(Weak Version) For any $f \in K_n$, there exists $q, f_1 \in K_n$ and a polynomial $r \in K_{n-1}[\upsilon_n^{1/p^\infty}]$ of degree less than s with

$$f = qg + r + f_1 \tag{6.9}$$

$$|q|, |r| \leqslant |f| \text{ and } |f_1| \leqslant \epsilon|f| \tag{6.10}$$

If we prove this weaker condition, we can prove our theorem. Proceed inductively starting with $f_0 = f$ and get

$$f_i = q_i g + r_i + f_{i+1}, i \in \mathbb{N}$$
$$|q_i|, |r_i| \leqslant \epsilon^i |f| \text{ and } |f_{i+1}| \leqslant \epsilon^{i+1}|f|$$

and thus we get the required result.

$$f = \left(\sum_{i=0}^{\infty} q_i\right) g + \left(\sum_{i=0}^{\infty} r_i\right) \tag{6.11}$$

We now prove the weak version. First we approximate $f \in K_n$ by a polynomial $f \in K_{n-1}[\upsilon^{1/p^\infty}]$. Set g' as a polynomials in υ_n^{1/p^∞} of distinguished order s and $g' = \sum_{i=0}^{s} g_i \left(\upsilon_n^{1/p^\infty}\right)^i$, where g' is a polynomial in υ_n^{1/p^∞} of distinguished order s with $|g'| = 1$. The Euclid's division algorithm in $K_{n-1}[\upsilon_n^{1/p^\infty}]$ yields

$$f = qg' + r, \quad g \in K_n \text{ and } r \in K_{n-1}[\upsilon_n^{1/p^\infty}] \quad \deg r < s \tag{6.12}$$

Since, $|f| = \max(|q|, |r|)$ we get

$$g = qg + r + f_1 \tag{6.13}$$

with $f_1 = qg' - qg$. As $|g - g'| = \epsilon$ and $|q| \leqslant |f|$, we get $|f_1| \leqslant \epsilon|f|$, giving us the required result.

□

6.4 Weierstraß Preparation Theorem

Theorem 6.4.1. *Consider $g \in K_n$ be υ_n distinguished of order s. Then there exists a unique monic polynomial $\phi \in K_{n-1}[\upsilon_n^{1/p^\infty}]$ of degree s such that $g = e\phi$ for $e \in K_n$ and e a unit. Additionally, $|\phi| = 1$ so that ϕ is υ_n distinguished of order s.*

Proof. By the Weierstraß division formula, we get an equation

$$\upsilon_n^s = qg + r \tag{6.14}$$

where $q \in K_n$ and there is a polynomial $r \in K_{n-1}[\upsilon_n^{1/p^\infty}]$ with $\deg r < s$ and $|r| \leqslant 1$. We can put $\phi = \upsilon_n^s - r$ to get $\phi = qg$ which satisfies $|\phi| = 1$ and is υ_n distinguished for order s. To show that g decomposes as qg, we need to show q is a unit of K_n. If we assume that $|g| = |q| = 1$, we can look at the reduced equation $\overline{\phi} = \overline{q} \cdot \overline{g}$.Since, both $\overline{\phi}$ and $\overline{g}$ are polynomials of degree s in υ_n, and since $\overline{\phi}$ is monic, it follows that $\overline{q} \in K^\times$. This implies that $q \in K_n$ is a unit.

To prove uniqueness, we start by defining $r = \upsilon_n^s - \phi$ and decomposing $g = e\phi$ to get

$$\upsilon_n^s = e^{-1}g + r \tag{6.15}$$

and the uniqueness of Weierstraß division shows us the uniqueness of e^{-1}

and r and, hence of e and ϕ.

□

Corollary 6.4.2. *The algebra* $K\langle \upsilon^{1/p^\infty}\rangle$ *of restricted power series in* υ_n *is a Bezout domain.*

Proof. We simply need to show that every finitely generated ideal is principally generated. We have defined degree for elements in $K\langle \upsilon^{1/p^\infty}\rangle$ at 7.2.2. Let ideal I be finitely generated and every element in I will have a degree, using Weierstraß division (Weierstraß division is used analogous to Euclidean Division in $k[X]$, using order instead of degree) we conclude that I is generated by element of lowest degree.

We need to note that some ideals in $K\langle \upsilon^{1/p^\infty}\rangle$ could be infinitely generated and hence it might not be possible to know the minimal degree in the ideal, thus leading to failure of PID condition. □

6.5 Maximum Principle

In this section we prove the maximum principle for perfectoid following the case of Tate Algebras as given in [Bosch, 2014, Proposition 5,p 15].

Proposition 6.5.1. *Let* $f \in K_n$*. Then* $|f(x)| \leqslant |f|$ *for all points* x *in the unit ball* $\mathbb{B}^n(\overline{K})$*, and there is a point in the unit ball such that the maximum is obtained, that is* $|f(x)| = |f|$.

Proof. The first claim follows from the definition of $|\ \ |$. For the second assertion assume that $|f| = 1$ and consider the projection map π given as

$\pi : K_n \to k[\upsilon_1^{1/p^\infty}, \ldots, \upsilon_n^{1/p^\infty}]$

$$\begin{array}{ccc} K_n & \xrightarrow{\pi} & k[\upsilon_1^{1/p^\infty}, \ldots, \upsilon_n^{1/p^\infty}] \\ \Big\downarrow{\scriptstyle \text{eval}} & & \Big\downarrow{\scriptstyle \text{eval}} \\ \overline{R} & \longrightarrow & \overline{k} \end{array}$$

Let $\pi(f) = \overline{f}$ be the non-trivial polynomial which will not be zero at some $\overline{x} \in \overline{k}^n$, where $\overline{k}$ is algebraic closure of k. Consider $\overline{R}$ as valuation ring of $\overline{K}$ and $\overline{k}$ as the residue field. We choose a lifting x of $\overline{x} \in \mathbb{B}^n(\overline{K})$, we get $\left|\overline{f}(x)\right| = 1 = |f|$.

□

6.6 Morphisms

In this section we make sense of morphisms of the form

(6.16) $$K_{n-1} \to K_n \to K_n/\langle g \rangle$$

(6.17) $$K_{n-1} \to K_n/\langle g \rangle \to K_n/\mathfrak{a}$$

Let $\mathfrak{a} \neq 0$ be an ideal in K_n, then we can choose an element $0 \neq g \in \mathfrak{a}$, such that g is υ_n-distinguished of order s (using lemma 6.2.2).

Given a $f \in K_n$ we have $f \cong r \mod g$ where $r \in K_{n-1}[\upsilon^{1/p^\infty}]$ and $\deg r < s$. This gives us the first morphism.

The second one is obtained in the same manner by considering the map $K_n/\langle g \rangle \to K_n/\langle \mathfrak{a} \rangle$ obtained by adding the elements to g which generate $\mathfrak{a}$. In the case of Tate Algebras we get Noether Normalization because of finiteness condition. Here the finiteness condition does not hold.

Chapter 7

Vector Bundles over Projectivoid Line

In this chapter we describe vector bundles over projectivoid line $\mathbb{P}_K^{1,\mathrm{ad},\mathrm{perf}}$ in Propostion 7.4.2. The description will be similar to vector bundles on $\mathbb{P}^1$ as described in [Hazewinkel and Martin, 1982, Proposition 2.3] . We reproduce the proposition here (word for word),

Proposition 7.0.1. Isomorphism classes of m dimensional algebraic vector bundles over $\mathbb{P}^1$ correspond bijectively to equivalence classes of polynomial $m \times m$ matrices $A(s, s^{-1})$ over $k[s, s^{-1}]$ such that $\det A(s, s^{-1}) = s^n, n \in \mathbb{Z}$ where equivalence relation is the following: $A(s, s^{-1}) \sim A'(s, s^{-1})$ iff there exist polynomial invertible $m \times m$ matrices $U(s), V(s^{-1})$ over $k[s]$ and $k[s^{-1}]$

respectively with constant determinant such that

$$A'(s, s^{-1}) = V(s^{-1})A(s, s^{-1})U(s). \tag{7.1}$$

7.1 Vector Bundles over $\mathbb{P}_K^{1,\mathrm{ad},\mathrm{perf}}$

The projectivoid line is covered by $\mathfrak{U} = U_1 \cup U_2$, with $\mathcal{O}(U_1) = K\left\langle \upsilon^{1/p^\infty} \right\rangle$ and $\mathcal{O}(U_2) = K\left\langle \upsilon^{-1/p^\infty} \right\rangle$. Where U_i is the perfectoid affine space $\mathbb{A}_K^{1,\mathrm{ad},\mathrm{perf}}$ and $\mathcal{O}(U_1 \cap U_2) = K\left\langle \upsilon^{\pm 1/p^\infty} \right\rangle$.

Let E be a m-bundle over $\mathbb{P}_K^{1,\mathrm{ad},\mathrm{perf}}$ defined over K.

There are two trivilizations of this bundle over the cover corresponding to U_1 and U_2. The trivialization is of the form $U_i \times \mathbb{A}_K^{m,\mathrm{ad},\mathrm{perf}} = \mathbb{A}_K^{1,\mathrm{ad},\mathrm{perf}} \times \mathbb{A}_K^{m,\mathrm{ad},\mathrm{perf}}$. Let $s \in \mathbb{A}_K^{1,\mathrm{ad},\mathrm{perf}}$ and $\nu \in \mathbb{A}_K^{m,\mathrm{ad},\mathrm{perf}}$. To construct the projectivoid we identify the perfectoid affine spaces via the map $s \mapsto 1/s, s \neq 0$. Now we can glue the two trivializations of the vector bundle to get a vector bundle over the projectivoid space.

$$U_1 \backslash \{0\} \times \mathbb{A}_K^{m,\mathrm{ad},\mathrm{perf}} \to U_2 \backslash \{0\} \times \mathbb{A}_K^{m,\mathrm{ad},\mathrm{perf}}$$
$$(s, \nu) \mapsto (s^{-1}, A(s, s^{-1})\nu)$$

where $A(s, s^{-1})$ is a matrix with coefficients in $\mathcal{O}(U_1 \cap U_2) = K\left\langle \upsilon^{\pm 1/p^\infty} \right\rangle$. For the correspondence to hold this matrix must have a determinant that is a unit in the ring. The determinant is a power series.

$$\det(A(s,s^{-1})) \in K\left\langle \upsilon^{\pm 1/p^{\infty}} \right\rangle \text{ and } \det(A(s,s^{-1})) \neq 0 \text{ for all } \quad \upsilon \tag{7.2}$$

As we see in the next section 7.2 the units of (7.2) are given as

$$\upsilon^{n/p^{b}} \cdot \nu, \quad n \in \mathbb{Z}, b \in \mathbb{Z}_{>0}, \tag{7.3}$$

where ν is degree zero term of $K\left\langle \upsilon^{\pm 1/p^{\infty}} \right\rangle$.

Notice that if we restrict to the case of $k[s,s^{-1}]$ we end up getting determinant as s^n, as in the proposition 7.0.1

7.2 Polynomials and Power Series

The units in the ring $K[X]$ are precisely $K^{\times}$, and for the laurent polynomials $K[X,X^{-1}]$ the units are $uX^n, u \in K^{\times}$.

In the case of power series $K[[X]]$ the units are formal power series with non zero constant term.

$$\sum_{n=0}^{\infty} a_n x^n \in K[[X]] \text{ is a unit iff } a_0 \neq 0. \tag{7.4}$$

In the case of formal Laurent series $K((X))$, we notice that X is a unit, since $X^{-1} \cdot X = 1$. The set of units is $K((X))\backslash 0$, the proof can be seen in [mo1,]. In [Schwaiger, 1985] we find the complete description of roots of power series.

For a series f in Tate Algebra T_n, the series is a unit iff the constant coefficient of f is bigger than all other coefficients of f [Bosch, 2014, Corollary 4,p 14] . For $T_n := K\langle \upsilon_1, \ldots, \upsilon_n \rangle$ the units are

$$T_n{}^{\times} = \left\{ \sum_{i \in \mathbb{N}^n} a_i \upsilon^i \in T_n \ : \ |a_0| > |a_i| \text{ for all } i \neq 0 \right\} \tag{7.5}$$

Equipping $f \in T_n$ with a Gauss norm, $|f| = 1$ is a unit iff the reduction of f denoted as $\widetilde{f}$ lies in $K^{\times}$ as described on [Bosch, 2014, pp 13-14].

7.2.1 Units of $K\left\langle \upsilon^{1/p^{\infty}} \right\rangle$

We now formally write down the units of $K\left\langle \upsilon^{1/p^{\infty}} \right\rangle$ which will be used for the description of vector bundles.

Proposition 7.2.1. $K\left\langle \upsilon^{1/p^{\infty}} \right\rangle$ *is complete with respect to Gauss Norm.*

Proof. This proof is an adaptation of a similar proposition for Tate Algebras as given in [Bosch, 2014, Proposition 3, p14]. We start with a Cauchy sequence $\sum_i f_i$ and end up showing that it lies in $K\left\langle \upsilon^{1/p^{\infty}} \right\rangle$. We will use ν as an index for (a, b), this will help as streamline the proof to make it closer to the proof of units of Tate Algebra.

(7.6) $$\lim_{i\to\infty} f_i = 0 \text{ where } f_i = \sum_{\nu} c_{i\nu}\upsilon^{\nu} \in K\left\langle \upsilon^{1/p^{\infty}} \right\rangle$$

First note that

(7.7) $$|c_{i\nu}| \leqslant |f_i| \text{ thus } \lim_{i\to\infty} |c_{i\nu}| = 0 \text{ for all } \nu.$$

Thus, the limit $c_\nu = \sum_{i=0}^{\infty} c_{i\nu}$ exists (note that we are using Gauß norm). To finish the proof we need to show that the series $f = \sum_\nu c_\nu \upsilon^\nu$ is strictly convergent and $f = \sum_i f_i$.

In the section 6.1.1 we put an order on $K\langle \upsilon^{1/p^{\infty}} \rangle$. In order to make our argument simpler, we jump a finite number of terms (we noticed in our ordering that there are only finite number of terms for every grading) in the order given in 6.1.1 and consider terms of the form $(a, 0)$ lying on the x axis. This helps us in thinking directly in terms of natural numbers $\mathbb{N}$.

For any given $\epsilon > 0$ there is an integer N such that $|c_{i\nu}| < \epsilon$ for $i \geqslant N$ and all ν. Since coefficients of the series $f_0, \ldots, f_{N-1}$ form a zero sequence, and almost all the coefficients of these sequences would have an absolute value less than ϵ. Thus, the elements $|c_{i\nu}|$ form a zero sequence in K. Since the non Archimedean triangle inequality generalizes for a convergent series to an inequality below

(7.8) $$\left| \sum_{i=0}^{\infty} \alpha_\nu \right| \leqslant \max_{i=0,\ldots,\infty} |\alpha_\nu|$$

we get that power series $f = \sum_i f_i$ and $f \in K\langle \upsilon^{1/p^\infty} \rangle$. □

Corollary 7.2.2. *A series* $f \in K\langle \upsilon^{1/p^\infty} \rangle$ *with* $|f| = 1$ *is a unit iff its reduction* $\widetilde{f} \in k^\times$.

Proof. Without loss of generality we can consider only elements with $f \in K\langle \upsilon^{1/p^\infty} \rangle$ with Gauß norm 1. If f is a unit in $K\langle \upsilon^{1/p^\infty} \rangle$it is also a unit in $R\langle \upsilon^{1/p^\infty} \rangle$, where

$$R = \{a \in K \mid |a| \leqslant 1\} \tag{7.9}$$

$$m = \{a \in K \mid |a| < 1\} \tag{7.10}$$

$$k = R/m \tag{7.11}$$

$$R\left\langle \upsilon^{1/p^\infty} \right\rangle \to k[\upsilon^{1/p^\infty}] \tag{7.12}$$

$$f \mapsto \widetilde{f} \tag{7.13}$$

Thus, $\widetilde{f}$ is a unit in $k[\upsilon^{1/p^\infty}]$ and hence in $f \in k^\times$.

Conversely, if $\widetilde{f} \in k^\times$, the constant term $f(0)$ satisfies $|f(0)| = 1$ (since $\widetilde{f} = 0$ iff $|f| < 1$). But then we can put $f = 1 - g$ with $|g| < 1$, giving us an inverse of f as a series $\sum_{i=0}^{\infty} g^i$.

□

In the above corollary we showed f is of the type $f = 1 - g$ with $|g| < 1$. Thus, we can restate the above corollary as

Corollary 7.2.3. *An arbitrary series* $f \in K\langle \upsilon^{1/p^\infty} \rangle$ *is a unit iff* $|f - f(0)| <$

$|f(0)|$. *In other words the absolute value of other coefficients of* f *are less than the absolute value of the constant coefficient.*

$$(7.14)\quad K\left\langle \upsilon^{1/p^\infty}\right\rangle^\times = \left\{ \sum_{i\in\mathbb{N}^n} a_i\upsilon^i \in K\left\langle \upsilon^{1/p^\infty}\right\rangle \ : \ |a_0| > |a_i| \textit{ for all } i \neq 0 \right\}$$

We can carry the exact same procedure as above for $K\left\langle \upsilon^{-1/p^\infty}\right\rangle$ to get an identical result as stated in 7.14.

7.2.2 Units of $K\left\langle \upsilon^{\pm 1/p^\infty}\right\rangle$

We can consider algebra of the form $K\left\langle X^{1/p^\infty}, Y^{1/p^\infty}\right\rangle$. An element $f \in K\left\langle X^{1/p^\infty}, Y^{1/p^\infty}\right\rangle$ is a series in which each individual term has a degree $= \deg X + \deg Y$, where X and Y occur in the term. Thus, we can put an order on these terms as given in section 6.1.1. If we have terms which have only X or only Y appearing in them, we can still arrange them by degree. In case the degree of X and Y term is same, we put an order by first writing the X term and then the Y terms of the same degree. The order simply comes from observing that rational numbers are countable.

Using the results (and procedure) from the previous section we get the units of $K\left\langle X^{1/p^\infty}, Y^{1/p^\infty}\right\rangle$ given below where ξ represents product of X and Y.

$$(7.15)\qquad \left\{ \sum_{i\in\mathbb{N}^n} a_i\xi^i \in K\left\langle X^{1/p^\infty}, Y^{1/p^\infty}\right\rangle \ : \ |a_0| > |a_i| \text{ for all } i \neq 0 \right\}$$

Setting $X = \upsilon$ and $Y = 1/\upsilon$ we know that elements of the form below

$$\upsilon^{n/p^b} \left\{ \sum_{i \in \mathbb{N}^n} a_i \upsilon^i \in K \left\langle \upsilon^{\pm 1/p^\infty} \right\rangle \ : \ |a_0| > |a_i| \text{ for all } i \neq 0 \right\} \tag{7.16}$$

are units in $K \left\langle \upsilon^{\pm 1/p^\infty} \right\rangle$. The units we are most interested in are of the form $\upsilon^{n/p^b} \cdot u$ where u is a degree zero term of $K \left\langle \upsilon^{\pm 1/p^\infty} \right\rangle$. We take the degree of the above element as n/p^b.

It might seem that there are other units of $K \left\langle \upsilon^{\pm 1/p^\infty} \right\rangle$ that might not have a clearly defined notion of degree.

For other units of $K \left\langle \upsilon^{\pm 1/p^\infty} \right\rangle$ we notice that the tail ends of series on both positive and negative side tend to zero. Thus, there are only finitely many terms that could be dominant. We can still define the degree to be maximum degree of all dominant terms (which are finite in number). In case we just have a polynomial with all the coefficients equal, then we have the degree is the power of the highest term, which is same as degree of polynomial in the classical case.

Hence we have a well defined notion of degree for units which might not be of the form (7.16). We can take the degree term out and write the unit as $\upsilon^{n/p^b} \cdot u'$, where the degree of u' is zero.

7.3 Isomorphism Classes of Vector Bundles over perfectoid affine $\mathbb{A}_K^{1,ad,perf}$

We now want to talk about vector bundle automorphism over the space $\mathbb{A}_K^{1,ad,perf} \times \mathbb{A}_K^{m,ad,perf}$.

$$U_1 \times \mathbb{A}_K^{m,ad,perf} \to U_1 \times \mathbb{A}_K^{m,ad,perf}$$

$$(s, v) \mapsto (s, U(s)v)$$

where $U(s)$ is a matrix with coefficients in $K\left\langle \upsilon^{1/p^\infty} \right\rangle$ and $\det(U(s)) \neq 0$. From the section 7.2 we get the units as

$$\left\{ \text{Elements of } K\left\langle \upsilon^{1/p^\infty} \right\rangle \text{ such that } |a_0| > |a_i| \text{ for all } i \neq 0 \right\} \tag{7.17}$$

Notice that there is no gluing condition just on the piece U_1, therefore can have $\upsilon = 0$. If we restrict this to $k[s]$ we just get $k\backslash\{0\}$ as in the proposition 7.0.1.

Similarly we have a correspondence on U_2

$$U_2 \times \mathbb{A}_K^{m,ad,perf} \to U_2 \times \mathbb{A}_K^{m,ad,perf}$$

$$(t, \nu) \mapsto (t, V(t)\nu)$$

where $V(t)$ is a matrix with coefficients in $K\left\langle \upsilon^{1/p^\infty} \right\rangle$ and $\det(V(t)) \neq 0$. Notice that to obtain the projectivoid space we will have $t = 1/s$. Thus, we write $V(s^{-1})$ in place of $V(t)$.

From the section 7.2 we get the units as

$$\left\{ \text{Elements of } K\left\langle \upsilon^{-1/p^\infty} \right\rangle \text{ such that } |a_0| > |a_i| \text{ for all } i \neq 0 \right\} \tag{7.18}$$

Notice that there is no gluing condition just on the piece U_2, therefore can have $\upsilon = 0$. If we restrict this to $k[t]$ we just get $k \backslash \{0\}$ as in the proposition 7.0.1.

We want an equivalence relation for transition matrix between two covers. This can be obtained modulo the automorphisms $U(s), V(s^{-1})$ and is given as (7.19).

$$U_1 \times \mathbb{A}_K^{m,ad,perf} \xrightarrow{U(s)} U_1 \times \mathbb{A}_K^{m,ad,perf} \xrightarrow{A(s,s^{-1})} U_2 \times \mathbb{A}_K^{m,ad,perf} \xrightarrow{V(s^{-1})} U_2 \times \mathbb{A}_K^{m,ad,perf}$$

$$U_1 \times \mathbb{A}_K^{m,ad,perf} \xrightarrow{A'(s,s^{-1})} U_2 \times \mathbb{A}_K^{m,ad,perf}$$

$$A'(s, s^{-1}) = V(s^{-1})A(s, s^{-1})U(s) \tag{7.19}$$

7.4 Degree of Vector Bundles

In this section we define the notion of degree of the vector bundles on the projectivoid line, which is motivated by proposition 7.0.1. In this proposition we have

$$\deg \det U(s) = 0 = \deg \det V(s^{-1}) \text{ which implies} \tag{7.20}$$

$$\deg \det A'(s, s^{-1}) = \deg \det A(s, s^{-1}) \tag{7.21}$$

Thus, isomorphic vector bundles on $\mathbb{P}^1$ have the same degree of the determinant

Definition 7.4.1. Degree of the vector bundle is the degree of zeroth term of the determinant of the vector bundle.

The consequence of the above definition is that (7.5) will have degree zero. Thus, (7.17) and (7.18) will also have degree zero. Furthermore the degree of (7.3) is n/p^∞. From, (7.19) and the observations just made

$$\deg \det A'(s, s^{-1}) = \deg \det A(s, s^{-1}) \tag{7.22}$$

Thus, isomorphic vector bundles on $\mathbb{P}_K^{1,\mathrm{ad,perf}}$ have the same degree of the determinant.

We have proved the following proposition

Proposition 7.4.2. *Isomorphism classes of* m *dimensional analytic vector* *bundles over* $\mathbb{P}_K^{1,\mathrm{ad},\mathrm{perf}}$ *correspond bijectively to equivalence classes of* $m \times m$ *matrices* $A(s, s^{-1})$ *over* $K\left\langle \upsilon^{\pm 1/p^\infty} \right\rangle$ *such that*

(7.23) $\det A(s, s^{-1}) = \upsilon^{n/p^b} \cdot \{$ *Elements of* $K\left\langle \upsilon^{\pm 1/p^\infty} \right\rangle$ *such that* $|a_0| > |a_i|$ *for all* $i \neq 0\}$

where equivalence relation is the following: $A(s, s^{-1}) \sim A'(s, s^{-1})$ *iff there exist invertible* $m \times m$ *matrices* $U(s), V(s^{-1})$ *over* $K\left\langle \upsilon^{1/p^\infty} \right\rangle$ *and* $K\left\langle \upsilon^{-1/p^\infty} \right\rangle$ *respectively with determinants of* $U(s)$ *and* $V(s)$ *given as*

(7.24)

$$\det U(s) = \left\{ \text{ Elements of } K\left\langle \upsilon^{1/p^\infty} \right\rangle \text{ such that } |a_0| > |a_i| \text{ for all } i \neq 0 \right\}$$

(7.25)

$$\det V(s^{-1}) = \left\{ \text{ Elements of } K\left\langle \upsilon^{-1/p^\infty} \right\rangle \text{ such that } |a_0| > |a_i| \text{ for all } i \neq 0 \right\}$$

such that

(7.26) $$A'(s, s^{-1}) = V(s^{-1}) A(s, s^{-1}) U(s).$$

and

(7.27) $$\deg \det A'(s, s^{-1}) = \deg \det A(s, s^{-1})$$

7.5 Classification of Vector Bundles on $\mathbb{P}_K^{1,\text{ad},\text{perf}}$

The classification of vector bundles over $\mathbb{P}^1$ depends upon the fact that there are only finitely many ways to partition an integer. But, this is no longer true for fractions. For example, consider the following non equivalent (and infinitely many) vector bundles with degree one.

$$\begin{pmatrix} X^a & 0 \\ 0 & X^b \end{pmatrix} \text{ such that } a+b=1 \text{ and } a,b \in [0,1] \cap \mathbb{Z}[1/p] \tag{7.28}$$

A more subtle question is whether every vector bundle on $\mathbb{P}_K^{1,\text{ad},\text{perf}}$ splits as a sum of line bundles. This question was answered in the negative by Prof Kiran Kedlaya and communicated to me via email. The counterexample is mentioned in full detail in Lecture 3 of [Kedlaya, 2017, pp.80-81].

Chapter 8

Line Bundles on $\mathbb{P}_K^{n,ad,perf}$

8.1 Proj of Graded Ring

Let $S = \oplus_d S_d$ be the graded homogeneous co-ordinate ring. Corresponding to the graded ring S we have a sheaf of rings $\mathcal{O}_{\mathrm{Proj}}(S)$ which will give the scheme $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$, as given in ([Stacks Project Authors, 2016, Tag 01M3] or [Hartshorne, 1977, p 117]).

Let $M = \oplus_{\mathbb{Z}} M_n$ be a graded S module, that is $S_n M_m \subset M_{n+m}$ then there is a sheaf $\widetilde{M}$ on the basis of standard open sets.

$$\widetilde{M}(U) = M_{(f)} = \{mf^{-d} \in M_f \text{ such that } m \in M_{d \cdot \deg f}\} \tag{8.1}$$

We want to construct a graded ring for fractional power series.

8.1.1 Grading

Analogous to the case of standard grading of homogeneous polynomials, we can construct a fractional grading with $d \in \mathbb{Z}[1/p]$ arranged in the canonical order.

$$S_d := \text{ homogneous polynomials of degree } d \tag{8.2}$$

8.2 Defining $\mathcal{O}(m)$

Let $B(n)$ denote the graded S-module defined by $B(n)_d = S_{n+d}$. This will be called twist of B. For a given $X = \text{Proj}(S)$, let $\mathcal{O}_X(n)$ denote the $\mathcal{O}_X$ module $\widetilde{B(n)}$. Let $f \in S$ be homogeneous of degree one (with affine open set $D_+(f)$), then we get

$$B(n)_{(f)} = f^n B_{(f)} \tag{8.3}$$

$$\mathcal{O}_X(n)|_{D_+(f)} = f^n \mathcal{O}|_{D_+(f)} \tag{8.4}$$

Notice that $n \in \mathbb{Z}$, but we could choose $n \in \mathbb{Z}[1/p]$ which would give us rational degrees.

Informally, we can define $\mathcal{O}(m)$ with $m \in \mathbb{Z}[1/p]$ as

$$\mathcal{O}(m) = \left\{ \frac{f}{g} \text{ where } \ f, g \in S \text{ and } \deg f - \deg g = m \right\} \tag{8.5}$$

Remark

1. Note that since, f and g are homogeneous, degree is well defined for the series (it is the same for all elements).

2. $\Gamma(\mathbb{P}_K^{1,\text{ad},\text{perf}}\mathcal{O}(1))$ is generated by $X, Y, \ldots, X^{q_i}Y^{q_j}, \ldots$ where $q_i + q_j = 1$. The basis reduces to X, Y if we take exponent of p in X^a/p^b as 0. Thus, $\dim \Gamma(\mathbb{P}_K^{1,\text{ad},\text{perf}}\mathcal{O}(1)) = \infty$ and reduces to 2 when we take exponent of p to be 0.

8.2.1 Twisting the sheaf $\mathcal{O}(m)$

We have an isomorphism between graded modules, which is given as

$$\oplus_d S_d \xrightarrow{\cdot S_n} \oplus_d S_d, \qquad n \in \mathbb{Z}[1/p] \tag{8.6}$$

$$S_d \mapsto S_{d+n} \tag{8.7}$$

We get the twist by tensoring with $\mathcal{O}(m)$ where $m \in \mathbb{Z}[1/p]$.

8.2.2 Injection into Picard Group

The tensor products of sheaves $\mathcal{O}(m) \otimes \mathcal{O}(n) = \mathcal{O}(m+n)$ gives us an injection into Picard group as $\mathbb{Z}[1/p]$. This can also be directly seen from the transition function of the vector bundles as shown in previous chapter. We can naturally extend the theory of ample vector bundles to the sheaf just described.

8.3 Weil Divisors

We have the notion of order in 6.2.1 (adapted from [Bosch, 2014, p 15]) and we want to utilize this for constructing Weil Divisors in $\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}$. Consider a rational function f on $\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}$, we can get the order for each piece

$$(f) = \sum \operatorname{ord}(f)[\mathbb{A}_K^{n,\mathrm{ad},\mathrm{perf}}] = \sum_i n_i[\mathbb{A}_K^{n,\mathrm{ad},\mathrm{perf}}]_i, n_i \in \mathbb{Z}[1/p] \tag{8.8}$$

Thus we have an abelian group just like in Weil divisors. This is a finite sum, since there are only finite number of Affine pieces to consider. We have an obvious homomorphisms from Weil divisors to $\mathbb{Z}[1/p]$ given as

$$\sum_i n_i[\mathbb{A}_K^{n,\mathrm{ad},\mathrm{perf}}]_i \to \sum_i n_i. \tag{8.9}$$

The rational functions of $\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}$ have degree zero, and thus these are precisely the functions that will get mapped to the kernel of the above morphism. The degree of the numerator will get mapped to positive order and degree in denominator will get mapped to negative order, and these will cancel each other out in the summation. In particular, the sheaf $\mathcal{O}(d)$ gets mapped to $d \in \mathbb{Z}[1/p]$ via (8.8).

We do not have a Krull Dimension for $K\left\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\right\rangle$, but we have this notion for the Tate Algebra. We want codimension one so that Weil Divisors can be defined for $K\left\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\right\rangle$.

Definition 8.3.1. The Krull Dimension for $K\left\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\right\rangle$ is set equal to the Krull Dimension of corresponding Tate Algebra $K\langle X_1, \ldots, X_n\rangle$. This definition is also carried over to any algebras obtained by modding out the ideals.

$$(8.10) \qquad (f) = \sum_Y \mathrm{ord}_Y(f)Y = \sum_Y n_Y Y, \qquad n_i \in \mathbb{Z}[1/p]$$

The above definition makes sense only if $\mathrm{ord}_Y(f) = 0$ except for finitely many values. Since, we consider irreducible elements in Tate Algebras, that the above definition makes sense. First we consider Y as $K\left\langle X_1^{1/p^\infty}, \ldots, X_{n-1}^{1/p^\infty}\right\rangle$ and then replace it with $K\langle X_1, \ldots, X_{n-1}\rangle$.

8.4 Cartier Divisors

We will represent cartier divisors as a system $D := \{(U_i, f_i)_i\}$ where U_i are the open sets forming a cover of the space, and f_i is the quotient of two regular elements of $\mathcal{O}_{\mathbb{P}_K^{n,ad,perf}}(U_i)$. On the intersection $U_i \cap U_j$ we have

$$(8.11) \qquad f_i|_{U_i \cap U_j} \in f_j|_{U_i \cap U_j} \mathcal{O}_X(U_i \cap U_j)^\times \text{ for every } i, j.$$

Two systems represent the same divisor if they differ by a multiplicative factor in $\mathcal{O}_X(U_i \cap U_j)^\times$. In other words, $f_i/g_j \in \mathcal{O}_X(U_i \cap U_j)^\times$ and the two equivalent divisors are $\{(U_i, f_i)_i\}$ and $\{(V_j, g_j)_j\}$. The sum of two divisors is given as

(8.12) $$D_1 + D_2 = \{(U_i \cap V_j)_{i,j}\}$$

The divisor is effective iff it can be represented by $f_i \in \mathcal{O}_X(U_i)$.

Alternatively we can represent Cartier Divisors as the global section of the sheaf $\mathcal{K}_X^\times / \mathcal{O}_X^\times$, arising from the short exact sequence of sheaves

(8.13) $$0 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \mathcal{K}_X^\times / \mathcal{O}_X^\times \to 0$$

where $\mathcal{O}_X^\times$ is subsheaf of groups of units of $\mathcal{O}_X$ and $\mathcal{K}_X^\times$ is subsheaf of groups of units of $\mathcal{K}_X$. The sheaf $\mathcal{K}_X$ comes from the presheaf $\mathrm{Frac}(\Gamma(U, \mathcal{O}_X))$.

The following result from [Liu, 2002, p 257] page 257 will hold here too.

Proposition 8.4.1. *1. The map* $\rho : D \mapsto \mathcal{O}(D)$ *is additive, that is*

(8.14) $$\rho(D_1 + D_2) = \mathcal{O}_X(D_1)\mathcal{O}_X(D_2) \simeq \mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2)$$

2. *We have an injective homomorphism from the isomorphism classes of cartier divisors (denoted by* $\mathrm{CaCl}(X)$*) to* $\mathrm{Pic}(X)$.

3. *The image of* ρ *corresponds to invertible sheaves contained in the field of fractions associated to the scheme.*

In particular the above proposition shows that there is an isomorphism

$$\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \simeq \text{Invertible sheaves of } \mathcal{K}_X \tag{8.15}$$

Furthermore we have from [Hartshorne, 1977, Remark 6.12.1, p 143] we have

$$\text{Pic}(X) \simeq H^1(X, \mathcal{O}_X^\times) \tag{8.16}$$

8.5 Equivalence of Cartier and Weil Divisors

We want to use [Hartshorne, 1977, Proposition 6.11, p 141]. But, the ring $K\left\langle X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}\right\rangle$ is not Notherian, thus, we cannot invoke the proposition. All is not lost, the space we are considering is $\mathbb{P}_K^{n,\text{ad},\text{perf}}$. It comes equipped with a finite standard affine cover which replaces Noetherian condition.

We can send a Weil divisor to a Cartier divisor by considering the cover by affine spaces, defined by $X_i = 0$.

$$\sum_i n_i[\mathbb{A}_K^{n,\text{ad},\text{perf}}]_i \mapsto \{\prod_i X_i^{n_i}, [\mathbb{A}_K^{n,\text{ad},\text{perf}}]_i\} \tag{8.17}$$

In the opposite direction , that is from Cartier to Weil Divisors we have the map

$$\{f_i, [\mathbb{A}_K^{n,\mathrm{ad},\mathrm{perf}}]_i\} \mapsto \sum_i \mathrm{ord}(f_i)[\mathbb{A}_K^{n,\mathrm{ad},\mathrm{perf}}]_i \tag{8.18}$$

where $\mathrm{ord}(f)$ is defined at the start of the section. The correspondence between cartier and weil divisors gives the isomorphism $\mathrm{Cl}(\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}) \cong \mathbb{Z}[1/p]$ with the map given in(8.9).

Chapter 9

Cohomology of Line Bundles on $\mathbb{P}_K^{n,\text{ad},\text{perf}}$

In this chapter we compute Čech cohomology of line bundles which agrees with the derived functor cohomology as given in [Vakil, 2017, p 631].

9.1 Cohomology

In the previous chapter we defined the notion on degree of of an element in $K\langle X^{\pm 1/\infty}\rangle$. Here, we use this notion to concretely define line bundles on the projective perfectoid.

Recall that global sections $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ are generated by homogeneous polynomials of degree m in $n+1$ variables. For example, $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$ is generated by x^2, xy, y^2. We can similarly, define the global sections on

$\mathbb{P}_K^{1,\mathrm{ad},\mathrm{perf}}$, of degree $d = 2$ being generated by $x^{a_1/p^{n_1}} \cdot y^{a_2/p^{n_2}}$ such that

$$\frac{a_1}{p^{n_1}} + \frac{a_2}{p^{n_2}} = 2 \text{ and } a_i \in \mathbb{Z}_{>0}, n_i \in \mathbb{Z}_{\geqslant 0} \tag{9.1}$$

Thus, we see that the dimension of $H^0(\mathbb{P}_K^{1,\mathrm{ad},\mathrm{perf}}, \mathcal{O}_{\mathbb{P}_K^{1,\mathrm{ad},\mathrm{perf}}}(2))$ is infinite, and $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \subset H^0(\mathbb{P}_K^{1,\mathrm{ad},\mathrm{perf}}, \mathcal{O}_{\mathbb{P}_K^{1,\mathrm{ad},\mathrm{perf}}}(2))$. For $n_i = 0$ we recover the global sections $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$.

$$\dim H^0(\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}, \mathcal{O}_{\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}}(m)) = \infty \tag{9.2}$$

The first theorem which just drops out of the Cech complex using the arguments in [Hartshorne, 1977, p 225] is the following

Theorem 9.1.1. *1. $H^0(\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}, \mathcal{O}_{\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}}(m))$ is a free module of infinite rank.*

2. $H^n(\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}, \mathcal{O}_{\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}}(-m))$ for $m > n$ is a free module of infinite rank.

We take the standard cover of $\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}$ by affine sets $\mathfrak{U} = \{U_i\}_i$ where each $U_i = D(x_i), i = 0, \dots, n$.

We get $H^0(\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}, \mathcal{F})$ as the kernel of the following map

$$\prod S_{x_{i_0}} \to \prod S_{x_{i_0} x_{i_1}} \tag{9.3}$$

The element mapping to the Kernel has to lie in all the intersections $S = \cap_i S_{x_i}$, as given on [Hartshorne, 1977, pp 118].

$H^n(\mathbb{P}_K^{n,\mathrm{ad,perf}}, \mathcal{F})$ is the cokernel of the map

$$d^{n-1} : \prod_k S_{x_0 \cdots \hat{x}_k \cdots x_n} \to S_{x_0 \cdots x_n} \tag{9.4}$$

$S_{x_0 \cdots x_n}$ is a free A module with basis $x_0^{l_0} \cdots x_n^{l_n}$ with each $l_i \in \mathbb{Z}[1/p]$. The image of d^{n-1} is the free submodule generated by those basis elements with atleast one $l_i \geqslant 0$. Thus H^n is the free module with basis as negative monomials

$$\{x_0^{l_0} \cdots x_n^{l_n}\} \text{ such that } l_i < 0 \tag{9.5}$$

The grading is given by $\sum l_i$ and there are infinitely many monomials with degree $-n - \epsilon$ where ϵ is something very small and $\epsilon \in \mathbb{Z}[1/p]$. Recall, that in the standard coherent cohomology there is only one such monomial $x_0^{-1} \cdots x_n^{-1}$. For example, in case of $\mathbb{P}^2$ we have $x_0^{-1} x_1^{-1} x_2^{-1}$ but in $\mathbb{P}_K^{2,\mathrm{ad,perf}}$ in addition to above we also have $x_0^{-1/2} x_1^{-1/2} x_2^{-2}$.

Recall that in coherent cohomology of $\mathbb{P}^n$ the dual basis of $x_0^{m_0} \cdots x_n^{m_n}$ is given by $x_0^{-m_0-1} \cdots x_n^{-m_n-1}$ and the operation of multiplication gives pairing. We do not have this pairing for $\mathbb{P}_K^{n,\mathrm{ad,perf}}$, but we can pair $x_0^{m_0}$ with $x_0^{-m_0}$.

Theorem 9.1.2. $H^i(\mathbb{P}_K^{n,\mathrm{ad,perf}}, \mathcal{O}_{\mathbb{P}_K^{n,\mathrm{ad,perf}}}) = 0$ *if* $0 < i < n$

We will use the proof from [Vakil, 2017, pp 474-475], using the convention that H^0 denotes global sections. We will work with $\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}, n = 2$, the case for general n is identical. The Čech complex is given in 9.1.

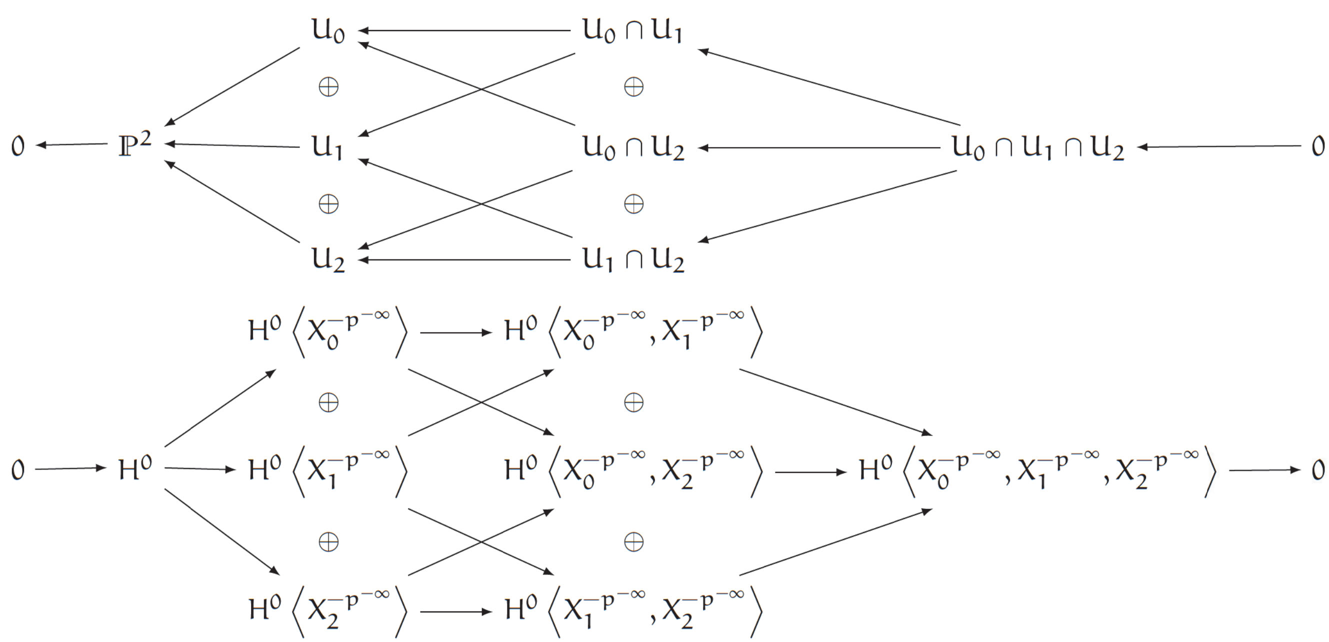


Figure 9.1: Čech Complex for $\mathbb{P}_K^{n,\text{ad,perf}}, n = 2$

3 negative exponents The monomial $X_0^{a_0} \cdot X_1^{a_1} \cdot X_2^{a_2}$ where $a_i < 0$. We cannot lift it to any of the coboundaries (that is lift only to 0 coefficients). If K_{012} denotes the coefficient of the monomial in the complex (Figure 9.2), we get zero cohomology except for the spot corresponding to $U_0 \cap U_1 \cap U_2$.

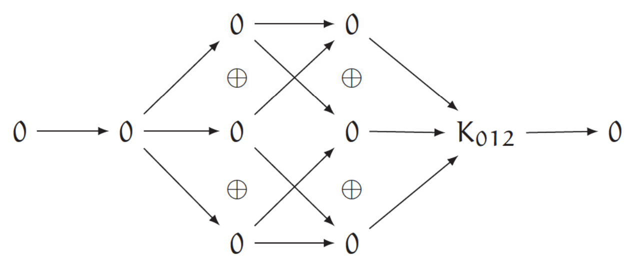


Figure 9.2: 3 negative exponents

2 negative exponents The monomial $X_0^{a_0} \cdot X_1^{a_1} \cdot X_2^{a_2}$ where two exponents are negative, say $a_0, a_1 < 0$. Then we can perfectly lift to coboundary coming from $U_0 \cap U_1$, which gives exactness.

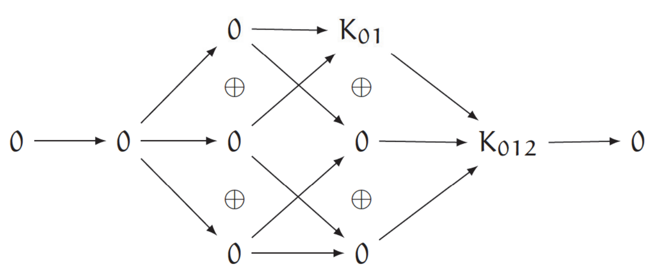


Figure 9.3: 2 negative exponents

1 negative exponent The monomial $X_0^{a_0} \cdot X_1^{a_1} \cdot X_2^{a_2}$ where one exponents is negative, say $a_0 < 0$, we get the complex (Figure 9.4). Notice that K_0 maps injectively giving zero cohomology group.

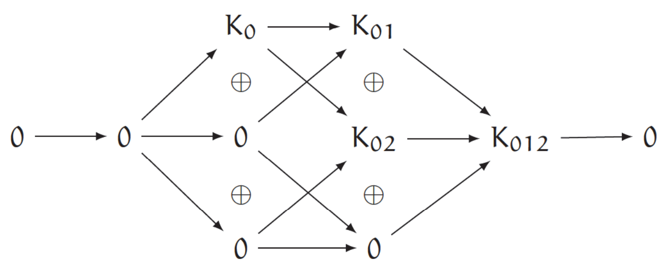


Figure 9.4: 1 negative exponent

Furthermore, the mapping in the Figure 9.5 gives Kernel when $f = g$ which is possible for zero only. Again giving us zero cohomology groups.

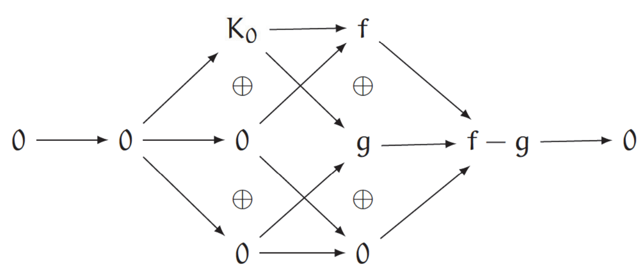


Figure 9.5: Mapping for 1 negative exponent

0 **negative exponent** The monomial $X_0^{a_0} \cdot X_1^{a_1} \cdot X_2^{a_2}$ where none of the exponents is negative $a_i > 0$, gives the complex Figure 9.6.

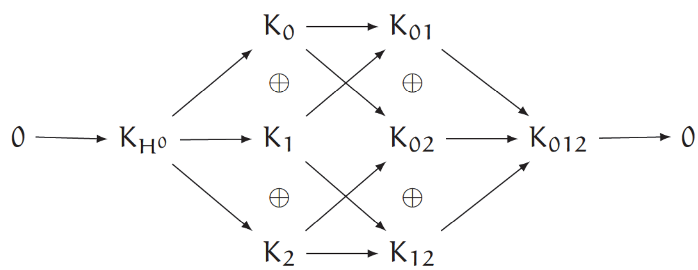


Figure 9.6: 0 negative exponent

Consider the SES of complex as in Figure 9.7 . The top and bottom row come from the 1 negative exponent case, thus giving zero cohomology. The SES of complex gives LES of cohomology groups, since top and bottom row have zero cohomology, so does the middle.

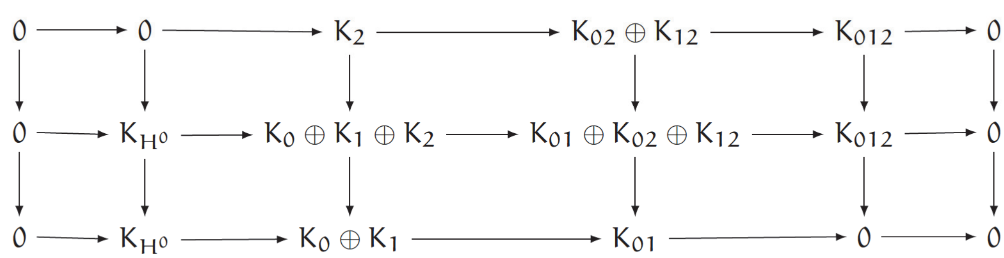


Figure 9.7: SES of Complex

Chapter 10

Module of Differentials

We want to define differentials for $K\left\langle X^{1/p^\infty}\right\rangle$ and the Laurent case $K\left\langle X^{\pm 1/p^\infty}\right\rangle$. The guiding case will be that of polynomials as given on page 172 of [Hartshorne, 1977] or page 559 of [Vakil, 2015].

Notice that for $K\left\langle X^{1/p^\infty}\right\rangle$ we cannot define differentials in the same way since,

$$dX = d(\sqrt{X})^2 = 2\sqrt{X}d\sqrt{X}. \tag{10.1}$$

Thus, we will have to modulo out such relations. In other words the basis is dX modulo $X^a = (X^{a/p^b})^{p^b}$.

We do not face any such problems for the laurent case since

$$dX = d(\sqrt{X})^2 = 2\sqrt{X}d\sqrt{X} = 2\sqrt{X}\frac{1}{2\sqrt{X}}dX. \tag{10.2}$$

But, there are convergence issues. After taking the derivative of an element

$f \in K\left\langle X^{\pm 1/p^{\infty}}\right\rangle$ we might have the problem that df might not be convergent. We want to wish this problem away.

Let us recall that a similar problem is faced in calculus with respect to continuous and differentiable functions, which is resolved by restricting to the case of C^{∞} functions, and we will adopt the same point of view here. Instead of considering the entire ring $K\left\langle X^{\pm 1/p^{\infty}}\right\rangle$ we will consider a subring whose elements are convergent after infinite differentiation. We will call this new algebra of convergent series after infinite differentiation as *desi algebra*.

Definition 10.0.1. We define the differential forms over the intersection of the open sets $\cap_i U_i$ (that are glued together to make $\mathbb{P}_K^{n,\mathrm{ad,perf}}$) to be generated by $\oplus_{i=0}^{n} f_i dX_i$ where $f_i \in K\left\langle X_0^{\pm 1/p^{\infty}}, \ldots, X_n^{\pm 1/p^{\infty}}\right\rangle$ are homogeneous power series such that the coefficients converge to zero on the unit disc after infinite differentiation.

$$\sum_I c_I X^I, \quad c_I \in K, \quad \lim_{|I|\to\infty} |c_I| = 0, \quad \text{where } I \text{ is multi-index} \tag{10.3}$$

The differential forms on the affine pieces are defined via pull back from the common intersection.

10.1 Čech Complex

Let us build a Čech Complex following of [Hartshorne, 1977, Example 4.0.3 pp 219-220]. Let Ω be the sheaf of differentials and $XY = 1$

$$C^0 = \Gamma(U, \Omega) \times \Gamma(V, \Omega) \tag{10.4}$$

$$C^1 = \Gamma(U \cap V, \Omega) \tag{10.5}$$

$$\Gamma(U, \Omega) = K \left\langle X^{1/p^\infty} \right\rangle dX \tag{10.6}$$

$$\Gamma(V, \Omega) = K \left\langle Y^{1/p^\infty} \right\rangle dY \tag{10.7}$$

$$\Gamma(U \cap V, \Omega) = K \left\langle X^{1/p^\infty}, X^{-1/p^\infty} \right\rangle dX \tag{10.8}$$

$$\tag{10.9}$$

The differential $d : C^0 \to C^1$ is given by the map

$$X \mapsto X \tag{10.10}$$

$$Y \mapsto \frac{1}{X} \tag{10.11}$$

$$dY \mapsto -\frac{1}{X^2} dX \tag{10.12}$$

$$\tag{10.13}$$

To compute $\check{H}^0$ we need to compute Ker d, which is a pair $(f(X)dX, g(Y)dY)$ such that

$$f(X) = -\frac{1}{X^2} g\left(\frac{1}{X}\right) \tag{10.14}$$

Left hand side is a series in $K \left\langle X^{1/p^\infty} \right\rangle$ and the Right hand side is a series in $K \left\langle X^{-1/p^\infty} \right\rangle \cdot X^{-2}$ and the equality would mean that $f = 0 = g$. Thus $\check{H}^0 = 0$.

$$\check{H}^1 = \frac{K\left\langle X^{\pm 1/p^\infty}\right\rangle}{\operatorname{Im} d} \text{ such that} \tag{10.15}$$

$$\operatorname{Im} d = \left(f(X) + \frac{1}{X^2} g\left(\frac{1}{X}\right)\right) dX \tag{10.16}$$

(10.17)

We can order the basis using first quadrant for the terms coming from $K\left\langle X^{1/p^\infty}\right\rangle$ and using the second quadrant for the terms coming from $K\left\langle X^{-1/p^\infty}\right\rangle$ as given in 6.1.1. We then see that $\operatorname{Im} d$ is missing the term corresponding to $X^{-i}, i \in (-2, 0) \cap \mathbb{Z}[1/p]$. Therefore $\check{H}^1$ is infinite dimensional generated by the image of $X^{-i} dX$.

10.2 Riemann Roch

The Riemann Roch theorem critically depends upon the relation between Euler Characteristic of SES of sheaves.

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \tag{10.18}$$

$$\chi(G) = \chi(F) + \chi(H) \tag{10.19}$$

But, we have infinite dimensional spaces for cohomology groups, which in turn makes Euler Characteristic infinite. Hence, we cannot get standard Reimann-Roch equation for projective perfectoid.

10.3 Euler Sequence

We want to show that that an Euler sequence exists on $\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}$ similar to the Euler sequence on $\mathbb{P}^n$. We can consider $\mathcal{O}(-1)$ as a line bundle whose elements have deg equal to -1. We have defined degree of elements in $K\left\langle X^{\pm 1/p^\infty}\right\rangle$ in 6.2.1.

Theorem 10.3.1. *The sheaf of differentials $\Omega_{\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}}$ satisfies the following*

$$0 \to \Omega_{\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}} \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}} \to 0 \tag{10.20}$$

Proof. For the sake of clarity we will consider $n = 2$, the proof of the general case is similar. We will follow page 578 [Vakil, 2015]. First we describe a degree one map $\mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}$.

$$\mathcal{O}(-1)^{\oplus(3)} \to \mathcal{O}_{\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}} \tag{10.21}$$

$$(s_0, s_1, s_2) \mapsto x_0 s_0 + x_1 s_1 + x_2 s_2 \tag{10.22}$$

We now want to describe the Kernel of the map above as sheaf of differentials. We consider the open set U_0 where $x_0 \neq 0$, and co-ordinates $x_{1/0} := x_1/x_0$ and $x_{2/0} := x_2/x_0$. We define the following injective map over the open set U_0

$$f_1 dx_{1/0} + f_2 dx_{2/0} \mapsto \left(\frac{-1}{x_0^2}(x_1 f_1 + x_2 f_2), \frac{1}{x_0} f_1, \frac{1}{x_0} f_2 \right) \tag{10.23}$$

This map lies in the kernel of $\mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}$. In fact this map is surjective. For any (g_0, g_1, g_2) in the Kernel take $f_1 = x_0 g_1$ and $f_2 = x_0 g_2$ to get

$$f_1 dx_{1/0} + f_2 dx_{2/0} \mapsto \left(\frac{-1}{x_0}(x_1 g_1 + x_2 g_2), g_1, g_2 \right) \tag{10.24}$$

which maps to zero when 10.21 is applied.

Let us verify the construction applied to co-ordinate patches U_0 and U_1. We will pass from $dx_{i/0}, i = 1, 2$ to $dx_{j/1}, j = 0, 2$

$$\begin{aligned} f_1 dx_{1/0} + f_2 dx_{2/0} &= f_1 d\frac{1}{x_{0/1}} + f_2 d\frac{x_{2/1}}{x_{0/1}} \\ &= -\frac{f_1 + f_2 x_{2/1}}{x_{0/1}^2} dx_{0/1} + \frac{f_2}{x_{0/1}} dx_{2/1} \\ &= \frac{-x_1}{x_0^2}(x_1 f_1 + x_2 f_2) dx_{0/1} + \frac{x_1}{x_0} f_2 dx_{2/1} \end{aligned} \tag{10.25}$$

In the co-ordinate patch $U_1 \neq 0$ with $x_1 \neq 0$ we have

$$g_0 dx_{0/1} + g_2 dx_{2/1} \mapsto \left(\frac{1}{x_1} g_0, \frac{-1}{x_1^2}(x_0 g_0 + x_2 g_2), \frac{1}{x_1} g_2 \right) \tag{10.26}$$

We define g_0, g_1 with reference to 10.25.

$$
\begin{aligned}
g_0 &:= \frac{-x_1}{x_0^2}(x_1 f_1 + x_2 f_2) \\
g_2 &:= \frac{x_1}{x_0} f_2 \\
\frac{1}{x_1} g_0 &= \frac{-1}{x_0^2}(x_1 f_1 + x_2 f_2) \\
\frac{-1}{x_1^2}(x_0 g_0 + x_2 g_2) &= \frac{-1}{x_1^2}\left(\frac{-x_1}{x_0}(x_1 f_1 + x_2 f_2) + \frac{x_2 x_1}{x_0} f_2\right) \\
&= \frac{1}{x_0} f_1 \\
\frac{1}{x_1} g_2 &= \frac{1}{x_0} f_2
\end{aligned}
$$

We see that the transition between patches works by mapping to (10.23).

(10.27)

$$
\left(\frac{1}{x_1} g_0, \frac{-1}{x_1^2}(x_0 g_0 + x_2 g_2), \frac{1}{x_1} g_2\right) \mapsto \left(\frac{-1}{x_0^2}(x_1 f_1 + x_2 f_2), \frac{1}{x_0} f_1, \frac{1}{x_0} f_2\right)
$$

□

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